Symmetries impact in chaotification of piecewise smooth systems

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Abstract

This paper is devoted to a mathematical analysis of a route to chaos for bounded piecewise smooth systems of dimension three subjected to symmetric non-smooth bifurcations. This study is based on period doubling method applied to the associated Poincaré maps. Those Poincaré maps are characterized taking into account the symmetry of the transient manifolds. The corresponding Poincaré sections are chosen to be transverse to these transient manifolds, this particular choice takes into account the fact that the system dynamics crosses the intersection of both manifolds. In this case, the dimension of the Poincaré map (defined as discrete map of dimension two) is reduced to dimension one in this particular neighborhood of transient points. This dimension reduction allows us to deal with the famous result "period three implies chaos". The approach is also highlighted by simulations results applied particularly to Chua circuit subjected to symmetric grazing bifurcations.

Keywords: Chaotification analysis, period doubling, Non-smooth bifurcations, Symmetries, Chua circuit

1 Introduction

In the literature, hybrid dynamic models can represent systems for which the behavior consists of continuous evolution interspersed by instantaneous jumps in the velocity. More precisely, those systems exhibit non-smoothness or discontinuities in the dynamics and this induces new dynamics phenomena witch are not present in smooth dynamics. However, the field of hybrid systems is not as mature as the smooth one. The corresponding fundamental theoretical concepts have not been so developed. The most known general textbook on hybrid systems is [46] and the book [40] contains qualitative analysis of some classes of hybrid systems. Recently, it was gradually recognized that a particular class of those

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systems exhibits many interesting phenomena because of the specific complex structure of the state space composed of some different vector fields. In this case, the dynamics of the system can be defined by an ordinary differential equation in each region and the associated Poincaré map is continuous across the border but its derivative is discontinuous. Those systems are called piecewise smooth systems (noted p.w.s systems), they occur naturally in the description of many physical processes as grazing, sliding, switching, friction and so on. This type of dynamics was introduced and studied in many seminal papers [2], [3], [17], [27], [18], [31], [38], [41], [42], [50]. Many books and monographs have been published on this topic. The analysis in [32] generalized several fundamental theories in smooth systems theory to this relevant class of hybrid systems. [12] gave a comprehensive treatment on the theory of p.w.s systems. The reader can also refer to recent overviews articles [13] for numerous references therein. Such class of p.w.s systems is common in the literature. Authors in [15], [16], [33] dealt with p.w.s systems from mechanical problems, other applications were performed in control in engineering [3], [48], [37] electromechanical systems [29] or in gene regulatory networks and neurons in computational neuroscience and biology [45]. In those applications, it is often essential to characterize its bifurcations. Those events , known as discontinuity induced bifurcations, occur when an invariant set of the system (as an equilibrium point or a limit cycle) crosses or hits tangentially the switching manifold in the phase space. A pioneering work was achieved by Feigin in [23], [24], [25] who introduced the notion of C-bifurcations and has recently re-evaluated it in [7]. Furthermore, symmetric bifurcations are widespread phenomena, one of the oldest known example is the Lorenz dynamics [47] for the smooth systems and the Chua circuit [21] for the piecewise smooth ones. This kind of symmetric non-smooth transients occurs for example in a multicell chopper coupled with nonlinear load and may generate a chaotic behavior [22] (see [1], [28] for a mathematical definitions and characterizations of chaos in dynamical systems). In fact, all those types of bifurcations can give rise to a chaotic behavior. Most notably, p.w.s systems can exhibit robust chaotic behavior that have been conjectured not to exist for smooth systems. This is due to the discontinuous dependence on initial conditions leading to chaotic behavior. Knowing that there exist three main branches of chaotic dynamic systems theory namely the symbolic dynamics, ergotic theory and bifurcation theory, we focus on the last one in this paper. Those notions can be found in references [28], [30], [43]. Author in [32] generalized several fundamental theories in smooth systems theory including Lyapounov exponents and Conley index to p.w.s systems. Some interesting results in [51] are dedicated to bifurcations and chaos analysis to p.w.s systems. P. Collins gives in [19] an overview of some chaotic hybrid systems. He proposed results on dynamics in switched arrival systems and in systems with periodic forcing.

Hereafter, we propose a mathematical analysis of way to chaos for bounded p.w.s systems of dimension three subjected to symmetric non-smooth bifurcations. We restrict our attention to bimodal p.w.s systems depending on a parameter ε. Such class of p.w.s systems is common in the literature due to its importance in many applications [44], [49],...

This work is an extension to symmetric case of the results obtained in [4] and [5] and associated to non-symmetric and non-smooth bifurcations. The suggested procedure is based on four main features: the first one is the Poincaré maps determination associated to p.w.s systems subjected to symmetric non-smooth transitions. It is an extension of the Poincaré Discontinuity Maps (noted P.D.M.) associated to p.w.s systems subjected to classic non-smooth transitions given in [8], [9], [10]. The Poincaré maps computed here are characterized by a composition of the previous Poincaré maps with some particular maps that take into account the symmetries of the dynamics. The second feature is the special choice of the Poincaré sections relatively to the switching manifolds. Those Poincaré sections are perpendicular to the switching manifolds, this permits to reduce the dimension of the Poincaré maps from two to one, this reduction being available only in a specific neighborhood of the bifurcation points. The third feature is the application of period doubling method based on the famous result of [35] "period three implies chaos". It is important to mention here that another choice of Poincaré sections will oblige us to be in dimension 2 and thus to use results of Marotto published in 1978 who generalized results of Li and Yorke to discrete systems of dimension greater than one. This result is summarized by "snap-back repealers imply chaos " [39] and was revisited by several authors, see for example [36], [34]. Note that a snap-back repealer is an expanding fixed point such that for a very small variations of the bifurcation parameter, the trajectory is repelled and for more larger deviations of this parameter, the process jumps onto the fixed point. As the determination of the snap-back repealer is difficult in general, our purpose is to avoid the corresponding approaches by considering specific choice of Poincaré sections. The fourth feature is the use of a simple and simultaneously powerful mathematical tool that is the Implicit Function theorem. It guaranties that the expected points for chaotifying the considered system defined on the Poincaré section are close to the bifurcation points and vary continuously with respect to the bifurcation parameter. This is primordial because on the one hand limitedness condition of the trajectories is respected (knowing that if it is not the case, study of chaos has no sense) and on the other hand, the processes of period doubling occurs until the dimension of the considered discrete map is reduced to one in the neighborhood of the bifurcation parameter permitting us to use the result "period three implies chaos".

