Brownian motion and Random Walk above Quenched Random Wall

Bastien Mallein, Piotr Miloś

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Abstract

We study the probability of a random walk staying above a trajectory of another random walk. More precisely, let $\{B_n\}_{n\in\mathbb{N}}$ and $\{W_n\}_{n\in\mathbb{N}}$ be two centered random walks (subject to moment conditions). We establish that $\mathbb{P}\left(\forall_{n\leq N} B_n \geq W_n | W\right) \sim N^{-\gamma}$, where γ is a non-random exponent and $\sim$ is understood on the log scale. In the classical setting (i.e. $W_n \equiv 0$) it is well-known that $\gamma = 1/2$. We prove that for any non-trivial wall W one has $\gamma > 1/2$ and the exponent γ depends only on $\mathrm{Var}(B_1)/\mathrm{Var}(W_1)$.

Further, we prove that these results still hold if B depends weakly on W, this problem naturally emerges in studies of branching random walks in a time-inhomogenous random environment. They are valid also in the continuous time setting, when B and W are (possibly perturbed) Brownian motions. Finally, we present an analogue for Ornstein-Uhlenbeck processes. This time the decay is exponential $\exp(-\gamma N)$.

1 Introduction and main results

We recall a classical result concerning a standard Brownian motion $\{B_t\}_{t\geq 0}$:

$$\mathbb{P}\left(\forall_{s\leq t} 1 + B_s \geq 0\right) = \mathbb{P}\left(\inf_{s\leq t} B_s \geq -1\right) = 1 - 2\mathbb{P}\left(B_t \leq -1\right) = \mathbb{P}\left(|B_t| < 1\right) \sim \frac{1}{\sqrt{2\pi}} t^{-1/2}, \qquad (1.1)$$

where in the second line we utilized the reflection principle. The above can be viewed as the probability of a Brownian motion staying above the wall $f(t) = 0$. It is well-known for any $f(t) = O(t^{1/2-\epsilon}), \epsilon > 0$ the order of decay remains the same:

$$\limsup_{t\to+\infty} t^{1/2}\mathbb{P}\left(\forall_{s\leq t} 1 + B_s \geq f(t)\right) < +\infty \text{ and } \liminf_{t\to+\infty} t^{1/2}\mathbb{P}\left(\forall_{s\leq t} 1 + B_s \geq f(t)\right) > 0.$$

We will concentrate on the case when the wall is sampled randomly and then kept frozen (i.e. a quenched result). In our typical example, when the wall is another Brownian motion, we prove that the order decay is $t^{-\gamma}$. Importantly, $\gamma > 1/2$, which can be interpreted as the relevance of the disorder introduced by the wall. As a benchmark we compare this result with a wall sampled from an i.i.d. sequence, which turns out be the same as for $f(t) = 0$, i.e. $\gamma = 1/2$.

This phenomenon is universal. An analogous result holds for the decay of probability of a random walk staying over a path of another random walk (even for random walks in time-changing random environment). We also extend this result to strongly ergodic diffusions, Ornstein-Uhlenbeck processes in our case. Below we present our results in separate subsections.

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1.1 Brownian motion over Brownian motion

Theorem 1.1. *Let B, W be two independent standard Brownian motions. There exists a continuous function $\gamma : \mathbb{R} \to \mathbb{R}$ such that for any $\beta \in \mathbb{R}$, $0 \le a < b \le +\infty$ and $x > 0$*

$$\lim_{t \to +\infty} \frac{\log \mathbb{P}\left(\forall_{s \le t} x + B_s \ge \beta W_s, B_t - \beta W_t \in (at^{1/2}, bt^{1/2})|W\right)}{\log t} = -\gamma(\beta) \quad a.s. \text{ and in } L^p, \ p \ge 1$$

Moreover, γ is symmetric, convex and for any $\beta \ne 0$,

$$\gamma(\beta) > \gamma(0) = 1/2. \tag{1.2}$$

Consequently, γ is strictly increasing and $\lim_{\beta \to +\infty} \gamma(\beta) = +\infty$.

Remark 1.2. We conjecture that $\beta \mapsto \gamma(\beta)$ is strictly convex.

Remark 1.3. Inequality (1.2) can be interpreted as the relevance of the disorder. Namely, for any $\beta > 0$ we have

$$\lim_{t \to +\infty} \frac{\mathbb{E}\left[-\log \mathbb{P}\left(\forall_{s \le t} x + B_s \ge \beta W_s|W\right)\right]}{\log t} > \lim_{t \to +\infty} \frac{-\log \mathbb{P}\left(\forall_{s \le t} x + B_s \ge \beta W_s\right)}{\log t} = \frac{1}{2}.$$

We compare Theorem 1.1 with a similar result for a random wall with fast decay of correlations. For simplicity, we choose an i.i.d. sequence but the result is still valid for other processes such as Ornstein-Uhlenbeck. In this case, the wall has no impact on the asymptotic behaviour of the probability.

Fact 1.4. *Let $\{X_i\}_{i \in \mathbb{N}}$ be an i.i.d. sequence of random variables such that $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i^2 < +\infty$ and B an independent Brownian motion, we have*

$$\lim_{N \to +\infty} \frac{\log \mathbb{P}\left(\forall_{n \in \{1, \dots, N\}} x + B_n \ge X_n|X\right)}{\log N} = -\frac{1}{2} \quad a.s.$$

The result of Theorem 1.1 is stable under some perturbations of the starting condition and the wall.

Theorem 1.5. *Let B, W be two independent Brownian motions, $f : \mathbb{R}_+ \to \mathbb{R}$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$f(0) = 0 \ , \ \lim_{t \to +\infty} \frac{|f(t)|}{t^{1/2 - \epsilon}} = 0, \ \inf_{t \ge 0} g(t) > 0 \ and \ \lim_{t \to +\infty} \frac{\log g(t)}{\log t} = 0.$$

for some $\epsilon > 0$. For any $\beta \in \mathbb{R}$ and $0 \le a < b \le +\infty$ we have

$$\lim_{t \to +\infty} \frac{\log \mathbb{P}\left(\forall_{s \le t} g(t) + B_s \ge \beta W_s + f(s), B_t - \beta W_t \in (at^{1/2}, bt^{1/2})|W\right)}{\log t}$$

$$= -\gamma(\beta), \quad a.s. \text{ and } L^p, \ p \ge 1.$$

1.2 Ornstein-Uhlenbeck process over Ornstein-Uhlenbeck process

We extend Theorem 1.1 to a more general setting. We recall that an Ornstein-Uhlenbeck process $\{X_t\}_{t \ge 0}$ with parameters $\sigma, \mu > 0$ is a diffusion fulfilling the stochastic differential equation

$$dX_t = \sigma dW_t - \mu X_t dt.$$

Remark 1.6. We recall a well-known relation. Let W be a standard Wiener process then

$$X_t := xe^{-\mu t} + \frac{\sigma}{\sqrt{2\mu}} e^{-\mu t} W_{e^{2\mu t} - 1} \tag{1.3}$$

is an Ornstein-Uhlenbeck process with parameters $\sigma, \mu > 0$ starting from $X_0 = x$.

The main result of this section is following.

Theorem 1.7. *Let X, Y be two independent Ornstein-Uhlenbeck processes with the parameters $\sigma_1 = \sigma_2 = 1$ and $\mu_1, \mu_2 > 0$. There exists two functions γ_{μ_1, μ_2} and δ_{μ_1, μ_2} such that for any $\beta \in \mathbb{R}$, $0 \le a < b \le +\infty$, if $X_0 > \beta Y_0$ then*

$$\lim_{t \to +\infty} \frac{\log \mathbb{P}\left(\forall_{s \le t} X_s \ge \beta Y_s, X_t - \beta Y_t \in (a, b) | Y\right)}{t} = -\gamma_{\mu_1, \mu_2}(\beta), \quad a.s. \text{ and in } L^p, p \ge 1, \qquad (1.4)$$

$$\lim_{t \to +\infty} \frac{\log \mathbb{P}\left(\forall_{s \le t} X_s \ge \beta Y_s, X_t - \beta Y_t \in (a, b)\right)}{t} = -\delta_{\mu_1, \mu_2}(\beta). \qquad (1.5)$$

Both functions are symmetric, convex, and for any $\beta \ne 0$,

$$\gamma_{\mu_1, \mu_2}(\beta) > \delta_{\mu_1, \mu_2}(\beta) > 0. \qquad (1.6)$$

Using (1.3) one checks that the assumption $\sigma_1 = \sigma_2 = 1$ is non-restrictive. Indeed, we set $g(\beta; \sigma_1, \sigma_2, \mu_1, \mu_2)$ be

$$g(\beta; \sigma_1, \sigma_2, \mu_1, \mu_2) := \lim_{t \to +\infty} \frac{-\log \mathbb{P}\left(\forall_{s \le t} X_s \ge \beta Y_s, X_t - \beta Y_t \in (a, b) | Y\right)}{t}$$

where X, Y are two independent Ornstein-Uhlenbeck processes with parameters (μ_1, σ_1^2) and (μ_2, σ_2^2) respectively. By scaling property of the Brownian motion, we have

$$g(\beta; \sigma_1, \sigma_2, \mu_1, \mu_2) = \gamma_{\mu_1, \mu_2}\left(\beta \frac{\sigma_2}{\sigma_1}\right) \text{ and } \gamma_{\mu_1, \mu_2} = \mu_2 \gamma_{\mu_1/\mu_2, 1}.$$

The same relations hold for δ_{μ_1, μ_2}.

Remark 1.8. By (1.3), one can see that Theorem 1.7 for $\mu = \mu_1 = \mu_2$ is equivalent to Theorem 1.1. Indeed, one checks that

$$\gamma_{\mu, \mu}(\beta) = 2\mu\gamma(\beta), \quad \delta_{\mu, \mu}(\beta) = \mu.$$

We stress however that the case of different μ's cannot be expressed in the terms of Theorem 1.1. Moreover, we suspect that $\max(\mu_1, \mu_2) > \delta_{\mu_1, \mu_2}(\beta) > \min(\mu_1, \mu_2)$.

1.3 Random walk in random environment

Results analogous to Theorem 1.1 hold for random walks. Let $\{B_n\}_{n \in \mathbb{N}}, \{W_n\}_{n \in \mathbb{N}}$ be two independent random walks. Some technical difficulties arise while dealing with these processes. In fact, there may exist n such that $x + B_n < W_n$ a.s. To resolve this issue we introduce the event

$$\mathcal{A}_x := \bigcap_{N \ge 0} \{\mathbb{P}\left(\forall_{n \le N} x + B_n \ge W_n | W\right) > 0\}, \qquad (1.7)$$

and study the asymptotic behaviour of the probability for B to stay above W on the event $\mathcal{A}_x$. We first briefly study this event.

Fact 1.9. *For any $x \le x'$ we have $\mathcal{A}_x \subset \mathcal{A}_{x'}$ and $\lim_{x \to +\infty} \mathbb{P}(\mathcal{A}_x) = 1$. Moreover the following conditions are equivalent:*

- *For any $x > 0$, $\mathbb{P}(\mathcal{A}_x) = 1$.*

- $\sup S_B \ge \sup S_W$*, where S_B, S_W are respectively the supports of the measures describing B_1 and W_1 (we allow both the sides to be infinite).*

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Theorem 1.10. *Let B, W be two independent random walks such that $\mathbb{E}B_1 = \mathbb{E}W_1 = 0$ and suppose that there exists $b > 0$ such that $\mathbb{E}e^{b|B_1|} < +\infty$ and $\mathbb{E}e^{b|W_1|} < +\infty$. Then for any $x > 0$ and $0 \leq a < b \leq +\infty$ we have*

$$\lim_{N \to +\infty} \frac{-\log \mathbb{P}\left(\forall_{n \leq N} x + B_n \geq W_n, B_N - W_N \in (aN^{1/2}, bN^{1/2})|W\right)}{\log N}$$

$$= \begin{cases} \gamma\left(\sqrt{\frac{\mathrm{Var}(W_1)}{\mathrm{Var}(B_1)}}\right) & \text{on } \mathcal{A}_x, \\ +\infty & \text{on } \mathcal{A}_x^c. \end{cases} \quad a.s.$$

Theorem 1.10 can be extended to a more general model of a random walk in random environment that we define now.

Let $\mu = \{\mu_n\}_{n \in \mathbb{N}}$ be an i.i.d. sequence with values in the space of probability laws on $\mathbb{R}$. Conditionally on μ we sample $\{X_n\}_{n \in \mathbb{N}}$ a sequence of independent random variables such that X_n has law μ_n. Moreover, we set

$$S_n := \sum_{j=1}^n X_j, \ W_n := -\sum_{j=1}^n \mathbb{E}(X_j|\mu) \text{ and } B_n := S_n + W_n.$$

Note that W is a random walk and conditionally on μ the process B is the sum of independent centred random variables. We make the following assumptions:

(A1) We have $\mathbb{E}W_1 = 0$, $\mathrm{Var}(W_1) \in [0, +\infty)$ and $\mathrm{Var}(B_1) = \mathbb{E}B_1^2 \in (0, +\infty)$.

(A2) There exist $C_1, C_2 > 0$ such that $\mathbb{E}(e^{C_1|B_1|}|\mu) \leq C_2$ a.s.

(A3) There exists $C > 0$ such that $\mathbb{E}e^{C|W_1|} < +\infty$.

We introduce a function $f : \mathbb{N} \to \mathbb{N}$ and we extend definition (1.7) as follows

$$\mathcal{A}_x := \bigcap_{N \geq 0} \{\mathbb{P}\left(\forall_{n \leq N} x + B_n \geq W_n + f(n)|W\right) > 0\}. \tag{1.8}$$

Our result states

Theorem 1.11. *Let S be a random walk in random environment, and B, W as described above. Let $f : \mathbb{N} \mapsto \mathbb{N}$ such that $|f(n)| = o(n^{1/2-\epsilon})$ for some $\epsilon > 0$. For any $x > 0$ and $0 \leq a < b \leq +\infty$ the following limit exists*

$$\lim_{N \to +\infty} \frac{-\log \mathbb{P}\left(\forall_{n \leq N} x + S_n \geq f(n), S_N \in (aN^{1/2}, bN^{1/2})|\mu\right)}{\log N}$$

$$= \begin{cases} \gamma\left(\sqrt{\frac{\mathrm{Var}(W_1)}{\mathrm{Var}(B_1)}}\right) & \text{on } \mathcal{A}_x, \\ +\infty & \text{on } \mathcal{A}_x^c. \end{cases} \quad a.s. \tag{1.9}$$

The previous result holds with some uniformity on the starting position. It is somewhat cumbersome to define an analogue $\mathcal{A}_x$ in this case. For this reason we state an example when $x_N \nearrow +\infty$ and this event is trivial.

Theorem 1.12. *Let S, B and W be as above. Let $f : \mathbb{N} \mapsto \mathbb{N}$ such that $|f(n)| = o(n^{1/2-\epsilon})$ for some $\epsilon > 0$ and $\{x_n\}_n \geq 0$ be such that $x_n \nearrow +\infty$ and $x_n = e^{o(\log n)}$. Then for any $0 \leq a < b \leq +\infty$ the following limit exists*

$$\lim_{N \to +\infty} \frac{\log \mathbb{P}\left(\forall_{n \leq N} x_N + S_n \geq f(n), S_N \in (aN^{1/2}, bN^{1/2})|W\right)}{\log N} = -\gamma\left(\sqrt{\frac{\mathrm{Var}(W_1)}{\mathrm{Var}(B_1)}}\right) \ a.s.$$

1.4 Related works

Our result can be understood from various perspectives. One of them is the so-called entropic repulsion. This question was asked in [2] in the context of the Gaussian free field for $d \geq 3$. Namely, the authors studied the repulsive effect on the interface of the wall which is a fixed realization of an i.i.d. field $\{\phi_x\}_{x \in \mathbb{Z}^d}$. They observe that the tail of ϕ_x plays a fundamental role. When it is subgaussian than the effect of the wall is essentially equivalent to the wall given by 0, while when the tail is heavier than Gaussian the interface is pushed much more upwards. It would be interesting to ask an analogous question in our case. By Fact 1.4 we know already that the disorder has a negligible effect when $\mathbb{E}X_i^{2+\epsilon} < +\infty$, for $\epsilon > 0$. We expect that when $\mathbb{E}X_i^2 = \infty$ the repulsion becomes much stronger.

The paper [2] was followed by [3] which could be seen as an analogue of our work. Namely, the topic of this paper is a Gaussian free field interface conditioned to be above the fixed realization of another Gaussian free field. The authors obtain the precise estimates for the probability of this event and the entropic repulsion induced by the conditioning.

A natural question arising in random walk theory is to study the probability for a random walk to stay non-negative during n units of time. Typically this probability decays as $n^{-1/2}$, which is known as the ballot theorem. Our result stated in Theorem 1.11 provides a version of this result for random walks in random environment. The decay is like $n^{-\gamma}$ for $\gamma \geq 1/2$. Moreover, $\gamma > 1/2$ whenever the quenched random walk is not centered.

This perspective was the initial motivation for analyzing the problems in this paper (more precisely the result given in Theorem 1.11). In fact, the question arises from studies of extremal particles of a branching random walk in a time-inhomogeneous random environment. In the companion paper [8] we show that the randomness of the environment has a slowing effect on the position of the maximal particle. Namely, the logarithmic correction to the speed is bigger than in the standard (time-homogenous) case, which is a consequence of (1.2).

1.5 Organization of the paper

The next section is a collection of preliminary results on the FKG inequality, Ornstein-Uhlenbeck processes and some technical results. We prove in Section 3 the convergence (1.4) using Kingman's theorem. Section 4 shows that the disorder of the wall has an effect (expressed by inequality (1.6)). Section 5 is devoted to a translation of the results from Ornstein-Uhlenbeck to Brownian motion settings, and generalize it to Theorem 1.5. This last theorem is used in Section 6 to study the analogue problem for random walks in random environment. The concluding Section 7 contains further discussion and open questions.

2 Preliminaries and Technical Results

In this section we list a collection of results that will be useful in the rest of the article. We first introduce the so-called FKG inequality for a Brownian motion and an Ornstein-Uhlenbeck process. It states that increasing events are positively correlated. We also list some facts concerning Ornstein-Uhlenbeck and derive technical consequences.

2.1 The FKG inequality for Brownian motion and Ornstein-Uhlenbeck processes

In the proofs we will often use the so-called FKG inequality. Let $\mathcal{C} := \mathcal{C}([0, T)\mathbb{R})$, for $T \geq 0$ be the usual space of continuous functions with the uniform norm topology. We introduce a partial ordering $\prec$ on this space. For two $f, g \in \mathcal{C}$ we set

$$f \prec g \text{ if and only if } \forall_{t \in [0,T]} f(t) \leq g(t). \tag{2.1}$$

By [1, Theorem 4 and Remark 2.1] we have

Fact 2.1. *(The FKG inequality) Let X be a Brownian motion or an Ornstein-Uhlenbeck process and $F, G : \mathcal{C} \to \mathbb{R}$ be bounded measurable functions, which are non-decreasing with respect to $\prec$ then*

$$\mathbb{E}\left[F(X)G(X)\right] \geq \left[\mathbb{E}F(X)\right]\left[\mathbb{E}G(X)\right]. \tag{2.2}$$

The result of [1] is stated for the Brownian motion. It can be transferred easily to the Ornstein-Uhlenbeck process as (1.3) preserves the order $\prec$ defined in (2.1). The same reasoning hold for other proofs in this section. To shorten and simplify proofs, we only work with Brownian motion.

We will often use the following corollary of Fact 2.1.

Corollary 2.2. *Let X be a Brownian motion or an Ornstein-Uhlenbeck process and $\mathcal{A}, \mathcal{B}$ be increasing events (i.e. such that the functions $1_\mathcal{A}$ and $1_\mathcal{B}$ are non-decreasing for $\prec$), then*

$$\mathbb{P}\left(\mathcal{A} \cap B\right) \geq \mathbb{P}\left(\mathcal{A}\right)\mathbb{P}\left(\mathcal{B}\right). \tag{2.3}$$

We also use the following property, sometimes called the strong FKG.

