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DOUBLY-RESONANT SADDLE-NODES IN $(\mathbb{C}^3, 0)$ AND THE FIXED SINGULARITY AT INFINITY IN THE PAINLEVÉ EQUATIONS. PART III: LOCAL ANALYTIC CLASSIFICATION

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ABSTRACT. In this work, following [Bit15] and [Bit16a], we consider analytic singular vector fields in \mathbb{C}^3 with an isolated and doubly-resonant singularity of saddle-node type at the origin. Such vector fields come from irregular two-dimensional differential systems with two opposite non-zero eigenvalues, and appear for instance when studying the irregular singularity at infinity in Painlevé equations $(P_j)_{j=I\dots V}$, for generic values of the parameters. Under suitable assumptions, we provide an analytic classification under the action of fibered diffeomorphisms, based on the study of the Stokes diffeomorphisms obtained by comparing consecutive sectorial normalizing maps *à la* Martinet-Ramis / Stolovitch [MR82, Sto96]. These normalizing maps over sectorial domains are obtained in the main theorem of [Bit16a], which is analogous to the classical one due to Hukuhara-Kimura-Matuda [HKM61] for saddle-nodes in \mathbb{C}^2 . We also prove that these maps are in fact the Gevrey-1 sums of the formal normalizing map, the existence of which has been proved in [Bit15].

1. INTRODUCTION

As in [Bit15] and [Bit16a], we consider (germs of) singular vector fields Y in \mathbb{C}^3 which can be written in appropriate coordinates $(x, \mathbf{y}) := (x, y_1, y_2)$ as

$$(1.1) \quad Y = x^2 \frac{\partial}{\partial x} + \left(-\lambda y_1 + F_1(x, \mathbf{y}) \right) \frac{\partial}{\partial y_1} + \left(\lambda y_2 + F_2(x, \mathbf{y}) \right) \frac{\partial}{\partial y_2} ,$$

where $\lambda \in \mathbb{C}^*$ and F_1, F_2 are germs of holomorphic function in $(\mathbb{C}^3, 0)$ of homogeneous valuation (order) at least two. They represent irregular two-dimensional differential systems having two opposite non-zero eigenvalues and a vanishing third eigenvalue. These we call doubly-resonant vector fields of saddle-node type (or simply **doubly-resonant saddle-nodes**). For a historical context, a presentation of the main motivations (the study of the irregular singularity at infinity in Painlevé equations $(P_j)_{j=I\dots V}$), and a review of some results linked with this study, we refer to the introduction of [Bit16a].

In this paper we provide an analytic classification under the action of fibered diffeomorphisms for a specific (to be defined later on) class of doubly-resonant saddle-nodes which contains the Painlevé case.

Several authors studied the problem of convergence of formal transformations putting vector fields as in (1.1) into “normal forms”. Shimomura, improving on a result of Iwano [Iwa80], shows in [Shi83] that analytic doubly-resonant saddle-nodes satisfying more restrictive conditions are conjugate (formally and over sectors) to vector fields of the form

$$x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x) y_1 \frac{\partial}{\partial y_1} + (\lambda + a_2 x) y_2 \frac{\partial}{\partial y_2}$$

via a diffeomorphism whose coefficients have asymptotic expansions as $x \rightarrow 0$ in sectors of opening greater than π .

Stolovitch then generalized this result to any dimension in [Sto96]. More precisely, Stolovitch’s work offers an analytic classification of vector fields in \mathbb{C}^{n+1} with an irregular singular point, without

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further hypothesis on eventual additional resonance relations between eigenvalues of the linear part. However, as Iwano and Shimomura did, he needed to impose other assumptions, among which the condition that the restriction of the vector field to the invariant hypersurface $\{x = 0\}$ is a linear vector field. In [BDM08], the authors obtain a *Gevrey-1 summable* “normal form”, though not as simple as Stolovitch’s one and not unique *a priori*, but for more general kind of vector field with one zero eigenvalue. However, the same assumption on hypersurface $\{x = 0\}$ is required (the restriction is a linear vector field). Yet from [Yos85] (and later [Bit15]) it stems the fact that this condition is not met in the case of Painlevé equations $(P_j)_{j=I\dots V}$.

In comparison, we merely ask here that this restriction be orbitally linearizable (see Definition 1.6), *i.e.* the foliation induced by Y on $\{x = 0\}$ (and not the vector field $Y|_{\{x=0\}}$ itself) be linearizable. The fact that this condition is fulfilled by the singularities of Painlevé equations formerly described is well-known. As discussed in Remark 1.16, this more general context also introduces new phenomena and technical difficulties as compared to prior classification results.

1.1. Scope of the paper.

The action of local analytic / formal diffeomorphisms Ψ fixing the origin on local holomorphic vector fields Y of type (1.1) by change of coordinates is given by

$$\Psi_* Y := (D\Psi.Y) \circ \Psi^{-1} .$$

In [Bit15] we performed the formal classification of such vector fields by exhibiting an explicit universal family of vector fields for the action of formal changes of coordinates at 0 (called a family of normal forms). Such a result seems currently out of reach in the analytic category: it is unlikely that an explicit universal family for the action of local analytic changes of coordinates be described anytime soon. If we want to describe the space of equivalent classes (of germs of a doubly-resonant saddle-node under local analytic changes of coordinates) with same formal normal form, we therefore need to find a complete set of invariants which is of a different nature. We call **moduli space** this quotient space and give it a (non-trivial) presentation based on functional invariants *à la* Martinet-Ramis [MR82, MR83].

In this paper we will therefore present only the x -fibred local analytic classification for vector fields of the form (1.1), with some additional assumptions detailed further down (see Definitions 1.1, 1.2 and 1.6). Importantly, these hypothesis are met in the case of Painlevé equations mentioned above. The full analytic classification (under the action of all local diffeomorphisms, not necessarily x -fibred) will be done in a forthcoming work.

In [Bit16a], we have proved the existence of analytic sectorial normalizing maps (over a pair of opposite “wide” sectors of opening greater than π whose union covers a full punctured neighborhood of $\{x = 0\}$). Then we attach to each vector field a complete set of invariants given as transition maps (over “narrow” sectors of opening less than π) between the sectorial normalizing maps. Although this viewpoint has become classical since the work of Martinet and Ramis, and has latter been generalized by Stolovitch as already mentioned, our approach has a more geometric flavor (for instance, we perform a precise study of the Stokes diffeomorphisms in the *space of leaves*).

As a by-product, we deduce that the normalizing sectorial diffeomorphisms are Gevrey-1 asymptotic to the normalizing formal power series of [Bit15], retrospectively proving their 1-summability. When the vector field additionally supports a symplectic transverse structure (which is again the case of Painlevé equations) we prove a theorem of analytic classification under the action of *transversally symplectic* diffeomorphisms.

1.2. Definitions and previous results.

To state our main results we need to introduce some notations and nomenclature.

- For $n \in \mathbb{N}_{>0}$, we denote by $(\mathbb{C}^n, 0)$ an (arbitrary small) open neighborhood of the origin in \mathbb{C}^n .
- We denote by $\mathbb{C}\{x, \mathbf{y}\}$, with $\mathbf{y} = (y_1, y_2)$, the \mathbb{C} -algebra of germs of holomorphic functions at the origin of \mathbb{C}^3 , and by $\mathbb{C}\{x, \mathbf{y}\}^\times$ the group of invertible elements for the multiplication (also called units), *i.e.* elements U such that $U(0) \neq 0$.

- $\chi(\mathbb{C}^3, 0)$ is the Lie algebra of germs of singular holomorphic vector fields at the origin \mathbb{C}^3 . Any vector field in $\chi(\mathbb{C}^3, 0)$ can be written as

$$Y = b(x, y_1, y_2) \frac{\partial}{\partial x} + b_1(x, y_1, y_2) \frac{\partial}{\partial y_1} + b_2(x, y_1, y_2) \frac{\partial}{\partial y_2}$$

with $b, b_1, b_2 \in \mathbb{C}\{x, y_1, y_2\}$ vanishing at the origin.

- $\text{Diff}(\mathbb{C}^3, 0)$ is the group of germs of holomorphic diffeomorphisms fixing the origin of \mathbb{C}^3 . It acts on $\chi(\mathbb{C}^3, 0)$ by conjugacy: for all

$$(\Phi, Y) \in \text{Diff}(\mathbb{C}^3, 0) \times \chi(\mathbb{C}^3, 0)$$

we define the push-forward of Y by Φ by

$$(1.2) \quad \Phi_*(Y) := (D\Phi \cdot Y) \circ \Phi^{-1} \quad ,$$

where $D\Phi$ is the Jacobian matrix of Φ .

- $\text{Diff}_{\text{fib}}(\mathbb{C}^3, 0)$ is the subgroup of $\text{Diff}(\mathbb{C}^3, 0)$ of fibered diffeomorphisms preserving the x -coordinate, *i.e.* of the form $(x, \mathbf{y}) \mapsto (x, \phi(x, \mathbf{y}))$.
- We denote by $\text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id})$ the subgroup of $\text{Diff}_{\text{fib}}(\mathbb{C}^3, 0)$ formed by diffeomorphisms tangent to the identity.

All these concepts have *formal* analogues, where we only suppose that the objects are defined with formal power series, not necessarily convergent near the origin.

Definition 1.1. A **diagonal doubly-resonant saddle-node** is a vector field $Y \in \chi(\mathbb{C}^3, 0)$ of the form

$$(1.3) \quad Y = x^2 \frac{\partial}{\partial x} + \left(-\lambda y_1 + F_1(x, \mathbf{y}) \right) \frac{\partial}{\partial y_1} + \left(\lambda y_2 + F_2(x, \mathbf{y}) \right) \frac{\partial}{\partial y_2} \quad ,$$

with $\lambda \in \mathbb{C}^*$ and $F_1, F_2 \in \mathbb{C}\{x, \mathbf{y}\}$ of order at least two. We denote by $\mathcal{SN}_{\text{diag}}$ the set of such vector fields.

Based on this expression, and considering the expansion

$$F_j(x, \mathbf{y}) = \sum_{\mathbf{k}=(k_0, k_1, k_2)} F_{j, \mathbf{k}} x^{k_0} y_1^{k_1} y_2^{k_2}$$

for $j = 1, 2$, we state:

Definition 1.2. The **residue** of $Y \in \mathcal{SN}_{\text{diag}}$ as in (1.3) is the complex number

$$\text{res}(Y) := F_{1, (1, 1, 0)} + F_{2, (1, 0, 1)} \quad .$$

We say that Y is **non-degenerate** (*resp.* **strictly non-degenerate**) if $\text{res}(Y) \notin \mathbb{Q}_{\leq 0}$ (*resp.* $\Re(\text{res}(Y)) > 0$).

Remark 1.3. It is obvious that there is an action of $\text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id})$ on $\mathcal{SN}_{\text{diag}}$. The residue is an invariant of each orbit of $\mathcal{SN}_{\text{diag}}$ under the action of $\text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id})$ by conjugacy (it is actually invariant by formal conjugacies, see [Bit15]).

The main result of [Bit15] can now be stated as follows:

Theorem 1.4. [Bit15] *Let $Y \in \mathcal{SN}_{\text{diag}}$ be non-degenerate. Then there exists a unique formal fibered diffeomorphism $\hat{\Phi}$ tangent to the identity such that:*

$$(1.4) \quad \begin{aligned} \hat{\Phi}_*(Y) &= x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x + c_1(y_1 y_2)) y_1 \frac{\partial}{\partial y_1} \\ &+ (\lambda + a_2 x + c_2(y_1 y_2)) y_2 \frac{\partial}{\partial y_2} \quad , \end{aligned}$$

where $\lambda \in \mathbb{C}^*$, $c_1, c_2 \in v\mathbb{C}[[v]]$ are formal power series in $v = y_1 y_2$ without constant term and $a_1, a_2 \in \mathbb{C}$ are such that $a_1 + a_2 = \text{res}(Y) \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0}$.

Definition 1.5. The vector field obtained in (1.4) is called the **formal normal form** of Y . The formal fibered diffeomorphism $\hat{\Phi}$ is called the **formal normalizing map** of Y .

The above result is valid for formal objects, without considering problems of convergence. The main result in [Bit16a] states that this formal normalizing map is analytic in sectorial domains, under some additional assumptions that we are now going to precise.

Definition 1.6.

- We say that a germ of a vector field X in $(\mathbb{C}^2, 0)$ is **orbitally linear** if

$$X = U(\mathbf{y}) \left(\lambda_1 y_1 \frac{\partial}{\partial y_1} + \lambda_2 y_2 \frac{\partial}{\partial y_2} \right),$$

for some $U(\mathbf{y}) \in \mathbb{C}\{\mathbf{y}\}^\times$ and $(\lambda_1, \lambda_2) \in \mathbb{C}^2$.

- We say that a germ of vector field X in $(\mathbb{C}^2, 0)$ is analytically (*resp.* formally) **orbitally linearizable** if X is analytically (*resp.* formally) conjugate to an orbitally linear vector field.
- We say that a diagonal doubly-resonant saddle-node $Y \in \mathcal{SN}_{\text{diag}}$ is **asymptotically orbitally linearizable** if $Y|_{\{x=0\}} \in \chi(\mathbb{C}^2, 0)$ is (analytically) orbitally linearizable.

Remark 1.7. Alternatively we could say that the foliation associated to $Y|_{\{x=0\}}$ is linearizable. Since $Y|_{\{x=0\}}$ is analytic at the origin of \mathbb{C}^2 and has two opposite eigenvalues, it follows from a classical result of Brjuno (see [Mar81]), that $Y|_{\{x=0\}}$ is analytically orbitally linearizable if and only if it is formally orbitally linearizable.

Definition 1.8. We denote by $\mathcal{SN}_{\text{diag},0}$ the set of strictly non-degenerate diagonal doubly-resonant saddle-nodes which are asymptotically orbitally linearizable.

The main result of [Bit16a] can now be stated (we refer to section 2. for precise definitions).

Theorem 1.9. *Let $Y \in \mathcal{SN}_{\text{diag},0}$ and let $\hat{\Phi}$ (given by Theorem 1.4) be the unique formal fibered diffeomorphism tangent to the identity such that*

$$\begin{aligned} \hat{\Phi}_*(Y) &= x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x + c_1(y_1 y_2)) y_1 \frac{\partial}{\partial y_1} + (\lambda + a_2 x + c_2(y_1 y_2)) y_2 \frac{\partial}{\partial y_2} \\ &=: Y_{\text{norm}}, \end{aligned}$$

where $\lambda \neq 0$ and $c_1(v), c_2(v) \in v\mathbb{C}[[v]]$ are formal power series without constant term. Then:

- (1) the normal form Y_{norm} is analytic (i.e. $c_1, c_2 \in \mathbb{C}\{v\}$), and it also is orbitally linear, i.e. $c_1 + c_2 = 0$;
- (2) the formal normalizing map $\hat{\Phi}$ is weakly 1-summable in every direction $\theta \in \mathbb{R}$ such that $\theta \neq \arg(\lambda) \pmod{\pi}$;
- (3) there exist analytic sectorial fibered diffeomorphisms Φ_+ and Φ_- , (asymptotically) tangent to the identity, defined in sectorial domains of the form $S_+ \times (\mathbb{C}^2, 0)$ and $S_- \times (\mathbb{C}^2, 0)$ respectively, where

$$\begin{aligned} S_+ &:= \left\{ x \in \mathbb{C} \mid 0 < |x| < r \text{ and } \left| \arg\left(\frac{x}{i\lambda}\right) \right| < \frac{\pi}{2} + \epsilon \right\} \\ S_- &:= \left\{ x \in \mathbb{C} \mid 0 < |x| < r \text{ and } \left| \arg\left(\frac{-x}{i\lambda}\right) \right| < \frac{\pi}{2} + \epsilon \right\} \end{aligned}$$

(for any $\epsilon \in]0, \frac{\pi}{2}[$ and some $r > 0$ small enough), which admit $\hat{\Phi}$ as weak Gevrey-1 asymptotic expansion in these respective domains, and which conjugate Y to Y_{norm} . Moreover Φ_+ and Φ_- are the unique such germs of analytic functions in sectorial domains (see Definition 2.2).

Definition 1.10. We call Φ_+ and Φ_- the **sectorial normalizing maps** of $Y \in \mathcal{SN}_{\text{diag},0}$.

They are the weak 1-sums of $\hat{\Phi}$ along the respective directions $\arg(i\lambda)$ and $\arg(-i\lambda)$. Notice that Φ_+ and Φ_- are *germs of analytic sectorial fibered diffeomorphisms*, i.e. they are of the form

$$\begin{aligned} \Phi_+ : S_+ \times (\mathbb{C}^2, 0) &\longrightarrow S_+ \times (\mathbb{C}^2, 0) \\ (x, \mathbf{y}) &\longmapsto (x, \Phi_{+,1}(x, \mathbf{y}), \Phi_{+,2}(x, \mathbf{y})) \end{aligned}$$

and

$$\begin{aligned} \Phi_- : S_- \times (\mathbb{C}^2, 0) &\longrightarrow S_- \times (\mathbb{C}^2, 0) \\ (x, \mathbf{y}) &\longmapsto (x, \Phi_{-,1}(x, \mathbf{y}), \Phi_{-,2}(x, \mathbf{y})) \end{aligned}$$

(see section 2. for a precise definition of *germ of analytic sectorial fibered diffeomorphism*). The fact that they are also (*asymptotically*) *tangent to the identity* means that we have:

$$\Phi_{\pm}(x, \mathbf{y}) = \text{Id} + O\left(\|(x, \mathbf{y})\|^2\right).$$

Another result proved in [Bit16a], is that the uniqueness of the sectorial normalizing maps holds in fact under weaker assumptions.

Proposition 1.11. *Let φ_+ and φ_- be two germs of sectorial fibered diffeomorphisms in $S_+ \times (\mathbb{C}^2, 0)$ and $S_- \times (\mathbb{C}^2, 0)$ respectively, where S_+ and S_- are as in Theorem 1.9, which are (*asymptotically*) *tangent to the identity* and such that*

$$(\varphi_{\pm})_*(Y) = Y_{\text{norm}}.$$

Then, they necessarily coincide with the weak 1-sums Φ_+ and Φ_- defined above.

1.3. Main results.

The first main result of this paper is the following.

Theorem 1.12. *Let $Y \in \mathcal{SN}_{\text{diag},0}$ and let $\hat{\Phi}$ (given by Theorem 1.4) be the unique formal fibered diffeomorphism tangent to the identity such that*

$$\begin{aligned} \hat{\Phi}_*(Y) &= x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x - c(y_1 y_2)) y_1 \frac{\partial}{\partial y_1} + (\lambda + a_2 x + c(y_1 y_2)) y_2 \frac{\partial}{\partial y_2} \\ &=: Y_{\text{norm}}, \end{aligned}$$

where $\lambda \neq 0$ and $c(v) \in v\mathbb{C}\{v\}$. Then $\hat{\Phi}$ is 1-summable (with respect to x) in every direction $\theta \in \mathbb{R}$ such that $\theta \neq \arg(\lambda) \pmod{\pi}$, and Φ_+, Φ_- in Theorem 1.9 are the 1-sums of $\hat{\Phi}$ in directions $\arg(i\lambda), \arg(-i\lambda)$ respectively.

Since two analytically conjugate vector fields are also formally conjugate, we fix now a normal form

$$Y_{\text{norm}} = x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x - c(v)) y_1 \frac{\partial}{\partial y_1} + (\lambda + a_2 x + c(v)) y_2 \frac{\partial}{\partial y_2},$$

with $\lambda \in \mathbb{C}^*$, $\Re(a_1 + a_2) > 0$ and $c \in v\mathbb{C}\{v\}$ vanishing at the origin.

Definition 1.13. We denote by $[Y_{\text{norm}}]$ the set of germs of holomorphic doubly-resonant saddle-nodes in $(\mathbb{C}^3, 0)$ which are formally conjugate to Y_{norm} by formal fibered diffeomorphisms tangent to the identity, and denote by $[Y_{\text{norm}}] / \text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id})$ the set of orbits of the elements in this set under the action of $\text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id})$.

