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Some results on exponential synchronization of nonlinear systems (long version)

Vincent Andrieu, Bayu Jayawardhana, Sophie Tarbouriech

Abstract—Based on recent works on transverse exponential stability, we establish some necessary and sufficient conditions for the existence of a (locally) exponential synchronizing control law. We show that the existence of a structured synchronizer is equivalent to the existence of a stabilizer for the individual linearized systems (on the synchronization manifold) by a linear state feedback. This, in turn, is also equivalent to the existence of a symmetric covariant tensor field, which satisfies a Control Matrix Function inequality. Based on this result, we provide the construction of such synchronizer via backstepping approaches. In some particular cases, we show how global exponential synchronization may be obtained.

I. INTRODUCTION

Controlled synchronization, as a coordinated control problem of a group of autonomous systems, has been regarded as one of important group behaviors. It has found its relevance in many engineering applications, such as, the distributed control of (mobile) robotic systems, the control and reconfiguration of devices in the context of internet-of-things, and the synchronization of autonomous vehicles (see, for example, [15]).

For linear systems, the solvability of this problem and, as well as, the design of controller, have been thoroughly studied in literature. To name a few, we refer to the classical work on the nonlinear Goodwin oscillators [12], to the synchronization of linear systems in [24], [22] and to the recent works in nonlinear systems [20], [10], [9], [8], [21]. For linear systems, the solvability of synchronization problem reduces to the solvability of stabilization of individual systems by either an output or state feedback. It has recently been established in [24] that for linear systems, the solvability of the output synchronization problem is equivalent to the existence of an internal model, which is a well-known concept in the output regulation theory.

The generalization of these results to the nonlinear setting has appeared in the literature (see, for example, [16], [7], [17], [14], [20], [10], [9], [8], [21], [13]). In these works, the synchronization of nonlinear systems with a fixed network topology can be solved under various different sufficient conditions.

For instance, the application of passivity theory plays a key role in [7], [17], [20], [8], [21], [13]. By using the input/output

passivity property, the synchronization control law in these works can simply be given by the relative output measurement. Another approach for synchronizing nonlinear systems is by using output regulation theory as pursued in [14], [10], [16]. In these papers, the synchronization problem is reformulated as an output regulation problem where the output of each system has to track an exogeneous signal driven by a common exosystem and the resulting synchronization control law is again given by relative output measurement. Lastly, another synchronization approach that has gained interest in recent years is via incremental stability [5] or other related notions, such as, convergent systems [16]. If we restrict ourselves to the class of incremental ISS, as discussed in [5], the synchronizer can again be based on the relative output/state measurement.

Despite assuming a fixed network topology, necessary and sufficient condition for the solvability of synchronization problem of nonlinear systems is not yet established. Therefore, one of our main contributions of this paper is the characterization of controlled synchronization for general nonlinear systems with fixed network topology. Using recent results on the transverse exponential contraction, we establish some necessary and sufficient conditions for the solvability of a (locally) exponential synchronization. It extends the work in [2] where only two interconnected systems are discussed. We show that a necessary condition for achieving synchronization is the existence of a symmetric covariant tensor field of order two whose Lie derivative has to satisfy a Control Matrix Function (CMF) inequality, which is similar to the Control Lyapunov Function and detailed later in Section III.

This paper extends our preliminary work presented in [4]. In particular, we provide detailed proofs for all main results (which were exempted from the aforementioned paper) and additionally, we present the backstepping approach that allows us to construct a CMF-based synchronizer, as well as, the extension of the local synchronization result to the global one for a specific case.

The paper is organized as follows. We present the problem formulation of synchronization in Section II. In Section III, we present our first main results on necessary conditions to the solvability of the synchronization problem. Some sufficient conditions for local or global synchronization are given in Section IV. A constructive synchronizer design is presented in Section V, where a backstepping procedure is given for designing a CMF-based synchronizing control law.

Notation. The vector of all ones with a dimension N is denoted by $\mathbb{1}_N$. We denote the identity matrix of dimension n by I_n or I when no confusion is possible. Given M_1, \dots, M_N

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square matrices, $\text{diag}\{M_1, \dots, M_N\}$ is the matrix defined as

$$\text{diag}\{M_1, \dots, M_N\} = \begin{bmatrix} M_1 & & \\ & \ddots & \\ & & M_N \end{bmatrix}.$$

Given a vector field f on \mathbb{R}^n and a covariant two tensor $P : \mathbb{R}^n \rightarrow \mathbb{R}^{m \times m}$, P is said to have a derivative along f denoted $\mathfrak{d}_f P$ if the following limit exists

$$\mathfrak{d}_f P(z) = \lim_{h \rightarrow 0} \frac{P(Z(z, h)) - P(z)}{h}, \quad (1)$$

where $Z(z, \cdot)$ is the flow of the vector field f with an initial state z in \mathbb{R}^n . In that case and, when $m = n$ and f is C^1 $L_f P$ is the Lie derivative of the tensor along f which is defined as

$$L_f P(z) = \mathfrak{d}_f P(z) + P(z) \frac{\partial f}{\partial z}(z) + \frac{\partial f}{\partial z}(z)^\top P(z). \quad (2)$$

II. PROBLEM DEFINITION

A. System description and communication topology

In this note, we consider the problem of synchronizing N identical nonlinear systems with $N \geq 2$. For every $i = 1, \dots, N$, the i -th system Σ_i is described by

$$\dot{x}_i = f(x_i) + g(x_i)u_i, \quad i = 1, \dots, N \quad (3)$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^p$ and the functions f and g are assumed to be C^2 . In this setting, all systems has the same drift vector field f and the same control vector field $g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times p}$, but not the same controls in \mathbb{R}^p . For simplicity of notation, we denote the complete state variables by $x = [x_1^\top \dots x_N^\top]^\top$ in \mathbb{R}^{Nn} .

The synchronization manifold \mathcal{D} , where the state variables of different systems agree with each other, is defined by

$$\mathcal{D} = \{(x_1, \dots, x_N) \in \mathbb{R}^{Nn} \mid x_1 = x_2 = \dots = x_N\}.$$

For every x in \mathbb{R}^{Nn} , we denote the Euclidean distance to the set \mathcal{D} by $|x|_{\mathcal{D}}$.

The communication graph \mathcal{G} , which is used for synchronizing the state through distributed control u_i , $i = 1, \dots, N$, is assumed to be an undirected graph and is defined by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where \mathcal{V} is the set of N nodes (where the i -th node is associated to the system Σ_i) and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is a set of M edges that define the pairs of communicating systems. Moreover we assume that the graph \mathcal{G} is connected.

Let us, for every edge k in \mathcal{G} connecting node i to node j , label one end (e.g., the node i) by a positive sign and the other end (e.g., the node j) by a negative sign. The incidence matrix D that corresponds to \mathcal{G} is an $N \times M$ matrix such that

$$d_{i,k} = \begin{cases} +1 & \text{if node } i \text{ is the positive end of edge } k \\ -1 & \text{if node } i \text{ is the negative end of edge } k \\ 0 & \text{otherwise} \end{cases}$$

Using D , the Laplacian matrix L can be given by $L = DD^\top$ whose kernel, by the connectedness of \mathcal{G} , is spanned by $\mathbf{1}_N$. We will need the following lemma on the property of L in some results.

Lemma 1: Let $L = \begin{bmatrix} L_{11} & L_{1,2:N} \\ L_{1,2:N}^\top & L_{2:N,2:N} \end{bmatrix}$ be a non-zero balanced Laplacian matrix associated to an undirected graph \mathcal{G} where L_{11} is a scalar. Then, the eigenvalues of the $(N-1) \times (N-1)$ matrix $\bar{L} := L_{2:N,2:N} - \mathbf{1}_{N-1} L_{1,2:N}$ are the same as the non-zero eigenvalues of L with the same multiplicity. Moreover, if the graph is connected then $-\bar{L}$ is Hurwitz.