Fact 2.3. *Let X be a Brownian motion or an Ornstein-Uhlenbeck process, $f, g : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ be measurable such that $f(t) \geq g(t)$ for $t \in \mathbb{R}_+$. We assume that $\mathbb{P}\left(\forall_{t \in [0,T]} X_t \geq f(t)\right) > 0$ and $\mathbb{P}\left(\forall_{t \in [0,T]} X_t \geq g(t)\right) > 0$. The probability distribution $\mathbb{P}\left(\cdot | \forall_{t \in [0,T]} X_t \geq f(t)\right)$ stochastically dominates $\mathbb{P}\left(\cdot | \forall_{t \in [0,T]} X_t \geq g(t)\right)$ with respect to $\prec$, in other words for any measurable function $h : \mathbb{R}_+ \to \mathbb{R}$, we have*

$$\mathbb{P}\left(\forall_{t \in [0,T]} X_t \geq h(t) | \forall_{t \in [0,T]} X_t \geq f(t)\right) \geq \mathbb{P}\left(\forall_{t \in [0,T]} X_t \geq h(t) | \forall_{t \in [0,T]} X_t \geq g(t)\right).$$

Proof. Let us assume that $\mathbb{P}\left(X_0 = f(0)\right) = 0$, and $\mathbb{P}\left(X_0 = g(0)\right) = 0$. We leave to the reader removing this condition. Using the Girsanov theorem it is easy to show that $\mathbb{P}\left(\forall_{t \in [0,T]} X_t \geq f(t)\right) = \mathbb{P}\left(\forall_{t \in [0,T]} X_t > f(t)\right)$. We will change $\geq$ to $>$ and vice-versa whenever convenient. Notice that $\left\{\forall_{t \in [0,T]} X_t \geq f(t)\right\} = \left\{\forall_{t \in [0,T]} X_t \geq \tilde{f}(t)\right\}$, where $\tilde{f} : [0,T] \mapsto \mathbb{R}$ is given by

$$\tilde{f}(x) := \inf_{\omega \in \mathcal{F}} \omega(x),$$

for $\mathcal{F} := \left\{\omega \in \mathcal{C} : \forall_{t \in [0,T]} \omega(t) > f(t)\right\}$. It is well-known that $\tilde{f}$ is upper semincontinous. Thus without loss of generality we assume that both f and g are upper semincontinous. By Baire's theorem there exists a sequence $\{f_n\}_n$ such that $f_n \in \mathcal{C}$ and $f_n(t) \searrow f(t)$ point-wise. This in particular implies that $\mathcal{A}_n := \left\{\forall_{t \in [0,T]} X_t > f_n(t)\right\}$ is an increasing sequence of events and $\bigcup_n \mathcal{A}_n = \left\{\forall_{t \in [0,T]} X_t > f(t)\right\}$ (we tacitly assume that we work on the canonical Wiener space). We have an analogous sequence $\{g_n\}$ for g. Taking $\min(f_n, g_n)$ we may assume that $g_n \leq f_n$.

For any continuous f_n and $\epsilon > 0$ we can find a finite set $0 \leq t_1 < t_2 \ldots < t_k \leq T$ such that

$$\mathbb{P}\left(\left\{\forall_{i \in \{1,\ldots,n\}} X_{t_i} \geq f(t_i)\right\} \setminus \left\{\forall_{t \in [0,T]} X_t \geq f(t)\right\}\right) \leq \epsilon.$$

Assume now that the statement of the fact is false. Then there exists a measurable, bounded non-decreasing function $F : \mathcal{C} \to \mathbb{R}$ such that

$$\mathbb{E}\left(F | \forall_{t \in [0,T]} X_t \geq f(t)\right) < \mathbb{E}\left(F | \forall_{s \in [0,T]} X_t \geq g(t)\right). \tag{2.4}$$

Using previous arguments we can find n and $0 \leq t_1 < \ldots < t_k \leq T$ such that

$$\mathbb{E}\left(F | \forall_{i \in \{1,\ldots,k\}} X_{t_i} \geq f_n(t_i)\right) < \mathbb{E}\left(F | \forall_{i \in \{1,\ldots,k\}} X_{t_i} \geq g_n(t_i)\right). \tag{2.5}$$

Using the same techniques as [5, B.6] one shows that $\mathbb{P}\left((X_{t_1}, \ldots, X_{t_k}) \in \cdot | \forall_{i \in \{1,\ldots,k\}} X_{t_i} \geq f_n(t_i)\right)$ stochastically dominates $\mathbb{P}\left((X_{t_1}, \ldots, X_{t_k}) \in \cdot | \forall_{i \in \{1,\ldots,k\}} X_{t_i} \geq g_n(t_i)\right)$. We notice that conditionally on $X_{t_i} = x$ and $X_{t_{i+1}} = y$ the process $\left\{X_t - \left[(t_{i+1} - t)x + (t - t)y\right] / (t_{i+1} - t_i)\right\}_{t \in [t_i, t_{i+1}]}$ is a Brownian bridge. Moreover if we condition on the whole vector $(X_{t_1}, X_{t_2}, \ldots, X_{t_k})$, by the Markov property the brides on the different intervals are independent. Finally by simple calculations we arrive at contradiction with (2.5) and consequently also with (2.4). □

2.2 Integrability estimates for Ornstein-Uhlenbeck processes

We list standard estimates on Ornstein-Uhlenbeck processes.

Fact 2.4. *Let X be an Ornstein-Uhlenbeck process with parameters $\sigma, \mu > 0$ starting from $X_0 = x$.*

1. *The process X is a strong Markov process with an invariant measure $\mathcal{N}\left(0, \sigma^2/(2\mu)\right)$. For any $t > 0$ the random variable X_t is distributed as $\mathcal{N}\left(xe^{-\mu t}, \frac{\sigma^2}{2\mu}(1 - e^{-2\mu t})\right)$.*

2. *For any $y \in \mathbb{R}$ the tail of $\inf\{t \geq 0 : X_t = y\}$ decays exponentially.*

3. *The process $\{\tilde{X}_t\}_{t \geq 0}$ given by $\tilde{X}_t := X_t - e^{-\mu t}X_0$ is an Ornstein-Uhlenbeck process with parameters $\sigma, \mu > 0$ starting from $\tilde{X}_0 = 0$.*

4. *The random variable $M := \sup_{t \leq 1} X_t$ has Gaussian concentration i.e. for some constants $C, c > 0$ there is*
$$\mathbb{P}\left(M > x\right) \leq C\exp(-cx^2), \quad x \geq 0.$$

We recall standard Gaussian tail estimates.

Fact 2.5. *Let Z be a standard Gaussian random variable and $x > 0$. Then we have*

$$\frac{1}{\sqrt{2\pi}}\frac{x}{1 + x^2}e^{-x^2/2} \leq \mathbb{P}\left(Z \geq x\right) \leq \frac{1}{\sqrt{2\pi}}\frac{1}{x}e^{-x^2/2}. \tag{2.6}$$

We present a convex analysis result.

Lemma 2.6. *Let X be a Brownian motion or Ornstein-Uhlenbeck process and $h_1, h_2 : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ be càdlàg functions such that*

$$\mathbb{P}\left(\forall_{s \geq 0} X_s \geq h_1(s)\right) > 0, \quad \mathbb{P}\left(\forall_{s \geq 0} X_s \geq h_2(s)\right) > 0.$$

Then the function

$$\begin{array}{rcl} [0, 1] & \longmapsto & \mathbb{R}_+ \\ \lambda & \longmapsto & -\log \mathbb{P}\left(\forall_{s \geq 0} X_s \geq \lambda h_1(s) + (1 - \lambda)h_2(s)\right) \end{array}$$

is convex.

Proof. By standard limit arguments it is enough to show that for any $n, N \in \mathbb{N}$ the function

$$\begin{array}{rcl} [0, 1] & \mapsto & \mathbb{R}_+ \\ \lambda & \mapsto & -\log \mathbb{P}\left(\forall_{k \leq N} X_{k/n} \geq \lambda h_1(k/n) + (1 - \lambda)h_2(k/n)\right) \end{array} \tag{2.7}$$

is convex. To this end we will use the Prekopa-Leindler inequality along the lines of the proof below [4, Theorem 7.1]. Let $H^\lambda(x) = d(x)1_{\forall_k x_k \geq \lambda h_1(k/n) + (1-\lambda)h_2(k/n)}(x)$, where by d we denote the joint density of $(X_{1/n}, X_{2/n}, \ldots, X_{N/n})$. The density d is log-concave i.e for any $\lambda \in (0, 1)$ and $x, y \in \mathbb{R}^N$ we have $d(\lambda x + (1 - \lambda)y) \geq d(x)^\lambda d(y)^{(1-\lambda)}$. Similarly

$$1_{\forall_k \lambda x_k + (1-\lambda)y_k \geq \lambda h_1(k/n) + (1-\lambda)h_2(k/n)} \geq \left(1_{\forall_k x_k \geq h_1(k/n)}\right)^\lambda \left(1_{\forall_k y_k \geq h_2(k/n)}\right)^{1-\lambda}.$$

Thus the assumption of the Prekopa-Leindler inequality is fulfilled i.e.

$$H^\lambda(\lambda x + (1 - \lambda)y) \geq \left(H^1(x)\right)^\lambda \left(H^0(y)\right)^{1-\lambda}.$$

Now [4, Theorem 7.1] implies (2.7). $\qquad\square$

Finally, we prove the random variable $-\log \mathbb{P}\left(\forall_{s \leq 1} X_s \geq Y_s | Y\right)$ is integrable.

Fact 2.7. *Let X, Y be Ornstein-Uhlenbeck processes.*

1. *Let $C, c > 0$ and $x \geq 0$. We assume that $X_0 \sim \mathcal{N}(x, c_1^2)$ with $c_1 \geq c$ and $Y_0 \sim \mathcal{N}(0, C_1)$ with $C_1 \leq C$. Then there exists a constant $\tilde{C} > 0$ such that*

$$\mathbb{E}\left[-\log \mathbb{P}\left(\forall_{s \leq 1} X_s \geq Y_s | Y\right)\right] \leq \tilde{C} \tag{2.8}$$

2. *Let $X_0 = x > 0$ and $Y_0 = 0$, we set $\rho = \inf\{t \geq 0 : Y_t = 0, \exists s < t : |Y_s| = 1\}$. Then*

$$\mathbb{E}\left[-\log \mathbb{P}\left(\forall_{s \leq \rho} X_s \geq Y_s | Y\right)\right] < +\infty. \tag{2.9}$$

3. *Let $X_0 = 0$, $Y_0 = 0$ and $a, b > 0$ then the random variable*

$$-\log \mathbb{P}\left(\forall_{s \leq 1} X_s \geq Y_s - a, X_1 \geq Y_1 + b | Y\right)$$

has exponential moments.

Proof. Let X, Y be two independent Ornstein-Uhlenbeck processes of parameters $(\mu_1, \sigma_1), (\mu_2, \sigma_2)$. We first prove point 3. By the FKG property (2.3) we have

$$-\log \mathbb{P}\left(\forall_{s \leq 1} X_s \geq Y_s - a, X_1 \geq Y_1 + b | Y\right) \leq -\log \mathbb{P}\left(\forall_{s \leq 1} X_s \geq Y_s - a | Y\right) - \log \mathbb{P}\left(X_1 \geq Y_1 + b | Y\right).$$

Proving the exponential integrability of the second summand is easy and left to the reader. Let us denote the first one by H and apply (1.3). We have

$$H \leq -\log \mathbb{P}\left(\forall_{s \leq 1} B_{t_1(s)} \geq \beta |W_{t_2(s)}| - a' | W\right).$$

where B, W are Brownian motions, $\beta = \frac{\sigma_2}{\sigma_1}$, $t_1(s) = e^{\mu_1 s} - 1$ and $t_2(s) = e^{\mu_2 s} - 1$ ((μ_1, μ_2 are parameters of the Ornstein-Uhlenbeck processes X and Y). The constants $a', \beta > 0$ can be calculated explicitly but do not matter for the calculations. We denote

$$\mathcal{A}_1 := \left\{\forall_{i \in \mathbb{N}} \forall_{s \in [\frac{3}{4} 2^{-i}, 2^{-i}]} B_{t_1(s)} \geq \beta |W_{t_2(s)}| - a'\right\},$$

$$\mathcal{A}_2 := \left\{\forall_{i \in \mathbb{N} \setminus \{0\}} \forall_{s \in [2^{-i}, \frac{3}{2} 2^{-i}]} B_{t_1(s)} \geq \beta |W_{t_2(s)}| - a'\right\}.$$

Using the FKG property (2.3) we write

$$H \leq -\log \mathbb{P}\left(\mathcal{A}_1 \cap \mathcal{A}_2 | W\right) \leq -\log \mathbb{P}\left(\mathcal{A}_1 | W\right) - \log \mathbb{P}\left(\mathcal{A}_2 | W\right).$$

Let us denote the first expression as H_1. We will prove that it is exponentially integrable. One can apply exactly the same argument to the second one. We notice that

$$\mathcal{B} := \left\{\forall_{i \in \mathbb{N}} \forall_{s \in [\frac{3}{4} 2^{-i}, 2^{-i}]} B_{t_1(s)} - B_{t_1(2^{-i}/2)} \geq \beta |W_{t_2(s)} - W_{t_2(2^{-i}/2)}| - 2^{-i/4} a'/8\right\} \subset \mathcal{A}_1.$$

Let $\theta > 0$, using the fact that the increments of a Brownian motion are independent we obtain

$$L(\theta) := \mathbb{E} e^{\theta H_1} = \mathbb{E}\left(\mathbb{P}\left(\mathcal{B}|W\right)^{-\theta}\right) = \prod_{i \in \mathbb{N}} L_i(\theta),$$

where

$$L_i(\theta) := \mathbb{E}\left(\mathbb{P}\left(\forall_{s \in [\frac{3}{2} 2^{-i}, 2^{-i}]} B_{t_1(s) - t_1(2^{-i}/2)} \geq \beta |W_{t_2(s) - t_2(2^{-i}/2)}| - 2^{-i/4} a'' | W\right)^{-\theta}\right),$$

and $a'' := a'/8$. By the Brownian scaling we get

$$L_i(\theta) = \mathbb{E}\left(\mathbb{P}\left(\forall_{s \in [\frac{3}{2} 2^{-i}, 2^{-i}]} B_{2^i [t_1(s) - t_1(2^{-i}/2)]} \geq \beta |W_{2^i [t_2(s) - t_2(2^{-i}/2)]}| - 2^{i/4} a'' | W\right)^{-\theta}\right).$$

It is easy to check that there exist $0 < c_1 < C_1$ such that for any $i \in \mathbb{N}$ we have $c_1 \geq 2^i[t_1(\frac{3}{4}2^{-i}) - t_1(2^{-i}/2)]$ and $C_1 \leq 2^i[t_1(2^{-i}) - t_1(2^{-i}/2)]$ analogously we define $0 < c_2 < C_2$ for t_2. We put

$$M := \sup_{s \in [c_2, C_2]} \beta W_s, \quad m := \inf_{s \in [c_1, C_1]} B_s.$$

With this notation we estimate

$$L_i(\theta) \leq \mathbb{E}\left(\mathbb{P}\left(m \geq M - 2^{i/4}a''|W\right)^{-\theta}\right) =: \tilde{L}_i(\theta).$$

We have

$$0 \leq \tilde{L}_i(\theta) - 1 = \mathbb{E}\left(\frac{1 - \mathbb{P}\left(m \geq M - 2^{i/4}a''|W\right)^{\theta}}{\mathbb{P}\left(m \geq M - 2^{i/4}a''|W\right)^{\theta}}\right).$$

It is well-known that for $x \geq 0$ we have $q(x) := \mathbb{P}(M > x) \leq C_3 e^{-c_3 x^2}$ for some $c_3, C_3 > 0$. We also prove a bound from below for the tail of m. Namely, we have

$$\mathbb{P}(m > x) \geq \mathbb{P}\left(\{B_{c_1} \geq x + 5\} \cap \left\{\sup_{s \in [c_1, C_1]} |B_s - B_{c_1}| \leq 5\right\}\right)$$

$$\geq \mathbb{P}(B_{c_1} \geq x + 5)\,\mathbb{P}\left(\sup_{s \in [c_1, C_1]} |B_s - B_{c_1}| \leq 5\right) \geq C_4 e^{-c_4 x^2},$$

for some $c_4, C_4 \geq 0$. We combine the estimates to get

$$\tilde{L}_i(\theta) - 1 \leq 2\theta\mathbb{P}\left(M \leq 2^{i/4}a''/2\right)\mathbb{P}\left(m \leq -2^{i/4}a''/2\right) + \mathbb{P}\left(M \geq 2^{i/4}a''/2\right)\mathbb{P}(m \geq 0)^{-\theta}$$

$$+ \int_0^{+\infty} \mathbb{P}(m \geq y)^{-\theta}\,\mathbb{P}\left(M \geq y + 2^{i/4}a''\right)\mathrm{d}y.$$

One easily checks that the first two terms estimate by $\leq C_5 e^{-c_5 i}$ for $c_5, C_5 > 0$. For the last one we have

$$\int_0^{+\infty} \mathbb{P}(m \geq y)^{-\theta}\,\mathbb{P}\left(M \geq y + 2^{i/4}a''\right)\mathrm{d}y \leq C_4^{-\theta}C_3 \int_0^{+\infty} e^{\theta c_4 y^2}e^{-c_3(y + 2^{i/4})^2}\,\mathrm{d}y \leq C_6 e^{-c_6 i},$$

for $c_6, C_6 > 0$, where the last estimate holds under assumption that θ is small (it is enough that $\theta c_4 < c_3$). Putting together we obtain that

$$0 \leq \tilde{L}_i(\theta) - 1 \leq C_5 e^{-c_5 i} + C_6 e^{-c_6 i}.$$

This is enough to claim that $\prod_{i \in \mathbb{N}} \tilde{L}_i(\theta) < +\infty$ and consequently also $\prod_{i \in \mathbb{N}} L_i(\theta) < +\infty$ and thus the exponential integrability of H_1.

Similar, but simpler, calculations prove that

$$\mathbb{E}\left[-\log \mathbb{P}\left(\forall_{s \leq 1} X_s \geq Y_s | Y, X_0 = x, Y_0 = 0\right)\right] < +\infty. \tag{2.10}$$

Let us now pass to the proof of (2.9). By the FKG inequality (2.3) we have

$$\mathbb{E}\left[-\log \mathbb{P}\left(\forall_{s \leq \rho} X_s \geq Y_s | Y\right)\right] \leq \mathbb{E}\left[-\log \mathbb{P}\left(\forall_{s \leq 1} X_s \geq Y_s | Y\right)\right]$$

$$+ \mathbb{E}\left[\sum_{i=1}^{\lceil \rho \rceil} -\log \mathbb{P}\left(\forall_{s \in [i, i+1]} X_s \geq \sup_{s \in [0, \rho]} Y_s | Y\right)\right].$$

The first term is finite by (2.10). To treat the second one we study

$$p_i(m) := -\log \mathbb{P}\left(\forall_{s \in [i,i+1]} X_s \geq m\right).$$

By point 3 of Fact 2.4 the process $\tilde{X}_t := X_{i+t} - X_{i+t}e^{-\mu t}$ is an Ornstein-Uhlenbeck process starting from $\tilde{X}_0 = 0$. Therefore we have

$$p_i(m) \leq -\log \mathbb{P}\left(X_i \geq me^{\mu} + 1\right) - \log \mathbb{P}\left(\forall_{s \in [0,1]} \tilde{X}_s \geq -1\right).$$

Clearly $-\log \mathbb{P}\left(\forall_{s \in [0,1]} \tilde{X}_s \geq -1\right) > -\infty$, using point 1 of Fact 2.4 and (2.6) one easily checks that

$$p_i(m) \leq C_9(m^2 + 1),$$

for $C_9 > 0$. Recalling that $M = \sup_{s \in [0,\rho]} Y_s$ and Fact 3.3 we conclude

$$\mathbb{E}\left(\sum_{i=1}^{\lfloor \rho \rfloor} p_i(M)\right) \leq \mathbb{E}\left(\rho \left[C_9(M^2 + 1)\right]\right) \leq \left(\mathbb{E}\left(\rho^2\right) \mathbb{E}\left[C_9(M^2 + 1)\right]^2\right)^{1/2} < +\infty.$$

The estimate (2.8) follows by similar calculations and Fact 2.4. $\qquad\square$

3   Existence and properties of the function γ

Let X, Y be two independent Ornstein-Uhlenbeck processes with parameters (μ_1, σ_1) and (μ_2, σ_2). The main result of the section is the existence of $\gamma > 0$ such that

$$\lim_{t \to +\infty} \frac{-\log \mathbb{P}\left(\forall_{s \leq t} X_s \geq \beta Y_s, X_t - \beta Y_t \in (a,b)|Y\right)}{t} = \gamma \text{ a.s.}$$

To make notation lighter we write γ instead of $\gamma_{\mu_1,\mu_2}(\sigma_2/\sigma_1)$. We start proving the annealed part of Theorem 1.7.