The paper is structured as follows: in Section 2 some preliminaries and statements on the characterization of symmetric non-smooth transitions are provided followed by the determination of the corresponding Poincaré maps. A route to chaos analysis is proposed in Section 3. Section 4 is dedicated to some simulation results: the first one concerns an academic example subjected to symmetric sliding bifurcations and the second one concerns Chua circuit subjected to symmetric grazing bifurcations [20]. The results obtained for both examples highlight the efficiency of the proposed approach. Finally, concluding remarks and some perspectives end the paper.

2 Symmetric non-smooth transitions and Poincaré maps characterization :

We propose, in this section, a characterization of symmetric non-smooth transitions and after the determination of the associated Poincaré maps.

2.1 Characterization of p.w.s systems subjected to symmetric non smooth transitions

Let us consider the following piecewise smooth system:

$$\dot{x} = \begin{cases} F_1(x,\varepsilon) \text{ if } x \in D_1 \\ F_2(x,\varepsilon) \text{ if } x \in D_2 \end{cases} \tag{1}$$

where x[1] $: I \longrightarrow D$, $I \subset R^+$ and $D \supset D_1 \cup D_2$ is an open bounded domain of R^3 with:

$$D_1 = \{x \in D : |H(x)| < E\}$$

$$D_2 = \{x \in D : |H(x)| > E\}$$

E is a positive fixed real number and ε is a real parameter defined on a neighborhood of 0 noted by V_ε.
$H : D \to R$ is a continuous function that characterizes the phase space boundary between two regions of smooth dynamics, H defines the two symmetric transient sets:

$$\Pi_1 := \{x \in D : H(x) = E\} \text{ and } \Pi_2 := \{x \in D : H(x) = -E\}$$

Π_1 and Π_2 are termed the switching manifolds and divide respectively the phase space into the following regions:

$$\Pi_1^+ = \{x \in D : H(x(t)) \geq E\}, \ \Pi_1^- = \{x \in D : H(x) < E\}$$

and $\Pi_2^+ = \{x(t) \in D : H(x(t)) \geq -E\}$, $\Pi_2^- = \{x(t) \in D : H(x(t)) < -E\}$.

$F_1, F_2 : C^1(I,D) \times V_\varepsilon \longrightarrow C^m(I,D), \qquad m \geq 4$, where $C^m(I,D)$ is the set of C^k functions defined on I and having values in R^3, $C^m(I,D)$ is provided with the following norm:

$$\|x\| = \sup_{t\in I} \|x(t)\|_e + \sup_{t\in I} \|\dot{x}(t)\|_e + ... + \sup_{t\in I} \left\| x^{(m)}(t) \right\|_e, \qquad \forall x \in C^m(I,D)$$

[2].

According to [14], $(C^m(I,D), \|.\|)$ is a Banach space.

The vector fields F_1 and F_2 are defined on both sides of Π_k, $k = 1,2$.

Moreover, the system (1) is assumed to depend smoothly on the parameter ε such that at $\varepsilon = 0$, there exists a periodic orbit $x(.)$ that intersects the switching manifolds Π_1 and Π_2 at two points $\overline{x}_1$ and $\overline{x}_2$

[1] For the sake of simplicity, we denote by x the function and also the value of x at time t when the context is without ambiguity.

[2] $x^{(m)}(.)$ denotes the m^{th} derivative of $x(.)$ and $||.||_e$ is a norm defined on R^3.

corresponding to $\overline{t}$ (where $\overline{t}$ is the period time associated to the system (1)) .

The assumptions given by [11], [8], [10], [13] to characterize the sliding and grazing non-smooth bifurcations are generalized to the symmetric non-smooth cases in the following subsections[3], notations will be more complicated because all types of grazing and sliding bifurcations are considered here at the same time with the symmetry phenomena.

2.1.1 First case: symmetric sliding bifurcations:

A symmetric sliding bifurcations occur on two transient surfaces $\Pi_1$and Π_2 at two sliding points $\overline{x}_k$, $k = 1, 2$ at time t_0 (taken for simplifying to be equal to 0) if the following general sliding conditions are satisfied for each function $H_1 := H - E$ and $H_2 := H + E$:

$C_1^{k,s}$) $< \nabla H_k(x(t)), F_2(x(t), 0) - F_1(x(t), 0) >\in R_+^*$ for all $x(t) \in v_s^k$, where v_s^k is a bifurcation neighborhood in Π_k.
$C_2^{k,s}$) $H_k(\overline{x}_k) = 0$ and $\nabla H_k(\overline{x}_k) \neq 0$.
$C_3^{k,s}$) for $i = 1, 2$ and $k = 1, 2 : < \nabla H_k(\overline{x}_k), F_{ki}^0 >= 0$.
where $F_{ki}^0 := F_i(\Phi_i(\overline{x}_k, 0), 0), i = 1, 2,$and Φ_i is the flow associated to F_i.

Moreover, each type of the four symmetric sliding bifurcations is characterized by specific assumptions noted $A_i^{k,s}$), $i = 1, 2, 3, 4$ and $k = 1, 2$:
A$_1^{k,s}$) $\left\langle \nabla H_k(\bar{x}_k), \frac{\partial F_1(\bar{x}_k,0)}{\partial x} F_{k1}^0 \right\rangle > 0$
A$_2^{k,s}$) $\left\langle \nabla H_k(\bar{x}_k), \frac{\partial F_2(\bar{x}_k,0)}{\partial x} F_{k2}^0 \right\rangle > 0$
A$_3^{k,s}$) $\left\langle \nabla H_k(\bar{x}_k), \frac{\partial F_1(\bar{x}_k,0)}{\partial x} F_{k1}^0 \right\rangle < 0.$
A$_4^{k,s}$) $\left\langle \nabla H_k(\bar{x}_k), (\frac{\partial F_1(\bar{x}_k,0)}{\partial x})^2 F_{k1}^0 \right\rangle < 0.$

