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On level-transitivity and exponential growth*

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Abstract

We prove that if the group generated by a Mealy automaton acts level-transitively on a regular rooted tree, then the semigroup generated by the dual automaton has exponential growth, hence giving a decision procedure of exponential growth for a restricted family of automaton semigroups.

The purpose of this note is to link up two classes of groups and semigroups highly studied for themselves: level-transitive (semi)groups and (semi)groups of exponential growth, through automaton (semi)groups.

On the one hand, level-transitive groups (or equivalently spherically transitive groups, depending on the authors) —*i.e.* groups acting transitively on any level of a regular rooted tree— have received special focus these last years because of branch groups, which form a particular class of level-transitive groups, one of the three classes into which the class of just infinite groups is naturally decomposed [7, 3].

On the other hand, the study on how (semi)groups grow has been highlighted since Milnor’s question on the existence of groups of intermediate growth in 1968 [12] and the very first example of such a group given by Grigorchuk [5].

In this note, we prove that no semigroup of polynomial or intermediate growth can be generated by a reversible Mealy automaton whose dual generates a level-transitive group. Even if the problem of deciding the level-transitivity of an automaton group is still open, there exist some families of Mealy automata for which the level-transitivity of an element in the generated semigroup is decidable [14].

1 Basic notions

1.1 Semigroups and groups generated by Mealy automata

We first recall the formal definition of an automaton. A (*finite, deterministic, and complete*) automaton is a triple $(Q, \Sigma, \delta = (\delta_i: Q \rightarrow Q)_{i \in \Sigma})$, where the *stateset* Q and the *alphabet* Σ are non-empty finite sets, and the δ_i are functions.

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A *Mealy automaton* is a quadruple $(Q, \Sigma, \delta, \rho)$, such that (Q, Σ, δ) and (Σ, Q, ρ) are both automata. In other terms, a Mealy automaton is a complete, deterministic, letter-to-letter transducer with the same input and output alphabet. Its *size* is the cardinal of its stateset.

The graphical representation of a Mealy automaton is standard, see Figure 1.

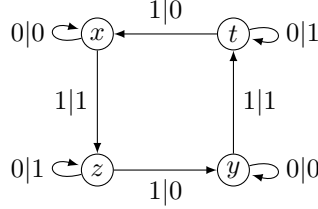


Figure 1: An example of a Mealy automaton which does not generate a free semigroup on its stateset but whose dual generates a level-transitive group.

Let $\mathcal{A} = (Q, \Sigma, \delta, \rho)$ be a Mealy automaton. Each state $x \in Q$ defines a mapping from Σ^* into itself recursively defined by:

$$\forall i \in \Sigma, \forall \mathbf{s} \in \Sigma^*, \quad \rho_x(i\mathbf{s}) = \rho_x(i)\rho_{\delta_i(x)}(\mathbf{s}).$$

The image of the empty word is itself. The mapping ρ_x for each $x \in Q$ is length-preserving and prefix-preserving. We say that ρ_x is the function *induced* by x . For $\mathbf{x} = x_1 \cdots x_n \in Q^n$ with $n > 0$, set $\rho_{\mathbf{x}}: \Sigma^* \rightarrow \Sigma^*$, $\rho_{\mathbf{x}} = \rho_{x_n} \circ \cdots \circ \rho_{x_1}$. The semigroup of mappings from Σ^* to Σ^* generated by $\{\rho_x, x \in Q\}$ is called the *semigroup generated by \mathcal{A}* and is denoted by $\langle \mathcal{A} \rangle_+$.

A Mealy automaton $\mathcal{A} = (Q, \Sigma, \delta, \rho)$ is *invertible* if the functions ρ_x are permutations of Σ . In this case, the functions induced by the states are permutations on words of the same length and thus we may consider the group of mappings from Σ^* to Σ^* generated by $\{\rho_x, x \in Q\}$: it is called the *group generated by \mathcal{A}* and is denoted by $\langle \mathcal{A} \rangle$.

In a Mealy automaton $\mathcal{A} = (Q, \Sigma, \delta, \rho)$, the sets Q and Σ play dual roles. So we may consider the *dual (Mealy) automaton* defined by $\mathfrak{d}(\mathcal{A}) = (\Sigma, Q, \rho, \delta)$. We extend to δ the former notations on ρ , in a natural way. Hence $\delta_i: Q^* \rightarrow Q^*$, $i \in \Sigma$, are the functions induced by the states of $\mathfrak{d}(\mathcal{A})$, and for $\mathbf{s} = s_1 \cdots s_n \in \Sigma^n$ with $n > 0$, we set $\delta_{\mathbf{s}}: Q^* \rightarrow Q^*$, $\delta_{\mathbf{s}} = \delta_{s_n} \circ \cdots \circ \delta_{s_1}$.