According to Theorem 1.9, to any $Y \in [Y_{\text{norm}}]$ we can associate two sectorial normalizing maps Φ_+, Φ_- , which can in fact extend analytically in domains $S_+ \times (\mathbb{C}^2, 0)$ and $S_- \times (\mathbb{C}^2, 0)$, where S_{\pm} is an asymptotic sector in the direction $\arg(\pm i\lambda)$ with opening 2π (see Definition 2.3):

$$(S_+, S_-) \in \mathcal{AS}_{\arg(i\lambda), 2\pi} \times \mathcal{AS}_{\arg(-i\lambda), 2\pi}.$$

Then, we consider two germs of sectorial fibered diffeomorphisms $\Phi_\lambda, \Phi_{-\lambda}$ analytic in $S_\lambda, S_{-\lambda}$, with

$$(1.5) \quad \begin{aligned} S_\lambda &:= S_+ \cap S_- \cap \left\{ \Re \left(\frac{x}{\lambda} \right) > 0 \right\} \in \mathcal{AS}_{\arg(\lambda), \pi} \\ S_{-\lambda} &:= S_+ \cap S_- \cap \left\{ \Re \left(\frac{x}{\lambda} \right) < 0 \right\} \in \mathcal{AS}_{\arg(-\lambda), \pi} , \end{aligned}$$

defined by:

$$\begin{cases} \Phi_\lambda := (\Phi_+ \circ \Phi_-^{-1})|_{S_\lambda \times (\mathbb{C}^2, 0)} \in \text{Diff}_{\text{fib}}(\mathcal{S}_{\arg(\lambda), \epsilon}, \text{Id}) & , \forall \epsilon \in [0, \pi[\\ \Phi_{-\lambda} := (\Phi_- \circ \Phi_+^{-1})|_{S_{-\lambda} \times (\mathbb{C}^2, 0)} \in \text{Diff}_{\text{fib}}(\mathcal{S}_{\arg(-\lambda), \epsilon}, \text{Id}) & , \forall \epsilon \in [0, \pi[. \end{cases}$$

Notice that $\Phi_\lambda, \Phi_{-\lambda}$ are *isotropies* of Y_{norm} , *i.e.* they satisfy:

$$(\Phi_{\pm\lambda})_*(Y_{\text{norm}}) = Y_{\text{norm}} .$$

Definition 1.14. With the above notations, we define $\Lambda_\lambda(Y_{\text{norm}})$ (*resp.* $\Lambda_{-\lambda}(Y_{\text{norm}})$) as the group of germs of sectorial fibered isotropies of Y_{norm} , tangent to the identity, and admitting the identity as Gevrey-1 asymptotic expansion (see Definition 2.4) in sectorial domains of the form $S_\lambda \times (\mathbb{C}^2, 0)$ (*resp.* $S_{-\lambda} \times (\mathbb{C}^2, 0)$), with $S_{\pm\lambda} \in \mathcal{AS}_{\arg(\pm\lambda), \pi}$.

The two sectorial isotropies Φ_λ and $\Phi_{-\lambda}$ defined above are called the **Stokes diffeomorphisms** associate to $Y \in [Y_{\text{norm}}]$.

Our second main result gives the moduli space for the analytic classification that we are looking for.

Theorem 1.15. *The map*

$$\begin{aligned} [Y_{\text{norm}}] / \text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id}) &\longrightarrow \Lambda_\lambda(Y_{\text{norm}}) \times \Lambda_{-\lambda}(Y_{\text{norm}}) \\ Y &\longmapsto (\Phi_\lambda, \Phi_{-\lambda}) \end{aligned}$$

is well-defined and bijective.

In particular, the result states that Stokes diffeomorphisms only depend on the class of $Y \in [Y_{\text{norm}}]$ in the quotient $[Y_{\text{norm}}] / \text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id})$. We will give a description of this set of invariants in terms of power series in the *space of leaves* in section 4.

Remark 1.16. In [Bit16a] we have proved a theorem of sectorial normalizing map analogous to the classical one due to Hukuhara-Kimura-Matuda for saddle-nodes in $(\mathbb{C}^2, 0)$ [HKM61], generalized later by Stolovitch in any dimension in [Sto96]. Unlike the method based on a fixed point theorem used by these authors, we have used a more geometric approach (following the works of Teyssier [Tey03, Tey04]) based on the resolution of an homological equation, by integrating a well chosen 1-form along asymptotic paths. This latter approach turned out to be more efficient to deal with the fact that $Y_{\{x=0\}}$ is not necessarily linearizable. Indeed, if we try to adapt the proof of [Sto96], one of the first new main difficulties is that in the irregular systems that needs to be solved by a fixed point method (for instance equation (2.7) in the cited paper), the non-linear terms would not be divisible by the independent variable (*i.e.* the time) in our situation. This would complicate the different estimates that are done later in the cited work. This was the first main new phenomena we have met.

In contrast to a result of [Bit16a] which states that the only sectorial isotropy (tangent to the identity) of the normal form over wide sectors (of opening $> \pi$) is the identity, we will see here that the situation is rather different over sector with narrow opening. In order to prove both Theorems 1.12 and 1.15, we will show that the Stokes diffeomorphisms Φ_λ and $\Phi_{-\lambda}$ obtained from the germs of sectorial normalizing maps Φ_+ and Φ_- , admit the identity as Gevrey-1 asymptotic expansion. In the cited reference we were only able to establish that fact with a weaker notion of Gevrey-1 expansion. The main difficulty is to prove that such a sectorial isotropy of Y_{norm} over the “narrow” sectors described above is necessarily exponentially close to the identity (see Proposition 3.5). This will be done by studying more generally germs of sectorial isotropies of the normal form Y_{norm} in sectorial domains

with “narrow” opening, and by considering these isotropies in the space of leaves. In fact, this is the second main new difficulties we have met, which is due to the presence of the “resonant” term

$$\frac{c_m (y_1 y_2)^m \log(x)}{x}$$

in the exponential expression of the first integrals of the vector field (see (4.1)). In [Sto96], similar computations are done in subsection 3.4.1. In this part of the paper, infinitely many irregular differential equations appear when identifying terms of same homogeneous degree. The fact that $Y|_{\{x=0\}}$ is linear implies that these differential equations are all linear and independent of each others. In our situation, this is not the case and then more complicated. Using a “non-abelian” version of the Ramis-Sibuya theorem due to Martinet and Ramis [MR82], we prove both sectorial normalizing maps Φ_+ and Φ_- admit the formal normalizing map $\hat{\Phi}$ as Gevrey-1 asymptotic expansion in the corresponding sectorial domains. This establishes the Gevrey-1 summability of $\hat{\Phi}$.

1.4. The transversally Hamiltonian case.

In order to motivate the following definition, we refer to [Bit16a] where the study of the Painlevé case is performed.

Definition 1.17. Consider the rational 1-form

$$\omega := \frac{dy_1 \wedge dy_2}{x}.$$

We say that vector field Y is **transversally Hamiltonian** (with respect to ω and dx) if

$$\mathcal{L}_Y(dx) \in \langle dx \rangle \quad \text{and} \quad \mathcal{L}_Y(\omega) \in \langle dx \rangle.$$

For any open sector $S \subset \mathbb{C}^*$, we say that a germ of sectorial fibered diffeomorphism Φ in $S \times (\mathbb{C}^2, 0)$ is **transversally symplectic** (with respect to ω and dx) if

$$\Phi^*(\omega) \in \omega + \langle dx \rangle$$

(here $\Phi^*(\omega)$ denotes the pull-back of ω by Φ).

We denote by $\text{Diff}_\omega(\mathbb{C}^3, 0, \text{Id})$ the group of transversally symplectic diffeomorphisms which are tangent to the identity.

Remark 1.18. A fibered sectorial diffeomorphism Φ is transversally symplectic if and only if $\det(D\Phi) = 1$.

Definition 1.19. A **transversally Hamiltonian doubly-resonant saddle-node** is a transversally Hamiltonian vector field which is conjugate, *via* a (fibered) transversally symplectic diffeomorphism, to one of the form

$$Y = x^2 \frac{\partial}{\partial x} + \left(-\lambda y_1 + F_1(x, y) \right) \frac{\partial}{\partial y_1} + \left(\lambda y_2 + F_2(x, y) \right) \frac{\partial}{\partial y_2},$$

with $\lambda \in \mathbb{C}^*$ and F_1, F_2 analytic in $(\mathbb{C}^3, 0)$ and of order at least 2.

Notice that a transversally Hamiltonian doubly-resonant saddle-node is necessarily strictly non-degenerate (since its residue is always equal to 1), and also asymptotically orbitally linearizable (see section 3).

We recall the second main result from [Bit15].

Theorem 1.20. [Bit15]

Let Y be a diagonal doubly-resonant saddle-node which is transversally Hamiltonian. Then, there exists a unique formal fibered transversally symplectic diffeomorphism $\hat{\Phi}$ tangent to identity such that:

$$\begin{aligned} \hat{\Phi}_*(Y) &= x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x - c(y_1 y_2)) y_1 \frac{\partial}{\partial y_1} + (\lambda + a_2 x + c(y_1 y_2)) y_2 \frac{\partial}{\partial y_2} \\ (1.6) \quad &=: Y_{\text{norm}}, \end{aligned}$$

where $\lambda \in \mathbb{C}^*$, $c(v) \in v\mathbb{C}[[v]]$ a formal power series in $v = y_1 y_2$ without constant term and $a_1, a_2 \in \mathbb{C}$ are such that $a_1 + a_2 = 1$.

The second main result in [Bit16a] is the following.

Theorem 1.21. *Let Y be a transversally Hamiltonian doubly-resonant saddle-node. Let $\hat{\Phi}$ be the unique formal normalizing map given by Theorem 1.20. Then the associate sectorial normalizing maps Φ_+ and Φ_- given by Theorem 1.9 are also transversally symplectic.*

Let us fix a normal form Y_{norm} as in Theorem 1.21, and consider two sectorial domains $S_\lambda \times (\mathbb{C}^2, 0)$ and $S_{-\lambda} \times (\mathbb{C}^2, 0)$ as in (1.5). Then, the Stokes diffeomorphisms $(\Phi_\lambda, \Phi_{-\lambda})$ defined in the previous subsection as

$$\begin{cases} \Phi_\lambda := (\Phi_+ \circ \Phi_-^{-1})|_{S_\lambda \times (\mathbb{C}^2, 0)} \\ \Phi_{-\lambda} := (\Phi_- \circ \Phi_+^{-1})|_{S_{-\lambda} \times (\mathbb{C}^2, 0)} \end{cases} ,$$

are also transversally symplectic.

Definition 1.22. We denote by $\Lambda_\lambda^\omega(Y_{\text{norm}})$ (*resp.* $\Lambda_{-\lambda}^\omega(Y_{\text{norm}})$) the group of germs of sectorial fibered isotropies of Y_{norm} , admitting the identity as Gevrey-1 asymptotic expansion in sectorial domains of the form $S_\lambda \times (\mathbb{C}^2, 0)$ (*resp.* $S_{-\lambda} \times (\mathbb{C}^2, 0)$), and which are transversally symplectic.

Let us denote by $[Y_{\text{norm}}]_\omega$ the set of germs of vector fields which are formally conjugate to Y_{norm} via (formal) transversally symplectic diffeomorphisms tangent to the identity. As a consequence of Theorems (1.15) and (1.21), we can now state our third main result:

Theorem 1.23. *The map*

$$\begin{aligned} [Y_{\text{norm}}]_\omega / \text{Diff}_\omega(\mathbb{C}^3, 0, \text{Id}) &\longrightarrow \Lambda_\lambda^\omega(Y_{\text{norm}}) \times \Lambda_{-\lambda}^\omega(Y_{\text{norm}}) \\ Y &\longmapsto (\Phi_\lambda, \Phi_{-\lambda}) \end{aligned}$$

is a well-defined bijection.

In a forthcoming paper we will compare our approach to the works of Joshi, Kapaev and Kitaev cited earlier in order to compute the Stokes diffeomorphisms for Painlevé equations $(\mathbb{P}_j)_{j=I\dots V}$.

1.5. Outline of the paper.

In section 2, we recall the different tools, notations and nomenclature we will need regarding asymptotic expansion, Gevrey-1 series, 1-summability and sectorial germs.

In section 3, we prove the main theorems presented above, assuming the Proposition 3.5 holds.

In section 4, we prove the key Proposition 3.5 by studying the automorphisms of the space of leaves.

In section 5, we give a description of the moduli space in Theorem 1.15 in terms of power series in the space of leaves, and present some applications.

In section 6, we present a generalization of Theorem 1.15 where we study the action of $\text{Diff}(\mathbb{C}^3, 0)$ instead of $\text{Diff}_{\text{fb}}(\mathbb{C}^3, 0)$.

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2. BACKGROUND

We refer the reader to [MR82], [Mal95] and [RS93] for a general and detailed introduction to the theory of asymptotic expansion, Gevrey series and summability (see also [Sto96] for a useful discussion of these concepts). We refer more precisely to [Bit16a] when it comes to the notion of *weak 1-summability*.

We call $x \in \mathbb{C}$ the *independent* variable and $\mathbf{y} := (y_1, \dots, y_n) \in \mathbb{C}^n$, $n \in \mathbb{N}$, the *dependent* variables. As usual we define $\mathbf{y}^{\mathbf{k}} := y_1^{k_1} \dots y_n^{k_n}$ for $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$, and $|\mathbf{k}| = k_1 + \dots + k_n$. The notions of asymptotic expansion, Gevrey series and 1-summability presented here are always considered with respect to the independent variable x living in (open) sectors

$$S(r, \alpha, \beta) = \{x \in \mathbb{C} \mid 0 < |x| < r \text{ and } \alpha < \arg(x) < \beta\} ,$$

the dependent variable \mathbf{y} belonging to poly-discs

$$\mathbf{D}(\mathbf{0}, \mathbf{r}) := \{\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{C}^n \mid |y_1| < r_1, \dots, |y_n| < r_n\} ,$$

of poly-radius $\mathbf{r} = (r_1, \dots, r_n) \in (\mathbb{R}_{>0})^n$. Given an open subset $\mathcal{U} \subset \mathbb{C}^{n+1}$, we denote by $\mathcal{O}(\mathcal{U})$ the algebra of holomorphic function in \mathcal{U} .

2.1. Sectorial germs.

Let $\theta \in \mathbb{R}$, $\eta \in \mathbb{R}_{\geq 0}$ and $n \in \mathbb{N}$.

Definition 2.1. (1) An x -sectorial neighborhood (or simply sectorial neighborhood) of the origin (in \mathbb{C}^{n+1}) in the direction θ with opening η is an open set $\mathcal{S} \subset \mathbb{C}^{n+1}$ such that

$$\mathcal{S} \supset S\left(r, \theta - \frac{\eta}{2} - \epsilon, \theta + \frac{\eta}{2} + \epsilon\right) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$$

for some $r > 0$, $\mathbf{r} \in (\mathbb{R}_{>0})^n$ and $\epsilon > 0$. We denote by $(\mathcal{S}_{\theta, \eta}, \leq)$ the directed set formed by all such neighborhoods, equipped with the order relation

$$S_1 \leq S_2 \iff S_1 \supset S_2 .$$

(2) The algebra of germs of holomorphic functions in a sectorial neighborhood of the origin in the direction θ with opening η is the direct limit

$$\mathcal{O}(\mathcal{S}_{\theta, \eta}) := \varinjlim \mathcal{O}(\mathcal{S})$$

with respect to the directed system defined by $\{\mathcal{O}(\mathcal{S}) : \mathcal{S} \in \mathcal{S}_{\theta, \eta}\}$.

We now give the definition of a (germ of a) sectorial diffeomorphism.

Definition 2.2. (1) Given an element $\mathcal{S} \in \mathcal{S}_{\theta, \eta}$, we denote by $\text{Diff}_{\text{fib}}(\mathcal{S}, \text{Id})$ the set of holomorphic diffeomorphisms of the form

$$\begin{aligned} \Phi : \mathcal{S} &\rightarrow \Phi(\mathcal{S}) \\ (x, \mathbf{y}) &\mapsto (x, \phi_1(x, \mathbf{y}), \phi_2(x, \mathbf{y})) , \end{aligned}$$

such that $\Phi(x, \mathbf{y}) - \text{Id}(x, \mathbf{y}) = O(\|x, \mathbf{y}\|^2)$, as $(x, \mathbf{y}) \rightarrow (0, \mathbf{0})$ in \mathcal{S} .¹

(2) The set of *germs of (fibered) sectorial diffeomorphisms in the direction θ with opening η , tangent to the identity*, is the direct limit

$$\text{Diff}_{\text{fib}}(\mathcal{S}_{\theta, \eta}, \text{Id}) := \varinjlim \text{Diff}_{\text{fib}}(\mathcal{S}, \text{Id})$$

with respect to the directed system defined by $\{\text{Diff}_{\text{fib}}(\mathcal{S}, \text{Id}) : \mathcal{S} \in \mathcal{S}_{\theta, \eta}\}$. We equip $\text{Diff}_{\text{fib}}(\mathcal{S}_{\theta, \eta}, \text{Id})$ of a group structure as follows: given two germs $\Phi, \Psi \in \text{Diff}_{\text{fib}}(\mathcal{S}_{\theta, \eta}, \text{Id})$ we chose corresponding representatives $\Phi_0 \in \text{Diff}_{\text{fib}}(\mathcal{S}, \text{Id})$ and $\Psi_0 \in \text{Diff}_{\text{fib}}(\mathcal{T}, \text{Id})$ with $\mathcal{S}, \mathcal{T} \in \mathcal{S}_{\theta, \eta}$ such that $\mathcal{T} \subset \Phi_0(\mathcal{S})$ and let $\Psi \circ \Phi$ be the germ defined by $\Psi_0 \circ \Phi_0$.²

We will also need the notion of *asymptotic sectors*.

Definition 2.3. An *(open) asymptotic sector of the origin in the direction θ and with opening η* is an open set $S \subset \mathbb{C}$ such that

$$S \in \bigcap_{0 \leq \eta' < \eta} \mathcal{S}_{\theta, \eta'} .$$

We denote by $\mathcal{AS}_{\theta, \eta}$ the set of all such (open) asymptotic sectors.

2.2. Gevrey-1 power series and 1-summability.

In this subsection we fix a formal power series

$$\hat{f}(x, \mathbf{y}) = \sum_{k \geq 0} f_k(\mathbf{y}) x^k = \sum_{(j_0, \mathbf{j}) \in \mathbb{N}^{n+1}} f_{j_0, \mathbf{j}} x^{j_0} \mathbf{y}^{\mathbf{j}} \in \mathbb{C}[[x, \mathbf{y}]] .$$

Definition 2.4.

- An analytic (and bounded) function f in a sectorial domain $S(r, \alpha, \beta) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ admits \hat{f} as *Gevrey-1 asymptotic expansion* in this domain, if for all closed sub-sector $S' \subset S(r, \alpha, \beta)$, there exists $A, C > 0$ such that:

$$\left| f(x, \mathbf{y}) - \sum_{k=0}^{N-1} f_k(\mathbf{y}) x^k \right| \leq AC^N (N!) |x|^N$$

for all $N \in \mathbb{N}$ and $(x, \mathbf{y}) \in S' \times \mathbf{D}(\mathbf{0}, \mathbf{r})$.

- A formal power series $\hat{f} \in \mathbb{C}[[x, \mathbf{y}]]$ is 1-summable in the direction θ if and only if there exists a germ of a sectorial holomorphic function $f_\theta \in \mathcal{O}(\mathcal{S}_{\theta, \pi})$ which admits \hat{f} as Gevrey-1 asymptotic expansion in some $S \in \mathcal{S}_{\theta, \pi}$.

Remark 2.5. In the definition above, f_θ is unique (as a germ in $\mathcal{O}(\mathcal{S}_{\theta, \pi})$), and is called the *1-sum of \hat{f} in the direction θ* .

