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Convergence of the consistent maximal displacement of the branching random walk

Bastien Mallein*

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Abstract

We consider a supercritical branching random walk on $\mathbb{R}$. The consistent maximal displacement is the smallest of the distances between the trajectories of individuals at the nth generation and the boundary of the process. It has been proved by Fang and Zeitouni [7] and by Faraud, Hu and Shi [8] that the consistent maximal displacement grows almost surely at rate $\lambda^* n^{1/3}$ for an explicit λ^*. We obtain here a necessary and sufficient condition for this asymptotic behaviour to hold.

1 Introduction

A branching random walk on $\mathbb{R}$ is a process defined as follows. It starts with one individual located at 0 at time 0. Its children are positioned on $\mathbb{R}$ according to the law $\mathcal{L}$ of a point process, and form the first generation of the process. Then for any $n \in \mathbb{N}$, each individual in the n-th generation makes children around its current position according to an independent point process with law $\mathcal{L}$. We write $\mathbf{T}$ for the genealogical tree of the population. For any $u \in \mathbf{T}$ we denote by $V(u)$ the position of the individual u and by $|u|$ the generation to which u belongs. The random marked tree $(\mathbf{T}, V)$ is the branching random walk with reproduction law $\mathcal{L}$. We assume the Galton-Watson tree $\mathbf{T}$ is supercritical:

$$\mathbb{E}\left(\sum_{|u|=1} 1\right) > 1, \tag{1.1}$$

and we write $S = \{\#\mathbf{T} = +\infty\}$ the survival event. We also assume the branching random walk $(\mathbf{T}, V)$ is in the boundary case (in the sense of [5])

$$\mathbb{E}\left(\sum_{|u|=1} e^{-V(u)}\right) = 1 \quad \text{and} \quad \mathbb{E}\left(\sum_{|u|=1} V(u)e^{-V(u)}\right) = 0. \tag{1.2}$$

Under these assumptions, Biggins [4] proved that $\frac{1}{n}\max_{|u|=n} V(u)$ converges to 0 almost surely on S. Any branching random walk with mild integrability

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assumption can be normalized to be in the boundary case, see e.g. Bérard and Gouéré [3]. We also assume that

$$\sigma^2 := \mathbb{E}\left(\sum_{|u|=1} V(u)^2 e^{-V(u)}\right) < +\infty. \tag{1.3}$$

Let $n \geq 0$. For any $u \in \mathbf{T}$ such that $|u| = n$ and $k \leq n$ we denote by u_k the ancestor of u alive at generation k. The consistent maximal displacement of the branching random walk is the quantity defined as

$$L_n := \min_{|u|=n} \max_{k \leq n} V(u_k).$$

It correspond to the distance between the boundary of the branching random walk and the individual that stayed as close as possible to it. The asymptotic behaviour of L_n has been studied by Fang and Zeitouni [7] and by Fauraud, Hu and Shi [8]. Under stronger integrability assumptions, they proved that L_n behaves as $\lambda^* n^{1/3}$ almost surely for some explicit λ^*. The main result of this article is a necessary and sufficient condition for this asymptotic behaviour to hold. Roberts [15] computed the second order of the asymptotic behaviour of L_n for the branching Brownian motion.

We now introduce the integrability assumption

$$\lim_{x \to +\infty} x^2\, \mathbb{E}\left(\sum_{|u|=1} e^{-V(u)} \mathbf{1}_{\left\{\log\left(\sum_{|v|=1} e^{-V(v)}\right) \geq x\right\}}\right) = 0. \tag{1.4}$$

Observe that (1.4) is strictly weaker than the classical integrability assumption that is [1, Assumption (1.4)]. This stronger assumption is necessary and sufficient to obtain the asymptotic behaviour of many quantities associated to the branching random walk, such as the minimal displacement, or the derivative martingale (see [1, 6]).

Theorem 1.1. *We assume* (1.1), (1.2) *and* (1.3) *hold. Then* (1.4) *is a necessary and sufficient condition for*

$$\lim_{n \to +\infty} \frac{L_n}{n^{1/3}} = \left(\frac{3\pi^2\sigma^2}{2}\right)^{1/3} \quad \textit{a.s. on } S.$$

If (1.4) is changed into

$$\lim_{x \to +\infty} x^2\, \mathbb{E}\left(\sum_{|u|=1} e^{-V(u)} \mathbf{1}_{\left\{\log\left(\sum_{|v|=1} e^{-V(v)}\right) \geq x\right\}}\right) = \rho \in (0, +\infty),$$

we were not able to compute the precise asymptotic behaviour of $\frac{L_n}{n^{1/3}}$. Using the methods developed in this article, it can be proved that a.s. on S, for n large enough we have

$$\left(3\frac{\pi^2\sigma^2}{2} + 3\rho\right)^{1/3} \leq \frac{L_n}{n^{1/3}} \leq \left(3\frac{\pi^2\sigma^2}{2}\right)^{1/3} + (3\rho)^{1/3}.$$

The rest of the article is organised as follows. In Section 2 we introduce the spinal decomposition of the branching random walk and the Mogul'skiĭ's small deviations estimate. These results are used to bound the left tail of L_n in Section 3, yielding to its a.s. asymptotic behaviour.

