

Convergence to Equilibrium in Free Fokker-Planck Equation With a Double-Well Potential

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Abstract

We consider the one-dimensional free Fokker-Planck equation

$$\frac{\partial \mu_t}{\partial t} = \frac{\partial}{\partial x} \left[\mu_t \left(\frac{1}{2} V' - H \mu_t \right) \right],$$

where H denotes the Hilbert transform and V is a particular double-well quartic potential, namely $V(x) = \frac{1}{4}x^4 + \frac{c}{2}x^2$, with $-2 \leq c < 0$. We prove that the solution $(\mu_t)_{t \geq 0}$ of this PDE converges to the equilibrium measure μ_V as t goes to infinity, which provides a first result of convergence in a non-convex setting. The proof involves free probability and quadratic differentials techniques.

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1 Introduction

We consider the following one-dimensional free Fokker-Planck equation

$$\frac{\partial \mu_t}{\partial t} = \frac{\partial}{\partial x} \left[\mu_t \left(\frac{1}{2} V' - H \mu_t \right) \right]. \quad (1)$$

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In this equation, μ_t denotes an unknown probability measure on \mathbb{R} , $V : \mathbb{R} \rightarrow \mathbb{R}$ is a given potential, and H denotes the Hilbert transform, that is, for any measure μ on \mathbb{R} and $x \in \mathbb{R}$,

$$H\mu(x) = \oint_{\mathbb{R}} \frac{1}{x-y} d\mu(y),$$

where \oint stands for the principal value of the integral. Partial differential equation (1), abbreviated PDE (1), must be understood in the sense of distributions, i.e. for any regular enough test function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$,

$$\frac{d}{dt} \int \varphi(x) d\mu_t(x) = -\frac{1}{2} \int V'(x) \varphi'(x) d\mu_t(x) + \frac{1}{2} \iint \frac{\varphi'(x) - \varphi'(y)}{x-y} d\mu_t(x) d\mu_t(y).$$

Under this form, it is sometimes called McKean-Vlasov equation with logarithmic interaction.

1.1 Existence and uniqueness

As far as we know, the problems of existence and uniqueness of the solution to this PDE are not completely solved. They have been tackled when the potential V satisfies some properties but a general result is not known to us.

Starting from the following classical result: if $(X_t)_{t \geq 0}$ denotes the solution of the stochastic differential equation (SDE)

$$dX_t = dB_t - \frac{1}{2} V'(X_t) dt,$$

then, by Itô's formula, the distribution of the solution X_t satisfies the linear Fokker-Planck equation

$$\frac{\partial \mu_t}{\partial t} = \Delta \mu_t - \frac{1}{2} \frac{\partial}{\partial x} (\mu_t V'),$$

Biane and Speicher [9] considered the following free SDE:

$$dX_t = dS_t - \frac{1}{2} V'(X_t) dt, \tag{2}$$

where S is a free Brownian motion and X is an unknown free diffusion process. They proved the following existence and uniqueness result using free stochastic calculus (see [7, 8] for an introduction).

Theorem 1.1 (see [9, Theorem 3.1]). *We assume that V is a \mathcal{C}^1 potential such that V' is locally Lipschitz, and that there exist $a \in \mathbb{R}$ and $b > 0$ such that for all $x \in \mathbb{R}$,*

$$-xV'(x) \leq ax^2 + b. \tag{3}$$

Then, for any given X_0 with compactly supported distribution, free SDE (2) admits a unique solution $(X_t)_{t \geq 0}$ starting from X_0 and the distribution of the solution X_t satisfies Equation (1).

This is why PDE (1) is also called *free Fokker-Planck equation*.

As for uniqueness, Li, Li, and Xie [23] proved the following, using free transportation techniques.

Theorem 1.2 (see [23, Theorem 1.3]). *Let us consider a \mathcal{C}^2 potential V such that:*

(i) *We have*

$$\lim_{|x| \rightarrow +\infty} V(x) - 2 \log |x| = +\infty.$$

(ii) *For all $R > 0$, there exists $K_R > 0$ such that for all $x, y \in [-R, R]$,*

$$(x - y)(V'(x) - V'(y)) \geq -K_R(x - y)^2.$$

(iii) *There exists $\gamma > 0$ such that for all $x \in \mathbb{R}$,*

$$-xV'(x) \leq \gamma(1 + x^2).$$

(iv) *There exists $K \in \mathbb{R}$ such that for all $x \in \mathbb{R}$,*

$$V''(x) \geq K.$$

Then, for any given μ_0 with compact support, free Fokker-Planck equation (1) admits a unique solution starting from μ_0 .

This solution obviously coincide with the one of Biane and Speicher.

In this paper, we are interested in free Fokker-Planck equation (1) for the particular potential

$$V(x) = \frac{1}{4}x^4 + \frac{c}{2}x^2, \quad -2 \leq c < 0. \quad (4)$$

Indeed, after the quadratic potential which is well understood, the quartic potential is the most simple example of potential for which everything is well defined.

Note that for this potential, Equation (1) admits a unique solution given an initial condition with compact support. Indeed, the assumptions of Theorem 1.2 are satisfied here, in particular, the potential V defined by (4) has a second derivative which is uniformly bounded below (by c). Consequently, we can identify the solution μ_t of free Fokker-Planck with the distribution of the solution X_t to free SDE (2).

The aim of this paper is to study the asymptotic behaviour of the solution $(\mu_t)_{t \geq 0}$ under the quartic potential (4).

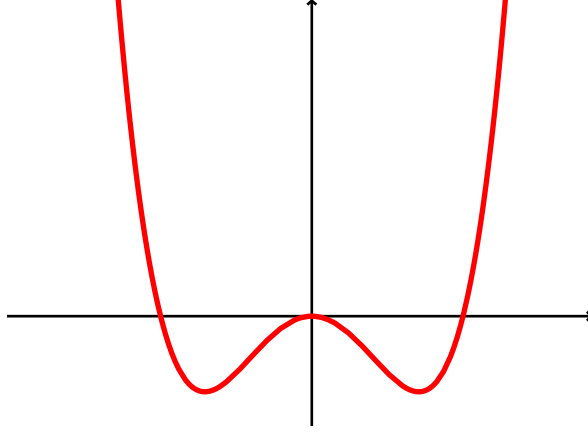


Figure 1: Potential V defined by (4).