The proof of Lemma 1 can be found in Appendix A.

B. Synchronization problem formulation

Using the description of the interconnected systems via \mathcal{G} , the state synchronization control problem is defined as follows.

Definition 1: The control laws $u_i = \phi_i(x)$, $i = 1, \dots, N$ solve the *local uniform exponential synchronization* problem for (3) if the following conditions hold:

- 1) For all non-communicating pair (i, j) (i.e., $(i, j) \notin \mathcal{E}$),

$$\frac{\partial \phi_i}{\partial x_j}(x) = \frac{\partial \phi_j}{\partial x_i}(x) = 0, \quad \forall x \in \mathbb{R}^{Nn};$$

- 2) For all $x \in \mathcal{D}$, $\phi(x) = 0$ (i.e., ϕ is zero on \mathcal{D}); and
- 3) The manifold \mathcal{D} of the closed-loop system

$$\dot{x}_i = f(x_i) + g(x_i)\phi_i(x), \quad i = 1, \dots, N \quad (4)$$

is uniformly exponentially stable, i.e., there exist positive constants r , k and $\lambda > 0$ such that for all x in \mathbb{R}^{Nn} satisfying $|x|_{\mathcal{D}} < r$,

$$|X(x, t)|_{\mathcal{D}} \leq k \exp(-\lambda t) |x|_{\mathcal{D}}, \quad (5)$$

where $X(x, t)$ denotes the solution initiated from x , holds for all t in the time domain of existence of solution.

When $r = \infty$, it is called the *global uniform exponential synchronization* problem. \triangle

In this definition, the condition 1) implies that the solution u_i is a distributed control law that requires only a local state measurement from its neighbors in the graph \mathcal{G} .

An important feature of our study is that we focus on exponential stabilization of the synchronizing manifold. This allows us to rely on the study developed in [2] (or [3]) in which an infinitesimal characterization of exponential stability of a transverse manifold is given. As it will be shown in the following section this allows us to formalize some necessary and sufficient conditions in terms of matrix functions ensuring the existence of a synchronizing control law.

III. NECESSARY CONDITIONS

A. Infinitesimal stabilizability conditions

In [2], a first attempt has been made to give necessary conditions for the existence of an exponentially synchronizing control law for only two agents. In [3], the same problem has been addressed for N agents but without any communication constraints (all agents can communicate with all others). In both cases, it is shown that assuming some bounds on derivatives of the vector fields and assuming that the synchronizing control law is invariant by permutation of agents, the following two properties are necessary conditions.

IS Infinitesimal stabilizability. The couple (f, g) is such that the n -dimensional manifold $\{\tilde{z} = 0\}$ of the transversally linear system

$$\dot{\tilde{z}} = \frac{\partial f}{\partial z}(z)\tilde{z} + g(z)\tilde{u} \quad (6a)$$

$$\dot{z} = f(z) \quad (6b)$$

with \tilde{z} in \mathbb{R}^n and z in \mathbb{R}^n is stabilizable by a state feedback that is linear in \tilde{z} (i.e., $\tilde{u} = h(z)\tilde{z}$ for some function $h : \mathbb{R}^n \rightarrow \mathbb{R}^{p \times n}$).

CMF Control Matrix Function. For all positive definite matrix $Q \in \mathbb{R}^{n \times n}$, there exist a continuous function $P : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$, which values are symmetric positive definite matrices and strictly positive real numbers \underline{p} and \bar{p} such that

$$\underline{p}I_n \leq P(z) \leq \bar{p}I_n \quad (7)$$

holds for all $z \in \mathbb{R}^n$, and the inequality (see (1) and (2))

$$v^\top L_f P(z)v \leq -v^\top Qv \quad (8)$$

holds for all (v, z) in $\mathbb{R}^n \times \mathbb{R}^n$ satisfying $v^\top P(z)g(z) = 0$.

An important feature of properties IS and CMF comes from the fact that they are properties of each individual agent, independent of the network topology. The first one is a local stabilizability property. The second one establishes that there exists a symmetric covariant tensor field of order two denoted by P whose Lie derivative satisfies a certain inequality in some specific directions. This type of condition can be related to the notion of control Lyapunov function, which is a characterization of stabilizability as studied by Artstein in [6] or Sontag in [23]. This property can be regarded as an Artstein like condition. The dual of the CMF property has been thoroughly studied in [18] when dealing with an observer design ([18, Eq. (8)], see also [2] or [1]).

B. Necessity of IS and CMF for exponential synchronization

We show that properties IS and CMF are still necessary conditions if one considers a network of agents with a communication graph \mathcal{G} as given in II-A. Hence, as this is already the case for linear system, we recover the paradigm, which establishes that a necessary condition for synchronization is a stabilizability property for each individual agent.

Theorem 1: Consider the interconnected systems in (3) with the communication graph \mathcal{G} and assume that there exists a control law $u = \phi(x)$ where $\phi(x) = [\phi_1^\top(x) \ \dots \ \phi_N^\top(x)]^\top$ in \mathbb{R}^{Np} that solves the local uniform exponential synchronization for (3). Assume moreover that g is bounded, f, g and the ϕ_i 's have bounded first and second derivatives and the closed-loop system is complete. Then properties IS and CMF hold.

Note that this theorem is a refinement of the result which is written in [4] since we have removed an assumption related to the structure of the control law.

C. Proof of Theorem 1

Proof : The first part of the proof is to show that the synchronizing manifold satisfies a transverse uniform exponential stability property. This allows us to use tools developed in [3] and show a stabilizability property for an Nn -dimensional. Employing some kind of Lyapunov projection, we are able to obtain the stabilizability properties for the n -dimensional transversally linear system (6a).

Let $e = [e_2^\top \ e_3^\top \ \dots \ e_N^\top]^\top$ with $e_i = x_i - x_1$, $i = 2, \dots, N$, and $z = x_1$. The closed-loop system (3) with the control law ϕ is given by

$$\dot{e} = F(e, z), \quad \dot{z} = G(e, z) \quad (9)$$

with e in $\mathbb{R}^{(N-1)n}$, z in \mathbb{R}^n and where

$$F = [F_2^\top \ F_3^\top \ \dots \ F_N^\top]^\top \quad (10)$$

$$F_i(e, z) = f(z + e_i) - f(z) + g(z + e_i)\bar{\phi}_i(e, z) - g(x_1)\bar{\phi}_1(e, z), \quad (11)$$

$$G(e, z) = f(z) + g(z)\bar{\phi}_1(e, z), \quad (12)$$

where we have used the notation

$$\bar{\phi}_i(e, z) = \phi_i(z, z + e_2, \dots, z + e_N). \quad (13)$$

Note that we have

$$|e|^2 = \sum_{i=2}^N |x_i - x_1|^2, \quad (14)$$

$$\leq 2 \sum_{i=2}^N |x_i - \bar{x}|^2 + 2(N-1)|\bar{x} - x_1|^2 \quad (14)$$

$$\leq 2(N-1)|x|_{\mathcal{D}}^2, \quad (15)$$

and,

$$|x|_{\mathcal{D}}^2 = \min_{z \in \mathbb{R}^n} \sum_{i=1}^N |z - x_i|^2 \leq \sum_{i=1}^N |x_1 - x_i|^2 = |e|^2. \quad (16)$$

Hence, if we denote $E(e, z, t)$ the e components of the solution to (9), then (5) implies for all (e, z) in $\mathbb{R}^{(N-1)n} \times \mathbb{R}^n$

$$|E(e, z, t)| \leq \sqrt{(N-1) \left(1 + \frac{N-1}{N^2}\right)} k \exp(-\lambda t) |e|.$$

It follows that the manifold $e = 0$ is locally uniformly (in z) exponentially stable for (9). In other words, property TULES-NL (see Section C in the Appendix) is satisfied. Employing the assumptions on the bounds on f, g, ϕ and its derivatives, we conclude with [3, Prop. 1] that the so-called Property ULMTE is satisfied (see Section C in the Appendix for the definition). Hence there exists a C^1 function with matrix valued $P_N : \mathbb{R}^n \rightarrow \mathbb{R}^{(N-1)n \times (N-1)n}$ and a positive definite matrix Q_N in $\mathbb{R}^{(N-1)n \times (N-1)n}$ such that for all z in \mathbb{R}^n

$$\partial_f P_N(z) + P_N(z) \frac{\partial F}{\partial e}(0, z) + \frac{\partial F}{\partial e}(0, z)^\top \leq -Q_N, \quad (17)$$

and

$$\underline{p}_N \mathbf{I} \leq P_N(z) \leq \bar{p}_N \mathbf{I}. \quad (18)$$

For each z , let us decompose

$$P_N(z) = \begin{bmatrix} S(z) & T(z) \\ T(z)^\top & R(z) \end{bmatrix},$$

with S taking value in $\mathbb{R}^{n \times n}$ and T and R of appropriate dimensions.