2 Preliminary results

2.1 Spinal decomposition of the branching random walk

For $n \in \mathbb{N}$, we write $W_n = \sum_{|u|=n} e^{-V(u)}$ and $\mathcal{F}_n = \sigma\left(u, V(u), |u| \leq n\right)$. Under assumption (1.2), (W_n) is a non-negative $(\mathcal{F}_n)$-martingale. We introduce the probability $\overline{\mathbb{P}}$ such that for any $n \in \mathbb{N}$, $\overline{\mathbb{P}}_{|\mathcal{F}_n} = W_n \cdot \mathbb{P}_{|\mathcal{F}_n}$.

The spinal decomposition consists in an alternative description of $\overline{\mathbb{P}}$ as a branching random walk with a distinguished individual with a different reproduction law. It generalizes a similar construction for Galton-Watson processes, that can be found in [11]. This result has been proved by Lyons in [10]. Let $\mathbf{T}$ be a tree, a spine of $\mathbf{T}$ is a sequence $w = (w_n) \in \mathbf{T}^{\mathbb{N}}$ such that $|w_n| = n$ and for any $k \leq n$, $(w_n)_k = w_k$. We write $\widehat{\mathcal{L}}$ for the law of the point process $(V(u), |u| = 1)$ under the law $\overline{\mathbb{P}}$.

We now define the law $\widehat{\mathbb{P}}$ of a branching random walk with spine $(\mathbf{T}, V, w)$. It starts with a unique individual w_0 located at 0 at time 0. Its children are positioned according to a point process of law $\widehat{\mathcal{L}}$. The individual w_1 is then chosen at random among these children u with probability proportional to $e^{-V(u)}$. Similarly at each generation n, every individual u makes children independently, according to law $\mathcal{L}$ if $u \neq w_n$ and $\widehat{\mathcal{L}}$ otherwise; and w_{n+1} is chosen at random among the children v of w_n with probability proportional to $e^{-V(v)}$.

Proposition 2.1 (Spinal decomposition, Lyons [10])**.** *Assuming* (1.2) *and* (1.3)*, for any $n \in \mathbb{N}$, we have $\widehat{\mathbb{P}}_{|\mathcal{F}_n} = \overline{\mathbb{P}}_{|\mathcal{F}_n}$, and for any $|u| = n$,*

$$\widehat{\mathbb{P}}\left(w_n = u \,|\mathcal{F}_n\right) = e^{-V(u)}/W_n,$$

and $(V(w_n), n \geq 0)$ is a centred random walk with variance σ^2.

2.2 Small deviations estimate for enriched random walk

Let $(X_n, \xi_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. vectors in $\mathbb{R}^2$ such that

$$\mathbb{E}(X_n) = 0 \quad \text{and} \quad \mathbb{E}(X_n^2) = \sigma^2 \in (0, +\infty). \tag{2.1}$$

We set $\rho_+ = \limsup_{x\to+\infty} x^2\mathbb{P}(\xi_1 \geq x)$ and $\rho_- = \liminf_{x\to+\infty} x^2\mathbb{P}(\xi_1 \geq x)$.

We write $T_n = T_0 + X_1 + \cdots + X_n$ and we call $((T_n, \xi_n), n \geq 0)$ an enriched random walk. For any $z \in \mathbb{R}$, $\mathbb{P}_z$ is the probability such that $\mathbb{P}_z(T_0 = z) = 1$. We simply write $\mathbb{P}$ for $\mathbb{P}_0$. We study in this section the probability that an enriched random walk stays during n unites of time in an interval of width $o(n^{1/2})$, generalizing the Mogul'skiĭ small deviations estimate [14].

Theorem 2.2. *Let (a_n) be a sequence of real non-negative numbers such that $\lim_{n\to+\infty} a_n = +\infty$ and $\lim_{n\to+\infty} \frac{a_n^2}{n} = 0$. Assuming* (2.1)*, for any continuous functions $f < g$ and $h > 0$, for any x, x' such that $f(0) < x \leq 0 \leq x' < g(0)$, we have*

$$\limsup_{n\to+\infty} \frac{a_n^2}{n} \log \sup_{z\in[x,x']} \mathbb{P}_{za_n}\left(\tfrac{T_j}{a_n} \in [f(j/n), g(j/n)], \tfrac{\xi_j}{a_n} \leq h(j/n), j \leq n\right)$$
$$= -\int_0^1 \frac{\pi^2\sigma^2}{2(g(s) - f(s))^2} + \frac{\rho_-}{h(s)^2} ds, \tag{2.2}$$

$$\liminf_{n\to+\infty}\frac{a_n^2}{n}\log\inf_{z\in[x,x']}\mathbb{P}_{za_n}\left(\tfrac{T_j}{a_n}\in[f(j/n),g(j/n)],\tfrac{\xi_j}{a_n}\le h(j/n),j\le n\right)$$
$$=-\int_0^1\frac{\pi^2\sigma^2}{2(g(s)-f(s))^2}+\frac{\rho_+}{h(s)^2}ds. \quad (2.3)$$

We prove this result using the same techniques as in [13, Lemma 2.6]. We prove in a first time the following result.