Lemma 3.1. *There exists $\delta > 0$ such that for any $0 \leq a < b \leq +\infty$,*

$$\lim_{t \to +\infty} \frac{-\log \mathbb{P}\left(\forall_{s \leq t} X_s \geq \beta Y_s, X_t - \beta Y_t \in (a,b)\right)}{t} = \delta.$$

Proof. This is a standard result from the spectral theory. We set $T = \inf\{t \geq 0 : X_t = Y_t\}$, the process $\{(X_{s \wedge T}, Y_{s \wedge T})\}_{s \geq 0}$ is a Markov process. We set L its infinitesimal operator. The exponent δ is the first eigenvalue of L on the domain $D := \{(x,y) \in \mathbb{R}^2 : x \geq y\}$. $\qquad\square$

3.1   Path decomposition

We provide a decomposition of the path Y. This decomposition is used both in proofs of (1.4) and (1.6). We define the random variables $\{\tau_i\}_{i \geq 0}, \{\rho_i\}_{i \geq 0}$ such that $\rho_0 = 0$ and
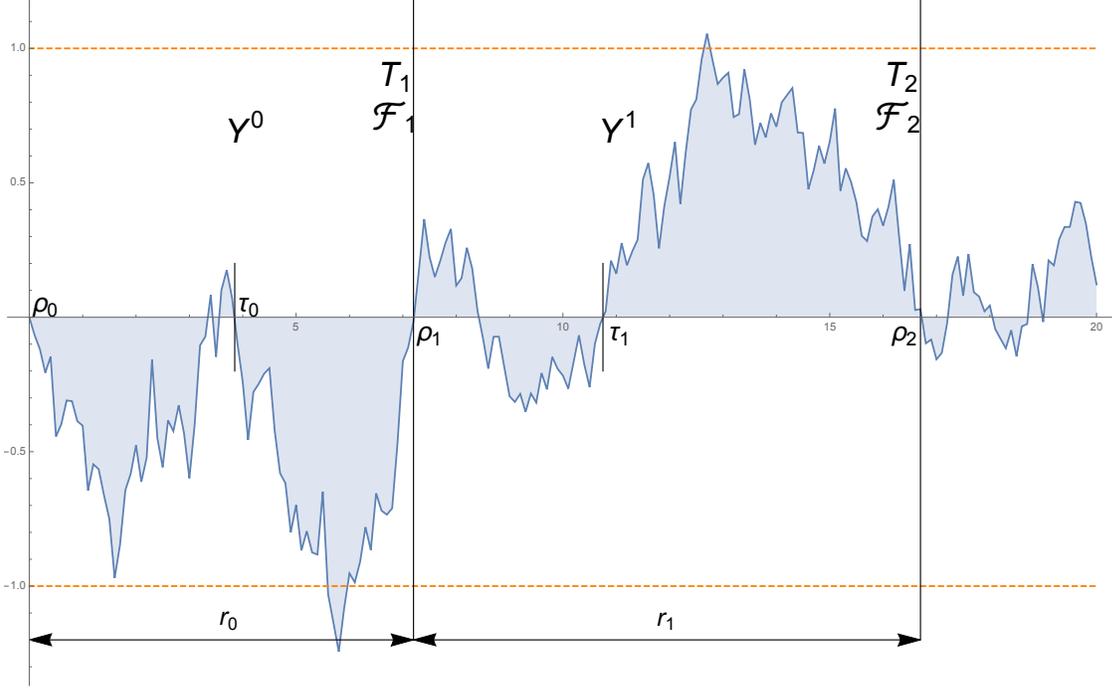
$$\rho_{i+1} := \inf\left\{t \geq \rho_i : Y_t = 0 \text{ and } \exists_{s \in (\rho_i, t)}|Y_s| = 1\right\} \quad \text{and} \quad \tau_i := \sup\left\{t < \rho_{i+1} : Y_t = 0\right\}.$$

We also define $r_i := \rho_{i+1} - \rho_i$ and denote

$$Y^i(t) := Y_{t+\rho_i}, \quad t \in [0, r_i]. \tag{3.1}$$

Remark 3.2. Note that ρ_i is a stopping time (contrary to τ_i). Details of the definition of ρ_i and τ_i are not important. What matters for our proofs is that on the interval $[\tau_i, \rho_{i+1}]$ the process performs a "macroscopic" excursion which is symmetric around 0.

Figure 3.1: Notation used in the paper.

Fact 3.3. *The sequence $\left\{(Y^i, r_i)\right\}_{i \geq 0}$ is i.i.d. and the random variables r_i and $M^i := \sup_{s \leq r_i} Y^i(s)$ have tails which decay exponentially.*

Proof. The first statement follows by the fact that $Y_{\rho_i} = 0$ and the strong Markov property of Y. We define $\tilde{\rho} := \inf \{t \geq 0 : |Y_t| = 1\}$ then $\rho_1 = \inf \{t \geq \tilde{\rho} : Y_t = 0\}$.

By point 2 of Fact 2.4 both $\tilde{\rho}$ and $\rho_1 - \tilde{\rho}$ have exponential tails, thus also $r_1 = \rho_1$ has an exponential tail. Let us now consider $x \geq 0$ and

$$\mathbb{P}\left(M^i \geq x\right) \leq \mathbb{P}\left(\sup_{s \leq x} Y^i(s) \geq x\right) + \mathbb{P}\left(r_i \geq x\right) \leq \sum_{k=1}^{\lceil x \rceil} \mathbb{P}\left(\sup_{s \in [k-1, k]} Y^i(s) \geq x\right) + \mathbb{P}\left(r_i \geq x\right).$$

We deduce the concentration of M^i using point 4 of Fact 2.4. $\qquad \square$

3.2 A modified version of Theorem 1.7

In a first time, we study the asymptotic behaviour of $\log \mathbb{P}(\forall_{u \in [0, \rho_n]} X_u \geq Y_u | Y)$ as $n \to +\infty$, using Kingman's subadditive ergodic theorem.

Lemma 3.4. *We assume that $Y_0 = 0$. For any $0 < a < b \leq +\infty$, there exists $\tilde{\gamma}_{a,b}$ such that*

$$\lim_{n \to +\infty} \frac{-\log \inf_{x \in (a,b)} \mathbb{P}(\forall_{u \in [0, \rho_n]} X_u \geq Y_u, X_{\rho_n} \in (a,b) | Y, X_0 = x)}{\log n} = \tilde{\gamma}_{a,b} \quad \text{a.s. and in } L^1.$$

Proof. Let $0 < a < b \leq +\infty$, we set $I = (a, b)$. For any $0 \leq m < n$, we set

$$p_{m,n} := \inf_{x \in (a,b)} \mathbb{P}(\forall_{u \in [\rho_m, \rho_n]} X_u \geq Y_u, X_{\rho_n} \in I | Y, X_{\rho_m} = x) \tag{3.2}$$

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and $q_{m,n} = -\log p_{m,n}$. Note that when $b = +\infty$, the FKG inequality (2.3) the minimal value of $p_{m,n}$ is attained at $x = a$. We prove that $\{q_{m,n}\}_{n > m \geq 1}$ fulfils the assumptions of Kingman's subadditive ergodic theorem as stated in [6, Theorem 9.14].

By the Markov property, as $Y_{\rho_n} = 0$ for any $1 \leq m < n$ we have

$$p_{0,n} = p_{0,m}\mathbb{P}(\forall_{u \in [\rho_m, \rho_n]} X_u \geq Y_u, X_{\rho_n} \in I | Y, \forall_{u \in [0, \rho_m]} X_u \geq Y_u, X_{\rho_m} \in I) \geq p_{0,m} p_{m,n},$$

thus $q_{0,n} \leq q_{0,m} + q_{m,n}$, which is the subadditivity condition [6, (9.9)].

We fix $k \geq 1$. We recall that $\{Y^l\}_{l \geq 0}$ is i.i.d. Consequently the sequence

$$\{q_{lk,(l+1)k}\}_{l \geq 0} \tag{3.3}$$

is i.i.d. and condition [6, (9.7)] is fulfilled. Further, condition [6, (9.8)] follows by the fact that the process $\{Y_{t+\rho_k}\}_{t \geq 0}$ is an Ornstein-Uhlenbeck process distributed as Y. As $q_{0,n} \geq 0$; Fact 2.7 implies that $\mathbb{E}q_{0,1} < +\infty$ thus [6, Theorem 9.14] applies and

$$\lim_{n \to +\infty} \frac{-\log p_{0,n}}{n} = \lim_{n \to +\infty} \frac{q_{0,n}}{n} =: \tilde{\gamma}_{a,b}, \quad \text{a.s. and } L^1. \tag{3.4}$$

The constant $\tilde{\gamma}_{a,b}$ is non-random since (3.3) is ergodic. $\qquad\square$

In a second time, we prove the constant $\tilde{\gamma}$ does not depend on (a, b).

Lemma 3.5. *There exists $\tilde{\gamma} > 0$ such that for any $0 < a < +\infty$ we have $\tilde{\gamma} = \tilde{\gamma}_{a,+\infty}$.*

Proof. For any $a \geq 0$ and $x > 0$, we write

$$p_n(x, a) := \mathbb{P}(\forall_{u \in [0, \rho_n]} X_u \geq Y_u, X_{\rho_n} > a | Y, X_0 = x),$$

and accordingly $q_n(x, a) := -\log p_n(x, a)$. We prove that

$$\lim_{n \to +\infty} \frac{q_n(x, a)}{n} = \tilde{\gamma}, \quad \text{a.s. and } L^1, \tag{3.5}$$

exists and is independent of $x > 0, a \geq 0$. Fix $x > 0$, by (3.4), we know that

$$\lim_{n \to +\infty} \frac{q_n(x, x)}{n} = \tilde{\gamma}_{x,+\infty}, \quad \text{a.s. and } L^1,$$

as the minimum in (3.2) is attained in $x = a$. We prove that $p_n(x, 0, +\infty)$ behaves similarly. As $p_n(x, x) \leq p_n(x, 0, +\infty)$, we have

$$\begin{aligned} 0 \leq d_n &:= \frac{q_n(x, x, +\infty)}{n} - \frac{q_n(x, 0, +\infty)}{n} = -\frac{1}{n} \log \frac{p_n(x, x, +\infty)}{p_n(x, 0, +\infty)} \\ &= -n^{-1} \log \mathbb{P}\left(X_{\rho_n} \geq x | \forall_{u \in [\rho_0, \rho_n]} X_u \geq Y_u, X_0 = x, Y\right) \\ &\leq -n^{-1} \log \mathbb{P}\left(X_{\rho_n} \geq x | X_0 = x, Y\right), \end{aligned}$$

by the FKG inequality. We conclude easily that $d_n \to 0$ a.s. and in L^1. By a simple monotonicity argument we conclude that convergence (3.5) holds for any pair (x, a), when $x > 0$ and $a \in [0, x]$ and the limit depends only on x.

We now fix $x_1 > x_2 > 0$, we have $p_n(x_1, 0, +\infty) \leq p_n(x_2, 0, +\infty)$. On the other hand

$$\frac{q_n(x_2, 0, +\infty)}{n} \leq \frac{-\log \mathbb{P}\left(\forall_{s \leq \rho_1} X_s \geq Y_s, X_{\rho_1} \geq x_1 | Y, X_0 = x_2\right)}{n} + \frac{q_n(x_1, 0, +\infty)}{n}. \tag{3.6}$$

This proves that $\tilde{\gamma} = \tilde{\gamma}_{x,+\infty}$ does not depend of x. $\qquad\square$

Lemma 3.6. *For any $0 < a < b \leq +\infty$, we have $\tilde{\gamma} = \tilde{\gamma}_{a,b}$*

Proof. Using the previous lemma, we set $\tilde{\gamma} = \gamma_{a,+\infty}$ for any $a > 0$. To show the claim it is enough to prove that for any $b < +\infty$ the limit cannot be smaller. We define $n_0 = \lceil n - C_1 \log n \rceil$ for $C_1 > 1$ to be fixed later and $n_1 = n - 1$. Using the Markov property we decompose

$$\mathbb{P}(\forall_{u \leq \rho_n} X_u \geq Y_u, X_{\rho_n} \in (a,b) | Y) \geq p_1(n) p_2(n) p_3(n),$$

where

$$p_1(n) := \inf_{x \in (a,b)} \mathbb{P}(\forall_{u \leq \rho_{n_0}} X_u \geq Y_u, X_{\rho_{n_0}} \in (a,n) | Y, X_0 = x),$$

$$p_2(n) := \inf_{x \in [a,n]} \mathbb{P}(\forall_{u \in [\rho_{n_0}, \rho_{n_1}]} X_u \geq Y_u, X_{\rho_{n_1}} \in (a, \log n) | Y, X_{\rho_{n_0}} = x),$$

$$p_3(n) := \inf_{x \in [a, \log n]} \mathbb{P}(\forall_{u \in [\rho_{n_1}, \rho_n]} X_u \geq Y_u, X_{\rho_n} \in (a,b) | Y, X_{\rho_{n_1}} = x).$$

We prove that

$$\lim_{n \to +\infty} \frac{-\log p_1(n)}{n} = \tilde{\gamma}, \quad \lim_{n \to +\infty} \frac{-\log p_2(n)}{n} = 0, \quad \liminf_{n \to +\infty} \mathbb{P}\left(-\log p_3 \leq n^{1/2}\right) > 0, \qquad (3.7)$$

where the first convergences hold in probability. This limit and (3.4) imply the claim of the lemma. Let us now treat the second convergence. Clearly we have

$$p_2(n) \geq \inf_{x \in [a,n]} \mathbb{P}(\forall_{u \in [\rho_{n_0}, \rho_{n_1}]} X_u \geq Y_u, X_{\rho_{n_1}} \geq a | Y, X_{\rho_{n_0}} = x) - \sup_{x \in [a,n]} \mathbb{P}(X_{\rho_{n_1}} \geq \log n | Y, X_{\rho_{n_0}} = x)$$

$$\geq \mathbb{P}(\forall_{u \in [\rho_{n_0}, \rho_{n_1}]} X_u \geq Y_u, X_{\rho_{n_1}} \geq a | Y, X_{\rho_{n_0}} = a) - \mathbb{P}(X_{\rho_{n_1}} \geq \log n | Y, X_{\rho_{n_0}} = n).$$

We fix C_1 such that with high probability $\mathbb{P}(X_{\rho_{n_1}} \geq \log n | Y, X_{\rho_{n_0}} = n) \leq \exp(-2C_1 \tilde{\gamma} \log n)$. This can be done since on the interval $[n_0, n_1]$ the drift of the Ornstein-Uhlenbeck process removes the starting condition $X_{n_0} = n$. Using (3.4) we conclude that the second term is negligible and in fact we have

$$\limsup_{n \to +\infty} \frac{-\log p_2(n)}{C_1 \log n} \leq \tilde{\gamma}, \quad \text{in probability.}$$

This yields the second convergence in (3.7). An analogous proof gives the first one. For the last one we consider an event $\mathcal{A}_n := \{\rho_n - \rho_{n_1} \in [1,2], \sup_{s \in [\rho_{n_1}, \rho_n]} |Y_s| \leq a/2\}$. Clearly,

$$p_3(n) \geq \inf_{x \in [1, \log n]} \mathbb{P}(X_{\rho_n} \in (a,b) | Y, X_{\rho_{n_1}} = x) \mathbb{P}(\forall_{u \in [\rho_{n_1}, \rho_n]} X_u \geq Y_u | Y, X_{\rho_{n_1}} = a, X_{\rho_n} = a).$$

Conditionally on $\mathcal{A}_n$ the second term is bounded from below by a constant and the first one by $\exp(-(\log n)^3)$. We conclude that for large n there is $\mathbb{P}(-\log p_3 \leq n^{1/2}) \geq \mathbb{P}(\mathcal{A}_n)$. This finishes the proof as the right-hand side is non-zero and does not depend on n. $\qquad \square$

Finally, we prove the limit in Lemma 3.4 holds for any starting position.

Lemma 3.7. *For any $x > y$ and $0 \leq a < b \leq +\infty$, we have*

$$\lim_{n \to +\infty} \frac{-\log \mathbb{P}(\forall_{u \in [0, \rho_n]} X_u \geq Y_u, X_{\rho_n} \in (a,b) | Y, X_0 = x, Y_0 = y)}{\log n} = \tilde{\gamma} \quad a.s \text{ and in } L^1,$$

Proof. We prove this result assuming $Y_0 = 0$, leaving to the reader removing this condition. We write

$$q_n(x, a, b) = -\log \mathbb{P}(\forall_{u \in [0, \rho_n]} X_u \geq Y_u, X_{\rho_n} \in (a,b) | Y, X_0 = x)$$

Using the two previous lemmas, we have

$$\limsup_{n\to\infty} \frac{q_n(x,a,b)}{n} \leq \lim_{n\to+\infty} \frac{\sup_{x\in(a,b)} q_n(x,a,b)}{n} = \tilde{\gamma},$$

as $\sup_{x\in(a,b)} q_n(x,a,b) = q_{0,n}$. Similarly, for any $x \geq a$, we have

$$\liminf_{n\to+\infty} \frac{q_n(x,a,b)}{n} \geq \liminf_{n\to+\infty} \frac{q_n(x,a,+\infty)}{n} \geq \liminf_{n\to+\infty} \frac{q_n(a,a,+\infty)}{n} \geq \tilde{\gamma},$$

by the FKG inequality. Finally, using a reasoning similar to (3.6), a similar inequality holds for $x \leq a$. Consequently, the convergence

$$\lim_{n\to+\infty} \frac{q_n(x,a,b)}{n} = \tilde{\gamma}, \quad \text{a.s. and } L^1, \tag{3.8}$$

holds for any a, b, x. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.3 Existence and basic properties of γ

We now prove that (1.4) holds. To this end we state an auxiliary fact, whose proof is postponed to the end of the section.

Fact 3.8. *For any $C \geq 0$, $a \geq C$ and $b \in (a, +\infty]$ the family of random variables $\{H_t\}_{t\geq 0}$ defined by*

$$H_t := \frac{-\log \mathbb{P}\left(\forall_{s\leq t} X_s \geq Y_s + C, X_t - Y_t \in (a,b) | Y\right)}{t}. \tag{3.9}$$

is L^p-uniformly integrable for any $p \geq 1$.

Lemma 3.9. *For any $0 \leq a < b \leq +\infty$ and $X_0 > Y_0$, we have*

$$\lim_{t\to+\infty} \frac{-\log \mathbb{P}\left(\forall_{s\leq t} X_s \geq Y_s, X_t - Y_t \in (a,b) | Y\right)}{t} = \frac{\tilde{\gamma}}{\mathbb{E}(r_1)} \quad \text{a.s. and in } L^1.$$

Consequence of this lemma, we set $\gamma_{\mu_1,\mu_2}(1) = \frac{\tilde{\gamma}}{\mathbb{E}(r_1)}$.

