On the trajectory of an individual chosen according to supercritical Gibbs measure in the branching random walk

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Abstract

Consider a branching random walk on the real line. In a recent article, Chen [9] proved that the renormalised trajectory leading to the leftmost individual at time n converges in law to a standard Brownian excursion. Besides, in [22], Madaule showed the renormalised trajectory of an individual selected according to the critical Gibbs measure converges in law to a Brownian meander. In this article, we prove that trajectory of individuals selected independently according to a supercritical Gibbs measure converge in law to Brownian excursions. Refinements of this results also enables to express the probability for the trajectory of two individuals selected according to the Gibbs measure to have split before time t, partially answering a question of [10].

1 Introduction

A branching random walk on the real line is a point process on $\mathbb{R}$ defined as follow : It starts with a unique individual sitting at the origin, forming the 0^{th} generation of the process. At time 1, this individual dies and gives birth to children, which are positioned on $\mathbb{R}$ according to a point process law $\mathcal{L}$. These children form the 1^{th} generation. Similarly, at each time $n \in \mathbb{N}$, every individual z of the $(n-1)^{\text{th}}$ generation dies, giving birth to children positioned according to an independent copy of $\mathcal{L}$ translated from the position of z.

We denote by $\mathbb{T}$ the genealogical tree of the process. Obviously $\mathbb{T}$ is a Galton-Watson tree. For any individual $z \in \mathbb{T}$, we write $|z|$ for the generation at which z belongs and $V(z) \in \mathbb{R}$ for the position of z. With these notations, $(V(z), |z| = 1)$ has the law of $\mathcal{L}$. The collection of positions $(V(z), z \in \mathbb{T})$, together with the genealogical information, defines the branching random walk.

If $z \in \mathbb{T}$, for all $k \leq |z|$, we denote by z_k the ancestor of z alive at generation k.

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Throughout this paper, we suppose that the point process law $\mathcal{L}$ verifies some integrability conditions. We assume that the Galton-Watson tree is supercritical, in other words,

$$\mathbf{E}\left[\sum_{|z|=1} 1\right] > 1. \tag{1.1}$$

Note that we do not assume that $\mathbf{P}\left(\sum_{|x|=1} 1 = +\infty\right) = 0$. Under this assumption, the survival set

$$S := \{\forall n \in \mathbb{N}, \exists u \in \mathbb{T}, |u| \geq n\} \tag{1.2}$$

is of positive probability.

Assume also that we are in the so-called "boundary case" defined in [3], i.e.

$$\mathbf{E}\left[\sum_{|z|=1} e^{-V(z)}\right] = 1 \quad \text{and} \quad \mathbf{E}\left[\sum_{|z|=1} V(z)e^{-V(z)}\right] = 0. \tag{1.3}$$

Under mild assumptions, a branching random walk can be reduced to this case by an affine transformation –see Appendix A in [17] for a detailed discussion. Furthermore, we assume that

$$\mathbf{E}\left[X(\log_+ X)^2\right] + \mathbf{E}\left[\widetilde{X}\log_+ \widetilde{X}\right] + \underbrace{\mathbf{E}\left[\sum_{|z|=1} V(z)^2 e^{-V(z)}\right]}_{=:\sigma^2} < +\infty \tag{1.4}$$

where

$$X := \sum_{|z|=1} e^{-V(z)} \quad \text{and} \quad \widetilde{X} := \sum_{|z|=1} V(z)_+ e^{-V(z)}. \tag{1.5}$$

It follows from (1.3) that

$$W_{n,1} = \sum_{|z|=n} e^{-V(z)} \quad \text{and} \quad Z_n = \sum_{|z|=n} V(z)e^{-V(z)}$$

are martingales. Chen [8] proved that (1.4) is a necessary and sufficient condition to obtain the almost sure convergence of (Z_n) to a non-negative random variable Z_∞. Moreover, $S = \{Z_\infty > 0\}$ a.s.

Let $\beta > 1$, we write $W_{n,\beta} = \sum_{|z|=n} e^{-\beta V(z)}$. It is proved in [14] that $W_{n,\beta}$ is of order $n^{-3\beta/2}$ in probability. In a recent article [21], Madaule proved the convergence in law of finite-dimensional distributions of

$$\left(n^{3\beta/2} W_{n,\beta}, \quad \beta > 1\right),$$

yielding the following fact: The point process $(V(z) - \frac{3}{2}\log n + \log Z_\infty; |z| = n)$ converges in law to a Poisson point process with intensity e^x, decorated by another point process D.

In this article, we consider a probability measure on the n^{th} generation of the branching random walk, defined on the set $\{W_{n,\beta} > 0\}$ by

$$\nu_{n,\beta} = \frac{1}{W_{n,\beta}} \sum_{|z|=n} e^{-\beta V(z)} \delta_z,$$

which is called the Gibbs measure in the literature. We prove that conditionally on the survival S, the trajectory followed by an individual chosen according to $\nu_{n,\beta}$ converges, when suitably rescaled, to a Brownian excursion. For a given individual $z \in \mathbb{T}$ such that $|z| \leq n$, we define

$$H^{(n)}(z) := \left(\frac{V(z_{\lfloor tn \rfloor})}{\sqrt{\sigma^2 n}}, 0 \leq t \leq \frac{|z|}{n} \right),$$

the Brownian normalisation of the trajectory followed by individual z up to time $|z|$. This is an element of $\mathcal{D}([0, \frac{|z|}{n}])$: the set of càdlàg functions –left-continuous functions with right limits at each point– equipped with the Skorokhod topology. For all $\beta > 1$ and $n \in \mathbb{N}$, on $\{W_{n,\beta} > 0\}$, we define the image measure of $\nu_{n,\beta}$ by $H^{(n)}(\cdot)$ by $\mu_{n,\beta}$ on $\mathcal{D}$ as, i.e.,

$$\mu_{n,\beta} = \frac{1}{W_{n,\beta}} \sum_{|u|=n} e^{-\beta V(u)} \delta_{H^n(u)}.$$

Let $\mathcal{C}_b(\mathcal{D})$ be the set of continuous bounded functions from $\mathcal{D}$ to $\mathbb{R}$, and $\mathcal{C}_b^u(\mathcal{D}) \subset \mathcal{C}_b(\mathcal{D})$ the collection of uniformly continuous functions. The following theorem gives the weak limit, as n goes to infinity, of $\mu_{n,\beta}(F)$ for $F \in \mathcal{C}_b^u(\mathcal{D})$.

Theorem 1.1. *For all $\beta > 1$ and $F \in \mathcal{C}_b^u(\mathcal{D})$, conditionally on the survival of the branching random walk, the following convergence in law holds:*

$$\mu_{n,\beta}(F) \underset{n \to +\infty}{\Longrightarrow} \sum_{k \in \mathbb{N}} p_k F(e_k),$$

where (e_k) is a sequence of i.i.d. normalised Brownian excursions, and $(p_k, k \in \mathbb{N})$ follows an independent Poisson-Dirichlet distribution with parameter $(\frac{1}{\beta}, 0)$.

Remark 1.2. *A direct consequence of this theorem and Theorem 5.2 in [4] is that the –annealed– measure $\mathbf{E}(\mu_{n,\beta}|S)$ converges weakly to the law of a normalised Brownian excursion.*

The case $\beta = 1$ has been investigated in [22]. It is proved that

$$\lim_{n \to +\infty} \mu_{n,1}(F) = F(\mathcal{M}) \text{ in probablity},$$

where $\mathcal{M}$ is a Brownian meander ; in the case $\beta = +\infty$, the measure $\mu_{n,\infty}$ is the uniform measure on the trajectories leading to the leftmost position at time n, which has been treated in [9]. It is believed that for $\beta < 1$, the trajectory of a particle chosen according to $\nu_{n,\beta}$ behaves as a random walk with positive drift.

Using the techniques leading to Theorem 1.1, we obtain information on the genealogy of two individuals sampled according to the Gibbs measure $\nu_{n,\beta}$. For $z, z' \in \mathbb{T}$, we set $\mathrm{MRCA}(z, z')$ to be the generation at which the most recent common ancestor of z and z' was alive, in other words,

$$\mathrm{MRCA}(z, z') = \max\{k \leq \min(|z|, |z'|) : z_k = z'_k\}.$$

Derrida and Spohn conjectured in [10] that for any $\beta > 1$

$$\nu_{n,\beta}^{\otimes 2}\left(\frac{\mathrm{MRCA}(z, z')}{n} \in dx\right) \underset{n\to+\infty}{\Longrightarrow} p_\beta \delta_1 + (1 - p_\beta)\delta_0,$$

where p_β is a random variable such that $\lim_{\beta\to+\infty} p_\beta = 1$ and $\lim_{\beta\to 1} p_\beta = 0$ in probability. The following immediate consequence of Theorem 1.1 strengthen this conjecture, as a similar behaviour holds for the probability that two trajectories split before time t.

Corollary 1.3. *For any $\beta > 1$ and $t \in (0, 1)$, conditionally on the survival of the branching random walk, we have*

$$\nu_{n,\beta}^{\otimes 2}\left(V(z_{\lfloor nt\rfloor}) \neq V(z'_{\lfloor nt\rfloor})\right) \underset{n\to+\infty}{\Longrightarrow} 1 - \pi_\beta. \tag{1.6}$$

where $(p_k, k \geq 1)$ has Poisson-Dirichlet distribution with parameter $(\frac{1}{\beta}, 0)$ and $\pi_\beta = \sum_{k\in\mathbb{N}} p_k^2$.

This corollary can be seen as an explicit computation of the well-known fact that in a branching random walk, two individuals within distance $O(1)$ from the leftmost position are either close relatives, or their ancestral lineages split early in the process. In the context of Gibbs measure, the probability of an early splitting is π_β.

In particular, we note that $\mathbf{E}(\pi_\beta) = 1 - \frac{1}{\beta}$, which is consistent with the probability of overlapping in a Gaussian free field, obtained by Arguin and Zindy in [2]. Moreover, we observe that when $\beta \to \infty$, $1 - \pi_\beta$ goes to zero. This is consistent with the fact that $\mu_{n,\infty}$ only puts mass on particles which are at the leftmost position at time n, which are eventually close relatives. Similarly, when $\beta \to 1$, $1 - \pi_\beta$ goes to 1 ; therefore the corresponding paths are asymptotically independent, which coincides with the weak convergence (3.3) in [22].

To prove Theorem 1.1 as well as Corollary 1.3, the main idea is to understand the tail decay of the random variable

$$\widetilde{\mu}_{n,\beta}(F) := n^{3\beta/2} W_{n,\beta} \times \mu_{n,\beta}(F) = n^{3\beta/2} \sum_{|u|=n} e^{-\beta V(u)} F(H_n(u))$$

by borrowing ideas from Chen [9] and Madaule [21]. This tail decay is used to obtain the limit Laplace transform of $\widetilde{\mu}_{n,\beta}(F)$, see Proposition 4.1 for details.

We introduce in Section 2 a general method linking additive moments of the branching random walk with the law of a random walk. This method is applied in Section 3 to obtain a tight estimate on the tail decay of $\widetilde{\mu}_{n,\beta}(F)$. Finally the proofs of Theorem 1.1 and Corollary 1.3 are given in Section 4.

2 Many-to-one lemma and random walk estimates

We introduce in a first time the Lyon's change of measure of the branching random walk and the spinal decomposition. This result enables to compute additive moments of the branching random walk using random walk estimates. In a second part, we consider a random walk, conditioned to stay above 0 until time n and ending at time n at distance $o(\sqrt{n})$. We prove that the endpoint is asymptotically independent with the rescaled shape of this random walk, which converges to a Brownian excursion.

2.1 Lyon's change of measures and the many-to-one lemma

For any $a \in \mathbb{R}$, let $\mathbf{P}_a$ be the probability measure of the branching random walk started from a, and let $\mathbf{E}_a$ be the corresponding expectation. We recall that $(W_{n,1}, n \in \mathbb{N})$ is a non-negative martingale with respect to the natural filtration $\mathcal{F}_n = \sigma(x, V(x), |x| \leq n)$. We define a new probability measure $\bar{\mathbf{P}}_a$ on $\mathcal{F}_\infty$ such that for all $n \in \mathbb{N}$,

$$\left.\frac{d\bar{\mathbf{P}}_a}{d\mathbf{P}_a}\right|_{\mathcal{F}_n} = e^a W_{n,1}. \tag{2.1}$$

The so-called spinal decomposition, introduced by Russell Lyons in [20] gives an alternative construction of the measure $\bar{\mathbf{P}}_a$, by introduction of a special individual, the "spine", which reproduction is modified.

We introduce another point process law $\widehat{\mathcal{L}}$ with Radon-Nikodým derivative $\sum_\ell e^{-\ell}$ with respect to the law of $\mathcal{L}$. The branching random walk with spine starts with one individual located at a at time 0, denoted by ω_0. It generates its children according to the law $\widehat{\mathcal{L}}$. Individual ω_1 is chosen among the children z of ω_0 with probability proportional to $e^{-V(z)}$. Then, for all $n \geq 1$, individuals at the n^{th} generation die, giving birth to children independently according to the law $\mathcal{L}$, except for the individual ω_n which uses the law $\widehat{\mathcal{L}}$. The individual ω_{n+1} is chosen at random among the children z of ω_n, proportionally to $e^{-V(z)}$. We denote by $\mathbb{T}$ the genealogical tree of this process, and by $\widehat{\mathbf{P}}_a$ the law of $(V(x), x \in \mathbb{T}, (\omega_n, n \geq 0))$ as we just defined.

Proposition 2.1. *For any $n \in \mathbb{N}$, we have $\widehat{\mathbf{P}}_a\big|_{\mathcal{F}_n} = \bar{\mathbf{P}}_a\big|_{\mathcal{F}_n}$. Moreover, for any $z \in \mathbb{T}$ such that $|z| = n$, we have $\widehat{\mathbf{P}}_a(\omega_n = z \,|\mathcal{F}_n) = \frac{e^{-V(z)}}{W_{n,1}}$, and $(V(\omega_n), n \geq 0)$ is a centred random walk under $\widehat{\mathbf{P}}_a$, starting from a, and with variance σ^2.*

In particular, this proposition implies the many-to-one lemma, which has been introduced for the first time by Peyrière in [23], and links additive moments of the branching random walks with random walk estimates.

Lemma 2.2. *There exists a centred random walk $(S_n, n \geq 0)$, starting from a under $\mathbf{P}_a$, with variance σ^2 such that for all $n \geq 1$ and $g : \mathbb{R}^n \to \mathbb{R}_+$ measurable, we have*

$$\mathbf{E}_a\left[\sum_{|z|=n} g(V(z_1), \cdots V(z_n))\right] = \mathbf{E}_a\left[e^{S_n - a} g(S_1, \cdots S_n)\right] \tag{2.2}$$

Proof. We use Proposition 2.1 to compute

$$\begin{aligned}\mathbf{E}_a\left[\sum_{|z|=n} g(V(z_1),\cdots V(z_n))\right] &= \bar{\mathbf{E}}_a\left[\frac{e^{-a}}{W_{n,1}}\sum_{|z|=n} g(V(z_1),\cdots V(z_n))\right]\\ &= e^{-a}\widehat{\mathbf{E}}_a\left[\sum_{|z|=n}\mathbf{1}_{\{z=\omega_n\}}e^{V(z)}g(V(z_1),\cdots V(z_n))\right]\\ &= \widehat{\mathbf{E}}_a\left[e^{V(\omega_n)-a}g(V(\omega_1),\cdots,V(\omega_n))\right].\end{aligned}$$

Therefore we define the random walk S under $\mathbf{P}_a$ to have the same law as $(V(\omega_n), n\geq 0)$ under $\widehat{\mathbf{P}}_a$, which ends the proof. □

2.2 Approximation of a random walk excursion

In this subsection, $(S_n, n\geq 0)$ is a centred random walk on $\mathbb{R}$ with finite variance σ^2. We write, for $0\leq m\leq n$, $\underline{S}_{[m,n]} = \min_{m\leq k\leq n} S_k$ and $\underline{S}_n = \underline{S}_{[0,n]}$ the minimal position of the random walk until time n. We introduce in a first time a piece of notation, before computing the probability for a random walk to make an excursion of length n above 0.

2.2.1 Some random walk notation and preliminary results

The ballot theorem We present the following estimates, which bounds the probability for a random walk to make an excursion of length n above a given level. Let $\lambda\in(0,1)$. There exists a constant $c_1>0$ such that for any $b\geq a\geq 0$, $x,y\geq 0$ and $n\geq 1$, we have

$$\mathbf{P}_x\Big(S_n\in[y+a,y+b],\underline{S}_n\geq 0,\underline{S}_{[\lambda n,n]}\geq y\Big)\leq c_1(1+x)(1+b-a)(1+b)n^{-3/2}. \tag{2.3}$$

Ladder epochs and height processes We denote by $(\tau_k^+, k\geq 0)$ and $(H_k^+, k\geq 0)$ the strictly ascending ladder epochs and the height process, writing $\tau_0^+=0, H_0^+=0$ and, for $k\geq 1$,

$$\tau_k^+ = \inf\{n>\tau_{k-1}^+ : S_n > H_{k-1}^+\} \quad\text{and}\quad H_k^+ = S_{\tau_k^+}. \tag{2.4}$$

Note that $(\tau_k^+, k\geq 0)$ and $(H_k^+, k\geq 0)$ are renewal processes, i.e., random walks with i.i.d. non-negative increments. Similarly, we write τ^- and H^- the strictly ascending ladder epoch and the height process associated to $-S$. It is given in Theorem A of [19] that there exist two constants $C_\pm>0$ such that

$$\mathbf{P}(\tau_1^\pm>n) = \mathbf{P}(\min_{k\leq n}(\pm S_k)\geq 0) = \frac{C_\pm}{\sqrt{n}} + o(n^{-1/2}). \tag{2.5}$$

Renewal function We write $V^-(\cdot)$ (respectively $V^+(\cdot)$) the renewal function associated to $(H_k^-, k \geq 0)$ (resp. $(H_k^+, k \geq 0)$), defined by

$$\forall x \geq 0, \quad V^-(x) = \sum_{k \geq 0} \mathbf{P}\left(H_k^- \leq x\right). \tag{2.6}$$

Observe that V^- is a non-decreasing, right-continuous function with $V^-(0) = 1$. We can rewrite V^- in the following way

$$V^-(x) = \sum_{k \geq 0} \mathbf{P}\left(-x \leq S_k < \underline{S}_{k-1}\right) \tag{2.7}$$

As a consequence of the Renewal Theorem in [13] (p. 360), there exist two constants $c_\pm > 0$ such that

$$V^\pm(x) \sim_{x \to +\infty} c_\pm x. \tag{2.8}$$

Local measure of the random walk staying non-negative We introduce, for $n \geq 1$, the measure

$$\pi_n^+(x, dy) := \mathbf{P}_x\Big(\underline{S}_n \geq 0, S_n \in dy\Big), \tag{2.9}$$

Let $K > 0$, it has been proved by Doney [12] that uniformly in $x = o(\sqrt{n})$ and $y = o(\sqrt{n})$,

$$\pi_n^+(x, [y, y+K]) = \frac{1}{\sigma\sqrt{2\pi}n^{3/2}} V^-(x) \int_{[y,y+K]} V^+(z)dz \left(1 + o_n(1)\right). \tag{2.10}$$

and that uniformly in $x = o(\sqrt{n})$ and $y \in [0, \infty)$,

$$\pi_n^+(x, [y, y+K]) = \frac{C_- y}{\sigma^2 n^{3/2}} e^{-\frac{y^2}{2n\sigma^2}} K V^-(x) + o(n^{-1}). \tag{2.11}$$

Obviously, similar estimates holds for π^- the measure associated to $-S$.