Lemma 2.3. *Under the assumptions of Theorem 2.2, if $\rho_+=\rho_-=:\rho\in[0,+\infty]$ then for any $a<0<b$, $\lambda>0$ and $a\le c<d\le b$, we have*

$$\lim_{n\to+\infty}\frac{a_n^2}{n}\log\mathbb{P}\left(\tfrac{T_n}{a_n}\in[c,d],\tfrac{T_j}{a_n}\in[a,b],\tfrac{\xi_j}{a_n}\le\lambda,j\le n\right)=-\frac{\pi^2\sigma^2}{2(b-a)^2}-\frac{\rho}{\lambda^2}.$$

Proof. Note that

$$\mathbb{P}\left(|T_j|<ha_n,\xi_j\le\lambda a_n,j\le n\right)\le\mathbb{P}(\xi_1\le\lambda a_n)^n\le\exp(-n\mathbb{P}(\xi_1>\lambda a_n)).$$

Consequently, if $\rho=+\infty$, the proof is immediate. We assume in the rest of the proof that $\rho<+\infty$.

Let B be a Brownian motion and P an independent Poisson process with intensity 1. For any $n\in\mathbb{N}$ and $k\le n$, we denote by $P_k^{(n)}=\sum_{j=1}^k\mathbf{1}_{\{\xi_j\ge\lambda a_n\}}$. By [16, Theorem 5.1], for any $T>0$ we have

$$\lim_{n\to+\infty}\left(\tfrac{T_{ta_n^2}}{a_n},P^{(n)}_{ta_n^2},t\le T\right)=\left(B_{\sigma^2t},P_{\rho t/\lambda^2},t\le T\right) \quad (2.4)$$

in the Skorohod sense.

Let $a<0<b$, $a\le c<d\le b$ and $T>0$. We set $r_n=\lfloor Ta_n^2\rfloor$. Applying the Markov property, for any $\epsilon>0$ small enough, we have

$$\left(\inf_{\substack{h\le b-a-4\epsilon\\|x|\le\epsilon}}\mathbb{P}_{xa_n}\left(|\tfrac{T_{r_n}}{a_n}-h|\le\epsilon,\tfrac{T_j}{a_n}\in[a,b],\tfrac{\xi_j}{a_n}\le\lambda,j\le r_n\right)\right)^{\lceil n/r_n\rceil}$$
$$\le\mathbb{P}\left(\tfrac{T_n}{a_n}\in[c,d],\tfrac{T_j}{a_n}\in[a,b],\tfrac{\xi_j}{a_n}\le\lambda,j\le n\right)$$
$$\le\left(\sup_{x\in[a,b]}\mathbb{P}_{xa_n}\left(\tfrac{T_j}{a_n}\in[a,b],\tfrac{\xi_j}{a_n}\le\lambda,j\le r_n\right)\right)^{\lfloor n/r_n\rfloor}.$$

Consequently, letting $n\to+\infty$ and using (2.4), we obtain:

$$\liminf_{n\to+\infty}\frac{a_n^2}{n}\log\mathbb{P}\left(\tfrac{T_n}{a_n}\in[c,d],\tfrac{T_j}{a_n}\in[a,b],\tfrac{\xi_j}{a_n}\le\lambda,j\le n\right)$$
$$\ge\frac{1}{T}\log\inf_{\substack{h\le b-a-4\epsilon\\|x|\le\epsilon}}\mathbb{P}_x\left(|B_{\sigma^2T}-h|\le\epsilon,P_{\rho T/\lambda^2}=0,B_{\sigma^2s}\in[a,b],s\le T\right)$$
$$\ge-\frac{\rho}{\lambda^2}+\frac{1}{T}\log\inf_{\substack{h\le b-a-4\epsilon\\|x|\le\epsilon}}\mathbb{P}_x\left(|B_{\sigma^2T}-h|\le\epsilon,B_{\sigma^2s}\in[a,b],s\le T\right),$$

$$\limsup_{n\to+\infty}\frac{a_n^2}{n}\log\mathbb{P}\left(\tfrac{T_j}{a_n}\in[a,b],\tfrac{\xi_j}{a_n}\le\lambda,j\le n\right)$$
$$\le\frac{1}{T}\log\sup_{x\in[a,b]}\mathbb{P}\left(P_{\rho T/\lambda^2}=0,B_{\sigma^2 s}\in[a,b],s\le T\right)$$
$$\le-\frac{\rho}{\lambda^2}+\frac{1}{T}\log\sup_{x\in[a,b]}\mathbb{P}\left(B_{\sigma^2 s}\in[a,b],s\le T\right).$$

Letting $T\to+\infty$, using e.g. [9, Chapter 1.7, Problem 8], we conclude that

$$\lim_{n\to+\infty}\frac{a_n^2}{n}\log\mathbb{P}\left(\tfrac{T_j}{a_n}\in[a,b],\tfrac{\xi_j}{a_n}\le\lambda,j\le n\right)=-\frac{\pi^2\sigma^2}{2(b-a)^2}-\frac{\rho}{\lambda^2}.$$

□

Corollary 2.4. *Under the assumptions of Theorem 2.2, for any $a<0<b$, $\lambda>0$ and $a\le c<d\le b$, we have*

$$\limsup_{n\to+\infty}\frac{a_n^2}{n}\log\mathbb{P}\left(\tfrac{T_j}{a_n}\in[a,b],\tfrac{\xi_j}{a_n}\le\lambda,j\le n\right)=-\frac{\pi^2\sigma^2}{2(b-a)^2}-\frac{\rho_-}{\lambda^2},$$

$$\liminf_{n\to+\infty}\frac{a_n^2}{n}\log\mathbb{P}\left(\tfrac{T_n}{a_n}\in[c,d],\tfrac{T_j}{a_n}\in[a,b],\tfrac{\xi_j}{a_n}\le\lambda,j\le n\right)=-\frac{\pi^2\sigma^2}{2(b-a)^2}-\frac{\rho_+}{\lambda^2}.$$

Proof. The two equations being proved the same way, we only prove the first one. As $\mathbb{P}\left(|T_j|<ha_n,\xi_j\le n,j\le n\right)\le\mathbb{P}(\xi_1\le n)^n\le\exp(-n\mathbb{P}(\xi_1>n))$, we only consider the case $\rho_-<+\infty$.