Consider the C^1 matrix function $P : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ defined as follows.

$$\begin{aligned} P(z) &= \begin{bmatrix} \mathbf{I} & -T(z)R(z)^{-1} \end{bmatrix} P_N(z) \begin{bmatrix} \mathbf{I} \\ -R(z)^{-1}T(z)^\top \end{bmatrix}, \\ &= S(z) - T(z)R(z)^{-1}T(z)^\top. \end{aligned}$$

We will show that this matrix function P satisfies all assumptions of property **CMF**. First of all, we show that P satisfies (7). Pre- and post- multiplying equation (18) by the two matrices $\begin{bmatrix} \mathbf{I} & -T(z)R(z)^{-1} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{I} \\ -R(z)^{-1}T(z)^\top \end{bmatrix}$, yields

$$\begin{aligned} \underline{p}_N (\mathbf{I} + T(z)R(z)^{-2}T(z)^\top) &\leq P(z) \\ &\leq \bar{p}_N (\mathbf{I} + T(z)R(z)^{-2}T(z)^\top). \end{aligned}$$

On another hand,

$$|T(z)T(z)^\top| = \left| \begin{bmatrix} 0 & \mathbf{I} \end{bmatrix} P(z) \begin{bmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{bmatrix} P(z) \begin{bmatrix} 0 \\ \mathbf{I} \end{bmatrix} \right| \leq \bar{p}_N^2.$$

Moreover,

$$\underline{p}_N \mathbf{I} \leq R(z) \leq \bar{p}_N \mathbf{I}.$$

Which gives by pre and post multiplying by $R(z)^{-\frac{1}{2}}$

$$R(z)^{-1} \underline{p}_N \leq \mathbf{I} \leq \bar{p}_N R(z)^{-1}.$$

Consequently, it yields equation (7) since we have

$$\underline{p}_N \mathbf{I} \leq P(z) \leq \bar{p}_N \left(1 + \frac{\bar{p}_N^2}{\underline{p}_N^2} \right) \mathbf{I}.$$

We now show that (8) holds. Note that we have

$$\begin{aligned} \mathfrak{d}_f P(z) &= \mathfrak{d}_f S(z) - \mathfrak{d}_f T(z)R(z)^{-1}T(z)^\top \\ &\quad + T(z)R(z)^{-1} \mathfrak{d}_f R(z)R(z)^{-1}T(z)^\top \\ &\quad - T(z)R(z)^{-1} \mathfrak{d}_f T(z)^\top, \end{aligned}$$

which gives

$$\mathfrak{d}_f P(z) = \begin{bmatrix} \mathbf{I} & -T(z)R(z)^{-1} \end{bmatrix} \mathfrak{d}_f P_N(z) \begin{bmatrix} \mathbf{I} \\ -R(z)^{-1}T(z)^\top \end{bmatrix}. \quad (19)$$

Note now that

$$\begin{aligned} &\begin{bmatrix} \mathbf{I} & -T(z)R(z)^{-1} \end{bmatrix} P_N(z) \frac{\partial F}{\partial \bar{e}}(0, z) \begin{bmatrix} \mathbf{I} \\ -R(z)^{-1}T(z)^\top \end{bmatrix} \\ &= \begin{bmatrix} P(z) & 0 \end{bmatrix} \frac{\partial F}{\partial \bar{e}}(0, z) \begin{bmatrix} \mathbf{I} \\ -R(z)^{-1}T(z)^\top \end{bmatrix} \\ &= P(z) \frac{\partial F_2}{\partial \bar{e}}(0, z) \begin{bmatrix} \mathbf{I} \\ -R(z)^{-1}T(z)^\top \end{bmatrix} \end{aligned} \quad (20)$$

On another hand, by the definition of $\bar{\phi}$ in (13) and the second point of Definition 1, it follows that $\bar{\phi}_i(0, z) = 0$. This implies that for every $i = 2, \dots, N$,

$$\begin{aligned} \frac{\partial F_i}{\partial \bar{e}_i}(0, z) &= \frac{\partial f}{\partial z}(z) \\ &\quad + g(z) \left[\frac{\partial \bar{\phi}_i}{\partial \bar{e}_i}(0, z) - \frac{\partial \bar{\phi}_1}{\partial \bar{e}_i}(0, z) \right] \end{aligned} \quad (21)$$

and for all $j \neq i$,

$$\frac{\partial F_i}{\partial \bar{e}_j}(0, z) = g(z) \left[\frac{\partial \bar{\phi}_i}{\partial \bar{e}_j}(0, z) - \frac{\partial \bar{\phi}_1}{\partial \bar{e}_j}(0, z) \right]. \quad (22)$$

Consequently,

$$\frac{\partial F_2}{\partial \bar{e}}(0, z) \begin{bmatrix} \mathbf{I} \\ -R(z)^{-1}T(z)^\top \end{bmatrix} = \frac{\partial f}{\partial z}(z) + g(z)h(z) \quad (23)$$

where

$$\begin{aligned} h(z) &= \begin{bmatrix} \frac{\partial \bar{\phi}_2}{\partial \bar{e}_2}(z) - \frac{\partial \bar{\phi}_1}{\partial \bar{e}_2}(z) & \dots & \frac{\partial \bar{\phi}_2}{\partial \bar{e}_N}(z) - \frac{\partial \bar{\phi}_1}{\partial \bar{e}_N}(z) \end{bmatrix} \\ &\quad \times \begin{bmatrix} \mathbf{I} \\ -R(z)^{-1}T(z)^\top \end{bmatrix}. \end{aligned} \quad (24)$$

Consequently, pre- and post- multiplying equation (17) by the two matrices $\begin{bmatrix} \mathbf{I} & -T(z)R(z)^{-1} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{I} \\ -R(z)^{-1}T(z)^\top \end{bmatrix}$, and employing equations (19), (20), (23) and (24) yield a positive definite matrix Q such that

$$\begin{aligned} \mathfrak{d}_f P(z) + P(z) \left[\frac{\partial f}{\partial x}(z) + g(z)h(z) \right] \\ + \left[\frac{\partial f}{\partial x}(z) + g(z)h(z) \right]^\top P(z) \leq -Q. \end{aligned} \quad (25)$$

From this equation, (8) is satisfied and Property **CMF** holds.

Moreover (25) implies that property **ULMTE** introduced in [3] (see Appendix C) is satisfied for the system

$$\dot{z} = \bar{F}(e, z), \quad \dot{z} = \bar{G}(e, z),$$

where

$$\bar{F}(e, z) = f(e + z) - f(z) + g(z)h(z)e, \quad \bar{G}(e, z) = f(z).$$

Hence, employing Proposition 2 in the appendix, one can conclude that property **IS** is satisfied with the control $\tilde{u} = h(z)\tilde{z}$. \square

In the following section, we discuss the possibility to design an exponential synchronizing control law based on these necessary conditions.