Random walk conditioned to stay non-negative We observe the renewal function V^- is invariant for the semigroup of the random walk killed when first enters the negative half-line $(-\infty, 0)$, i.e.

$$\forall x \geq 0, \ \forall N \in \mathbb{N}, \ V^-(x) = \mathbf{E}\left[V^-(x + S_N)\mathbf{1}_{\left\{\underline{S}_N \geq -x\right\}}\right] \tag{2.12}$$

This estimate can be found in [19].

Using (2.12), for all $x \geq 0$, we define the probability measure $\mathbf{P}_x^\uparrow$ by

$$\mathbf{P}_x^\uparrow(B) := \frac{1}{V^-(x)} \mathbf{E}_x\left(1_B V^-(S_N); \underline{S}_N \geq 0\right). \tag{2.13}$$

for $N \geq 1$ and $B \in \sigma(S_0, \dots S_N)$. We call $\mathbf{P}_x^{\uparrow}$ the law of the random walk conditioned to stay positive. For any positive sequence (x_n) such that $\frac{x_n}{\sqrt{\sigma^2 n}} \to x \geq 0$, we have the following invariance principle, Theorem 1.1 of [6]

$$\forall F \in \mathcal{C}_b(\mathcal{D}), \quad \mathbf{E}_{x_n}^{\uparrow}\Big(F(\frac{S_{\lfloor nt \rfloor}}{\sqrt{\sigma^2 n}}; t \in [0,1])\Big) \xrightarrow[n\to\infty]{} \mathbf{E}_x\Big(F(R(t), t \in [0,1])\Big), \tag{2.14}$$

where $R = (R(t); t \geq 0)$ is a three-dimensional Bessel process.

We also state another functional central limit theorem related to (2.14), which has been proved by Iglehart [15], Bolthausen [5] and Doney [11].

$$\forall F \in \mathcal{C}_b(\mathcal{D}), \quad \mathbf{E}\Big(F(\frac{S_{\lfloor nt \rfloor}}{\sqrt{\sigma^2 n}}; t \in [0,1]) \Big| \underline{S}_n \geq 0\Big) \xrightarrow[n\to\infty]{} \mathbf{E}\Big(F(\mathcal{M})\Big), \tag{2.15}$$

where $\mathcal{M} = (\mathcal{M}(t); t \in [0,1])$ is a Brownian meander process. The following equality from Imhof [16] reveals the relation between these two limit processes. For any $t \in (0,1]$,

$$\forall \Phi \in \mathcal{C}_b(\mathcal{D}[0,t]), \quad \mathbf{E}\left[\Phi(\mathcal{M}(u), u \leq t)\right] = \sqrt{\frac{\pi}{2}} \mathbf{E}\left[\frac{\sqrt{t}}{R(t)} \Phi(R(u), u \leq t)\right]. \tag{2.16}$$

Decomposition of the excursions We write $\rho_{x,y}^t = (\rho_{x,y}^t(s), s \in [0,t])$ for a 3-dimensional Bessel bridge of length $t \in [0,1]$ between x and y, where $x, y \in \mathbb{R}_+$. Intuitively,

$$\forall F \in \mathcal{C}_b(\mathcal{D}), \quad \mathbf{E}\Big(F(\rho_{x,y}^t)\Big) = \mathbf{E}_x\Big(F(R(s), s \in [0,t]) \Big| R(t) = y\Big). \tag{2.17}$$

For all $\lambda \in (0,1)$, $G_1 \in \mathcal{C}(\mathcal{D}([0,\lambda]))$, $G_2 \in \mathcal{C}(\mathcal{D}([0,1-\lambda]))$ and $x \in \mathcal{D}$ we set

$$G_1 \star G_2(f) = G_1(x_s, s \leq \lambda) G_2(x_{s+\lambda}, s \leq 1-\lambda).$$

Lemma 2.3. *Let $(e_t, t \in [0,1])$ be a normalised Brownian excursion. We write $(\mathcal{M}_t, t \in [0,\lambda])$ and $(\rho_{x,0}^{1-\lambda}(t), x \in \mathbb{R}^+, t \in [0, 1-\lambda]$ two independent processes, with $\mathcal{M}$ a Brownian meander of length λ and $\rho_{x,0}^{1-\lambda}$ a Bessel bridge between x and 0 of length $1-\lambda$. We have*

$$\mathbf{E}\left[G_1 \star G_2(e)\right] = \sqrt{\frac{2}{\pi}} \frac{1}{\lambda^{1/2}(1-\lambda)^{3/2}} \mathbf{E}\left[\mathcal{M}(\lambda) e^{-\frac{\mathcal{M}(\lambda)^2}{2(1-\lambda)}} G_1\left(\mathcal{M}\right) G_2\left(\rho_{\mathcal{M}(\lambda),0}^{1-\lambda}\right)\right]. \tag{2.18}$$

Proof. We show that both sides in (2.18) are equal to

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x^2}{\lambda^{3/2}(1-\lambda)^{3/2}} e^{-\frac{x^2}{2\lambda(1-\lambda)}} \mathbf{E}\left[G_1\left(\rho_{0,x}^{\lambda}\right)\right] \mathbf{E}\left[G_2\left(\rho_{x,0}^{1-\lambda}\right)\right] dx. \tag{2.19}$$

Recall that e has the same law as $\rho_{0,0}^1$ a 3-dimensional Bessel bridge of length 1. Conditioning on the value of $\rho_{0,0}^1(\lambda)$, we have

$$\begin{aligned}
\mathbf{E}\left[G_1 \star G_2(e)\right] &= \mathbf{E}\left[G_1\left(\rho_{0,0}^1(s), s \leq \lambda\right) G_2\left(\rho_{0,0}^1(s), \lambda \leq s \leq 1\right)\right] \\
&= \int_0^\infty \mathbf{P}\left(\rho_{0,0}^1(\lambda) \in dx\right) \mathbf{E}\left[G_1\left(\rho_{0,0}^1(s), s \leq \lambda\right) G_2\left(\rho_{0,0}^1(s), \lambda \leq s \leq 1\right) \Big| \rho_{0,0}^1(\lambda) = x\right],
\end{aligned}$$

where the density of $\rho^1_{0,0}(\lambda)$ is $\mathbf{P}\left(\rho^1_{0,0}(\lambda)\in dx\right) = \sqrt{\frac{2}{\pi}}\frac{1}{\lambda^{3/2}(1-\lambda)^{3/2}}x^2e^{-\frac{x^2}{2\lambda(1-\lambda)}}1_{x\geq 0}dx$. It hence follows that

$$\begin{aligned}&\mathbf{E}\left[G_1\star G_2(e)\right]\\ =&\sqrt{\frac{2}{\pi}}\int_0^{+\infty}dx\frac{x^2}{\lambda^{3/2}(1-\lambda)^{3/2}}e^{-\frac{x^2}{2\lambda(1-\lambda)}}\mathbf{E}\left[G_1\left(\rho^1_{0,0}(s),s\leq\lambda\right)G_2\left(\rho^1_{0,0}(s),\lambda\leq s\leq 1\right)\Big|\rho^1_{0,0}(\lambda)=x\right].\end{aligned}$$

Applying the Markov property at time λ yields to

$$\begin{aligned}&\mathbf{E}\left[G_1\left(\rho^1_{0,0}(s),s\leq\lambda\right)G_2\left(\rho^1_{0,0}(s),\lambda\leq s\leq 1\right)\Big|\rho^1_{0,0}(\lambda)=x\right]\\ &=\mathbf{E}\left[G_1\left(\rho^1_{0,0}(s),s\leq\lambda\right)\Big|\rho^1_{0,0}(\lambda)=x\right]\mathbf{E}\left[G_2\left(\rho^{1-\lambda}_{x,0}\right)\right]\\ &=\mathbf{E}\left[G_1\left(\rho^{\lambda}_{0,x}\right)\right]\mathbf{E}\left[G_2\left(\rho^{1-\lambda}_{x,0}\right)\right].\end{aligned}$$

As a consequence

$$\mathbf{E}\left[G_1\star G_2(e)\right]=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{x^2}{\lambda^{3/2}(1-\lambda)^{3/2}}e^{-\frac{x^2}{2\lambda(1-\lambda)}}\mathbf{E}\left[G_1\left(\rho^{\lambda}_{0,x}\right)\right]\mathbf{E}\left[G_2\left(\rho^{1-\lambda}_{x,0}\right)\right]dx.$$

On the other hand, writing

$$\Gamma(G_1,G_2,\lambda)=\sqrt{\frac{2}{\pi}}\frac{1}{\lambda^{1/2}(1-\lambda)^{3/2}}\mathbf{E}\left[\mathcal{M}(\lambda)e^{-\frac{\mathcal{M}(\lambda)^2}{2(1-\lambda)}}G_1\left(\mathcal{M}\right)G_2\left(\rho^{\mathcal{M}(\lambda)}\right)\right],$$

by (2.16) we have

$$\begin{aligned}\Gamma(G_1,G_2,\lambda)&=\sqrt{\frac{2}{\pi}}\frac{1}{\lambda^{1/2}(1-\lambda)^{3/2}}\mathbf{E}\left[G_1\left(\mathcal{M}\right)\mathcal{M}(\lambda)e^{-\frac{\mathcal{M}(\lambda)^2}{2(1-\lambda)}}G_2\left(\rho^{1-\lambda}_{\mathcal{M}(\lambda),0}\right)\right]\\ &=\frac{1}{(1-\lambda)^{3/2}}\mathbf{E}\left[G_1\left(R(s);s\in[0,\lambda]\right)e^{-\frac{R(\lambda)^2}{2(1-\lambda)}}G_2\left(\rho^{1-\lambda}_{R(\lambda),0}\right)\right],\end{aligned}$$

where $(R(s);0\leq s\leq\lambda)$ is a Bessel process independent with $(\rho^{1-\lambda}_{x,0})$. We now condition on the value of $R(\lambda)$ –recall that the law of $R(\lambda)$ is $\mathbf{P}(R(\lambda)\in dx)=\sqrt{\frac{2}{\pi\lambda^3}}x^2e^{-x^2/(2\lambda)}1_{x\geq 0}dx$– to obtain

$$\begin{aligned}&\Gamma(G_1,G_2,\lambda)\\ =&\frac{1}{(1-\lambda)^{3/2}}\int_0^{\infty}\mathbf{P}\Big(R(\lambda)\in dx\Big)\mathbf{E}\left[G_1\left(R(s);s\in[0,\lambda]\right)e^{-\frac{R(\lambda)^2}{2(1-\lambda)}}\mathbf{E}\left[G_2\left(\rho^{1-\lambda}_{R(\lambda),0}\right)\right]\Big|R(\lambda)=x\right]\\ =&\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{x^2}{\lambda^{3/2}(1-\lambda)^{3/2}}e^{-\frac{x^2}{2\lambda(1-\lambda)}}\mathbf{E}\left[G_1\left(\rho^{\lambda}_{0,x}\right)\right]\mathbf{E}\left[G_2\left(\rho^{1-\lambda}_{x,0}\right)\right]dx.\end{aligned}$$

We conclude that $\mathbf{E}\left[G_1\star G_2(e)\right]=\Gamma(G_1,G_2,\lambda)$. □

2.2.2 Asymptotic independence of the endpoint and the shape of the trajectory in a random walk excursion

For $n \in \mathbb{N}$, let $\mathbf{S}^{(n)}$ be the normalized path, defined, for $t \in [0,1]$ by

$$\mathbf{S}_t^{(n)} := \frac{S_{\lfloor nt \rfloor}}{\sqrt{\sigma^2 n}}, \tag{2.20}$$

also written $\mathbf{S}$ when the value of n is unambiguous. Clearly, $(\mathbf{S}_t^{(n)}, t \in [0,1]) \in \mathcal{D}$. We prove in this section that conditionally on $\{\underline{S}_n \geq 0\}$ and $\{S_n = o(\sqrt{n})\}$, the endpoint S_n is asymptotically independent of $\mathbf{S}$ the shape of the excursion.

We begin with the following estimate, for a random walk which is at time 0 within distance $O(\sqrt{n})$.

Lemma 2.4. *Let $(y_n)_{n\geq 1}$ be a non-negative sequence such that $\lim_{n\to+\infty} \frac{y_n}{n^{1/2}} = 0$. There exists $C_\star = \frac{C_+}{\sigma}$ such that for all $K \in \mathbb{R}^+$ and $F \in \mathcal{C}_b^u(\mathcal{D})$, we have*

$$\lim_{n\to+\infty} \sup_{x\in\mathbb{R}_+} \Big| n\mathbf{E}_{x\sigma\sqrt{n}}\left[F(\mathbf{S}^{(n)}); \underline{S}_n \geq y_n, S_n \in [y_n, y_n + K]\right] - C_\star g(x)\mathbf{E}\left(F(\rho^1_{x,0})\right)\int_0^K V^+(z)dz \Big| = 0, \tag{2.21}$$

where $g : x \mapsto xe^{-\frac{x^2}{2}}\mathbf{1}_{\{x\geq 0\}}$ and $\rho^1_{x,0}$ is a three-dimensional Bessel bridge of length 1 from x to 0.

Proof. The proof of this lemma largely depends on the arguments in [7].

Let $n \in \mathbb{N}$ and F uniformly continuous, we have, in terms of the local measure

$$\begin{aligned}\mathbf{E}_{x\sigma\sqrt{n}}\left(F(\mathbf{S}); \underline{S}_n \geq y_n, S_n \in [y_n, y_n + K]\right) \\ = \left\{\mathbf{E}_{x\sigma\sqrt{n}-y_n}\left(F(\mathbf{S}) | \, \underline{S}_n \geq 0, S_n \in [0,K]\right) + o_n(1)\right\}\pi_n^+(x\sigma\sqrt{n} - y_n, [0,K]).\end{aligned} \tag{2.22}$$

Recall that (2.10) and (2.11) give estimates of $\pi_n^+(x,[y,y+K])$ when $x = o(\sqrt{n})$. We first show that there exists $C_\star > 0$ such that uniformly in $x \geq 0$,

$$\pi_n^+(x,[0,K]) = \frac{C_\star}{n}\int_0^K V^+(z)dz\, g\left(\frac{x}{\sigma\sqrt{n}}\right) + o\left(\frac{1}{n}\right). \tag{2.23}$$

Let $n \in \mathbb{N}$, we write $S_k^- = S_{n-k} - S_n$ for $0 \leq k \leq n$, the "time-reversal random walk", which has the same law as $-S$. We observe that

$$\begin{aligned}\pi_n^+(x,[0,K]) =& \mathbf{P}_x(\underline{S}_n \geq 0, S_n \in [0,K]) = \mathbf{P}\left(\min_{0\leq k\leq n} S_k^- \geq S_n^- - x \geq -K\right) \\ =& \sum_{j=0}^n \mathbf{P}\left(T = j, \min_{0\leq k\leq n} S_k^- \geq S_n^- - x \geq -K\right),\end{aligned}$$

where $T := \min\{j \leq n : S_j^- = \min_{0\leq k\leq n} S_k^-\}$. Applying the Markov property at time $T = j$ yields to

$$\pi_n^+(x,[0,K]) = \sum_{j=0}^{n} \underbrace{\mathbf{E}\left(\mathbf{1}_{\left\{-K\leq S_j^- < \min_{0\leq k\leq j-1} S_k^-\right\}} \pi_{n-j}^-(0,[x-K-S_j^-,x])\right)}_{\tau_j}. \tag{2.24}$$

Applying (2.11) to π^-, uniformly in $y \geq 0$, we have

$$\pi_n^-(0,[y,y+K]) = \frac{C_+}{\sigma n}\left(g(\tfrac{y}{\sigma\sqrt{n}})K + o_n(1)\right). \tag{2.25}$$

Therefore,

$$\begin{aligned}\sum_{0\leq j\leq\sqrt{n}} \tau_j &= \frac{C_+}{\sigma n}\left(g(\tfrac{x}{\sigma\sqrt{n}}) + o_n(1)\right) \sum_{0\leq j\leq\sqrt{n}} \mathbf{E}\left[\mathbf{1}_{\left\{-K\leq S_j^- < \min_{0\leq k\leq j-1} S_k^-\right\}}(K + S_j^-)\right] \\ &= \frac{C_+}{\sigma n}\left(g(\tfrac{x}{\sigma\sqrt{n}}) + o_n(1)\right)\left(\int_0^K V^+(z)dz + o_n(1)\right). \end{aligned} \tag{2.26}$$

Using (2.25), we observe there exists $c_2 > 0$ such that for all $n \in \mathbb{N}$ and $y \geq 0$, $\pi_n^-(0,[y,y+K]) \leq \frac{c_2(1+K)}{n+1}$, which implies

$$\begin{aligned}\sum_{\sqrt{n}<j\leq n} \tau_j &\leq \sum_{\sqrt{n}<j\leq n} \frac{c_2}{n-j+1}\mathbf{E}\left[\mathbf{1}_{\left\{-K\leq S_j^- < \min_{0\leq k\leq j-1} S_k^-\right\}}(1 + K + S_j^-)\right] \\ &\leq \sum_{\sqrt{n}<j\leq n} \frac{c_2(1+K)}{n-j+1}\mathbf{P}(\underline{S}_j \geq 0, S_j \leq K) && \text{by time-reversal} \\ &\leq \sum_{\sqrt{n}<j\leq n} \frac{c_3(1+K)^2}{(n-j+1)j^{3/2}} && \text{using (2.3)},\end{aligned}$$

so $\sum_{\sqrt{n}<j\leq n} \tau_j = o(n^{-1})$. As a consequence, writing $C_\star = \frac{C_+}{\sigma}$ uniformly in $x \geq 0$, (2.24) becomes

$$\pi_n^+(x,[0,K]) = \frac{C_\star}{n} g(\tfrac{x}{\sigma\sqrt{n}}) \int_0^K V^+(z)dz + o(n^{-1}). \tag{2.27}$$