There exists a subsequence (n_k) such that $a_{n_k}^2\mathbb{P}(\xi_1>\lambda a_{n_k})$ converges to ρ_-/λ^2, therefore by Lemma 2.3,

$$\limsup_{n\to+\infty}\frac{a_n^2}{n}\log\mathbb{P}\left(\tfrac{T_j}{a_n}\in[a,b],\tfrac{\xi_j}{a_n}\le\lambda,j\le n\right)$$
$$\ge\lim_{k\to+\infty}\frac{a_{n_k}^2}{n_k}\log\mathbb{P}\left(\tfrac{T_j}{a_{n_k}}\in[a,b],\tfrac{\xi_j}{a_{n_k}}\le\lambda,j\le n_k\right)=-\frac{\pi^2\sigma^2}{2(b-a)^2}-\frac{\rho_-}{\lambda^2}.$$

Moreover, we may couple the random variable ξ_j with a random variable $\widetilde{\xi}_j$ such that $\widetilde{\xi}_j\le\xi_j$ and $\lim_{n\to+\infty}x^2\mathbb{P}(\widetilde{\xi}_1\ge x)=\rho_-$. By Lemma 2.3 again,

$$\limsup_{n\to+\infty}\frac{a_n^2}{n}\log\mathbb{P}\left(\tfrac{T_j}{a_n}\in[a,b],\tfrac{\xi_j}{a_n}\le\lambda,j\le n\right)$$
$$\le\limsup_{n\to+\infty}\frac{a_n^2}{n}\log\mathbb{P}\left(\tfrac{T_j}{a_n}\in[a,b],\tfrac{\widetilde{\xi}_j}{a_n}\le\lambda,j\le n\right)=-\frac{\pi^2\sigma^2}{2(b-a)^2}-\frac{\rho_-}{\lambda^2},$$

concluding the proof. □

Proof of Theorem 2.2. We first prove that for any $a<b$ and $\lambda>0$,

$$\limsup_{n\to+\infty}\frac{a_n^2}{n}\log\sup_{x\in[a,b]}\mathbb{P}_{xa_n}\left(\tfrac{T_j}{a_n}\in[a,b],\tfrac{\xi_j}{a_n}\le\lambda,j\le n\right)=-\frac{\pi^2\sigma^2}{2(b-a)^2}-\frac{\rho_-}{\lambda^2}. \tag{2.5}$$

Let $\delta > 0$, we write $M = \lceil \frac{b-a}{\delta} \rceil$. We observe that

$$\begin{aligned}
&\sup_{x\in[a,b]} \mathbb{P}_{xa_n}\left(\tfrac{T_j}{a_n} \in [a,b], \tfrac{\xi_j}{a_n} \le \lambda, j \le n\right) \\
&\le \max_{0\le m<M} \sup_{x\in[a+m\delta, a+(m+1)\delta]} \mathbb{P}_{xa_n}\left(\tfrac{T_j}{a_n} \in [a,b], \tfrac{\xi_j}{a_n} \le \lambda, j \le n\right) \\
&\le \max_{0\le m<M} \mathbb{P}\left(\tfrac{T_j}{a_n} \in [-(m+1)\delta, b-a-m\delta], \tfrac{\xi_j}{a_n} \le \lambda, j \le n\right).
\end{aligned}$$

We apply Corollary 2.4 and let $\delta \to 0$ to obtain (2.5). Then we approach functions f, g and h by staircase functions, using (2.5) we obtain (2.2).

For any $a < 0 < b$, $\lambda > 0$, $y \in [a,b]$ and $\epsilon > 0$ small enough, we prove in a second time that

$$\begin{aligned}
\liminf_{n\to+\infty} \frac{a_n^2}{n} \log \inf_{|x|<\epsilon} \mathbb{P}_{xa_n}\left(\left|\tfrac{T_n}{a_n} - y\right| \le \epsilon, \tfrac{T_j}{a_n} \in [a,b], \tfrac{\xi_j}{a_n} \le \lambda, j \le n\right) \\
= -\frac{\pi^2\sigma^2}{2(b-a)^2} - \frac{\rho_+}{\lambda^2}. \quad (2.6)
\end{aligned}$$

Let $\epsilon > 0$ small enough such that $[-2\epsilon, 2\epsilon] \subset [a,b]$, let $0 < \delta < \epsilon$, we set $M = \lceil \frac{2\delta}{\epsilon} \rceil$. We have again

$$\begin{aligned}
&\inf_{|x|\le\epsilon} \mathbb{P}_{xa_n}\left(\left|\tfrac{T_n}{a_n} - y\right| \le \epsilon, \tfrac{T_j}{a_n} \in [a,b], \xi_j \le n, j \le n\right) \\
&\ge \min_{0\le m<M} \inf_{x\in[-\epsilon+m\delta, -\epsilon+(m+1)\delta]} \mathbb{P}_{xa_n}\left(\left|\tfrac{T_n}{a_n} - y\right| \le \epsilon, \tfrac{T_j}{a_n} \in [a,b], \xi_j \le n, j \le n\right) \\
&\ge \min_{0\le m<M} \mathbb{P}\left(\begin{array}{l} \frac{T_j}{a_n} \in [a+\epsilon-m\delta, b+\epsilon-(m+1)\delta], \xi_j \le n, j \le n \\ \frac{T_n}{a_n} \in [y-m\delta, y+2\epsilon-(m+1)\delta] \end{array}\right).
\end{aligned}$$

Applying again Corollary 2.4, (2.6) holds. We finally approach functions f, g and h by staircase functions and use the Markov property, to obtain the lower bound (2.3).