2.1.2 Second case: symmetric grazing bifurcations:

A symmetric grazing bifurcations occur on the two transient surfaces $\Pi_1$and Π_2 at two grazing points (denoted also for simplicity) $\overline{x}_k$, $k = 1, 2$ at time $t_0 = 0$ if the following general grazing conditions are satisfied on a bifurcation neighborhood v_s^k of Π_k. for each function $H_1 := H - E$ and $H_2 := H + E$:

$C_1^{k,g}$) $H_k(\overline{x}_k) = 0$ and $\nabla H_k(\overline{x}_k) \neq 0$.
$C_2^{k,g}$) for $i = 1, 2$ and $k = 1, 2 : < \nabla H_k(\overline{x}_k), F_{ki}^0 >= 0,$
$C_3^{k,g}$) for $i = 1, 2.$ and $k = 1, 2$: $\frac{\partial^2 H_k(\bar{x}_k,0)}{\partial x^2} \in R_+^*,$
$C_4^{k,g}$) $(< L_k, F_{k1}^0 >< L_k, F_{k2}^0 >) \in R_+^*$ for each $k = 1, 2$.
where L_k is the unit vector perpendicular to $\nabla H(\overline{x}_k)$ at point $\overline{x}_k$.

2.2 Determination of Poincaré maps associated to symmetric non smooth transitions:

As it is assumed that at $\varepsilon = 0$ then there exists a periodic orbit $x(.)$ that intersects symmetrically at two points the two symmetric manifolds Π_1

[3]In this paper, indexes s and g are relied respectively to sliding and grazing cases.

and Π_2.It is also requested that this orbit is hyperbolic and hence isolated. This implies that there is no points of sliding (respectively grazing) along the orbit other than $\overline{x}_k$, $k = 1, 2$. Those conditions are defined on an open set such that there exist sufficiently small neighborhoods V_ε of $\varepsilon = 0$ and $v_{\overline{x}_k}$ of $\overline{x}_k$ such that assumptions $C_j^{k,s}$, $j = 1, 2, 3$ associated to symmetric sliding bifurcations (respectively $C_j^{k,g}$, $j = 1, 2, 3$ associated to symmetric grazing) bifurcations are satisfied.

At this step, in order to compute the corresponding Poincaré maps, let's begin by choosing specially two symmetric Poincaré sections noted Λ_1 and Λ_2 to be perpendicular to Π_1 and Π_2 and consider the following diffeomorphism defined by:

$$S : R^2 \times S^1 \to R^2 \times S^1 \qquad (x_1, x_2, t) \to S(x_1, x_2, t) = (-x_1, -x_2, t + 2p\pi)$$

where S^1 is the unit circle and $p \epsilon Z$ (the set of relative numbers).

The Poincaré maps noted P^s (for non-symmetric sliding case) and P^g (for the non-symmetric grazing case) are given in details in [8] and [10].

The procedure for computing the Poincaré map being the same for the symmetric sliding and the symmetric grazing case, we directly deal with notation $P^{s,g}$,where following the cases, this map corresponds to the sliding or the grazing Poincaré one.

Now, let's consider $P_1^{s,g}$ the part of Poincaré map including sliding (respectively grazing) bifurcation on the transient surface Π_1 going from Λ_1 to Λ_2 and consider $P_2^{s,g}$ the other part of Poincaré map including sliding (respectively grazing) bifurcation on the transient surface Π_2 going from Λ_2 to Λ_1, then the global Poincaré map of the system subjected to symmetric sliding (respectively symmetric grazing) are given by :

$$P^{s,g} : \Lambda_1 \to \Lambda_2 \quad \text{such that} \quad P^{s,g} = P_2^{s,g} \circ P_1^{s,g}$$

However, due to the symmetry of the trajectory, maps $P_1^{s,g}$ and $P_2^{s,g}$ are relied by the following relation:

$$S \circ P_2^{s,g} \quad = \quad P_1^{s,g} \circ S$$

this implies that

$$P^{s,g} \quad = \quad S^{-1} \circ P_1^{s,g} \circ S \circ P_1^{s,g}$$

Taking this fact into account, the Poincaré maps have the following form:

$$P^{s,g}(x, \varepsilon) = \begin{cases} S^{-1} \circ P_1^{s,g} \circ S \circ P_1^{s,g}(x, \varepsilon) & \text{if} < \nabla H_1, x > \in R_+ \text{ or } < \nabla H_2, x > \in R_- \\ S^{-1} \circ P_2^{s,g} \circ S \circ P_2^{s,g}(x, \varepsilon) & \text{if} < \nabla H_1, x > \in R_-^* \text{ and } < \nabla H_2, x > \in R_+^* \end{cases} \tag{2}$$

In the next section, a rigorous approach of a route to chaos for p.w.s systems subjected to those symmetric non-smooth bifurcations is proposed.

3 Analysis of route to Chaos for p.w.s systems subjected to symmetric non smooth transitions:

A mathematical analysis to generate chaos for bounded piecewise smooth systems of dimension 3, subjected to symmetric sliding or the grazing bifurcations is now presented. This approach is based on the period doubling method applied to the corresponding Poincaré maps given by (2). Note that those Poincaré maps are discrete maps defined in dimension 2 and thus at this step, the result of Li and Yorke "Period three implies chaos" can't be used because period three does not imply necessarily chaos for continuous flows of dimension three (and so for their corresponding Poincaré maps that are discrete maps of dimension 2). In fact, determinism (non intersection of trajectories) and continuity requirement set constraints on how points of period doubling are defined on the corresponding Poincaré maps and move around the associated orbit. In other part, many simulation results show that period doubling can imply chaos for discrete systems of dimension greatest than one. This is possible for specific cases as when the multidimensional map is described in one direction by a particular map (as the saw-tooth one or the logistic one) while the others directions are characterized by strong contractions or if the processes of squeezing and stretching is chosen for particular systems defined in dimension three. Moreover, the processes corresponding to a pure rotation doesn't imply a chaotic attractor but those corresponding to braid implies chaos. In this work, a more general case of dynamic systems is considered and the trick proposed here is to reduce the dimension of the Poincaré map to one in the neighborhood of the transient points. This is possible by choosing a convenient Poincaré map section's that is transversal to the switching surface, this considered neighborhood of x is noted $v_x^{s,g}$.This main idea is supported by applying the Implicit Function theorem on $v_x^{s,g}$. It is a simple and a powerful mathematical tool allowing us to generate a "branch" of continuous solutions x with respect to the bifurcation parameter ε defined in some neighborhood of $\varepsilon = 0$ noted $v_{\varepsilon=0}^{s,g} \subset V_\varepsilon$. In this context, the dimension of the discrete map $P^{s,g}$ defined on $v_x^{s,g} \times v_{\varepsilon=0}^{s,g}$ is reduced to 1, without confusion and only for simplifying we note it also by $P^{s,g}$. Now, the famous result of Li and Yorke can be applied to $P^{s,g}$.