Plugging this result into (2.22), we obtain, uniformly in $x \geq 0$

$$\begin{aligned}&\mathbf{E}_{x\sigma\sqrt{n}}\left(F(\mathbf{S}); \underline{S}_n \geq y_n, S_n \in [y_n, y_n + K]\right) \\ &\qquad = \frac{C_\star g(x)}{n}\int_0^K V^+(z)dz\mathbf{E}_{(x\sigma\sqrt{n}-y_n)}\left[(F(\mathbf{S})\,|\underline{S}_n \geq 0, S_n \in [0,K]\right] + o(n^{-1}).\end{aligned}$$

As $\lim_{x\to+\infty} g(x) = 0$, it remains to prove that for any $K_0 > 0$ fixed,

$$\lim_{n\to+\infty} \sup_{x\in[0,K_0]} \left|\mathbf{E}_{(x\sigma\sqrt{n}-y_n)}\left[F(\mathbf{S})\,|\underline{S}_n \geq 0, S_n \in [0,K]\right] - \mathbf{E}(F(\rho^{x,0}))\right| = 0. \tag{2.28}$$

Let $K_0 > 0$ and $\varepsilon > 0$, we prove that (2.28) holds for any $F \in \mathcal{C}_b(\mathcal{D}([0, 1-\varepsilon]))$.
Let $M := \lfloor (1-\varepsilon)n \rfloor$. For any $x \geq 0$, applying the Markov property at time M gives

$$\begin{aligned}\mathbf{E}_x\left(F(\mathbf{S})\,|\underline{S}_n \geq 0, S_n \in [0,K]\right) &= \frac{\mathbf{E}_x\left[F(\mathbf{S}); \underline{S}_n \geq 0, S_n \in [0,K]\right]}{\mathbf{P}_x\left[\underline{S}_n \geq 0, S_n \in [0,K]\right]} \\ &= \frac{\mathbf{E}_x\left[F(\mathbf{S})\mathbf{1}_{\left\{\underline{S}_M \geq 0\right\}}\mathbf{P}_{S_M}\left(\underline{S}_{n-M} \geq 0, S_{n-M} \in [0,K]\right)\right]}{\pi_n^+(x,[0,K])} \\ &= \frac{\mathbf{E}_x\left[F(\mathbf{S})\mathbf{1}_{\left\{\underline{S}_M \geq 0\right\}}\pi_{n-M}^+(S_M,[0,K])\right]}{\pi_n^+(x,[0,K])}.\end{aligned}$$

We set $x_n := x\sigma\sqrt{n} - y_n$. By change of measures introduced in (2.13), we observe that

$$\begin{aligned}\mathbf{E}_{x_n}\left[F(\mathbf{S})\,|\underline{S}_n \geq 0, S_n \in [0,K]\right] =& \frac{\mathbf{E}^{\uparrow}_{x_n}\left[F(\mathbf{S})\frac{V^-(x_n)}{V^-(S_M)}\pi_{n-M}^+(S_M,[0,K])\right]}{\pi_n^+(x_n,[0,K])} \\ =& \mathbf{E}^{\uparrow}_{x_n}\left[F(\mathbf{S})f^n_{\varepsilon,x_n}(\tfrac{S_M}{\sigma\sqrt{n}})\right],\end{aligned}$$

where we write (recalling that $M = \lfloor n(1-\varepsilon) \rfloor$)

$$f^n_{\varepsilon,x_n}(z) := \frac{V^-(x_n)}{\pi_n^+(x_n,[0,K])}\frac{\pi_{n-M}^+(z\sigma\sqrt{n},[0,K])}{V^-(z\sigma\sqrt{n})}. \tag{2.29}$$

On the other hand, for a Bessel bridge $\rho^1_{x,0}$, by the Markov property at time $1-\varepsilon$,

$$\mathbf{E}\left(F(\rho^1_{x,0})\right) = \mathbf{E}_x\left[F\Big(R(s); s \in [0,1-\varepsilon]\Big)f_{\varepsilon,x}\Big(R(1-\varepsilon)\Big)\right], \tag{2.30}$$

where

$$f_{\varepsilon,x}(z) := \frac{e^{-z^2/(2\varepsilon)}}{\varepsilon^{3/2}e^{-x^2/2}}. \tag{2.31}$$

As a result,

$$\begin{aligned}&\left|\mathbf{E}_{x_n}\left[F(\mathbf{S})\,|\underline{S}_n \geq 0, S_n \in [0,K]\right) - \mathbf{E}\left(F(\rho^1_{x,0})\right)\right| \\ &\qquad \leq \left|\mathbf{E}^{\uparrow}_{x_n}\left[F(\mathbf{S})f^n_{\varepsilon,x_n}(\tfrac{S_M}{\sigma\sqrt{n}})\right] - \mathbf{E}^{\uparrow}_{x_n}\left(F(\mathbf{S})f_{\varepsilon,x}(\tfrac{S_M}{\sigma\sqrt{n}})\right)\right| \\ &\qquad + \left|\mathbf{E}^{\uparrow}_{x_n}\left(F(\mathbf{S})f_{\varepsilon,x}(\tfrac{S_M}{\sigma\sqrt{n}})\right) - \mathbf{E}_x\left[F\Big(R(s); s \in [0,1-\varepsilon]\Big)f_{\varepsilon,x}\Big(R(1-\varepsilon)\Big)\right]\right|,\end{aligned}$$

which leads to

$$\begin{aligned}&\left|\mathbf{E}_{x_n}\left[F(\mathbf{S})\,|\underline{S}_n \geq 0, S_n \in [0,K]\right) - \mathbf{E}\left(F(\rho^1_{x,0})\right)\right| \\ &\qquad \leq \sup_{z\geq 0, x\in[0,K]}\left|f^n_{\varepsilon,x_n}(z) - f_{\varepsilon,x}(z)\right| ||F||_\infty \\ &\qquad + \left|\mathbf{E}^{\uparrow}_{x_n}\left(F(\mathbf{S})f_{\varepsilon,x}(\tfrac{S_M}{\sigma\sqrt{n}})\right) - \mathbf{E}_x\left[F\left(R(s); s \in [0,1-\varepsilon]\right)f_{\varepsilon,x}\left(R(1-\varepsilon)\right)\right]\right|.\end{aligned}$$

By use of (2.23) and (2.10), we have

$$\lim_{n\to\infty}\sup_{z\geq 0, x\in[0,K]}\left|f^n_{\varepsilon,x_n}(z)-f_{\varepsilon,x}(z)\right|=0. \tag{2.32}$$

It follows from (2.14) that

$$\lim_{n\to+\infty}\sup_{x\in[0,K]}\left|\mathbf{E}^{\uparrow}_{x_n}\left(F(\mathbf{S})f_{\varepsilon,x}(\frac{S_M}{\sigma\sqrt{n}})\right)-\mathbf{E}_x\left[F\Big(R(s);s\in[0,1-\varepsilon]\Big)f_{\varepsilon,x}\Big(R(1-\varepsilon)\Big)\right]\right|=0. \tag{2.33}$$

We end up with checking the tightness of $\mathbf{S}$ under $\mathbf{P}_{x_n}\left(\cdot\,|\underline{S}_n\geq 0, S_n\in[0,K]\right)$ to conclude Lemma 2.4. By Theorem 15.3 of [4], for any $\eta>0$, it suffices to say that

$$\lim_{\delta\to 0}\lim_{n\to\infty}\sup_{x\in[0,K]}\mathbf{P}_{x_n}\left(\sup_{0\leq k\leq\delta n}S_{n-k}\geq\eta\sigma\sqrt{n}\,\middle|\,\underline{S}_n\geq 0, S_n\in[0,K]\right)=0, \tag{2.34}$$

which holds immediately by time reversal properties. □

Using Lemma 2.4, we obtain the main result of this section, the joint convergence of this normalized path and the terminal position in a random walk excursion.

Lemma 2.5. *Let $f:\mathbb{R}_+\to\mathbb{R}$ be a Riemann-integrable function such that there exists a non-increasing function $\widehat{f}$ verifying $|f(x)|\leq\widehat{f}(x)$ and $\int_{\mathbb{R}_+}x\widehat{f}(x)dx<+\infty$. Let (r_n) be a non-negative sequence such that $\limsup_{n\to+\infty}\frac{r_n}{\log n}<+\infty$. There exists a constant $C_1>0$ such that for all such functions f, $\lambda\in(0,1)$ and $F\in\mathcal{C}^u_b(\mathcal{D})$,*

$$\lim_{n\to+\infty}\sup_{y\in[0,r_n]}\left|n^{3/2}\mathbf{E}\left[F\left(\mathbf{S}\right)f(S_n-y);\underline{S}_n\geq 0,\underline{S}_{[\lambda n,n]}\geq y\right]-C_1\mathbf{E}\left[F(e_1)\right]\int_{\mathbb{R}_+}f(x)V^+(x)dx\right|=0. \tag{2.35}$$

Proof. This lemma is a slight refinement of Lemma 2.4 in [9], which proved the convergence when the function $F(\mathbf{S})=\Phi(\mathbf{S}_t,t\in[0,\alpha])$ for some $\alpha<1$. Without loss of generality, we assume $0\leq F\leq 1$. For convenience, we set

$$\chi(F,f):=\mathbf{E}\left[F(\mathbf{S})f(S_n-y);\underline{S}_n\geq 0,\underline{S}_{[\lambda n,n]}\geq y\right]. \tag{2.36}$$

For any $K>0$, writing $f_K(x)=f(x)\mathbf{1}_{\{x\in[0,K]\}}$, we observe that

$$\chi(F,f)=\chi\left(F,f_K\right)+\chi\left(F,f-f_K\right).$$

As $0\leq F\leq 1$, we have $\chi\left(F,f-f_K\right)\leq\chi\left(1,f-f_K\right)$, and

$$\begin{aligned}\chi\left(1,f-f_K\right)&\leq\sum_{j=K}^{+\infty}\mathbf{E}\left(f(S_n-y);\underline{S}_n\geq 0,\underline{S}_{[m_n,n]}\geq y,S_n\in[y+j,y+j+1]\right)\\&\leq\sum_{j=K}^{+\infty}\widehat{f}(j)\underbrace{\mathbf{P}\left(\underline{S}_n\geq 0,\underline{S}_{[m_n,n]}\geq y,S_n\in[y+j,y+j+1]\right)}_{\leq c_1(2+j)n^{-3/2}}.\end{aligned}$$

by use of (2.3). As $\int_0^{+\infty} x\widehat{f}(x)dx < +\infty$, we have $\lim_{K\to+\infty}\sum_{j=K}^{+\infty}(2+j)\widehat{f}(j) = 0$.

Therefore, we only need to estimate $\chi(F, f_K)$, and, as f is Riemann-integrable, it is enough to consider functions of the form $\mathbf{1}_{\{[0,K]\}}$, for $K \in \mathbb{R}$. We now compute an equivalent of

$$\chi(F,K) := \chi(F, \mathbf{1}_{\{[0,K]\}}) = \mathbf{E}\left[F(\mathbf{S}); \underline{S}_n \geq 0, \underline{S}_{[m,n]} \geq y, S_n \leq y + K\right].$$

We choose $F = G_1 \star G_2$ where $G_1 \in \mathcal{C}(\mathcal{D}([0,\lambda]))$ and $G_2 \in \mathcal{C}(\mathcal{D}([0,1-\lambda]))$ are two uniformly continuous functions. We prove that uniformly in $y \in [0, r_n]$, we have

$$\lim_{n\to+\infty}\left|n^{3/2}\chi(G_1\star G_2(\mathbf{S}),K) - C_1\mathbf{E}\left[G_1\star G_2(e_1)\right]\int_0^K V^+(x)dx\right| = 0 \tag{2.37}$$

This result implies (2.35), by monotone classes. Applying the Markov property at time $m = m_n := \lfloor \lambda n \rfloor$, we have

$$\chi\left(G_1\star G_2, K\right) = \mathbf{E}\left[G_1\left(\mathbf{S}_t; t\in[0,\lambda]\right)\Psi_{K,G_2}\left(\tfrac{S_m}{\sigma\sqrt{n}}\right); \underline{S}_m \geq 0\right], \tag{2.38}$$

where for $x \geq 0$,

$$\Psi_{K,G_2}(x) := \mathbf{E}_{x\sigma\sqrt{n}}\left[G_2\left(\tfrac{S_{\lfloor n(t+\lambda)\rfloor - m}}{\sigma\sqrt{n}}; t\in[0,1-\lambda]\right); S_{n-m} \leq y+K, \underline{S}_{n-m} \geq y\right].$$

Recall that $\rho^t_{x,y}$ is a 3-dimensional Bessel bridge of length t between x and y. Using Lemma 2.4, uniformly in $x \geq 0$ and $y \in [0, r_n]$, we have

$$\Psi_{K,G_2}(x) = \frac{C_\star}{(1-\lambda)n}\int_0^K V^+(z)dz\psi(x) + o(n^{-1}),$$

where $\psi(x) := g\left(\tfrac{x}{\sqrt{1-\lambda}}\right)\mathbf{E}\left[G_2\left(\rho^{1-\lambda}_{x,0}\right)\right]$ and $C_\star = \tfrac{C_+}{\sigma}$. As a consequence, (2.38) becomes

$$\begin{aligned}
&\chi\left(G_1\star G_2, K\right)\\
&\quad\frac{C_\star}{(1-\lambda)n}\int_0^K V^+(z)dz\mathbf{E}\left[G_1\left(\mathbf{S}_t; t\in[0,\lambda]\right)\psi\left(\tfrac{S_m}{\sigma\sqrt{n}}\right); \underline{S}_m \geq 0\right] + o(n^{-1})\mathbf{P}\left(\underline{S}_m \geq 0\right)\\
&\quad= \frac{C_+C_-}{\sigma(1-\lambda)\sqrt{\lambda}n^{3/2}}\int_0^K V^+(z)dz\mathbf{E}\left[G_1\left(\tfrac{S_{\lfloor nt\rfloor}}{\sigma\sqrt{n}}; t\in[0,\lambda]\right)\psi\left(\tfrac{S_m}{\sigma\sqrt{n}}\right)\Big|\, \underline{S}_m \geq 0\right] + o(n^{-3/2}),
\end{aligned}$$

where the last equality is a consequence of (2.5).

Using (2.15), conditionally on $\{\underline{S}_m = \underline{S}_{\lfloor\lambda n\rfloor} \geq 0\}$, the normalised random walk $\mathbf{S}^{(n)}$ converges in law to a Brownian meander of length λ, written $\mathcal{M} = (\mathcal{M}(t), 0 \leq r \leq \lambda)$. Therefore, uniformly in $y \in [0, r_n]$,

$$\begin{aligned}
\chi\left(G_1\star G_2, K\right) &= \frac{C_+C_-}{\sigma(1-\lambda)\sqrt{\lambda}n^{3/2}}\int_0^K V^+(z)dz\mathbf{E}\left[G_1\left(\mathcal{M}\right)\psi\left(\mathcal{M}(\lambda)\right)\right] + o(n^{-3/2})\\
&= \frac{C_+C_-}{\sigma n^{3/2}}\int_0^K V^+(z)dz\Gamma(G_1,G_2,\lambda) + o(n^{-3/2}),
\end{aligned}$$

where we write

$$\Gamma(G_1, G_2, \lambda) = \frac{1}{\lambda^{1/2}(1-\lambda)^{3/2}} \mathbf{E}\left[G_1\left(\mathcal{M}\right) \mathcal{M}(\lambda) e^{-\frac{\mathcal{M}(\lambda)^2}{2(1-\lambda)}} \mathbf{E}\left[G_2\left(\rho^{1-\lambda}_{\mathcal{M}(\lambda),0} \right) \right] \right],$$

with $(\mathcal{M}(t); 0 \leq r \leq \lambda)$ and $(\rho^{1-\lambda}_{x,0}(t); t \in [0,1], x \in \mathbb{R}_+)$ two independent processes. Applying Lemma 2.3, we have, uniformly in $y \in [0, r_n]$,

$$\chi(G_1 \star G_2; K) = \frac{C_1}{n^{3/2}} \int_0^K V^+(z) dz \times \mathbf{E}\left(G_1 \star G_2(e_1)\right) + o(n^{-3/2}), \tag{2.39}$$

where $C_1 := \frac{C_+ C_-}{\sigma}\sqrt{\frac{\pi}{2}}$, which leads to (2.37), hence (2.35). Finally, we can express any uniformly continuous functional F as a mixture of $G_1 \star G_2$, which concludes the proof. □

This lemma can be extended, using standard computations, to the following estimate, which enables to choose the starting point uniformly in $[0, r_n]$.

Lemma 2.6. *Under the hypotheses of Lemma 2.5, we have*

$$\lim_{n\to+\infty} \sup_{a,y\in[0,r_n]} \Big| n^{3/2} \mathbf{E}\left[F\left(\mathbf{S}\right) f(S_n - y); \underline{S}_n \geq -a, \underline{S}_{[\lambda n, n]} \geq y \right] - C_1 V^-(a) \mathbf{E}\left[F(e_1)\right] \int_{\mathbb{R}_+} f(x) V^+(x) dx \Big| = 0. \tag{2.40}$$

Proof. We set

$$\chi_a(F, f) := \mathbf{E}\left[F(\mathbf{S}) f(S_n - y); \underline{S}_n \geq -a, \underline{S}_{[\lambda n, n]} \geq y \right]. \tag{2.41}$$

Decomposing with respect to the first time at which the random walk hits its minimum, we prove that uniformly in $a \in [0, r_n]$, $\chi_a(F, f) \approx V^-(a)\chi(F, f)$. Let $\tau := \inf\{0 \leq k \leq m : S_k = \underline{S}_n\}$, we show that $\tau \leq \sqrt{n}$ with high probability. By Markov property at time k, we have

$$\begin{aligned}
\chi_a(F, f) &= \mathbf{E}\left(F(\mathbf{S}) f(S_n - y); \tau = k, \underline{S}_n \geq -a, \underline{S}_{[m,n]} \geq y \right) \\
&\leq \mathbf{E}\left(f(S_n - y); \tau = k, \underline{S}_n \geq -a, \underline{S}_{[\lambda n, n]} \geq y \right) \\
&\leq \mathbf{E}\left(\zeta(S_k, n-k) \mathbf{1}_{\{\underline{S}_{k-1} > S_k \geq -a\}} \right),
\end{aligned}$$

where $\zeta(x, n-k) := \mathbf{E}(f(S_{n-k} - y + x); \underline{S}_{n-k} \geq 0, \underline{S}_{[\lambda n - k, n-k]} \geq y - x)$.