□

3 Tail of the consistent maximal displacement

For any $u \in \mathbf{T}$, we write πu for the parent of u, $\Omega(u)$ for the set of children of u,

$$\widetilde{\xi}(u) = \log \sum_{v\in\Omega(u)} e^{V(u)-V(v)} \text{ and } \xi(u) = \widetilde{\xi}(\pi u).$$

Note that by (1.2), we have $\mathbb{P}(\widetilde{\xi}(u) \ge x) \le e^{-x}$ for any $x \ge 0$. We introduce

$$\rho_+ = \limsup_{x\to+\infty} x^2 \widehat{\mathbb{P}}\left(\widetilde{\xi}(w_0) \ge x\right) \text{ and } \rho_- = \liminf_{x\to+\infty} x^2 \widehat{\mathbb{P}}\left(\widetilde{\xi}(w_0) \ge x\right) \quad (3.1)$$

$$\lambda_+^* = \left(\tfrac{3\pi^2\sigma^2}{2} + 3\rho_+\right)^{1/3}, \ \lambda_-^* = \left(\tfrac{3\pi^2\sigma^2}{2} + 3\rho_-\right)^{1/3} \text{ and } \lambda^* = \left(\tfrac{3\pi^2\sigma^2}{2}\right)^{1/3}. \quad (3.2)$$

We use here the results of the previous section to obtain upper and lower bounds for the left tail of L_n. We first provide an upper bound.

Lemma 3.1. *We assume* (1.1), (1.2) *and* (1.3). *For any* $\lambda \in (0, \lambda_-^*)$, *we have*

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbb{P}\left(L_n \le \lambda n^{1/3}\right) \le \lambda - \lambda_-^*, \tag{3.3}$$

and for any $\lambda \in (0, \lambda_+^*)$, *we have*

$$\liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbb{P}\left(L_n \le \lambda n^{1/3}\right) \le \lambda - \lambda_+^*. \tag{3.4}$$

Proof. Let $\lambda > 0$, f be a continuous increasing function such that $f(0) < 0$ and $f(1) = \lambda$, and $h = \lambda - f$. We set $I_k^{(n)} = \left[f(k/n)n^{1/3}, \lambda n^{1/3}\right]$ for $k \le n$ and we denote by

$$G_n = \left\{u \in \mathbf{T} : |u| \le n, V(u_j) \in I_j^{(n)}, \xi(u_j) \le h(j/n)n^{1/3}, j \le |u|\right\}.$$

We introduce the quantities

$$\begin{aligned} X_k^{(n)} &= \sum_{|u|=k} \mathbf{1}_{\left\{V(u)<f(k/n)n^{1/3}\right\}} \mathbf{1}_{\{\pi u \in G_n\}} \quad \text{and} \\ Y_k^{(n)} &= \sum_{|u|=k-1} \mathbf{1}_{\left\{\widetilde{\xi}(u)>h(k/n)n^{1/3}\right\}} \mathbf{1}_{\{u \in G_n\}}. \end{aligned}$$

We observe that

$$\begin{aligned} \mathbb{P}\left(L_n \le \lambda n^{1/3}\right) =& \mathbb{P}\left(\exists |u| = n : V(u_j) \le \lambda n^{1/3}, j \le n\right) \\ \le& \mathbb{P}\left(\sum_{j=1}^n X_j^{(n)} + Y_j^{(n)} \ge 1\right) \le \sum_{j=1}^n \mathbb{E}\left(X_j^{(n)} + Y_j^{(n)}\right). \end{aligned} \tag{3.5}$$

For any $k \le n$, using the spinal decomposition we have

$$\begin{aligned} \mathbb{E}\left(X_k^{(n)}\right) =& \widehat{\mathbb{E}}\left(\sum_{|u|=k} \frac{e^{-V(u)}}{W_k} e^{V(u)} \mathbf{1}_{\left\{V(u)<f(k/n)n^{1/3}\right\}} \mathbf{1}_{\{\pi u \in G_n\}}\right) \\ =& \widehat{\mathbb{E}}\left(e^{V(w_k)} \mathbf{1}_{\left\{V(w_k)<f(k/n)n^{1/3}\right\}} \mathbf{1}_{\{w_{k-1} \in G_n\}}\right) \\ \le& e^{f(k/n)n^{1/3}} \widehat{\mathbb{P}}\left(w_{k-1} \in G_n\right). \end{aligned}$$