Lemma 2.6. *The set $\Sigma_\theta \subset \mathbb{C}[[x, \mathbf{y}]]$ of 1-summable power series in the direction θ is an algebra closed under differentiation. Moreover the map*

$$\begin{aligned} \Sigma_\theta &\longrightarrow \mathcal{O}(\mathcal{S}_{\theta, \pi}) \\ \hat{f} &\longmapsto f_\theta \end{aligned}$$

is an injective morphism of differential algebras.

¹This condition implies in particular that $\Phi(\mathcal{S}) \in \mathcal{S}_{\theta, \eta}$.

²One can prove that this definition is independent of the choice of the representatives

Proposition 2.7. *Let $\hat{\Phi}(x, \mathbf{y}) \in \mathbb{C}[[x, \mathbf{y}]]$ be 1-summable in directions θ and $\theta - \pi$, and let $\Phi_+(x, \mathbf{y})$ and $\Phi_-(x, \mathbf{y})$ be its 1-sums directions θ and $\theta - \pi$ respectively. Let also $\hat{f}_1(x, \mathbf{z}), \dots, \hat{f}_n(x, \mathbf{z})$ be 1-summable in directions $\theta, \theta - \pi$, and $f_{1,+}, \dots, f_{n,+}$, and $f_{1,-}, \dots, f_{n,-}$ be their 1-sums in directions θ and $\theta - \pi$ respectively. Assume that*

$$(2.1) \quad \hat{f}_j(0, \mathbf{0}) = 0, \text{ for all } j = 1, \dots, n .$$

Then

$$\hat{\Psi}(x, \mathbf{z}) := \hat{\Phi}\left(x, \hat{f}_1(x, \mathbf{z}), \dots, \hat{f}_n(x, \mathbf{z})\right)$$

is 1-summable in directions $\theta, \theta - \pi$, and its 1-sum in the corresponding direction is

$$\Psi_{\pm}(x, \mathbf{z}) := \Phi_{\pm}(x, f_{1,\pm}(x, \mathbf{z}), \dots, f_{n,\pm}(x, \mathbf{z})) ,$$

which is a germ of a sectorial holomorphic function in directions θ and $\theta - \pi$.

Proof. See [Bit16a]. □

Consider \hat{Y} a formal singular vector field at the origin and a formal fibered diffeomorphism $\hat{\varphi} : (x, \mathbf{y}) \mapsto (x, \hat{\varphi}(x, \mathbf{y}))$. Assume that both \hat{Y} and $\hat{\varphi}$ are 1-summable in directions θ and $\theta - \pi$, for some $\theta \in \mathbb{R}$, and denote by Y_+, Y_- (resp. φ_+, φ_-) their 1-sums in directions θ and $\theta - \pi$ respectively. As a consequence of Proposition 2.7 and Lemma 2.6, we can state the following:

Corollary 2.8. *Under the assumptions above, $\hat{\varphi}_*\left(\hat{Y}\right)$ is 1-summable in both directions θ and $\theta - \pi$, and its 1-sums in these directions are $\varphi_+(Y_+)$ and $\varphi_-(Y_-)$ respectively.*

2.3. An important result by Martinet and Ramis.

We are going to make an essential use of an isomorphism theorem proved in [MR82]. This result is of paramount importance in the present paper since it will be a key tool in the proofs of both Theorems 1.9 and 1.15 (see section 3).

Let us consider two open asymptotic sectors \mathcal{S} and \mathcal{S}' at the origin in directions θ and $\theta - \pi$ respectively, both of opening π :

$$\begin{aligned} \mathcal{S} &\in \mathcal{AS}_{\theta, \pi} \\ \mathcal{S}' &\in \mathcal{AS}_{\theta - \pi, \pi} \end{aligned}$$

(see Definition 2.3). In this particular setting, the cited theorem can be stated as follows.

Theorem 2.9. [MR82, Théorème 5.2.1] *Consider a pair of germs of sectorial diffeomorphisms*

$$(\varphi, \varphi') \in \text{Diff}_{\text{fib}}(\mathcal{S}_{\theta, 0}, \text{Id}) \times \text{Diff}_{\text{fib}}(\mathcal{S}_{\theta - \pi, 0}, \text{Id})$$

such that φ and φ' extend analytically and admit the identity as Gevrey-1 asymptotic expansion in $S \times (\mathbb{C}^2, 0)$ and $S' \times (\mathbb{C}^2, 0)$ respectively. Then, there exists a pair (ϕ_+, ϕ_-) of germs of sectorial fibered diffeomorphisms

$$(\phi_+, \phi_-) \in \text{Diff}_{\text{fib}}(\mathcal{S}_{\theta + \frac{\pi}{2}, \eta}, \text{Id}) \times \text{Diff}_{\text{fib}}(\mathcal{S}_{\theta - \frac{\pi}{2}, \eta}, \text{Id})$$

with $\eta \in]\pi, 2\pi[$, which extend analytically in $S_+ \times (\mathbb{C}^2, 0)$ and $S_- \times (\mathbb{C}^2, 0)$ respectively, for some $S_+ \in \mathcal{AS}_{\theta + \frac{\pi}{2}, 2\pi}$ and $S_- \in \mathcal{AS}_{\theta - \frac{\pi}{2}, 2\pi}$, such that:

$$\begin{cases} \phi_+ \circ (\phi_-)^{-1}|_{S \times (\mathbb{C}^2, 0)} = \varphi \\ \phi_+ \circ (\phi_-)^{-1}|_{S' \times (\mathbb{C}^2, 0)} = \varphi' \end{cases} .$$

There also exists a formal diffeomorphism $\hat{\phi}$ which is tangent to the identity, such that ϕ_+ and ϕ_- both admit $\hat{\phi}$ as Gevrey-1 asymptotic expansion in $S_+ \times (\mathbb{C}^2, 0)$ and $S_- \times (\mathbb{C}^2, 0)$ respectively.

In particular, in the theorem above $\hat{\phi}$ is 1-summable in every direction except θ and $\theta - \pi$, and its 1-sums in directions $\theta + \frac{\pi}{2}$ and $\theta - \frac{\pi}{2}$ respectively are ϕ_+ and ϕ_- . For future use, we are going to prove a “transversally symplectic” version of the above theorem.

Corollary 2.10. *With the assumptions and notations of Theorem 2.9, if φ and φ' both are transversally symplectic (see Definition 1.17), then there exists a germ of an analytic fibered diffeomorphism $\psi \in \text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id})$ (tangent to the identity), such that*

$$\sigma_+ := \phi_+ \circ \psi \text{ and } \sigma_- := \phi_- \circ \psi$$

both are transversally symplectic. Moreover we also have:

$$\begin{cases} \sigma_+ \circ (\sigma_-)^{-1}|_{S \times (\mathbb{C}^2, 0)} = \varphi \\ \sigma_+ \circ (\sigma_-)^{-1}|_{S' \times (\mathbb{C}^2, 0)} = \varphi' \end{cases} .$$

Proof. We recall that for any germ φ of a sectorial fibered diffeomorphism which is tangent to the identity, φ is transversally symplectic if and only if $\det(\text{D}\varphi) = 1$.

First of all, let us show that

$$\det(\text{D}\phi_+) = \det(\text{D}\phi_-) \text{ in } (S_+ \cap S_-) \times (\mathbb{C}^2, 0) .$$

Since ϕ_+ and ϕ_- both are sectorial fibered diffeomorphism which are tangent to the identity and transversally symplectic, then

$$\det\left(\phi_+ \circ (\phi_-)^{-1}|_{(S_+ \cap S_-) \times (\mathbb{C}^2, 0)}\right) = 1 .$$

The *chain rule* implies immediately that

$$\det(\text{D}\phi_+) = \det(\text{D}\phi_-) \text{ in } (S_+ \cap S_-) \times (\mathbb{C}^2, 0) .$$

Thus, this equality allows us to define (thanks to the Riemann's Theorem of removable singularities) a germ of analytic function $f \in \mathcal{O}(\mathbb{C}^3, 0)$. Notice that $f(0, 0, 0) = 1$ because ϕ_+ and ϕ_- are tangent to the identity. Now, let us look for an element $\psi \in \text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id})$ of the form

$$(2.2) \quad \psi : (x, y_1, y_2) \mapsto (x, \psi_1(x, \mathbf{y}), y_2)$$

such that

$$\sigma_+ := \phi_+ \circ \psi \text{ and } \sigma_- := \phi_- \circ \psi$$

both be transversally symplectic. An easy computation gives:

$$\det(\sigma_{\pm}) = (\det(\text{D}\phi_{\pm}) \circ \psi) \det(\text{D}\psi) = 1$$

i.e.

$$(f \circ \psi) \det(\text{D}\psi) = 1 .$$

According to (2.2), we must have:

$$(2.3) \quad (f \circ \psi) \frac{\partial \psi_1}{\partial y_1} = 1 .$$

Let us define

$$F(x, y_1, y_2) := \int_0^{y_1} f(x, z, y_2) dz ,$$

so that (2.3) can be integrated as

$$F \circ \psi = y_1 + h(x, y_2) ,$$

for some $h \in \mathbb{C}\{x, y_2\}$. Notice that

$$\frac{\partial F}{\partial y_1}(0, 0, 0) = 1$$

since $f(0, 0, 0) = 1$. Let us chose $h = 0$. Then, we have to solve

$$F \circ \psi = y_1 ,$$

with unknown $\psi \in \text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id})$ as in (2.3). If we define

$$\Phi : (x, \mathbf{y}) \mapsto (x, F(x, \mathbf{y}), y_2) ,$$

the latter problem is equivalent to find ψ as above such that:

$$\Phi \circ \psi = \text{Id} .$$

Since $D\Phi_0 = \text{Id}$, the inverse function theorem gives us the existence of the germ $\psi = \Phi^{-1} \in \text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id})$. \square

2.4. 1-summability implies weakly 1-summability.

Any function $f(x, \mathbf{y})$ analytic in a domain $\mathcal{U} \times \mathbf{D}(\mathbf{0}, \mathbf{r})$, where $\mathcal{U} \subset \mathbb{C}$ is open, and bounded in any domain $\mathcal{U} \times \overline{\mathbf{D}}(\mathbf{0}, \mathbf{r}')$ with $r'_1 < r_1, \dots, r'_n < r_n$, can be written

$$(2.4) \quad f(x, \mathbf{y}) = \sum_{\mathbf{j} \in \mathbb{N}^n} F_{\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}} \quad ,$$

where for all $\mathbf{j} \in \mathbb{N}^n$, $F_{\mathbf{j}}$ is analytic and bounded on \mathcal{U} , and defined *via* the Cauchy formula:

$$F_{\mathbf{j}}(x) = \frac{1}{(2i\pi)^n} \int_{|z_1|=r'_1} \cdots \int_{|z_n|=r'_n} \frac{f(x, \mathbf{z})}{(z_1)^{j_1+1} \cdots (z_n)^{j_n+1}} dz_n \cdots dz_1 \quad .$$

Notice that the convergence of the series above is uniform on every compact with respect to x and \mathbf{y} .

In the same way, any formal power series $\hat{f}(x, \mathbf{y}) \in \mathbb{C}[[x, \mathbf{y}]]$ can be written as

$$\hat{f}(x, \mathbf{y}) = \sum_{\mathbf{j} \in \mathbb{N}^n} \hat{F}_{\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}} \quad .$$

We present here a weaker notion of 1-summability that we will also need.

Definition 2.11.

- A function

$$f(x, \mathbf{y}) = \sum_{\mathbf{j} \in \mathbb{N}^n} F_{\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}}$$

analytic and bounded in a domain $S(r, \alpha, \beta) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$, admits \hat{f} as **weak Gevrey-1 asymptotic expansion** in $x \in S(r, \alpha, \beta)$, if for all $\mathbf{j} \in \mathbb{N}^n$, $F_{\mathbf{j}}$ admits $\hat{F}_{\mathbf{j}}$ as Gevrey-1 asymptotic expansion in $S(r, \alpha, \beta)$.

- The formal power series \hat{f} is said to be **weakly 1-summable in the direction** $\theta \in \mathbb{R}$, if the following conditions hold:
 - for all $\mathbf{j} \in \mathbb{N}^n$, $\hat{F}_{\mathbf{j}}(x) \in \mathbb{C}[[x]]$ is 1-summable in the direction θ , whose 1-sum in the direction θ is denoted by $F_{\mathbf{j}}$;
 - the series $f_{\theta}(x, \mathbf{y}) := \sum_{\mathbf{j} \in \mathbb{N}^n} F_{\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}}$ defines a germ of a sectorial holomorphic function in a domain of the form

$$S\left(r, \theta - \frac{\pi}{2} - \epsilon, \theta + \frac{\pi}{2} + \epsilon\right) \times \mathbf{D}(\mathbf{0}, \mathbf{r}) \quad .$$

In this case, $f_{\theta}(x, \mathbf{y})$ is called **the weak 1-sum of \hat{f} in the direction θ** .

The following proposition is an analogue of Proposition 2.7 for weak 1-summable formal power series, with the a stronger condition instead of (2.1).

Proposition 2.12. *Let*

$$\hat{\Phi}(x, \mathbf{y}) = \sum_{\mathbf{j} \in \mathbb{N}^n} \hat{\Phi}_{\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}} \in \mathbb{C}[[x, \mathbf{y}]]$$

and

$$\hat{f}^{(k)}(x, \mathbf{y}) = \sum_{\mathbf{j} \in \mathbb{N}^n} \hat{F}_{\mathbf{j}}^{(k)}(x) \mathbf{y}^{\mathbf{j}} \in \mathbb{C}[[x, \mathbf{y}]] \quad ,$$

for $k = 1, \dots, n$, be $n+1$ formal power series which are weakly 1-summable in directions θ and $\theta - \pi$, and let us denote by $\Phi_+, f_+^{(1)}, \dots, f_+^{(n)}$ (resp. $\Phi_-, f_-^{(1)}, \dots, f_-^{(n)}$) their respective weak 1-sums in the direction θ (resp. $\theta - \pi$). Assume that $\hat{F}_{\mathbf{0}}^{(k)} = 0$ for all $k = 1, \dots, n$. Then,

$$\hat{\Psi}(x, \mathbf{y}) := \hat{\Phi}\left(x, \hat{f}^{(1)}(x, \mathbf{y}), \dots, \hat{f}^{(n)}(x, \mathbf{y})\right)$$

is weakly 1-summable directions θ and $\theta - \pi$, and its 1-sum in the corresponding direction is

$$\Psi_{\pm}(x, \mathbf{y}) = \Phi_{\pm} \left(x, f_{\pm}^{(1)}(x, \mathbf{y}), \dots, f_{\pm}^{(n)}(x, \mathbf{y}) \right) ,$$

which is a germ of a sectorial holomorphic function in the direction θ (resp. $\theta - \pi$) with opening π .

Proof. See [Bit16a]. \square

As proved in [Bit16a], the next corollary gives the link between 1-summability in some direction and weak 1-summability in the same direction (we refer to [Bit16a], Definition 2.8, or to [MR82], section IV, for a definition of the norm $\|\cdot\|_{\lambda, \theta, \delta, \rho}$ associate to the space of 1-summable formal power series in the direction θ).

Corollary 2.13. *Let*

$$\hat{f}(x, \mathbf{y}) = \sum_{\mathbf{j} \in \mathbb{N}^n} \hat{F}_{\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}} \in \mathbb{C}[[x, \mathbf{y}]]$$

be a formal power series. Then, \hat{f} is 1-summable in the direction $\theta \in \mathbb{R}$, of 1-sum $f \in \mathcal{O}(\mathcal{S}_{\theta, \pi})$, if and only if the following two conditions hold:

- \hat{f} is weakly 1-summable in the direction θ , i.e. there exists λ, δ, ρ such that $\forall \mathbf{j} \in \mathbb{N}^n$, $\|\hat{F}_{\mathbf{j}}\|_{\lambda, \theta, \delta, \rho} < \infty$
- the power series $\sum_{\mathbf{j} \in \mathbb{N}^n} \|\hat{F}_{\mathbf{j}}\|_{\lambda, \theta, \delta, \rho} \mathbf{y}^{\mathbf{j}}$ is convergent in some polydisc $\mathbf{D}(\mathbf{0}, \mathbf{r})$.

Proof. See [Bit16a]. \square

3. PROOFS OF THE MAIN THEOREMS

The aim of this section is to prove the main results of this paper, assuming Proposition 3.5 holds.

3.1. Analytic invariants: Stokes diffeomorphisms.

From now on, we fix a normal form

$$Y_{\text{norm}} = x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x - c(y_1 y_2)) y_1 \frac{\partial}{\partial y_1} + (\lambda + a_2 x + c(y_1 y_2)) y_2 \frac{\partial}{\partial y_2} ,$$

with $\lambda \in \mathbb{C}^*$, $\Re(a_1 + a_2) > 0$ and $c \in v\mathbb{C}\{v\}$ vanishing at the origin. We denote by $[Y_{\text{norm}}]$ the set of germs of holomorphic doubly-resonant saddle-nodes in $(\mathbb{C}^3, 0)$, formally conjugate to Y_{norm} by formal fibered diffeomorphisms tangent to the identity. We refer the reader to Definition 2.1 for notions relating to sectors.

Definition 3.1.

- We define $\text{Isot}_{\arg(\lambda), \eta}(Y_{\text{norm}})$ (resp. $\text{Isot}_{\arg(-\lambda), \eta}(Y_{\text{norm}})$), for all $\eta \in [0, 2\pi]$, as the group of germs of sectorial fibered isotropies of Y_{norm} in sectorial domains in $\mathcal{S}_{\arg(\lambda), \eta}$ (resp. $\mathcal{S}_{\arg(-\lambda), \eta}$) (see Definition 2.3).
- We define $\tilde{\Lambda}_{\lambda}(Y_{\text{norm}})$ (resp. $\tilde{\Lambda}_{-\lambda}(Y_{\text{norm}})$) as the group of germs of sectorial fibered isotropies of Y_{norm} , admitting the identity as **weak** Gevrey-1 asymptotic expansion in sectorial domains of the form $S_{\lambda} \times (\mathbb{C}^2, 0)$ (resp. $S_{-\lambda} \times (\mathbb{C}^2, 0)$), where:

$$\begin{aligned} S_{\lambda} &\in \mathcal{AS}_{\arg(\lambda), \pi} \\ S_{-\lambda} &\in \mathcal{AS}_{\arg(-\lambda), \pi} \end{aligned}$$

(see Definition 2.3).

We recall the notations given in the introduction: we have defined $\Lambda_{\lambda}(Y_{\text{norm}})$ (resp. $\Lambda_{-\lambda}(Y_{\text{norm}})$) as the group of germs of sectorial fibered isotropies of Y_{norm} , admitting the identity as Gevrey-1 asymptotic expansion in sectorial domains of the form $S_{\lambda} \times (\mathbb{C}^2, 0)$ (resp. $S_{-\lambda} \times (\mathbb{C}^2, 0)$). It is clear that we have:

$$\Lambda_{\pm\lambda}(Y_{\text{norm}}) \subset \tilde{\Lambda}_{\pm\lambda}(Y_{\text{norm}}) \subset \text{Isot}_{\arg(\pm\lambda), \eta}(Y_{\text{norm}}), \quad \forall \eta \in]0, \pi[.$$

According to Theorem 1.9, to any $Y \in [Y_{\text{norm}}]$, we can associate a pair of germs of sectorial fibered isotropies in $S_\lambda \times (\mathbb{C}^2, 0)$ and $S_{-\lambda} \times (\mathbb{C}^2, 0)$ respectively, denoted by $(\Phi_\lambda, \Phi_{-\lambda})$:

$$\begin{cases} \Phi_\lambda := (\Phi_+ \circ \Phi_-^{-1})|_{S_\lambda \times (\mathbb{C}^2, 0)} \in \text{Isot}_{\arg(\lambda), \eta}(Y_{\text{norm}}) & , \forall \eta \in]0, \pi[\\ \Phi_{-\lambda} := (\Phi_- \circ \Phi_+^{-1})|_{S_{-\lambda} \times (\mathbb{C}^2, 0)} \in \text{Isot}_{\arg(-\lambda), \eta}(Y_{\text{norm}}) & , \forall \eta \in]0, \pi[\end{cases} ,$$

where (Φ_+, Φ_-) is the pair of the sectorial normalizing maps given by Theorem 1.9.