Proof. Let $m(t) := \lfloor t/\mathbb{E}\rho_1 - t^{2/3} \rfloor$ and $M(t) := \lfloor t/\mathbb{E}\rho_1 + t^{2/3} \rfloor$ and

$$\mathcal{A}_t := \left\{ t \in [\rho_{m(t)}, \rho_{M(t)}] \right\}.$$

Clearly, $\rho_n = \sum_{k=0}^{n-1} r_k$, using Fact 3.3 one checks that $1_{\mathcal{A}_t} \to 1$ a.s. By Fact 3.8 it follows that

$$\lim_{t\to+\infty} H_t 1_{\mathcal{A}_t^c} = 0, \quad \text{a.s and } L^p. \tag{3.10}$$

By (3.8) we have

$$\liminf_{t\to+\infty} H_t \geq \lim_{t\to+\infty} 1_{\mathcal{A}_t} \frac{-\log \mathbb{P}\left(\forall_{s\leq\rho_{m(t)}} X_s \geq Y_s | Y\right)}{m(t)} \frac{m(t)}{t} = \frac{\tilde{\gamma}}{\mathbb{E}r_1}, \quad \text{a.s and } L^p. \tag{3.11}$$

The bound from above is slightly more involved

$$1_{\mathcal{A}_t} H_t \leq 1_{\mathcal{A}_t} \frac{-\log \mathbb{P}\left(\forall_{s\leq\rho_{m(t)}} X_s \geq Y_s, \forall_{s\in[\rho_{m(t)},\rho_{M(t)}]} X_s - Y_s \in (a,b) | Y\right)}{M(t)} \frac{M(t)}{t}.$$

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Let us denote the probability in the expression above by p. We fix a', b' such that $a < a' < b' < b$ and use the Markov property

$$\log p \geq \log \mathbb{P}\left(\forall_{s \leq \rho_{m(t)}} X_s \geq Y_s, X_{\rho_{m(t)}} - Y_{\rho_{m(t)}} \in (a', b') | Y\right)$$
$$+ \log\left[\inf_{x \in [a', b']} \log \mathbb{P}\left(\forall_{s \in [\rho_{m(t)}, \rho_{M(t)}]} X_s - Y_s \in (a, b) | Y, X_{\rho_{m(t)}} = x\right)\right].$$

It is easy to check that the second term divided by t converges to 0 (which essentially follows by the fact that $M(t) - m(t) = o(t)$). Thus using (3.8) again we obtain

$$\liminf_{t \to +\infty} H_t \leq \frac{\tilde{\gamma}}{\mathbb{E} r_1}, \quad \text{a.s and } L^p.$$

This together with (3.11) concludes the proof of (1.4). $\qquad \square$

Lemma 3.10. *The function γ is symmetric and convex.*

Proof. We recall that for any $\beta \in \mathbb{R}$,

$$\gamma(\beta) = -\frac{\log \mathbb{P}(\forall_{s \leq t} X_s \geq \beta Y_s | Y)}{t} \quad \text{a.s.}$$

As the law of Y is symmetric, γ is symmetric. To prove convexity we use Lemma 2.6. To this end we fix $t > 0$ and $\lambda \in (0, 1)$. Applied conditionally on W the lemma implies

$$-\frac{\log \mathbb{P}(\forall_{s \leq t} X_s \geq (\lambda a + (1 - \lambda) b) Y_s | Y)}{t} \leq -\lambda \frac{\log \mathbb{P}(\forall_{s \leq t} X_s \geq a Y_s | Y)}{t} - (1 - \lambda) \frac{\log \mathbb{P}(\forall_{s \leq t} X_s \geq b Y_s | Y)}{t}.$$

Taking $t \to +\infty$ we obtain $\gamma(\lambda a + (1 - \lambda) b) \leq \lambda \gamma(a) + (1 - \lambda) \gamma(b)$. $\qquad \square$

Proof of Fact 3.8. Without loss of generality it is enough to work with integer times and assume that $a > C$. Denoting the probability in (3.9) by p_t we estimate

$$-\log p_n \leq -\log \mathbb{P}\left(\forall_{s \leq n} X_s \geq Y_s + C, \forall_{k \in \{1, \dots, n\}} X_k - Y_k \in (a, b) | Y\right)$$
$$\leq -\log \mathbb{P}\left(\forall_{s \in [0,1]} X_s \geq Y_s + C, X_1 - Y_1 \in (a, b) | Y\right) + \sum_{k \in \{1, \dots, n-1\}} q_k,$$

where $q_k := -\log\left[\inf_{x \in (a,b)} \mathbb{P}\left(\forall_{s \in [k, k+1]} X_s \geq Y_s + C, X_{k+1} - Y_{k+1} \in (a, b) | Y, X_k - Y_k = x\right)\right]$.

By the Markov property the random variables $\{q_k\}_{k \geq 1}$ are independent and identically distributed thus, by Fact 2.7, the sequence $\left\{\frac{1}{n} \sum_{k=1}^{n} q_k\right\}_n$ is L^p-uniformly integrable. Further, the proof follows by standard arguments. $\qquad \square$

4 Relevance of the Disorder

Thanks to Lemmas 3.1, 3.9 and 3.10, the only thing left to prove Theorem 1.7 is the strict inequality (1.6). Observe that for any fixed $t > 0$, by Jensen's inequality we have

$$\mathbb{E}\left[-\log \mathbb{P}\left(\forall_{s \leq t} X_s \geq Y_s | Y\right)\right] > -\log \mathbb{P}\left(\forall_{s \leq t} X_s \geq Y_s\right),$$

which implies $\gamma_{\mu_1, \mu_2} \geq \delta_{\mu_1, \mu_2}$. Obtaining the strict inequality is much harder. We recall the path decomposition from Section 3.1. The key observation, on which the proof strategy hinges on, is that Jensen's inequality applied on each interval $[\rho_i, \rho_{i+1}]$ separately is strict. The main technical difficulty will be to control its "gap" uniformly in i. This control is established in Proposition 4.5.

In the whole section, unless specified otherwise, we assume that $Y_0 = 0$ and $X_0 = 1$. Before the main proof we present three technical lemmas. The first one is a concentration inequality for a conditioned Ornstein-Uhlenbeck process

Lemma 4.1. *Let X be an Ornstein-Uhlenbeck process. For any $C_1 > 0$ there exist $C_2, C_3 > 0$ such that for any $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ being a C_1-Lipschitz function we have*

$$\mathbb{P}\left(X_t \geq x + f(t) | \forall_{s \leq t} X_s \geq f(s)\right) \leq \exp\left(-C_2 x^2\right), \quad x \geq C_3 f(t), \tag{4.1}$$

as soon as $X_0 \in [f(0) + 1, (C_1 + 1)f(0))$.

Proof. To avoid cumbersome notation we assume that $t \in \mathbb{N}$. The proof for general t follows similar lines and is left to the reader. Further we assume that

$$\forall_{s \leq t} f(s) \geq \min\left\{(s+1)^{1/3}, (t-s+1)^{1/3}\right\}, \tag{4.2}$$

If it is not the case using Fact 2.3 we freely can change f by $s \mapsto f(s) + \min\left\{(s+1)^{1/3}, (t-s+1)^{1/3}\right\}$ which is $(C_1 + 1)$-Lipschitz.

We shorten $x_t := x + f(t)$ and let $c_1 > 1$. Using Fact 2.3 we estimate

$$\mathbb{P}\left(X_t \geq x_t | \forall_{s \leq t} X_s \geq f(s)\right) \leq \mathbb{P}\left(X_t \geq x_t \,|\, \forall_{s \leq t} X_s \geq f(s), \forall_{n \in \{1,\ldots,t\}} X_n \geq c_1 f(n)\right) \tag{4.3}$$

$$= \frac{\mathbb{P}\left(X_t \geq x_t, \forall_{s \leq t} X_s \geq f(s) \,|\, \forall_{n \in \{1,\ldots,t\}} X_n \geq c_1 f(n)\right)}{\mathbb{P}\left(\forall_{s \leq t} X_s \geq f(s) \,|\, \forall_{n \in \{1,\ldots,t\}} X_n \geq c_1 f(n)\right)}$$

$$\leq \frac{\mathbb{P}\left(X_t \geq x_t \,|\, \forall_{n \in \{1,\ldots,t\}} X_n \geq c_1 f(n)\right)}{\mathbb{P}\left(\forall_{s \leq t} X_s \geq f(s) \,|\, \forall_{n \in \{1,\ldots,t\}} X_n \geq c_1 f(n)\right)}.$$

Let us first treat the denominator denoted by I_d. We use (4.2) and choose c_1 sufficiently large so that I_d is bounded from below by a constant independent on t and f. Using Fact 2.3 and the Markov property we obtain

$$I_d = \mathbb{P}\left(\forall_{s \in [0,1]} X_s \geq f(s) \,|\, \forall_{n \in \{1,\ldots,t\}} X_n \geq c_1 f(n)\right)$$
$$\times \mathbb{P}\left(\forall_{s \in [1,t]} X_s \geq f(s) \,|\, \forall_{n \in \{1,\ldots,t\}} X_n \geq c_1 f(n), \forall_{s \in [0,1]} X_s \geq f(s)\right)$$
$$\geq \mathbb{P}\left(\forall_{s \in [0,1]} X_s \geq f(s) | X_1 \geq c_1 f(1)\right) \mathbb{P}\left(\forall_{s \in [1,t]} X_s \geq f(s) \,|\, \forall_{n \in \{1,\ldots,t\}} X_n \geq c_1 f(n)\right).$$

Continuing in a same manner we obtain that

$$I_d \geq \mathbb{P}\left(\forall_{s \in [0,1]} X_s \geq f(s) | X_1 \geq c_1 f(1)\right) \times \prod_{n \in \{1,\ldots,t\}} \mathbb{P}\left(\forall_{s \in [n-1,n]} X_s \geq f(s) \,|\, X_{n-1} = c_1 f(n-1)\right).$$
$$\tag{4.4}$$

By point 3 of Fact 2.4 the process $\{X_t\}_{t \in [0,1]}$ defined by $\tilde{X}_t := X_{n-1+t} - c_1 e^{-\mu t} f(n-1)$ is an Ornstein-Uhlenbeck process starting from $\tilde{X}_0 = 0$. Thus

$$\mathbb{P}\left(\forall_{s \in [n-1,n]} X_s \geq f(s) \,|\, X_{n-1} = c_1 f(n-1)\right) \geq \mathbb{P}\left(\forall_{s \in [0,1]} \tilde{X}_s \geq \sup_{s \in [n-1,n]} f(s) - c_1 e^{-\mu} f(n-1)\right)$$

$$\geq \mathbb{P}\left(\forall_{s \in [0,1]} \tilde{X}_s \geq -c_1 e^{-\mu} f(n)/2\right).$$

We used inequality $\sup_{s \in [n-1,n]} f(s) - c_1 e^{-\mu} f(n-1) \leq -c_1 e^{-\mu} f(x)/2$ which can be easily verified by (4.2) and Lipschitz property as soon as c_1 is large enough. By point 4 of Fact 2.4 we conclude that

$$\mathbb{P}\left(\forall_{s \in [n-1,n]} X_s \geq f(s) \,|\, X_{n-1} = c_1 f(n-1)\right) \geq 1 - C \exp\left(-c \left[c_1 e^{-\mu} f(n)/2\right]^2\right).$$

By this estimate, (4.4), (4.2) and increasing c_1 if necessary we obtain

$$\mathbb{P}\left(\forall_{s \leq t} X_s \geq f(s) \,|\, \forall_{n \leq t} X_{n-1} \geq c_1 f(n)\right) \geq p,$$

for some $p > 0$. From now on c_1 is fixed. Now in order to show (4.1) it is enough to prove that the nominator in (4.3) decays in a Gaussian fashion. This is the aim for the rest of the proof. We define a sequence $\{G_n\}_{n\geq 0}$ by putting $G_0 = X_0 > 0$ and

$$G_n := X_n - c_n X_{n-1}, \quad n \geq 1 \tag{4.5}$$

where $c_n := \frac{Cov(X_n, X_{n-1})}{Cov(X_n, X_n)}$. It is easy to check that in fact $c_n = c \in (0,1)$ and moreover the random variables $\{G_n\}_{n\geq 0}$ are independent, distributed according to $\mathcal{N}(0, b^2)$ where b is a function of the parameters of the process X. We will prove that there exist $c_2, C_2 > 0$ such that for any $x > C_2 f(t)$ and $t \in \mathbb{N}$ we have

$$\mathbb{P}\left(X_t \geq x + f(t) \,\middle|\, \forall_{n\in\{1,\ldots,t\}} X_n \geq c_1 f(n)\right) \leq e^{-c_2 x^2}. \tag{4.6}$$

We start by choosing constants $B, c_2 > 0$ so that they fulfill

$$B \in (c, 1), \quad c_2 < \frac{(1-B)^2}{2b^2}. \tag{4.7}$$

Let $L \geq 0$, without loss of generality we assume that $f(t) \geq L$. This assumption with the Lipschitz property yields that

$$A_L^{-1} \leq \frac{f(t+1)}{f(t)} \leq A_L. \tag{4.8}$$

for A_L such that $A_L \searrow_L 1$. We fix L such that $B/(cA_L) > 1$. We proceed inductively. The constants L and C_2 potentially may be increased during the further proof (the other constants stay fixed). We stress that this increase happens once and later the constants are valid for all steps of the induction.

Checking the base case is an easy exercise left to the reader. Let us assume that (4.6) holds for $t \geq 0$. Let x be such that $x + f(t+1) \geq c_1 f(t+1)$, we have

$$\mathbb{P}\left(X_{t+1} \geq x + f(t+1) \,\middle|\, \forall_{n\in\{1,\ldots,t+1\}} X_n \geq c_1 f(n)\right)$$
$$= \frac{\mathbb{P}\left(X_{t+1} \geq x + f(t+1), \forall_{n\in\{1,\ldots,t\}} X_n \geq c_1 f(n)\right)}{\mathbb{P}\left(\forall_{n\in\{1,\ldots,t+1\}} X_n \geq c_1 f(n)\right)}$$
$$= \frac{\mathbb{P}\left(X_{t+1} \geq x + f(t+1) \,\middle|\, \forall_{n\in\{1,\ldots,t\}} X_n \geq c_1 f(n)\right)}{\mathbb{P}\left(X_{t+1} \geq c_1 f(t+1) \,\middle|\, \forall_{n\in\{1,\ldots,t\}} X_n \geq c_1 f(n)\right)}.$$

We denote the denominator by I_d. By (4.5) and (4.8) we have

$$I_d \geq \mathbb{P}\left(G_{t+1} \geq c_1 f(t+1) - cc_1 f(t)\right) \geq \mathbb{P}\left(G_{t+1} \geq c_1(A_L - c)f(t)\right). \tag{4.9}$$

By (4.5) and the union bound we conclude that the nominator is smaller than $I_n^1 + I_n^2$, where

$$I_n^1 := \mathbb{P}\left(X_t \geq \frac{B}{c}(x + f(t+1)) \,\middle|\, \forall_{n\in\{1,\ldots,t\}} X_n \geq c_1 f(n)\right), \quad I_n^2 := \mathbb{P}\left(G_{t+1} \geq (1-B)(x + f(t+1))\right).$$

Let $x > C_2 f(t+1)$ then

$$\frac{B}{c}(x + f(t+1)) - c_1 f(t) \geq \frac{B}{c}(C_2 + 1)f(t+1) - c_1 f(t) \geq f(t)\left[\frac{B}{cA_L}(C_2 + 1) - c_1\right] \geq C_2 f(t).$$

We assumed that $B/(cA_L) > 1$ thus the last inequality holds if we choose C_2 large enough. We can thus use the induction hypothesis (4.6) for t. We have

$$I_n^1 \leq \exp\left\{-c_2\left(\frac{B}{c}x + \frac{B}{c}f(t+1) - f(t)\right)^2\right\} \leq \exp\left\{-c_2\left(\frac{B}{c}x + [B/(cA_L) - 1]f(t)\right)^2\right\}.$$

Recalling (4.9) and increasing L so that $c_1(A_L - c)f(t) \geq c_1(A_L - c)L \geq 2$ holds we can use the Gaussian tail estimate (2.6) as follows

$$\frac{I_n^1}{I_d} \leq \frac{5c_1(A_L - c)f(t)}{b} \exp\left\{\frac{c_1^2(A_L - c)^2 f(t)^2}{2b^2}\right\} \exp\left\{-c_2\left(\frac{B}{c}x + [B/(cA_L) - 1]f(t)\right)^2\right\}$$

$$\leq \frac{5c_1(A_L - c)f(t)}{b} \exp\left\{\frac{c_1^2(A_L - c)^2}{2b^2}f(t)^2 - \frac{2c_2 B[B/(cA_L) - 1]}{c}xf(t)\right\} \exp\left\{-\frac{c_2 B^2}{c^2}x^2\right\}.$$

We increase C_2 (we recall that $x \geq C_2 f(t + 1)$) so that $\frac{c_1^2(A_L - c)^2}{2b^2} < C_2 \frac{2c_2 B[B/(cA_L) - 1]}{cA_L}$. Then we increase L if necessary so that the first two factors are bounded by $1/2$. Finally

$$\frac{I_n^1}{I_d} \leq \frac{1}{2} \exp\left\{-\frac{c_2 B^2}{c^2}x^2\right\},$$

which by (4.7) implies $I_n^1/I_d \leq \exp\left\{-c_2 x^2\right\}/2$. We perform similar calculations for I_n^2:

$$\frac{I_n^2}{I_d} \leq \frac{5c_1(A_L - c)f(t)}{b} \exp\left\{\frac{c_1^2(A_L - c)^2 f(t)^2}{2b^2}\right\} \exp\left\{-\frac{(1 - B)^2}{2b^2}(x + f(t)/A_L)^2\right\}$$

$$\leq \frac{5c_1(A_L - c)f(t)}{b} \exp\left\{\frac{c_1^2(A_L - c)^2}{2b^2}f(t)^2 - \frac{(1 - B)^2}{b^2 A_L}xf(t)\right\} \exp\left\{-\frac{(1 - B)^2}{2b^2}x^2\right\}$$

$$\leq \frac{1}{2} \exp\left\{-\frac{(1 - B)^2}{2b^2}x^2\right\},$$

where the last estimates follows by increasing C_2 and L if necessary (analogously to the previous case). Now, by (4.7) follows $I_n^2/I_d \leq \exp(-c_2 x^2)/2$. Recalling the previous step we obtain $(I_n^1 + I_n^2)/I_d \leq \exp(-c_2 x^2)$ which establish (4.6) for $t + 1$. □

A similar property holds for conditioning in future.