IV. SUFFICIENT CONDITION

A. Sufficient conditions for local exponential synchronization

The interest of the Property **CMF** given in Subsection III-A is to use the symmetric covariant tensor P in the design of a local synchronizing control law. Indeed, following one of the main results in [3], we get the following sufficient condition for the solvability of (local) uniform exponential synchronization problem. The first assumption is that, up to a scaling factor, the control vector field g is a gradient field

with P as a Riemannian metric (see also [11] for similar integrability assumption). The second one is related to the CMF property.

Theorem 2 (Local sufficient condition): Assume that g is bounded and that f and g have bounded first and second derivatives. Assume that there exists a C^2 function $P : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ which values are symmetric positive definite matrices and with a bounded derivative that satisfies the following two conditions.

1. There exist a C^2 function $U : \mathbb{R}^n \rightarrow \mathbb{R}$ which has bounded first and second derivatives, and a C^1 function $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^p$ which has bounded first and second derivatives such that

$$\frac{\partial U}{\partial z}(z)^\top = P(z)g(z)\alpha(z), \quad (26)$$

holds for all z in \mathbb{R}^n ; and

2. There exist a symmetric positive definite matrix Q and positive constants \underline{p}, \bar{p} and $\rho > 0$ such that (7) holds and

$$L_f P(z) - \rho \frac{\partial U}{\partial z}(z)^\top \frac{\partial U}{\partial z}(z) \leq -Q, \quad (27)$$

hold for all z in \mathbb{R}^n .

Then, given a connected graph \mathcal{G} with associated Laplacian matrix $L = (L_{ij})$, there exists a constant $\underline{\ell}$ such that the control law $u = \phi(x)$ with $\phi = [\phi_1^\top \dots \phi_N^\top]^\top$ given by

$$\phi_i(x) = -\ell \alpha(x_i) \sum_{j=1}^N L_{ij} U(x_j) \quad (28)$$

with $\ell \geq \underline{\ell}$ solves the local uniform exponential synchronization of (3).

Remark 1: Assumption (27) is stronger than the necessary condition CMF. Note however, that employing some variation on Finsler Lemma (see [3] for instance) it can be shown that these assumptions are equivalent when x remains in a compact set.

Remark 2: Note that for all $x = \mathbf{1}_N \otimes z = (z, \dots, z)$ in \mathcal{D} and for all (i, j) with $i \neq j$

$$\frac{\partial \phi_i}{\partial x_j}(x) = -\ell \alpha(z) L_{ij} \frac{\partial U}{\partial z}(z). \quad (29)$$

Hence, for all $x = \mathbf{1}_N \otimes z$ in \mathcal{D} , we get

$$\frac{\partial \phi}{\partial x}(x) = -\ell L \otimes \alpha(z) \frac{\partial U}{\partial z}(z). \quad (30)$$

Proof : First of all, note that the control law ϕ satisfies the condition 1) and 2) in Definition 1. Indeed, for all x and all (i, j) with $i \neq j$

$$\frac{\partial \phi_i}{\partial x_j}(x) = -\ell \alpha(x_i) L_{ij} \frac{\partial U}{\partial z}(x_j).$$

If $(i, j) \notin \mathcal{E}$, it yields $L_{ij} = 0$ and consequently $\frac{\partial \phi_i}{\partial x_j}(x) = 0$.

Moreover, when x is in \mathcal{D} , i.e., $x = \mathbf{1}_N \otimes z = (z, \dots, z)$ for all i

$$\phi_i(x) = -\ell \alpha(z) \left(\sum_{j=1}^N L_{ij} \right) U(z) = 0.$$

It remains to show that condition 3) of Definition 1 holds. More precisely, we need to prove that the manifold \mathcal{D} is locally exponentially stable along the solution of the closed-loop system.

As in the proof of Theorem 1, let us denote $e = (e_2, \dots, e_N)$ with $e_i = x_1 - x_i$ and $z = x_1$. Note that the closed-loop system may be rewritten as in (9) with the vector fields F and G as defined in (10)–(12) with ϕ as the control law.

The rest of the proof is to apply [3, Proposition 3]. For this purpose, we need to show that for closed-loop system (10)–(12) the property ULMTE introduced in [3] and given in Section C is satisfied.

By the assumption on the graph being connected and together with Lemma 1, we have that the matrix $A = -(L_{2:N,2:N} - \mathbf{1}_{N-1} L_{1,2:N})$ is Hurwitz. Let S in $\mathbb{R}^{(N-1) \times (N-1)}$ be a symmetric positive definite matrix solution to the Lyapunov equation

$$SA + A^\top S \leq -\nu S \quad (31)$$

where ν is a positive real number.

Consider the C^1 function $P_N : \mathbb{R}^n \rightarrow \mathbb{R}^{(N-1)n \times (N-1)n}$ defined as

$$P_N(z) = S \otimes P(z).$$

Our aim is to show that the closed loop system satisfies property ULMTE given in Section C. First of all, note that S being symmetric positive definite, with (7), it yields the existence of positive real numbers $\underline{p}_N, \bar{p}_N$ such that

$$\underline{p}_N \mathbf{I}_{N-1} \leq P_N(z) \leq \bar{p}_N \mathbf{I}_{N-1}.$$

Hence, equation (53) is satisfied.

Note that we have $G(0, z) = f(z)$. Moreover we have

$$\mathfrak{d}_{G(0,z)} P_N(z) = S \otimes \mathfrak{d}_f P(z).$$

Note that with properties (21), (22) and (30), it follows that

$$\begin{aligned} \frac{\partial F}{\partial \tilde{e}}(0, z) &= \mathbf{I}_{N-1} \otimes \frac{\partial f}{\partial z}(z) \\ &\quad + \ell A \otimes \left(\alpha(z) g(z) \frac{\partial U}{\partial z}(z) \right). \end{aligned} \quad (32)$$

Hence,

$$\begin{aligned} \mathfrak{d}_{G(0,z)} P_N(z) + P_N(z) \frac{\partial F}{\partial \tilde{e}}(0, z) + \frac{\partial F}{\partial \tilde{e}}(0, z)^\top P_N(z) \\ = S \otimes \left(\mathfrak{d}_f P(z) + P(z) \frac{\partial f}{\partial z}(z) + \frac{\partial f}{\partial z}(z)^\top P(z) \right) \\ + \ell (SA + A^\top S) \otimes \left(\frac{\partial U}{\partial z}(z)^\top \frac{\partial U}{\partial z}(z) \right). \end{aligned}$$

With (31) and (27) this implies that

$$\begin{aligned} \mathfrak{d}_f P_N(z) + P_N(z) \frac{\partial F}{\partial \tilde{e}}(0, z) + \frac{\partial F}{\partial \tilde{e}}(0, z)^\top P_N(z) \\ \leq S \otimes \left(-Q + (\rho - \ell \nu) \frac{\partial U}{\partial z}(z)^\top \frac{\partial U}{\partial z}(z) \right). \end{aligned}$$

Hence, by choosing $\underline{\ell} \geq \frac{\rho}{\nu}$, inequality (52) holds and consequently Property ULMTE holds. The last part of the proof is to make sure that the vector field F has bounded first and

second derivatives and that the vector field G has bounded first derivative. Note that by employing the bounds on the functions P , f , g , α and their derivatives, the result immediately follows from Proposition 3 in Section C. Indeed, this implies that Property TULES-NL holds and consequently, $e = 0$ is (locally) exponentially stable manifold for system (10)–(12) in closed loop with the control (28). With inequalities (15) and (16), it implies that inequality (5) holds for r sufficiently small. \square

B. Sufficient conditions for global exponential synchronization

Note that in [3] with an extra assumption related to the metric (the level sets of U are totally geodesic sets with respect to the Riemannian metric obtained from P), it is shown that global synchronization may be achieved when considering only two agents which are connected. It is still an open question to know if global synchronization may be achieved in the general nonlinear context with more than two agents. However in the particular case in which the matrix $P(z)$ and the vector field g are constant, then global synchronization may be achieved as this is shown in the following theorem.