Proposition 3.2. *For any given $\eta \in]0, \pi[$ the map*

$$\begin{aligned} [Y_{\text{norm}}] &\longrightarrow \text{Isot}_{\arg(\lambda), \eta}(Y_{\text{norm}}) \times \text{Isot}_{\arg(-\lambda), \eta}(Y_{\text{norm}}) \\ Y &\longmapsto (\Phi_\lambda, \Phi_{-\lambda}) \end{aligned} ,$$

actually ranges in $\tilde{\Lambda}_\lambda(Y_{\text{norm}}) \times \tilde{\Lambda}_{-\lambda}(Y_{\text{norm}})$.

Proof. The fact that the sectorial normalizing maps Φ_+, Φ_- given by Theorem 1.9 both conjugate $Y \in [Y_{\text{norm}}]$ to Y_{norm} in the corresponding sectorial domains proves that the arrow above is well-defined, with values in $\text{Isot}_{\arg(\lambda), \eta}(Y_{\text{norm}}) \times \text{Isot}_{\arg(-\lambda), \eta}(Y_{\text{norm}})$, for all $\eta \in]0, \pi[$. The fact that $\Phi_{\pm\lambda}$ admits the identity as weak Gevrey-1 asymptotic expansion in $S_{\pm\lambda} \times (\mathbb{C}^2, 0)$ comes from the fact that Φ_+ and Φ_- admits the same weak Gevrey-1 asymptotic expansion in $S_\lambda \times (\mathbb{C}^2, 0)$ and $S_{-\lambda} \times (\mathbb{C}^2, 0)$, and from Proposition 2.12. \square

The subgroup $\text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id}) \subset \text{Diff}_{\text{fib}}(\mathbb{C}^3, 0)$ formed by fibered diffeomorphisms tangent to the identity acts naturally on $[Y_{\text{norm}}]$ by conjugacy. Now we show that uniqueness of germs of sectorial normalizing maps (Φ_+, Φ_-) implies that the Stokes diffeomorphisms $(\Phi_\lambda, \Phi_{-\lambda})$ of a vector field $Y \in [Y_{\text{norm}}]$ is invariant under the action of $\text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id})$. Furthermore, this map is one-to-one.

Proposition 3.3. *The map*

$$\begin{aligned} [Y_{\text{norm}}] &\longrightarrow \tilde{\Lambda}_\lambda(Y_{\text{norm}}) \times \tilde{\Lambda}_{-\lambda}(Y_{\text{norm}}) \\ Y &\longmapsto (\Phi_\lambda, \Phi_{-\lambda}) \end{aligned}$$

factorizes through a one-to-one map

$$\begin{aligned} [Y_{\text{norm}}] / \text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id}) &\longrightarrow \tilde{\Lambda}_\lambda(Y_{\text{norm}}) \times \tilde{\Lambda}_{-\lambda}(Y_{\text{norm}}) \\ Y &\longmapsto (\Phi_\lambda, \Phi_{-\lambda}) . \end{aligned}$$

Remark 3.4. This very result means that the Stokes diffeomorphisms encode completely the class of Y in the quotient $[Y_{\text{norm}}] / \text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id})$ as they separate conjugacy classes.

Proof. First of all, let us prove that the latter map is well-defined. Let $Y, \tilde{Y} \in [Y_{\text{norm}}]$ and $\Theta \in \text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id})$ be such that $\Theta_*(Y) = \tilde{Y}$. We denote by Φ_\pm (*resp.* $\tilde{\Phi}_\pm$) the sectorial normalizing maps of Y (*resp.* \tilde{Y}), and $(\Phi_\lambda, \Phi_{-\lambda})$ (*resp.* $(\tilde{\Phi}_\lambda, \tilde{\Phi}_{-\lambda})$) the Stokes diffeomorphisms of Y (*resp.* \tilde{Y}). By assumption, $\tilde{\Phi}_\pm \circ \Theta$ is also a germ of a sectorial fibered normalization of Y in $S_\pm \times (\mathbb{C}^2, 0)$, which is tangent to the identity. Thus, according to the uniqueness statement in Theorem 1.9:

$$\Phi_\pm = \tilde{\Phi}_\pm \circ \Theta .$$

Consequently, in $S_{\pm\lambda} \times (\mathbb{C}^2, 0)$ we have

$$\begin{aligned} \Phi_\lambda &= (\Phi_+ \circ \Phi_-^{-1})|_{S_\lambda \times (\mathbb{C}^2, 0)} \\ &= \tilde{\Phi}_+ \circ \Theta \circ \Theta^{-1} \circ \tilde{\Phi}_- \\ &= \tilde{\Phi}_\lambda , \end{aligned}$$

and similarly

$$\begin{aligned}\Phi_{-\lambda} &= (\Phi_- \circ \Phi_+^{-1})|_{S_{-\lambda} \times (\mathbb{C}^2, 0)} \\ &= \tilde{\Phi}_- \circ \Theta \circ \Theta^{-1} \circ (\tilde{\Phi}_+)^{-1} \\ &= \tilde{\Phi}_{-\lambda} .\end{aligned}$$

Let us prove that the map is one-to-one. Let $Y, \tilde{Y} \in [Y_{\text{norm}}]$ share the same Stokes diffeomorphisms $(\Phi_\lambda, \Phi_{-\lambda})$. We denote by Φ_\pm (*resp.* $\tilde{\Phi}_\pm$) the germ of a sectorial fibered normalizing map of Y (*resp.* \tilde{Y}) $S_\pm \times (\mathbb{C}^2, 0)$. We have:

$$\begin{cases} \Phi_+ \circ (\Phi_-)^{-1} = \Phi_\lambda = \tilde{\Phi}_+ \circ (\tilde{\Phi}_-)^{-1} & \text{in } S_\lambda \times (\mathbb{C}^2, 0) \\ \Phi_- \circ (\Phi_+)^{-1} = \Phi_{-\lambda} = \tilde{\Phi}_- \circ (\tilde{\Phi}_+)^{-1} & \text{in } S_{-\lambda} \times (\mathbb{C}^2, 0) . \end{cases}$$

Thus:

$$\begin{cases} (\tilde{\Phi}_+)^{-1} \circ \Phi_+ = (\tilde{\Phi}_-)^{-1} \circ \Phi_- & \text{in } S_\lambda \times (\mathbb{C}^2, 0) \\ (\tilde{\Phi}_+)^{-1} \circ \Phi_+ = (\tilde{\Phi}_-)^{-1} \circ \Phi_- & \text{in } S_{-\lambda} \times (\mathbb{C}^2, 0) . \end{cases}$$

We can then define a map φ analytic in a domain of the form $(D(0, r) \setminus \{0\}) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ by setting:

$$\begin{cases} \varphi|_{S_+} = (\tilde{\Phi}_+)^{-1} \circ \Phi_+ & \text{in } S_+ \\ \varphi|_{S_-} = (\tilde{\Phi}_-)^{-1} \circ \Phi_- & \text{in } S_- . \end{cases}$$

This map is analytic and bounded in $(D(0, r) \setminus \{0\}) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$, and the Riemann singularity theorem tells us that this map can be analytically extended to the entire poly-disc $D(0, r) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$. As a conclusion, $\varphi \in \text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id})$, $\Phi_\pm = \tilde{\Phi}_\pm \circ \varphi$ and $\varphi_*(Y) = \tilde{Y}$. \square

3.2. Proof of Theorem 1.12: 1-summability of the formal normalization.

We fix a normal form

$$Y_{\text{norm}} = x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x - c(y_1 y_2)) y_1 \frac{\partial}{\partial y_1} + (\lambda + a_2 x + c(y_1 y_2)) y_2 \frac{\partial}{\partial y_2} ,$$

with $\lambda \in \mathbb{C}^*$, $\Re(a_1 + a_2) > 0$ and $c \in v\mathbb{C}\{v\}$ vanishing at the origin. In section 4 we will prove the following result.

Proposition 3.5. *Any $\psi \in \tilde{\Lambda}_{\pm\lambda}(Y_{\text{norm}})$ admits the identity as Gevrey-1 asymptotic expansion in $S_{\pm\lambda} \times (\mathbb{C}^2, 0)$. In other words:*

$$\tilde{\Lambda}_{\pm\lambda}(Y_{\text{norm}}) = \Lambda_{\pm\lambda}(Y_{\text{norm}}) .$$

As a first consequence of Proposition 3.5, we obtain Theorem 1.12 which states that the formal normalizing map from [Bit15] is in fact 1-summable.

Proof. (of Theorem 1.12) Let us consider the unique germs of a sectorial normalizing map Φ_+ and Φ_- in $S_+ \times (\mathbb{C}^2, 0)$ and $S_- \times (\mathbb{C}^2, 0)$ respectively, and their associated Stokes diffeomorphisms:

$$\begin{cases} \Phi_\lambda = (\Phi_+ \circ \Phi_-^{-1})|_{S_\lambda \times (\mathbb{C}^2, 0)} \in \tilde{\Lambda}_\lambda(Y_{\text{norm}}) \\ \Phi_{-\lambda} = (\Phi_- \circ \Phi_+^{-1})|_{S_{-\lambda} \times (\mathbb{C}^2, 0)} \in \tilde{\Lambda}_{-\lambda}(Y_{\text{norm}}) \end{cases} .$$

According to Proposition 3.5,

$$\tilde{\Lambda}_{\pm\lambda}(Y_{\text{norm}}) = \Lambda_{\pm\lambda}(Y_{\text{norm}}) ,$$

so that Φ_λ and $\Phi_{-\lambda}$ both admit the identity as Gevrey-1 asymptotic expansion, in $S_\lambda \times (\mathbb{C}^2, 0)$ and $S_{-\lambda} \times (\mathbb{C}^2, 0)$ respectively. Then, Theorem 2.9 gives the existence of

$$(\phi_+, \phi_-) \in \text{Diff}_{\text{fib}}(\mathcal{S}_{\arg(i\lambda), \eta}, \text{Id}) \times \text{Diff}_{\text{fib}}(\mathcal{S}_{\arg(-i\lambda), \eta}, \text{Id})$$

for all $\eta \in]\pi, 2\pi[$, such that:

$$\begin{cases} \phi_+ \circ (\phi_-)^{-1}|_{S_\lambda \times (\mathbb{C}^2, 0)} = \Phi_\lambda \\ \phi_- \circ (\phi_+)^{-1}|_{S_{-\lambda} \times (\mathbb{C}^2, 0)} = \Phi_{-\lambda} \end{cases} ,$$

and the existence of a formal diffeomorphism $\hat{\phi}$ which is tangent to the identity, such that ϕ_+ and ϕ_- both admit $\hat{\phi}$ as Gevrey-1 asymptotic expansion in $S_+ \times (\mathbb{C}^2, 0)$ and $S_- \times (\mathbb{C}^2, 0)$ respectively. In particular, we have:

$$\left((\Phi_+)^{-1} \circ \phi_+ \right) |_{(S_\lambda \cup S_{-\lambda}) \times (\mathbb{C}^2, 0)} = \left((\Phi_-)^{-1} \circ \phi_- \right) |_{(S_\lambda \cup S_{-\lambda}) \times (\mathbb{C}^2, 0)} .$$

This proves that the function Φ defined by $(\Phi_+)^{-1} \circ \phi_+$ in $S_+ \times (\mathbb{C}^2, 0)$ and by $(\Phi_-)^{-1} \circ \phi_-$ in $S_- \times (\mathbb{C}^2, 0)$ is well-defined and analytic in $D(0, r) \setminus \{0\} \times \mathbf{D}(\mathbf{0}, \mathbf{r})$. Since it is also bounded, it can be extended to an analytic map Φ in $D(0, r) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ by Riemann's theorem. Hence:

$$\begin{cases} \phi_+ = \Phi_+ \circ \Phi \\ \phi_- = \Phi_- \circ \Phi \end{cases} .$$

In particular, by composition, Φ_+ and Φ_- both admit $\hat{\phi} \circ \Phi^{-1}$ as Gevrey-1 asymptotic expansion in $S_\lambda \times (\mathbb{C}^2, 0)$ and $S_{-\lambda} \times (\mathbb{C}^2, 0)$ respectively. Since Φ_+ and Φ_- conjugates Y to Y_{norm} and since the notion of asymptotic expansion commutes with the partial derivative operators, the formal diffeomorphism $\hat{\phi} \circ \Phi^{-1}$ formally conjugates Y to Y_{norm} . Finally, notice that $\hat{\phi} \circ \Phi^{-1}$ is necessarily tangent to the identity. Hence, by uniqueness of the formal normalizing map given by Theorem 1.4, we deduce that $\hat{\phi} \circ \Phi^{-1} = \hat{\Phi}$, the unique formal normalizing map tangent to the identity. \square

3.3. Proofs of Theorems 1.15 and 1.23.

Let us now present the proofs of Theorems 1.15 and 1.23, assuming Proposition 3.5.

3.3.1. Proof of Theorem 1.15.

Propositions 3.3, together with Proposition 3.5, tell us that the considered map is well-defined and one-to-one. It remains to prove that this map is onto. Let

$$\begin{cases} \Phi_\lambda \in \Lambda_\lambda(Y_{\text{norm}}) \\ \Phi_{-\lambda} \in \Lambda_{-\lambda}(Y_{\text{norm}}) \end{cases} .$$

According to Theorem 2.9, there exists

$$(\phi_+, \phi_-) \in \text{Diff}_{\text{fib}}(\mathcal{S}_{\arg(i\lambda), \eta}, \text{Id}) \times \text{Diff}_{\text{fib}}(\mathcal{S}_{\arg(-i\lambda), \eta}, \text{Id})$$

with $\eta \in]\pi, 2\pi[$, which extend analytically to $S_+ \times (\mathbb{C}^2, 0)$ and $S_- \times (\mathbb{C}^2, 0)$ respectively, such that:

$$\phi_\pm \circ (\phi_\mp)^{-1}|_{S_\pm \times (\mathbb{C}^2, 0)} = \Phi_{\pm\lambda}$$

and there also exists a formal diffeomorphism $\hat{\phi}$ which is tangent to the identity, such that ϕ_\pm both admit $\hat{\phi}$ as asymptotic expansion in $S_\pm \times (\mathbb{C}^2, 0)$. Let us consider the two germs of sectorial vector fields obtained as

$$Y_\pm := (\phi_\pm^{-1})_* (Y_{\text{norm}})$$

In particular, since $\hat{\phi}$ is the Gevrey-1 asymptotic expansion of ϕ_\pm , the vector fields Y_\pm both admit $(\hat{\phi})_*(Y_{\text{norm}})$ as Gevrey-1 asymptotic expansion. The fact that $\phi_+ \circ (\phi_-)^{-1}$ is an isotropy of Y_{norm} implies immediately that $Y_+ = Y_-$ on

$$(S_+ \cap S_-) \times (\mathbb{C}^2, 0) = (S_\lambda \cup S_{-\lambda}) \times (\mathbb{C}^2, 0) .$$

Then, the vector field Y , which coincides with Y_\pm in $S_\pm \times (\mathbb{C}^2, 0)$, defines a germ of analytic vector field in $(\mathbb{C}^3, 0)$ by Riemann's theorem. By construction, $Y \in \text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id})_*(Y_{\text{norm}})$ and admits $(\Phi_\lambda, \Phi_{-\lambda})$ as Stokes diffeomorphisms.

3.3.2. Proof of Theorem 1.23.

In a similar way, we prove now Theorem 1.23. Let $Y_{\text{norm}} \in \mathcal{SN}_{\text{diag},0}$ be a normal form which is also transversally symplectic. We refer to subsection 1.4 for the notations. It is clear from Theorems 1.15 and 1.21 that the mapping is well-defined and one-to-one. It remains to prove that it is also onto. Let

$$\begin{cases} \Phi_{\lambda} \in \Lambda_{\lambda}^{\omega}(Y_{\text{norm}}) \\ \Phi_{-\lambda} \in \Lambda_{-\lambda}^{\omega}(Y_{\text{norm}}) \end{cases} .$$

Since $\Lambda_{\lambda}^{\omega}(Y_{\text{norm}}) \subset \Lambda_{\lambda}(Y_{\text{norm}})$ and $\Lambda_{-\lambda}^{\omega}(Y_{\text{norm}}) \subset \Lambda_{-\lambda}(Y_{\text{norm}})$, according to Theorem 2.9 there exists

$$(\phi_{+}, \phi_{-}) \in \text{Diff}_{\text{fib}}(\mathcal{S}_{\arg(i\lambda),\eta}, \text{Id}) \times \text{Diff}_{\text{fib}}(\mathcal{S}_{\arg(-i\lambda),\eta}, \text{Id})$$

with $\eta \in]\pi, 2\pi[$, which extend analytically in $S_{+} \times (\mathbb{C}^2, 0)$ and $S_{-} \times (\mathbb{C}^2, 0)$ respectively, such that:

$$\phi_{\pm} \circ (\phi_{\mp})|_{S_{\pm\lambda} \times (\mathbb{C}^2, 0)}^{-1} = \Phi_{\pm\lambda}$$

and there also exists a formal diffeomorphism $\hat{\phi}$ which is tangent to the identity, such that ϕ_{\pm} both admit $\hat{\phi}$ as Gevrey-1 asymptotic expansion in $S_{\pm} \times (\mathbb{C}^2, 0)$. According to Corollary 2.10, there exists a germ of an analytic fibered diffeomorphism $\psi \in \text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id})$ (tangent to the identity), such that

$$\sigma_{\pm} := \phi_{\pm} \circ \psi$$

both are transversally symplectic. Then, we have:

$$\sigma_{\pm} \circ (\Psi_{\mp})|_{S_{\pm\lambda} \times (\mathbb{C}^2, 0)}^{-1} = \Phi_{\pm\lambda} .$$

The end of the proof goes exactly as at the end of the proof of the previous theorem.

4. SECTORIAL ISOTROPIES AND SPACE OF LEAVES: PROOF OF PROPOSITION 3.5

A normal form

$$Y_{\text{norm}} = x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x - c(y_1 y_2)) y_1 \frac{\partial}{\partial y_1} + (\lambda + a_2 x + c(y_1 y_2)) y_2 \frac{\partial}{\partial y_2}$$

is fixed for some $\lambda \in \mathbb{C}^*$, $\Re(a_1 + a_2) > 0$ and $c \in v\mathbb{C}\{v\}$ (vanishing at the origin). The aim of this section is to prove Proposition 3.5 stated in Section 3.

4.1. Sectorial first integrals and the space of leaves.

In a sectorial neighborhood of the origin of the form $S_{\lambda} \times (\mathbb{C}^2, 0)$ (*resp.* $S_{-\lambda} \times (\mathbb{C}^2, 0)$), we can give two particular analytic first integrals of Y_{norm} . First of all one checks easily the following:

Fact 4.1. *The quantity*

$$w := \frac{y_1 y_2}{x^a}$$

is a first integral of Y_{norm} , i.e.

$$\mathcal{L}_{Y_{\text{norm}}}(w) = 0$$

(here Y_{norm} acts on $w = \frac{y_1 y_2}{x^a}$ as a Lie derivative).

Let us write

$$c(v) = \sum_{k=1}^{+\infty} c_k v^k ,$$

$a := a_1 + a_2$ and $m = \frac{1}{a}$. If $m \notin \mathbb{N}$, we set $c_m := 0$. We also define:

$$\tilde{c}(v) = m \sum_{k \neq m} \frac{c_k}{k - m} v^k .$$

Notice that $\tilde{c}(v) \in \mathbb{C}\{v\}$ and $\tilde{c}(0) = 0$. As above, an elementary computation immediately proves the existence of two other first integrals of Y_{norm} .