Lemma 4.2. *Let X be an Ornstein-Uhlenbeck process starting from $X_0 = 1$ and $u \in [c, C]$, for $C > c > 0$. Then there exist $C_1, c_1 > 0$ such that for any $t > 1$ we have*

$$\mathbb{P}\left(X_u \geq x | \forall_{s \in [u, u+t]} X_s \geq (1 + s - u)^{1/3}\right) \leq C_1 e^{-c_1 x^2}.$$

Proof. We set $f(s) := (1 + \lceil s \rceil)^{1/3}$, by Fact 2.3 it is enough show the claim with $f(s)$ instead of $(1 + s)^{1/3}$. Using the Markov property we write

$$\mathbb{P}\left(X_u \geq x | \forall_{s \in [u, u+t]} X_s > f(s - u)\right) = \frac{\mathbb{P}\left(\{X_u \geq x\} \cap \{\forall_{s \in [u, u+t]} X_s > f(s - u)\}\right)}{\mathbb{P}\left(\forall_{s \in [u, u+t]} X_s > f(s - u)\right)}$$

$$= \frac{\int_x^{+\infty} w(y, t)\mathbb{P}(X_u \in dy)}{\int_0^{+\infty} w(y, t)\mathbb{P}(X_u \in dy)},$$

where

$$w(y, t) = \mathbb{P}\left(\forall_{s \leq t} X_t \geq f(s) | X_0 = y\right).$$

The function is increasing with respect to y. By the Gaussian concentration of X_u, one checks that to show the claim it is enough to that

$$\frac{w(x, t)}{w(3, t)} \leq \exp\left(C_1(\log x)^{5/3}\right), \tag{4.10}$$

for $C_1 > 0$ for $x \geq 3$. It will be easier to rewrite w as $w(x,t) = \mathbb{P}\left(\forall_{s \leq t} X_t \geq f(s) - xe^{-\mu s}\right)$ with the assumption that $X_0 = 0$. Let us set $t_x := \lceil C_t \log x \rceil$, where $C_t > 0$ will be adjusted later. For $x > 3$ we have

$$w(x,t) = \mathbb{P}\left(\forall_{s \in [t_x, t]} X_s \geq f(s) - xe^{-\mu s} | \forall_{s \leq t_x} X_s \geq f(s) - xe^{-\mu s}\right) \mathbb{P}\left(\forall_{s \leq t_x} X_s \geq f(s) - xe^{-\mu s}\right)$$
$$\leq \mathbb{P}\left(\forall_{s \in [t_x, t]} X_s \geq f(s) - xe^{-\mu s} | \forall_{s \leq t_x} X_s \geq f(s) - xe^{-\mu s}\right)$$
$$\leq \mathbb{P}\left(\forall_{s \in [t_x, t]} X_s \geq f(s) - xe^{-\mu s} | \forall_{s \leq t_x} X_s \geq f(s) + 1 - 3e^{-\mu s}\right),$$

where in the last line we used Fact 2.3. Moreover, by convention we assume that the probability above is 1 if $t_x \geq t$. Similarly we estimate

$$w(3,t) \geq \mathbb{P}\left(\forall_{s \leq t_x} X_s \geq f(s) + 1 - 3e^{-\mu s}, \forall_{s \in [t_x, t]} X_s \geq f(s)\right)$$
$$\geq \mathbb{P}\left(\forall_{s \leq t_x} X_s \geq f(s) + 1 - 3e^{-\mu s}\right) \mathbb{P}\left(\forall_{s \in [t_x, t]} X_s \geq f(s) | \forall_{s \leq t_x} X_s \geq f(s) + 1 - 3e^{-\mu s}\right).$$

Using calculations similar to (4.4) and $f(t_x) = O((\log x)^{1/3})$ one can show that

$$\mathbb{P}\left(\forall_{s \leq t_x} X_s \geq f(s) + 1 - 3e^{-\mu t}\right) \geq c_3 e^{-C_3(\log x)^{2/3} t_x},$$

for some $c_3, C_3 > 0$. Now we will show that for C_t (recall that $t_x := \lceil C_t \log x \rceil$) large enough and $y \geq f(t_x) + 1 - 3e^{-\mu t_x}$ there exists a constant $c > 0$ such that

$$\frac{\mathbb{P}\left(\forall_{s \in [t_x, t]} X_s \geq f(s) | X_{t_x} = y\right)}{\mathbb{P}\left(\forall_{s \in [t_x, t]} X_s \geq f(s) - xe^{-\mu t} | X_{t_x} = y\right)}$$
$$= \mathbb{P}\left(\forall_{s \in [t_x, t]} X_s \geq f(s) | \forall_{s \in [t_x, t]} X_s \geq f(s) - xe^{-\mu s}, X_{t_x} = y\right) \geq c.$$

One verifies that this is enough to conclude the proof of (4.10) and consequently the proof of the lemma. Equivalently we will show that

$$H := \mathbb{P}\left(\exists_{s \in [t_x, t]} X_s \leq f(s) | \forall_{s \in [t_x, t]} X_s \geq f(s) - xe^{-\mu s}, X_{t_x} = y\right) \leq 1 - c. \tag{4.11}$$

We consider

$$H \leq \sum_{k=t_x}^{\lceil t \rceil} \mathbb{P}\left(\exists_{s \in [k, k+1]} X_s \leq f(s) | \forall_{s \leq t} X_s \geq f(s) - xe^{-\mu s}, X_{t_x} = y\right)$$
$$\leq \sum_{k=t_x}^{\lceil t \rceil} \mathbb{P}\left(\exists_{s \in [k, k+1]} X_s \leq f(k) | \forall_{s \in [k, k+1]} X_s \geq f(k) - xe^{-\mu s}, X_{t_x} = y\right).$$

The first inequality follows by the union bound and the second one by the assumption on f and Fact 2.3. The first term (i.e. $k = t_x$) can be made arbitrarily small by choosing C_t (and thus t_x) large. To estimate the other terms we define a function $p_k : \mathbb{R}_+ \mapsto \mathbb{R}$ by

$$p_k(A) = -\log \mathbb{P}\left(\forall_{s \in [k, k+1]} X_s \geq A | X_{t_x} = y\right).$$

By Lemma 2.6 one deduces that p is convex. Further we notice

$$\mathbb{P}\left(\forall_{s \in [k, k+1]} X_s \geq A | X_{t_x} = y\right) \leq \mathbb{P}\left(X_k \geq Ae^{\mu} | X_{t_x} = y\right) \mathbb{P}\left(\forall_{s \in [k, k+1]} X_s \geq A | X_k = e^{\mu} A\right).$$

The second factor can be easily bounded from below by a strictly positive constant uniform in A, k. Thus for some $C_4 > 0$ we have $p_k(A) \leq C_4(x+1)^2$. Using the convexity of p_k it is easy to deduce that for some $C_5 > 0$ we have $p_k'(A) \leq C_5(x+1)$, where p_k' denotes the left derivative of p_k. Thus

$$\log \mathbb{P}\left(\forall_{s \in [k, k+1]} X_t \geq f(k) | \forall_{s \in [k, k+1]} X_t \geq f(k) - xe^{-\mu k}\right)$$
$$= p_k(f(k) - xe^{-\mu k}) - p_k(f(k)) \geq -C_5 xe^{-\mu k} f(k).$$

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Now we can make the final estimate. We write

$$\sum_{k=t_x+1}^{\lceil t \rceil} \left(1 - \exp(-C_5 x e^{-\mu k} f(k))\right) \leq C_6 \sum_{k=\lfloor t_x \rfloor}^{\lceil t \rceil} f(k) e^{-\mu(k - \lceil \log k/\mu \rceil)}.$$

for some $C_6 > 0$. Increasing C_t (recall that $t_x := \lceil C_t \log x \rceil$) if necessary, we can make the sum arbitrarily small, proving (4.11) and concluding the proof. $\square$

We introduce $b_t : [0, t] \mapsto \mathbb{R}_+$ by

$$b_t(s) := \min \left\{ (s+1)^{1/3}, (t - s + 1)^{1/3} \right\}. \tag{4.12}$$

Let us recall the notation of Section 3.1. We have

Lemma 4.3. *There exist $C, c > 0$ such that for any $n, k \in \mathbb{N}$ we have*

$$\mathbb{P}\left(\forall_{s \leq \rho_k} Y_s \leq C b_{\rho_k}(s)\right) > c, \quad \mathbb{P}\left(\{\mathbb{P}\left(\mathcal{B}_{n,k} | \{r_i\}\right) \geq 1/10\}\right) > c, \tag{4.13}$$

where $\mathcal{B}_{n,k} := \left\{\forall_{\rho_{k+1} \leq s \leq \rho_n} Y_s \leq C(s - \rho_{k+1} + 1)^{1/3}\right\}$.

Proof. The proof is rather standard. We present a sketch, leaving details to the reader. We denote $f(s) := C(s+1)^{1/3}$. We consider

$$\mathbb{P}\left(\forall_{s \leq \rho_k} Y_s \leq C b_{\rho_k}(s)\right) = 1 - \mathbb{P}\left(\exists_{s \leq \rho_k} Y_s \geq C b_{\rho_s}(s)\right) \tag{4.14}$$
$$\geq 1 - \mathbb{P}\left(\exists_{s \leq \rho_k} Y_s \geq C f(s)\right) - \mathbb{P}\left(\exists_{s \leq \rho_k} Y_s \geq C f(\rho_k - s)\right).$$

Let us treat the second term. Let $l \in \mathbb{N}$, we have

$$\mathbb{P}\left(\exists_{s \leq \rho_k} Y_s \geq C f(s)\right) \leq \mathbb{P}\left(\exists_{s \geq 0} Y_s \geq C f(s)\right) \leq \mathbb{P}\left(\exists_{s \in [0, \rho_l]} Y_s \geq C f(s)\right) + \sum_{i=l}^{+\infty} \mathbb{P}\left(\exists_{s \in [\rho_i, \rho_{i+1}]} Y_s \geq C f(s)\right). \tag{4.15}$$

We recall Fact 3.3 and the notation there. For large enough i and some $c > 0$ we have

$$\mathbb{P}\left(\exists_{s \in [\rho_i, \rho_{i+1}]} Y_s \geq C f(s)\right) \leq \mathbb{P}\left(\{M^i \geq C f(\rho_i)\} \cap \{\rho_i \geq ci\}\right) + \mathbb{P}\left(\rho_i < ci\right)$$
$$\leq \mathbb{P}\left(M^i \geq C f(ci)\right) + \mathbb{P}\left(\rho_i < ci\right)$$
$$\leq e^{-C_1 C f(ci)} + e^{-C_2 i},$$

where $C_1, C_2 > 0$. Increasing l and C one can make (4.15) as small as we want. Treating the third of term (4.14) similarly we obtain the first statement of (4.13). We set $\mathcal{A}_{n,k} := \{\mathbb{P}\left(\mathcal{B}_{n,k} | \{r_i\}\right) \geq 1/10\}$ and $p := \mathbb{P}\left(\mathcal{A}_{n,k}\right)$. We have

$$\mathbb{P}\left(\mathcal{B}_{n,k}\right) = \mathbb{E}\mathbb{E}[1_{\mathcal{B}_{n,k}} | \{r_i\}] = \mathbb{E}\left[1_{\mathcal{A}_{n,k}} \mathbb{E}[1_{\mathcal{B}_{n,k}} | \{r_i\}]\right] + \mathbb{E}\left[1_{\mathcal{A}_{n,k}^c} \mathbb{E}[1_{\mathcal{B}_{n,k}} | \{r_i\}]\right]$$
$$\leq p + (1-p)/10 = \frac{1}{10} + \frac{9}{10} p.$$

By the first argument we choose C such that $\mathbb{P}\left(\mathcal{B}_{n,k}\right) > 1/10$ which implies $p > 0$ (uniformly in n and k). $\square$

4.1 Reformulation of the problem

We introduce necessary notions and reformulate the problem. Let $\mathcal{M}$ be the space of finite measures on $\mathbb{R}_+$. Given $m \in \mathcal{M}$ we denote $\|m\| := \int_{\mathbb{R}_+} m(\mathrm{d}x)$. Let $\mathcal{P}$ be a functional space

$$\mathcal{P} := \{f : f \text{ is a continuous function from an interval } [0, t] \text{ to } \mathbb{R} \text{ such that } f(0) = f(t) = 0\}. \tag{4.16}$$

Let us define an operator $T : \mathcal{M} \times \mathcal{P} \mapsto \mathcal{M}$. Given a measure $m \in \mathcal{M}$ and $f \in \mathcal{P}$ such that $f : [0, t] \mapsto \mathbb{R}$ we set $T(m, f) := \tilde{m}$ defined by

$$\tilde{m}(\mathrm{d}x) := \|m\| \mathbb{P}_{m/\|m\|} \left(\forall_{s \leq t} X_s \geq f(s), X_t \in \mathrm{d}x \right),$$

where under $\mathbb{P}_{m/\|m\|}$ the process X is an Ornstein-Uhlenbeck process such that $X_0 =^d m/\|m\|$.

For $n \in \mathbb{N}$ we define iteratively $\mathcal{M}$-valued random variables T_n by

$$T_n := \begin{cases} \delta_1 & n = 0 \\ T(T_{n-1}, Y^{n-1}) & n > 0 \end{cases}, \tag{4.17}$$

where Y^n is given by (3.1). Using the Markov property one proves by induction that

$$T_n = \mathbb{P}\left(\forall_{s \leq \rho_n} X_s \geq Y_s, X_{\rho_n} \in \mathrm{d}x | Y \right). \tag{4.18}$$

Let us denote $\mathcal{F} := \sigma(\rho_i, i \in \mathbb{N})$. The following lemma relates T_n to our original problem.

Lemma 4.4. *Let γ_{μ_1,μ_2} and δ_{μ_1,μ_2} be the same as in Theorem 1.7. Then*

$$\gamma_{\mu_1,\mu_2} = \lim_{n \to +\infty} \frac{\mathbb{E}\left[-\log \|T_n\|\right]}{n \mathbb{E} r_1}, \quad \delta_{\mu_1,\mu_2} \leq \liminf_{n \to +\infty} \frac{\mathbb{E}\left[-\log \mathbb{E}\left(\|T_n\| | \mathcal{F}\right)\right]}{n \mathbb{E} r_1}. \tag{4.19}$$

The following proposition is the main technical result of this proof

Proposition 4.5. *There exist $c > 0$ and $n_0 \in \mathbb{N}$ such that*

$$\mathbb{E}\log\|T_n\| - \mathbb{E}\log \mathbb{E}(\|T_n\| | \mathcal{F}) \leq -nc, \tag{4.20}$$

for any $n \geq n_0$.

We observe that this proposition together with Lemma 4.4 imply (1.6).

Proof of Lemma 4.4. The first convergence in (4.19) holds by (3.8) (recall also relation between $\tilde{\gamma}$ and γ_{μ_1,μ_2} given in (3.11)). We observe that (4.18) yields $\mathbb{E}\left(\|T_n\| | \mathcal{F}\right) = \mathbb{P}\left(\forall_{s \leq \rho_n} X_s \geq Y_s | \mathcal{F}\right)$. We note that methods of Section 3 imply that

$$\frac{-\log \mathbb{P}\left(\forall_{s \leq \rho_n} X_s \geq Y_s | \mathcal{F}\right)}{n} \tag{4.21}$$

converges a.s. and in L^1 (details are left to the reader). We define $r(n) := \mathbb{E}\rho_n - n^{2/3} = n\mathbb{E}r_1 - n^{2/3}$ and a sequence of events $\mathcal{A}_n := \{\rho_n \geq r(n)\}$. Using Fact 3.3 one proves $1_{\mathcal{A}_n^c} \to 0$ a.s. Consequently the convergence of (4.21) implies

$$\lim_{n \to +\infty} \frac{\mathbb{E}1_{\mathcal{A}_n^c} \log \mathbb{P}\left(\forall_{s \leq r(n)} X_s \geq Y_s | \mathcal{F}\right)}{n} = 0.$$

Using $\mathbb{E}\left(\|T_n\| | \mathcal{F}\right) \leq 1$ we estimate

$$\mathbb{E}\log \mathbb{E}\left(\|T_n\| | \mathcal{F}\right) \leq \mathbb{E}1_{\mathcal{A}_n} \log \mathbb{E}\left(\|T_n\| | \mathcal{F}\right) \tag{4.22}$$
$$\leq \mathbb{E}1_{\mathcal{A}_n} \log \mathbb{P}\left(\forall_{s \leq r(n)} X_s \geq Y_s | \mathcal{F}\right)$$
$$= \mathbb{E}\log \mathbb{P}\left(\forall_{s \leq r(n)} X_s \geq Y_s | \mathcal{F}\right) - \mathbb{E}1_{\mathcal{A}_n^c} \log \mathbb{P}\left(\forall_{s \leq r(n)} X_s \geq Y_s | \mathcal{F}\right).$$

We denote the first term the right-hand side of (4.22) by J_n. Applying Jensen's inequality we get

$$J_n \leq \log \mathbb{P}\left(\forall_{s \leq r(n)} X_s \geq Y_s\right).$$

By (1.5) and the definition of $r(n)$ we have $\limsup_n J_n/(n\mathbb{E}r_1) \leq -\delta_{\mu_1,\mu_2}$ and the second term (4.22) can be shown to converge to 0. We conclude that the second claim of (4.19) holds. $\qquad\square$

Remark 4.6. We believe that the second convergence in (4.19) can be improved to

$$\delta_{\mu_1,\mu_2} = \lim_{n\to+\infty} \mathbb{E}\left[-\log\mathbb{E}\left(\|T_n\|\,|\,\mathcal{F}\right)\right]/(n\mathbb{E}r_1).$$

This stronger result is not needed to our applications thus we skip proving it. A careful reader will notice some technical complications in the proof of Proposition 4.5 coming from the conditioning on $\mathcal{F}$ and might be tempted to replace $\mathbb{E}\left(\|T_n\|\,|\,\mathcal{F}\right)$ with $\mathbb{E}\|T_n\|$.

4.2 Proof of Proposition 4.5

Proof of Proposition 4.5. We recall $\mathcal{F} = \sigma(\rho_i, i \in \mathbb{N})$ and define a filtration $\{\mathcal{F}_k\}_{k\geq 0}$ by putting $\mathcal{F}_0 := \{\emptyset, \Omega\}$ and

$$\mathcal{F}_k := \sigma\left\{Y^i : i < k\right\}, \quad k > 0,$$

(see also Figure 3.1). We recall (4.17) and for $k \in \{0, 1, \ldots, n\}$ define $\mathcal{M}$-valued random variables T_n^k by

$$T_n^k := \mathbb{E}(T_n|\mathcal{F}_k, \mathcal{F}).$$

This definition and (4.18) imply that

$$T_n^k = \mathbb{P}\left(\forall_{s\leq\rho_n} X_s \geq Y_s, X_{\rho_n} \in \mathrm{d}x\,\big|\,\left\{Y^i\right\}_{i<k}, \mathcal{F}\right), \quad T_n^n = T_n. \tag{4.23}$$

By the Markov property of X we have

$$\log\|T_n^{k+1}\| = \log\|T_k\| + \log\mathbb{P}_{T_k/\|T_k\|}\left(\forall_{\rho_k\leq s\leq\rho_n} X_s \geq Y_s | Y^k, \mathcal{F}\right). \tag{4.24}$$

This expression requires some comment. We recall that T_k is a random measure, conditionally on $m = T_k/\|T_k\|$ we understand X to be an Ornstein-Uhlenbeck process starting from m at time ρ_k. Let us now denote

$$G_{n,k} := \mathbb{E}\left[\log\mathbb{P}_{T_k/\|T_k\|}\left(\forall_{\rho_k\leq s\leq\rho_n} X_s \geq Y_s | Y^k, \mathcal{F}\right)|\mathcal{F}_k, \mathcal{F}\right] - \log\mathbb{P}_{T_k/\|T_k\|}\left(\forall_{\rho_k\leq s\leq\rho_n} X_s \geq Y_s | \mathcal{F}\right).$$

We notice that $G_{n,k}$ is a random variable, which by Jensen's inequality fulfills $G_{n,k} \leq 0$ (we will prove strict inequality later). In this notation (4.24) yields

$$\mathbb{E}\left[\log\|T_n^{k+1}\|\,|\,\mathcal{F}_k, \mathcal{F}\right] = \log\|T_k\| + \log\mathbb{P}_{T_k/\|T_k\|}\left(\forall_{\rho_k\leq s\leq\rho_n} X_s \geq Y_s | \mathcal{F}\right) + G_{n,k} = \log\|T_n^k\| + G_{n,k}.$$

We apply this relation iteratively

$$\begin{aligned}
\mathbb{E}\left[\log\|T_n\|\,|\,\mathcal{F}\right] = \mathbb{E}\left[\log\|T_n^n\|\,|\,\mathcal{F}\right] &= \mathbb{E}\left[\mathbb{E}\left[\log\|T_n^n\|\,|\,\mathcal{F}_{n-1}, \mathcal{F}\right]|\mathcal{F}\right] \\
&= \mathbb{E}\left[\log\|T_n^{n-1}\|\,|\,\mathcal{F}\right] + \mathbb{E}[G_{n,n-1}|\mathcal{F}] \\
&= \mathbb{E}\left[\mathbb{E}\left[\log\|T_n^{n-1}\|\,|\,\mathcal{F}_{n-2}, \mathcal{F}\right]\right] + \mathbb{E}[G_{n,n-1}|\mathcal{F}] \\
&= \mathbb{E}\left[\log\|T_n^{n-2}\|\,|\,\mathcal{F}\right] + \mathbb{E}[G_{n,n-2}|\mathcal{F}] + \mathbb{E}[G_{n,n-1}|\mathcal{F}] \\
&= \ldots \\
&= \mathbb{E}\left[\log\|T_n^0\|\,|\,\mathcal{F}\right] + \sum_{k=0}^{n-1}\mathbb{E}[G_{n,k}|\mathcal{F}].
\end{aligned}$$