Theorem 3 (Global sufficient condition): Assume that $g(z) = G$ and there exists a symmetric positive definite matrix P in $\mathbb{R}^{n \times n}$, a symmetric positive definite matrix Q and $\rho > 0$ such that

$$P \frac{\partial f}{\partial z}(z) + \frac{\partial f}{\partial z}(z)^\top P - \rho P G G^\top P \leq -Q. \quad (33)$$

Assume moreover that the graph is connected with Laplacian matrix L . Then there exist constants $\underline{\ell}$ and positive real numbers c_1, \dots, c_N such that the control law $u = \phi(x)$ with $\phi = [\phi_1^\top \ \dots \ \phi_N^\top]^\top$ given by

$$\phi_i(x) = -\ell c_i \sum_{j=1}^N L_{ij} G^\top P x_j \quad (34)$$

with $\ell \geq \underline{\ell}$, solves the global uniform exponential synchronization for (3).

Proof : Let $c_j = 1$ for $j = 2, \dots, N$. Hence only c_1 is different from 1 and remains to be selected. As in the proof of Theorem 1, let us denote $e = (e_2, \dots, e_N)$ with $e_i = x_1 - x_i$ and $z = x_1$. Note that for $i = 2, \dots, N$, we have along the solution of the system (3) with u defined in (34),

$$\begin{aligned} \dot{e}_i &= f(z) - \ell c_1 \sum_{j=1}^N L_{1j} G G^\top P x_j \\ &\quad - f(z + e_i) + \ell \sum_{j=1}^N L_{ij} G G^\top P x_j. \end{aligned}$$

Note that L being a Laplacian, we have for all i in $[1, N]$ the equality $\sum_{j=1}^N L_{ij} = 0$. Consequently, we can add the term $\ell c_1 \sum_{j=1}^N L_{1j} G G^\top P x_1$ and subtract the term

$\ell \sum_{j=1}^N L_{ij} G G^\top P x_1$ in the preceding equation above so that for $i = 2, \dots, N$

$$\begin{aligned} \dot{e}_i &= f(z) - \ell c_1 \sum_{j=1}^N L_{1j} G G^\top P (x_j - x_1) \\ &\quad - f(z + e_i) + \ell \sum_{j=1}^N L_{ij} G G^\top P (x_j - x_1), \\ &= f(z) - f(z + e_i) - \ell \sum_{j=2}^N (L_{ij} - c_1 L_{1j}) G G^\top P e_j. \end{aligned}$$

One can check that these equations can be written compactly as

$$\dot{e} = \left[\int_0^1 \Delta(z, e, s) ds + \ell (A(c_1) \otimes G G^\top P) \right] e,$$

with $A(c_1)$ is matrix in $\mathbb{R}^{(N-1) \times (N-1)}$, which depends on the parameter c_1 and is obtained from the Laplacian as :

$$A(c_1) = -[L_{2:N,2:N} - c_1 L_{1,2:N} \mathbf{1}_{N-1}],$$

where $L = \begin{bmatrix} L_{11} & L_{1,2:N} \\ L_{1,2:N}^\top & L_{2:N,2:N} \end{bmatrix}$ and Δ is the $(N-1)n \times n$ matrix valued function defined as

$$\Delta(z, e, s) = \text{Diag} \left\{ \frac{\partial f}{\partial z}(z - s e_2), \dots, \frac{\partial f}{\partial z}(z - s e_N) \right\}.$$

The following Lemma shows that by selecting c_1 sufficiently small the matrix A satisfies the following property. Its proof is given in the Appendix.

Lemma 2: If the communication graph is connected then there exist sufficiently small c_1 and $\mu > 0$ such that

$$A(c_1) + A(c_1)^\top \leq -\mu I$$

With this lemma in hand, we consider now the candidate Lyapunov function $V(e) = e^\top P_N e$, where $P_N = (I_{N-1} \otimes P)$. Note that along the solution, the time derivative of this function satisfies :

$$\dot{\widehat{V}}(e) = 2e^\top P_N \left[\int_0^1 \Delta(z, e, s) ds + \ell (A(c_1) \otimes G G^\top P) \right] e.$$

Note that we have

$$\begin{aligned} P_N \Delta(z, e, s) &= \text{Diag} \left\{ P \frac{\partial f}{\partial z}(z - s e_2), \dots, P \frac{\partial f}{\partial z}(z - s e_N) \right\}, \end{aligned}$$

and

$$\begin{aligned} 2e^\top (I_{N-1} \otimes P) (A(c_1) \otimes G G^\top P) e &= e^\top ([A(c_1) + A(c_1)^\top] \otimes P G G^\top P) e \\ &\leq -e^\top (\mu I_{N-1} \otimes P G G^\top P) e. \end{aligned}$$

Hence, we get $\dot{\widehat{V}}(e) \leq \int_0^s e^\top M(e, z, s) e ds$, where M is the $(N-1)n \times (N-1)n$ matrix defined as

$$M(e, z, s) = \text{Diag} \{ M_2(e, z, s), \dots, M_N(e, z, s) \},$$

with, for $i = 2, \dots, N$

$$M_i(e, z, s) = P \frac{\partial f}{\partial z}(z - se_i) + \frac{\partial f}{\partial z}(z - se_i)^\top P - 2\ell\mu PGG^\top P.$$

Note that by taking ℓ sufficiently large, with (33) this yields $M_i(e, z, s) \leq -Q$. This immediately implies that $\overline{V(e)} \leq -e^\top (I_{N-1} \otimes Q) e$. This ensures exponential convergence of e to zero on the time of existence of the solution. With (15) and (16), this yields global exponential synchronization of the closed-loop system. \square

In the following section, we show that the property CMF required to design a distributed synchronizing control law can be obtained for a large class of nonlinear systems. This is done via backstepping design.

V. CONSTRUCTION OF AN ADMISSIBLE TENSOR VIA BACKSTEPPING

A. Adding derivative (or backstepping)

As proposed in Theorem 2, a distributed synchronizing control law can be designed using a symmetric covariant tensor field of order 2, which satisfies (8). Given a general nonlinear system, the construction of such a matrix function P may be a hard task. In [19], a construction of the function P for observer based on the integration of a Riccati equation is introduced. Similar approach could be used in our synchronization problem. Note however that in our context an integrability condition (i.e. equation (26)) has to be satisfied by the function P . This constraint may be difficult to address when considering a Riccati equation approach.

In the following we present a constructive design of such a matrix P that resembles the backstepping method. This approach can be related to [26], [25] in which a metric is also constructed iteratively. We note that one of the difficulty we have here is that we need to propagate the integrability property given in equation (26).

For outlining the backstepping steps for designing P , we consider the case in which the vector fields (f, g) can be decomposed as follows

$$f(z) = \begin{bmatrix} f_a(z_a) + g_a(z_a)z_b \\ f_b(z_a, z_b) \end{bmatrix},$$

and,

$$g(z) = \begin{bmatrix} 0 \\ g_b(z) \end{bmatrix}, \quad 0 < \underline{g}_b \leq g_b(z) \leq \bar{g}_b$$

with $z = [z_a^\top \quad z_b]^\top$, z_a in \mathbb{R}^{n_a} and z_b in \mathbb{R} . In other words,

$$\dot{z}_a = f_a(z_a) + g_a(z_a)z_b, \quad \dot{z}_b = f_b(z) + g_b(z)u. \quad (35)$$

Let \mathcal{C}_a be a compact subset of \mathbb{R}^{n_a} . As in the standard backstepping approach, we make the following assumptions on the z_a -subsystem where z_b is treated as a control input to this subsystem.

Assumption 1 (z_a -Synchronizability): Assume that there exists a C^∞ function $P_a : \mathbb{R}^{n_a} \rightarrow \mathbb{R}^{n_a \times n_a}$ that satisfies the following conditions.