Proposition 4.2. *Let h_1 and h_2 be defined as*

$$(4.1) \quad \begin{cases} h_1(x, \mathbf{y}) = y_1 \exp\left(\frac{-\lambda}{x} + \frac{c_m(y_1 y_2)^m \log(x)}{x} + \frac{\tilde{c}(y_1 y_2)}{x}\right) x^{-a_1} \\ h_2(x, \mathbf{y}) = y_2 \exp\left(\frac{\lambda}{x} - \frac{c_m(y_1 y_2)^m \log(x)}{x} - \frac{\tilde{c}(y_1 y_2)}{x}\right) x^{-a_2} \end{cases} .$$

They both are first integrals of Y_{norm} . Moreover, h_1 and h_2 satisfy:

$$h_1 h_2 = w \quad \left(= \frac{y_1 y_2}{x^a} \right) .$$

Remark 4.3. In other words, we can parametrize a leaf of the foliation associated to Y_{norm} in sectorial domains as follows:

$$(4.2) \quad \begin{cases} y_1(x) = h_1 \exp\left(\frac{\lambda}{x} - c_m(h_1 h_2)^m \log(x) - \frac{\tilde{c}(h_1 h_2 x^a)}{x}\right) x^{a_1} \\ y_2(x) = h_2 \exp\left(-\frac{\lambda}{x} + c_m(h_1 h_2)^m \log(x) + \frac{\tilde{c}(h_1 h_2 x^a)}{x}\right) x^{a_2} \end{cases} \\ (h_1, h_2) \in \mathbb{C}^2$$

for these mappings are functionally independent.

Corollary 4.4. *The map*

$$\begin{aligned} \mathcal{H} : S_{\pm\lambda} \times (\mathbb{C}^2, 0) &\rightarrow S_{\pm\lambda} \times \mathbb{C}^2 \\ (x, \mathbf{y}) &\mapsto (x, h_1(x, \mathbf{y}), h_2(x, \mathbf{y})) , \end{aligned}$$

where h_1, h_2 are defined in (4.1), is a sectorial germ of an analytic and one-to-one map in $S_{\pm\lambda} \times (\mathbb{C}^2, 0)$. Moreover, there exists an open neighborhood of the origin $\Gamma_{\arg(\pm\lambda)} \subset \mathbb{C}^2$ such that:

$$\mathcal{H}(S_{\pm\lambda} \times (\mathbb{C}^2, 0)) = S_{\pm\lambda} \times \Gamma_{\arg(\pm\lambda)} .$$

In particular, \mathcal{H} induces a fibered biholomorphism

$$S_{\pm\lambda} \times (\mathbb{C}^2, 0) \xrightarrow{\mathcal{H}} S_{\pm\lambda} \times \Gamma_{\arg(\pm\lambda)}$$

which conjugates Y_{norm} to $x^2 \frac{\partial}{\partial x}$, i.e.

$$\mathcal{H}_*(Y_{\text{norm}}) = x^2 \frac{\partial}{\partial x} .$$

Definition 4.5. We call $\Gamma_{\arg(\pm\lambda)}$ the **space of leaves** of Y_{norm} in $S_{\pm\lambda} \times (\mathbb{C}^2, 0)$.

Remark 4.6. $\Gamma_{\arg(\pm\lambda)}$ depends on the size and shape of the neighborhood $(\mathbb{C}^2, 0)$.

4.2. Sectorial isotropies in the space of leaves. Now, we consider a germ of a sectorial isotropy $\psi_{\pm} \in \tilde{\Lambda}_{\pm\lambda}(Y_{\text{norm}})$ and we denote by $\Gamma'_{\arg(\pm\lambda)}$ the (germ of an) open subset of \mathbb{C}^2 such that:

$$\mathcal{H} \circ \psi_{\pm}(S_{\pm\lambda} \times (\mathbb{C}^2, 0)) = S_{\pm\lambda} \times \Gamma'_{\arg(\pm\lambda)} .$$

Proposition 4.7. *With the notations and assumptions above, the map*

$$\Psi_{\pm} := \mathcal{H} \circ \psi_{\pm} \circ \mathcal{H}^{-1} : S_{\pm\lambda} \times \Gamma_{\arg(\pm\lambda)} \longrightarrow S_{\pm\lambda} \times \Gamma'_{\arg(\pm\lambda)}$$

is a sectorial germ of a fibered biholomorphism from $S_{\pm\lambda} \times \Gamma_{\arg(\pm\lambda)}$ to $S_{\pm\lambda} \times \Gamma'_{\arg(\pm\lambda)}$, which is of the form:

$$\Psi_{\pm}(x, h_1, h_2) = (x, \Psi_{1,\pm}(h_1, h_2), \Psi_{2,\pm}(h_1, h_2)) .$$

In particular, $\Psi_{1,\pm}$ and $\Psi_{2,\pm}$ are analytic and depend only on $(h_1, h_2) \in \Gamma_{\arg(\pm\lambda)}$, while Ψ_{\pm} induces a biholomorphism (still written Ψ_{\pm}):

$$\begin{aligned} \Psi_{\pm} : \Gamma_{\arg(\pm\lambda)} &\rightarrow \Gamma'_{\arg(\pm\lambda)} \\ (h_1, h_2) &\mapsto (\Psi_{1,\pm}(h_1, h_2), \Psi_{2,\pm}(h_1, h_2)) . \end{aligned}$$

Proof. We only have to prove that $\Psi_{1,\pm}$ and $\Psi_{2,\pm}$ depend only on $(h_1, h_2) \in \Gamma_{\arg(\pm\lambda)}$. By assumption, Ψ_{\pm} is an isotropy of $x^2 \frac{\partial}{\partial x}$:

$$(\Psi_{\pm})_* \left(x^2 \frac{\partial}{\partial x} \right) = x^2 \frac{\partial}{\partial x} .$$

We immediately obtain:

$$\frac{\partial \Psi_{1,\pm}}{\partial x} = \frac{\partial \Psi_{2,\pm}}{\partial x} = 0 .$$

□

In the space of leaves $\Gamma_{\arg(\pm\lambda)}$ equipped with coordinates (h_1, h_2) , we denote by w the product of h_1 and h_2 :

$$w(h_1, h_2) := h_1 h_2 .$$

We define the two following quantities:

$$(4.3) \quad \begin{cases} f_1(x, w) := \exp \left(\frac{\lambda}{x} - c_m w^m \log(x) - \frac{\tilde{c}(w x^a)}{x} \right) x^{a_1} \\ f_2(x, w) := \exp \left(-\frac{\lambda}{x} + c_m w^m \log(x) + \frac{\tilde{c}(w x^a)}{x} \right) x^{a_2} \end{cases} ,$$

such that the leaves of the foliations are parametrized by:

$$\begin{cases} y_1(x) = h_1 f_1(x, h_1 h_2) \\ y_2(x) = h_2 f_2(x, h_1 h_2) \end{cases} , (h_1, h_2) \in \mathbb{C}^2 .$$

Notice that:

$$f_1(x, w) f_2(x, w) = x^a .$$

Moreover, one checks immediately the following statement.

Fact 4.8. *For all $w \in \mathbb{C}$:*

$$\begin{cases} \lim_{\substack{x \rightarrow 0 \\ x \in S_{\lambda}}} |f_1(x, w)| = \lim_{\substack{x \rightarrow 0 \\ x \in S_{-\lambda}}} |f_2(x, w)| = +\infty \\ \lim_{\substack{x \rightarrow 0 \\ x \in S_{-\lambda}}} |f_1(x, w)| = \lim_{\substack{x \rightarrow 0 \\ x \in S_{\lambda}}} |f_2(x, w)| = 0 \end{cases} .$$

Using notations of Proposition 4.7, we also assume from now on that $(\mathbb{C}^2, 0) = \mathbf{D}(\mathbf{0}, \mathbf{r})$, with $\mathbf{r} = (r_1, r_2) \in (\mathbb{R}_{>0})^2$ small enough so that

$$\psi_{\pm}(S_{\pm\lambda} \times \mathbf{D}(\mathbf{0}, \mathbf{r})) \subset S_{\pm\lambda} \times \mathbf{D}(\mathbf{0}, \mathbf{r}')$$

for some $\mathbf{r}' = (r'_1, r'_2) \in (\mathbb{R}_{>0})^2$. Then

$$\Gamma_{\arg(\pm\lambda)} = \bigcup_{x \in S_{\pm\lambda}} \Gamma_{\arg(\pm\lambda), x} ,$$

where for all $x \in S_{\pm\lambda}$ we define

$$\Gamma_{\arg(\pm\lambda), x} := \left\{ (h_1, h_2) \in \mathbb{C}^2 \text{ such that } \begin{cases} |h_1| < \frac{r_1}{|f_1(x, h_1 h_2)|} \\ |h_2| < \frac{r_2}{|f_2(x, h_1 h_2)|} \end{cases} \right\}$$

We define similarly the sets $\Gamma'_{\arg(\pm\lambda), x}$ for $x \in S_{\pm\lambda}$ and using the polyradius \mathbf{r}' instead of \mathbf{r} :

$$\Gamma'_{\arg(\pm\lambda)} \subset \bigcup_{x \in S_{\pm\lambda}} \Gamma'_{\arg(\pm\lambda), x} ,$$

Remark 4.9.

(1) It is important to notice that the particular form of Ψ_{\pm} implies that the image of any fiber

$$\{x = x_0\} \times \Gamma_{\arg(\pm\lambda), x_0}$$

by Ψ_{\pm} is included in a fiber of the form

$$\{x = x_0\} \times \Gamma'_{\arg(\pm\lambda), x_0} .$$

(2) If $(h_1, h_2) \in \Gamma_{\arg(\pm\lambda), x}$, then

$$|h_1 h_2| < \frac{r_1 r_2}{|x^a|} .$$

(3) As $(h_1, h_2) \in \Gamma_{\arg(\pm\lambda)}$ varies the values of $w = h_1 h_2$ cover the whole \mathbb{C} .

4.3. Action of sectorial isotropies on the resonant monomial in the space of leaves.

Let us study the the action of Ψ on the resonant monomial $w = h_1 h_2$ in the space of leaves.

Lemma 4.10. *With the notations and assumptions above, let us define $\Psi_{w, \pm} := \Psi_{1, \pm} \Psi_{2, \pm}$. Then, for all $n \in \mathbb{N}$, there exists entire functions $\Psi_{w, +, n}$ and $\Psi_{w, -, n}$ (analytic in \mathbb{C}) such that*

$$\begin{cases} \Psi_{w, +}(h_1, h_2) = \sum_{n \geq 0} \Psi_{w, +, n}(h_1 h_2) h_1^n \\ \Psi_{w, -}(h_1, h_2) = \sum_{n \geq 0} \Psi_{w, -, n}(h_1 h_2) h_2^n \end{cases} ,$$

and these series converge (for the sup-norm) in every subset of $\Gamma_{\arg(\pm\lambda)}$ of the form

$$\bigcup_{x \in S_{\arg(\pm\lambda), 0}} \left\{ (h_1, h_2) \in \mathbb{C}^2 \text{ such that } \begin{cases} |h_1| \leq \frac{\tilde{r}_1}{|f_1(x, h_1 h_2)|} \\ |h_2| \leq \frac{\tilde{r}_2}{|f_2(x, h_1 h_2)|} \end{cases} \right\}$$

for $0 < \tilde{r}_1 < r_1$ and $0 < \tilde{r}_2 < r_2$. More precisely, for all $\tilde{r}_1, \tilde{r}_2, \delta > 0$ such that

$$0 < \tilde{r}_j + \delta < r_j \quad , \quad j \in \{1, 2\}$$

for all $x \in S_{\pm\lambda}$ and $w \in \mathbb{C}$ we have

$$|wx^a| \leq \tilde{r}_1 \tilde{r}_2 \implies \begin{cases} |\Psi_{w, +, n}(w)| \leq \frac{r'_1 r'_2}{|x^a|} \left| \frac{f_1(x, w)}{\tilde{r}_1 + \delta} \right|^n \\ |\Psi_{w, -, n}(w)| \leq \frac{r'_1 r'_2}{|x^a|} \left| \frac{f_2(x, w)}{\tilde{r}_2 + \delta} \right|^n \end{cases} , \quad \forall n \geq 0 .$$

Proof. Let us give the proof for $\Psi_{w, +}, \Psi_{1, +}$ and $\Psi_{2, +}$ in $\Gamma_{\arg(\lambda)}$ (the same proof applies also for $\Psi_{w, -}$ in $\Gamma_{\arg(-\lambda)}$ by exchanging the role played by h_1 and h_2). We fix some $0 < \tilde{r}_j < r_j$, $j \in \{1, 2\}$, and $\delta > 0$ such that

$$0 < \tilde{r}_j + \delta < r_j \quad , \quad j \in \{1, 2\}$$

For a fixed value $w \in \mathbb{C}$, we consider the restriction of $\Psi_{w, +}$ to the hypersurface $M_w := \{h_1 h_2 = w\} \cap \Gamma_{\arg(\lambda)}$, which is analytic on M_w . The map

$$\varphi_w : h_1 \mapsto \Psi_{w, +} \left(h_1, \frac{w}{h_1} \right)$$

is analytic in

$$M_{w, 1} := \bigcup_{\substack{x \in S_{\lambda} \\ |wx^a| < r_1 r_2}} \Omega_{x, w} ,$$

where for all $x \in S_{\lambda}$ with $|wx^a| < r_1 r_2$, the set $\Omega_{x, w}$ is the following annulus:

$$\Omega_{x, w} := \left\{ h_1 \in \mathbb{C} \mid \left| \frac{w f_2(x, w)}{r_2} \right| < |h_1| < \left| \frac{r_1}{f_1(x, w)} \right| \right\} .$$

In particular, φ_w admits a Laurent expansion

$$\varphi_w(h_1) = \Psi_{w, +} \left(h_1, \frac{w}{h_1} \right) = \sum_{n \in \mathbb{Z}} \Psi_{w, +, n}(w) h_1^n$$

in any annulus $\Omega_{x,w}$, with $x \in S_\lambda$ such that $|wx^a| < r_1 r_2$. In particular, for all $x \in S_\lambda$ such that $|wx^a| < r_1 r_2$, Cauchy's formula gives

$$\Psi_{w,+n}(w) = \frac{1}{2i\pi} \oint_{\gamma(x,w)} \frac{\Psi_{w,+}\left(h_1, \frac{w}{h_1}\right)}{h_1^{n+1}} dh_1, \text{ for all } n \in \mathbb{Z},$$

where $\gamma(x, w)$ is any circle centered at the origin with a radius $\rho(x, w)$ satisfying

$$\left| \frac{w f_2(x, w)}{r_2} \right| < \rho(x, w) < \left| \frac{r_1}{f_1(x, w)} \right|.$$

If $|wx^a| < (\tilde{r}_1 + \delta)(\tilde{r}_2 + \delta)$, we can take for instance

$$\rho(x, w) = \left| \frac{\tilde{r}_1 + \delta}{f_1(x, w)} \right|.$$

Therefore, for all $x \in S_\lambda$ and all $w \in \mathbb{C}$ such that $|wx^a| \leq \tilde{r}_1 \tilde{r}_2$, for all $\xi \in \mathbb{C}$ with $|\xi| < \delta$, we also have:

$$\Psi_{w,+n}(w + \xi) = \frac{1}{2i\pi} \oint_{\gamma(x,w)} \frac{\Psi_{w,+}\left(h_1, \frac{w+\xi}{h_1}\right)}{h_1^{n+1}} dh_1, \text{ for all } n \in \mathbb{Z},$$

where $\gamma(x, w)$ is the same circle (of radius $\rho(x, w) = \left| \frac{\tilde{r}_1 + \delta}{f_1(x, w)} \right|$) for all $|\xi| < \delta$. Moreover, since for all $x \in S_\lambda$, we have

$$\begin{aligned} \Gamma'_{\arg(\pm\lambda),x} &\subset \left\{ (h'_1, h'_2) \in \mathbb{C}^2 \text{ such that } |h'_j| < \frac{r'_j}{|f_j(x, h'_1 h'_2)|}, j \in \{1, 2\} \right\} \\ &\subset \left\{ (h'_1, h'_2) \in \mathbb{C}^2 \mid |h'_1 h'_2 x^a| < r'_1 r'_2 \right\}, \end{aligned}$$

then for all $x \in S_\lambda$ and $w \in \mathbb{C}$ such that $|wx^a| \leq \tilde{r}_1 \tilde{r}_2$, the following inequality holds for all h_1 with $|h_1| < \frac{r_1}{f_1(x,w)}$:

$$\left| \Psi_{w,+}\left(h_1, \frac{w}{h_1}\right) \right| < \frac{r'_1 r'_2}{|x^a|}.$$

The well-known theorem regarding integrals depending holomorphically on a parameter asserts that for all $n \in \mathbb{Z}$ the mapping $\Psi_{w,+n}$ is analytic near any point $w \in \mathbb{C}$. Hence, it is an entire function. Moreover, the inequality above and the Cauchy's formula together imply that for all $n \in \mathbb{Z}$ and for all $(x, w) \in S_\lambda \times \mathbb{C}$ such that $|wx^a| \leq \tilde{r}_1 \tilde{r}_2$, we have:

$$|\Psi_{w,+n}(w)| < \frac{r'_1 r'_2}{|x^a| \rho(x, w)^n} = \frac{r'_1 r'_2}{|x^a|} \left| \frac{f_1(x, w)}{\tilde{r}_1 + \delta} \right|^n.$$

According to Fact 4.8, for a fixed value $w \in \mathbb{C}$, if $n < 0$, the right hand-side tends to 0 as x tends to 0 in S_λ . This implies in particular that for all $n < 0$, $\Psi_{w,+n} = 0$. Consequently:

$$\Psi_{w,+}\left(h_1, \frac{w}{h_1}\right) = \sum_{n \geq 0} \Psi_{w,+n}(w) h_1^n.$$

Moreover, for all $w \in \mathbb{C}$ the series converges normally in every domain of the form

$$\Omega_{x,w} := \left\{ h_1 \in \mathbb{C} \mid |h_1| \leq \left| \frac{\tilde{r}_1}{f_1(x, w)} \right| \right\}, \text{ for all } x \in S_\lambda, 0 < \tilde{r}_1 < r_1,$$

since the Laurent expansion's range is $n \geq 0$. This actually means that the series converges normally in an entire neighborhood of the origin in \mathbb{C} . In particular, for all fixed $w \in \mathbb{C}$, the function

$$h_1 \mapsto \Psi_{w,+}\left(h_1, \frac{w}{h_1}\right) = \sum_{n \geq 0} \Psi_{w,+n}(w) h_1^n$$

is analytic in a neighborhood of the origin. Finally, the series

$$\Psi_{w,+}(h_1, h_2) = \sum_{n \geq 0} \Psi_{w,+,n}(h_1 h_2) h_1^n$$

converges normally, and hence its sum is analytic in every domain of the form

$$\bigcup_{x \in S_\lambda} \left\{ (h_1, h_2) \in \mathbb{C}^2 \text{ such that } |h_j| \leq \frac{\tilde{r}_j}{|f_j(x, h_1 h_2)|}, j \in \{1, 2\} \right\}, 0 < \tilde{r}_1 < r_1, 0 < \tilde{r}_2 < r_2.$$

□

4.4. Action of sectorial isotropies on the resonant monomial.

Since $\psi_\pm \in \tilde{\Lambda}_{\pm\lambda}(Y_{\text{norm}})$, the mapping ψ_\pm is of the form

$$\psi_\pm(x, \mathbf{y}) = (x, \psi_{1,\pm}(x, \mathbf{y}), \psi_{2,\pm}(x, \mathbf{y})),$$

with $\psi_{1,\pm}, \psi_{2,\pm}$ analytic and bounded in $S_{\pm\lambda} \times \mathbf{D}(\mathbf{0}, \mathbf{r})$. Moreover, by assumption ψ_\pm admits the identity as weak Gevrey-1 asymptotic expansion, *i.e.*

$$\psi_{i,\pm}(x, \mathbf{y}) = y_i + \sum_{\mathbf{k} \in \mathbb{N}^2} \psi_{i,\pm,\mathbf{k}}(x) \mathbf{y}^{\mathbf{k}},$$

where for $i = 1, 2$ and for all $\mathbf{k} = (k_1, k_2) \in \mathbb{N}^2$, $\psi_{i,\pm,\mathbf{k}}$ is holomorphic in $S_{\pm\lambda}$ and admits 0 as Gevrey-1 asymptotic expansion.