1.2 Main result of the paper

We recall that for $p \geq 1$, the Wasserstein distance of order p is defined on the set $\mathcal{P}_p(\mathbb{R})$ of real probability measures having a finite p -th moment by

$$W_p(\mu, \nu) = \left(\inf_{\substack{(X,Y) \text{ r.v.} \\ X \sim \mu, Y \sim \nu}} \mathbb{E} |X - Y|^p \right)^{1/p}.$$

We also recall that if $(\mu_n), \mu$ are probability measures in $\mathcal{P}_p(\mathbb{R})$ such that $W_p(\mu_n, \mu) \rightarrow 0$ as $n \rightarrow +\infty$, then (μ_n) converges in distribution to μ . The converse is true if in addition, the p -th moment of μ_n converges to the p -th moment of μ , or equivalently, if we have

$$\lim_{A \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \int_{\mathbb{R} \setminus [-A, A]} |x|^p d\mu_n(x) = 0.$$

See [33, Theorem 7.12] for instance.

Moreover, we recall from potential theory that, for a given admissible domain $D \subset \mathbb{C}$ and a given potential $V : D \rightarrow \mathbb{C}$ satisfying

$$\lim_{|z| \rightarrow +\infty, z \in D} \operatorname{Re} V(z) - 2 \log |z| = +\infty,$$

the functional

$$\Sigma_V : \mu \mapsto - \iint_{\mathbb{C}^2} \log |z - t| d\mu(z) d\mu(t) + \int_{\mathbb{C}} \operatorname{Re} V(z) d\mu(z), \quad (5)$$

called *Voiculescu free entropy*, admits a unique minimizer among probability measures supported on D . This minimizer is called the *equilibrium measure*

associated to V and D , and is denoted by μ_V . Note that when $D \subset \mathbb{R}$ and V is real-valued, we have

$$\Sigma_V(\mu) = - \iint_{\mathbb{R}^2} \log |x - y| d\mu(x) d\mu(y) + \int_{\mathbb{R}} V(x) d\mu(x).$$

See Saff and Totik's book [28] for a development on this topic.

For the quartic potential

$$V(x) = \frac{1}{4}x^4 + \frac{c}{2}x^2$$

and $D = \mathbb{R}$, the equilibrium measure is explicitly known (see [21, Example 3.2] for instance):

- when $c \geq -2$, it is given by the density

$$\rho_V(x) = \frac{1}{\pi} \left(\frac{1}{2}x^2 + b_0 \right) \sqrt{a^2 - x^2} \mathbf{1}_{[-a, a]}(x) \quad (6)$$

where

$$a^2 = \frac{2}{3} \left(\sqrt{c^2 + 12} - c \right), \quad b_0 = \frac{1}{3} \left(c + \sqrt{\frac{c^2}{4} + 3} \right),$$

- when $c < -2$, it is given by the density

$$\rho_V(x) = \frac{1}{2\pi} |x| \sqrt{(x^2 - a^2)(b^2 - x^2)} \mathbf{1}_{[-b, -a] \cup [a, b]}(x) \quad (7)$$

where $a^2 = -2 - c$, $b^2 = 2 - c$.

We are now able to present some existing results about the asymptotic behaviour of the solution μ_t to free Fokker-Planck equation (1) and to state our main result.

First, Li, Li, and Xie described the asymptotic behaviour of μ_t under a convex potential V .

Theorem 1.3 (see [23, Theorem 1.6 (ii)]). *Assume that V is \mathcal{C}^2 and strictly convex and μ_0 has compact support. Then we have*

$$\lim_{t \rightarrow +\infty} W_2(\mu_t, \mu_V) = 0.$$

This result can be proved using free transportation inequalities, and an explicit rate of convergence can be obtained:

$$W_2(\mu_t, \mu_V) \leq e^{-Kt} W_2(\mu_0, \mu_V),$$

where $K > 0$ is such that for all $x \in \mathbb{R}$, $V''(x) \geq K$. Note that this result applies for the potential V defined by (4) when $c > 0$. Using the fact $\mathcal{P}_2(\mathbb{R})$ is a nonpositively curved space, it is also possible to prove convergence when V is \mathcal{C}^2 and convex but not necessarily strictly convex, see [23, Theorem 1.6 (i)]. Besides, for the quartic potential, Li, Li, and Xie conjectured the convergence of μ_t towards the equilibrium measure when $c < 0$ is close to zero.

In this paper, we will focus on the case when $c \in [-2, 0)$, in which the equilibrium measure has a connected support and we expect the wells of the potential are small enough in order to get the convergence of μ_t towards the equilibrium measure μ_V even though.

Here is the main result of this paper.

Theorem 1.4. *Let $V(x) = \frac{1}{4}x^4 + \frac{c}{2}x^2$ with $-2 \leq c < 0$. The solution $(\mu_t)_{t \geq 0}$ of free Fokker-Planck equation (1) starting from any compactly supported μ_0 satisfies*

$$\lim_{t \rightarrow +\infty} W_p(\mu_t, \mu_V) = 0$$

for all $p \geq 1$, where μ_V is given by (6).

This result solves the conjecture raised by Li, Li, and Xie. Furthermore, as we will explain in Section 1.4, it provides a first case of convergence in granular media equation (10) with both a singular interaction potential \mathcal{W} and a non-convex confinement potential \mathcal{V} .

In addition to this, when c is very negative, the support of μ_V has two connected components (see Formula (7)) and because of a result of Biane and Speicher [9, Section 7.1], the solution $(\mu_t)_{t \geq 0}$ can not converge towards the equilibrium measure μ_V if the filling fractions of μ_0 and μ_V do not coincide, which means a neighbourhood of any connected component of $\text{supp}(\mu_V)$ must have the same mass for both measures. Consequently, Theorem 1.4 fills a part of the gap between the result of Li, Li, and Xie on the one hand, and the result of Biane and Speicher on the other hand.

Let us comment the choice of this special potential V with a varying parameter c , which is of particular interest. First, it is convex for a nonnegative c but it presents two wells when c is negative. Besides, the behaviour of the equilibrium measure of V presents a phase transition at $c = -2$, see Formulas (6) and (7). Finally, as we will discuss in Section 4, another phase transition probably occurs at $c = -\sqrt{15}$ when considering critical measures instead of equilibrium measure. When we know that these objects play a fundamental role in the convergence of the solution of free Fokker-Planck equation, this explains why different long-time behaviours can happen, de-

pending on the value of the parameter c .

We will end this introduction giving two motivations for the study of the long-time behaviour in free Fokker-Planck equation and explaining why this particular quartic potential is of natural interest.

1.3 Generalized Dyson Brownian motion

A first motivation comes from random matrix theory (RMT). In [19], Dyson showed that the eigenvalues of a $N \times N$ Hermitian matrix $(H_t^N)_{t \geq 0}$ with Brownian entries form a diffusive Coulomb gas, which we call *Dyson Brownian motion*. More precisely, the ordered eigenvalues $(\lambda_1^N(t), \dots, \lambda_N^N(t))_{t \geq 0}$ of the rescaled matrix H_t^N / \sqrt{N} live almost surely in the open simplex

$$\Delta_N = \{(x_1, \dots, x_N) \in \mathbb{R}^N \mid x_1 < x_2 < \dots < x_N\},$$

and satisfy the following system of SDEs:

$$\forall i \in \llbracket 1, N \rrbracket, d\lambda_i^N(t) = \frac{1}{\sqrt{N}} dB_i(t) + \frac{1}{N} \sum_{j \neq i} \frac{1}{\lambda_i^N(t) - \lambda_j^N(t)} dt,$$

where the B_i 's are independent one-dimensional standard Brownian motions. See also [3, Section 4.3].