On the one hand, observe that

$$\begin{aligned}
\zeta(x, n-k) \leq& \mathbf{E}\left(f(S_{n-k} - y + x) \mathbf{1}_{\{S_{n-k} \geq y - x\}}; \underline{S}_{n-k} \geq 0 \right) \\
&\leq \sum_{j=0}^{+\infty} \mathbf{E}\left(f(S_{n-k} - y + x) \mathbf{1}_{\{S_{n-k} \in [y-x+j, y-x+j+1]\}}; \underline{S}_{n-k} \geq 0 \right) \\
&\leq \sum_{j=0}^{+\infty} \widehat{f}(j) \mathbf{P}\left(S_{n-k} \in [y-x+j, y-x+j+1], \underline{S}_{n-k} \geq 0 \right),
\end{aligned}$$

which, by (2.3), is bounded by

$$c_1 \sum_{j=0}^{\infty} \widehat{f}(j) \frac{(j+y-x+2)}{(n-k)^{3/2}} \leq \frac{c_1}{(1-\lambda)^{3/2} n^{3/2}} 2(1+y-x) \sum_{j=0}^{\infty} (1+j)\widehat{f}(j).$$

As $\int x\widehat{f}(x)dx < \infty$, uniformly in $a, y \in [0, r_n]$, $x \in [-a, 0]$ and $k \leq \lambda n$, we have

$$\zeta(x, n-k) \leq c_2(1+y+a)n^{-3/2}. \tag{2.42}$$

On the other hand, by (2.3), $\mathbf{P}(\underline{S}_{k-1} > S_k \geq -a) \leq c_1(1+a)^2 k^{-3/2}$. As a consequence, writing $k_n = \lfloor \sqrt{n} \rfloor$, we have

$$\begin{aligned}\sum_{k=k_n+1}^{\lambda n} \mathbf{E}\left(F(\mathbf{S})f(S_n - y); \tau = k, \underline{S}_n \geq -a, \underline{S}_{[\lambda n, n]} \geq y\right) \leq & c_3 n^{-3/2}(1+y+a)(1+a)^2 \sum_{k=k_n+1}^{\lambda n} k^{-3/2} \\ \leq & c_4 \frac{(1+\log n)^3}{k_n^{1/2}} n^{-3/2}.\end{aligned}$$

Thus

$$\begin{aligned}\chi_a(F, f) &= \sum_{k=0}^{\lambda n} \mathbf{E}\left(F(\mathbf{S})f(S_n - y); \tau = k, \underline{S}_n \geq -a, \underline{S}_{[\lambda n, n]} \geq y\right) \\ &= \sum_{k=0}^{k_n} \mathbf{E}\left(F(\mathbf{S})f(S_n - y); \tau = k, \underline{S}_n \geq -a, \underline{S}_{[\lambda n, n]} \geq y\right) + o(n^{-3/2}).\end{aligned}$$

We now prove that $\max_{k \leq \tau} S_k \leq n^{1/4}$. Let $M > 0$, by Markov property and (2.42),

$$\begin{aligned}&\sum_{k=0}^{k_n} \mathbf{E}\left(F(\mathbf{S})f(S_n - y); \tau = k, \max_{j \leq k} S_j \geq M, \underline{S}_n \geq -a, \underline{S}_{[\lambda n, n]} \geq y\right) \\ \leq &\sum_{k=0}^{k_n} \mathbf{E}\left(\zeta(S_k, n-k)\mathbf{1}_{\left\{\underline{S}_{k-1} > S_k \geq -a; \max_{j \leq k} S_j \geq M\right\}}\right) \\ \leq &c_2 n^{-3/2}(1+y+a) \sum_{k=0}^{k_n} \mathbf{P}\left(\underline{S}_{k-1} > S_k \geq -a; \max_{j \leq k} S_j \geq M\right).\end{aligned}$$

We recall that $(\tau_k^-, H_k^-)_{k \geq 0}$ are the strict descending epochs and heights of (S_n). For all $k \geq 1$, the sequence $\left(\left\{S_{n-\tau_{k-1}^-} + H_{k-1}^-, n \leq \tau_k^- - \tau_{k-1}^-\right\}, k \geq 0\right)$ is i.i.d. Let

$$M_k = \max\{S_n + H_{k-1}^-, \tau_{k-1}^- \leq n \leq \tau_k^-\},$$

we deduce that $(H_k^- - H_{k-1}^-, M_k)$ is i.i.d. Consequently,

$$\begin{aligned}
&\sum_{k=0}^{k_n} \mathbf{P}\left(\underline{S}_{k-1} > S_k \geq -a; \max_{j\leq k} S_j \geq M\right) \\
&\quad \leq \mathbf{P}(M_1 \geq M) + \sum_{k\geq 1} \mathbf{P}(H_k^- \leq a, M_1 < M, \ldots, M_k < M + H_{k-1}^-, M_{k+1} > M + H_k^-) \\
&\quad \leq \mathbf{P}(M_1 \geq M) + \sum_{k\geq 1} \mathbf{P}(H_k^- \leq a, M_{k+1} > M + H_k^-) \\
&\quad \leq \mathbf{P}(M_1 \geq M) + \sum_{k\geq 1} \mathbf{P}(H_k^- \leq a)\mathbf{P}(M_1 > M) \\
&\quad \leq R(a)\mathbf{P}(M_1 > M).
\end{aligned}$$

According to Corollary 3 in [11], $\mathbf{P}(M_1 > n) = \frac{c}{n} + o(n^{-1})$. Taking $M = n^{1/4}$ yields to

$$\sum_{k=0}^{k_n} \mathbf{E}\left(F(\mathbf{S})f(S_n - y); \tau = k, \max_{j\leq k} S_j \geq n^{1/4}, \underline{S}_n \geq -a, \underline{S}_{[\lambda n,n]} \geq y\right) = o(n^{-3/2}).$$

Finally, by uniform continuity of F, we have

$$\begin{aligned}
\chi_a(F,f) &= \sum_{k=0}^{k_n} \mathbf{E}\Big(F(\mathbf{S})f(S_n - y); \tau = k, \max_{j\leq k} S_j \leq n^{1/4}, \underline{S}_n \geq -a, \underline{S}_{[\lambda n,n]} \geq y\Big) + o(n^{-3/2}) \\
&= \sum_{k=0}^{k_n} \mathbf{E}\Big(\zeta(S_k, n-k, F)\mathbf{1}_{\{\underline{S}_{k-1} > S_k \geq -a\}}\Big) + o(n^{-3/2}),
\end{aligned} \tag{2.43}$$

where

$$\zeta(x, n-k, F) := \mathbf{E}\left(F\Big(\frac{S_{\lfloor (n-k)t\rfloor}}{\sqrt{\sigma^2(n-k)}}, t \in [0,1]\Big) f(S_{n-k} - y + x); \underline{S}_{n-k} \geq 0, \underline{S}_{[\lambda n-k, n-k]} \geq y - x\right).$$

Observe that for $k \leq \sqrt{n}$, the asymptotic behaviour of $\zeta(x, n-k, F)$ follows from that of $\chi(F,f)$. It follows from (2.35) that uniformly in $k \leq k_n$, $x \in [-a, 0]$ and $a, y \in [0, r_n]$,

$$\zeta(x, n-k, F) = \frac{C_1}{n^{3/2}} \mathbf{E}\left(F(e)\right) \int_0^\infty f(z)V^+(z)dz + o(n^{-3/2}).$$

Going back to (2.43), we have

$$\begin{aligned}
\chi_a(F,f) &= \sum_{k=0}^{k_n} \mathbf{E}\left(\zeta(S_k, n-k, F)\mathbf{1}_{\{mS_{k-1} > S_k \geq -a\}}\right) + o(n^{-3/2}) \\
&= \frac{C_1}{n^{3/2}} \mathbf{E}\left(F(e)\right) \int_0^\infty f(z)V^+(z)dz \sum_{k=0}^{k_n} \mathbf{P}\left(\underline{S}_{k-1} > S_k \geq -a\right) + o(n^{-3/2}).
\end{aligned}$$

Observe also that $\sum_{k=0}^{+\infty} \mathbf{P}\left(\underline{S}_{k-1} > S_k \geq -a\right) = V^-(a)$ and that uniformly in $a \in [0, r_n]$,

$$\sum_{k=k_n+1}^{\infty} \mathbf{P}\left(\underline{S}_{k-1} > S_k \geq -a\right) = o_n(1).$$

We conclude that uniformly in $y, a \in [0, r_n]$,

$$\lim_{n\to+\infty} n^{3/2}\chi(F,f) = C_1 V^-(a)\mathbf{E}\left(F(e_1)\right)\int_0^\infty f(z)V^+(z)dz, \tag{2.44}$$

which ends the proof. □

3 Laplace transform of the Gibbs measure

We recall that for a branching random walk $(V(u), u \in \mathbb{T})$ and $\beta > 1$,

$$\widetilde{\mu}_{n,\beta}(F) = n^{\frac{3\beta}{2}} \sum_{|u|=n} e^{-\beta V(u)} F(H_n(u)).$$

This section is devoted to the computation of the Laplace transform of $\widetilde{\mu}_{n,\beta}(F)$, which is closely related to already known estimates on the minimal displacement of the branching random walk. Therefore, we define M_n as the smallest occupied position in the n-th generation, i.e.,

$$M_n := \inf_{|u|=n} V(u), \tag{3.1}$$

with the convention $\inf \emptyset := +\infty$. We denote by $m^{(n)}$ an individual uniformly chosen in $\{u : |u| = n, V(u) = M_n\}$ the set of leftmost individuals at time n.

The rest of this section is devoted to the proof of the following result.

Proposition 3.1. *Let $\beta > 1$, under (1.3) and (1.4), there exists $C_\beta > 0$ such that for all non-negative $F \in \mathcal{C}_b^u(\mathcal{D})$ and $\varepsilon > 0$, there exists $(A, N) \in \mathbb{R}^+ \times \mathbb{N}$ such that*

$$\sup_{n\geq N} \sup_{x\in[A,\frac{3}{2}\log n - A]} \left| \frac{e^x}{x}\mathbf{E}\left[1 - \exp\left(-e^{-\beta x}\widetilde{\mu}_{n,\beta}(F)\right)\right] - C_\beta \mathbf{E}\left(F(e_1)^{\frac{1}{\beta}}\right)\right| \leq \varepsilon, \tag{3.2}$$

where e_1 is a standard Brownian excursion.

Observe that if $F = \theta \in \mathbb{R}^+$ is a constant, Proposition 3.1 is: For all $\varepsilon > 0$, there exists $(A, N) \in \mathbb{R}^+ \times \mathbb{N}$ such that

$$\sup_{n\geq N} \sup_{x\in[A,\frac{3}{2}\log n - A]} \left| \frac{e^x}{x}\mathbf{E}\left(1 - \exp\left(-\theta e^{-\beta x}\widetilde{\mu}_{n,\beta}(1)\right)\right) - C_\beta \theta^{\frac{1}{\beta}}\right| \leq \varepsilon, \tag{3.3}$$

which has been proved in [21]. Therefore, it is enough to prove, using Lemma 2.5 that

$$\mathbf{E}\left[\exp\left(-e^{-\beta x}\widetilde{\mu}_{n,\beta}(F)\right)\right] \approx \mathbf{E}\left[\exp\left(-e^{-\beta x}\widetilde{\mu}_{n,\beta}(1)F(e_1)\right)\right]$$

where e_1 is a Brownian excursion independent of the branching random walk.

For all $n \in \mathbb{N}$, following [1], we write $a_n = \frac{3}{2}\log n$ and $a_n(z) = a_n - z$. For all $x \in \mathbb{R}$, $F \in \mathcal{C}_b^u(\mathcal{D}, \mathbb{R}^+)$ and E a measurable event, we write

$$\Sigma(n,x,F) := \mathbf{E}\left[\exp\left(-e^{-\beta x}\widetilde{\mu}_{n,\beta}(F)\right)\right] \quad \text{and} \quad \Sigma_E(n,x,F) := \mathbf{E}\left[\exp\left(-e^{-\beta x}\widetilde{\mu}_{n,\beta}(F)\mathbf{1}_E\right)\right].$$

For $\lambda \in (0,1)$, $L, L_0 \geq 0$ and $z > K_0 > 0$, we define the set of individuals

$$J^L_{\lambda,z,K_0,L_0}(n) = \left\{ u \in \mathbb{T} : \begin{array}{c} |u| = n, V(u) \leq a_n(z-L), \min_{k\leq n} V(u_k) \geq -z + K_0, \\ \min_{\lambda n \leq k \leq n} V(u_k) \geq a_n(z+L_0) \end{array} \right\}, \tag{3.4}$$

For simplicity, we often write $J_{\lambda,z,K_0,L_0}(n)$ instead of $J^0_{\lambda,z,K_0,L_0}(n)$. We now consider the following event

$$E_n := \{m^{(n)} \in J_{\lambda,x-\Delta,K_0,L_0}(n)\}. \tag{3.5}$$

At the end of the section, we will choose $\Delta < L_0 \ll K_0 \ll x$, and $L \in \{0, L_0\}$. We prove in a first time that Σ and Σ_{E_n} are close to each other.

Lemma 3.2. *There exists $\alpha_1 > 0$ small enough such that for all $\varepsilon > 0$, there exists $\Delta_{\varepsilon,1} \geq 1$ such that such that for all $\Delta \geq \Delta_{\varepsilon,1}$, $L_0 \geq 2\Delta/\alpha_1$, $x \geq 2e^{K_0+\Delta}/\varepsilon$ and $n \geq 1$, we have*

$$0 \leq \Sigma_{E_n}(n,x,F) - \Sigma(n,x,F) \leq \varepsilon x e^{-x}. \tag{3.6}$$

Proof. Observe that

$$\begin{aligned} \Sigma_{E_n}(n,x,F) =& \mathbf{E}\left[\exp\left(-e^{-\beta x}\widetilde{\mu}_{n,\beta}(F)\right); E_n\right] + \mathbf{P}\left(E_n^c\right), \\ \Sigma(n,x,F) =& \mathbf{E}\left[\exp\left(-e^{-\beta x}\widetilde{\mu}_{n,\beta}(F)\right); E_n\right] + \mathbf{E}\left[\exp\left(-e^{-\beta x}\widetilde{\mu}_{n,\beta}(F)\right); E_n^c\right]. \end{aligned}$$

As a consequence,

$$0 \leq \Sigma_{E_n}(n,x,F) - \Sigma(n,x,F) = \mathbf{E}\left(1 - \exp\{-e^{-\beta x}\widetilde{\mu}_{n,\beta}(F)\}; E_n^c\right). \tag{3.7}$$

We observe that $1 - e^{-W} = \int_0^\infty e^{-u}\mathbf{1}_{\{W \geq u\}}du$, thus

$$\begin{aligned} \Sigma_{E_n}(n,x,F) - \Sigma(n,x,F) &= \mathbf{E}\left[\int_0^\infty e^{-u}\mathbf{1}_{\left\{e^{-\beta x}\widetilde{\mu}_{n,\beta}(F) \geq u\right\}}du; E_n^c\right] \\ &= \int_0^\infty e^{-u}\mathbf{P}\left(e^{-\beta x}\widetilde{\mu}_{n,\beta}(F) \geq u; E_n^c\right)du. \end{aligned}$$

Using the fact that F is non-negative bounded, we have

$$\Sigma_{E_n}(n,x,F) - \Sigma(n,x,F) \leq \int_0^{+\infty} e^{-u}\mathbf{P}\left(\widetilde{\mu}_{n,\beta}(1) \geq \tfrac{u}{\|F\|_\infty}e^{\beta x}; E_n^c\right).$$

Let $\Delta \in (1, x-1)$, as $E_n^c \subset \{M_n \geq a_n(x-\Delta)\} \cup (E_n^c \cap \{M_n \leq a_n(x-\Delta)\})$, we have

$$\begin{aligned}(3.8)\quad \Sigma_{E_n}(n,x,F) - \Sigma(n,x,F) &\leq \underbrace{\int_0^\infty e^{-u}\mathbf{P}\left(\widetilde{\mu}_{n,\beta}(1) \geq \tfrac{u}{\|F\|_\infty}e^{\beta x}; M_n \geq a_n(x-\Delta)\right)du}_{\mathbf{P}_\dagger} \\ &\quad + \int_0^\infty e^{-u}\underbrace{\mathbf{P}\left(M_n \leq a_n(x-\Delta); E_n^c\right)}_{\mathbf{P}_\ddagger}du.\end{aligned}$$

On the one hand,

$$\begin{aligned}\mathbf{P}_\ddagger =& \mathbf{P}\left(m^{(n)} \notin J_{\lambda,x-\Delta,K_0,L_0}(n); M_n \leq a_n(x-\Delta)\right) \\ \leq& \mathbf{P}\left(\exists z : |z| = n, V(z) \leq a_n(x-\Delta), z \notin J_{\lambda,x-\Delta,K_0,L_0}(n)\right) \\ (3.9)\qquad \leq& \left(e^{K_0} + e^{-c_6L_0}x\right)e^{-x+\Delta}\end{aligned}$$

applying Lemma 3.3 in Aïdékon [1].