Lemma 4.11. *With the notations and assumptions above, let us define $\psi_{v,\pm} := \psi_{1,\pm} \psi_{2,\pm}$. Then $\psi_{v,+}$ and $\psi_{v,-}$ can be expanded as the series*

$$\begin{cases} \psi_{v,+}(x, \mathbf{y}) = y_1 y_2 + x^a \sum_{n \geq 1} \Psi_{w,+,n} \left(\frac{y_1 y_2}{x^a} \right) \left(\frac{y_1}{f_1 \left(x, \frac{y_1 y_2}{x^a} \right)} \right)^n \\ \psi_{v,-}(x, \mathbf{y}) = y_1 y_2 + x^a \sum_{n \geq 1} \Psi_{w,-,n} \left(\frac{y_1 y_2}{x^a} \right) \left(\frac{y_2}{f_2 \left(x, \frac{y_1 y_2}{x^a} \right)} \right)^n \end{cases}$$

which are normally convergent in every subset of $S_{\pm\lambda} \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ of the form $S_{\pm\lambda} \times \overline{\mathbf{D}}(\mathbf{0}, \tilde{\mathbf{r}})$, where $\overline{\mathbf{D}}(\mathbf{0}, \tilde{\mathbf{r}})$ is a closed poly-disc with $\tilde{\mathbf{r}} = (\tilde{r}_1, \tilde{r}_2)$ such that

$$0 < \tilde{r}_j < r_j, \quad j \in \{1, 2\}$$

Here $\Psi_{w,+,n}$ and $\Psi_{w,-,n}$ for $n \in \mathbb{N}$ are the ones appearing in Lemma 4.10. Moreover, for all closed sub-sector $S' \subset S_{\pm\lambda}$ and for all closed poly-disc $\overline{\mathbf{D}} \subset \mathbf{D}(\mathbf{0}, \mathbf{r})$, there exists $A, B > 0$ such that:

$$|\psi_{v,\pm}(x, y_1, y_2) - y_1 y_2| \leq A \exp\left(-\frac{B}{|x|}\right), \quad \forall (x, \mathbf{y}) \in S' \times \overline{\mathbf{D}}.$$

In particular, $\psi_{v,\pm}$ admits $y_1 y_2$ as Gevrey-1 asymptotic expansion in $S_{\pm\lambda} \times \mathbf{D}(\mathbf{0}, \mathbf{r})$.

Proof. By definition, we have

$$\Psi_\pm \circ \mathcal{H} = \mathcal{H} \circ \psi_\pm.$$

In particular, for all $(x, \mathbf{y}) \in S_{\pm\lambda} \times \mathbf{D}(\mathbf{0}, \mathbf{r})$:

$$\Psi_{w,\pm} \left(x, \frac{y_1}{f_1 \left(x, \frac{y_1 y_2}{x^a} \right)}, \frac{y_2}{f_2 \left(x, \frac{y_1 y_2}{x^a} \right)} \right) = \frac{\psi_{v,\pm}(x, y_1, y_2)}{x^a}.$$

Thus, according to Lemma 4.10 we have:

$$(4.4) \quad \begin{cases} \psi_{v,+}(x, \mathbf{y}) = x^a \sum_{n \geq 0} \Psi_{w,+,n} \left(\frac{y_1 y_2}{x^a} \right) \left(\frac{y_1}{f_1 \left(x, \frac{y_1 y_2}{x^a} \right)} \right)^n \\ \psi_{v,-}(x, \mathbf{y}) = x^a \sum_{n \geq 0} \Psi_{w,-,n} \left(\frac{y_1 y_2}{x^a} \right) \left(\frac{y_2}{f_2 \left(x, \frac{y_1 y_2}{x^a} \right)} \right)^n \end{cases}.$$

Besides we know that $\psi_{v,\pm}$ admits y_1y_2 as weak Gevrey-1 asymptotic expansion in $S_{\pm\lambda} \times \mathbf{D}(\mathbf{0}, \mathbf{r})$:

$$(4.5) \quad \psi_{v,\pm}(x, y_1, y_2) = y_1y_2 + \sum_{\mathbf{k} \in \mathbb{N}^2} \psi_{v,+, \mathbf{k}}(x) \mathbf{y}^{\mathbf{k}},$$

where for all $\mathbf{k} = (k_1, k_2) \in \mathbb{N}^2$ the mapping $\psi_{v,\pm, \mathbf{k}}$ is holomorphic in $S_{\pm\lambda}$ and admits 0 as Gevrey-1 asymptotic expansion. Let us compare both expressions of $\psi_{v,\pm}$. Looking at monomials $\mathbf{y}^{\mathbf{k}}$ with $k_1 = k_2$ in (4.5), and at terms corresponding to $n = 0$ on the right-hand side of (4.4), we must have for all $x \in S_{\pm\lambda}$ and $v \in \mathbb{C}$ with $|v| < r_1r_2$:

$$v + \sum_{k \geq 0} \psi_{v,+, (k,k)}(x) v^k = x^a \Psi_{w,+,0} \left(\frac{v}{x^a} \right).$$

Since $\Psi_{w,\pm,0}$ is analytic in \mathbb{C} , there exists $(\alpha_{\pm,k})_{k \in \mathbb{N}} \subset \mathbb{C}$ such that

$$\Psi_{w,\pm,0} \left(\frac{v}{x^a} \right) = \sum_{k \geq 0} \alpha_{\pm,k} \left(\frac{v}{x^a} \right)^k.$$

This can only happen if $\alpha_{\pm,k} = 0$ whenever $k \neq 1$, for $\psi_{v,\pm, \mathbf{k}}$ is holomorphic in $S_{\pm\lambda}$ and admits 0 as Gevrey-1 asymptotic expansion. A further immediate identification yields

$$\Psi_{v,\pm,0}(w) = w.$$

Thus

$$\begin{cases} \psi_{v,+}(x, \mathbf{y}) = y_1y_2 + x^a \sum_{n \geq 1} \Psi_{w,+,n} \left(\frac{y_1y_2}{x^a} \right) \left(\frac{y_1}{f_1(x, \frac{y_1y_2}{x^a})} \right)^n \\ \psi_{v,-}(x, \mathbf{y}) = y_1y_2 + x^a \sum_{n \geq 1} \Psi_{w,-,n} \left(\frac{y_1y_2}{x^a} \right) \left(\frac{y_2}{f_2(x, \frac{y_1y_2}{x^a})} \right)^n \end{cases}.$$

Let us prove that $\psi_{v,\pm}$ admits y_1y_2 as Gevrey-1 asymptotic expansion in $S_{\pm\lambda} \times (\mathbb{C}^2, 0)$. We have to show that $|\psi_{v,\pm}(x, y_1, y_2) - y_1y_2|$ is exponentially small with respect to $x \in S_{\pm\lambda}$, uniformly in $\mathbf{y} \in \mathbf{D}(\mathbf{0}, \mathbf{r})$. As for the previous lemma, we perform the proof for $\psi_{v,+}$ only (the same proof applies for $\psi_{v,-}$ by exchanging y_1 and y_2).

From the computations above we derive

$$|\psi_{v,+}(x, y_1, y_2) - y_1y_2| \leq \sum_{n \geq 1} \left| x^a \Psi_{w,+,n} \left(\frac{y_1y_2}{x^a} \right) \left(\frac{y_1}{f_1(x, \frac{y_1y_2}{x^a})} \right)^n \right|.$$

Let us fix $\tilde{r}_1, \tilde{r}_2, \delta > 0$ in such a way that

$$0 < \tilde{r}_j + \delta < r_j, \quad j \in \{1, 2\}.$$

Let us take $|x|, |y_1|$ and $|y_2|$ small enough so that

$$2x \in S_\lambda$$

and

$$|y_1y_2| < \frac{\tilde{r}_1\tilde{r}_2}{|2^a|} < r_1r_2.$$

According to Lemma 4.10, for all $\tilde{x} \in S_\lambda$ and all $w \in \mathbb{C}$:

$$|w\tilde{x}^a| \leq \tilde{r}_1\tilde{r}_2 \implies |\Psi_{w,+,n}(w)| \leq \frac{r'_1r'_2}{|\tilde{x}^a|} \left| \frac{f_1(\tilde{x}, w)}{\tilde{r}_1 + \delta} \right|^n.$$

In particular for $\tilde{x} = 2x$ and $w = \frac{y_1y_2}{x^a}$ we derive $|w\tilde{x}^a| < \tilde{r}_1\tilde{r}_2$, from which we conclude

$$\left| \Psi_{w,+,n} \left(\frac{y_1y_2}{x^a} \right) \right| \leq \frac{r'_1r'_2}{|2^a x^a|} \left| \frac{f_1(2x, \frac{y_1y_2}{x^a})}{\tilde{r}_1 + \delta} \right|^n.$$

Consequently, for all $(x, y_1, y_2) \in S_\lambda \times \mathbf{D}(\mathbf{0}, \tilde{\mathbf{r}})$ with

$$\begin{cases} 2x \in S_\lambda \\ |y_1 y_2| < \frac{\tilde{r}_1 \tilde{r}_2}{|2^a|} < r_1 r_2 \end{cases}$$

we have

$$\begin{aligned} |\psi_{v,+}(x, y_1, y_2) - y_1 y_2| &\leq \sum_{n \geq 1} \left| x^a \frac{r'_1 r'_2}{2^a x^a} \left(\frac{f_1(2x, \frac{y_1 y_2}{x^a})}{\tilde{r}_1 + \delta} \right)^n \left(\frac{y_1}{f_1(x, \frac{y_1 y_2}{x^a})} \right)^n \right| \\ &\leq \frac{r'_1 r'_2}{|2^a|} \sum_{n \geq 1} \left| \left(\frac{y_1}{\tilde{r}_1 + \delta} \right)^n \left(\frac{f_1(2x, \frac{y_1 y_2}{x^a})}{f_1(x, \frac{y_1 y_2}{x^a})} \right)^n \right|. \end{aligned}$$

Let us bound

$$\left| \frac{f_1(2x, \frac{y_1 y_2}{x^a})}{f_1(x, \frac{y_1 y_2}{x^a})} \right| = \left| 2^{a_1} \exp \left(-\frac{\lambda}{2x} - c_m \frac{(y_1 y_2)^m}{x} \log(2) - \frac{\tilde{c}(y_1 y_2 2^a)}{2x} + \frac{\tilde{c}(y_1 y_2)}{x} \right) \right|.$$

Since \tilde{c} is analytic in a neighborhood of the origin and null at the origin, we can take $r_1, r_2 > 0$ small enough in order that for all closed sub-sector $S' \subset S_\lambda$, for all $0 < \tilde{r}_1 < r_1$ and $0 < \tilde{r}_2 < r_2$, there exist $B > 0$ satisfying:

$$(x, y_1, y_2) \in S' \times \mathbf{D}(\mathbf{0}, \tilde{\mathbf{r}}) \implies \left| \frac{f_1(2x, \frac{y_1 y_2}{x^a})}{f_1(x, \frac{y_1 y_2}{x^a})} \right| \leq |2^{a_1}| \exp \left(-\frac{B}{|x|} \right) < 1.$$

Finally:

$$\begin{aligned} |\psi_{v,+}(x, y_1, y_2) - y_1 y_2| &\leq \frac{r'_1 r'_2}{|2^a|} \sum_{n \geq 1} \left| \frac{2^{a_1} y_1}{\tilde{r}_1 + \delta} \exp \left(-\frac{B}{|x|} \right) \right|^n \\ &\leq \frac{r'_1 r'_2}{|2^a|} \frac{\left| \frac{2^{a_1} y_1}{\tilde{r}_1 + \delta} \exp \left(-\frac{B}{|x|} \right) \right|}{1 - \left| \frac{2^{a_1} y_1}{\tilde{r}_1 + \delta} \exp \left(-\frac{B}{|x|} \right) \right|} \\ &\leq A \exp \left(-\frac{B}{|x|} \right), \end{aligned}$$

for a convenient $A > 0$. □

The latter lemma implies $\Psi_{v,\pm,0}(w) = w$, having for consequence the next result.

Corollary 4.12. *For all closed sub-sector $S' \subset S_{\pm\lambda}$ and for all $0 < \tilde{r}_1 < r_1$ and $0 < \tilde{r}_2 < r_2$, there exists $A, B > 0$ such that for all $x \in S'$:*

$$\begin{cases} |h_1| \leq \frac{\tilde{r}_1}{|f_1(x, h_1 h_2)|} \\ |h_2| \leq \frac{\tilde{r}_2}{|f_2(x, h_1 h_2)|} \end{cases} \implies |\Psi_{w,\pm}(x, h_1, h_2) - h_1 h_2| \leq \frac{A \exp \left(-\frac{B}{|x|} \right)}{|x^a|}.$$

In particular, there exists $C > 0$ such that:

$$\begin{cases} |h_1| \leq \frac{\tilde{r}_1}{|f_1(x, h_1 h_2)|} \\ |h_2| \leq \frac{\tilde{r}_2}{|f_2(x, h_1 h_2)|} \end{cases} \implies \frac{\left| \exp \left(c_m (h_1 h_2)^m \log(x) + \frac{\tilde{c}(x^a (h_1 h_2)^m)}{x} \right) \right|}{\left| \exp \left(c_m (\Psi_w(x, h_1, h_2))^m \log(x) + \frac{\tilde{c}(x^a (\Psi_w(x, h_1, h_2))^m)}{x} \right) \right|} < C.$$

4.5. Power series expansion of sectorial isotropies in the space of leaves.

Now, we give a power series expansion of $\Psi_{1,\pm}$ and $\Psi_{2,\pm}$ in the space of leaves. Let us introduce the following notations:

$$\begin{cases} N(1, +) := N(2, -) := 1 \\ N(1, -) := N(2, +) := -1 \end{cases}.$$

Lemma 4.13. *With the notations and assumptions above, there exists entire functions (analytic in \mathbb{C}) denoted by $\Psi_{j,\nu,n}$, $j \in \{1, 2\}$, $\nu \in \{+, -\}$, $n \geq N(j, \nu)$, such that for $j \in \{1, 2\}$:*

$$\begin{cases} \Psi_{j,+,n}(h_1, h_2) = \sum_{n \geq N(j,+)} \Psi_{j,+,n}(h_1 h_2) h_1^n \\ \Psi_{j,-,n}(h_1, h_2) = \sum_{n \geq N(j,-)} \Psi_{j,-,n}(h_1 h_2) h_2^n \end{cases} .$$

These series converge normally in every subset of $\Gamma_{\arg(\pm\lambda)}$ of the form

$$\bigcup_{x \in S_{\pm\lambda}} \left\{ (h_1, h_2) \in \mathbb{C}^2 \mid |h_j| \leq \frac{\tilde{r}_j}{|f_j(x, h_1 h_2)|} , j \in \{1, 2\} \right\} ,$$

where $0 < \tilde{r}_1 < r_1$ and $0 < \tilde{r}_2 < r_2$. More precisely, for all $\tilde{r}_1, \tilde{r}_2, \delta > 0$ such that

$$0 < \tilde{r}_j + \delta < r_j , j \in \{1, 2\}$$

there exists $C > 0$ such that for all $x \in S_{\pm\lambda}$ and for all $w \in \mathbb{C}$:

$$|wx^a| \leq \tilde{r}_1 \tilde{r}_2 \implies \begin{cases} |\Psi_{1,+,n}(w)| < Cr'_1 \frac{|f_1(x, w)|^{n-1}}{(\tilde{r}_1 + \delta)^n} , n \geq 1 \\ |\Psi_{2,+,n}(w)| < \frac{Cr'_2}{|x^a|} \frac{|f_1(x, w)|^{n+1}}{(\tilde{r}_1 + \delta)^{n+1}} , n \geq -1 \\ |\Psi_{1,-,n}(w)| < \frac{Cr'_1}{|x^a|} \frac{|f_2(x, w)|^{n+1}}{(\tilde{r}_2 + \delta)^n} , n \geq -1 \\ |\Psi_{2,-,n}(w)| < Cr'_2 \frac{|f_2(x, w)|^{n-1}}{(\tilde{r}_2 + \delta)^n} , n \geq 1 . \end{cases}$$

Moreover:

$$\Psi_{1,-,-1}(0) = \Psi_{2,+,-1}(0) = 0 .$$

Proof. We use the same notations as in the proof of Lemma 4.10. For fixed $w \in \mathbb{C}$, the maps

$$\varphi_1 : h_1 \mapsto \Psi_{1,+} \left(h_1, \frac{w}{h_1} \right)$$

and

$$\varphi_2 : h_1 \mapsto \Psi_{2,+} \left(h_1, \frac{w}{h_1} \right)$$

are analytic in

$$M_{w,1} = \bigcup_{\substack{x \in S_\lambda \\ |wx^a| < r_1 r_2}} \Omega_{x,w}$$

(see the proof of Lemma 4.10). In particular, φ_1 and φ_2 admit Laurent expansions

$$\begin{cases} \varphi_1(h_1) = \Psi_{1,+} \left(h_1, \frac{w}{h_1} \right) = \sum_{n \in \mathbb{Z}} \Psi_{1,+,n}(w) h_1^n \\ \varphi_2(h_1) = \Psi_{2,+} \left(h_1, \frac{w}{h_1} \right) = \sum_{n \in \mathbb{Z}} \Psi_{2,+,n}(w) h_1^n \end{cases}$$

in any annulus $\Omega_{x,w}$, with $x \in S_\lambda$ such that $|wx^a| < r_1 r_2$. Using the same method as in the proof of Lemma 4.10, we prove without additional difficulties that for all $n \in \mathbb{Z}$, $\Psi_{1,+,n}$ and $\Psi_{2,+,n}$ are analytic in any point $w \in \mathbb{C}$, and thus are entire functions. Moreover, we also show in the same way as earlier that for all $\tilde{r}_1, \tilde{r}_2, \delta > 0$ with

$$0 < \tilde{r}_j + \delta < r_j , j \in \{1, 2\} ,$$

for all $n \in \mathbb{Z}$ and for all $(x, w) \in S_\lambda \times \mathbb{C}$ such that $|wx^a| \leq \tilde{r}_1 \tilde{r}_2$, we have:

$$\begin{cases} |\Psi_{1,+}(w)| < \frac{r'_1}{\left| f_1 \left(x, \Psi_{w,+} \left(x, h_1, \frac{w}{h_1} \right) \right) \right|} \left| \frac{f_1(x, w)}{\tilde{r}_1 + \delta} \right|^n \\ |\Psi_{2,+}(w)| < \frac{r'_2}{\left| f_2 \left(x, \Psi_{w,+} \left(x, h_1, \frac{w}{h_1} \right) \right) \right|} \left| \frac{f_1(x, w)}{\tilde{r}_1 + \delta} \right|^n \end{cases} .$$

According to Corollary 4.12, there exists $C > 0$ such that for all $(x, w) \in S_\lambda \times \mathbb{C}$ with $|wx^a| \leq \tilde{r}_1 \tilde{r}_2$, we have:

$$\begin{cases} |\Psi_{1,+}(w)| < Cr'_1 \frac{|f_1(x, w)|^{n-1}}{(\tilde{r}_1 + \delta)^n} \\ |\Psi_{2,+}(w)| < \frac{Cr'_2}{|x^a|} \frac{|f_1(x, w)|^{n+1}}{(\tilde{r}_1 + \delta)^n} \end{cases} .$$

According to the statement in Fact 4.8, for a fixed value $w \in \mathbb{C}$, if we look at the limit as x tends to 0 in S_λ of the right hand-sides above we deduce that:

$$\begin{cases} |\Psi_{1,+}(w)| = 0 \quad , \forall n \leq 0 \\ |\Psi_{2,+}(w)| = 0 \quad , \forall n \leq -2 \end{cases} .$$

Consequently:

$$\begin{cases} \Psi_{1,+}(h_1, h_2) = \sum_{n \geq 1} \Psi_{1,+}(h_1 h_2) h_1^n \\ \Psi_{2,+}(h_1, h_2) = \sum_{n \geq -1} \Psi_{2,+}(h_1 h_2) h_1^n \end{cases} .$$

These function series converges normally (and are analytic) in every domain of the form

$$\bigcup_{x \in S_\lambda} \left\{ (h_1, h_2) \in \mathbb{C}^2 \mid \begin{cases} |h_1| \leq \frac{\tilde{r}_1}{|f_1(x, h_1 h_2)|} \\ |h_2| \leq \frac{\tilde{r}_2}{|f_2(x, h_1 h_2)|} \end{cases} \right\} , \quad 0 < \tilde{r}_1 < r_1, \quad 0 < \tilde{r}_2 < r_2 .$$

Moreover, for any fixed value of h_2 , on the one hand the function series

$$h_1 \mapsto \Psi_{2,+}(h_1, h_2) = \sum_{n \geq -1} \Psi_{2,+}(h_1 h_2) h_1^n$$

is analytic in a punctured disc, since

$$|f_2(x, h_1, h_2)| \xrightarrow[x \in S_\lambda]{x \rightarrow 0} 0 ,$$

and on the other hand, we already know that the function $h_1 \mapsto \Psi_{2,+}(h_1, h_2)$ is analytic in a neighborhood of the origin. Thus, we must have $\Psi_{2,+,-1}(0) = 0$.