1. There exist a C^∞ function $U_a : \mathbb{R}^{n_a} \rightarrow \mathbb{R}$ and a C^∞ function $\alpha_a : \mathbb{R}^{n_a} \rightarrow \mathbb{R}$ such that

$$\frac{\partial U_a}{\partial z_a}(z_a)^\top = \alpha_a(z_a)P_a(z_a)g_a(z_a) \quad (36)$$

holds for all z_a in \mathcal{C}_a ;

2. There exist a symmetric positive definite matrix Q_a and positive constants $\underline{p}_a, \bar{p}_a$ and $\rho_a > 0$ such that

$$\underline{p}_a I_{n_a} \leq P_a(z_a) \leq \bar{p}_a I_{n_a}, \quad \forall z_a \in \mathbb{R}^{n_a}, \quad (37)$$

holds and

$$L_{f_a}P_a(z_a) - \rho_a \frac{\partial U_a}{\partial z_a}(z_a)^\top \frac{\partial U_a}{\partial z_a}(z_a) \leq -Q_a, \quad (38)$$

holds for all z_a in \mathcal{C}_a .

As a comparison to the standard backstepping method for stabilizing nonlinear systems in the strict-feedback form, the z_a -synchronizability conditions above are akin to the stabilizability condition of the upper subsystem via a control Lyapunov function. However, for the synchronizer design as in the present context, we need an additional assumption to allow the recursive backstepping computation of the tensor P . Roughly speaking, we need the existence of a mapping q_a such that the metric P_a becomes invariant along the vector field $\frac{g_a}{q_a}$. In other words, $\frac{g_a}{q_a}$ is a Killing vector field.

Assumption 2: There exists a non-vanishing smooth function $q_a : \mathbb{R}^{n_a} \rightarrow \mathbb{R}$ such that the metric obtained from P_a on \mathcal{C}_a is invariant along $\frac{g_a(z_a)}{q_a(z_a)}$. In other words, for all z_a in \mathcal{C}_a

$$L_{\frac{g_a(z_a)}{q_a(z_a)}}P_a(z_a) = 0. \quad (39)$$

Similar assumption can be found in [11] in the characterization of differential passivity.

Based on the Assumptions 1 and 2, we have the following theorem on the backstepping method for constructing a symmetric covariant tensor field P_b of the complete system (35).

Theorem 4: Assume that the z_a -subsystem satisfies Assumption 1 and Assumption 2 in the compact set \mathcal{C}_a with a $n_a \times n_a$ symmetric covariant tensor field P_a of order two and a non-vanishing smooth mapping $q_a : \mathbb{R}^{n_a} \rightarrow \mathbb{R}$. Then for all positive real number M_b , the system (35) with the state variables $z = (z_a, z_b) \in \mathbb{R}^{n_a+1}$ satisfies the Assumption 1 in the compact set $\mathcal{C}_a \times [-M_b, M_b] \subset \mathbb{R}^{n_a+1}$ with the symmetric covariant tensor field P_b be given by

$$P_b(z) = \begin{bmatrix} P_a(z_a) + S_a(z)S_a(z)^\top & S_a(z)q_a(z_a) \\ S_a(z)^\top q_a(z_a) & q_a(z_a)^2 \end{bmatrix}$$

where $S_a(z) = \frac{\partial q_a}{\partial z_a}(z_a)^\top z_b + \eta \alpha_a(z_a)P_a(z_a)g_a(z_a)$ and η is a positive real number. Moreover, there exists a non-vanishing mapping $q_b : \mathbb{R}^{n_a+1} \rightarrow \mathbb{R}$ such that P_b is invariant along $\frac{g}{q_b}$. In other words, Assumptions 1 and 2 hold for the complete system (35).

Remark 3: Note that with this theorem, since we propagate the required property we are able to obtain a synchronizing control law for any triangular nonlinear system.

Proof : Let M_b be a positive real number and let $\mathcal{C}_b = \mathcal{C}_a \times [-M_b, M_b]$. Let $U_b : \mathbb{R}^{n_a+1} \rightarrow \mathbb{R}$ be the function defined by

$$U_b(z_a, z_b) = \eta U_a(z_a) + q_a(z_a)z_b.$$

where η is a positive real number that will be selected later on. It follows from (36) that for all $(z_a, z_b) \in \mathcal{C}_b$, we have

$$\begin{aligned} \frac{\partial U_b}{\partial z}(z)^\top &= \begin{bmatrix} \eta \frac{\partial U_a}{\partial z_a}(z_a)^\top + \frac{\partial q_a}{\partial z_a}(z_a) z_b \\ q_a(z_a) \end{bmatrix} \\ &= \frac{1}{q_a(z_a)} P_b(z) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \alpha_b(z) P_b(z) g(z) \end{aligned}$$

with $\alpha_b(z) = \frac{1}{q_a(z_a)g_b(z)}$. Hence, the first condition in Assumption 1 is satisfied.

Consider z in \mathcal{C}_b and let $v = [v_a^\top \ v_b]^\top$ in \mathbb{R}^{n_a+1} be such that

$$v^\top P_b(z) g(z) = 0. \quad (40)$$

Note that this implies that

$$v_b = -v_a^\top \frac{S_a(z)}{q_a(z_a)}. \quad (41)$$

In the following, we compute the expression :

$$v^\top L_f P_b(z) v = v^\top \mathfrak{d}_f P_b(z) v + 2v^\top P_b(z) \frac{\partial f}{\partial z}(z) v.$$

For the first term, we have

$$\begin{aligned} v^\top \mathfrak{d}_f P_b(z) v &= v_a^\top \mathfrak{d}_{f_a} P_a(z_a) v_a + z_b v_a^\top \mathfrak{d}_{g_a} P_a(z_a) v_a \\ &\quad + v_a^\top \mathfrak{d}_f S_a(z) S_a(z)^\top v_a + 2v_a^\top \mathfrak{d}_f S_a(z) q_a(z_a) v_b \\ &\quad + \mathfrak{d}_{f_a+g_a z_b} q_a(z_a)^2 v_b^2 \end{aligned}$$

With (41), it yields

$$\begin{aligned} v_a^\top \mathfrak{d}_f S_a(z) S_a(z)^\top v_a + 2v_a^\top \mathfrak{d}_f S_a(z) q_a(z_a) v_b \\ + \mathfrak{d}_{f_a+g_a z_b} q_a(z_a)^2 v_b^2 = 0 \end{aligned}$$

Hence

$$v^\top \mathfrak{d}_f P_b(z) v = v_a^\top \mathfrak{d}_{f_a} P_a(z_a) v_a + z_b v_a^\top \mathfrak{d}_{g_a} P_a(z_a) v_a.$$

On the other hand, for the second term we have

$$P_b(z) = \begin{bmatrix} P_a(z_a) & 0 \\ 0 & 0 \end{bmatrix} + \frac{P_b(z) g(z) g(z)^\top P_b(z)}{(q_a(z_a) g_b(z))^2}$$

Hence, with (40), it yields

$$\begin{aligned} v^\top P_b(z) \frac{\partial f}{\partial z}(z) v &= \begin{bmatrix} v_a^\top & -v_a^\top \frac{S_a(z)}{q_a(z_a)} \end{bmatrix} P(z) \\ &\quad \begin{bmatrix} \frac{\partial f_a}{\partial z_a}(z_a) + \frac{\partial q_a}{\partial z_a}(z_a) z_b & g_a(z_a) \\ \frac{\partial f_b}{\partial z_a}(z_a, z_b) & \frac{\partial f_b}{\partial z_b}(z_a, z_b) \end{bmatrix} \begin{bmatrix} v_a \\ -\frac{S_a(z)^\top}{q_a(z_a)} v_a \end{bmatrix} \\ &= v_a^\top P_a(z_a) \frac{\partial f_a}{\partial z_a}(z_a) v_a + z_b v_a^\top P_a(z_a) \frac{\partial g_a}{\partial z_a}(z_a) v_a \\ &\quad - \frac{\eta}{\alpha_a(z_a) q_a(z_a)} \left| \frac{\partial U_a}{\partial z_a}(z_a) v_a \right|^2 \\ &\quad - \frac{z_b}{q_a(z_a)} v_a^\top P_a(z_a) g(z_a) \frac{\partial q_a}{\partial z_a}(z_a) \end{aligned}$$