To propose the main result of this paper, we set the following assumptions:

$\mathrm{B}_1^{s,g}$) $\frac{\partial P^{s,g}}{\partial x}(0,0) - 1 \neq 0.$

$\mathrm{B}_2^{s,g}$) $-\frac{\partial P^{s,g}}{\partial x}(0,0)(\frac{\partial P^{s,g}}{\partial x}(0,0) - 1)^{-1} + (\frac{\partial P^{s,g}}{\partial x}(0,0) - 1)^{-1} - 1 \neq 0.$

$\mathrm{B}_3^{s,g}$) $\frac{\partial P^{s,g}}{\partial x}(\frac{\partial P^{s,g}}{\partial x}(0,0)-1)^{-1}(\frac{\partial P^{s,g}}{\partial x}(0,0)(\frac{\partial P^{s,g}}{\partial x}(0,0)-1))^{-1} - (\frac{\partial P^{s,g}}{\partial x}(0,0)-1)^{-1}+1) - (\frac{\partial P^{s,g}}{\partial x}(0,0)-1)^{-1}(\frac{\partial P^{s,g}}{\partial x}(0,0)\ (\frac{\partial P^{s,g}}{\partial x}(0,0)-1))^{-1} - (\frac{\partial P^{s,g}}{\partial x}(0,0)-1)^{-1} + 1) - 1 \neq 0$

1. Symmetric sliding case: Under conditions $C_j^{k,s}$) $j = 1,2,3$, $A_i^{k,s}$), $i = 1,2,3,4$, $k = 1,2$ and $B_i^{s,g}$, $i = 1,2,3$ the bounded p.w.s system

(1) admits a chaotic behavior associated to specific type of symmetric sliding transitions.

2. Symmetric grazing case: Under conditions $C_j^{k,g})$ $j = 1,2,3,4,$ $k = 1,2$ and $B_i^{s,g}$, $i = 1,2,3$ the bounded p.w.s system (1) admits a chaotic behavior associated to symmetric grazing transitions.

According to period doubling method, the problem is to determine three distinct points noted respectively by x, y and z that satisfy: $P^{s,g}(x,\varepsilon) = y$, $P^{s,g}(y,\varepsilon) = z$ and $P^{s,g}(z,\varepsilon) = x$.

So this procedure will be done in three steeps, each step corresponds to the determination of one of the 3 previous searched points:

First step of the period doubling procedure: it is traduced by the analysis of the following equation:

$$P^{s,g}(x,\varepsilon) = y \tag{3}$$

$$y := x + \eta \tag{4}$$

where η is a real parameter defined in the neighborhood of x.

The equation (3) is equivalent to the following one:

$$\Psi^{s,g}(x,\varepsilon,\eta) := P^{s,g}(x,\varepsilon) - x - \eta = 0 \tag{5}$$

Under assumption $\frac{\partial \Psi^{s,g}}{\partial x}(0,0,0) \neq 0$, (that is equivalent to assumption $\mathrm{B}_1^{s,g})$), and using the Implicit Functions Theorem, one obtains that $\exists$ a neighborhood of the parameter ε noted $\vartheta_{\varepsilon=0}^{s,g} \subset \upsilon_{\varepsilon=0}^{s,g}$ in R, a neighborhood of the parameter η noted $\upsilon_{\eta=0}^{s,g} \subset R$, a neighborhood of x noted $\upsilon_{x=0}^{s,g} \subset \upsilon_x^{s,g} \subset R$ and an unique application x^*: $\vartheta_{\varepsilon=0}^{s,g} \times \upsilon_{\eta=0}^{s,g} \longrightarrow \upsilon_{x=0}^{s,g}$ solution of $\Psi^{s,g}(x^*(\varepsilon,\eta),\varepsilon,\eta) = 0$ such that $x^*(0,0) = 0$. Furthermore, x^* depends continuously on ε and η.

Second step of the period doubling procedure: it is equivalent to the analysis of the following equation:

$$P^{s,g}(P^{s,g}(x,\varepsilon),\varepsilon) = z \tag{6}$$

$$\text{where } z := y + \mu \tag{7}$$

where μ stands for a real parameter defined in the neighborhood of x.

Taking into account results of the previous step, the equation (6) becomes equivalent to:

$$\Gamma^{s,g}(\varepsilon,\eta,\mu) := P^{s,g}(x^*(\varepsilon,\eta) + \eta, \varepsilon) - x^*(\varepsilon,\eta) - \eta - \mu = 0 \tag{8}$$

for $(\varepsilon,\eta,\mu) \epsilon \vartheta_{\varepsilon=0}^{s,g} \times \upsilon_{\eta=0}^{s,g} \times R$.