On the other hand, by change of variables,

$$(3.10)\qquad \mathbf{P}_\dagger = \int_{\mathbb{R}} \beta e^{-e^{\beta y}+\beta y}\underbrace{\mathbf{P}\left(\widetilde{\mu}_{n,\beta}(1) \geq \tfrac{1}{\|F\|_\infty}e^{\beta(x+y)}; M_n \geq a_n(x-\Delta)\right)}_{\mathbf{P}_\dagger(x,y)}dy.$$

To bound $\mathbf{P}_\dagger(x,y)$, we use Proposition 2.1 of [21]. For all $0 \leq K \leq \Delta$, one sees immediately that, for $|y| \leq K$,

$$\begin{aligned}\mathbf{P}_\dagger(x,y) &= \mathbf{P}\left(\widetilde{\mu}_{n,\beta}(1) \geq \frac{1}{\|F\|_\infty}e^{\beta(x+y)}; M_n \geq a_n(x-\Delta)\right) \\ &\leq \sum_{j \geq \Delta+y}\mathbf{P}\left(\widetilde{\mu}_{n,\beta}(1) \geq \frac{1}{\|F\|_\infty}e^{\beta(x+y)}; M_n - a_n(0) \in [j-(x+y); (j+1)-(x+y)]\right) \\ &\leq c_7(x+y)e^{-(x+y)}e^{-\alpha(\Delta+y)} \\ (3.11)\qquad &\leq c_8 x e^{-x}e^{(1+\alpha)K-\alpha\Delta}.\end{aligned}$$

In the same way, for $|y| > K$, we have

$$\begin{aligned}\mathbf{P}_\dagger(x,y) &\leq \mathbf{P}\left(\widetilde{\mu}_{n,\beta}(1) \geq \frac{1}{\|F\|_\infty}e^{\beta(x+y)}\right) \\ &\leq c_9(x+y)e^{-(x+y)}\mathbf{1}_{\{x+y\geq 1\}} + \mathbf{1}_{\{y+x\leq 1\}} \\ (3.12)\qquad &\leq c_9 x e^{-x}\mathbf{1}_{\{y>K\}} + c_9 e^{-(x+y)}\mathbf{1}_{\{-K>y\geq 1-x\}} + \mathbf{1}_{\{x+y\leq 1\}}.\end{aligned}$$

Combining (3.11) with (3.12) yields to

$$\begin{aligned}\mathbf{P}_\dagger &\leq c_8 x e^{-x}e^{(1+\alpha)K-\alpha\Delta} + c_9 x e^{-x}\int_K^{+\infty}\beta e^{-e^{\beta y}+\beta y}dy \\ &\quad + c_9 x e^{-x}\int_{\{1-x\leq y<-K\}}\beta e^{-e^{\beta y}+\beta y-y}dy + \int_{-\infty}^{1-x}\beta e^{-e^{\beta y}+\beta y}dy \\ &\leq c_8 x e^{-x}e^{(1+\alpha)K-\alpha\Delta} + c_{10}x e^{-x}\left(e^{(1-\beta)K} + e^{(1-\beta)x}\right).\end{aligned}$$

Taking $K = \frac{\alpha\Delta}{\alpha+\beta}$, we obtain

$$\mathbf{P}_\dagger \leq c_{11} x e^{-x} e^{-\frac{(\beta-1)\alpha\Delta}{\alpha+\beta}}. \tag{3.13}$$

Using (3.9) and (3.13), inequality (3.8) becomes

$$\Sigma_{E_n}(n,x,F) - \Sigma(n,x,F) \leq c_{11} x e^{-x} e^{-\frac{(\beta-1)\alpha\Delta}{\alpha+\beta}} + \left(e^{K_0} + e^{-c_6 L_0} x\right) e^{-x+\Delta}. \tag{3.14}$$

We set $\alpha_1 := \min\{\frac{(\beta-1)\alpha}{\alpha+\beta}, c_6\}$ and $L_0 \geq 2\Delta/\alpha_1$, we have

$$\Sigma_{E_n}(n,x,F) - \Sigma(n,x,F) \leq c_{12} x e^{-x} e^{-\alpha_1\Delta} + \frac{e^{K_0+\Delta}}{x} x e^{-x}.$$

Since $\alpha_1 > 0$, for all $\varepsilon > 0$, there exists $\Delta_{\varepsilon,1} > 1$ such that $c_{12} e^{-\alpha_1 \Delta_{\varepsilon,1}} \leq \varepsilon/2$. For all $\Delta \geq \Delta_{\varepsilon,1}$ and $x \geq 2e^{K_0+\Delta}/\varepsilon$ we obtain finally

$$\Sigma_{E_n}(n,x,F) - \Sigma(n,x,F) \leq \varepsilon x e^{-x}, \tag{3.15}$$

which ends the proof. □

In what follows, we prove that on the set E_n, individuals who make the most important contribution to $\widetilde{\mu}_{n,\beta}(F)$ are the ones who are geographically close to $m^{(n)}$. For any $L \geq 1$, let

$$\widetilde{\mu}^L_{n,\beta}(F) := n^{3\beta/2} \sum_{u \in J^L_{\lambda,x-\Delta,K_0,L_0}(n)} e^{-\beta V(u)} F(H_n(u)) \quad \text{and} \quad \widetilde{W}^L_{n,\beta} := \widetilde{\mu}^L_{n,\beta}(1). \tag{3.16}$$

In the same way as above, for all measurable event E, we denote by

$$\Sigma^L_E(n,x,F) := \mathbf{E}\left[\exp\left(-e^{-\beta x}\widetilde{\mu}^L_{n,\beta}(F)\mathbf{1}_E\right)\right]. \tag{3.17}$$

We now prove the following lemma.

Lemma 3.3. *There exists $\alpha_2 > 0$ such that for all $\varepsilon > 0$ there exists $\Delta_{\varepsilon,2} \geq 1$ such that for all $\Delta \geq \Delta_{\varepsilon,2}$, $L = L_0 \geq 2\Delta/\alpha_2$, $x \geq 2e^{K_0+\Delta}/\varepsilon$ and $n \geq 1$, we have*

$$0 \leq \Sigma^L_{E_n}(n,x,F) - \Sigma_{E_n}(n,x,F) \leq \varepsilon x e^{-x}. \tag{3.18}$$

Proof. As $\widetilde{\mu}_{n,\beta}(F) \geq \widetilde{\mu}^L_{n,\beta}(F)$, we have

$$\Sigma^L_{E_n}(n,x,F) - \Sigma_{E_n}(n,x,F) = \mathbf{E}\left[\exp\left(-e^{-\beta x}\widetilde{\mu}^L_{n,\beta}(F)\right) - \exp\left(-e^{-\beta x}\widetilde{\mu}_{n,\beta}(F)\right); E_n\right] \geq 0,$$

We observe that, for all $0 \leq W_1 \leq W_2$,

$$\begin{aligned}\left(e^{-W_1} - e^{-W_2}\right)\mathbf{1}_{\{W_2-W_1\geq 0\}} \leq & |W_1 - W_2|\mathbf{1}_{\{0\leq W_2-W_1\leq\delta\}} + \mathbf{1}_{\{W_2-W_1>\delta\}} \\ \leq & \delta + \mathbf{1}_{\{W_2-W_1>\delta\}}.\end{aligned}$$

Applying this inequality with $\delta = e^{-\beta\Delta}$, $W_1 = \widetilde{\mu}^L_{n,\beta}(F)$ and $W_2 = \widetilde{\mu}_{n,\beta}(F)$ gives

$$\Sigma^L_{E_n}(n,x,F) - \Sigma_{E_n}(n,x,F) \leq e^{-\beta\Delta}\mathbf{P}\left(E_n\right) + \mathbf{P}\left(\widetilde{\mu}_{n,\beta}(F) - \widetilde{\mu}^L_{n,\beta}(F) \geq e^{\beta(x-\Delta)}; E_n\right).$$

As $E_n \subset \{M_n \leq a_n(x-\Delta)\}$, we have

$$\Sigma^L_{E_n}(n,x,F) - \Sigma_{E_n}(n,x,F) \leq e^{-\beta\Delta}\mathbf{P}\Big(M_n \leq a_n(x-\Delta)\Big) + \mathbf{P}_\diamond, \tag{3.19}$$

where

$$\mathbf{P}_\diamond := \mathbf{P}\left(n^{3\beta/2}\sum_{|u|=n}\mathbf{1}_{\left\{u\notin J^L_{\lambda,x-\Delta,K_0,L_0}(n)\right\}}e^{-\beta V(u)} \geq e^{\beta(x-\Delta)}; M_n \leq a_n(x-\Delta)\right).$$

From (3.19), on the one hand we recall that

$$\mathbf{P}\left(M_n \leq a_n(x-\Delta)\right) \leq c_9(x-\Delta)e^{-(x-\Delta)}.$$

On the other hand, by Proposition 4.6 of [21], there exists $\alpha_2 \in (0,\beta-1)$ such that for $L = L_0$,

$$\mathbf{P}_\diamond \leq e^{K_0+\Delta}e^{-x} + c_{13}xe^{-x}e^{-\alpha_2 L_0+\Delta}. \tag{3.20}$$

As a consequence,

$$\Sigma^L_{E_n}(n,x,F) - \Sigma_{E_n}(n,x,F) \leq c_{14}xe^{-x}\left(e^{-\alpha_2\Delta} + e^{-\alpha_2 L_0+\Delta}\right) + e^{K_0+\Delta}e^{-x}.$$

For any $\varepsilon > 0$, there exists $\Delta_{\varepsilon,2} > 0$ such that $c_{14}e^{-\alpha_2\Delta_{\varepsilon,2}} \leq \varepsilon/4$. We set $\Delta \geq \Delta_{\varepsilon,2}$, $L_0 \geq 2\Delta/\alpha_2$ and $x \geq 2e^{K_0+\Delta}/\varepsilon$, and obtain that

$$0 \leq \Sigma^{L_0}_{E_n}(n,x,F) - \Sigma_{E_n}(n,x,F) \leq \varepsilon xe^{-x} \tag{3.21}$$

which ends the proof. □

Recall that $m^{(n)}$ is uniformly chosen from the set of leftmost individuals at time n. For any $1 \leq k \leq n$, we use $m^{(n)}_k$ to represent the ancestor of $m^{(n)}$ at generation k. We prove now that the individuals who make significant contributions to $\widetilde{\mu}$ are the close relatives of $m^{(n)}$. We write, for $k \leq n$

$$\widehat{\mu}^L_{n,k,\beta}(F) := n^{3\beta/2}\sum_{u\in J^L_{\lambda,x-\Delta,K_0,L_0}(n)}\mathbf{1}_{\left\{u\geq m^{(n)}_k\right\}}e^{-\beta V(u)}F(H_n(u)), \tag{3.22}$$

and for E a measurable event

$$\widehat{\Sigma}^L_E(n,k,x,F) := \mathbf{E}\left[\exp\left(-e^{-\beta x}\widehat{\mu}^L_{n,k,\beta}(F)\mathbf{1}_E\right)\right]. \tag{3.23}$$

Lemma 3.4. *For all $\varepsilon > 0$ and $L_0 \geq 1$, there exist $K = K_{\varepsilon,L_0} > 0$, $B = B_{\varepsilon,L_0} \geq 1$ and $N = N_{\varepsilon,L_0} \geq 1$ such that for all $K_0 \geq K + L_0$, $n \geq N$ and $b \geq B$,*

$$0 \leq \widehat{\Sigma}^{L_0}_{E_n}(n,n-b,x,F) - \Sigma^{L_0}_{E_n}(n,x,F) \leq \varepsilon xe^{-x}. \tag{3.24}$$

Before giving the proof of Lemma 3.4, we state a result about the branching random walk under $\widehat{\mathbf{P}}$. Recall that $(\omega_k; k \geq 0)$ is the spine of $\mathbb{T}$. For any integer $b \geq 0$, we define

$$\xi_n(z, L, b) := \{\forall k \leq n - b, \min_{u \geq \omega_k; |u|=n} V(u) \geq a_n(z) + L\}. \tag{3.25}$$

Fact 3.5. *For any $\eta > 0$ and $L > 0$, there exists $K(\eta) > 0$, $B(L, \eta) \geq 1$ and $N(\eta) \geq 1$ such that for any $b \geq B(L, \eta) \geq 1$, $n \geq N(\eta)$ and $z \geq K \geq K(\eta) + L$,*

$$\widehat{\mathbf{P}}\left(\xi_n(z, L, b)^c, \omega_n \in J_{\lambda,z,K,L}(n)\right) \leq \eta(1 + L)^2(1 + z - K)n^{-3/2}. \tag{3.26}$$

Fact 3.5 is a slight refinement of Lemma 3.8 in [1], so we feel free to omit its proof. Using this result, we prove Lemma 3.4 as follows.

Proof. As $\widehat{\mu}^{L_0}_{n,k,\beta}(F) \leq \widetilde{\mu}^{L_0}_{n,\beta}(F)$, we have $\widehat{\Sigma}^{L_0}_{E_n}(n, n-b, x, F) - \Sigma^{L_0}_{E_n}(n, x, F) \geq 0$. We also observe that

$$\widehat{\Sigma}^{L_0}_{E_n}(n, n-b, x, F) - \Sigma^{L_0}_{E_n}(n, x, F) = \mathbf{E}\left[\exp\left(-e^{-\beta x}\widehat{\mu}^{L_0}_{n,n-b,\beta}(F)\right) - \exp\left(-e^{-\beta x}\widetilde{\mu}^{L_0}_{n,\beta}(F)\right); E_n\right].$$

By change of measures, we have

$$\begin{aligned}
&\widehat{\Sigma}^{L_0}_{E_n}(n, n-b, x, F) - \Sigma^{L_0}_{E_n}(n, x, F)\\
=&\widehat{\mathbf{E}}\left[\frac{\exp\left(-e^{-\beta x}\widehat{\mu}^{L_0}_{n,n-b,\beta}(F)\right) - \exp\left(-e^{-\beta x}\widetilde{\mu}^{L_0}_{n,\beta}(F)\right)}{W_n}; m^{(n)} \in J_{\lambda,x-\Delta,K_0,L_0}(n)\right]\\
=&\widehat{\mathbf{E}}\left[\frac{e^{V(\omega_n)}\mathbf{1}_{\left\{V(\omega_n)=M_n, \omega_n \in J_{\lambda,x-\Delta,K_0,L_0}(n)\right\}}}{\sum_{|u|=n}\mathbf{1}_{\{V(u)=M_n\}}}\left[\exp\left(-e^{-\beta x}\mathring{\mu}^{L_0}_{n,n-b,\beta}(F)\right) - \exp\left(-e^{-\beta x}\widetilde{\mu}^{L_0}_{n,\beta}(F)\right)\right]\right],
\end{aligned}$$

where

$$\mathring{\mu}^{L_0}_{n,k,\beta}(F) := n^{3\beta/2} \sum_{u \in J^{L_0}_{\lambda,x-\Delta,K_0,L_0}(n)} \mathbf{1}_{\{u \geq \omega_k\}} e^{-\beta V(u)} F(H_n(u)).$$

Observe that $0 \leq \exp\{-e^{-\beta x}\mathring{\mu}^{L_0}_{n,n-b,\beta}(F)\} - \exp\{-e^{-\beta x}\widetilde{\mu}^{L_0}_{n,\beta}(F)\} \leq 1$. Moreover, on the event $\xi_n(x - \Delta, L_0, b)$, we have $\mathring{\mu}^{L_0}_{n,n-b,\beta}(F) = \widetilde{\mu}^{L_0}_{n,\beta}(F)$. Therefore,

$$\begin{aligned}
\widehat{\Sigma}^{L_0}_{E_n}(n, n-b, x, F) - \Sigma^{L_0}_{E_n}(n, x, F) \leq&\widehat{\mathbf{E}}\left[\frac{e^{V(\omega_n)}\mathbf{1}_{\left\{V(\omega_n)=M_n, \omega_n \in J_{\lambda,x-\Delta,K_0,L_0}(n)\right\}}}{\sum_{|u|=n}\mathbf{1}_{\{V(u)=M_n\}}}; \xi^c_n(x - \Delta, L_0, b)\right]\\
\leq&\widehat{\mathbf{E}}\left[e^{V(\omega_n)}\mathbf{1}_{\left\{V(\omega_n)=M_n, \omega_n \in J_{\lambda,x-\Delta,K_0,L_0}(n)\right\}}; \xi^c_n(x - \Delta, L_0, b)\right]\\
\leq&n^{3/2}e^{-x+\Delta}\widehat{\mathbf{P}}\left(\xi^c_n(x - \Delta, L_0, b), \omega_n \in J_{\lambda,x-\Delta,K_0,L_0}(n)\right).
\end{aligned} \tag{3.27}$$

Applying Fact 3.5 to $\eta = \varepsilon e^{-\Delta}/(1 + L_0)^2$ shows that

$$\widehat{\Sigma}^{L_0}_{E_n}(n, n-b, x, F) - \Sigma^{L_0}_{E_n}(n, x, F) \leq \varepsilon x e^{-x}, \tag{3.28}$$

holds for $n \geq N(\eta)$, $b \geq B(L_0, \eta)$ and $x > K_0 \geq K(\eta) + L_0$, which ends the proof. □

We now study $\widehat{\Sigma}^{L_0}_{E_n}(n, n-b, x, F)$, to prove Proposition 3.1. We begin with the following estimate, which brings out the Brownian excursion.

Lemma 3.6. *For any $\varepsilon > 0$, set $\Delta = \Delta_\varepsilon := \Delta_{\varepsilon,1} \vee \Delta_{\varepsilon,2}$ and $L_0 = \frac{2\Delta}{\alpha_1 \wedge \alpha_2}$. Let $K = K_{\varepsilon,L_0} > 0$, $B = B_{\varepsilon,L_0} \geq 1$ and $N = N_{\varepsilon,L_0} \geq 1$ as in Lemma 3.4. For all $K_0 \geq K + L_0$, $n \geq N$ and $b \geq B$, there exists $n_\varepsilon \geq N$ such that for all $n \geq n_\varepsilon$ and $x \geq 2e^{K_0+\Delta}/\varepsilon$,*

$$\left|\widehat{\Sigma}^{L}_{E_n}(n, n-b, x, F) - \mathbf{E}\left[\exp\left(-e^{-\beta x}\widetilde{\mu}_{n,\beta}(1)F(e_1)\right)\right]\right| \leq \varepsilon x e^{-x}. \tag{3.29}$$

Proof. By change of measures, we have

$$\begin{aligned}\widehat{\Sigma}^{L_0}_{E_n}(n, n-b, x, F) &= \mathbf{E}\left[\exp\left(-e^{-\beta x}\widehat{\mu}^{L_0}_{n,n-b,\beta}(F)\mathbf{1}_{E_n}\right)\right] \\ &= \widehat{\mathbf{E}}\left[\frac{e^{V(\omega_n)}\mathbf{1}_{\left\{V(\omega_n)=M_n, \omega_n \in J_{\lambda,x-\Delta,K_0,L_0}(n)\right\}}}{\sum_{|u|=n}\mathbf{1}_{\{V(u)=M_n\}}}\exp\left(-e^{-\beta x}\mathring{\mu}^{L_0}_{n,n-b,\beta}(F)\right)\right] + \mathbf{P}(E_n^c).\end{aligned}$$

First, we are going to compare it with $\mathbf{E}\left[\exp\left(-e^{-\beta x}\widehat{\mu}^{L_0}_{n,n-b,\beta}(1)F(e_1)\mathbf{1}_{E_n}\right)\right]$, which equals to

$$\widehat{\mathbf{E}}\left[\frac{e^{V(\omega_n)}\mathbf{1}_{\left\{V(\omega_n)=M_n, \omega_n \in J_{\lambda,x-\Delta,K_0,L_0}(n)\right\}}}{\sum_{|u|=n}\mathbf{1}_{\{V(u)=M_n\}}}\exp\left(-e^{-\beta x}\mathring{\mu}^{L_0}_{n,n-b,\beta}F(e_1)\right)\right] + \mathbf{P}(E_n^c).$$

The strategy is to show that

$$\begin{aligned}\widehat{\Upsilon}^{L_0}_{E_n}(n, n-b, x, F) :=& \widehat{\Sigma}^{L_0}_{E_n}(n, n-b, x, F) - \mathbf{P}(E_n^c) \\ \text{and } \widehat{\Upsilon}^{L_0}_{E_n}(n, n-b, x, F(e_1)) :=& \mathbf{E}\left[\exp\left(-e^{-\beta x}\widehat{\mu}^{L_0}_{n,n-b,\beta}(1)F(e_1)\mathbf{1}_{E_n}\right)\right] - \mathbf{P}(E_n^c)\end{aligned}$$

are both close to the same quantity as $n \to \infty$. Then we compare

$$\mathbf{E}\left[\exp\left(-e^{-\beta x}\widehat{\mu}^{L_0}_{n,n-b,\beta}(1)F(e_1)\mathbf{1}_{E_n}\right)\right] \text{ with } \mathbf{E}\left[\exp\left(-e^{-\beta x}\widetilde{\mu}_{n,\beta}(1)F(e_1)\right)\right].$$

We set

$$Z := \frac{e^{V(\omega_n)}\mathbf{1}_{\left\{V(\omega_n)=M_n, \omega_n \in J_{\lambda,x-\Delta,K_0,L_0}(n)\right\}}}{\sum_{|u|=n}\mathbf{1}_{\{V(u)=M_n\}}} \quad \text{and} \quad Z_b := \frac{e^{V(\omega_n)}\mathbf{1}_{\left\{V(\omega_n)=M_n, \omega_n \in J_{\lambda,x-\Delta,K_0,L_0}(n)\right\}}}{\sum_{|u|=n}\mathbf{1}_{\{V(u)=M_n, u \geq \omega_{n-b}\}}}, \tag{3.30}$$

so that $\widehat{\Upsilon}^{L_0}_{E_n}(n, n-b, x, F) = \widehat{\mathbf{E}}\left[Ze^{-e^{-\beta x}\mathring{\mu}^{L_0}_{n,n-b,\beta}(F)}\right]$.