We notice that $\|T_n^0\| = \mathbb{P}\left(\forall_{s\leq\rho_n} X_s \geq Y_s | \mathcal{F}\right) = \mathbb{E}\left[\|T_n\|\,|\,\mathcal{F}\right]$. Thus

$$\mathbb{E}\left[\log\|T_n\|\right] = \mathbb{E}\log\mathbb{E}\left[\|T_n\|\,|\,\mathcal{F}\right] + \sum_{k=0}^{n-1}\mathbb{E}G_{n,k}.$$

One easily sees that an inequality

$$\mathbb{E}G_{n,k} \leq c, \tag{4.25}$$

for some $c < 0$, is sufficient to conclude the proof of the proposition. Proving (4.25) is our aim now. To avoid heavy notation we denote $\tilde{\mathbb{E}}(\cdot) := \mathbb{E}(\cdot|\mathcal{F})$ and $\tilde{\mathbb{E}}_k(\cdot) := \tilde{\mathbb{E}}(\cdot|\mathcal{F}_k)$. Further, we introduce additional randomization: a probability measure $\mathbb{P}_\pm$ and the random variable η such that $\mathbb{P}_\pm(\eta = 1) = \mathbb{P}_\pm(\eta = -1) = 1/2$. For $k \in \mathbb{N}$ we define $\{\tilde{Y}_s^k(\eta)\}_{s \geq 0}$ by

$$\tilde{Y}_s^k(\eta) := \begin{cases} \eta|Y_s| & \text{if } s \in [\tau_k, \rho_{k+1}] \\ Y_s & \text{otherwise.} \end{cases}.$$

There are two easy but crucial observations to be made at this point. Firstly,

$$\forall_{s\geq 0}\tilde{Y}_s^k(1) \geq \tilde{Y}_s^k(-1) \text{ and } \forall_{s\in(\tau_k,\rho_{k+1})}\tilde{Y}_s^k(1) > \tilde{Y}_s^k(-1). \tag{4.26}$$

Secondly, the excursions of an Ornstein-Uhlenbeck process are symmetric around 0. Formally, under $\mathbb{E}_\pm \otimes \tilde{\mathbb{E}}_k$ the process $\tilde{Y}^k(\eta)$ has the same law as Y under $\tilde{\mathbb{E}}_k$. Let us shorten $m := T_k/\|T_k\|$ and denote "the gap"

$$\Delta_{n,k} := \mathbb{E}_\pm\tilde{\mathbb{E}}_k \log \tilde{\mathbb{P}}_m \left(\forall_{\rho_k \leq s \leq \rho_n} X_s \geq \tilde{Y}_s^k(\eta)|Y^k\right) - \tilde{\mathbb{E}}_k \log \mathbb{E}_\pm\tilde{\mathbb{P}}_m \left(\forall_{\rho_k \leq s \leq \rho_n} X_s \geq \tilde{Y}_s^k(\eta)|Y^k\right). \tag{4.27}$$

By Jensen's inequality we have $G_{n,k} \leq \Delta_{n,k} \leq 0$. In oder to show (4.25) we will obtain a bound from above on $\Delta_{n,k}$ which is strictly negative and uniform in n, k. We define

$$g_{n,k} := \frac{\tilde{\mathbb{P}}_m \left(\forall_{\rho_k \leq s \leq \rho_n} X_s \geq \tilde{Y}_s^k(1)|Y^k\right)}{\tilde{\mathbb{P}}_m \left(\forall_{\rho_k \leq s \leq \rho_n} X_s \geq \tilde{Y}_s^k(-1)|Y^k\right)}, \tag{4.28}$$

and $z_{n,k} := \tilde{\mathbb{P}}_m \left(\forall_{\rho_k \leq s \leq \rho_n} X_s \geq \tilde{Y}_s^k(-1)|Y^k\right)$. In this notation (4.27) writes as

$$\Delta_{n,k} = \tilde{\mathbb{E}}_k \left[\frac{1}{2}\left(\log(g_{n,k}z_{n,k}) + \log z_{n,k}\right) - \log\left(\frac{g_{n,k}z_{n,k} + z_{n,k}}{2}\right)\right] = \tilde{\mathbb{E}}_k \left[\frac{1}{2}\log g_{n,k} - \log\left(\frac{g_{n,k}+1}{2}\right)\right] \tag{4.29}$$

$$\leq -\frac{1}{8}\tilde{\mathbb{E}}_k(g_{n,k} - 1)^2.$$

To explain the last inequality we observe that (4.26) yields $g_{n,k} \leq 1$ and that for $x \in (0,1]$ we have and elementary inequality $\frac{1}{2}\log x - \log\left(\frac{x+1}{2}\right) \leq -\frac{1}{8}(x-1)^2$. Now we concentrate on proving that in fact, uniformly in n, k we have $g_{n,k} < 1$. Let us analyze the expressions appearing in (4.28). We denote $\mathbb{Q}_{m,k}(\cdot) := \mathbb{P}(\cdot|Y^k)$, $\mathcal{A}_i := \{\forall_{\rho_k \leq s \leq \rho_{k+1}} X_s \geq \tilde{Y}_s^k(i)\}$ and $\mathcal{B} := \{\forall_{\rho_{k+1} \leq s \leq \rho_n} X_s \geq Y_s\}$. We fix $x \in \mathbb{R}_+$ and we want to find a formula for $p_i := \mathbb{Q}_{\delta_x,k}(\mathcal{A}_i \cap \mathcal{B})$. By the Markov property we get

$$p_i = \mathbb{Q}_{\delta_x,k}\left[\mathbb{Q}_{\delta_x,k}(1_{\mathcal{A}_i}|X_{\rho_{k+1}})\mathbb{Q}_{\delta_x,k}(1_{\mathcal{B}}|X_{\rho_{k+1}})\right].$$

We denote $L_k(x,y;i) := \mathbb{Q}_{\delta_x,k}(1_{\mathcal{A}_i}|X_{\rho_{k+1}} = y)$, which expresses more explicitly as

$$L_k(x,y;i) = \tilde{\mathbb{P}}\left(\forall_{\rho_k \leq s \leq \rho_{k+1}} X_s \geq \tilde{Y}_s^k(i)|Y^k, X_{\rho_k} = x, X_{\rho_{k+1}} = y\right). \tag{4.30}$$

We write

$$p_i = \mathbb{Q}_{\delta_x,k}\left[L(x, X_{\rho_{k+1}}; i)\frac{\mathbb{Q}_{\delta_x,k}(1_{\mathcal{B}}|X_{\rho_{k+1}})}{\mathbb{Q}_{\delta_x,k}(1_{\mathcal{B}})}\right] \times \mathbb{Q}_{\delta_x,k}(\mathcal{B}).$$

Let us now consider a measure defined by

$$\mu_{x,n,k}(X_{\rho_{k+1}} \in D) := \mathbb{Q}_{\delta_x,k}\left[1_{X_{\rho_{k+1}} \in D}\frac{\mathbb{Q}_{\delta_x,k}(1_{\mathcal{B}}|X_{\rho_{k+1}})}{\mathbb{Q}_{\delta_x,k}(1_{\mathcal{B}})}\right],$$

where $D \subset \mathbb{R}$ is a Borel set. One easily verifies that it is a probability measure. Removing the conditional expectation we get

$$\mu_{x,n,k}(X_{\rho_{k+1}} \in D) = \mathbb{Q}_{\delta_x,k} \left[\frac{\mathbb{Q}_{\delta_x,k}(1_{X_{\rho_{k+1}} \in D} 1_{\mathcal{B}} | X_{\rho_{k+1}})}{\mathbb{Q}_{\delta_x,k}(1_{\mathcal{B}})} \right] = \frac{\mathbb{Q}_{\delta_x,k}(1_{X_{\rho_{k+1}} \in D} 1_{\mathcal{B}})}{\mathbb{Q}_{\delta_x,k}(1_{\mathcal{B}})}$$

$$= \mathbb{Q}_{\delta_x,k}\left(X_{\rho_{k+1}} \in D | \mathcal{B} \right).$$

Again, writing more explicitly we have

$$\mu_{x,n,k}(\mathrm{d}y) = \tilde{\mathbb{P}}\left(X_{\rho_{k+1}} \in \mathrm{d}y | \forall_{\rho_{k+1} \leq s \leq \rho_n} X_s \geq Y_s, X_{\rho_k} = x \right). \tag{4.31}$$

Finally, concluding the above calculations we obtain that for $i \in \{-1, 1\}$ we have

$$\tilde{\mathbb{P}}_m\left(\forall_{\rho_k \leq s \leq \rho_n} X_s \geq \tilde{Y}_s^k(i) | Y^k \right) = \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} L_k(x, y; i) \mu_{x,n,k}(\mathrm{d}y) m(\mathrm{d}x) \tag{4.32}$$

$$\times \tilde{\mathbb{P}}_m\left(\forall_{\rho_{k+1} \leq s \leq \rho_n} X_s \geq Y_s \right).$$

Before going further let us comment on the further strategy. It is easy to see that for any fixed $x, y \in \mathbb{R}_+$ we have $L_k(x, y; 1) < L_k(x, y; -1)$. The gap vanished however smaller when $x, y \to +\infty$. The uniform inequality $g_{n,k} < 1$ can be obtained by by showing that with positive probability the measure $\mu_{x,n,k}(\mathrm{d}y) m(\mathrm{d}x)$ is uniformly concentrated in a box.

Let $C_1 > 0$ be a constant as in Fact 4.3. We denote sequences of events

$$\mathcal{A}_{n,k} := \mathcal{A}_k^1 \cap \{r_k \in [1, 10]\} \cap \mathcal{A}_{n,k}^2,$$
$$\mathcal{A}_k^1 := \{\rho_k \geq 1\} \cap \{\forall_{s \leq \rho_k} Y_s \leq C_1 b_{\rho_k}(s)\},$$
$$\mathcal{A}_{n,k}^2 := \left\{ \tilde{\mathbb{P}}\left(\mathcal{B}_{n,k} | \mathcal{F} \right) \geq 1/10 \right\},$$

where $\mathcal{B}_{n,k} := \left\{ \forall_{\rho_{k+1} \leq s \leq \rho_n} Y_s \leq C_1(s - \rho_{k+1} + 1)^{1/3} \right\}$. We first prove concentration of $m = T_k / \|T_k\|$ (recall (4.18)). Let $R > 0$, by the FKG property stated in Fact 2.3, conditionally on the event $\mathcal{A}_k^1$ we have

$$(T_k / \|T_k\|)\left([R^{-1}, R]\right) = \tilde{\mathbb{P}}\left(X_{\rho_k} \in [0, R] | \forall_{s \leq \rho_k} X_s \geq Y_s, Y \right) - \tilde{\mathbb{P}}\left(X_{\rho_k} \leq R^{-1} | \forall_{s \leq \rho_k} X_s \geq Y_s, Y \right)$$
$$\geq \tilde{\mathbb{P}}\left(X_{\rho_k} \in [R^{-1}, R] | \forall_{s \leq \rho_k} X_s \geq C_1 b_{\rho_k}(s) \right) - \tilde{\mathbb{P}}\left(X_{\rho_k} \leq R^{-1} | X_{\rho_k} \geq 0 \right).$$

Using Lemma 4.1 we can choose $R > 0$ such that the first term is arbitrarily close to 1. By easy calculations the second term can be made arbitrarily close to 0. We fix R such that

$$(T_k / \|T_k\|)\left([R^{-1}, R]\right) \geq 1_{\mathcal{A}_k^1}(1/2). \tag{4.33}$$

Our next aim is to study concentration of (4.31). To this end we denote

$$p(x) := \tilde{\mathbb{P}}\left(X_{\rho_{k+1}} \in [0, C_e] | \forall_{\rho_{k+1} \leq s \leq \rho_n} X_s \geq Y_s, X_{\rho_k} = x \right),$$

where $C_e > 0$ is to be fixed later. Using Fact 2.4 we estimate

$$p(x) \geq \tilde{\mathbb{P}}\left(\{X_{\rho_{k+1}} \in [0, C_e]\} \cap \mathcal{B}_{n,k} | \forall_{\rho_{k+1} \leq s \leq \rho_n} X_s \geq Y_s, X_{\rho_k} = x \right)$$
$$= \tilde{\mathbb{P}}\left(X_{\rho_{k+1}} \in [0, C_e] | \mathcal{B}_{n,k} \cap \{\forall_{\rho_{k+1} \leq s \leq \rho_n} X_s \geq Y_s, X_{\rho_k} = x\} \right)$$
$$\times \tilde{\mathbb{P}}\left(\mathcal{B}_{n,k} | \forall_{\rho_{k+1} \leq s \leq \rho_n} X_s \geq Y_s, X_{\rho_k} = x \right)$$
$$\geq \tilde{\mathbb{P}}\left(X_{\rho_{k+1}} \in [0, C_e] | \forall_{\rho_{k+1} \leq s \leq \rho_n} X_s \geq C_1(s - \rho_{k+1} + 1)^{1/3}, X_{\rho_k} = x \right) \tilde{\mathbb{P}}\left(\mathcal{B}_{n,k} \right).$$

Assuming that $r_k = \rho_{k+1} - \rho_k \in [1, 10]$, we can set C_e such that Lemma 4.2 implies that the first term is bounded away from 0 uniformly in n, k and $x \in [R^{-1}, R]$. Together with (4.33) this implies

$$\int_{R^{-1}}^{R} \mu_{x,n,k}([0, C_e]) m(\mathrm{d}x) \geq C 1_{\mathcal{A}_{n,k}},$$

for some $C > 0$. One further finds $c_e > 0$ (we skip details) such that

$$\int_{R^{-1}}^{R} \mu_{x,n,k}([c_e, C_e]) m(\mathrm{d}x) \geq (C/2) 1_{\mathcal{A}_{n,k}}. \tag{4.34}$$

We are now ready to come back to (4.28). We recall (4.30) and denote

$$r_k := \sup_{x \in [R^{-1}, R], y \in [c_e, C_e]} \frac{L_k(x, y; 1)}{L_k(x, y; -1)}, \quad L_k := \inf_{x \in [R^{-1}, R], y \in [c_e, C_e]} L_k(x, y; -1). \tag{4.35}$$

We have $L_k(x, y; -1) > L_k(x, y; 1)$. Moreover, one verifies that for $\eta \in \{-1, 1\}$ the functions $L_k(\cdot, \cdot; \eta)$ are continuous and $L_k(x, y; \eta) > 0$ if $x, y > 0$. These imply $r_k < 1$ (note that r_k is a random variable since it depends on Y^k). Similarly we have $L_k > 0$. One checks that

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} L_k(x, y; i) \mu_{x,k,n}(\mathrm{d}y) m(\mathrm{d}x) \leq 1.$$

Using the elementary inequality $(a + c)/(b + d) \leq (a + 1)/(b + 1)$ valid for $0 < a \leq b$ and $0 < c \leq d \leq 1$ we get

$$g_{n,k} \leq \frac{\int_{R^{-1}}^{R} \int_{c_e}^{C_e} L_k(x, y; 1) \mu_{x,n,k}(\mathrm{d}y) m(\mathrm{d}x) + 1}{\int_{R^{-1}}^{R} \int_{c_e}^{C_e} L_k(x, y; -1) \mu_{x,n,k}(\mathrm{d}y) m(\mathrm{d}x) + 1} \leq \frac{r_k \int_{R^{-1}}^{R} \int_{c_e}^{C_e} L_k(x, y; -1) \mu_{x,n,k}(\mathrm{d}y) m(\mathrm{d}x) + 1}{\int_{R^{-1}}^{R} \int_{c_e}^{C_e} L_k(x, y; -1) \mu_{x,n,k}(\mathrm{d}y) m(\mathrm{d}x) + 1}.$$

Further, we notice that for $r_k \in [0, 1]$ we have $(r_k a + 1)/(a + 1) \leq (r_k b + 1)/(b + 1)$ if $b \leq a$. Applying (4.34) we get

$$g_{n,k} \leq \frac{r_k L_k \int_{R^{-1}}^{R} \mu_{x,n,k}([c_e, C_e]) m(\mathrm{d}x) + 1}{L_k \int_{R^{-1}}^{R} \mu_{x,n,k}([c_e, C_e]) m(\mathrm{d}x) + 1} \leq \frac{r_k L_k C/2 + 1}{L_k C/2 + 1} \leq 1 + \frac{C L_k (r_k - 1)}{4},$$

where the last estimate follows by $(ab + 1)/(a + 1) \leq 1 + a(b - 1)/2$ valid for $a, b \in [0, 1]$. Combining the last inequality with (4.29) we arrive at

$$G_{n,k} \leq -C_3 \tilde{\mathbb{E}}_k \left[1_{\mathcal{A}_{n,k}} L_k^2 (r_k - 1)^2 \right]. \tag{4.36}$$

for a constant $C_3 > 0$.

We are now ready to show (4.25) (which concludes the proof of the proposition). By the strong Markov property we have

$$\mathbb{E} G_{n,k} \leq -C_3 \mathbb{E} \left[1_{\mathcal{A}_{n,k}} L_k^2 (r_k - 1)^2 \right] \leq -C_3 \mathbb{P} \left(\mathcal{A}_k^1 \right) \mathbb{E} \left[L_k^2 (r_k - 1)^2 1_{r_k \in [1, 10]} \right] \mathbb{P} \left(\mathcal{A}_{n,k}^2 \right).$$

We notice that the law of $L_k^2 (r_k - 1)^2 1_{r_k \in [1, 10]}$ does not depend on k and it is not concentrated on 0, thus $\mathbb{P} \left(\mathcal{A}_k^1 \right) \mathbb{E} \left[L_k^2 (r_k - 1)^2 1_{r_k \in [1, 10]} \right] > 0$. The other two terms are uniformly bounded away from 0 by Fact 4.3. $\qquad \square$

5 Proofs for Brownian motion

As explained in Remark 1.8 Theorem 1.7 implies Theorem 1.1. We will now show how to prove its extension given in Theorem 1.5.