In order to continue the process with the same arguments (i.e. the Implicit function theorem applied to $\Gamma^{s,g}$), the following hypothesis is necessary:

$\frac{\partial \Gamma^{s,g}}{\partial \eta}(0,0,0) \neq 0$ that is written in details as:

$\frac{\partial P^{s,g}}{\partial x^*}(0,0)\frac{\partial x^*}{\partial \eta}(0,0) - \frac{\partial x^*}{\partial \eta}(0,0) - 1 \neq 0$

knowing that $\frac{\partial x^*}{\partial \eta}(0,0) = -(\frac{\partial P^{s,g}}{\partial x^*}(0,0) - 1)^{-1}$, this is exactly the stated assumption B$_2^{s,g}$) and thus, $\exists$ a neighborhood $\upsilon_{\varepsilon=0}^{s,g} \subset \vartheta_{\varepsilon=0}^{s,g}$, a neighborhood $\nu_{\eta=0}^{s,g} \subset \upsilon_{\eta=0}^{s,g}$, a neighborhood of μ noted $\nu_{\mu=0}^{s,g} \subset R$ and an unique application η^*:$\upsilon_{\varepsilon=0}^{s,g} \times \nu_{\mu=0}^{s,g} \longrightarrow \nu_{\eta=0}^{s,g}$ solution of $\Gamma^{s,g}(\varepsilon, \eta^*(\varepsilon,\mu), \mu) = 0$ such that $\eta^*(0,0) = 0$. Furthermore, η^* depends continuously on ε and μ.

Third step of the period doubling procedure: the last step of the period doubling is reduced to the analysis of the following equation:

$$P^{s,g}(P^{s,g}(P^{s,g}(x(\varepsilon,\eta),\varepsilon),\varepsilon),\varepsilon) = x \tag{9}$$

Taking into account the results obtained from the 2 previous steps, the analysis of this equation (9) becomes equivalent to the analysis of the following one:
for $(\varepsilon,\mu)\epsilon\upsilon_{\varepsilon=0}^{s,g} \times \nu_{\mu=0}^{s,g}$:

$$\Pi^{s,g}(\varepsilon,\mu) := P^{s,g}(x^*(\varepsilon,\eta^*(\varepsilon,\mu)) + \eta^*(\varepsilon,\mu) + \mu, \varepsilon) - x^*(\varepsilon,\eta^*(\varepsilon,\mu)) = 0 \tag{10}$$

In this case, the following hypothesis is required to apply the Implicit Function Theorem to $\Pi^{s,g}$:

$\frac{\partial \Pi^{s,g}}{\partial \mu}(0,0) \neq 0$ that is equivalent in details to:

$\frac{\partial P^{s,g}}{\partial x^*}\frac{\partial x^*}{\partial \eta}\frac{\partial \eta}{\partial \mu}(0,0) - \frac{\partial x^*}{\partial \eta}\frac{\partial \eta}{\partial \mu}(0,0) - 1 \neq 0$

and as $\frac{\partial \eta}{\partial \mu}(0,0) = -(\frac{\partial \Gamma^{s,g}}{\partial \eta}(0,0,0))^{-1}$, this is exactly the stated assumption B$_3^{s,g}$).
This permits us to affirm that: $\exists$ a neighborhood $\omega_{\varepsilon=0}^{s,g} \subset \upsilon_{\varepsilon=0}^{s,g}$, a neighborhood $\theta_{\mu=0}^{s,g} \subset \nu_{\mu=0}^{s,g}$ and an unique application μ^*: $\omega_{\varepsilon=0}^{s,g} \longrightarrow \theta_{\mu=0}^{s,g}$ solution of $\Pi^{s,g}(\varepsilon,\mu^*(\varepsilon)) = 0$ such that $\mu^*(0) = 0$. Furthermore, μ^* depends continuously on ε .

Thus the period doubling procedure applied to the Poincaré map (2), associated to p.w.s system (1) (reduced to a discrete map of dimension 1 on the neighborhood $\upsilon_x^{s,g} \times \upsilon_{\varepsilon=0}^{s,g}$) is constructed step by step and this system becomes chaotic according to the well-known result "period 3 implies chaos" applied to the discrete map $P^{s,g}$. □

4 Simulations results

4.1 Symmetric sliding case:

Let's consider an academic model subjected to symmetric sliding bifurcations given by:

$$\dot{x} = \begin{cases} F_1(x,\varepsilon) & for\ \ x \in D_1 \\ F_2(x,\varepsilon) & for\ \ x \in D_2 \end{cases} \tag{11}$$

where $D_1 := \left\{x\epsilon R^3 : x_3 - \frac{44}{3}x_1^3 - \frac{41}{2}x_1^2 - 5.3x_1 > 0\right\}$
and $D_2 := \left\{x\epsilon R^3 : x_3 - \frac{44}{3}x_1^3 - \frac{41}{2}x_1^2 - 5.3x_1 \leq 0\right\}$

$$
\begin{aligned}
F_1(x,\varepsilon) &= \begin{pmatrix} 100 \\ -x_3 \\ -0.7x_1 + x_2 + 0.24x_3 - (\varepsilon x_3)^3 \end{pmatrix} \\
F_2(x,\varepsilon) &= \begin{pmatrix} -100 \\ -x_3 \\ -0.7x_1 + x_2 + 0.24x_3 - (\varepsilon x_3)^3 \end{pmatrix}
\end{aligned}
$$

ε is the bifurcation parameter defined near 0.

Applying the procedure presented in section 2 in order to compute the Poincaré map associated to (11) and the method of chaotification given in section 3, we obtain the following results:

• For $\varepsilon = 0.4$, there is a limit cycle between the two sides Π_1 and Π_2, see fig 1.

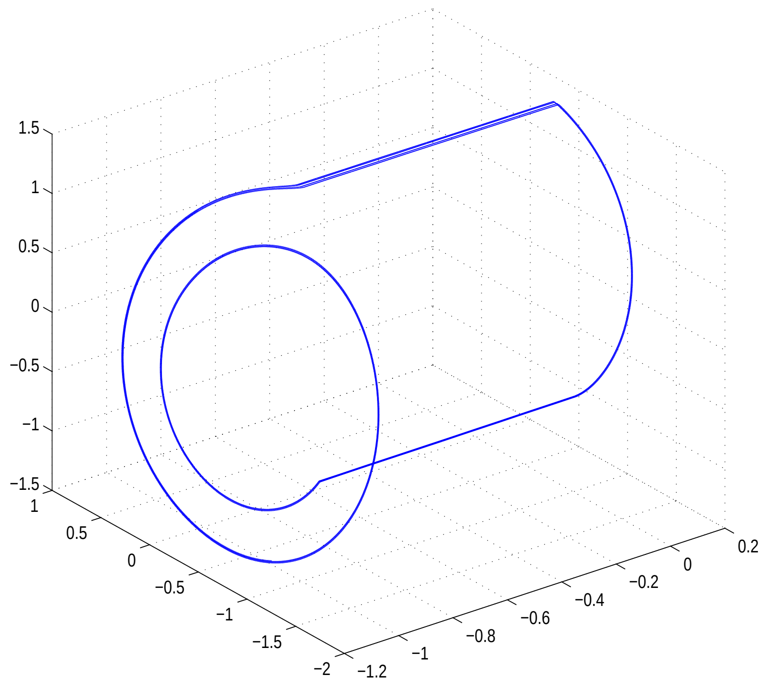

Figure 1: Symmetric sliding case: Limit Cycle for $\varepsilon = 0.4$

• For $\varepsilon = 0.2$, a symmetric sliding period doubling appears, see fig 2.