Moreover, as $\xi(u)$ is independent of $\mathcal{F}_{|u|}$, we also note that

$$\begin{aligned} \mathbb{E}\left(Y_k^{(n)}\right) &= \mathbb{E}\left(\sum_{|u|=k-1} \mathbf{1}_{\{u \in G_n\}}\right) \mathbb{P}\left(\widetilde{\xi}(u) > h(k/n)n^{1/3}\right) \\ &\le e^{-h(k/n)n^{1/3}} \widehat{\mathbb{E}}\left(e^{V(w_{k-1})} \mathbf{1}_{\{w_{k-1} \in G_n\}}\right) \le e^{f(k/n)n^{1/3}} \widehat{\mathbb{P}}\left(w_{k-1} \in G_n\right). \end{aligned}$$

Consequently, (3.5) becomes

$$\mathbb{P}\left(L_n \le \lambda n^{1/3}\right) \le 2 \sum_{k=1}^n e^{f(k/n)n^{1/3}} \widehat{\mathbb{P}}\left(w_{k-1} \in G_n\right).$$

We set $A > 0$, for any $a \leq A$, we write $m_a = \lfloor na/A \rfloor$. As f is increasing, for any $k \in (m_a, m_{a+1}]$, we have

$$e^{f(k/n)n^{1/3}} \widehat{\mathbb{P}}\left(w_{k-1} \in G_n\right) \leq e^{f((a+1)/A)n^{1/3}} \widehat{\mathbb{P}}\left(w_{m_a} \in G_n\right).$$

Moreover, by the spinal decomposition, $(V(w_j), \xi(w_j))$ is an enriched random walk under law $\widehat{\mathbb{P}}$. We apply Theorem 2.2 to obtain for any $a \leq A$,

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \widehat{\mathbb{P}}\left(w_{m_a} \in G_n\right) = -\int_0^{a/A} \frac{3(\lambda_-^*)^3}{(\lambda - f(s))^2} ds,$$
$$\liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \widehat{\mathbb{P}}\left(w_{m_a} \in G_n\right) = -\int_0^{a/A} \frac{3(\lambda_+^*)^3}{(\lambda - f(s))^2} ds.$$

We conclude that, letting $n \to +\infty$ then $A \to +\infty$ we have

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbb{P}\left(L_n \leq \lambda n^{1/3}\right) \leq \sup_{t\in[0,1]} f(t) - \int_0^t \frac{3(\lambda_-^*)^3}{(\lambda - f(s))^2} ds, \tag{3.6}$$

$$\liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbb{P}\left(L_n \leq \lambda n^{1/3}\right) \leq \sup_{t\in[0,1]} f(t) - \int_0^t \frac{3(\lambda_+^*)^3}{(\lambda - f(s))^2} ds. \tag{3.7}$$

Optimizing these equations in f concludes the proof.

Indeed, to prove (3.3), we consider the following two cases. If $\rho_- = +\infty$ then $\lambda_-^* = +\infty$. Choosing a sequence of functions f_M such that $f_M(0) < -M$, and letting $M \to +\infty$, (3.6) yields

$$\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbb{P}\left(L_n \leq \lambda n^{1/3}\right) = -\infty.$$

If $\rho_- < +\infty$, then setting $f(t) = \lambda - \lambda_-^*(1-t)^{1/3}$ we have

$$\begin{aligned}\limsup_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbb{P}\left(L_n \leq \lambda n^{1/3}\right) &\leq \sup_{t\in[0,1]} f(t) - \int_0^t \frac{3(\lambda_-^*)^3}{(\lambda - f(s))^2} ds \\ &\leq \sup_{t\in[0,1]} f(t) - \int_0^t \frac{3\lambda_-^*}{(1-s)^{2/3}} ds = \lambda - \lambda_-^*.\end{aligned}$$

A similar reasoning leads to (3.4). □

Corollary 3.2. *Under assumptions* (1.1), (1.2) *and* (1.3), *we have*

$$\liminf_{n\to+\infty} \frac{L_n}{n^{1/3}} \geq \lambda_-^* \quad \textit{and} \quad \limsup_{n\to+\infty} \frac{L_n}{n^{1/3}} \geq \lambda_+^* \quad \textit{a.s.}$$

In particular, if $\rho_+ > 0$ *then* $\limsup_{n\to+\infty} \frac{L_n}{n^{1/3}} > \lambda^*$ *a.s.*

Proof. By Lemma 3.1, for any $\epsilon > 0$, $\sum_{n=1}^{+\infty} \mathbb{P}(\frac{L_n}{n^{1/3}} \geq (\lambda_-^* + \epsilon)) < +\infty$, and there exists an increasing sequence (n_k) such that $\sum_{k=1}^{+\infty} \mathbb{P}(\frac{L_{n_k}}{n_k^{1/3}} \geq (\lambda_+^* + \epsilon)) < +\infty$. We conclude the proof by Borel-Cantelli lemma. □

In a second time, to bound from above the consistent maximal displacement, we prove that with high probability there exists an individual staying below $\lambda n^{1/3}$ for n units of time, as soon as λ is large enough. To do so, we compute the first two moments of the number of individuals staying in two well-chosen lines, while making "not too many children".

Lemma 3.3. *Assuming* (1.1), (1.2), (1.3) *and* (1.4) *(i.e.* $\rho_+ = \rho_- = 0$*), for any* $0 < \lambda < \lambda^*$*, we have* $\liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbb{P}\left(L_n \le \lambda n^{1/3}\right) \ge \lambda - \lambda^*$.