Proof of Theorem 1.5. For simplicity we assume that $a = 0$ and $b = +\infty$, therefore we ignore the condition $B_t - \beta W_t \in (at^{1/2}, bt^{1/2})$. Denoting $x := \inf_{t \geq 0} g(t)$, one can find $A > 0$ and $\epsilon > 0$ such that

$$-x/2 - Aj(t) \leq f(t) \leq x/2 + Aj(t), \quad t \geq 0,$$

where $j(t) := \min(t, t^{1/2-\epsilon})$. We have

$$
\begin{aligned}
\mathbb{P}\left(\forall_{s \leq t} x/2 + B_s \geq W_s + Aj(s) | W\right) &\leq \mathbb{P}\left(\forall_{s \leq t} x + B_s \geq W_s + f(s) | W\right) \\
&\leq \mathbb{P}\left(\forall_{s \leq t} g(t) + B_s \geq W_s + f(s) | W\right) \leq \mathbb{P}\left(\forall_{s \leq t} g(t) + B_s \geq W_s - Aj(s) | W\right) \\
&\leq \mathbb{P}\left(\forall_{h(t) \leq s \leq t} g(t) + x/2 + B_s \geq W_s - Aj(s) | W\right), \quad (5.1)
\end{aligned}
$$

where $h(t)$ is any function such that $(h(t))^{1/2-\epsilon} \geq g(t) + x + 1$ and $h(t) = e^{o(\log t)}$. The right-hand side of the last expression is bounded from above by $\mathbb{P}\left(\forall_{h(t) \leq s \leq t} 1 + B_s \geq W_s - (A+1)j(s) | W\right)$. We will show that the event

$$\mathcal{A} := \left\{ \lim_{t \to +\infty} -\frac{\log \mathbb{P}\left(\forall_{h(t) \leq s \leq t} 1 + B_s \geq W_s - Aj(s) | W\right)}{\log t} = \gamma(\beta) \right\}, \quad (5.2)$$

fulfills $\mathbb{P}(\mathcal{A}) = 1$. The same method can be used to show the almost sure convergence of the left-hand side of (5.1) (we skip the details). These will conclude the proof. We define a stochastic process $\{Z_t\}_{t \geq 0}$ by

$$Z_t := \exp\left(\int_0^t Aj'(s) dW_s - \frac{1}{2} \int_0^t [Aj'(s)]^2 ds \right).$$

This process is an uniformly integrable martingale (since $\int_0^{+\infty} [j'(s)]^2 ds \leq +\infty$). We denote its limit by Z_∞ and define the measure (on the Wiener space) $d\mathbb{Q} = Z_\infty d\mathbb{P}$. By the Girsanov theorem under this measure $\{W_s - Aj(s)\}_{s \geq 0}$ is a standard Wiener process. We will show that

$$\mathbb{Q}(\mathcal{A}) = 1, \quad (5.3)$$

which is enough to conclude the proof. Indeed, one easily verifies that $\mathbb{P}(Z_\infty > 0) = 1$, which together with $\mathbb{E} 1_{\mathcal{A}} Z_\infty = \mathbb{Q}(\mathcal{A}) = 1$ implies $\mathbb{P}(\mathcal{A}) = 1$. We are now going to show (5.3). By Theorem 1.1 it is enough to prove

$$0 \leq \delta_t := -\left(\frac{\log \mathbb{P}\left(\forall_{0 \leq s \leq t} 1 + B_s \geq W_s | W\right)}{\log t} - \frac{\log \mathbb{P}\left(\forall_{h(t) \leq s \leq t} 1 + B_s \geq W_s | W\right)}{\log t} \right) \to 0, \quad \mathbb{Q} - \text{a.s.}$$

We have

$$
\begin{aligned}
\delta_t &= -\frac{\log \mathbb{P}\left(\forall_{s \leq h(t)} 1 + B_s \geq W_s | \forall_{h(t) \leq s \leq t} 1 + B_s \geq W_s, W\right)}{\log t} \\
&\leq -\frac{\log \mathbb{P}\left(\forall_{s \leq h(t)} 1 + B_s \geq W_s | W\right)}{\log t},
\end{aligned}
$$

where the last inequality holds by Fact 2.3. Now the proof is straightforward, indeed

$$-\frac{\log \left(\forall_{0 \leq s \leq h(t)} 1 + B_s \geq W_s, W\right)}{\log t} = -\frac{\log \mathbb{P}\left(\forall_{0 \leq s \leq h(t)} 1 + B_s \geq W_s, W\right)}{\log h(t)} \frac{\log h(t)}{\log t} \to 0, \quad \mathbb{Q} - \text{a.s.}$$

Theorem 1.1 and $\log h(t)/\log t \to 0$.

In order to prove the L^p convergence it is enough to show that that the family

$$\left\{ -\log \mathbb{P}\left(\forall_{s\le t} x/2 + B_s \ge W_s + Aj(s)|W\right)/t \right\}_{t\ge 0},$$

is L^p-uniformly integrable. This follows by easy calculations from Fact 3.8 using relation (1.3). □

We end this section with a proof of Fact 1.4.

Proof of Fact 1.4. Let $(X_k)_{k\ge 0}$ be a sequence of i.i.d. random variables such that $\mathbb{E}(X_k^{2+\epsilon}) < +\infty$. By Borel-Cantelli lemma, we have

$$L := \max\left\{ n \in \mathbb{N} : |X_n| \ge n^{1/(2+\epsilon)} \right\} < +\infty \text{ a.s.}$$

Therefore, we have

$$\limsup_{N\to+\infty} \frac{\log \mathbb{P}\left(\forall_{n\le N} B_n \ge X_n | X\right)}{\log n} \le \limsup_{N\to+\infty} \frac{\log \mathbb{P}\left(\forall_{L<n\le N} B_n \ge n^{1/(2+\epsilon)} | L\right)}{\log n} \le \frac{-1}{2} \text{a.s.}$$

Similarly, setting $M = \max_{n\le L} X_n$, we have

$$\liminf_{N\to+\infty} \frac{\log \mathbb{P}\left(\forall_{n\le N} B_n \ge X_n | X\right)}{\log n} \ge \liminf_{N\to+\infty} \frac{\log \mathbb{P}\left(\forall_{n\le N} B_n \ge -M - n^{1/(2+\epsilon)} | L, M\right)}{\log n} \ge \frac{-1}{2} \text{a.s.}$$

□

6 Proof of the facts for random walks

We now use definitions of Section 1.3. We write S a random walk in random environment μ, and set

$$W_n = -\mathbb{E}(S_n|\mu) \quad \text{and} \quad B_n = S_n + W_n.$$

To make the notation more clear in this section we assume that we have two probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$, $(\hat{\Omega}, \mathcal{G}, \mathbb{Q})$ which supports B and μ respectively. The measure $\mathbb{P}$ depends on the realization of μ, which is made implicit in the notation. Thus we are going to prove

$$\lim_{N\to+\infty} \frac{\log \mathbb{P}\left(\forall_{n\le N} x + B_n \ge W_n + f(n), B_N - W_N \in (aN^{1/2}, bN^{1/2})\right)}{\log N}$$

$$= \begin{cases} -\gamma\left(\sqrt{\frac{\text{Var}(W_1)}{\text{Var}(B_1)}}\right) & \text{on } \mathcal{A}_x, \\ -\infty & \text{on } \mathcal{A}_x^c. \end{cases} \quad \mathbb{Q}\text{-a.s.} \quad (6.1)$$

Further to simplify the notation we put $\gamma := \gamma\left(\sqrt{\frac{\text{Var}(W_1)}{\text{Var}(B_1)}}\right)$ and

$$p_N := \mathbb{P}\left(\forall_{n\le N} x + B_n \ge W_n + f(n), B_N - W_N \in (aN^{1/2}, bN^{1/2})\right). \quad (6.2)$$

We need a bound that the inhomogeneous random walk B grows fast. This will be contained in the first two lemmas of this section. We will use tilting of measure. Let us denote the increments of B by

$$X_n := B_n - B_{n-1}.$$

Let us recall C_1 from the assumption (A2). For any $\theta \in [0, C_1]$ and $n \in \mathbb{N}$ we define a probability measure $\mathbb{H}_{n,\theta}$ by

$$\frac{d\mathbb{H}_{n,\theta}}{d\mathbb{P}} := e^{\theta X_n}\psi_n^{-1}(\theta), \quad \psi_n(\theta) = \mathbb{E}e^{\theta X_n}. \quad (6.3)$$

The tilting is supposed to "increase" X_n's. The following lemma quantifies this

Lemma 6.1. There exist $\theta_0 \in (0, C_1)$ and $0 < c \le \tilde{c}$ such that for any $\theta \le \theta_0$ and $n \in \mathbb{N}$ we have

$$\mathbb{H}_{n,\theta}(X_n) \in [c\theta \mathbb{E} X_n^2, \tilde{c}\theta \mathbb{E} X_n^2].$$

Proof. By (6.3) we have

$$\mathbb{H}_{n,\theta}(X_n) = \psi_n^{-1}(\theta)\mathbb{E} X_n e^{\theta X_n}.$$

The proof will be finished once we show that for any n and small enough θ we have

$$\mathbb{E} X_n e^{\theta X_n} = (\mathbb{E} X_n^2)\theta + o(\theta), \quad \psi_n(\theta) = 1 + O(\theta^2). \tag{6.4}$$

By the assumption of the uniform exponential integrability in (A2) and Cauchy's estimate [9, Theorem 10.26] for any n and $0 \le \theta \le C_1/2$ we get

$$\psi_n''(\theta) \le C, \quad |\psi_n'''(\theta)| \le C,$$

for some $C > 0$. By the assumptions $\psi_n(0) = 1$ and $\psi_n'(0) = \mathbb{E} X_n = 0$, thus the second statement of (6.4) follows by the Taylor formula (with the Lagrange reminder). For the first one we notice that $\mathbb{E} X_n e^{\theta X_n} = \psi_n'(\theta)$, $\psi_n''(0) = \mathbb{E} X_n^2$ and again apply the Taylor expansion. $\qquad\square$

We present now the aforementioned bound.

Lemma 6.2. *There exist $c, C > 0$ such that for large enough N on the event*

$$\left\{ \sum_{n=1}^{N} \mathbb{E}(X_n^2) \ge N\mathbb{Q}(\mathbb{E} X_i^2)/2 \right\}$$

we have

$$\mathbb{P}\left(B_N \ge c\sqrt{N} \log\log N \right) \ge N^{-C(\log\log N)^2}.$$

Proof. We define $a_n := \frac{1}{4}\theta_0 c n^{1/2} \log\log n$, where θ_0, c are given by Lemma 6.1 and consider the events $\mathcal{A}_N := \{ S_N \ge a_N \}$. We denote also $b_n := (\theta_0 n^{-1/2} \log\log n) \vee 0$ and let us define the tilted measure $\mathbb{P}_N$ by

$$\frac{d\mathbb{P}_N}{d\mathbb{P}} := \Lambda_N, \quad \Lambda_N := \exp\left(\sum_{n=1}^{N} b_n X_n \right) \prod_{n=1}^{N} \psi(b_n)^{-1}. \tag{6.5}$$

Further, we write $q_N := \mathbb{P}(\mathcal{A}_N) = \mathbb{P}_N\left(1_{\mathcal{A}_N} \Lambda_N^{-1} \right)$. We have to estimate

$$q_N = \mathbb{P}_N\left(1_{\mathcal{A}_N} \Lambda_N^{-1} \right) = \left(\prod_{n=1}^{N} \psi(b_n) \right) \mathbb{P}_N\left(1_{\mathcal{A}_N} \exp\left(-\sum_{n=1}^{N} b_n X_n \right) \right) \ge \mathbb{P}_N\left(1_{\mathcal{A}_N} \exp\left(-\sum_{n=1}^{N} b_n X_n \right) \right).$$

We introduce $\tilde{X}_n := X_n - \mathbb{E}_N X_n$ and accordingly $\tilde{B}_n := \sum_{i=1}^{n} \tilde{X}_i$. In our notation

$$q_N \ge \exp\left(-\sum_{i=1}^{N} (\mathbb{E}_N X_n) b_n \right) \mathbb{P}_N\left(1_{\mathcal{A}_N} \exp\left(-\sum_{n=1}^{N} b_n \tilde{X}_n \right) \right).$$

Now, by Lemma 6.1 and the assumption (A2) we obtain

$$\sum_{n=1}^{N} (\mathbb{E}_N X_n) b_n \le C_1 \sum_{n=1}^{N} b_n^2 \le 2C_1 (\log N)(\log\log N)^2,$$

for $C_1 > 0$. Next, we apply the Abel transform

$$\sum_{n=1}^{N} b_n \tilde{X}_n = \sum_{n=1}^{N} b_n (\tilde{B}_n - \tilde{B}_{n-1}) = \tilde{B}_N b_N + \sum_{n=1}^{N-1} (b_n - b_{n+1}) \tilde{B}_n.$$

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We define events $\mathcal{B}_N := \{\forall_{n \leq N} |\tilde{B}_n| \leq C_2 a_n\}$, for some $C_2 > 0$. We have an elementary estimation $|b_n - b_{n+1}| \leq C_3 n^{-3/2} \log \log n$, $C_3 > 0$. Putting things together we obtain

$$q_N \geq N^{-C_1(\log \log N)^2} \mathbb{P}_N \left(1_{\mathcal{A}_N \cap \mathcal{B}_N} \exp \left(-|\tilde{B}_N b_N| - C_3 \sum_{n=1}^{N-1} |n^{-3/2} \log \log n| |\tilde{B}_n| \right) \right)$$

$$\geq N^{-C_1(\log \log N)^2} \mathbb{P}_N \left(1_{\mathcal{A}_N \cap \mathcal{B}_N} \exp \left(-\tilde{B}_N b_N - C_4 \sum_{n=1}^{N-1} n^{-1} (\log \log n)^2 \right) \right)$$

$$\geq N^{-C_5(\log \log N)^2} \mathbb{P}_N \left(\mathcal{A}_N \cap \mathcal{B}_N \right),$$

where we introduced $C_4, C_5 > 0$. We notice that

$$\mathcal{A}_N \supseteq \left\{ \tilde{B}_N \geq a_N - \sum_{n=1}^{N} \mathbb{E}_N X_i \right\} \supseteq \left\{ \tilde{B}_N \geq -a_N/2 \right\}.$$

Finally, we leave to the reader proving that $\liminf_{N \to \infty} \mathbb{P}_N(\mathcal{A}_N \cap \mathcal{B}_N) > 0$, which concludes the proof. $\square$

Let us recall the event $\mathcal{A}_x$ defined in (1.8). Let us denote by $\tilde{p}_N$ the version of p_N from (6.2) without condition $B_N - W_N \in (aN^{1/2}, bN^{1/2})$ i.e.

$$\tilde{p}_N := \mathbb{P} \left(\forall_{n \leq N} x + B_n \geq W_n + f(n) \right). \tag{6.6}$$

In the following lemma we prove a crude bound corresponding to the bound from above in Theorem 1.11. Namely

Lemma 6.3. *We have*
$$\liminf_{N \to +\infty} \frac{\log \tilde{p}_N}{N^2} \geq \begin{cases} 0 & \text{on } \mathcal{A}_x, \\ -\infty & \text{on } \mathcal{A}_x^c. \end{cases} \quad \mathbb{Q} - a.s.$$

Proof. The proof will follow again by the change of measure techniques. Due to a very big normalization the proof can be somewhat brutal. We fix $b_N = b \in (0, C_1)$ (C_1 as in (A2)) and use Λ_N and $\mathbb{P}_N$ as in (6.5). We denote $\mathcal{B}_N := \{\forall_{n \leq N} x + B_n \geq W_n + f(n)\}$ and calculate

$$\tilde{p}_N = \mathbb{P}_N (1_{\mathcal{B}_N} \Lambda_N^{-1}) \geq \mathbb{P}_N \left(1_{\mathcal{B}_N} \exp \left(-\sum_{n=1}^{N} b X_n \right) \right).$$

We introduce also $\mathcal{B}_N^1 := \{\forall_{n \in \{1, \dots, N\}} X_n \leq N^{1/2}\}$. Trivially we have

$$\tilde{p}_N \geq \mathbb{P}_N \left(1_{\mathcal{B}_N \cap \mathcal{B}_N^1} \exp \left(-\sum_{n=1}^{N} b X_n \right) \right) \geq e^{-bN^{3/2}} \mathbb{P}_N (\mathcal{B}_N \cap \mathcal{B}_N^1) \tag{6.7}$$

$$\geq e^{-bN^{3/2}} \left[\mathbb{P}_N(\mathcal{B}_N) - \mathbb{P}_N((\mathcal{B}_N^1)^c) \right] \geq e^{-bN^{3/2}} \left[\mathbb{P}_N(\mathcal{B}_N) - N e^{-cN^{1/2}} \right],$$

where the last inequality follows by the union bound and the fact that exponential moments of X_n are uniformly bounded, see (A2). Let us concentrate on $\mathbb{P}_N(\mathcal{B}_N)$. We denote $v = \mathbb{Q} \left[\mathbb{E}(X_i^2) \right]$ and define

$$L := \inf \left\{ n \geq 0 : \forall_{k \geq n} \sum_{i=0}^{k} \mathbb{E}(X_i^2) \geq kv/2 \right\}.$$

Clearly $\mathbb{Q}(L < +\infty) = 1$. Fix $K > 0$ and denote the following events in $\mathcal{G}$ (i.e. describing conditions on W)

$$\mathcal{A}_K := \mathcal{A}_K^1 \cap \mathcal{A}_K^2, \quad \mathcal{A}_K^1 := \{L < K\}, \quad \mathcal{A}_K^2 := \left\{ \forall_{k \geq K} W_k \leq k^{2/3} - f(k) \right\}.$$

Using the Markov property we get

$$\mathbb{P}_N(\mathcal{B}_N) \geq \mathbb{P}_N \left(\forall_{n \in \{1,\dots,K\}} x + B_n \geq W_n + f(n), B_K \geq Kv/2 \right) \tag{6.8}$$
$$\times \, \mathbb{P}_N \left(\forall_{n \in \{K,\dots,N\}} B_n \geq W_n + f(n) | B_K = Kv/2 \right).$$

We denote the first term by $\tilde{p}_K$. It is easy to check that the law of B_n under $\mathbb{P}_N$ stochastically dominates the one under $\mathbb{P}$ thus $\{p_K = 0\} \subset \mathcal{A}_x^c$. Conditionally on $\mathcal{A}_K^2$ we have

$$\mathbb{P}_N \left(\forall_{n \in \{K,\dots,N\}} B_n \geq W_n + f(n) | B_K = Kv/2 \right) \geq 1 - \sum_{n=K}^{+\infty} \mathbb{P}_N \left(B_n \leq nv/4 | B_K = Kv/2 \right).$$

We denote $\tilde{B}_n := B_n - \mathbb{E}_N B_n$. By Lemma 6.1, conditionally on $\mathcal{A}_K^1$, we have $\mathbb{E}_N B_k \geq ckv/2$, thus

$$\mathbb{P}_N \left(\forall_{n \in \{K,\dots,N\}} B_n \geq W_n + f(n) | B_K = Kv/2 \right) \geq 1 - \sum_{n=0}^{+\infty} \mathbb{P}_N \left(\tilde{B}_n \leq -cnv/4 | B_0 = Kv/2 \right).$$

Observing that the random variables X_n are uniformly exponentially integrable we get a constant $c_1 > 0$ such that

$$\mathbb{P}_N \left(\forall_{n \in \{K,\dots,N\}} B_n \geq W_n + f(n) | B_K = Kv/2 \right) \geq 1 - \sum_{n=K}^{+\infty} e^{-c_1 n} > 0,$$

for K large enough. Putting the above estimates to (6.8) we obtain that for some $C > 0$

$$1_{\mathcal{A}_K} \mathbb{P}_N(\mathcal{B}_N) \geq 1_{\mathcal{A}_K} C \tilde{p}_K.$$

Using this in (6.7) we have

$$\liminf_{N \to +\infty} \frac{\log \tilde{p}_N}{N^2} \geq \begin{cases} 0 & \text{on } \mathcal{A}_K \cap \mathcal{A}_x, \\ -\infty & \text{on } \mathcal{A}_K^c \cup \mathcal{A}_x^c. \end{cases} \quad \mathbb{Q} - a.s.$$

The proof is concluded passing $K \nearrow +\infty$ and by observing that $1_{\mathcal{A}_K} \to 1$, $\mathbb{Q}$-a.s. $\qquad \square$

We finally pass to the proof of Theorem 1.11. We notice that by the very definition of $\mathcal{A}_x$ (see 1.8) it is obvious that the convergence holds on $\mathcal{A}_x^c$. Thus in the proofs below we concentrate on proving the convergence on the event $\mathcal{A}_x$. The instrumental tool of this proof will be the so-called KMT coupling. We choose the measure $\mathbb{Q}$ to be a special one. By [7, Corollary 2.3] we can find a probability space $(\tilde{\Omega}, \mathcal{G}, \mathbb{Q})$ with processes $\{W_k\}_{k \geq 0}$ and $\left\{ \hat{W}_k \right\}_{k \geq 0}$, which is a random walk with the increments distributed according to $\mathcal{N}(0, \sigma_W^2)$, where $\sigma_W^2 = \text{Var}(\overline{W}_1)$ and

$$\limsup_{k \to +\infty} \frac{|\hat{W}_k - W_k|}{(\log k)^2} = 0, \quad \mathbb{Q} - \text{a.s.} \tag{6.9}$$

Further we can extend the measure $\mathbb{Q}$ so that $\hat{W}$ is a marginal of a Brownian motion, also denoted by $\hat{W}$. This is rather standard and left to the reader also we keep $\mathbb{Q}$ to denote the extended measure and $\hat{W}$ for the Brownian motion.