We are interested here in *generalized Dyson Brownian motion (GDBM)*, defined as the solution of the following system of SDEs:

$$\forall i \in \llbracket 1, N \rrbracket, d\lambda_i^N(t) = \sqrt{\frac{2}{\beta N}} dB_i(t) + \frac{1}{N} \sum_{j \neq i} \frac{1}{\lambda_i^N(t) - \lambda_j^N(t)} dt - \frac{1}{2} V'(\lambda_i^N(t)) dt, \quad (8)$$

where the B_i 's are independent one-dimensional Brownian motions, $\beta > 0$ is the standard Dyson parameter in RMT, and V is a general confinement potential for particles. When V is zero or quadratic, this system has been much studied, see [17, 27] for instance. Let us also remark that for $\beta = 2$ and a general V , the eigenvalues of the $N \times N$ Hermitian diffusion process X_t^N defined as the solution to

$$dX_t^N = \frac{1}{\sqrt{N}} dH_t^N - \frac{1}{2} V'(X_t^N) dt, \quad (9)$$

where H_t^N is a $N \times N$ Hermitian matrix with Brownian entries, satisfy the SDEs (8), and that for $\beta \in [0, 2]$ and a quadratic V , GDBM also represents the eigenvalues of a solution to a matricial SDE, see [2]. However, we do not know whether in general, GDBM represents the eigenvalues of an explicit matrix model.

Adapting the original proof for Dyson Brownian motion, Li, Li, and Xie [23, Theorem 1.1] showed that if $\beta \geq 1$ and V satisfies Assumptions (i)-(iii) of Theorem 1.2, then GDBM is well-defined, i.e. given an initial value $\lambda^N(0)$ in Δ_N , the system of SDEs (8) admits a unique strong solution $(\lambda^N(t))_{t \geq 0}$ taking values in Δ_N . Note that in [1], Allez and Dumaz showed GDBM can also be defined in the case of a non-confining cubic potential.

For this process of N particles, we define the empirical spectral measure at time t by

$$L_N(t) = \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i^N(t)}.$$

On the one hand, for a fixed N , it is known that when $t \rightarrow +\infty$, the measure $L_N(t)$ converges to the empirical measure L_N of N particles distributed according to

$$\frac{1}{Z_N} \prod_{1 \leq i < j \leq N} |x_i - x_j|^\beta \exp \left(-\frac{\beta N}{2} \sum_{i=1}^N V(x_i) \right) dx_1 \dots dx_N.$$

This is because this measure is the Gibbs measure associated to the system of SDEs (8) and because we can apply an ergodic theorem. Moreover, when N then goes to infinity, this measure L_N converges to the equilibrium measure μ_V associated to V .

On the other hand, Li, Li, and Xie [23, Theorem 1.4] proved that, if $L_N(0)$ converges to a compactly supported $\mu(0)$ and if V'' is uniformly bounded below, then the limit as $N \rightarrow +\infty$ of $L_N(t)$ is μ_t , given by the solution of free Fokker-Planck equation (1) with initial condition $\mu(0)$.

Consequently, it is natural to ask whether the following diagram is commutative

$$\begin{array}{ccc} L_N(t) & \rightarrow & \mu_t \\ \downarrow & & \downarrow ? \\ L_N & \rightarrow & \mu_V \end{array}$$

i.e. in which cases μ_t converges to μ_V as t grows to infinity.

As we mentioned earlier, Li, Li, and Xie proved that this is indeed the case for a convex potential V . They also raised the conjecture of the commutativity of the previous diagram for the non-convex potential

$$V(x) = \frac{1}{4}x^4 + \frac{c}{2}x^2$$

when $c < 0$ is close to zero. We also recall that when c is very negative, the support of μ_V has two connected components and because of a result

of Biane and Speicher [9, Section 7.1], the solution $(\mu_t)_{t \geq 0}$ can not converge towards the equilibrium measure in general. As a consequence, in this paper, we focus on the intermediate case of the quartic potential with $c \in [-2, 0)$, when the potential is not convex but the equilibrium measure is still one-cut.

1.4 Granular media equation

Free Fokker-Planck equation (1) also appears as a particular case of granular media equation. This equation generally writes

$$\frac{\partial \mu_t}{\partial t} = \nabla \cdot [\mu_t \nabla (\mathcal{U}'(\mu_t) + \mathcal{V} + \mathcal{W} * \mu_t)] \quad (10)$$

where

- $\mathcal{U} : \mathbb{R}_+ \rightarrow \mathbb{R}$ is an internal energy,
- $\mathcal{V} : \mathbb{R}^d \rightarrow \mathbb{R}$ is a confinement potential,
- $\mathcal{W} : \mathbb{R}^d \rightarrow \mathbb{R}$ is an interaction potential,
- μ_t is an unknown probability measure on \mathbb{R}^d .

Free Fokker-Planck equation (1) corresponds to $d = 1$, $\mathcal{U}(s) = 0$, $\mathcal{V}(x) = \frac{1}{2}V(x)$, and $\mathcal{W}(x) = -\log|x|$.

Granular media equation contains several classes of classical partial differential equations arising from physics, such as heat equation for $\mathcal{U}(s) = s \log(s)$, $\mathcal{V} = 0$, $\mathcal{W} = 0$. See [33, Chapter 9.1] for more examples. In many cases, conditions are known to ensure that Equation (10) admits a unique solution, but we will not discuss this point here. We will rather review some existing results in the literature about the behaviour of the solution $(\mu_t)_{t \geq 0}$ as $t \rightarrow \infty$.

To study the long-time behaviour of the solution, a classical technique is to define an entropy associated to Equation (10) by

$$F(\mu) = \int_{\mathbb{R}^d} \mathcal{U}(\mu(x)) dx + \int_{\mathbb{R}^d} \mathcal{V}(x) d\mu(x) + \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{W}(x-y) d\mu(x) d\mu(y),$$

as the sum of an internal energy, a potential energy, and an interaction energy associated to a given measure μ , and to show that this entropy is strictly decreasing along the trajectory $(\mu_t)_{t \geq 0}$. Under reasonable assumptions we will not discuss here, F admits a unique minimizer μ_∞ , which provides a natural candidate for the limit of μ_t as $t \rightarrow \infty$.