Under the measure $\widehat{\mathbf{P}}$, on the set $\xi_n(x-\Delta, L_0, b)$, we have $Z = Z_b$, thus

$$\begin{aligned}&\widehat{\Upsilon}^{L_0}_{E_n}(n, n-b, x, F) \\ &\quad = \widehat{\mathbf{E}}\left[Z_b e^{-e^{-\beta x}\mathring{\mu}^{L_0}_{n,n-b,\beta}(F)}\right] + \widehat{\mathbf{E}}\left[(Z - Z_b)\exp\left(-e^{-\beta x}\mathring{\mu}^{L_0}_{n,n-b,\beta}(F)\right); \xi_n^c(x-\Delta, L_0, b)\right].\end{aligned} \tag{3.31}$$

Recall that under $\widehat{\mathbf{P}}$, we have

$$\mathring{\mu}^{L_0}_{n,n-b,\beta}(F) = n^{3\beta/2} \sum_{u\in J^{L_0}_{\lambda,x-\Delta,K_0,L_0}(n)} \mathbf{1}_{\{u\geq\omega_{n-b}\}} e^{-\beta V(u)} F(H^n(u)).$$

For $n \gg b$ large and $|u| = n$, we define the path $\widetilde{H}^n_s(u) = \frac{V(u_{\lfloor ns\wedge(n-b)\rfloor})}{\sigma\sqrt{n}}, \forall s \in [0,1]$. Observe that, for all $u \geq \omega_{n-b}$, $\widetilde{H}^n(u)$ is identical to $\widetilde{H}^n(\omega_n)$. For all $\varepsilon_0 > 0$, let

$$X_{F,\varepsilon_0} := F\left(\widetilde{H}^n_s(\omega_n); s \in [0,1]\right) \vee \varepsilon_0. \tag{3.32}$$

We prove that $\widehat{\Upsilon}^{L_0}_{E_n}(n, n-b, x, F)$ is close to $\widehat{\mathbf{E}}\left[Z_b \exp\left(-e^{-\beta x}\mathring{\mu}^{L_0}_{n,n-b,\beta}(1) \times X_{F,\varepsilon_0}\right)\right]$. It follows from (3.31) that

$$\begin{aligned}\left|\widehat{\Upsilon}^{L_0}_{E_n}(n, n-b, x, F) - \widehat{\mathbf{E}}\left[Z_b \exp\left(-e^{-\beta x}\mathring{\mu}^{L_0}_{n,n-b,\beta}(1) \times X_{F,\varepsilon_0}\right)\right]\right| \\ \leq \left|\widehat{\mathbf{E}}\left[Z_b\left(\exp\left(-e^{-\beta x}\mathring{\mu}^{L_0}_{n,n-b,\beta}(F)\right) - \exp\left(-e^{-\beta x}\mathring{\mu}^{L_0}_{n,n-b,\beta}(1) \times X_{F,\varepsilon_0}\right)\right)\right]\right| \\ + \widehat{\mathbf{E}}\left[|Z - Z_b| \exp\left(-e^{-\beta x}\mathring{\mu}^{L_0}_{n,n-b,\beta}(F)\right); \xi^c_n(x-\Delta, L_0, b)\right].\end{aligned} \tag{3.33}$$

As $|Z - Z_b| \leq e^{V(\omega_n)}\mathbf{1}_{\left\{V(\omega_n)=M_n, \omega_n\in J_{\lambda,x-\Delta,K_0,L_0}(n)\right\}}$, by (3.27) and Fact 3.5 applied to $\eta = \frac{\varepsilon e^{\Delta}}{(1+L_0)^2}$, this quantity is bounded from above by

$$\left|\widehat{\mathbf{E}}\left[Z_b\left(\exp\left(-e^{-\beta x}\mathring{\mu}^{L_0}_{n,n-b,\beta}(F)\right) - \exp\left(-e^{-\beta x}\mathring{\mu}^{L_0}_{n,n-b,\beta}(1) \times X_{F,\varepsilon_0}\right)\right)\right]\right| + \varepsilon x e^{-x}. \tag{3.34}$$

It remains to bound the first term of (3.34). We compare $\mathring{\mu}^{L_0}_{n,n-b,\beta}(F)$ with $\mathring{\mu}^{L_0}_{n,n-b,\beta}(1) \times X_{F,\varepsilon_0}$, by comparing $F(\widetilde{H}_s(u); s \in [0,1])$ with $F(H^n_s(u); s \in [0,1])$.

Since F is uniformly continuous, for $\varepsilon_0 > 0$, there exists $\delta_0 > 0$ such that on the set $\{\max_{n-b\leq k\leq n} V(u_k) \leq V(u) + \delta_0\sqrt{n}\}$,

$$|F(\widetilde{H}_s(u); s \in [0,1]) - F(H^n_s(u); s \in [0,1])| \leq \varepsilon_0.$$

And on the complement set, $|F(\widetilde{H}_s(u); s \in [0,1]) - F(H^n_s(u); s \in [0,1])| \leq 1$ as $0 \leq F \leq 1$. One then observes that

$$\begin{aligned}&\left|\widehat{\mathbf{E}}\left[Z_b\left(\exp\left(-e^{-\beta x}\mathring{\mu}^{L_0}_{n,n-b,\beta}(F)\right) - \exp\left(-e^{-\beta x}\mathring{\mu}^{L_0}_{n,n-b,\beta}(1) \times X_{F,\varepsilon_0}\right)\right)\right]\right| \\ &\quad\leq \widehat{\mathbf{E}}\left[Z_b e^{-\beta x}\left|\mathring{\mu}^{L_0}_{n,n-b,\beta}(F) - \mathring{\mu}^{L_0}_{n,n-b,\beta}(1) \times X_{F,\varepsilon_0}\right|\right] \\ &\quad\leq 2\widehat{\mathbf{E}}\left[Z_b e^{-\beta x} n^{3\beta/2} \sum_{u\in J^{L_0}_{\lambda,x-\Delta,K_0,L_0}(n)} \mathbf{1}_{\{u\geq\omega_{n-b}\}} e^{-\beta V(u)}\left(\varepsilon_0 + \mathbf{1}_{\left\{\max_{n-b\leq k\leq n} V(u_k)\geq V(u)+\delta_0\sqrt{n}\right\}}\right)\right],\end{aligned}$$

which by the Markov property at time $n-b$ is equal to

$$\begin{aligned}2n^{3/2}e^{-x+(\beta-1)(L_0-\Delta)}\widehat{\mathbf{E}}\Big[&G_{L_0,b,\varepsilon_0}\left(V(\omega_{n-b}) - a_n(z+L_0)\right); \\ &\min_{0\leq k\leq n-b} V(\omega_k) \geq -z + K_0, \min_{\lambda n\leq k\leq n-b} V(\omega_k) \geq a_n(z+L_0)\Big],\end{aligned} \tag{3.35}$$

where $G_{L_0,b,\varepsilon_0}(t)$ is defined as

$$\widehat{\mathbf{E}}_t\left(\frac{e^{V(\omega_b)}\mathbf{1}_{\left\{V(\omega_b)=M_b;\min_{0\le k\le b}V(\omega_k)\ge 0,V(\omega_b)\le L_0\right\}}}{\sum_{|u|=b}\mathbf{1}_{\{V(u)=M_b\}}}\times \right. \tag{3.36}$$
$$\left.\sum_{|u|=b} e^{-\beta V(u)}\mathbf{1}_{\left\{\min_{0\le k\le b}V(u_k)\ge 0,V(u)\le 2L_0\right\}}\left(\varepsilon_0+\mathbf{1}_{\left\{\max_{0\le k\le b}V(u_k)\ge V(u)+\delta_0\sqrt{n}\right\}}\right)\right).$$

To bound $G_{L_0,b,\varepsilon_0}(t)$, we return to the probability $\mathbf{P}$ and observe that

$$\begin{aligned}&G_{L_0,b,\varepsilon_0}(t)\\ &\quad= e^{(1-\beta)t}\mathbf{E}\left[\sum_{|u|=b} e^{-\beta V(u)}\mathbf{1}_{\left\{\min_{0\le k\le b}V(u_k)\ge -t,V(u)\le 2L_0-t\right\}}\left(\varepsilon_0+\mathbf{1}_{\left\{\max_{0\le k\le b}V(u_k)\ge V(u)+\delta_0\sqrt{n}\right\}}\right);\right.\\ &\qquad\qquad\left.\min_{0\le k\le b}V(m_k^{(b)})\ge -t, V(m^{(b)})\le L_0-t\right]\end{aligned}$$

thus

$$\begin{aligned}&G_{L_0,b,\varepsilon_0}(t)\\ &\quad\le e^{(1-\beta)t}\mathbf{E}\left[\sum_{|u|=b} e^{-\beta V(u)}\mathbf{1}_{\left\{\min_{0\le k\le b}V(u_k)\ge -t,V(u)\le 2L_0-t\right\}}\left(\varepsilon_0+\mathbf{1}_{\left\{\max_{0\le k\le b}V(u_k)\ge V(u)+\delta_0\sqrt{n}\right\}}\right)\right].\end{aligned}$$

By Many-to-one lemma,

$$\begin{aligned}G_{L_0,b,\varepsilon_0}(t)&\le \mathbf{E}_t\left(e^{(1-\beta)S_b}(\varepsilon_0+\mathbf{1}_{\left\{\max_{0\le k\le b}S_k-S_b\ge\delta_0\sqrt{n}\right\}});S_b\le 2L_0,\underline{S}_b\ge 0\right)\\ &\le 2\mathbf{P}_t(S_b\le 2L_0)\le 2\mathbf{P}(2L_0-S_b\ge t).\end{aligned}$$

We observe that the function $t\mapsto\mathbf{P}(2L_0-S_b\ge t)$ is non-increasing, and

$$\int_0^{+\infty} t\mathbf{P}(2L_0-S_b\ge t)dt\le\frac{1}{2}\mathbf{E}((2L_0-S_b)^2)<+\infty.$$

Using the dominated convergence theorem, we have

$$\lim_{n\to+\infty}\int_{\mathbb{R}_+} G_{L_0,b,\varepsilon_0}(t)tdt\le\frac{1}{2}\mathbf{E}((2L_0-S_b)^2)\varepsilon_0. \tag{3.37}$$

Moreover, the function G_{L_0,b,ε_0} is Riemann-integrable. Therefore, using Lemma 2.5 proves that for all n sufficiently large,

$$\begin{aligned}&\widehat{\mathbf{E}}\left[Z_b e^{-\beta x}n^{3\beta/2}\sum_{u\in J^{L_0}_{\lambda,x-\Delta,K_0,L_0}(n)}\mathbf{1}_{\{u\ge\omega_{n-b}\}}e^{-\beta V(u)}\left(\varepsilon_0+\mathbf{1}_{\left\{\max_{n-b\le k\le n}V(u_k)\ge V(u)+\delta_0\sqrt{n}\right\}}\right)\right]\\ &\qquad\qquad\le c_{16}\varepsilon_0 x e^{-x+(\beta-1)(L_0-\Delta)},\end{aligned}$$

plugging this result in (3.35), for all n large enough, we have

$$\left|\widehat{E}\left[Z_b\left(\exp\left(-e^{-\beta x}\mathring{\mu}^{L_0}_{n,n-b,\beta}(F)\right)-\exp\left(-e^{-\beta x}\mathring{\mu}^{L_0}_{n,n-b,\beta}(1)\times X_{F,\varepsilon_0}\right)\right)\right]\right|\leq 2c_{16}\varepsilon_0 e^{(\beta-1)(L_0-\Delta)}xe^{-x}.$$

In view of (3.33) and (3.34), we can choose $\varepsilon_0>0$ sufficiently small so that

$$\left|\widehat{\Upsilon}^{L_0}_{E_n}(n,n-b,x,F)-\widehat{\mathbf{E}}\left[Z_b\exp\left(-e^{-\beta x}\mathring{\mu}^{L_0}_{n,n-b,\beta}(1)\times X_{F,\varepsilon_0}\right)\right]\right|\leq 2\varepsilon xe^{-x}. \tag{3.38}$$

In the similar way, we get that

$$\left|\widehat{\Upsilon}^{L_0}_{E_n}(n,n-b,x,F(e_1))-\widehat{\mathbf{E}}\left[Z_b\exp\left(-e^{-\beta x}\mathring{\mu}^{L_0}_{n,n-b,\beta}(1)\times\left(F(e_1)\vee\varepsilon_0\right)\right)\right]\right|\leq 2\varepsilon xe^{-x}. \tag{3.39}$$

We now consider the quantity $\frac{e^x}{x}\widehat{\mathbf{E}}\left[Z_b\exp\left(-e^{-\beta x}\mathring{\mu}^{L_0}_{n,n-b,\beta}(1)\times X_{F,\varepsilon_0}\right)\right]$ and show that it is close to $\frac{e^x}{x}\widehat{\mathbf{E}}\left[Z_b\exp\left(-e^{-\beta x}\mathring{\mu}^{L_0}_{n,n-b,\beta}(1)\times(F(e_1)\vee\varepsilon_0)\right)\right]$. However, we can not compare these quantities directly, thus we prove that

$$\begin{aligned}&\frac{e^x}{x}\widehat{\mathbf{E}}\left[Z_b\left(1-\exp\left(-e^{-\beta x}\mathring{\mu}^{L_0}_{n,n-b,\beta}(1)\times X_{F,\varepsilon_0}\right)\right)\right]\\&\qquad\sim_{n\to+\infty}\frac{e^x}{x}\widehat{\mathbf{E}}\left[Z_b\left(1-\exp\left(-e^{-\beta x}\mathring{\mu}^{L_0}_{n,n-b,\beta}(1)\times(F(e_1)\vee\varepsilon_0)\right)\right)\right].\end{aligned}$$

Applying the equation

$$1-e^{-\lambda W}=\int_{\mathbb{R}_+}\lambda e^{-\lambda t}\mathbf{1}_{\{W\geq t\}}dt$$

with $\lambda=X_{F,\varepsilon_0}$ leads to

$$\begin{aligned}\frac{e^x}{x}\widehat{\mathbf{E}}\left[Z_b\left(1-\exp\left(-e^{-\beta x}X_{F,\varepsilon_0}\mathring{\mu}^{L_0}_{n,n-b,\beta}(1)\right)\right)\right]&=\int_{\mathbb{R}_+}\frac{e^x}{x}\widehat{\mathbf{E}}\left[Z_bX_{F,\varepsilon_0}e^{-tX_{F,\varepsilon_0}};\mathring{\mu}^{L_0}_{n,n-b,\beta}(1)\geq te^{\beta x}\right]dt\\&=\int_{\mathbb{R}}\beta\frac{e^x}{x}\widehat{\mathbf{E}}\left[Z_be^{\beta y}X_{F,\varepsilon_0}e^{-e^{\beta y}X_{F,\varepsilon_0}};\mathring{\mu}^{L_0}_{n,n-b,\beta}(1)\geq e^{\beta(x+y)}\right]dy,\end{aligned} \tag{3.40}$$

by change of variables $t=e^{\beta y}$. Applying the Markov property at time $n-b$ implies that

$$\begin{aligned}&\frac{e^x}{x}\widehat{\mathbf{E}}\left[Z_be^{\beta y}X_{F,\varepsilon_0}e^{-e^{\beta y}X_{F,\varepsilon_0}};\mathring{\mu}^{L_0}_{n,n-b,\beta}(1)\geq e^{\beta(x+y)}\right]\\&\quad=n^{3/2}\frac{e^\Delta}{x}\widehat{\mathbf{E}}\Bigg[e^{\beta y}X_{F,\varepsilon_0}e^{-e^{\beta y}X_{F,\varepsilon_0}}f_{L_0,b}\left(V(\omega_{n-b})-a_n(z+L_0),y+\Delta\right);\\&\qquad\qquad\min_{0\leq k\leq n-b}V(\omega_k)\geq -z+K_0,\min_{\lambda n\leq k\leq n-b}\geq a_n(z+L_0)\Bigg],\end{aligned} \tag{3.41}$$

where $z = x - \Delta$ and

$$f_{L_0,b}(z,y) := \widehat{\mathbf{E}}_z\Bigg[\frac{e^{V(\omega_b)-L_0}\mathbf{1}_{\{V(\omega_b)=M_b\}}}{\sum_{|u|=b}\mathbf{1}_{\{V(u)=M_b\}}}\mathbf{1}_{\left\{\min_{0\le k\le b}V(\omega_k)\ge 0, V(\omega_b)\le L_0\right\}} \times \mathbf{1}_{\left\{\sum_{|u|=b}e^{-\beta V(u)}\mathbf{1}_{\left\{\min_{0\le k\le b}V(u_k)\ge 0, V(u)\le 2L_0\right\}}\ge e^{\beta(y-L_0)}\right\}}\Bigg]. \tag{3.42}$$

According to Lemma 2.23 in [21], $f_{L_0,b}$ is Riemann integrable and bounded by $\mathbf{P}(S_b \le L_0 - z)$. For all $y \in \mathbb{R}_+$ and $n \ge 10b$, we have

$$\frac{e^x}{x}\widehat{\mathbf{E}}\left[Z_b e^{\beta y}X_{F,\varepsilon_0}e^{-e^{\beta y}X_{F,\varepsilon_0}}; \mathring{\mu}^{L_0}_{n,n-b,\beta}(1) \ge e^{\beta(x+y)}\right] \le c_{17}e^{\beta y}e^{-\varepsilon_0 e^{\beta y}}, \tag{3.43}$$

which is integrable with respect to the Lebesgue measure. Using (3.41) then applying Lemma 2.5 show that for any $y \in \mathbb{R}$, as $n \to \infty$,

$$\begin{aligned}\lim_{n\to\infty}\frac{e^x}{x}\widehat{\mathbf{E}}&\left[Z_b e^{\beta y}X_{F,\varepsilon_0}e^{-e^{\beta y}X_{F,\varepsilon_0}}; \mathring{\mu}^{L_0}_{n,n-b,\beta}(1) \ge e^{\beta(x+y)}\right]\\ &= C_1\frac{e^{\Delta}V^-(z-K_0)}{x}\int_{\mathbb{R}_+} f_{L_0,b}(z,y+\Delta)V^+(z)dz\mathbf{E}\left[e^{\beta y}\left(F(e_1)\vee\varepsilon_0\right)e^{-e^{\beta y}(F(e_1)\vee\varepsilon_0)}\right].\end{aligned}$$