• For $\varepsilon = -0.05$, a symmetric sliding multi period doubling appear, see fig 3.

• For $\varepsilon = -0.23$, a chaotic behaviors appears, see fig 4.

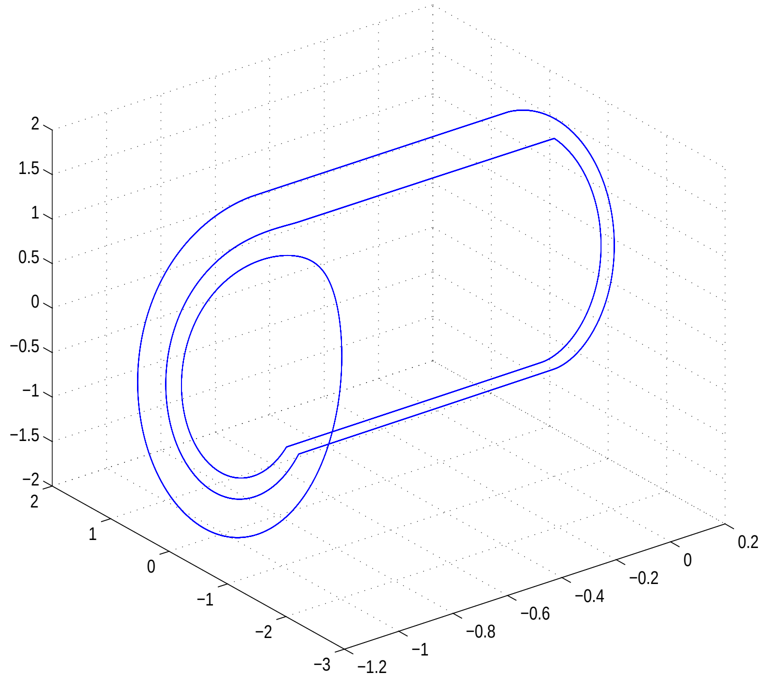

Figure 2: Symmetric sliding case: Period doubling for $\varepsilon = 0.2$

4.2 Symmetric grazing case (Chua Circuit):

Let's consider the Chua model subjected to symmetric grazing bifurcations given by:

$$\left\{ \begin{array}{lcl} \dot{x}_1 & = & \frac{-1}{C_1 R}(x_1 - x_2) + \frac{f(x_1,\varepsilon)}{C_1} \\ \dot{x}_2 & = & \frac{1}{C_2 R}(x_1 - x_2) + \frac{x_3}{C_2} \\ \dot{x}_3 & = & \frac{-x_2}{L} \end{array} \right. \tag{12}$$

with $f(x_1, \varepsilon) = G_b x_1 + 0.5(G_a(1+\varepsilon) - G_b)(|x_1 + E| - |x_1 - E|$, $R = 2.115K\Omega$, $E = 5.75V$, $C_1 = 10nF$, $C_2 = 100nF$, $G_a(\varepsilon) = \frac{1+\varepsilon}{0.999R}$, $\mathrm{G}_b = \frac{1}{2R}$ and the following initial conditions $(E + 0.3V, 0, -\frac{E}{R})$.
The system (12) can be rewritten according to the general form of systems considered in this paper as:

$$\dot{x} = \left\{ \begin{array}{ll} F_1(x, \varepsilon) & for \;\; x \in D_1 \\ F_2(x, \varepsilon) & for \;\; x \in D_2 \end{array} \right.$$

with $D_1 = \{x \epsilon R^3 : -E \leq x_1 \leq E\}$
$D_2 = \{x \epsilon R^3 : x_1 > E \text{ or } x_1 < -E\}$

$$F_1(x, \varepsilon) = \left(\begin{array}{c} [\alpha_1 + \frac{1}{C_1} G_a(1+\varepsilon)]x_1 - \alpha_1 x_2 \\ \alpha_2 x_1 - \alpha_2 x_2 + \frac{x_3}{C_2} \\ \alpha_3 x_2 \end{array} \right)$$

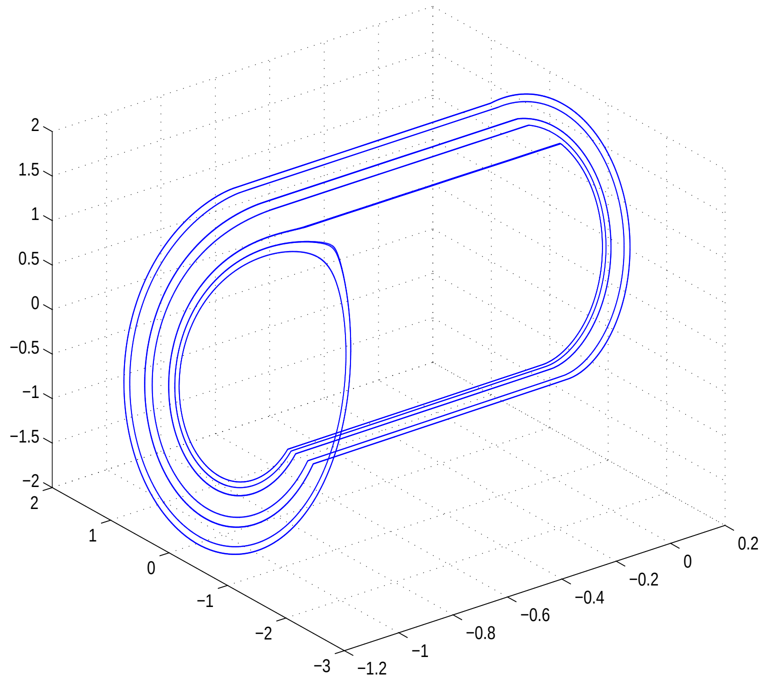

Figure 3: Symmetric sliding case: multi period doubling for $\varepsilon = -0.05$