Most of works in the literature deal with the case where \mathcal{V}, \mathcal{W} are zero or polynomials, and $\mathcal{U}(s)$ is 0 or $s \log(s)$. The first results we could find are

due to Benedetto, Caglioti, Pulvirenti [5] and Benedetto, Caglioti, Carrillo, Pulvirenti [4] respectively. These works provide two examples of granular media equations arising from physics, for which we can make explicit calculations on F in order to get the convergence of μ_t towards the minimizing measure μ_∞ .

Later, Carrillo, McCann, and Villani [15] managed to prove convergence results for general potentials \mathcal{V}, \mathcal{W} under convexity and positivity assumptions. Their powerful method relies on mass transportation methods and on the Riemannian structure of the set of probability measures $\mathcal{P}_2(\mathbb{R})$ equipped with the Wasserstein distance W_2 . Furthermore, this method allows to get explicit rates of convergence. Some of their results have been also proved by Malrieu [24] using an approximating particle system and propagation of chaos. Various improvements of Carrillo, McCann, and Villani's results exist in the literature, see for instance [16, 12, 10, 11].

Let us also mention a series of works by Tugaut, see in particular [32, 31]. In these works, granular media equation with polynomial potentials \mathcal{V}, \mathcal{W} is considered, but a Laplacian term multiplied by a small coefficient is added, i.e. $\mathcal{U}(s) = \sigma s \log(s)$ with a small $\sigma > 0$. Convergence results can be obtained in this setting, even when \mathcal{V} is not convex but a double-well potential, similar to the one we are studying here.

Most of the results we have discussed about concern a polynomial or a convex interaction potential \mathcal{W} . Li, Li, and Xie [23] recently tackled the problem of a logarithmic repulsion potential, namely $\mathcal{W}(x) = -\log|x|$. Note that this potential is singular in 0, which makes granular media equation more complicated to study, since the previous methods do not apply any longer. However, using free probability and more precisely free transportation methods, Li, Li, and Xie have adapted Carrillo, McCann, and Villani's method to get similar results when \mathcal{V} is a convex potential, such as Theorem 1.3. We also mention the recent work [14] by Carrillo, Castorina, and Volzone, in which a two-dimensional logarithmic interaction is considered, corresponding to Keller-Segel model.

Consequently, the issue of the long-time behaviour of the solution of granular media equation for this singular interaction $\mathcal{W}(x) = -\log|x|$ and a non-convex confinement potential \mathcal{V} is of natural interest. As suggested by the conjecture of Li, Li, and Xie mentioned above, we will focus in this paper on the non-convex potential V given by (4) with $-2 \leq c < 0$.

The rest of the paper is organized as follows. Several tools, such as free diffusions and the description of critical measures via quadratic differentials, are introduced in Section 2 and are used in Section 3, which consists in the

achievement of the proof of Theorem 1.4. Section 4 is the final section of this paper, in which we present some perspectives for future works.

2 Free probability and quadratic differentials tools

2.1 Some properties of the solution of free Fokker-Planck equation

As we explained in Introduction, the solution of free Fokker-Planck equation (1) can be interpreted as the distribution of the solution to free SDE

$$dX_t = dS_t - \frac{1}{2}V'(X_t)dt,$$

where S is a free Brownian motion. This approach requires to work with free probability, however, instead of studying a singular PDE, we are working with a free SDE whose drift is locally Lipschitz. In this free probability context, classical tools such as Picard iteration, Euler scheme, etc. can be adapted. As a consequence, the solution of free Fokker-Planck equation inherits from properties of free diffusions, studied by Biane and Speicher [9].

First, as in classical works around granular media equation (see [15] for instance), we have a formula for the derivative of free entropy (5) along solutions, showing it decreases.

Lemma 2.1 (see [9, Proposition 6.1]). *Let $(\mu_t)_{t \geq 0}$ be the solution of free Fokker-Planck equation (1). We have*

$$\frac{d}{dt}\Sigma_V(\mu_t) = -2 \int \left| \frac{1}{2}V' - H\mu_t \right|^2 d\mu_t. \quad (11)$$

The next statement summarizes two important strong properties of free diffusions and a corollary of these properties.

Proposition 2.2 (see [9, Theorems 3.1 and 5.2]). *Let V be a potential satisfying the assumptions of Theorem 1.1 with $a < 0$. Let $(\mu_t)_{t \geq 0}$ be the solution of free Fokker-Planck equation (1) starting from a compactly supported μ_0 .*

(i) *There exists $M > 0$ such that for every $t > 0$,*

$$\text{supp}(\mu_t) \subset [-M, M]. \quad (12)$$

(ii) *There exist $K_1, K_2 > 0$ depending only on V such that for every $t > 0$, the density ρ_t of μ_t satisfies*

$$\|\rho_t\|_\infty \leq \frac{K_1}{\sqrt{t}} + K_2, \quad \|D^{1/2}\rho_t\|_2 \leq \frac{K_1}{t} + K_2, \quad (13)$$

where $D^{1/2}$ is the fractional derivative of order $1/2$.

(iii) The family $\{\rho_t\}_{t \geq 1}$ lives in a subset \mathcal{A} of $L^2([-M, M])$ which is compact for the topology induced by $\|\cdot\|_2$.

In the statement of Point (ii), the notion of half-derivative appeared. It can be defined by several ways, we will just mention that for $u \in L^2$, the derivative of order $1/2$ of u is the inverse Fourier transform of $\xi \mapsto (1 + \xi^2)^{1/4} \hat{u}(\xi)$, where \hat{u} is the Fourier transform of u . See [18, Chapter 4] for instance.

Let us make some remarks about Proposition 2.2, which can be applied to the quartic potential and which is fundamental in our study.

First, Point (i) asserts that the μ_t 's have a uniformly bounded support. Not only does it imply tightness, but it also means that the trajectory lives in a compact set of probability measures for weak topology. Besides, uniform estimates of Point (ii) also imply compactness for L^2 -topology, see Point (iii). Consequently, we will be able to extract from $(\mu_t)_{t \geq 0}$ a converging subsequence in a strong sense (convergence in distribution and convergence of the densities in L^2) and the limit of this "good" converging subsequence will keep the properties of the μ_t 's, such as a support in $[-M, M]$ and a bounded density.

Moreover, the estimates in Point (ii) make some singular functionals continuous along the trajectory. For instance, the map $t \mapsto \Sigma_V(\mu_t)$ is continuous even if free entropy Σ_V is not continuous at all, and the integrals $\int |\frac{1}{2}V' - H\mu_t|^2 d\mu_t$ converge to $\int |\frac{1}{2}V' - H\mu|^2 d\mu$, as we will prove in Section 3.