A Mealy automaton $(Q, \Sigma, \delta, \rho)$ is *reversible* if its dual is invertible, that is if the functions δ_i are permutations of Q .

An automaton group or semigroup can be seen as acting on a regular root tree representing the language of all words on its alphabet.

1.2 Growth of a semigroup or of a group

Let H be a semigroup generated by a finite set S . The *length* of an element g of the semigroup, denoted by $|g|$, is the length of its shortest decomposition:

$$|g| = \min\{n \mid \exists s_1, \dots, s_n \in S, g = s_1 \cdots s_n\}.$$

The *growth function* γ_H^S of the semigroup H with respect to the generating set S enumerates the elements of H with respect to their length:

$$\gamma_H^S(n) = \#\{g \in H; |g| \leq n\}.$$

The *growth functions* of a group are defined similarly by taking symmetrical generating sets.

The growth functions corresponding to two generating sets are equivalent [11], so we may define the *growth* of a group or a semigroup as the equivalence class of its growth functions. Hence, for example, a finite (semi)group has a bounded growth, while an infinite abelian (semi)group has a polynomial growth, and a non-abelian free (semi)group has an exponential growth.

It is quite easy to obtain groups of polynomial or exponential growth. Answering a question of Milnor [12], Grigorchuk gave an example of an automaton group of intermediate growth [5]: more than any polynomial, less than any exponential, opening thus a new classification criterium for groups, that has been deeply studied since this seminal article (see [6] and references therein). Besides, intermediate growth and automaton groups seem to have a very strong link, as the only known examples of intermediate growth groups are automaton groups.

1.3 Level-Transitivity

The action of a (semi)group generated by an invertible Mealy automaton $\mathcal{A} = (Q, \Sigma, \delta, \rho)$ is *level-transitive* if its restriction to Σ^n has a unique orbit, for any n (this notion is equivalently called *spherically transitive* [8]). From a dual point of view it means that the powers of the dual $\mathfrak{d}(\mathcal{A})$ are connected, the n -th power of the automaton $\mathfrak{d}(\mathcal{A})$ being the Mealy automaton

$$\mathfrak{d}(\mathcal{A})^n = (\Sigma^n, Q, (\rho_x: \Sigma^n \rightarrow \Sigma^n)_{x \in Q}, (\delta_s: Q \rightarrow Q)_{s \in \Sigma^n}) .$$

Note that all the powers of a reversible Mealy automaton are reversible. The next theorem is proved in [9]:

Theorem 1. *Let \mathcal{A} be a reversible automaton with a prime number of states. If the action of $\mathfrak{d}(\mathcal{A})$ is level-transitive, then the semigroup $\langle \mathcal{A} \rangle_+$ is free on the automaton stateset.*

In [9] the hypothesis of the prime number of states was erroneously conjectured to be not mandatory. In fact, the Mealy automaton of Figure 1 given by Laurent Bartholdi (personal communication) does not generate a free semigroup on its stateset, even though its dual generates a level-transitive group.

Although deciding the level transitivity of an automaton group or of an element of an automaton group are open problems [8, Problems 7.2.1(e+f)], this former problem has received a solution in some cases [14] and it is even implemented in the GAP packages `FR` and `automgrp` [4, 2, 13].

1.4 Minimization and Nerode classes

Let $\mathcal{A} = (Q, \Sigma, \delta, \rho)$ be a Mealy automaton.

The *Nerode equivalence* \equiv on Q is the limit of the sequence of increasingly finer equivalences (\equiv_k) recursively defined by:

$$\begin{aligned} \forall x, y \in Q, \quad x \equiv_0 y &\iff \rho_x = \rho_y, \\ \forall k \geq 0, x \equiv_{k+1} y &\iff (x \equiv_k y \quad \wedge \quad \forall i \in \Sigma, \delta_i(x) \equiv_k \delta_i(y)). \end{aligned}$$

Since the set Q is finite, this sequence is ultimately constant. For every element x in Q , we denote by $[x]$ the class of x w.r.t. the Nerode equivalence, called the *Nerode class* of x . Extending to the n -th power of \mathcal{A} , we denote by $[\mathbf{x}]$ the Nerode class in Q^n of $\mathbf{x} \in Q^n$.