$$F_2(x,\varepsilon) = \begin{cases} F_{2,E}(x,\varepsilon) & \text{for} if\ x_1 > E \\ F_{2,-E}(x,\varepsilon) & \text{for} if\ x_1 < -E \end{cases}$$

where:

$$F_{2,E}(x,\varepsilon) = \begin{pmatrix} [\alpha_1 + \frac{1}{C_1}G_b]x_1 - \alpha_1 x_2 + \frac{1}{C_1}[G_a(1+\varepsilon)G_b]E \\ \alpha_2 x_1 - \alpha_2 x_2 + \frac{x_3}{C_2} \\ \alpha_3 x_2 \end{pmatrix}$$

and by symmetry:

$$F_{2,-E}(x,\varepsilon) = \begin{pmatrix} [\alpha_1 + \frac{1}{C_1}G_b]x_1 - \alpha_1 x_2 + \frac{1}{C_1}[G_a(1+\varepsilon)G_b](-E) \\ \alpha_2 x_1 - \alpha_2 x_2 + \frac{x_3}{C_2} \\ \alpha_3 x_2 \end{pmatrix}$$

where $\alpha_1 = \frac{-1}{C_1 R}$, $\alpha_2 = \frac{1}{C_2 R}$ and $\alpha_3 = \frac{-1}{L}$, ε is the parameter bifurcation. So applying the method presented in section 2 as for the first example, one determines the Poincaré map associated to this system when a symmetric grazing occurs. The procedure of chaotification given in section 3 and applied to this Poincaré map gives us the following results:

• <u>For $\varepsilon = 0.1$</u> (this corresponds to the initial value of G_a), there is a limit cycle between the two sides Π_1 and Π_2, see fig 5.

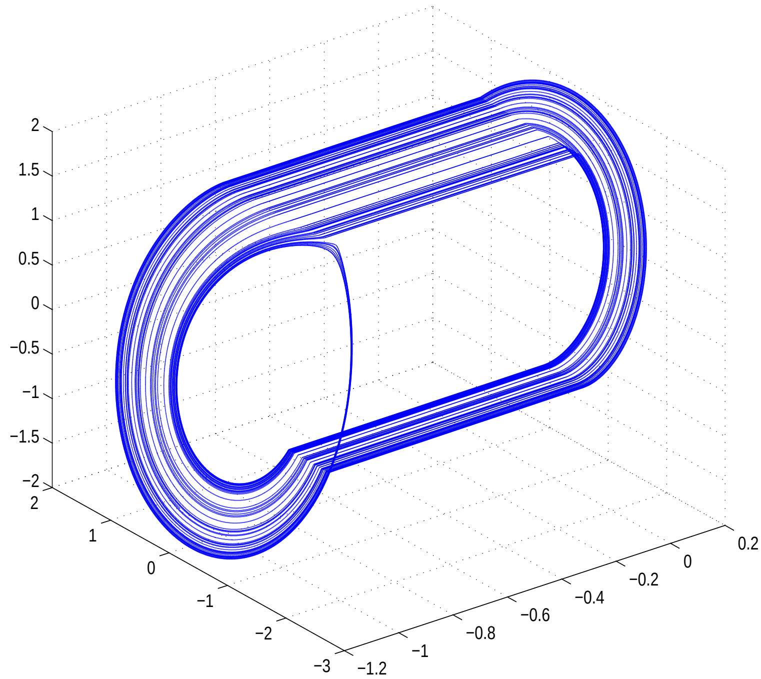

Figure 4: Symmetric sliding case: A chaotic behavior for $\varepsilon = -0.23$

- For $\varepsilon = 0.2$, a period doubling appears, see fig 6.
- For $\varepsilon = 0.3$, a Rössler behavior appears, see fig 7.
- For $\varepsilon = 0.4$, a double scroll behavior appears, see fig 8.

5 Conclusion:

In this letter, we have proposed a mathematical approach of route to chaos for bounded p.w.s systems of dimension three subjected to symmetric grazing or sliding bifurcations. This approach highlights the fact that it is possible to extend the procedure given in [4, 5] to the interesting case of symmetric non-smooth bifurcations. Moreover, simulations results show that it is less complicated to deal with symmetric non-smooth than non-symmetric non-smooth transitions. Simulations results were proposed for academic example subjected to symmetric sliding bifurcations and an application of this approach is also done for the well-known Chua Circuit where two grazing bifurcations associated to two symmetric transient surfaces appear simultaneously and symmetrically. Many possible perspectives can be investigated as to generalize those results to other forms of non-smooth transitions as corner one or to deal with multimodal p.w.s

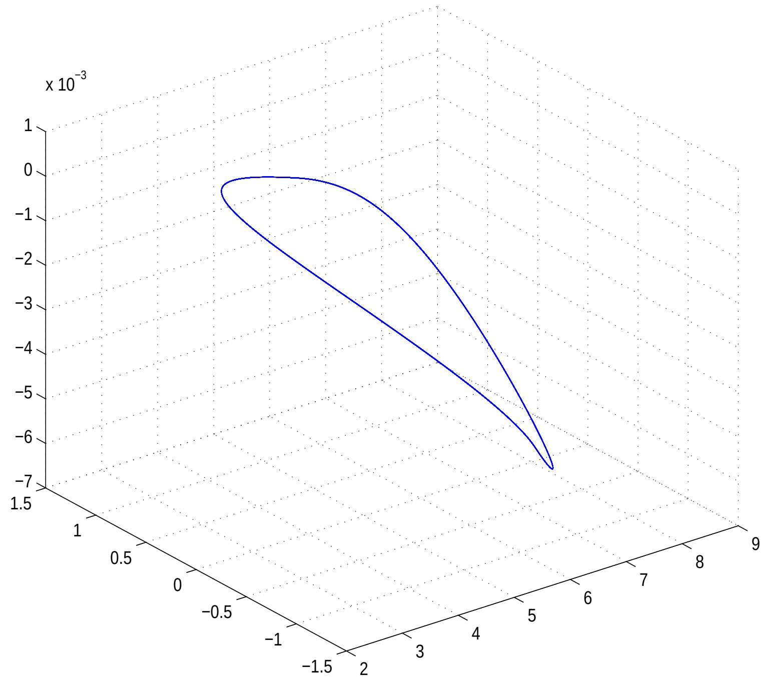

Figure 5: Symmetric grazing case (Chua Circuit): limit Cycle for $\varepsilon = 0.1$

systems....