Proof (of Proposition 2.2 (iii)). By Proposition 2.2 (i)-(ii), there exist $M, K_1, K_2 > 0$ such that for every $t > 0$, (12) and (13) hold. For every $t > 0$, we denote by \mathcal{A}_t the set of probability density functions f with support in $[-M, M]$ which satisfy $\|f\|_\infty \leq \frac{K_1}{\sqrt{t}} + K_2$ and $\|D^{1/2}f\|_2 \leq \frac{K_1}{t} + K_2$. Note that, for $t > 0$, \mathcal{A}_t contains all the ρ_{t+s} 's, $s \geq 0$, where ρ_{t+s} denotes the density of the measure μ_{t+s} as in Point (ii).

Furthermore, for every $t > 0$, \mathcal{A}_t is a subset of the Sobolev space $H^{1/2}([-M, M])$, defined as the set of L^2 -probability density functions whose derivative of order $1/2$ belongs to L^2 . Because the injection of $H^{1/2}([-M, M])$ in $L^p([-M, M])$ is compact for every $p \in [1, \infty)$ (see [18, Theorem 4.54] for instance) and \mathcal{A}_t is bounded in $H^{1/2}([-M, M])$, we can deduce that the set \mathcal{A}_t is relatively compact in $L^2([-M, M])$. Hence, we can choose for \mathcal{A} the closure of \mathcal{A}_1 . \square

Remark. Following arguments from [34], as a consequence of Point (iii), convergence of $(\mu_t)_{t \geq 0}$ in distribution and convergence of $(\rho_t)_{t \geq 0}$ in L^p are equivalent, but we will not prove this fact here.

2.2 Equilibrium, stationary, and critical measures

Three kinds of measures appear in our problem. First, as we have ever discussed in Section 1.2, the *equilibrium measure* of a potential V , well known in potential theory, is the unique minimizer over all probability measures on \mathbb{R} of Voiculescu free entropy Σ_V defined in (5). We refer again to Saff and Totik's book [28] for more details.

Second, we consider stationary measures. We say a real probability measure μ is a *stationary measure* for a real potential V if it satisfies Euler-Lagrange equation

$$H\mu = \frac{1}{2}V' \quad \mu\text{-a.e.} \quad (14)$$

We call these measures stationary measures because they are exactly the stationary solutions of Equation (1) in the sense of PDEs, i.e. they are the constant (in time) solutions of Equation (1).

Finally, as the map $t \mapsto \Sigma_V(\mu_t)$ decreases, it admits a limit and we have in mind that the solution of free Fokker-Planck equation $(\mu_t)_{t \geq 0}$ will converge to a local minimum of free entropy. But this functional is strictly convex, thus we need to precise what we mean by a local extremum of Σ_V . This leads to the notion of critical measure, defined as follows by Martínez-Finkelshtein and Rakhmanov [25].

A probability measure μ on \mathbb{C} such that $\Sigma_V(\mu) < +\infty$ is called a *critical measure* associated to V if for every $h : \mathbb{C} \rightarrow \mathbb{C}$ regular enough, the quantity

$$D_h \Sigma_V(\mu) = \lim_{s \rightarrow 0} \frac{\Sigma_V(\mu^{h,s}) - \Sigma_V(\mu)}{s}$$

is zero, where $\mu^{h,s}$ is the push-forward measure of μ by the deformation of identity $z \mapsto z + sh(z)$, $s \in \mathbb{C}$.

By [25, Lemma 3.7], we have

$$D_h \Sigma_V(\mu) = \operatorname{Re} \left(\int V'(x)h(x) d\mu(x) - \iint \frac{h(x) - h(y)}{x - y} d\mu(x)d\mu(y) \right),$$

hence for a μ supported on \mathbb{R} , the previous condition is equivalent to Euler-Lagrange equation (14). As a result, critical measures supported on \mathbb{R} are exactly stationary measures, and we will be able to use some tools developed to identify critical measures in order to identify stationary measures.

Note that in general, several critical measures may exist while there is only one equilibrium measure. This is the case for a potential satisfying the conditions given in [9, Section 7.1] for instance.

In the proof of Theorem 1.4, we will be interested in real critical measures associated to the quartic potential. A key point in the proof will be to show that for the quartic potential (4), there is no other critical measure than the equilibrium measure, since we expect the convergence towards this one.

The following statement gives the most important properties of real critical measures we will use in the sequel. The key point is that the Stieltjes transform of a critical measure μ , defined on $\mathbb{C} \setminus \mathbb{R}$ by

$$G^\mu(z) = \int_{\mathbb{R}} \frac{1}{z - x} d\mu(x),$$

satisfies an algebraic equation. This allows to recover a critical measure from an associated polynomial.

Proposition 2.3 (see [22, 20]). *Let V be a polynomial and μ be a critical measure supported on \mathbb{R} .*

(i) *There exists a polynomial R of degree $2 \deg(V) - 2$ such that*

$$R(z) = \left(\frac{1}{2} V'(z) - G^\mu(z) \right)^2 \quad (15)$$

almost everywhere for Lebesgue measure on \mathbb{C} . Moreover, we have

$$R(z) = \frac{1}{4} V'(z)^2 - \int_{\mathbb{R}} \frac{V'(x) - V'(z)}{x - z} d\mu(x). \quad (16)$$

(ii) *Every non-real root of R has even multiplicity.*

(iii) *The support of μ is a finite union of intervals connecting zeros of R .*

Point (i) combines Proposition 3.7 and Formula (3.31) from [22]. Point (ii) is an easy consequence of analyticity of Stieltjes transform, see [20, Lemma 2.6]. At last, Point (iii) comes from [22, Proposition 3.9].

Let us remark that a critical measure μ is completely determined by the associated polynomial R . To see this, without entering into the details of the theory of quadratic differentials (see [29] for an introduction), we can say that the support of a critical measure is a union of analytic arcs, which are maximal trajectories of the quadratic differential $-R(z) dz^2$. Moreover, in the interior of each arc, μ admits a density with respect to the arclength measure, which is given by

$$d\mu(s) = \frac{1}{i\pi} \sqrt{R(s)} ds.$$

Hence, to find critical measures boils down to determining all possible polynomials R . For the quartic potential and other polynomials with few monics,

this is possible to do so. However, in the quartic case, we will only use R in order to show that a critical measure has a connected support. Indeed, as soon as this is the case, we can just recover μ by solving a singular integral equation, as we will do in the next subsection.

See [25] and [22] for more details on the technique using quadratic differentials and [25, 20] for bringing it into play in order to study the asymptotics of the distribution of the zeros of some polynomials, such as some orthogonal polynomials.