Two states of a Mealy automaton belong to the same Nerode class if and only if they represent the same element in the generated semigroup, *i.e.* if and only if they induce the same action on Σ^* .

The *minimization* of \mathcal{A} is the Mealy automaton $\mathbf{m}(\mathcal{A}) = (Q/\equiv, \Sigma, \tilde{\delta}, \tilde{\rho})$, where for every (x, i) in $Q \times \Sigma$, $\tilde{\delta}_i([x]) = [\delta_i(x)]$ and $\tilde{\rho}_{[x]} = \rho_x$. This definition is consistent with the standard minimization of “deterministic finite automata” where instead of considering the mappings $(\rho_x : \Sigma \rightarrow \Sigma)_x$, the computation is initiated by the separation between terminal and non-terminal states.

The following remarks will be useful for the rest of the paper:

Remark 2. *If two words of Q^* are equivalent, so are their images under the action of any element of $\langle \mathfrak{d}(\mathcal{A}) \rangle_+$.*

Remark 3. *The Nerode classes of a connected reversible Mealy automaton have the same cardinal.*

Remark 4. *It is known from [10] that a reversible automaton generates a finite semigroup if and only if the connected components of its powers are bounded. It is straightforward to adapt the proof to show that a reversible automaton generates a finite semigroup if and only if the minimizations of the connected components of its powers are bounded.*

Remark 5. *Let \mathcal{A} and \mathcal{B} be two reversible Mealy automata on the same alphabet Σ , x some state of \mathcal{A} , and y some state of \mathcal{B} . If x and y have the same action on Σ^* , then $\mathbf{m}(\mathcal{A})$ and $\mathbf{m}(\mathcal{B})$ are isomorphic; in particular they have the same size.*

2 Main result

In this section we prove the following generalization of Theorem 1:

Theorem 6. *Let \mathcal{A} be a reversible Mealy automaton. If the action of $\mathfrak{d}(\mathcal{A})$ is level-transitive, then the semigroup $\langle \mathcal{A} \rangle_+$ has exponential growth.*

Let us first look at the structure of the Nerode classes of two consecutive powers of the stateset, in the case of a reversible Mealy automaton whose dual generates a level-transitive group.

Lemma 7. *Let $\mathcal{A} = (Q, \Sigma, \delta, \rho)$ be a reversible automaton whose dual generates a level-transitive group.*

Let $(C_i)_{1 \leq i \leq k}$ be the Nerode classes of Q^n for some n , and D be a Nerode class of Q^{n+1} . We have

$$D = \bigcup_{q \in Q_D} C_{i_{q,D}} q \quad \text{and} \quad D = \bigcup_{q \in Q'_D} q C_{i'_{q,D}},$$

where $Q_D \subseteq Q$ and $Q'_D \subseteq Q$ have the same cardinal, and the $(i_{q,D})_{q \in Q_D}$ on the one hand and the $(i'_{q,D})_{q \in Q'_D}$ on the other are pairwise distinct.

The automata $\mathbf{m}(\mathcal{A}^n)$ and $\mathbf{m}(\mathcal{A}^{n+1})$ have the same size if and only if $Q_D = Q'_D = Q$.

Proof. We prove the first decomposition.

Let $\mathbf{u} \in Q^n$ and C be the Nerode class of \mathbf{u} . If $\mathbf{v} \in C$, then $\mathbf{u}q$ and $\mathbf{v}q$ are in the same Nerode class of Q^{n+1} . So $\mathbf{u}q \in D$ implies $Cq \subseteq D$.

Let C_i and C_j be two different Nerode classes of Q^n , then it is impossible that $C_i q \subseteq D$ and $C_j q \subseteq D$.

As a consequence, the ratio between the cardinal of a Nerode class of Q^{n+1} and the cardinal of a Nerode class of Q^n (which does not depend on these classes by Remark 3) is the integer $\#Q_D = \#Q'_D$ which is less than or equal to $\#Q$. It is equal to $\#Q$ if and only if $\mathbf{m}(\mathcal{A}^n)$ and $\mathbf{m}(\mathcal{A}^{n+1})$ have the same size. \square \square

Proposition 8. *Let \mathcal{A} be a reversible Mealy automaton whose dual generates a level-transitive group, and n be an integer.*

If $\#\mathbf{m}(\mathcal{A}^{n+1}) = \#\mathbf{m}(\mathcal{A}^n)$, then $\#\mathbf{m}(\mathcal{A}^{n+2}) = \#\mathbf{m}(\mathcal{A}^{n+1})$. \square

Proof. Let us denote by Q the stateset of \mathcal{A} , by C_1, \dots, C_k the Nerode classes of Q^n , and by D_1, \dots, D_k the Nerode classes of Q^{n+1} (by hypothesis Q^n has as many Nerode classes as Q^{n+1}).