Hence, we get

$$\begin{aligned} v^\top L_f P_b(z) v &= v_a^\top L_{f_a} P_a(z_a) v_a \\ &\quad - \frac{2\eta}{\alpha_a(z_a) q_a(z_a)} \left| \frac{\partial U_a}{\partial z_a}(z_a) v_a \right|^2 \\ &\quad + z_b v_a^\top \left[\mathfrak{d}_{g_a} P_a(z_a) + P_a(z_a) \frac{\partial g_a}{\partial z_a}(z_a) \right. \\ &\quad \left. - 2z_b v_a^\top P_a(z_a) \frac{g(z_a)}{q_a(z_a)} \frac{\partial q_a}{\partial z_a}(z_a) \right] v_a. \end{aligned}$$

Let η be a positive real number such that

$$\rho_a \leq \frac{2\eta}{\alpha_a(z_a) q_a(z_a)}, \quad \forall z_a \in \mathcal{C}_a.$$

Using (38) in Assumption 1 and (39) in Assumption 2, it follows that for all z in \mathcal{C}_b and all v in \mathbb{R}^{n_a+1}

$$\begin{aligned} v^\top P_b(z_a) g(z) &= 0 \\ \Rightarrow v^\top \mathfrak{d}_f P_b(z) v + 2v^\top P_b(z) \frac{\partial f}{\partial x}(z) v &\leq -v^\top Q_a v. \end{aligned}$$

Employing Finsler theorem and the fact that \mathcal{C}_b is a compact set, it is possible to show that this implies the existence of a positive real number ρ_b such that for all z in \mathcal{C}_b

$$L_f P(z) - \rho_b \frac{\partial U_b}{\partial z}(z)^\top \frac{\partial U_b}{\partial z}(z) \leq -Q_b. \quad (42)$$

where Q_b is a symmetric positive definite matrix.

To finish the proof it remains to show that the metric is invariant along g with an appropriate control law. Note that if $q_b(z) = q_a(z_a) g_b(z)$ then it follows that this function is also non-vanishing. Moreover, we have

$$\begin{aligned} L_{\frac{g}{q_b}} P_b(z) &= \mathfrak{d}_{\frac{g}{q_b}} P_b(z) - \frac{P(z)}{q_a(z_a)^2} \begin{bmatrix} 0 & 0 \\ \frac{\partial q_a}{\partial z_a}(z_a) & 0 \end{bmatrix} \\ &\quad - \begin{bmatrix} 0 & \frac{\partial q_a}{\partial z_a}(z_a)^\top \\ 0 & 0 \end{bmatrix} \frac{P(z)}{q_a(z_a)^2}. \end{aligned}$$

However, since we have

$$\mathfrak{d}_g P_b(z) = \begin{bmatrix} \frac{\partial q_a}{\partial z_a}(z_a)^\top \frac{S_a(z)}{q_a(z_a)} + \frac{S_a(z)^\top}{q_a(z_a)} \frac{\partial q_a}{\partial z_a}(z_a) & \frac{\partial q_a}{\partial z_a}(z_a)^\top \\ \frac{\partial q_a}{\partial z_a}(z_a) & 0 \end{bmatrix}$$

and

$$\begin{aligned} \frac{P_b(z)}{q_a(z_a)^2} \begin{bmatrix} 0 & 0 \\ \frac{\partial q_a}{\partial z_a}(z_a) & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{\partial q_a}{\partial z_a}(z_a)^\top \\ 0 & 0 \end{bmatrix} \frac{P_b(z)}{q_a(z_a)^2} &= \\ \begin{bmatrix} \frac{\partial q_a}{\partial z_a}(z_a)^\top \frac{S_a(z)}{q_a(z_a)} + \frac{S_a(z)^\top}{q_a(z_a)} \frac{\partial q_a}{\partial z_a}(z_a) & \frac{\partial q_a}{\partial z_a}(z_a)^\top \\ \frac{\partial q_a}{\partial z_a}(z_a) & 0 \end{bmatrix} & \end{aligned}$$

then the claim holds. \square

B. Illustrative example

As an illustrative example, consider the case in which the vector fields f and g are given by

$$f(z) = \begin{bmatrix} -z_{a1} + \sin(z_{a2}) \cos(z_{a1}) + z_{a2} \\ [2 + \sin(z_{a1})]z_b \\ 0 \end{bmatrix}, g(z) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

This system may be rewritten with $z_a = (z_{a1}, z_{a2})$ as

$$\dot{z}_a = f_a(z_a) + g_a(z_a)z_b, \dot{z}_b = u$$

with

$$f_a(z_a) = \begin{bmatrix} -z_{a1} + \sin(z_{a2}) \cos(z_{a1}) + z_{a2} \\ 0 \end{bmatrix},$$

$$g_a(z_a) = \begin{bmatrix} 0 \\ 2 + \sin(z_{a1}) \end{bmatrix}$$

Consider the matrix $P_a = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Note that if we consider

$U_a(z_a) = z_{a1} + 2z_{a2}$, then equation (36) is satisfied with $\alpha_a = \frac{1}{2 + \sin(z_{a1})}$. Moreover, note that we have $v^\top \frac{\partial U_a}{\partial z_a}(z_a) = 0 \Leftrightarrow v_1 + 2v_2 = 0$. Moreover, we have

$$\begin{bmatrix} -2 & 1 \end{bmatrix} P_a \frac{\partial f_a}{\partial z_a}(z_a) \begin{bmatrix} -2 \\ 1 \end{bmatrix} = -3 \left[-2 \frac{\partial f_{a1}}{\partial z_{a1}} + \frac{\partial f_{a1}}{\partial z_{a2}} \right] = -3.$$

$$\begin{aligned} & [-2(-1 + \sin(z_{a2}) \sin(z_{a1})) - \cos(z_{a1}) \cos(z_{a2}) + 1] \\ & = -3. \end{aligned}$$

$$\begin{aligned} & [3 - \sin(z_{a2}) \sin(z_{a1}) - \cos(z_{a1} - z_{a2})] \\ & \leq -3 \end{aligned}$$

The function $\frac{\partial f_a}{\partial z_a}(z_a)$ being periodic in z_{a1} and z_{a2} we can assume that z_{a1} and z_{a2} are in a compact subset denoted \mathcal{C}_a . This implies employing Finsler Lemma that there exists ρ_a and Q_a such that inequality (38) holds. Consequently, the z_a subsystem satisfies Assumption 1. Finally note that Assumption 2 is also trivially satisfied by taking $q_a(z_a) = 2 + \sin(z_{a1})$. From Theorem 4, it implies that there exist positive real numbers ρ_b and η such that with $U(z) = \eta(z_{a1} + 2z_{a2}) + \frac{z_b}{2 + \sin(z_{a1})}$ with $\alpha(z) = 2 + \sin(z_{a1})$, equations (26) and (27) are satisfied. Hence from Theorem 2, the control law given in (34) solves the local exponential synchronization problem for the N identical systems that exchange information via any undirected communication graph \mathcal{G} , which is connected.

VI. CONCLUSION

In this paper, based on recent results in [3], we have presented necessary and sufficient conditions for the solvability of local exponential synchronization of N identical affine nonlinear systems through a distributed control law. In particular, we have shown that the necessary condition is linked to the infinitesimal stabilizability of the individual system and is independent of the network topology. The existence of a symmetric covariant tensor of order two, as a result of the infinitesimal stabilizability, has allowed us to design a distributed synchronizing control law. When the tensor and when the controlled vector field g are both constant

it is shown that global exponential synchronization may be achieved. Finally, a recursive computation of the tensor has been also discussed.