2.3 A uniqueness result about stationary measures for the quartic potential

As we explained in Section 2.1, in the proof of Theorem 1.4, we will extract from $(\mu_t)_{t \geq 0}$ a "good" converging subsequence whose limit μ has a compact support, a bounded density, and satisfies Euler-Lagrange equation

$$H\mu = \frac{1}{2}V' \quad \mu\text{-a.e.}$$

where

$$H\mu(x) = \oint_{\mathbb{R}} \frac{1}{x-y} d\mu(y).$$

This singular integral equation can be easily solved when we know a priori that μ has a connected support. In the proof of Theorem 1.4, we will show this additional property, using the quadratic differentials tools introduced above (see Proposition 2.3). Then, μ will be identified thanks to the following result, whose proof is given at the end of this subsection.

Proposition 2.4. *For the potential $V(x) = \frac{1}{4}x^4 + \frac{c}{2}x^2$ with $-2 \leq c < 0$, the only stationary probability measure with bounded density and connected support is the equilibrium measure μ_V , which is defined by (6).*

In order to establish this result, we will use a result due to Muskhelishvili [26], which allows to solve singular integral equations once we know the support of the solution, or at least its number of connected components. For a slightly different approach, see Tricomi [30].

Theorem 2.5 (see [26, §88]). *Let L be a finite union $\bigcup_{j=1}^p [a_{2j-1}, a_{2j}]$ and f a given Hölder continuous function on L . The singular integral equation*

$$\forall x \in L, \quad \oint_L \frac{\varphi(t)}{t-x} dt = f(x)$$

admits a Hölder continuous, bounded solution φ if and only if f satisfies the p following conditions:

$$\forall k \in \llbracket 0, p-1 \rrbracket, \quad \int_L \frac{t^k f(t)}{\prod_{j=1}^{2p} \sqrt{|t-a_j|}} dt = 0.$$

In this case, the solution is unique and it is given by

$$\forall x \in L, \quad \varphi(x) = -\frac{1}{\pi^2} \prod_{j=1}^{2p} \sqrt{|x - a_j|} \int_L \frac{f(t)}{(t-x) \prod_{j=1}^{2p} \sqrt{|t - a_j|}} dt.$$

We now prove that for the quartic potential with $-2 \leq c < 0$, the only suitable stationary measure is the equilibrium measure.

Proof (of Proposition 2.4). Let μ be a stationary probability measure with bounded density, denoted by ρ , and with connected support, denoted by $[a, b]$. By Theorem 2.5 applied to $f(x) = -\frac{1}{2}V'(x)$ and $p = 1$, the existence of μ is ensured by the condition

$$\int_a^b \frac{t^3 + ct}{\sqrt{(t-a)(b-t)}} dt = 0. \quad (17)$$

An elementary computation leads to

$$\int_a^b \frac{t^3 + ct}{\sqrt{(t-a)(b-t)}} dt = \frac{\pi}{16} (5b^3 + 3ab^2 + 3a^2b + 5a^3) + c \frac{\pi}{2} (a + b),$$

thus condition (17) reads

$$(a + b)(5b^2 - 2ab + 5a^2 + 8c) = 0. \quad (18)$$

Moreover, by Theorem 2.5 again, the density of μ is given by

$$\begin{aligned} \rho(x) &= \frac{\sqrt{(x-a)(b-x)}}{2\pi^2} \int_a^b \frac{t^3 + ct}{(t-x)\sqrt{(t-a)(b-t)}} dt \\ &= \frac{1}{2\pi} \sqrt{(x-a)(b-x)} \left(x^2 + \frac{a+b}{2}x + \frac{3}{8}b^2 + \frac{1}{4}ab + \frac{3}{8}a^2 + c \right) \end{aligned} \quad (19)$$

This result has been obtained by standard integral computations. By integrating this expression between a and b , since ρ is a probability density function, we get a new equation on a and b :

$$\frac{(b-a)^2}{256} (15a^2 + 18ab + 15b^2 + 16c) = 1. \quad (20)$$

The two equations (18) and (20) allow us to determine a and b . First, Equation (18) gives three families of possible solutions:

$$a = -b, \quad a = \frac{1}{5}b + \frac{2}{5}\sqrt{-10c - 6b^2}, \quad a = \frac{1}{5}b - \frac{2}{5}\sqrt{-10c - 6b^2}.$$

Equation (20) will then allow to eliminate some cases. Note before that if c is nonnegative, then only the first case would be possible, and that the same

situation occurs when c is negative but $b^2 > -\frac{5}{3}c$.

- Case 1: $a = -b$.

In this case, Equation (20) gives

$$b = \sqrt{\frac{2}{3} \left(\sqrt{c^2 + 12} - c \right)},$$

so the density given by (19) becomes

$$\rho(x) = \frac{1}{2\pi} \sqrt{b^2 - x^2} \left(x^2 + \frac{2}{3}c + \frac{1}{3}\sqrt{c^2 + 12} \right).$$

This is exactly the equilibrium measure of V for $c \geq -2$, see (6).

- Case 2: $a = \frac{1}{5}b + \frac{2}{5}\sqrt{-10c - 6b^2}$.

Equation (20) now implies that

$$45b^8 + 156cb^6 + (182c^2 - 552)b^4 + (76c^3 - 880c)b^2 + 5c^4 - 200c^2 + 2000 = 0.$$

We will show this is not possible under the conditions $-2 \leq c \leq 0$ and $0 \leq b^2 \leq -\frac{5}{3}c$. Indeed, we can study the polynomial function

$$f : (x, c) \mapsto 45x^4 + 156cx^3 + (182c^2 - 552)x^2 + (76c^3 - 880c)x + 5c^4 - 200c^2 + 2000$$

on the compact set

$$K = \left\{ (x, c) \in \mathbb{R}^2 \mid -2 \leq c \leq 0, 0 \leq x \leq -\frac{5}{3}c \right\}.$$



Figure 2: Compact set K .

The resolution of $\frac{\partial f}{\partial x}(x, c) = \frac{\partial f}{\partial c}(x, c) = 0$ shows that the only critical point of f in K is $(0, 0)$. Consequently, f attains its minimum on the boundary of K . The study of the three functions

$$c \mapsto f(0, c) = 5(c^2 - 20)^2,$$

$$x \mapsto f(x, -2) = 45x^4 - 312x^3 + 176x^2 + 1152x + 1280,$$

and

$$c \mapsto f\left(-\frac{5}{3}c, c\right) = \frac{80}{9}(c^2 - 15)^2$$

allows to conclude that the minimum of f on K is attained in $(\frac{10}{3}, -2)$ and is equal to $\frac{9680}{9}$. Consequently, f does not vanish on K and Case 2 does not lead to a suitable solution μ .

- Case 3: $a = \frac{1}{5}b - \frac{2}{5}\sqrt{-10c - 6b^2}$.

Very similar computations lead to the fact that the same function f must vanish on the same compact K , and thus to the same conclusion.