A similar proof holds for $\Psi_{1,-}$ and $\Psi_{2,-}$. □

4.6. Sectorial isotropies: proof of Proposition 3.5.

The following lemma is a more precise version of Proposition 3.5. We recall the notations:

$$\begin{cases} N(1, +) = N(2, -) = 1 \\ N(1, -) = N(2, +) = -1 \end{cases} .$$

Lemma 4.14. *With the notations and assumptions above, we consider $\psi_\pm \in \tilde{\Lambda}_{\pm\lambda}(Y_{\text{norm}})$, with*

$$\psi_\pm(x, \mathbf{y}) = (x, \psi_{1,\pm}(x, \mathbf{y}), \psi_{2,\pm}(x, \mathbf{y})) .$$

Then, for $i \in \{1, 2\}$, $\psi_{i,+}$ and $\psi_{i,-}$ can be written as series:

$$\begin{cases} \psi_{i,+}(x, \mathbf{y}) = y_i + f_i \left(x, \frac{\psi_{v,+}(x, \mathbf{y})}{x^a} \right) \sum_{n \geq N(i,+)+1} \Psi_{i,+,n} \left(\frac{y_1 y_2}{x^a} \right) \left(\frac{y_1}{f_1 \left(x, \frac{y_1 y_2}{x^a} \right)} \right)^n \\ \psi_{i,-}(x, \mathbf{y}) = y_i + f_i \left(x, \frac{\psi_{v,+}(x, \mathbf{y})}{x^a} \right) \sum_{n \geq N(i,-)+1} \Psi_{i,-,n} \left(\frac{y_1 y_2}{x^a} \right) \left(\frac{y_2}{f_2 \left(x, \frac{y_1 y_2}{x^a} \right)} \right)^n \end{cases} .$$

which are normally convergent in every subset of $S_{\pm\lambda} \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ of the form $S_{\pm\lambda} \times \overline{\mathbf{D}}(\mathbf{0}, \tilde{\mathbf{r}})$, where $\overline{\mathbf{D}}(\mathbf{0}, \tilde{\mathbf{r}})$ is a closed poly-disc with $\tilde{\mathbf{r}} = (\tilde{r}_1, \tilde{r}_2)$ such that

$$0 < \tilde{r}_j < r_j \quad , \quad j \in \{1, 2\} .$$

Here $\Psi_{i,+,n}$, $\Psi_{i,-,n}$ (for $i = 1, 2$ and $n \in \mathbb{N}$) are given in Lemma 4.13. Moreover, for all closed sub-sector $S' \subset S_{\pm\lambda}$ and for all closed poly-disc $\overline{\mathbf{D}} \subset \mathbf{D}(\mathbf{0}, \mathbf{r})$, there exists $A, B > 0$ such that for $j = 1, 2$:

$$|\psi_{j,\pm}(x, y_1, y_2) - y_j| \leq A \exp\left(-\frac{B}{|x|}\right) , \quad \forall (x, \mathbf{y}) \in S' \times \overline{\mathbf{D}} .$$

As a consequence, $\psi_{j,\pm}$ admits y_j as Gevrey-1 asymptotic expansion in $S_{\pm\lambda} \times \mathbf{D}(\mathbf{0}, \mathbf{r})$.

Remark 4.15. In particular, $\Psi_{1,+,1}(w) = \Psi_{2,-,1}(w) = 1$ and $\Psi_{1,-,-1}(w) = \Psi_{2,+,-1}(w) = w$.

Proof. By definition, we have

$$\Psi_{\pm} \circ \mathcal{H} = \mathcal{H} \circ \psi_{\pm} .$$

In particular, for $j = 1, 2$ and all $(x, \mathbf{y}) \in S_{\pm\lambda} \times \mathbf{D}(\mathbf{0}, \mathbf{r})$:

$$\Psi_{j,\pm} \left(x, \frac{y_1}{f_1 \left(x, \frac{y_1 y_2}{x^a} \right)}, \frac{y_2}{f_2 \left(x, \frac{y_1 y_2}{x^a} \right)} \right) = \frac{\psi_{j,\pm}(x, y_1, y_2)}{f_j \left(x, \frac{\psi_v(x, y_1, y_2)}{x^a} \right)} .$$

Thus, according to Lemma 4.13 we have for $i = 1, 2$:

$$(4.6) \quad \begin{cases} \psi_{i,+}(x, \mathbf{y}) = f_i \left(x, \frac{\psi_{v,+}(x, \mathbf{y})}{x^a} \right) \sum_{n \geq N(i,+)} \Psi_{i,+,n} \left(\frac{y_1 y_2}{x^a} \right) \left(\frac{y_1}{f_1 \left(x, \frac{y_1 y_2}{x^a} \right)} \right)^n \\ \psi_{i,-}(x, \mathbf{y}) = f_i \left(x, \frac{\psi_{v,+}(x, \mathbf{y})}{x^a} \right) \sum_{n \geq N(i,-)} \Psi_{i,-,n} \left(\frac{y_1 y_2}{x^a} \right) \left(\frac{y_2}{f_2 \left(x, \frac{y_1 y_2}{x^a} \right)} \right)^n \end{cases} ,$$

and these series are normally convergent in any domain of the form $S' \times \overline{\mathbf{D}}(\mathbf{0}, \tilde{\mathbf{r}})$, where S' is a closed sub-sector of $S_{\pm\lambda}$ and $\overline{\mathbf{D}}(\mathbf{0}, \tilde{\mathbf{r}})$ is a closed poly-disc with $\tilde{\mathbf{r}} = (\tilde{r}_1, \tilde{r}_2)$ such that

$$0 < \tilde{r}_j < r_j \quad , \quad j \in \{1, 2\} .$$

Let us compare the different expressions of $\psi_{j,\pm}$, $j = 1, 2$. We also know that $\psi_{j,\pm}(x, y_1, y_2)$ admits y_j as weak Gevrey-1 asymptotic expansion in $S_{\pm\lambda} \times \mathbf{D}(\mathbf{0}, \mathbf{r})$. Thus, we can write:

$$\psi_{j,\pm}(x, y_1, y_2) = y_j + \sum_{\mathbf{k} \in \mathbb{N}^2} \psi_{j,\pm,\mathbf{k}}(x) \mathbf{y}^{\mathbf{k}} ,$$

where for all $\mathbf{k} = (k_1, k_2) \in \mathbb{N}^2$, $\psi_{j,\pm,\mathbf{k}}$ is holomorphic in $S_{\pm\lambda}$ and admits 0 as Gevrey-1 asymptotic expansion. As usual, let us deal with the case of $\psi_{1,+}$ and $\psi_{2,+}$ (the other one being similar by exchanging y_1 and y_2).

According to the expressions given by Lemmas 4.10 and 4.13, we can be more precise on the index sets in the sums above:

$$(4.7) \quad \begin{cases} \psi_{1,+}(x, y_1, y_2) = y_1 + \sum_{\substack{\mathbf{k}=(k_1, k_2) \in \mathbb{N}^2 \\ k_1 \geq k_2+1}} \psi_{1,+, \mathbf{k}}(x) y_1^{k_1} y_2^{k_2} \\ \psi_{2,+}(x, y_1, y_2) = y_2 + \sum_{\substack{\mathbf{k}=(k_1, k_2) \in \mathbb{N}^2 \\ k_1 \geq k_2}} \psi_{2,+, \mathbf{k}}(x) y_1^{k_1} y_2^{k_2} \end{cases} .$$

Let us deal with $\psi_{1,+}$ (a similar proof holds for $\psi_{2,+}$). Looking at terms for $n = 1$ in (4.6) and at monomials terms $\mathbf{y}^{\mathbf{k}}$ such that $k_1 \leq k_2 + 1$ in (4.7), we must have for all $x \in S_\lambda$, $y_1, y_2 \in \mathbb{C}$ with $|y_1| < r_1$, $|y_2| < r_2$:

$$1 + \sum_{k \geq 0} \psi_{1,+,(k+1,k)}(x) y_1^k y_2^k = \frac{f_1\left(x, \frac{\psi_{v,+}(x,\mathbf{y})}{x^a}\right)}{f_1\left(x, \frac{y_1 y_2}{x^z}\right)} \Psi_{1,+,1}\left(\frac{y_1 y_2}{x^a}\right).$$

According to Lemma 4.11 and Corollary 4.12, we have:

$$\begin{aligned} \frac{f_1\left(x, \frac{\psi_{v,+}(x,\mathbf{y})}{x^a}\right)}{f_1\left(x, \frac{y_1 y_2}{x^z}\right)} &= 1 + \sum_{j_1 \geq j_2 + 1 \geq 1} F_{j_1, j_2}(x) y_1^{j_1} y_2^{j_2} \\ &= 1 + \underset{\substack{(x,\mathbf{y}) \rightarrow 0 \\ (x,\mathbf{y}) \in S_\lambda \times \mathbf{D}(\mathbf{0}, \mathbf{r})}}{O}(|y_1|), \end{aligned}$$

for some analytic and bounded functions $F_{j_1, j_2}(x)$, $j_1 \geq j_2$. As in the proof of Lemma 4.11, using the fact that ψ admit the identity as weak Gevrey-1 asymptotic expansion, we prove deduce that $\Psi_{1,+,1}(w) = 1$, and then:

$$\begin{aligned} \psi_{1,+}(x, \mathbf{y}) &= y_1 + f_1\left(x, \frac{\psi_{v,+}(x,\mathbf{y})}{x^a}\right) \sum_{n \geq 2} \Psi_{1,+,n}\left(\frac{y_1 y_2}{x^a}\right) \left(\frac{y_1}{f_1\left(x, \frac{y_1 y_2}{x^a}\right)}\right)^n \\ &= y_1 + \sum_{\substack{\mathbf{k}=(k_1, k_2) \in \mathbb{N}^2 \\ k_1 \geq k_2 + 2}} \psi_{1,+, \mathbf{k}}(x) y_1^{k_1} y_2^{k_2}. \end{aligned}$$

It remains to show that $\psi_{1,+}$ admits y_1 as Gevrey-1 asymptotic expansion in $S_\lambda \times \mathbf{D}(\mathbf{0}, \mathbf{r})$. From the computations above:

$$|\psi_{1,+}(x, y_1, y_2) - y_1| \leq \sum_{n \geq 2} \left| \Psi_{1,+,n}\left(\frac{y_1 y_2}{x^a}\right) \left(\frac{y_1}{f_1\left(x, \frac{y_1 y_2}{x^a}\right)}\right)^{n-1} \frac{f_1\left(x, \frac{\psi_{v,+}(x,\mathbf{y})}{x^a}\right)}{f_1\left(x, \frac{y_1 y_2}{x^a}\right)} y_1 \right|.$$

Using Lemma 4.13, Corollary 4.12 and the same method as at the end of the proof of Lemma 4.11, we can show the following: we can take $r_1, r_2 > 0$ small enough such that for all closed sub-sector S' of S_λ for all $0 < \tilde{r}_1 < r_1$ and $0 < \tilde{r}_2 < r_2$, there exists $A, B > 0$ satisfying:

$$(x, y_1, y_2) \in S' \times \mathbf{D}(\mathbf{0}, \tilde{\mathbf{r}}) \implies |\psi_{1,+}(x, y_1, y_2) - y_1| \leq A \exp\left(-\frac{B}{|x|}\right).$$

A similar proof holds for $\psi_{2,-}$, $\psi_{2,+}$ and $\psi_{1,-}$. □

Remark 4.16. It should be noticed that in the expressions

$$\begin{cases} \psi_{1,+}(x, \mathbf{y}) = y_1 + f_1\left(x, \frac{\psi_{v,+}(x,\mathbf{y})}{x^a}\right) \sum_{n \geq 2} \Psi_{1,+,n}\left(\frac{y_1 y_2}{x^a}\right) \left(\frac{y_1}{f_1\left(x, \frac{y_1 y_2}{x^a}\right)}\right)^n \\ \psi_{1,-}(x, \mathbf{y}) = y_1 + f_1\left(x, \frac{\psi_{v,+}(x,\mathbf{y})}{x^a}\right) \sum_{n \geq 0} \Psi_{1,-,n}\left(\frac{y_1 y_2}{x^a}\right) \left(\frac{y_2}{f_2\left(x, \frac{y_1 y_2}{x^a}\right)}\right)^n \end{cases}$$

given by Lemma 4.14, the expansion of $\psi_{1,+}$ with respect to $\mathbf{y} = (y_1, y_2)$ starts with a term of order 1, namely y_1 , followed by terms of order at least 2, while in the expansion of $\psi_{1,-}$, the term of lowest order is a constant, namely $\Psi_{1,-,0}(0)$. Similarly, the expansion of $\psi_{2,-}$ (with respect to $\mathbf{y} = (y_1, y_2)$) starts with y_2 , while the expansion of $\psi_{1,-}$ starts with the constant $\Psi_{2,+,0}(0)$.

5. DESCRIPTION OF THE MODULI SPACE AND SOME APPLICATIONS

5.1. A power series presentation of the moduli space.

From Lemmas 4.13 and 4.14, we can give a description of the moduli space $\Lambda_\lambda(Y_{\text{norm}}) \times \Lambda_{-\lambda}(Y_{\text{norm}})$ of a fixed analytic normal form Y_{norm} . We use the notations introduced in section 4. We denote by $\mathcal{O}(\mathbb{C})$ the set of entire functions, *i.e.* of functions holomorphic in \mathbb{C} . We consider the functions f_1 and f_2 defined in (4.3) and introduce four subsets of $(\mathcal{O}(\mathbb{C}))^{\mathbb{N}}$, denoted by $\mathcal{E}_{1,\lambda}(Y_{\text{norm}})$, $\mathcal{E}_{2,\lambda}(Y_{\text{norm}})$, $\mathcal{E}_{1,-\lambda}(Y_{\text{norm}})$ and $\mathcal{E}_{2,-\lambda}(Y_{\text{norm}})$, defined as follows:

Definition 5.1. For $j \in \{1, 2\}$, a sequence $(\psi_n(w))_{n \geq 2} \in (\mathcal{O}(\mathbb{C}))^{\mathbb{N}_{\geq 2}}$ belongs to $\mathcal{E}_{j,\pm\lambda}(Y_{\text{norm}})$ if there exists an open polydisc $\mathbf{D}(\mathbf{0}, \mathbf{r})$ and an open asymptotic sector

$$S_{\pm\lambda} \in \mathcal{AS}_{\arg(\pm\lambda), 2\pi}$$

such that for all $\tilde{r}_1, \tilde{r}_2, \delta > 0$ with

$$0 < \tilde{r}_i + \delta < r_i \quad , \quad i \in \{1, 2\}$$

there exists $C > 0$ such that for all $x \in S_\lambda$ (*resp.* $x \in S_{-\lambda}$) and for all $w \in \mathbb{C}$:

$$|wx^a| \leq \tilde{r}_1 \tilde{r}_2 \implies \begin{cases} |\psi_n(w)| < C \frac{|f_1(x, w)|^{n-1}}{(\tilde{r}_1 + \delta)^n} , \forall n \geq 2, & \text{if } (\psi_n(w))_{n \geq 2} \in \mathcal{E}_{1,\lambda}(Y_{\text{norm}}) \\ |\psi_n(w)| < \frac{C}{|x^a|} \frac{|f_1(x, w)|^{n+1}}{(\tilde{r}_1 + \delta)^n} , \forall n \geq 0, & \text{if } (\psi_n(w))_{n \geq 2} \in \mathcal{E}_{2,\lambda}(Y_{\text{norm}}) \\ |\psi_n(w)| < \frac{C}{|x^a|} \frac{|f_2(x, w)|^{n+1}}{(\tilde{r}_2 + \delta)^n} , \forall n \geq 0, & \text{if } (\psi_n(w))_{n \geq 2} \in \mathcal{E}_{1,-\lambda}(Y_{\text{norm}}) \\ |\psi_n(w)| < C \frac{|f_2(x, w)|^{n-1}}{(\tilde{r}_2 + \delta)^n} , \forall n \geq 2, & \text{if } (\psi_n(w))_{n \geq 2} \in \mathcal{E}_{2,-\lambda}(Y_{\text{norm}}) . \end{cases}$$

As explained in section 4, we can associate to any pair

$$(\psi_\lambda, \psi_{-\lambda}) \in \Lambda_\lambda(Y_{\text{norm}}) \times \Lambda_{-\lambda}(Y_{\text{norm}})$$

two germs of sectorial biholomorphisms of the space of leaves corresponding to each ‘‘narrow’’ sector, which we denote by Ψ_λ and $\Psi_{-\lambda}$, defined by:

$$(5.1) \quad \Psi_{\pm\lambda} := \mathcal{H} \circ \psi_{\pm\lambda} \circ \mathcal{H}^{-1} ,$$

where \mathcal{H} is given by Corollary 4.4. According to Lemmas 4.13 and 4.14, if we write $\Psi_{\pm\lambda} = (x, \Psi_{1,\pm}, \Psi_{2,\pm})$, then for $j = 1, 2$ we have:

$$(5.2) \quad \Psi_{j,\pm}(h_1, h_2) = h_j + \sum_{n \geq N(j,\pm)+1} \Psi_{j,\pm,n}(h_1 h_2) h_1^n$$

where $(\Psi_{j,\pm,n})_n \in \mathcal{E}_{j,\pm\lambda}$ and

$$\begin{cases} N(1, +) := N(2, -) := 1 \\ N(1, -) := N(2, +) := -1 \end{cases} .$$

Conversely, given $(\Psi_{j,\pm})_n \in \mathcal{E}_{j,\pm\lambda}$ for $j = 1, 2$, the estimates made in section 4 show that

$$\psi_{\pm\lambda} := \mathcal{H}^{-1} \circ \Psi_{\pm\lambda} \circ \mathcal{H} ,$$

where $\Psi_{\pm\lambda}(x, \mathbf{h}) = (x, \Psi_{1,\pm}(\mathbf{h}), \Psi_{2,\pm}(\mathbf{h}))$, belongs to $\Lambda_{\pm\lambda}(Y_{\text{norm}})$. Consequently, we can state:

Proposition 5.2. *We have the following bijections:*

$$\begin{aligned} \Lambda_\lambda(Y_{\text{norm}}) &\xrightarrow{\sim} \mathcal{E}_{1,\lambda}(Y_{\text{norm}}) \times \mathcal{E}_{2,\lambda}(Y_{\text{norm}}) \\ \psi_\lambda &\mapsto (\Psi_{1,+}, \Psi_{2,+}) \end{aligned}$$

and

$$\begin{aligned} \Lambda_{-\lambda}(Y_{\text{norm}}) &\xrightarrow{\sim} \mathcal{E}_{1,-\lambda}(Y_{\text{norm}}) \times \mathcal{E}_{2,-\lambda}(Y_{\text{norm}}) \\ \psi_{-\lambda} &\mapsto (\Psi_{1,-}, \Psi_{2,-}) \end{aligned}$$

(we identify here $\Psi_\pm(x, \mathbf{h}) = (x, \Psi_{1,\pm}(\mathbf{h}), \Psi_{2,\pm}(\mathbf{h}))$ with $(\Psi_{1,\pm}(\mathbf{h}), \Psi_{2,\pm}(\mathbf{h}))$).