First we prove (6.1) for this special measure. At the end of the proof we argue how this statement implies the thesis of Theorem 1.11. We start with the bound from above. We recall p_N defined in (6.2). One finds A and $\epsilon > 0$ such that for any $n \in \mathbb{N}$ we have $f(n) \geq -An^{1/2-\epsilon}$. Further we set $\tilde{p}_N$ of (6.6) with this function, i.e.

$$\tilde{p}_N := \log \mathbb{P} \left(\forall_{n \leq N} x + B_n \geq W_n - An^{1/2-\epsilon} \right).$$

In this part we will show

$$\limsup_{N\to+\infty} \frac{\log p_N}{\log N} \leq \limsup_{N\to+\infty} \frac{\log \tilde{p}_N}{\log N} \leq -\gamma, \quad \mathbb{Q}\text{-a.s.} \tag{6.10}$$

We define a function $f : \mathbb{N} \times \mathcal{M}(\mathbb{R})^{\mathbb{N}} \mapsto \mathbb{R}$ by

$$f(n,\mu) := \sum_{i=1}^{n} \mathbb{E}X_i^2. \tag{6.11}$$

We recall that the measure $\mathbb{P}$ depends on realization of W and that $\{Z_i := \mathbb{E}X_i^2\}_{i\geq 0}$ is a sequence of i.i.d variables with respect to $\mathbb{Q}$. It is easy to check, using the exponential Chebyshev inequality, that (A2) implies existence of $\tilde{C}_1 > 0$ such that $Z_i \leq \tilde{C}_1$. We define a sequence of events belonging to $\mathcal{G}$ given by

$$\mathcal{A}_N := \left\{\forall_{k\leq N}|\hat{W}_k - W_k| \leq (\log N)^2\right\} \cap \left\{\forall_{k\leq N}\sup_{t\in[k,k+1]}|\hat{W}_t - \hat{W}_k| \leq (\log N)^2\right\} \tag{6.12}$$

$$\cap \left\{\forall_{k\geq \log N}\max_{l\in\left\{-k^{2/3},\ldots,k^{2/3}\right\}}|\hat{W}_k - \hat{W}_{k+l}| \leq k^{4/9}\right\}$$

$$\cap \left\{\forall_{k\geq \log N}|f(k,\mu) - k\mathbb{Q}(\mathbb{E}(X_1^2))| \leq k^{1/2}\log k\right\}.$$

We have $1_{\mathcal{A}_N} \to 1$ $\mathbb{Q}$-a.s. The convergence of the first term follows by (6.9). The proof of the others are rather standard (we note that exponents $2/3$ and $4/9$ can be made smaller but this is not relevant for our proof). As an example we treat the last but one term. We set

$$q_k := \mathbb{P}\left(\max_{l\in\left\{-k^{2/3},\ldots,k^{2/3}\right\}}|\hat{W}_k - \hat{W}_{k+l}| \leq k^{4/9}\right).$$

By the properties of a Wiener process we know that $\max_{l\in\left\{-k^{2/3},\ldots,k^{2/3}\right\}}|\hat{W}_k - \hat{W}_{k+l}|$ has the tails decaying faster than $\exp\left(-t^2/(4k^{2/3})\right)$, for t large enough. Thus

$$1 - q_k \leq \mathbb{P}\left(\max_{l\in\left\{-k^{2/3},\ldots,k^{2/3}\right\}}|\hat{W}_k - \hat{W}_{k+l}| \geq k^{4/9}\right) \leq \exp\left(-k^{2/9}/4\right).$$

This quantity is summable thus the proof follows by the standard application of the Borel-Cantelli lemma. From now on, we will work conditionally on $\mathcal{A}_N$. Using its first condition we have

$$\tilde{p}_N \leq \log \mathbb{P}\left(\forall_{\log N\leq n\leq N}2(\log N)^2 + B_n \geq \hat{W}_n - An^{1/2-\epsilon}\right). \tag{6.13}$$

We use the coupling techniques also for $\mathbb{P}$. Namely, by [7, Theorem 3.1] on a common probability space (denoted still by $\mathbb{P}$), we have processes $\{B_k\}_{k\geq 0}$, distributed as the random walk from our theorem and $\left\{\hat{B}_t\right\}_{t\geq 0}$ a Brownian motion which approximates B. Recalling (6.11) we define

$$\mathcal{B}_N := \left\{\forall_{k\leq N}|B_k - \hat{B}_{f(k,\mu)}| \leq (\log N)^2\right\} \cap \left\{\forall_{k\leq N}\sup_{t\in[k,k+1]}(\hat{B}_t - \hat{B}_k) \leq (\log N)^2\right\}. \tag{6.14}$$

Applying [7, Theorem 3.1] to the first term and and standard considerations to the second one we obtain $\log \mathbb{P}\left(\mathcal{B}_N^c\right)/\log N \to_{N\to+\infty} -\infty$. We continue estimations of (6.13) as follows

$$\tilde{p}_N \leq \mathbb{P}\left(\forall_{\log N\leq n\leq N}4(\log N)^2 + \hat{B}_{f(n,\mu)} \geq \hat{W}_n - An^{1/2-\epsilon}, \mathcal{B}_N\right) + \mathbb{P}\left(\mathcal{B}_N^c\right).$$

We extend, in the piece-wise linear fashion, the function f to the whole line with respect to its first argument. This function is non-decreasing and we denote its generalized inverse by $g(t, \mu) := \inf \{s \geq 0 : f(s, \mu) \geq t\}$. We change to the continuous time (writing t instead of n). By the second and last condition of (6.12) for some $C > 0$ we have

$$\tilde{p}_N \leq \mathbb{P} \left(\forall_{C \log N \leq t \leq N/C} 5(\log N)^2 + \hat{B}_t \geq \hat{W}_{g(t,\mu)} - A(g(t,\mu))^{1/2-\epsilon} \right) + \mathbb{P}\left(\mathcal{B}_N^c\right).$$

Using two last conditions of (6.12) one checks that $\left\{ \forall_{t \geq C \log N} \hat{W}_{g(t,\mu)} \geq W_{t/\mathbb{Q}(\mathbb{E}(X_1^2))} - t^{4/9} \right\}$ and thus conditionally on $\mathcal{A}_N$ we have

$$\tilde{p}_N - \mathbb{P}\left(\mathcal{B}_N^c\right) \leq \mathbb{P} \left(\forall_{C \log N \leq t \leq N/C} 5(\log N)^2 + \hat{B}_t \geq \hat{W}_{t/\mathbb{Q}(\mathbb{E}(X_1^2))} - t^{4/9} - 2At^{1/2-\epsilon} \right) =: \hat{p}_N.$$

Utilizing Theorem 1.5 one gets

$$\lim_{N \to +\infty} \frac{\log \hat{p}_N}{\log N} = -\gamma \left(\sqrt{\frac{\mathbb{E}W_1^2}{\mathbb{Q}(\mathbb{E}(X_1^2))}} \right).$$

We recall that in our notation $\mathbb{Q}(\mathbb{E}(X_i^2))$ is the same as $\mathbb{E}\left[\mathbb{E}\left(B_1^2|\mu\right)\right] = \mathbb{E}B_1^2$ in the standard notation. Recalling that $\mathbb{P}\left(\mathcal{B}_N^c\right)$ is negligible we obtain (6.10).

Before passing further let us state a simple conditioning fact.

Lemma 6.4. *Let $\{T_n\}_{n \geq 0}$ be a random walk and $\{a_n\}_{n \geq 0}$ be a sequence. Then for any N the law $\mathbb{P}\left(T_N \in \cdot | \forall_{n \in \{1,\dots,N\}} T_n \geq a_n\right)$ stochastically dominates the law of T_N.*

The proof follows by inductive application of the Markov property and is left to the reader.

We pass to the bound from below. We recall (6.2). Our aim is to prove

$$\liminf_{N \to +\infty} \frac{\log p_N}{\log N} \geq \begin{cases} -\gamma & \text{on } \mathcal{A}_x, \\ -\infty & \text{on } \mathcal{A}_x^c. \end{cases} \tag{6.15}$$

We denote $K_N := \lfloor (\log N)^6 \rfloor$ and $A_N := c K_N^{1/2} \log \log K_N$ (c is as in Lemma 6.2). Utilizing the Markov property we obtain

$$\log p_N \geq \log q_N + \log \hat{p}_N,$$

where

$$q_N := \mathbb{P}\left(\forall_{n \in \{0,\dots,K_N\}} x + B_n \geq W_n + f(n), B_{K_N} \geq A_N, B_{K_N} \leq N^{1/3}\right),$$

$$\hat{p}_N := \inf_{x \in [A_N, N^{1/3}]} \mathbb{P}\left(\forall_{n \in \{K_N,\dots,N\}} x + B_n^{K_N} \geq W_n + f(n), x + B_N^{K_N} - W_N \in (aN^{1/2}, bN^{1/2})\right),$$

where for l we denote $B_k^l = B_k - B_l$, $k \geq l$. For q_N we utilize Lemma 6.4 as follows

$$q_N \geq \mathbb{P}\left(\forall_{n \in \{0,\dots,K_N\}} x + B_n \geq W_n\right) \mathbb{P}\left(B_{K_N} \geq A_N\right) - \mathbb{P}\left(B_{K_N} \geq N^{1/3}\right). \tag{6.16}$$

Let us further denote $k_N := \lfloor (\log K_N)^6 \rfloor$ and $a_N := k_N^{1/2} \log \log k_N$. Applying a similar procedure we get

$$\mathbb{P}\left(\forall_{n \in \{0,\dots,K_N\}} x + B_n \geq W_n + f(n)\right) \geq \log \mathbb{P}\left(B_{k_N} \geq a_N\right) + \log p(x, 0, k_N) + \log p(a_N, k_N, K_N),$$

where $p(x, k, l) := \mathbb{P}\left(\forall_{n \in \{k,\dots,l\}} x + B_n^k \geq W_n + f(n)\right)$. We will prove that

$$\liminf_{N \to +\infty} \frac{\hat{p}_N}{\log N} \geq -\gamma, \quad \mathbb{Q} - a.s. \tag{6.17}$$

Lemma 6.3 and Lemma 6.2 imply

$$\liminf_{N\to+\infty} \frac{\log p(x,0,k_N)}{\log N} \geq \begin{cases} 0 & \text{on } \mathcal{A}_x \\ -\infty & \text{on } \mathcal{A}_x^c \end{cases}, \quad \text{and} \quad \liminf_{N\to+\infty} \frac{\log \mathbb{P}\left(B_{K_N} \geq A_N\right)}{\log N} = 0, \quad \mathbb{Q} - a.s. \quad (6.18)$$

By simple scaling arguments we notice that (6.17) and (6.18) imply

$$\liminf_{N\to+\infty} \frac{\log p(a_N, k_N, K_N)}{\log N} \geq 0, \quad \text{and} \quad \liminf_{N\to+\infty} \frac{\log \mathbb{P}\left(B_{k_N} \geq a_N\right)}{\log N} = 0, \quad \mathbb{Q} - a.s.$$

We notice that by assumption (A2) for large N we have $\mathbb{P}\left(B_{K_N} \geq N^{1/3}\right) \leq \exp(-N^{1/4})$. Thus this term is negligible in (6.16) and we get

$$\liminf_{N\to+\infty} \frac{\log q_N}{\log N} \geq \begin{cases} 0 & \text{on } \mathcal{A}_x \\ -\infty & \text{on } \mathcal{A}_x^c \end{cases}.$$

This together with (6.17) implies (6.15). For (6.17) we will apply coupling arguments similar to the ones in the previous proof. We keep the notation $(W, \hat{W})$ and $(B, \hat{B})$. We will also use the events of (6.12). Finally, we know that for some $\epsilon > 0$ we have $f(n) \leq n^{1/2-\epsilon}/2$ for n large enough. We set a', b' such that $a < a' < b' < b$. Conditionally on $\mathcal{A}_N$ for N large enough

$$\hat{p}_N \geq \mathbb{P}\left(\forall_{K_N \leq n \leq N} A_N + B_n^{K_N} \geq \hat{W}_n + n^{1/2-\epsilon}/2, B_N^{K_N} - \hat{W}_N \in (a'N^{1/2}, b'N^{1/2})\right).$$

Further, recalling (6.11) and (6.14) for $a' < a'' < b'' < b'$ we we have

$$\hat{p}_N \geq \mathbb{P}\left(\begin{array}{c} \forall_{K_N \leq n \leq N} A_N/2 + \hat{B}_{f(n,\mu)-f(K_N,\mu)} \geq \hat{W}_n + n^{1/2-\epsilon} \\ \hat{B}_{f(N,\mu)-f(K_N,\mu)} - \hat{W}_N \in (a''N^{1/2}, b''N^{1/2}) \end{array} \right) - \mathbb{P}\left(\mathcal{B}_N^c\right).$$

Similarly as in the previous case the second term will be negligible. Let $f_N(\cdot, \mu)$ be the piece-wise linearization of $\{K_N, \ldots, N\} \ni n \mapsto f(n, \mu) - f(K_N, \mu)$. It is non-decreasing thus we may define its inverse by $g_N(t, \mu) := \inf\{s \geq 0 : f_N(s, \mu) \geq t\}$. We set $v = \mathbb{Q}(\mathbb{E}(X_1^2))$ (we recall that $X_1 = B_1$)

$$\mathcal{C}_N := \left\{\forall_{t\geq 0}|g_N(t,\mu) - t/v| \leq [(\log N)^3 \vee t^{2/3}]\right\}.$$

We leave to the reader verifying that $1_{\mathcal{C}_N} \to 1$ $\mathbb{Q}$-a.s. Now conditionally on $\mathcal{C}_N$ we have

$$\hat{p}_N \geq \mathbb{P}\left(\forall_{vK_N/2 \leq t \leq M_N} A_N/2 + \hat{B}_t \geq \hat{W}_{g_N(t,\mu)} + (g_N(t,\mu))^{1/2-\epsilon}, \mathcal{D}_N\right) - \mathbb{P}\left(\mathcal{B}_N^c\right),$$

where $M_N = vN - N^{3/4}$ and

$$\mathcal{D}_N := \left\{\forall_{M_N < t < vN + N^{3/4}} \hat{B}_t - \hat{W}_{g_N(t,\mu)} \in (a''N^{1/2}, b''N^{1/2})\right\}.$$

Using the third condition of (6.12) and performing simple calculations we have

$$\mathcal{A}_N \cap \mathcal{C}_N \subset \left\{\forall_{t\geq \log N} \hat{W}_{g_N(t,\mu)} \leq \hat{W}_{t/v} + t^{4/9} + (\log N)^3\right\}.$$

Therefore on $\mathcal{A}_N \cap \mathcal{C}_N$, for N large enough, we get

$$\hat{p}_N \geq \mathbb{P}\left(\forall_{vK_N/2 \leq t \leq M_N} A_N/4 + \hat{B}_t \geq \hat{W}_{t/v} + t^{4/9}, \mathcal{D}_N\right) - \mathbb{P}\left(\mathcal{B}_N^c\right).$$

We choose a''', b''' such that $a'' < a''' < b''' < b''$ and apply the Markov property

$$\mathbb{P}\left(\forall_{vK_N/2 \leq t \leq M_N} A_N/4 + \hat{B}_t \geq \hat{W}_{t/v} + t^{4/9}, \mathcal{D}_N\right)$$

$$\geq \mathbb{P}\left(\forall_{K_N/2 \leq t \leq M_N} A_N/4 + \hat{B}_t \geq \hat{W}_{t/v} + t^{4/9}, \hat{B}_{M_N} - \hat{W}_{M_N/v} \in (a'''N^{1/2}, b'''N^{1/2})\right)$$

$$\times \inf_{x \in (a'''N^{1/2}, b'''N^{1/2})} \mathbb{P}\left(\mathcal{D}_N | \hat{B}_{M_N} - \hat{W}_{M_N/v} = x\right).$$

It is easy to check that with high probability (with respect to $\mathbb{Q}$) the last term is bigger than $1/2$. Recalling that $\mathbb{P}\left(\mathcal{B}_N^c\right)$ is negligible and utilizing Theorem 1.5 we obtain (6.17). This together with (6.10) implies (6.1) for the special choice of the realization of W (i.e. we worked with the measure $\mathbb{Q}$ on which we had the coupling $(W, \hat{W})$). To remove this assumption let us consider l be the space with $\mathbb{R}$-valued sequences with the product topology. Given any other probability measure $\mathbb{P}$ supporting W and B we have $\mathbb{P}\left(W \in \mathcal{A}\right) = \mathbb{Q}\left(W \in \mathcal{A}\right)$ for any $\mathcal{A}$ in the Borel σ-field of l. One checks that

$$\mathcal{A}_0 := \left\{ g \in l : \lim_{N \to +\infty} \frac{\log \mathbb{P}\left(\forall_{n \leq N} x + B_n \geq g_n + f(n)\right), B_N - W_N \in (aN^{1/2}, bN^{1/2})}{\log N} = -\gamma \right\}$$

belongs to this σ-field. Now we have

$$\mathbb{P}\left(\mathcal{A}_0\right) = \mathbb{Q}(\mathcal{A}_0) = 1.$$

This concludes the proof of Theorem 1.11. We skip the proof of Theorem 1.12, it follows by rather simple modifications of the above proof.

We are still left with

Proof of Fact 1.9. The first part of the fact is easy (e.g. by the Hsu–Robbins theorem) and is left to the reader. For the second part let us consider first that $\sup S_B = +\infty$. Then every step of B can be bigger than the one of W thus clearly for any N we have $\mathbb{P}\left(\forall_{n \leq N} x + B_n \geq W_n | W\right) > 0$. Now, we assume $S := \sup S_B < +\infty$. For any fixed N we have $\mathbb{P}\left(B_1 \geq S - x/(2N)\right) > 0$. Further, one verifies that

$$\left\{\forall_{n \leq N} : B_n - B_{n-1} \geq S - x/(2N)\right\} \subset \left\{\mathbb{P}\left(\forall_{n \leq N} x + B_n \geq W_n | W\right) > 0\right\}.$$

Therefore, one obtains $\mathbb{P}\left(\mathcal{A}_x\right) = 1$. The second part of the proof goes easily by contradiction. If the condition does not hold then there exists S and $\epsilon > 0$ such that $\mathbb{P}\left(W_1 \geq S + \epsilon\right) > 0$ and $\mathbb{P}\left(B_1 \geq S - \epsilon\right) = 0$. From this we see that $\mathbb{P}\left(W_{\lceil 2x/\epsilon \rceil} \geq \lceil 2x/\epsilon \rceil S + 2x\right) > 0$ while $\mathbb{P}\left(B_{\lceil 2x/\epsilon \rceil} \geq \lceil 2x/\epsilon \rceil S\right) = 0$. $\square$

7 Discussion and Open Questions

In the concluding section we discuss some open questions and further areas of research.

- The function γ introduced in Theorem 1.1 calls for better understanding. We are convinced that it is strictly convex. It should be possible to obtain its asymptotics when $\beta \to +\infty$, we expect that $\gamma(\beta)/\beta^2 \to C$, for $C > 0$.

- The qualitative results of our paper should hold in a much greater generality. Let us illustrate that on an example. We expect that the convergence in Theorem 1.10 stays valid for any processes $\{W_n\}_{n \in \mathbb{N}}, \{B_n\}_{n \in \mathbb{N}}$ whose increments are weakly correlated (for example with the exponential decay of correlations like $\mathrm{Cov}(W_{n+1} - W_n, W_{k+1} - W_k) \sim \exp(-c|n-k|)$). Similarly the qualitative statement of Theorem 1.1 should be valid if processes $\{B_t\}_{t \geq 0}, \{W_t\}_{t \geq 0}$ are diffusions without strong drift (possibly the proper condition to assume is that the spectral gap is 0).

- The case $\beta = 0$ in Theorem 1.1 is well-studied, in particular it is known that conditioning a Brownian motion to stay above the line has a repelling effect and such a process escapes to infinity as $t^{1/2}$ as $t \to +\infty$. Our result $\gamma(\beta) > \gamma(0)$ for $\beta \neq 0$ suggests that the repelling effect is stronger when the disorder is present. Quantifying this effect would be an interesting research question.

Acknowledgments

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