Finally, the only stationary probability measure with bounded density and connected support is indeed the equilibrium measure μ_V . \square

Remark. The previous calculations show that the same conclusion holds when $c \geq 0$. However, there does not exist a stationary probability measure with bounded density and connected support when $-\sqrt{15} < c < -2$ because in this situation, Case 1 of the proof leads to a density taking negative values, and Cases 2 and 3 still lead to unsuitable solutions.

In addition to this, the same technique allows to prove that when $c < -2$, the only symmetric stationary probability measure having a bounded density and a support with two cuts is the equilibrium measure. Besides, when $c \geq -2$, there does not exist such a symmetric stationary measure.

3 Proof of Theorem 1.4

We are now able to prove Theorem 1.4. In Section 2, we introduced all the necessary tools and we gave some ideas of the proof we now summarize.

First, thanks to properties of free diffusions, the solution $(\mu_t)_{t \geq 0}$ of free Fokker-Planck equation (1) lives in compact sets for various topologies. Using in addition that free entropy Σ_V decreases along the trajectory, we can extract a particular converging subsequence (μ_{t_k}) . Its limit is stationary and has a bounded density, but using quadratic differentials properties, we show that it also has a connected support, which makes it is necessarily the equilibrium measure μ_V .

Moreover, using again the estimates on the solution $(\mu_t)_{t \geq 0}$, we show that free entropy Σ_V is continuous along the solution, thus accumulation

points of $(\mu_t)_{t \geq 0}$ must have the same entropy. As μ_V is the unique minimizer of Σ_V , this ends the proof.

From now, for every $t \geq 0$, we denote by ρ_t the density of μ_t .

Proof (of Theorem 1.4). By (11), the function $t \mapsto \Sigma_V(\mu_t)$ is decreasing on $[0, +\infty)$. As it is also bounded below (by $\Sigma_V(\mu_V)$), this function admits a finite limit as t goes to infinity and there exists a sequence $(t_k)_{k \in \mathbb{N}}$ such that $t_k \rightarrow \infty$ and $\frac{d}{dt} \Sigma_V(\mu_{t_k}) \rightarrow 0$ when $k \rightarrow \infty$.

By Proposition 2.2 (iii), even if it means to extract a subsequence again, we can assume that the densities ρ_{t_k} converge w.r.t. L^2 -topology to a limit ρ . As the ρ_{t_k} 's are defined on the compact set $[-M, M]$, L^2 -convergence implies that $\int \rho_{t_k}(x) dx$ converges to $\int \rho(x) dx$, hence the limit ρ is a density probability function defined on $[-M, M]$. We denote by μ the probability measure associated to ρ . By Scheffé's lemma, μ_{t_k} also converges in distribution towards μ .

We will now prove that μ is a stationary probability measure with a bounded density. First, as the densities ρ_{t_k} converge in $L^2([-M, M])$, even if it means to extract a subsequence again, we can assume that they converge almost everywhere on $[-M, M]$. Thus, we have $\|\rho\|_\infty \leq K_2$.

Furthermore, for all $k \in \mathbb{N}$, we can decompose

$$\begin{aligned} & \left| \int \left| H\mu_{t_k} - \frac{1}{2}V' \right|^2 d\mu_{t_k} - \int \left| H\mu - \frac{1}{2}V' \right|^2 d\mu \right| \\ & \leq \left| \int \left| H\mu_{t_k} - \frac{1}{2}V' \right|^2 d\mu_{t_k} - \int \left| H\mu - \frac{1}{2}V' \right|^2 d\mu_{t_k} \right| \\ & \quad + \left| \int \left| H\mu - \frac{1}{2}V' \right|^2 d\mu_{t_k} - \int \left| H\mu - \frac{1}{2}V' \right|^2 d\mu \right| \quad (21) \end{aligned}$$

where the integrals are taken over $[-M, M]$.

The first term in the right-hand side goes to 0 as $k \rightarrow +\infty$. Indeed, denoting by K a uniform bound on the densities and using the Cauchy-Schwarz inequality, we have for all $k \in \mathbb{N}^*$,

$$\begin{aligned} & \left| \int \left| H\mu_{t_k} - \frac{1}{2}V' \right|^2 d\mu_{t_k} - \int \left| H\mu - \frac{1}{2}V' \right|^2 d\mu_{t_k} \right| \\ & \leq K \left| \int |H\mu_{t_k}(x) - H\mu(x)| (|H\mu_{t_k}(x)| + |H\mu(x)| + |V'(x)|) dx \right| \\ & \leq K \|H\mu_{t_k} - H\mu\|_2 \times (\|H\mu_{t_k}\|_2 + \|H\mu\|_2 + \|V'\|_2) \end{aligned}$$

and by continuity of Hilbert transform from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$, $H\mu_{t_k}$ converges to $H\mu$ in L^2 .

On the other hand, it follows from similar arguments and from $\rho \in L^4$ that the second term in (21) also tends to 0 when $k \rightarrow +\infty$. By (11), we finally have

$$0 = \lim_{k \rightarrow +\infty} \frac{d}{dt} \Sigma_V(\mu_{t_k}) = \lim_{k \rightarrow +\infty} -2 \int \left| H\mu_{t_k} - \frac{1}{2}V' \right|^2 d\mu_{t_k} = -2 \int \left| H\mu - \frac{1}{2}V' \right|^2 d\mu.$$

The limit measure μ is thus stationary, in the sense of Section 2.2.

Moreover, we can show that μ has a connected support. Indeed, for the potential $V(x) = \frac{1}{4}x^4 + \frac{c}{2}x^2$, by (16), the polynomial R defined in (15) is

$$R(z) = \frac{1}{4}z^6 + \frac{c}{2}z^4 + \frac{1}{4}(c^2 - 4)z^2 - \int x d\mu(x).z - \int x^2 d\mu(x) - c.$$

We can not find the roots of this polynomial because the two first moments of μ are unknown, however, we will be able to count its real roots applying Descartes' rule of signs.

Proposition 3.1 (Descartes' rule of signs). *Let*

$$P(X) = a_n X^n + \dots + a_1 X + a_0$$

be a polynomial. We denote by p , resp. q , the number of sign changes in the sequence (a_n, \dots, a_1, a_0) , resp. $((-1)^n a_n, \dots, -a_1, a_0)$, in which we have removed the zeros. Then, the number of positive, resp. negative, roots of P is at most p , resp. q , and has the same parity as p , resp. q .

If we distinguish the four possible cases, it easily follows that, due to the inequalities $-2 \leq c < 0$, the polynomial R admits 0, 2, or 4 non-zero real roots, whatever the signs of the quantities $\int x d\mu(x)$ and $\int x^2 d\mu(x) + c$ are.