5.2. Analytic invariant varieties and two-dimensional saddle-nodes.

We can give a necessary and sufficient condition for the existence of analytic invariant varieties in terms of the moduli space described above.

We recall that for any vector field $Y \in [Y_{\text{norm}}]$ as in (1.1) (cf. Definition 1.13), there always exist three formal invariant varieties: $\mathcal{C} = \{(y_1, y_2) = (g_1(x), g_2(x))\}$, $\mathcal{H}_1 = \{y_1 = f_1(x, y_2)\}$ and $\mathcal{H}_2 = \{y_2 = f_2(x, y_1)\}$, where g_1, g_2, f_1, f_2 are formal power series with null constant term. The first one is classically called the *center variety*, and we have $\mathcal{C} = \mathcal{H}_1 \cap \mathcal{H}_2$. If $Y = Y_{\text{norm}}$, then:

$$\begin{cases} \mathcal{C} = \{y_1 = y_2 = 0\} \\ \mathcal{H}_1 = \{y_1 = 0\} \\ \mathcal{H}_2 = \{y_2 = 0\} \end{cases} .$$

Proposition 5.3. *Let $Y \in [Y_{\text{norm}}]$ and $(\Phi_\lambda, \Phi_{-\lambda}) \in \Lambda_\lambda(Y_{\text{norm}}) \times \Lambda_{-\lambda}(Y_{\text{norm}})$ be its Stokes diffeomorphisms. We consider $\Psi_\pm = \mathcal{H} \circ \Phi_{\pm\lambda} \circ \mathcal{H}^{-1}$ as above. Then:*

- (1) *the center variety \mathcal{C} is convergent (analytic near the origin) if and only if $\Psi_{2,+0}(0) = \Psi_{1,-0}(0) = 0$;*
- (2) *the invariant hypersurface \mathcal{H}_1 is convergent (analytic near the origin) if and only if for all $n \geq 0$, we have $\Psi_{1,-,n}(0) = 0$;*
- (3) *the invariant hypersurface \mathcal{H}_2 is convergent (analytic near the origin) if and only if for all $n \geq 0$, we have $\Psi_{2,+,n}(0) = 0$.*

Proof. It is a direct consequence of the power series representation (5.2) of the Stokes diffeomorphisms $(\Phi_\lambda, \Phi_{-\lambda})$. Let us explain point 2. (the same arguments hold for 1. and 3. with minor adaptation). The fact that $\Psi_{1,-,n}(0) = 0$ for all $n \geq 0$ means that $\Psi_{1,-}$ is divisible by h_1 . Equivalently, both $\Phi_{1,+}$ and $\Phi_{1,-}$ are divisible by y_1 , so that the analytic hypersurface $\{y_1 = 0\}$ has the same pre-image by Φ_+ and Φ_- . These pre-images glue together in order to define an analytic invariant hypersurface \mathcal{H}_1 . \square

Notice that if we consider the restriction of a formal normal form Y_{norm} to one of the formal invariant hypersurfaces, we obtain precisely the normal form for two-dimensional saddle-nodes as given in [MR82]. When one of these hypersurfaces is convergent, we recover the Martinet-Ramis invariants by restriction to this hypersurface, as we present below.

Proposition 5.4. *Suppose that the formal invariant hypersurface \mathcal{H}_1 is convergent. Then, the Martinet-Ramis invariants for the saddle-node $Y|_{\mathcal{H}_1}$ are given by:*

$$\begin{cases} \Psi_{2,+}(0, h_2) = h_2 + \Psi_{2,+0}(0) \in \text{Aff}(\mathbb{C}) \\ \Psi_{2,-}(0, h_2) = h_2 + \sum_{n \geq 2} \Psi_{2,-,n}(0) h_2^n \in \text{Diff}(\mathbb{C}, 0) . \end{cases}$$

Similar result holds for the hypersurface \mathcal{H}_2 .

5.3. The transversally symplectic case and quasi-linear Stokes phenomena in the first Painlevé equation.

Let us now focus on the transversally symplectic case studied in Theorem 1.23. Let $Y_{\text{norm}} \in \mathcal{SN}_{\text{diag},0}$ be transversally symplectic (i.e. its residue is $\text{res}(Y_{\text{norm}}) = 1$). Using the notations introduced in paragraph 5.1, we define the following sets:

$$\begin{aligned} (\mathcal{E}_{1,\lambda}(Y_{\text{norm}}) \times \mathcal{E}_{2,\lambda}(Y_{\text{norm}}))_\omega &:= \left\{ \begin{array}{l} \Psi_+ = (\Psi_{1,+}, \Psi_{2,+}) \in \mathcal{E}_{1,\lambda}(Y_{\text{norm}}) \times \mathcal{E}_{2,\lambda}(Y_{\text{norm}}) \\ \text{such that: } \det(D\Psi_+) = 1 \end{array} \right\} \\ (\mathcal{E}_{1,-\lambda}(Y_{\text{norm}}) \times \mathcal{E}_{2,-\lambda}(Y_{\text{norm}}))_\omega &:= \left\{ \begin{array}{l} \Psi_- = (\Psi_{1,-}, \Psi_{2,-}) \in \mathcal{E}_{1,-\lambda}(Y_{\text{norm}}) \times \mathcal{E}_{2,-\lambda}(Y_{\text{norm}}) \\ \text{such that: } \det(D\Psi_-) = 1 \end{array} \right\} . \end{aligned}$$

We recall that Proposition 5.2 tells us that the map

$$\begin{aligned} \Lambda_{\pm\lambda}(Y_{\text{norm}}) &\longrightarrow \mathcal{E}_{1,\pm\lambda}(Y_{\text{norm}}) \times \mathcal{E}_{2,\pm\lambda}(Y_{\text{norm}}) \\ \psi_\pm &\longmapsto \Psi_{\pm\lambda} := \mathcal{H} \circ \psi_{\pm\lambda} \circ \mathcal{H}^{-1} \end{aligned}$$

given in (5.1) is a bijection (we identify here $\Psi_{\pm}(x, \mathbf{h}) = (x, \Psi_{1,\pm}(\mathbf{h}), \Psi_{2,\pm}(\mathbf{h}))$ with $(\Psi_{1,\pm}(\mathbf{h}), \Psi_{2,\pm}(\mathbf{h}))$). An easy computation based on (4.1) gives:

$$(\mathcal{H}^{-1})^* \left(\frac{dy_1 \wedge dy_2}{x} \right) = dh_1 \wedge dh_2 \pmod{\langle dx \rangle}.$$

This means in particular that $\psi_{\pm\lambda}$ is transversally symplectic with respect to $\omega = \frac{dy_1 \wedge dy_2}{x}$, i.e.

$$(\psi_{\pm\lambda})^*(\omega) = \omega \pmod{\langle dx \rangle},$$

if and only if $\Psi_{\pm} = (\Psi_{1,\pm}, \Psi_{2,\pm})$ preserves the standard symplectic form $dh_1 \wedge dh_2$ in the space of leaves, i.e. $\det(D\Psi_{\pm}) = 1$. In other words:

Proposition 5.5. *We have the following bijections:*

$$\begin{aligned} \Lambda_{\lambda}^{\omega}(Y_{\text{norm}}) &\xrightarrow{\sim} (\mathcal{E}_{1,\lambda}(Y_{\text{norm}}) \times \mathcal{E}_{2,\lambda}(Y_{\text{norm}}))_{\omega} \\ \psi_{\lambda} &\mapsto (\Psi_{1,+}, \Psi_{2,+}) \end{aligned}$$

and

$$\begin{aligned} \Lambda_{-\lambda}^{\omega}(Y_{\text{norm}}) &\xrightarrow{\sim} (\mathcal{E}_{1,-\lambda}(Y_{\text{norm}}) \times \mathcal{E}_{2,-\lambda}(Y_{\text{norm}}))_{\omega} \\ \psi_{-\lambda} &\mapsto (\Psi_{1,-}, \Psi_{2,-}) \end{aligned}$$

(we identify here $\Psi_{\pm}(x, \mathbf{h}) = (x, \Psi_{1,\pm}(\mathbf{h}), \Psi_{2,\pm}(\mathbf{h}))$ with $(\Psi_{1,\pm}(\mathbf{h}), \Psi_{2,\pm}(\mathbf{h}))$).

In a forthcoming paper, we will link the study of *quasi-linear Stokes phenomena* (see [Kap04] for the first Painlevé equation) and *non-linear Stokes phenomena* (see [Nov93] for the second Painlevé equations) to our Stokes diffeomorphisms. For instance, in the case of the first Painlevé equation, we will prove that the quasi-linear Stokes phenomena formula found by Kapaev in [Kap04] allows to compute the terms $\Psi_{2,\lambda,0}(0)$ and $\Psi_{1,-\lambda,0}(0)$ in (5.2). For instance, we find:

$$\Psi_{2,\lambda,0}(0) = \Psi_{1,-\lambda,0}(0) = \frac{2e^{-\frac{i\pi}{8}}}{\sqrt{\pi}} 2^{-\frac{19}{8}} 3^{-\frac{1}{8}}.$$

Moreover, our description of the Stokes diffeomorphisms implies a more precise estimate of the order of the rest terms in Kapaev's formula.

6. ANALYTIC CLASSIFICATION UNDER GENERAL (NON-FIBERED) LOCAL ANALYTIC TRANSFORMATIONS

Based on the analytic classification given in Theorem 1.15 for the action of $\text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id})$, it is not difficult to establish the classification for the more general action of $\text{Diff}(\mathbb{C}^3, 0)$. We refer the reader to [Bit16b] (thesis in progress) for the details. We only discuss here the case of the action of germs of diffeomorphisms tangent to the identity (whose set is denoted by $\text{Diff}(\mathbb{C}^3, 0, \text{Id}) \subset \text{Diff}(\mathbb{C}^3, 0)$) for the sake of clarity.

The first result deals with formal classification under the action of $\widehat{\text{Diff}}(\mathbb{C}^3, 0, \text{Id})$, the group of formal diffeomorphisms tangent to the identity. For $\mathbf{a} := (a_1, a_2) \in \mathbb{C}^2 \setminus \{(0, 0)\}$, if $m := \frac{1}{a_1 + a_2} \in \mathbb{N}$, we define the following vector field

$$(6.1) \quad V_{\mathbf{a}} := v^m \left(x \frac{\partial}{\partial x} + a_1 y_1 \frac{\partial}{\partial y_1} + a_2 y_2 \frac{\partial}{\partial y_2} \right).$$

For all $\beta \in \mathbb{C}$, we also define the following diffeomorphism:

$$(6.2) \quad F_{\mathbf{a},\beta} := \exp(\beta V_{\mathbf{a}}).$$

Notice that $(F_{\mathbf{a},\beta})^{-1} = F_{\mathbf{a},-\beta}$. Such a diffeomorphism acts on the monomial $v := y_1 y_2$, and then on any formal power series $c(v) \in \mathbb{C}[[v]]$, by pull-back:

$$(6.3) \quad \begin{aligned} (F_{\mathbf{a},\beta})^*(c) &= c \circ F_{\mathbf{a},\beta}(v) , \\ &= c \left(\frac{v}{(1 - \beta v^m)^{\frac{1}{m}}} \right) . \end{aligned}$$

Let $Y_{\text{diag}}, Y'_{\text{diag}} \in \mathcal{SN}_{\text{diag},0}$ whose formal invariants (under the action of $\widehat{\text{Diff}}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id})$) are (λ, a_1, a_2, c) and $(\lambda', a'_1, a'_2, c')$ respectively (see Theorems 1.4 and 1.9): their formal normal forms are respectively

$$\begin{aligned} Y_{\text{norm}} &= x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x - c(y_1 y_2)) y_1 \frac{\partial}{\partial y_1} + (\lambda + a_2 x + c(y_1 y_2)) y_2 \frac{\partial}{\partial y_2} \\ Y'_{\text{norm}} &= x^2 \frac{\partial}{\partial x} + (-\lambda' + a'_1 x - c'(y_1 y_2)) y_1 \frac{\partial}{\partial y_1} + (\lambda' + a'_2 x + c'(y_1 y_2)) y_2 \frac{\partial}{\partial y_2} . \end{aligned}$$

Let us write $m := \frac{1}{a_1 + a_2}$, $m' := \frac{1}{a'_1 + a'_2}$, $c(v) = \sum_{k \geq 1} c_k v^k$, $c'(v) = \sum_{k \geq 1} c'_k v^k$, $k_0 := \text{ord}(c)$ and $k'_0 := \text{ord}(c')$. Notice that if $m = m'$ and $c' = (F_{\mathbf{a},\beta})^*(c)$, then $k_0 = k'_0$, and for all $j \leq k_0 + m - 1$, we have $c_j = c'_j$.

Theorem 6.1. *Two vector fields $Y_{\text{diag}}, Y'_{\text{diag}} \in \mathcal{SN}_{\text{diag},0}$ as above are $\widehat{\text{Diff}}(\mathbb{C}^3, 0, \text{Id})$ -conjugate if and only if the following two conditions hold:*

- (1) $(\lambda, a_1, a_2) = (\lambda, a'_1, a'_2)$ (in particular $m = m'$)
- (2) $[c = c']$ or $\left[m \in \mathbb{N}_{>0}, c \neq 0 \text{ and } c' = (F_{\mathbf{a},\beta})^*(c), \text{ with } \beta = \frac{m}{k_0} \left(\frac{c_{k_0+m} - c'_{k_0+m}}{c_{k_0}} \right) \right]$.

The second result that we state here (and which will be proved in [Bit16b]) deals with analytic classification under the action of $\text{Diff}(\mathbb{C}^3, 0, \text{Id})$.

Theorem 6.2. *Consider $Y_{\text{diag}}, Y'_{\text{diag}} \in \mathcal{SN}_{\text{diag},0}$ as above, satisfying both conditions 1. and 2. in the above theorem. Consider their Stokes diffeomorphisms denoted by $(\Phi_\lambda, \Phi_{-\lambda})$ and $(\Phi'_\lambda, \Phi'_{-\lambda})$ respectively (see Definition 1.14 and Theorem 1.15). Then, Y_{diag} and Y'_{diag} are $\text{Diff}(\mathbb{C}^3, 0, \text{Id})$ -conjugate if and only if one of the following conditions holds:*

- $m := \frac{1}{a_1 + a_2} \notin \mathbb{N}_{>0}$ and $(\Phi_\lambda, \Phi_{-\lambda}) = (\Phi'_\lambda, \Phi'_{-\lambda})$;
- $m \in \mathbb{N}_{>0}$ and there exists $\beta \in \mathbb{C}$ such that

$$\begin{cases} c' = (F_{\mathbf{a},\beta})^*(c) \\ \Phi_\lambda = F_{\mathbf{a},\beta} \circ \Phi'_\lambda \circ F_{\mathbf{a},-\beta} \\ \Phi_{-\lambda} = F_{\mathbf{a},\beta} \circ \Phi'_{-\lambda} \circ F_{\mathbf{a},-\beta} \end{cases} .$$

Remark 6.3. In the above theorem, if $m \in \mathbb{N}_{>0}$ and $c \neq 0$, then $\beta = \frac{m}{k_0} \left(\frac{c_{k_0+m} - c'_{k_0+m}}{c_{k_0}} \right)$.

REFERENCES

- [BDM08] P. Bonckaert and P. De Maesschalck. Gevrey normal forms of vector fields with one zero eigenvalue. *J. Math. Anal. Appl.*, 344(1):301–321, 2008.
- [Bit15] Amaury Bittmann. Doubly-resonant saddle-nodes in $(\mathbb{C}^3, 0)$ and the fixed singularity at infinity in the painlevé equations: formal classification. working paper or preprint, October 2015.
- [Bit16a] Amaury Bittmann. Doubly-resonant saddle-nodes in $(\mathbb{C}^3, 0)$ and the fixed singularity at infinity in the Painlevé equations (part II): sectorial normalization. working paper or preprint, May 2016.
- [Bit16b] Amaury Bittmann. *Doubly-resonant saddle-nodes in $(\mathbb{C}^3, 0)$ and the fixed singularity at infinity in the Painlevé equations*. PhD thesis, writing in progress, 2016.
- [HKM61] Masuo Hukuhara, Tosihusa Kimura, and Tizuko Matuda. *Equations différentielles ordinaires du premier ordre dans le champ complexe*. Publications of the Mathematical Society of Japan, 7. The Mathematical Society of Japan, Tokyo, 1961.

- [Iwa80] Masahiro Iwano. On a general solution of a nonlinear 2-system of the form $x^2 dw/dx = \Lambda w + xh(x, w)$ with a constant diagonal matrix Λ of signature $(1, 1)$. *Tohoku Math. J. (2)*, 32(4):453–486, 1980.
- [Kap04] A. A. Kapaev. Quasi-linear Stokes phenomenon for the Painlevé first equation. *J. Phys. A*, 37(46):11149–11167, 2004.
- [Mal95] Bernard Malgrange. Somme des séries divergentes. In *Exposition. Math*, volume 13, pages 163–222, 1995.
- [Mar81] Jean Martinet. Normalisation des champs de vecteurs holomorphes (d’après A.-D. Brjuno). In *Bourbaki Seminar, Vol. 1980/81*, volume 901 of *Lecture Notes in Math.*, pages 55–70. Springer, Berlin-New York, 1981.
- [MR82] Jean Martinet and Jean-Pierre Ramis. Problèmes de modules pour des équations différentielles non linéaires du premier ordre. *Inst. Hautes Études Sci. Publ. Math.*, (55):63–164, 1982.
- [MR83] Jean Martinet and Jean-Pierre Ramis. Classification analytique des équations différentielles non linéaires résonnantes du premier ordre. *Ann. Sci. École Norm. Sup. (4)*, 16(4):571–621 (1984), 1983.
- [Nov93] V. Yu. Novokshënov. Nonlinear Stokes phenomenon for the second Painlevé equation. *Phys. D*, 63(1-2):1–7, 1993.
- [RS93] Jean Pierre Ramis and L Stolovitch. Divergent series and holomorphic dynamical systems. *Unpublished lecture notes*, 1:993, 1993.
- [Shi83] Shun Shimomura. Analytic integration of some nonlinear ordinary differential equations and the fifth Painlevé equation in the neighbourhood of an irregular singular point. *Funkcial. Ekvac.*, 26(3):301–338, 1983.
- [Sto96] Laurent Stolovitch. Classification analytique de champs de vecteurs 1-résonnants de $(\mathbf{C}^n, 0)$. *Asymptotic Anal.*, 12(2):91–143, 1996.
- [Tey03] Loïc Teyssier. Analytical classification of saddle-node vector fields. *C. R. Math. Acad. Sci. Paris*, 336(8):619–624, 2003.
- [Tey04] Loïc Teyssier. Équation homologique et cycles asymptotiques d’une singularité nœud-col. *Bulletin des Sciences Mathématiques*, 128(3):167–187, 2004.
- [Yos85] Setsuji Yoshida. 2-parameter family of solutions for Painlevé equations (I)–(V) at an irregular singular point. *Funkcial. Ekvac.*, 28(2):233–248, 1985.

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