Proof. Let $\lambda \in (0, \lambda^*)$, $\delta > 0$, and $f : t \in [0,1] \mapsto \lambda - \lambda^*(1+\delta-t)^{1/3}$. We denote by $I_j^{(n)} = \left[f(j/n)n^{1/3}, \lambda n^{1/3}\right]$ for $j \le n$. We set

$$Z_n = \sum_{|u|=n} \mathbf{1}_{\left\{V(u_j)\in I_j^{(n)}, \xi(u_j)\le \delta n^{1/3}, j\le n\right\}}.$$

We compute the first two moments of Z_n to bound from below $\mathbb{P}(Z_n > 0)$.

By spinal decomposition, we have

$$\begin{aligned}\mathbb{E}(Z_n) &= \widehat{\mathbb{E}}\left(e^{V(w_n)}\mathbf{1}_{\left\{V(w_j)\in I_j^{(n)}, \xi(w_j)\le \delta n^{1/3}, j\le n\right\}}\right)\\ &\ge e^{f(1)n^{1/3}}\widehat{\mathbb{P}}\left(V(w_j)\in I_j^{(n)}, \xi(w_j)\le \delta n^{1/3}, j\le n\right).\end{aligned}$$

As $\lim_{n\to+\infty} n^{2/3}\widehat{\mathbb{P}}(\xi(w_1)\ge \delta n^{1/3}) = 0$ by (1.4), Theorem 2.2 yields

$$\liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbb{E}(Z_n) \ge f(1) - \frac{\pi^2\sigma^2}{2}\int_0^1 \frac{ds}{(\lambda - f(s))^2} \ge \lambda - \lambda^*(1+\delta)^{1/3}.$$

Similarly, to compute the second moment we observe that

$$\begin{aligned}\mathbb{E}(Z_n^2) &= \widehat{\mathbb{E}}\left(Z_n \sum_{|u|=n} \frac{e^{-V(u)}}{W_n} e^{V(u)}\mathbf{1}_{\left\{V(u_j)\in I_j^{(n)}, \xi(u_j)\le \delta n^{1/3}, j\le n\right\}}\right)\\ &= \widehat{\mathbb{E}}\left(Z_n e^{V(w_n)}\mathbf{1}_{\left\{V(w_j)\in I_j^{(n)}, \xi(w_j)\le \delta n^{1/3}, j\le n\right\}}\right)\\ &\le e^{\lambda n^{1/3}}\widehat{\mathbb{E}}\left(Z_n\mathbf{1}_{\left\{V(w_j)\in I_j^{(n)}, \xi(w_j)\le \delta n^{1/3}, j\le n\right\}}\right).\end{aligned}$$

Under the law $\widehat{\mathbb{P}}$, Z_n can be decomposed as follows

$$Z_n = \mathbf{1}_{\left\{V(w_j)\in I_j^{(n)}, \xi(w_j)\le \delta n^{1/3}, j\le n\right\}} + \sum_{k=0}^{n-1} \sum_{\substack{u\in\Omega(w_k)\\ u\ne w_{k+1}}} Z_n(u),$$

where $Z_n(u) = \sum_{|v|=n, v>u} \mathbf{1}_{\left\{V(u_j)\in I_j^{(n)}, \xi(u_j)\le \delta n^{1/3}, j\le n\right\}}$. We denote by

$$\mathcal{G} = \sigma\left(w_n, \Omega(w_n), V(u), u\in\Omega(w_n), n\ge 0\right).$$

Observe that conditionally on $\mathcal{G}$, for any $u \in \Omega(w_k)$ such that $u \ne w_{k+1}$, the subtree of the descendants of u has the law of a branching random walk starting

from $V(u)$. Therefore, writing $\mathbb{P}_x$ for the law of $(\mathbf{T}, V+x)$, for any $k<n$ and $u \in \Omega(w_k)$ such that $u \neq w_{k+1}$, we have

$$\begin{aligned}\widehat{\mathbb{E}}\left(Z_n(u)|\mathcal{G}\right) &\leq \mathbb{E}_{V(u)}\left(\sum_{|v|=n-k-1} \mathbf{1}_{\left\{V(v_j)\in I^{(n)}_{k+j+1}, \xi(v_j)\leq \delta n^{1/3}, j\leq n-k-1\right\}}\right)\\ &\leq \mathbb{E}_{V(u)}\left(\sum_{|v|=n-k-1} \mathbf{1}_{\left\{V(v_j)\in I^{(n)}_{k+j+1}, j\leq n-k-1\right\}}\right).\end{aligned}$$

Applying spinal decomposition, for any $x \in \mathbb{R}$ and $p \leq n$,

$$\begin{aligned}\mathbb{E}_x\left(\sum_{|v|=n-p} \mathbf{1}_{\left\{V(v_j)\in I^{(n)}_{p+j}, j\leq n-p\right\}}\right) &= \widehat{\mathbb{E}}\left(e^{V(w_{n-p}+x}\mathbf{1}_{\left\{V(w_j)+x\in I^{(n)}_{p+j}, j\leq n-p\right\}}\right)\\ &\leq e^{\lambda n^{1/3}-x}\widehat{\mathbb{P}}\left(V(w_j)+x\in I^{(n)}_{p+j}, j\leq n-p\right).\end{aligned}$$

Let $A>0$, for any $a \leq A$ we set $m_a = \lfloor na/A \rfloor$ and

$$\Psi^{(n)}_{a,A} = \sup_{y\in I^{(n)}_{m_a}} \widehat{\mathbb{P}}\left(V(w_j)+y\in I^{(n)}_{m_a+j}, j\leq n-m_a\right).$$