In addition to this, every non-real root of R has even multiplicity by Proposition 2.3 (ii). Since R admits 6 roots, it means that the multiplicity of 0 is necessarily even.

- If 0 is not a root of R , then R admits at most 4 real roots, thus at least two conjugate non-real roots. But, by Proposition 2.3 (ii), every non-real root is at least a double root, thus R has in fact at most two real roots. By Proposition 2.3 (iii), μ has a connected support in this case.
- If 0 is a root of R , then it is at least a double root. Thus R is explicit and we have $R(z) = \frac{1}{4}z^2(z^2 + c + 2)(z^2 + c - 2)$. This is impossible for $c > -2$ by Proposition 2.3 (ii). For $c = -2$, this leads to $R(z) = \frac{1}{4}z^4(z - 2)(z + 2)$, thus by Proposition 2.3 (iii), the support of μ is $[-2, 0]$, $[0, 2]$, or $[-2, 2]$.

In both cases, we showed that μ has a connected support.

If we summarize, μ is a stationary probability measure with bounded density and connected support so, by Proposition 2.4, it is the equilibrium measure μ_V .

We will now conclude the proof by showing that the density ρ_V of μ_V is the only possible accumulation point for $(\rho_t)_{t \geq 0}$ in L^2 -topology.

Indeed, let $(\rho_{s_k})_{k \in \mathbb{N}}$ and $(\rho_{s'_k})_{k \in \mathbb{N}}$ be two convergent subsequences from $(\rho_t)_{t \geq 0}$ for L^2 -topology. We denote by ρ and ρ' their respective limit. Clearly, as the ρ_t 's for $t \geq 1$, these limits are density probability functions supported in $[-M, M]$ and they are bounded by $K_1 + K_2$. We denote by μ and μ' the associated probability measures.

By the Cauchy-Schwarz inequality, we have

$$\left| \int V(x) d\mu_{s_k}(x) - \int V(x) d\mu(x) \right| \leq \|\rho_{s_k} - \rho\|_2 \|V\|_2$$

and

$$\begin{aligned} & \left| \iint \log|x-y| \rho_{s_k}(x) \rho_{s_k}(y) dx dy - \iint \log|x-y| \rho(x) \rho(y) dx dy \right| \\ & \leq \left| \iint \log|x-y| \rho_{s_k}(x) (\rho_{s_k}(y) - \rho(y)) dx dy \right| \\ & \quad + \left| \iint \log|x-y| \rho(y) (\rho_{s_k}(x) - \rho(x)) dx dy \right| \\ & \leq 2(K_1 + K_2) \cdot \sqrt{2M} \|\rho_{s_k} - \rho\|_2 \left(\iint \log^2|x-y| dx dy \right)^{1/2} \end{aligned}$$

for k large enough. Therefore, we get

$$\lim_{k \rightarrow +\infty} \Sigma_V(\mu_{s_k}) = \Sigma_V(\mu).$$

Similarly, we have $\lim_{k \rightarrow +\infty} \Sigma_V(\mu_{s'_k}) = \Sigma_V(\mu')$. Using that the function $t \mapsto \Sigma_V(\mu_t)$ is decreasing, we thus have $\Sigma_V(\mu) = \Sigma_V(\mu')$.

Consequently, two accumulation points of $(\rho_t)_{t \geq 0}$ in L^2 -topology lead to the same entropy. Since we proved that ρ_V is an accumulation point and since μ_V the unique minimizer of free entropy Σ_V , we conclude that the only possible accumulation point in L^2 -topology is ρ_V . But, by Proposition 2.2 (iii), the ρ_t 's, $t \geq 1$, are contained in a compact set \mathcal{A} for this topology, so ρ_t converges towards ρ_V in L^2 -topology. As we explained at the beginning of this proof, this implies that μ_t converges in distribution towards μ_V . Since weak convergence and W_p -convergence, $p \in [1, +\infty)$, coincide for distributions on a given compact set, the conclusion of Theorem 1.4 follows. \square

4 Conclusion and perspectives

We have obtained in Theorem 1.4 that the solution $(\mu_t)_{t \geq 0}$ of free Fokker-Planck equation (1) for the particular potential

$$V(x) = \frac{1}{4}x^4 + \frac{c}{2}x^2, \quad -2 \leq c < 0,$$

converges as t grows to infinity towards the equilibrium measure μ_V associated to V . This fills a part of the gap between a result by Li, Li, and Xie [23], stating that the same conclusion holds for $c \geq 0$, and a remark by Biane and Speicher [9], asserting that when c is very negative, the convergence towards μ_V is not possible if μ_0 has not the same filling fractions as μ_V . Moreover, this is the first convergence result we know when both the interaction is singular and the confinement potential is non-convex in granular media equation (10).

In addition to this, the proof we have proposed also applies in the case when $c \geq 0$. Hence, we can precise the result obtained by Li, Li, and Xie [23]. Indeed, we have showed that the solution $(\mu_t)_{t \geq 0}$ of free Fokker-Planck equation satisfies

$$\lim_{t \rightarrow +\infty} W_p(\mu_t, \mu_V) = 0$$

for every $p \geq 1$. Li, Li, and Xie proved only the case $p = 2$, and thus also $p \leq 2$. However, they obtained an exponential convergence rate when $c > 0$, which is not possible with our method.

Many natural questions follow this work:

- Our result uses the fact that we have only one suitable critical measure when $c \in [-2, 0)$. Can we describe the suitable critical measures when $c < -2$? For instance, there is no critical measure with bounded density and connected support when $-\sqrt{15} < c < -2$. In this case, is the equilibrium measure μ_V the only suitable critical measure, and is the convergence of the solution of (1) towards μ_V possible?
- The value $c = -\sqrt{15}$ appears as the value under which the existence of unilateral critical measures for the quartic potential becomes possible. This threshold also appears in a paper by Bertola and Tovbis [6] in a slightly different context. Are the measures described in [6] the only critical measures?
- When c is very negative, can we describe the basins of attraction associated to each possible limit for the solution of free Fokker-Planck equation?

- We used the special form of the quartic potential in order to show that a critical measure has a connected support and in order to compute it, but nowhere else. Do our methods apply in other cases? For instance, can we change the potential V , take a higher degree, or consider higher dimensions?
- Several works deal with non-confining potentials. For instance, Allez and Dumaz [1] studied a cubic potential, and Brézin, Itzykson, Parisi, and Zuber [13] considered the quartic potential $V(x) = \frac{1}{2}x^2 + \frac{g}{4}x^4$ with $g < 0$. For these potentials, once the problems of definitions are solved, we can tackle the problem of long-time behaviour. Can we prove a convergence result for the cubic potential or for the quartic potential with $-\frac{1}{12} < g < 0$, as Biane and Speicher conjectured for the latter?

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