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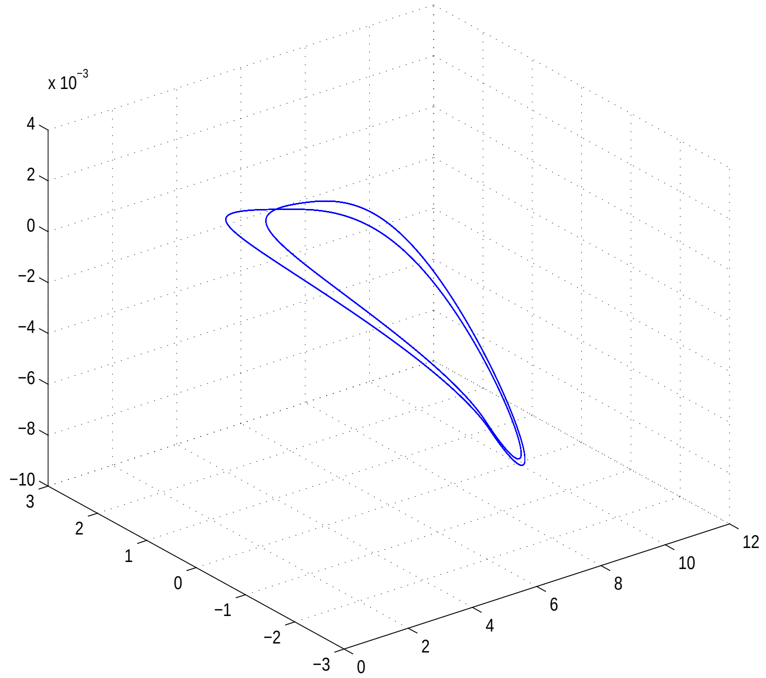

Figure 6: Symmetric grazing case (Chua Circuit): Period doubling for $\varepsilon = 0.2$

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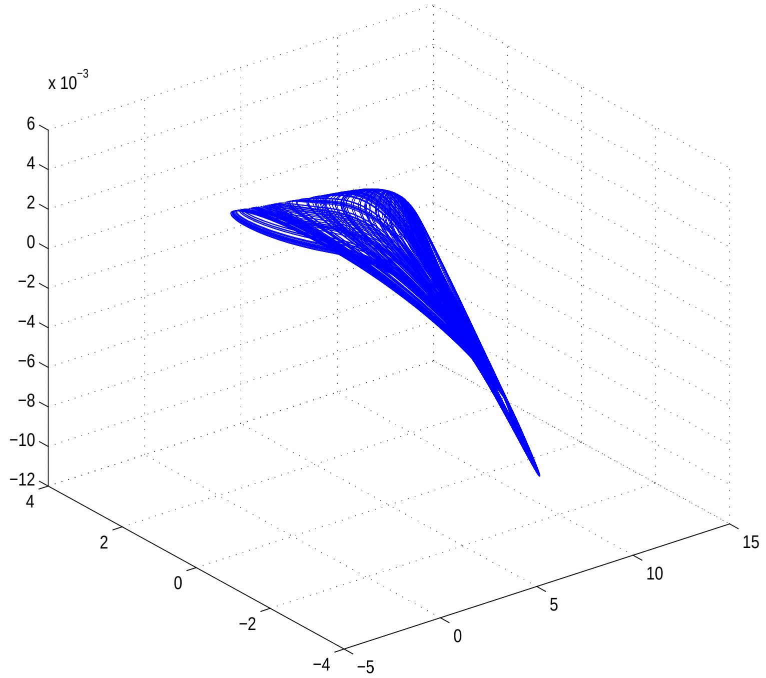

Figure 7: Symmetric grazing case (Chua Circuit): Rössler attractor for $\varepsilon = 0.3$

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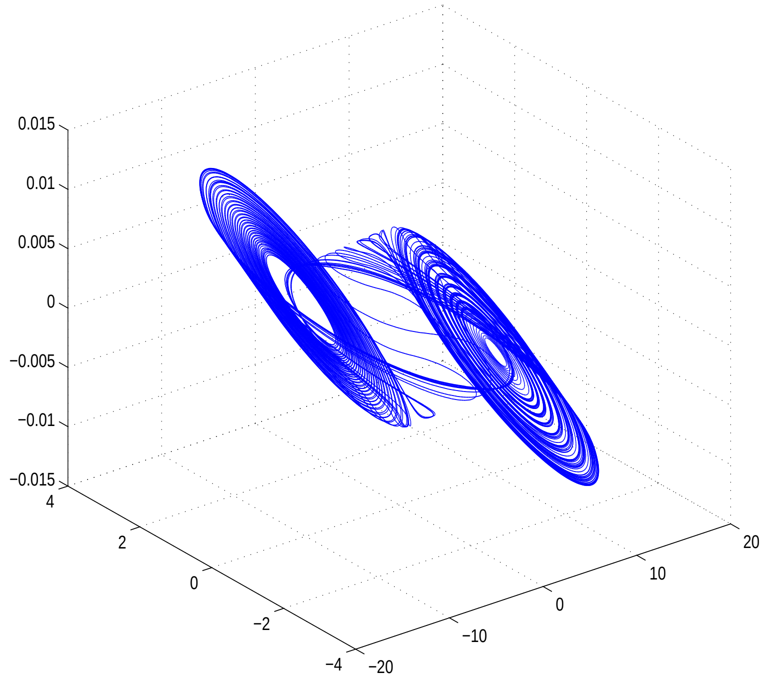

Figure 8: Symmetric grazing case (Chua Circuit): Double scroll attractor for $\varepsilon = 0.4$

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