Using the previous equation, for any $m_a \leq k < m_{a+1}$, we have

$$\begin{aligned}\sum_{\substack{u\in\Omega(w_k)\\ u\neq w_{k+1}}} \widehat{\mathbb{E}}\left(Z_n(u)|\mathcal{G}\right) &\leq e^{\lambda n^{1/3}}\Psi^{(n)}_{a+1,A}\sum_{u\in\Omega(w_k)} e^{V(u)}\\ &\leq e^{\lambda n^{1/3}+V(w_k)+\xi(w_{k+1})}\Psi^{(n)}_{a+1,A}.\end{aligned}$$

As $\xi(w_{k+1}) \leq \delta n^{1/3}$, we obtain

$$\begin{aligned}\mathbb{E}(Z_n^2) &\leq e^{\lambda n^{1/3}}\widehat{\mathbb{P}}\left(V(w_j)\in I^{(n)}_j, j\leq n\right)\\ &\quad + e^{(2\lambda+\delta)n^{1/3}}\sum_{a=0}^{A-1} n\Psi^{(n)}_{a+1,A}e^{-f(a/A)n^{1/3}}\widehat{\mathbb{P}}\left(V(w_j)\in I^{(n)}_j, j\leq n\right).\end{aligned}$$

Therefore, applying Theorem 2.2, we have

$$\begin{aligned}\limsup_{n\to+\infty}\frac{1}{n^{1/3}}\log\mathbb{E}(Z_n^2) &\leq 2\lambda+\delta-\frac{\pi^2\sigma^2}{2}\int_0^1\frac{ds}{(\lambda-f(s))^2}\\ &\quad -\min_{a<A} f(\tfrac{a}{A})-\frac{\pi^2\sigma^2}{2}\int_{(a+1)/A}^1\frac{ds}{(\lambda-f(s))^2}.\end{aligned}$$

Letting $A\to+\infty$, we obtain

$$\limsup_{n\to+\infty}\frac{1}{n^{1/3}}\log\mathbb{E}(Z_n^2) \leq \lambda+\delta-\lambda^*(1+\delta)^{1/3}.$$

By Cauchy-Schwarz inequality, $\mathbb{P}\left(L_n\leq\lambda n^{1/3}\right)\geq\mathbb{P}\left(Z_n>0\right)\geq\frac{\mathbb{E}(Z_n)^2}{\mathbb{E}(Z_n^2)}$, thus

$$\liminf_{n\to+\infty}\frac{1}{n^{1/3}}\log\mathbb{P}\left(L_n\leq\lambda n^{1/3}\right)\geq\lambda-\lambda^*(1+\delta)^{1/3}-\delta.$$

Letting $\delta\to 0$, we conclude the proof. □

Note that Lemmas 3.1 and 3.3 imply that, assuming (1.4)

$$\lim_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbb{P}(L_n \leq \lambda n^{1/3}) = \lambda - \lambda^*. \tag{3.8}$$

We use Lemma 3.3 to obtain an a.s. behaviour of L_n.

Lemma 3.4. *Assuming* (1.1), (1.2), (1.3) *and* (1.4), *we have* $\lim_{n\to+\infty} \frac{L_n}{n^{1/3}} = \lambda^*$ *almost surely on* S.

Proof. We have $\liminf_{n\to+\infty} \frac{L_n}{n^{1/3}} \geq \lambda^*$ by Corollary 3.2. We now turn to the upper bound of L_n. By Lemma 3.3, for any $\delta > 0$,

$$\liminf_{n\to+\infty} \frac{1}{n^{1/3}} \log \mathbb{P}(L_n \leq \lambda^* n^{1/3}) > -\delta.$$

We work in the rest of the proof conditionally on the survival event S. We write $\widehat{\mathbf{T}}$ for the subtree of $\mathbf{T}$ consisting of individuals having an infinite line of descent. By [2, Chapter 1, Theorem 12.1], $\widehat{\mathbf{T}}$ is a supercritical Galton-Watson process that never dies out. Applying [12, Lemma 2.4] to the branching random walk $(\widehat{\mathbf{T}}, V)$, there exists $a > 0$ and $\rho > 1$ such that the event

$$\mathcal{A}(p) = \{\#\{|u| = p : \forall j \leq p, V(u_j) \geq -pa\} \geq \rho^p\}$$

is verified a.s. for $p \geq 1$ large enough. Let $\eta > 0$, we set $p = \lfloor \eta n^{1/3} \rfloor$. Applying the Markov property at time p, we have

$$\mathbb{P}\left(L_{n+p} \geq (\lambda^* + A\eta) n^{1/3} | \mathcal{A}(p)\right) \leq \left(1 - \mathbb{P}\left(L_n \leq \lambda^* n^{1/3}\right)\right)^{\rho^p}.$$

Therefore by Borel-Cantelli lemma, $\limsup_{n\to+\infty} \frac{L_n}{n^{1/3}} \leq \lambda^* + A\eta$ a.s. on S. We let $\eta \to 0$ to conclude the proof. □

Theorem 1.1 is a consequence of Lemma 3.4 and Corollary 3.2: if (1.4) does not hold, then $\rho_+ > 0$ and $\frac{L_n}{n^{1/3}}$ does not converge toward λ^*.

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