Let $r \in Q$: rD_1 is included in some Nerode class of Q^{n+2} . But by Lemma 7 we have

$$rD_1 = r \bigcup_{q \in Q} C_{i_q} q = \bigcup_{q \in Q} rC_{i_q} q,$$

where C_{i_q} is written for $C_{i_{q,D_1}}$. Now, for a fixed $q \in Q$, rC_{i_q} is included in some Nerode class of Q^{n+1} , say D_{j_q} . Hence

$$\bigcup_{q \in Q} D_{j_q} q$$

is included in a Nerode class of Q^{n+2} and by Lemma 7 we obtain the result. \square \square

We can now prove Theorem 6.

Proof. If $\mathfrak{d}(\mathcal{A})$ is level-transitive, then $\langle \mathfrak{d}(\mathcal{A}) \rangle$ is infinite and so is $\langle \mathcal{A} \rangle_+$ (see for example [1, 10]). But if there exists n such that $\mathbf{m}(\mathcal{A}^n)$ and $\mathbf{m}(\mathcal{A}^{n+1})$ have the same size, then for any $m \geq n$, $\mathbf{m}(\mathcal{A}^m)$ has this same size, and the generated semigroup should be finite by Remark 4.

So for any n , the ratio of the size of $\mathbf{m}(\mathcal{A}^{n+1})$ and the size of $\mathbf{m}(\mathcal{A}^n)$, which is known to be an integer by Lemma 7, is at least 2. So the sequence $(\#\mathbf{m}(\mathcal{A}^n))_n$ increases and is componentwise greater than or equal to $(2^n)_n$. By Remark 5, this implies that, for any integer n , the actions induced by the states of \mathcal{A}^n cannot be induced by previous powers of \mathcal{A} and hence $\langle \mathcal{A} \rangle_+$ has an exponential growth. \square \square

Open problems For this result to be fully applicable, it remains of course to find a procedure to decide if an invertible Mealy automaton is level-transitive. Another interesting question would be: can a reversible Mealy automaton generate a semigroup of polynomial or of intermediate growth?

References

- [1] A. Akhavi, I. Klimann, S. Lombardy, J. Mairesse, and M. Picantin, *On the finiteness problem for automaton (semi)groups*, Int. J. Algebr. Comput. **22** (2012), no. 6, 26p.
- [2] L. Bartholdi, *FR functionally recursive groups, self-similar groups*, 2014.
- [3] L. Bartholdi, R.I. Grigorchuk, and Z. Šuník, *Handbook of algebra*, vol. 3, ch. Branch groups, pp. 989–1112, Elsevier BV, 2003.
- [4] The GAP Group, *GAP Groups, Algorithms, and Programming*, 2015.
- [5] R. Grigorchuk, *Degrees of growth of finitely generated groups and the theory of invariant means*, Izv. Akad. Nauk SSSR Ser. Mat. **48-5** (1984), 939–985.
- [6] R. Grigorchuk and I. Pak, *Groups of intermediate growth: an introduction for beginners*, 2006, arXiv:math/0607384.
- [7] R. I. Grigorchuk, *New horizons in pro-p groups*, ch. Just Infinite Branch Groups, pp. 121–179, Birkhäuser Boston, Boston, MA, 2000.
- [8] R.I. Grigorchuk, V.V. Nekrashevich, and V.I. Sushchanskiĭ, *Automata, dynamical systems, and groups*, Tr. Mat. Inst. Steklova **231** (2000), 134–214.
- [9] I. Klimann, *Automaton semigroups: The two-state case*, Theor. Comput. Syst. (special issue STACS’13) **58** (2014), no. 4, 664–680.
- [10] I. Klimann, M. Picantin, and D. Savchuk, *A connected 3-state reversible Mealy automaton cannot generate an infinite Burnside group*, DLT’ 15, LNCS, vol. 9168, 2015, pp. 313–325.
- [11] A. Mann, *How groups grow*, Lecture Note Series, vol. 395, The London Mathematical Society, 2012.
- [12] J. Milnor, *Advanced problem 5603*, MAA Monthly **75** (1968), 685–686.
- [13] Y. Muntyan and D. Savchuk, *automgrp automata groups*, 2014.
- [14] B. Steinberg, *Testing spherical transitivity in iterated wreath products of cyclic groups*, arXiv:math/0607563, 2006.