By the exact same argument, we have

$$\begin{aligned}C_1\frac{e^{\Delta}V^-(z-K_0)}{x}&\int_{\mathbb{R}_+} f_{L_0,b}(z,y+\Delta)V^+(z)dz\mathbf{E}\left[e^{\beta y}\left(F(e_1)\vee\varepsilon_0\right)e^{-e^{\beta y}(F(e_1)\vee\varepsilon_0)}\right]\\ &= \lim_{n\to\infty}\frac{e^x}{x}\widehat{\mathbf{E}}\left[Z_b e^{\beta y}\left(F(e_1)\vee\varepsilon_0\right)e^{-e^{\beta y}(F(e_1)\vee\varepsilon_0)}; \mathring{\mu}^{L_0}_{n,n-b,\beta}(1) \ge e^{\beta(x+y)}\right].\end{aligned}$$

Therefore, applying the dominated convergence theorem, (3.40) becomes

$$\begin{aligned}&\frac{e^x}{x}\widehat{\mathbf{E}}\left[Z_b\left(1-\exp\left(-e^{-\beta x}X_{F,\varepsilon_0}\mathring{\mu}^{L_0}_{n,n-b,\beta}(1)\right)\right)\right]\\ &\quad= \int_{\mathbb{R}}\beta\frac{e^x}{x}\widehat{\mathbf{E}}\left[Z_b e^{\beta y}X_{F,\varepsilon_0}e^{-e^{\beta y}X_{F,\varepsilon_0}}; \mathring{\mu}^{L_0}_{n,n-b,\beta}(1) \ge e^{\beta(x+y)}\right]dy\\ &\quad= \int_{\mathbb{R}}\beta\frac{e^x}{x}\widehat{\mathbf{E}}\left[Z_b e^{\beta y}\left(F(e_1)\vee\varepsilon_0\right)e^{-e^{\beta y}(F(e_1)\vee\varepsilon_0)}; \mathring{\mu}^{L_0}_{n,n-b,\beta}(1) \ge e^{\beta(x+y)}\right]dy + o_n(1)\\ &\quad= \frac{e^x}{x}\widehat{\mathbf{E}}\left[Z_b\left(1-\exp\left(-e^{-\beta x}\left(F(e_1)\vee\varepsilon_0\right)\mathring{\mu}^{L_0}_{n,n-b,\beta}(1)\right)\right)\right] + o_n(1).\end{aligned} \tag{3.44}$$

Thus, we obtain that for all n sufficiently large,

$$\begin{aligned}\Bigg|\frac{e^x}{x}\widehat{\mathbf{E}}&\left[Z_b\exp\left(-e^{-\beta x}\mathring{\mu}^{L_0}_{n,n-b,\beta}(1)\times X_{F,\varepsilon_0}\right)\right]\\ &- \frac{e^x}{x}\widehat{\mathbf{E}}\left[Z_b\exp\left(-e^{-\beta x}\mathring{\mu}^{L_0}_{n,n-b,\beta}(1)\times\left(F(e_1)\vee\varepsilon_0\right)\right)\right]\Bigg| \le \varepsilon.\end{aligned}$$

In view of (3.38) and (3.39), we check that for all n sufficiently large,

$$\left|\widehat{\Upsilon}^{L_0}_{E_n}(n,n-b,x,F)-\widehat{\Upsilon}^{L_0}_{E_n}(n,n-b,x,F(e_1))\right|\le 5\varepsilon x e^{-x}. \tag{3.45}$$

It hence follows that for all n sufficiently large,

$$\left|\widehat{\Sigma}^{L_0}_{E_n}(n,n-b,x,F)-\mathbf{E}\left[\exp\left(-e^{-\beta x}\widehat{\mu}^{L_0}_{n,n-b,\beta}(1)F(e_1)\mathbf{1}_{E_n}\right)\right]\right|\le 5\varepsilon x e^{-x}. \tag{3.46}$$

It remains to compare $\mathbf{E}\left[\exp\left(-e^{-\beta x}\widehat{\mu}^{L_0}_{n,n-b,\beta}(1)F(e_1)\mathbf{1}_{E_n}\right)\right]$ with $\mathbf{E}\left[\exp\left(-e^{-\beta x}\widetilde{\mu}_{n,\beta}F(e_1)\right)\right]$. Applying Lemmas 3.2, 3.3 and 3.4 to $\mathbf{E}\left(\exp\{-e^{-\beta x}\widetilde{\mu}_{n,\beta}(1)F(e_1)\}\right)$ implies that

$$0\le \mathbf{E}\left[\exp\left(-e^{-\beta x}\widehat{\mu}^{L_0}_{n,n-b,\beta}(1)F(e_1)1_{E_n}\right)\right]-\mathbf{E}\left[\exp\left(-e^{-\beta x}\widetilde{\mu}_{n,\beta}(1)F(e_1)\right)\right]\le 3\varepsilon x e^{-x}. \tag{3.47}$$

As a consequence, for all n sufficiently large,

$$\left|\widehat{\Sigma}^{L_0}_{E_n}(n,n-b,x,F)-\mathbf{E}\left[\exp\left(-e^{-\beta x}\widetilde{\mu}_{n,\beta}F(e_1)\right)\right]\right|\le 8\varepsilon x e^{-x}, \tag{3.48}$$

which completes the proof. □

We now prove Proposition 3.1.

Proof of Proposition 3.1. For any non-negative $F\in\mathcal{C}_b^u(\mathcal{D})$ and $\varepsilon>0$, we choose $\Delta=\Delta_\varepsilon:=\Delta_{\varepsilon,1}\vee\Delta_{\varepsilon,2}$ and $L=L_0=\frac{2\Delta}{\alpha_1\wedge\alpha_2}$. Set $K_0=K_{\varepsilon,L_0}+L_0$, $n\ge n_\varepsilon$ and $A\ge 2e^{K_0+\Delta}/\varepsilon$, we observe that

$$\begin{aligned}
&\left|\Sigma(n,x,F)-\mathbf{E}\left[\exp\left(-e^{-\beta x}\widetilde{\mu}_{n,\beta}(1)F(e_1)\right)\right]\right|\\
&\qquad\le|\Sigma(n,x,F)-\Sigma_{E_n}(n,x,F)|+\left|\Sigma^{L_0}_{E_n}(n,x,F)-\Sigma_{E_n}(n,x,F)\right|\\
&\qquad\quad+\left|\widehat{\Sigma}^{L_0}_{E_n}(n,n-b,x,F)-\Sigma^{L_0}_{E_n}(n,x,F)\right|\\
&\qquad\quad+\left|\widehat{\Sigma}^{L_0}_{E_n}(n,n-b,x,F)-\mathbf{E}\left(\exp\left(-e^{-\beta x}\widetilde{\mu}_{n,\beta}(1)F(e_1)\right)\right]\right|.
\end{aligned}$$

Using Lemmas 3.2, 3.3, 3.4 and 3.6, we have

$$\left|\Sigma(n,x,F)-\mathbf{E}\left[\exp\left(-e^{-\beta x}\widetilde{\mu}_{n,\beta}(1)F(e_1)\right)\right]\right|\le 4\varepsilon x e^{-x}, \tag{3.49}$$

where $\widetilde{\mu}_{n,\beta}(1)$ and $F(e_1)$ are independent. Recall that $\Sigma(n,x,F)=\mathbf{E}\left[\exp\left(-e^{-\beta x}\widetilde{\mu}_{n,\beta}(F)\right)\right]$. It hence follows that

$$\begin{aligned}
&\left|\frac{e^x}{x}\mathbf{E}\left[1-\exp\left(-\theta e^{-\beta x}\widetilde{\mu}_{n,\beta}(F)\right)\right]-\frac{e^x}{x}\mathbf{E}\left[1-\exp\left(-\theta e^{-\beta x}\widetilde{\mu}_{n,\beta}(1)F(e_1)\right)\right]\right|\\
&\qquad\le\frac{e^x}{x}\left|\Sigma(n,x,F)-\mathbf{E}\left[\exp\left(-e^{-\beta x}\widetilde{\mu}_{n,\beta}(1)F(e_1)\right)\right]\right|\le 4\varepsilon.
\end{aligned} \tag{3.50}$$

We replace θ by $\theta F(e_1)$, and then deduce from (3.3) that for all n sufficiently large,

$$\left|\frac{e^x}{x}\mathbf{E}\left[1-\exp\left(-\theta e^{-\beta x}\widetilde{\mu}_{n,\beta}(1)F(e_1)\right)\middle|\, e_1\right]-C_\beta\theta^{\frac{1}{\beta}}F(e_1)^{\frac{1}{\beta}}\right|\le\varepsilon.$$

In particular, for all n sufficiently large,

$$\left|\frac{e^x}{x}\mathbf{E}\left[1-\exp\left(-\theta e^{-\beta x}\widetilde{\mu}_{n,\beta}(1)F(e_1)\right)\right]-C_\beta\theta^{\frac{1}{\beta}}\mathbf{E}\left[F(e_1)^{\frac{1}{\beta}}\right]\right|\leq\varepsilon.$$

Going back to (3.50), we have

$$\left|\frac{e^x}{x}\mathbf{E}\left(1-\exp\{-\theta e^{-\beta x}\widetilde{\mu}_{n,\beta}(F)\}\right)-C_\beta\theta^{\frac{1}{\beta}}\mathbf{E}[F(e_1)^{\frac{1}{\beta}}]\right|\leq 5\varepsilon,$$

which completes the proof and gives Proposition 3.1. □

4 Proof of Theorem 1.1 and Corollary 1.3

4.1 Proof of Theorem 1.1

We apply the Laplace transform estimates obtained in the previous section to prove the main results of this article. We first study the convergence of the Laplace transform of $\widetilde{\mu}_{n,\beta}(F)$. We recall that $Z_n=\sum_{|u|=n}V(u)e^{-V(u)}$ is a martingale, and that $Z_\infty=\lim_{n\to+\infty}Z_n$ a.s.

Proposition 4.1. *Under (1.1), (1.3) and (1.4), for any $\alpha\geq 0$ and any non-negative $F\in\mathcal{C}_b^u(\mathcal{D})$,*

$$\begin{aligned}(4.1)\quad \lim_{l\to+\infty}\limsup_{n\to+\infty}\mathbf{E}\left[\mathbf{1}_{\{Z_l>0\}}e^{-\alpha Z_l-\widetilde{\mu}_{n+l,\beta}(F)}\right]&=\lim_{l\to+\infty}\liminf_{n\to+\infty}\mathbf{E}\left[\mathbf{1}_{\{Z_l>0\}}e^{-\alpha Z_l-\widetilde{\mu}_{n+l,\beta}(F)}\right]\\&=\mathbf{E}\left[\exp\left(-C_\beta Z_\infty\mathbf{E}\left(F(e_1)^{\frac{1}{\beta}}\right)\right)e^{-\alpha Z_\infty}\mathbf{1}_{\{Z_\infty>0\}}\right].\end{aligned}$$

In particular, conditionally on the survival event S, we have

$$(4.2)\qquad \lim_{n\to+\infty}\mathbf{E}\left[e^{-\widetilde{\mu}_{n,\beta}(F)}\middle|S\right]=\mathbf{E}\left[\exp\left(-C_\beta Z_\infty\mathbf{E}\left(F(e_1)^{\frac{1}{\beta}}\right)\right)\middle|S\right].$$

Remark 4.2. *Theorem 1.1 is a consequence of (4.2). However (4.1) enlightens the appearance of Z_∞.*

Proof. Note that (4.2) is a direct consequence of (4.1) as $S=\{Z_\infty>0\}$. We observe that

$$(4.3)\qquad \widetilde{\mu}_{n+l,\beta}(F)=\left(\frac{n+l}{n}\right)^{\frac{3\beta}{2}}\sum_{|u|=l}e^{-\beta V(u)}n^{\frac{3\beta}{2}}\sum_{\substack{|v|=n+l\\v\geq u}}e^{-\beta(V(v)-V(u))}F(H^{n+l}(v)).$$

For $|u|=l$, $v\geq u$ with $|v|=n+l$ and $t\in[0,1]$, we write $H^{(n),u}(v)_t:=\frac{V(v_{l+\lfloor nt\rfloor})-V(u)}{\sqrt{\sigma^2 n}}$ and

$$\widetilde{\mu}^{(u)}_{n,\beta}(F):=n^{3\beta/2}\sum_{\substack{|v|=n+l\\v\geq u}}e^{-\beta(V(v)-V(u))}F\left(H^{(n),u}(v)\right),$$

By uniform continuity of F, we have

$$\widetilde{\mu}_{n+l,\beta}(F) = \sum_{|u|=l} e^{-\beta V(u)} \widetilde{\mu}^{(u)}_{n,\beta}(F)(1+o(1)), \tag{4.4}$$

where $o(1) \to 0$ as $\frac{l}{n} \to 0$. Therefore (4.1) is a consequence of

$$\begin{aligned} \lim_{l\to+\infty} \limsup_{n\to+\infty} \mathbf{E}\left[e^{-\alpha Z_l} \mathbf{1}_{\{Z_l>0\}} e^{-\sum_{|u|=l} e^{-\beta V(u)} \widetilde{\mu}^{(u)}_{n,\beta}(F)}\right] \\ = \lim_{l\to+\infty} \liminf_{n\to+\infty} \mathbf{E}\left[e^{-\alpha Z_l} \mathbf{1}_{\{Z_l>0\}} e^{-\sum_{|u|=l} e^{-\beta V(u)} \widetilde{\mu}^{(u)}_{n,\beta}(F)}\right] \\ = \mathbf{E}\left[\exp\left(-C_\beta \mathbf{E}\left(F(e_1)^{\frac{1}{\beta}}\right) Z_\infty\right) e^{-\alpha Z_\infty} \mathbf{1}_{\{Z_\infty>0\}}\right]. \end{aligned} \tag{4.5}$$

Applying the Markov property at time l, we have

$$\mathbf{E}\left[e^{-\alpha Z_l} \mathbf{1}_{\{Z_l>0\}} e^{-\sum_{|u|=l} e^{-\beta V(u)} \widetilde{\mu}^{(u)}_{n,\beta}(F)}\right] = \mathbf{E}\left[e^{-\alpha Z_l} \mathbf{1}_{\{Z_l>0\}} \prod_{|u|=l} \Psi(V(u))\right],$$

where $\Psi : x \mapsto \mathbf{E}\left[e^{-e^{-\beta x} \widetilde{\mu}_{n,\beta}(F)}\right]$. For any $1 \le l \le n$, we set $\Xi_{n,l} := \left\{|u| = l : \frac{\log l}{3} \le V(u) \le \log n\right\}$, and we prove that $\prod_{|u|=l} \Psi(V(u)) \approx \prod_{u\in\Xi_{n,l}} \Psi(V(u))$ with high probability. Note that

$$\lim_{n\to+\infty} \sum_{u\in\Xi_{n,l}} V(u) e^{-V(u)} = Z_l a.s. \text{for } l \text{ large enough}, \tag{4.6}$$

as $\liminf_{l\to+\infty} \frac{M_l}{\log l} = \frac{1}{2} > \frac{1}{3}$ (see [14]).

We first observe that

$$\mathbf{E}\left[e^{-\alpha Z_l} \mathbf{1}_{\{Z_l>0\}} e^{-\sum_{|u|=l} e^{-\beta V(u)} \widetilde{\mu}^{(u)}_{n,\beta}(F)}\right] \le \mathbf{E}\left[e^{-\alpha Z_l} \mathbf{1}_{\{Z_l>0\}} \prod_{u\in\Xi_{n,l}} \Psi(V(u))\right].$$

By Proposition 3.1, for any $\varepsilon > 0$, there exist L, N such that for any $n \ge N$, $l \ge L$ and $u \in \Xi_{n,l}$,

$$\left|\Psi(V(u)) - 1 + \psi_\beta V(u) e^{-V(u)}\right| \le \varepsilon V(u) e^{-V(u)} n, \tag{4.7}$$

where $\psi_\beta = C_\beta \mathbf{E}\left[F(e_1)^{\frac{1}{\beta}}\right]$. This yields

$$\begin{aligned} \mathbf{E}\left[e^{-\alpha Z_l} \mathbf{1}_{\{Z_l>0\}} e^{-\sum_{|u|=l} e^{-\beta V(u)} \widetilde{\mu}^{(u)}_{n,\beta}(F)}\right] &\le \mathbf{E}\left[e^{-\alpha Z_l} \mathbf{1}_{\{Z_l>0\}} \prod_{u\in\Xi_{n,l}} \left(1 - (\psi_\beta - \varepsilon) V(u) e^{-V(u)}\right)\right] \\ &\le \mathbf{E}\left[e^{-\alpha Z_l} \mathbf{1}_{\{Z_l>0\}} e^{-\sum_{u\in\Xi_{n,l}} V(u) e^{-V(u)} (\psi_\beta - \varepsilon)}\right], \end{aligned}$$

where the last inequality follows from the fact that for any $x \ge 0$, $1 - x \le e^{-x}$. By (4.6) and the convergence of the derivative martingale, we have

$$\limsup_{n\to+\infty} \mathbf{E}\left[e^{-\alpha Z_l} \mathbf{1}_{\{Z_l>0\}} e^{-\sum_{|u|=l} e^{-\beta V(u)} \widetilde{\mu}^{(u)}_{n,\beta}(F)}\right] \le \mathbf{E}\left[e^{-(\alpha+\psi_\beta-\varepsilon) Z_l} \mathbf{1}_{\{Z_l>0\}}\right].$$

Letting $l \to +\infty$ then $\varepsilon \to 0$, it leads to

$$\limsup_{n\to+\infty} \mathbf{E}\left[e^{-\alpha Z_l}\mathbf{1}_{\{Z_l>0\}}e^{-\sum_{|u|=l} e^{-\beta V(u)}\widetilde{\mu}^{(u)}_{n,\beta}(F)}\right] \leq \mathbf{E}\left[e^{-\left(\alpha+C_\beta \mathbf{E}\left(F(e_1)^{\frac{1}{\beta}}\right)Z_\infty\right)}\mathbf{1}_{\{Z_\infty>0\}}\right]. \tag{4.8}$$

The lower bound follows from similar arguments. Applying once again the Markov property at time l,

$$\mathbf{E}\left[e^{-\alpha Z_l}\mathbf{1}_{\{Z_l>0\}}e^{-\sum_{|u|=l} e^{-\beta V(u)}\widetilde{\mu}^{(u)}_{n,\beta}(F)}\right] \geq \mathbf{E}\left[e^{-\alpha Z_l}\mathbf{1}_{\{Z_l>0,3M_l\geq \log l\}}\prod_{|u|=l}\Psi(V(u))\right].$$