APPENDIX

A. Proof of Lemma 1

From the property of Laplacian matrix, the eigenvalues of L are real and satisfy $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$. Let us take the non-zero eigenvalue $\nu > 0$ of L and its corresponding eigenvector v in \mathbb{R}^N . Note that we can decompose

$$v = \begin{bmatrix} v_a \\ v_b \end{bmatrix}, L = \begin{bmatrix} L_{11} & L_{1,2:N} \\ L_{1,2:N}^\top & L_{2:N,2:N} \end{bmatrix}$$

with v_a and L_{11} in \mathbb{R} . It follows that

$$L_{11}v_a + L_{1,2:N}v_b = \nu v_a \quad (43)$$

$$L_{1,2:N}^\top v_a + L_{2:N,2:N}v_b = \nu v_b. \quad (44)$$

Moreover, since $\mathbf{1}_N$ is an eigenvector associated to the eigenvalue 0,

$$L_{11} + L_{1,2:N}\mathbf{1}_{N-1} = 0 \quad (45)$$

$$L_{1,2:N}^\top + L_{2:N,2:N}\mathbf{1}_{N-1} = 0 \quad (46)$$

Consider now a vector in \mathbb{R}^{N-1} defined by

$$\tilde{v} = v_b - \mathbf{1}_{N-1}v_a$$

Note that \tilde{v} is non zero since v is not colinear to $\mathbf{1}_N$. By a routine algebraic computation, it follows that this vector satisfies

$$\begin{aligned} [L_{2:N,2:N} - \mathbf{1}_{N-1}L_{1,2:N}]\tilde{v} &= L_{2:N,2:N}v_b - \mathbf{1}_{N-1}L_{1,2:N}v_b \\ &+ [\mathbf{1}_{N-1}L_{1,2:N}\mathbf{1}_{N-1} - L_{2:N,2:N}\mathbf{1}_{N-1}]v_a \\ &= \nu v_b - L_{1,2:N}^\top v_a \\ &- \mathbf{1}_{N-1}L_{1,2:N}v_b + [-\mathbf{1}_{N-1}L_{11} + L_{1,2:N}]v_a \\ &= \nu \tilde{v}. \end{aligned}$$

This shows that \tilde{v} is an eigenvector with the same non-zero eigenvalue of L . It proves the first claim of the lemma.

Note that the multiplicity of the eigenvalue ν is the same for both matrices. Also, if the graph is connected, then the 0 eigenvalue of the Laplacian matrix L is of multiplicity 1 and the other eigenvalues are positive and distinct. Hence the matrix $-\bar{L}$ is Hurwitz. \square

B. Proof of Lemma 2

The matrix L being a balanced Laplacian matrix is positive semi-definite and its eigenvalues are real and satisfy $0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$. Consequently, the principal sub-matrix $L_{2:N,2:N}$ of L is also symmetric positive semi-definite (by the Cauchy's interlacing theorem). Moreover, by Kirchoff's theorem, the matrix $L_{2:N,2:N}$, which is a minor of the Laplacian, has a determinant strictly larger than 0 since the graph is connected. Hence, $L_{2:N,2:N}$ is positive definite. Consequently, there exists c_1 sufficiently small such that $A(c_1)$ is negative definite.

C. Some results from [3]

Throughout this section, we give some of the results of [3]. Hence, we consider a system in the form

$$\dot{e} = F(e, z), \quad \dot{z} = G(e, z) \quad (47)$$

where e is in \mathbb{R}^{n_e} , z is in \mathbb{R}^{n_z} and the functions $F : \mathbb{R}^{n_e} \times \mathbb{R}^{n_z} \rightarrow \mathbb{R}^{n_e}$ and $G : \mathbb{R}^{n_e} \times \mathbb{R}^{n_z} \rightarrow \mathbb{R}^{n_z}$ are C^2 . We denote by $(E(e_0, x_0, t), X(e, z, t))$ the (unique) solution which goes through (e, z) in $\mathbb{R}^{n_e} \times \mathbb{R}^{n_z}$ at $t = 0$. We assume it is defined for all positive times, i.e. the system is *forward complete*.

In the following, to simplify our notations, we denote by $B_e(a)$ the open ball of radius a centered at the origin in \mathbb{R}^{n_e} .

In [3], the following three notions are introduced.

TULES-NL (*Transversal uniform local exponential stability*)

There exist strictly positive real numbers r, k and λ such that we have, for all (e, x, t) in $\mathbb{R}^{n_e} \times \mathbb{R}^{n_z} \times \mathbb{R}_{\geq 0}$ with $|e| \leq r$,

$$|E(e, x, t)| \leq k|e| \exp(-\lambda t). \quad (48)$$

UES-TL (*Uniform exponential stability for the transversally linear system*)

The system

$$\dot{z} = \bar{G}(z) := G(0, z) \quad (49)$$

is forward complete and there exist strictly positive real numbers \tilde{k} and $\tilde{\lambda}$ such that any solution $(\tilde{E}(\tilde{e}, z, t), Z(z, t))$ of the transversally linear system

$$\dot{\tilde{e}} = \frac{\partial F}{\partial e}(0, z)\tilde{e}, \quad \dot{z} = \bar{G}(z) \quad (50)$$

satisfies, for all (\tilde{e}, z, t) in $\mathbb{R}^{n_e} \times \mathbb{R}^{n_z} \times \mathbb{R}_{\geq 0}$,

$$|\tilde{E}(\tilde{e}, z, t)| \leq \tilde{k} \exp(-\tilde{\lambda} t) |\tilde{e}|. \quad (51)$$

ULMTE (*Uniform Lyapunov matrix transversal equation*)

For all positive definite matrix Q , there exists a continuous function $P : \mathbb{R}^{n_z} \rightarrow \mathbb{R}^{n_e \times n_e}$ and strictly positive real numbers \underline{p} and \bar{p} such that for all z in \mathbb{R}^{n_z} ,

$$\partial_{\bar{G}} P(z) + P(z) \frac{\partial F}{\partial e}(0, z) + \frac{\partial F}{\partial e}(0, z)' P(z) \leq -Q \quad (52)$$

$$\underline{p} I \leq P(z) \leq \bar{p} I. \quad (53)$$

From these definitions and in the same spirit as Lyapunov second method, the following relationships have been established in [3].

Proposition 1 ([3], TULES-NL “ \Rightarrow ” UES-TL): If Property TULES-NL holds and there exist positive real number c such that, for all z in \mathbb{R}^{n_z} ,

$$\left| \frac{\partial F}{\partial e}(0, z) \right| \leq c, \quad \left| \frac{\partial G}{\partial x}(0, x) \right| \leq c \quad (54)$$

and, for all (e, x) in $B_e(kr) \times \mathbb{R}^{n_z}$,

$$\left| \frac{\partial^2 F}{\partial e \partial e}(e, z) \right| \leq c, \quad \left| \frac{\partial^2 F}{\partial z \partial e}(e, z) \right| \leq c, \quad \left| \frac{\partial G}{\partial e}(e, z) \right| \leq c, \quad (55)$$

then Property UES-TL holds.

Proposition 2 (UES-TL “ \Rightarrow ” ULMTE): If Property UES-TL holds, P is C^1 and there exists a positive real number c such that

$$\left| \frac{\partial F}{\partial e}(0, z) \right| \leq c \quad \forall z \in \mathbb{R}^{n_z}, \quad (56)$$

then Property ULMTE holds.

Proposition 3 (ULMTE “ \Rightarrow ” TULES-NL): If Property ULMTE holds and there exist positive real numbers η and c such that, for all (e, x) in $B_e(\eta) \times \mathbb{R}^{n_z}$,

$$\left| \frac{\partial P}{\partial x}(x) \right| \leq c, \quad (57)$$

$$\left| \frac{\partial^2 F}{\partial e \partial e}(e, x) \right| \leq c, \quad \left| \frac{\partial^2 F}{\partial x \partial e}(e, x) \right| \leq c, \quad \left| \frac{\partial G}{\partial e}(e, x) \right| \leq c, \quad (58)$$

then Property TULES-NL holds.

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