Therefore,

$$\begin{aligned}
&\mathbf{E}\left[e^{-\alpha Z_l}\mathbf{1}_{\{Z_l>0\}}e^{-\sum_{|u|=l} e^{-\beta V(u)}\widetilde{\mu}^{(u)}_{n,\beta}(F)}\right] \\
&\qquad \geq \mathbf{E}\left[e^{-\alpha Z_l}\mathbf{1}_{\{Z_l>0,3M_l\geq \log l\}}\prod_{u\in\Xi_{n,l}}\Psi(V(u))\times\prod_{|u|=l,u\notin\Xi_{n,l}}\Psi(V(u))\right] \\
&\qquad = \mathbf{E}\left[e^{-\alpha Z_l}\mathbf{1}_{\{Z_l>0,3M_l\geq \log l\}}\prod_{u\in\Xi_{n,l}}\Psi(V(u))\times\prod_{|u|=l,V(u)\geq \log n}\Psi(V(u))\right].
\end{aligned}$$

For $n \geq 1$ large enough, by [21, Proposition 2.1], there exists $c > 0$ such that

$$\forall x \geq \log n, 1-\Psi(x) \leq cxe^{-x}. \tag{4.9}$$

Consequently, (4.7) yields

$$\begin{aligned}
&\mathbf{E}\left[e^{-\alpha Z_l}\mathbf{1}_{\{Z_l>0\}}e^{-\sum_{|u|=l} e^{-\beta V(u)}\widetilde{\mu}^{(u)}_{n,\beta}(F)}\right] \\
\geq&\mathbf{E}\left[e^{-\alpha Z_l}\mathbf{1}_{\{Z_l>0,3M_l\geq \log l\}}\prod_{|u|=l}\left(1-\left[\mathbf{1}_{\{V(u)\leq \log n\}}\psi_\beta + c\mathbf{1}_{\{V(u)\geq \log n\}}+\varepsilon\right]V(u)e^{-V(u)}\right)\right] \\
\geq&\mathbf{E}\left[e^{-\alpha Z_l}\mathbf{1}_{\{Z_l>0,3M_l\geq \log l\}}e^{-(1+\varepsilon)\sum_{|u|=l}\left(\mathbf{1}_{\{V(u)\leq \log n\}}\psi_\beta + c\mathbf{1}_{\{V(u)\geq \log n\}}+\varepsilon\right)V(u)e^{-V(u)}}\right],
\end{aligned}$$

for $l \geq 1$ large enough, as for any $x > 0$ small enough, $1 - x \geq e^{-(1+\varepsilon)x}$. Letting $n \to +\infty$, we have

$$\liminf_{n\to+\infty} E\left[e^{-\alpha Z_l}\mathbf{1}_{\{Z_l>0\}}e^{-\sum_{|u|=l} e^{-\beta V(u)}\widetilde{\mu}^{(u)}_{n,\beta}(F)}\right] \geq \mathbf{E}\left[\mathbf{1}_{\{Z_l>0,3M_l\geq \log l\}}e^{-\left(\alpha+(1+\varepsilon)(\psi_\beta+\varepsilon)\right)}\right].$$

Finally, using the fact that $S = \{Z_\infty > 0\}$ and that $\liminf_{l\to+\infty} M_l/\log l > 1/3$, we obtain, letting $l \to +\infty$ then $\varepsilon \to 0$

$$\liminf_{l\to\infty}\liminf_{n\to\infty} \mathbf{E}\left[e^{-\sum_{|u|=l} e^{-\beta V(u)}\widetilde{\mu}^{(u)}_{n,\beta}(F)}e^{-\alpha Z_l}\mathbf{1}_{\{Z_l>0\}}\right] \geq \mathbf{E}\left[e^{-\left[\alpha+C_\beta \mathbf{E}\left(F(e_1)^{\frac{1}{\beta}}\right)\right]Z_\infty}\mathbf{1}_{\{Z_\infty>0\}}\right]$$

which ends the proof of (4.5). □

We now prove that for any $F \in \mathcal{C}_b^u(\mathcal{D})$, we have

$$\mu_{n,\beta}(F) \underset{n\to+\infty}{\Longrightarrow} \sum_{k\in\mathbb{N}} p_k F(e_k), \tag{4.10}$$

where (e_k) is a sequence of i.i.d. normalised Brownian excursions, and $(p_k, k \in \mathbb{N})$ follows an independent Poisson-Dirichlet distribution with parameter $(\frac{1}{\beta}, 0)$.

Proof of Theorem 1.1. We recall that $\mu_{n,\beta}$ is defined on S by $\mu_{n,\beta}(F) = \frac{\widetilde{\mu}_{n,\beta}(F)}{\widetilde{\mu}_{n,\beta}(1)}$, for $F \in \mathcal{C}_b(\mathcal{D})$. To prove the convergence in law of $\mu_{n,\beta}$, we start by identifying the limit law of $\widetilde{\mu}_{n,\beta}$.

Let $(\Delta_k, k \in \mathbb{N})$ be a Poisson point process on $\mathbb{R}$ with intensity $e^x dx$ and $(e_k, k \in \mathbb{N})$ be an independent sequence of i.i.d. normalised Brownian excursions, independent from the branching random walk. We introduce a random measure $\widetilde{\mu}_{\infty,\beta}$ on $\mathcal{D}$ by

$$\widetilde{\mu}_{\infty,\beta} = Z_\infty^\beta \sum_{k=1}^{+\infty} e^{-\beta(\Delta_k - c_\star(\beta))} \delta_{e_k},$$

where $c_\star(\beta) := \log \frac{C_\beta}{-\int_{\mathbb{R}} \left(e^{-e^{-\beta u}} - 1\right) e^u du}$. We first prove that for any non-negative $F \in \mathcal{C}_b^u(\mathcal{D})$,

$$\widetilde{\mu}_{n,\beta}(F) \underset{n\to+\infty}{\Longrightarrow} \widetilde{\mu}_{\infty,\beta}(F). \tag{4.11}$$

We compute the Laplace transform of $\widetilde{\mu}_{\infty,\beta}(F)$. As $S = \{Z_\infty > 0\}$, for any $\theta > 0$, we have

$$\begin{aligned}
\mathbf{E}\left[\exp\left(-\theta \widetilde{\mu}_{\infty,\beta}(F)\right) \mathbf{1}_{\{S\}}\right] &= \mathbf{E}\left[\exp\left(-\theta Z_\infty^\beta \sum_{k=1}^{+\infty} e^{-\beta[\Delta_k - c_\star(\beta)]} F(e_k)\right) \mathbf{1}_{\{Z_\infty > 0\}}\right] \\
&= \mathbf{E}\left[\exp\left(-\sum_{k=1}^{+\infty} \phi\left(\theta e^{-\beta[\Delta_k - \log Z_\infty - c_\star(\beta)]}\right)\right) \mathbf{1}_{\{Z_\infty > 0\}}\right],
\end{aligned}$$

where $\phi : x \mapsto \log \mathbf{E}\left[\exp\left(-uF(e_1)\right)\right]$. By Campbell's formula,

$$\mathbf{E}\left[\exp\left(-\theta \widetilde{\mu}_{\infty,\beta}(F)\right) \mathbf{1}_{\{S\}}\right] = \mathbf{E}\left[\exp\left(\int_{\mathbb{R}} e^{\phi\left(\theta e^{-\beta[x - \log Z_\infty - c_\star(\beta)]}\right) + x - 1} dx\right) \mathbf{1}_{\{Z_\infty > 0\}}\right].$$

As $\phi\left(\theta e^{-\beta[x - \log Z_\infty - c_\star(\beta)]}\right) = \log \mathbf{E}\left[\exp\left(-\theta e^{-\beta[x - \log Z_\infty - c_\star(\beta)]} F(e_1)\right) \middle| Z_\infty\right]$, it yields

$$\begin{aligned}
&\mathbf{E}\left[\exp\left(-\theta \widetilde{\mu}_{\infty,\beta}(F)\right) \mathbf{1}_{\{S\}}\right] \\
&= \mathbf{E}\left[\exp\left(\int_{\mathbb{R}} \mathbf{E}\left[\exp\left(-\theta e^{-\beta[x - \log Z_\infty - c_\star(\beta)]} F(e_1)\right) \middle| Z_\infty\right] - 1\right) e^x dx \mathbf{1}_{\{Z_\infty > 0\}}\right] \\
&= \mathbf{E}\left[\exp\left(\mathbf{E}\left[\int_{\mathbb{R}} \left(\exp\left(-\theta e^{-\beta[x - \log Z_\infty - c_\star(\beta)]} F(e_1)\right) - 1\right) e^x dx \middle| Z_\infty\right]\right) \mathbf{1}_{\{Z_\infty > 0\}}\right].
\end{aligned}$$

By change of variables $u = x - \log Z_\infty - c_\star(\beta) - \frac{1}{\beta}\left[\log\theta + \log F(e_1)\right]$, we obtain that

$$\begin{aligned}
&\mathbf{E}\left[\int_{\mathbb{R}} \left(\exp\left(-\theta e^{-\beta[x - \log Z_\infty - c_\star(\beta)]} F(e_1)\right) - 1\right) e^x dx \middle| Z_\infty\right] \\
=&\mathbf{E}\left[\int_{\mathbb{R}} \left(e^{-e^{-\beta u}} - 1\right) e^{u + \log Z_\infty + c_\star(\beta) + \frac{1}{\beta}[\log\theta + \log F(e_1)]} du \middle| Z_\infty\right] \\
=&e^{c_\star(\beta)} Z_\infty \mathbf{E}\left[(\theta F(e_1))^{\frac{1}{\beta}}\right] \int_{\mathbb{R}} \left(e^{-e^{-\beta u}} - 1\right) e^u du = -C_\beta Z_\infty \mathbf{E}\left((F(e_1)\theta)^{\frac{1}{\beta}}\right),
\end{aligned}$$

since $c_\star(\beta) = \log \frac{C_\beta}{-\int_{\mathbb{R}}\left(e^{-e^{-\beta u}}-1\right)e^u du}$. We thus end up with

$$\mathbf{E}\left[\exp\left(-\theta\widetilde{\mu}_{\infty,\beta}(F)\right)\mathbf{1}_{\{S\}}\right] = \mathbf{E}\left[\exp\left\{-C_\beta Z_\infty \mathbf{E}\left((F(e_1)\theta)^{\frac{1}{\beta}}\right)\right\}\mathbf{1}_{\{Z_\infty>0\}}\right].$$

Consequently, by Proposition 4.1, for any $\theta > 0$ we have

$$\lim_{n\to+\infty} \mathbf{E}\left[\mathbf{1}_{\{Z_\infty>0\}}e^{-\theta\widetilde{\mu}_{n,\beta}(F)}\right] = \mathbf{E}\left[\mathbf{1}_{\{Z_\infty>0\}}e^{-\theta\widetilde{\mu}_{\infty,\beta}(F)}\right],$$

which concludes (4.11) by Lévy's theorem.

Furthermore, for any $F \in \mathcal{C}_b^u(\mathcal{D})$ and $\theta_1, \theta_2, \theta_3 \in (0, +\infty)$, we have

$$\lim_{n\to+\infty} \mathbf{E}\left[\mathbf{1}_{\{Z_\infty>0\}}e^{-\theta\widetilde{\mu}_{n,\beta}(\theta_1 F_+ + \theta_2 F_- + \theta_3)}\right] = \mathbf{E}\left[\mathbf{1}_{\{Z_\infty>0\}}e^{-\theta\widetilde{\mu}_{\infty,\beta}(\theta_1 F_+ + \theta_2 F_- + \theta_3)}\right],$$

yielding

$$(\widetilde{\mu}_{n,\beta}(F_+), \widetilde{\mu}_{n,\beta}(F_-), \widetilde{\mu}_{n,\beta}(1)) \underset{n\to+\infty}{\Longrightarrow} (\widetilde{\mu}_{\infty,\beta}(F_+), \widetilde{\mu}_{\infty,\beta}(F_-), \widetilde{\mu}_{\infty,\beta}(1)).$$

Using the fact that $\widetilde{\mu}_{\infty,\beta}(1) > 0$ a.s. on S, we have

$$\mu_{n,\beta}(F) = \frac{\widetilde{\mu}_{n,\beta}(F_+) - \widetilde{\mu}_{n,\beta}(F_-)}{\widetilde{\mu}_{n,\beta}(1)} \underset{n\to+\infty}{\Longrightarrow} \sum_{k\geq 0} \frac{e^{-\beta\Delta_k}}{\sum_{j=0}^{+\infty} e^{-\beta\Delta_j}} F(e_k) \quad \text{on} \quad S.$$

Remark 4.3. *We obtain in a similar way the joint convergence of $(\mu_{n,\beta}(F_1), \cdots \mu_{n,\beta}(F_k))$, for any $(F_1, \ldots, F_k) \in \mathcal{C}_b^u(\mathcal{D})^k$.*

Using[24, Proposition 10], for a Poisson point process $(\Delta_k, k \geq 0)$ of intensity e^x, we have

$$\left(\frac{e^{-\beta\Delta_k}}{\sum_{j=0}^{+\infty} e^{-\beta\Delta_j}}, k \geq 0\right) \overset{(d)}{=} (p_k, k \geq 0),$$

where (p_k) is a process of Poisson-Dirichlet distribution with parameters $(\frac{1}{\beta}, 0)$, ending the proof. □

Using Theorem 1.1, we can compute a variety of quantities related to the trajectory of individuals chosen according to the Gibbs measure. In this theorem, we proved the convergence of $\mu_{n,\beta}(F)$ for F a uniformly continuous bounded function on $\mathcal{D}$. Unfortunately, this is not enough to conclude to the convergence of the random measure $\mu_{n,\beta}$. However, the convergence can be proved for several restrictions of this measure.

We first prove the finite dimensional distributions of $\mu_{n,\beta}$ converge, as $n \to +\infty$ to the finite dimensional distributions of $\mu_{\infty,\beta}$. For $0 \leq t_1 < t_2 < \cdots < t_k \leq 1$, we write

$$\Pi^{t_1,\cdots,t_k} : \begin{array}{ccc} \mathcal{C}(\mathcal{D}) & \longrightarrow & \mathbb{R}^d \\ f & \longmapsto & (f(t_1), \ldots, f(t_k)). \end{array}$$

Corollary 4.4. *Let $0 < t_1 < \cdots < t_k < 1$ and $n \in \mathbb{N} \cup \{+\infty\}$, we define on S the random measure $\mu_{n,\beta}^{FD}$ as the image measure of $\mu_{n,\beta}$ by $\Pi^{t_1,\ldots t_k}$. We have $\mu_{n,\beta}^{FD} \underset{n\to+\infty}{\Longrightarrow} \mu_{\infty,\beta}^{FD}$.*

Remark 4.5. *Note that a direct application of Corollary 4.4 proves Corollary 1.3.*

Proof. As $\mathbb{R}^d$ is a locally compact, second-countable and Hausdorff, by [18, Theorem 14.16] it is enough to prove that for any continuous function f with compact support on $\mathbb{R}^d$, we have

$$\mu_{n,\beta}^{FD}(f) \underset{n\to+\infty}{\Longrightarrow} \mu_{\infty,\beta}^{FD}(f).$$

As $f \circ \Pi^{t_1,\dots,t_k}$ is uniformly continuous, Theorem 1.1 leads to

$$\mu_{n,\beta}(f \circ \Pi^{t_1,\dots,t_k}) \underset{n\to+\infty}{\Longrightarrow} \mu_{\infty,\beta}(f \circ \Pi^{t_1,\dots,t_k}),$$

which concludes the proof. □

We also observe that the (non-random) measure on $\mathcal{D}^k$ defined by integrating $\mu_{n,\beta}^{\otimes k}$ with respect to the branching random walk law converge weakly too.

Corollary 4.6. *Let $k \geq 1$ and $n \in \mathbb{N} \cup \{+\infty\}$, we set $M_{n,\beta}^k = \frac{1}{\mathbf{P}(S)}\mathbf{E}\left[\mathbf{1}_{\{S\}}\mu_{n,\beta}^{\otimes k}\right]$. We have*

$$M_{n,\beta}^k \underset{n\to+\infty}{\Longrightarrow} M_{\infty,\beta}^k,$$

with $M_{\infty,\beta}^k(f_1,\dots f_k) = \mathbf{E}\left[\prod_{j=1}^k \left(\sum_{n\geq 1} p_n f_j(e_n)\right)\right]$.

Proof. By Prokhorov theorem, to prove this convergence, we prove the convergence of finite-dimensional distributions and the tension of the sequence of measures $(M_{n,\beta}^k, n \geq 1)$. Let $h \in \mathbb{N}$, we introduce $0 \leq t_1 < t_2 < \ldots < t_h \leq 1$ and $F : (\mathbb{R}^h)^k \to \mathbb{R}_+$ a uniformly continuous bounded function, we observe that

$$F \circ (\Pi^{t_1,\dots,t_h})^{\otimes k} : \begin{array}{ccc} \mathcal{D}^k & \longrightarrow & \mathbb{R}_+ \\ (f^1,\dots f^k) & \longmapsto & F\left(\left(f^1(t_1),\dots f^1(t_h)\right),\dots,\left(f^k(t_1),\dots f^k(t_h)\right)\right) \end{array}$$

is uniformly continuous. By Theorem 1.1, we have

$$\lim_{n\to+\infty} M_{n,\beta}^k\left(F \circ (\Pi^{t_1,\dots,t_h})^{\otimes k}\right) = M_{\infty,\beta}^k\left(F \circ (\Pi^{t_1,\dots,t_h})^{\otimes k}\right),$$

proving the convergence in finite-dimensional distributions convergence.

Let $\delta > 0$ and $f^1,\dots f^k$ be continuous functions on $[0,1]$, we set

$$\omega_\delta(f^1,\dots,f^k) = \max_{j\leq k} \sup_{|t-s|<\delta} |f^j(t) - f^j(s)| \wedge 1.$$

We observe that for any $\delta > 0$, ω_δ is a 2-Lipschitz function, thus uniformly continuous. By Theorem 1.1, we have

$$\lim_{n\to+\infty} M_{n,\beta}^k(\omega_\delta) = M_{\infty,\beta}^k(\omega_\delta).$$

Moreover by continuity of the Brownian excursions, we have $\lim_{\delta\to 0} M_{\infty,\beta}^k(\omega_\delta) = 0$, yielding

$$\lim_{\delta\to 0} \limsup_{n\to+\infty} M_{n,\beta}^k(\omega_\delta) = 0.$$

As $M_{n,\beta}^k\left(\left\{f_0^1 = 0,\dots,f_0^k = 0\right\}\right) = 1$, we conclude that $(M_{n,\beta}^k, n \in \mathbb{N})$ is tensed, ending the proof. □

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