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Could the Darboux's forces be an alternative to the dark matter/energy?

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Abstract: We investigate the possibility that the Newtonian potential becomes progressively Harmonic when distances increases, as suggested in the Friedman'equations. Transition between the two potentials is describe by the forces of Darboux. Model doesn't need dark matter or dark energy, allows to conserve energy and angular momentum and leads to flat curves of rotation at the periphery of galaxies. Dark matter is replaced by an increasing of ratio gravitational/inertial mass linked to the modification of potential.

Keywords: Central force; conic; dark matter; force of gravitation; galaxies; modified gravitation; PACS Number:04.70.Kd

1. Introduction

It is well known that several theoretical questions have recurred again and again in physics of Gravity, for the reason that they are not always well explained by classical theories. One of these questions concern the expansion of Universe, its rate, and the validity of cosmological models. Another is about the curve of rotation of galaxies and the possible existence of the "dark matter" [1]. Therefore important experiments are regularly in progress (see for example references [2,3]). It is also the reason for what theories of modified Gravity are regularly published. Generally these theories assume that the laws of Gravity are progressively modified when distance to the center of mass increases. An important part of them investigate the possibility that the Newtonian potential changes progressively, for example using a logarithmic corrected Newtonian potential, a Yukawa-like potential (see for example references [4], [5] where these possibilities are discussed), etc..

It was in line with this kind of thinking that we have investigated another possible evolution of the Newtonian potential. We have studied the possibility that this potential becomes gradually harmonic, when distances to the center of mass increases and we present our results in this paper.

This initial assumption wasn't entirely arbitrary, but based on the first question we have mentioned, i.e. regarding the expansion of the Universe. Indeed if we know since the beginning of the 20th that Universe isn't static, we recently learnt that it expands at an increasing rate [6]. The common explanation consists to introduce a constant, called "cosmological constant" in the Einstein's field of equations. This solution is described in the Friedman's equations [7], and consists schematically to add a harmonic potential to the Newtonian. Indeed, the second of these equation given by

$$\dot{a}_1^2 - \frac{8\pi G}{3} \rho(t) a_1^2 - \frac{\Lambda c^2}{3} a_1^2 = kc^2 \quad (1)$$

Where a_1 is the scale factor, G the Newton's gravitational constant, Λ the cosmological constant, c the speed of light in vacuum, $\rho(t)$ the density of mass of the universe and $\frac{k}{a_1^2}$ the spatial curvature in any time slice of the Universe. Considering $\rho(t) \propto a_1^{-3}$ [8] the equation of Friedman becomes

$$\dot{a}_1^2 - \frac{m}{a_1} - \frac{\Lambda c^2}{3} a_1^2 = kc^2 \quad (1')$$

Where m is constant. The first term of this equation can be compared with a Kinetic energy, the second with a Newtonian potential and the third with an isotropic harmonic oscillator potential. The right-hand side of this equation is constant which means this equation can be understood as the conservation of the Mechanical energy. This equation led several researchers to investigate the possibility that the Universe is a harmonic oscillator (see for example reference [9]).

The nature, the value and the sign of the cosmological constant remained a subject of theoretical interest (see for example references [10]). However it is generally accepted that this constant schematically represents the value of the energy density of the vacuum of space. This unknown form of mass/energy (called "dark energy") could be an important amount of the total mass of the Universe (around 70%) [1].

It is the reason for what we have investigated the possibility that Newtonian potential becomes gradually harmonic: indeed this second limiting case will be in agreement with the first question we have mentioned, and could consequently explain the expansion of the Universe. Goal of the paper is thus to see if using this initial assumption we can build a model able to solve another problems of the contemporary physics of Gravity, and in particular the second question we have evoked (i.e. the anomalous speed of the stars at the periphery of galaxies). Indeed the "dark matter hypothesis" which is generally evoked ([1], [11]) as the explanation of this anomaly seems really shaky, for the reason that experiments to detect this matter have failed [2].

To build this model we had several mathematical possibilities. Indeed if we know the two limiting cases we doesn't know the mathematical form of the evolution of the potential. For example we could study an evolution of the force such

$$F = \frac{A}{r^2} + Br \quad (2)$$

Indeed this old idea (it have been mentioned in the Principia of Newton) is an interesting candidate. But in this paper we have chosen another way: We have considered the possibility that the force of gravity be the "forces of Darboux". This choice isn't again absolutely arbitrary. One reason is that these classical forces of the physics, which has been discovered in 1877 by this well-known mathematician [12] admit our two limiting forces, the Newton's and the Hooke's. But we have a second reason to choose them, as we present it now.

Indeed we know that if we modify the mathematical form of the force of gravity we will induce perturbations on trajectories of planets inside our solar system (generally a modification of the perihelion precession). But these perturbations are not always seen by astronomers. It is the reason for what attempts to modify Newtonian potential are generally strongly constraint with observational data or simply failed. We can evoke attempts to build a theory with a logarithmic corrected Newtonian

potential [4,5] or with a Yukawa-like potential [5], MOND [13], Dark matter theory [14] or other examples [15, 16, 17].

Consequently, if we want minimize the expected perturbations inside our solar system, the forces of Darboux are solid candidates. Reason is that these forces lead to conic trajectories, i.e. trajectories of planets won't be modified. Trajectories of celestial bodies (without external perturbation) will always be conic, whatever the distance to the center of force.

The model we present is thus based on the following assumptions:

- Force of gravity is gradually modified when distances increase
- Second limiting potential is Harmonic
- Transition between the two potentials (Newtonian to Harmonic) is due to the forces of Darboux.

And the paper is organized as follows:

- A brief presentation of the Darboux's forces
- A mathematical study of central accelerations and forces which allow to obtain conic trajectories
- The utilization of these results to build the theory, based on the observed speed of stars at the periphery of Galaxies. In particular we present the expression of the ratio gravitational/inertial mass we expect.
- The numerical comparison of the model with others approaches (in particular with MOND)

2. Central forces which lead to conic trajectories: The Darboux's

The Darboux's forces are central and lead to conic trajectories. Darboux have presented in his paper two expressions forces. To build our model we use the second which is given [12] by

$$F = \frac{\mu}{R^2 \left(\frac{1}{R} - a_1 \cos(w) - b_1 \sin(w) \right)^3} \quad (1)$$

Where F is the magnitude of the force, μ , a_1 and b_1 are constant, R is the distance between the center of force (called in the paper l) and the point-particle which orbits around it (called in the paper M) and w is the angle of rotation measured from the periapsis of the conic.

It is important to note that the point l can be located anywhere in the plane, inside or outside the conic. However to do our demonstration we will consider a particular case: l is located at the periapsis, between the foci and the origin of the conic (see Figure 1.).

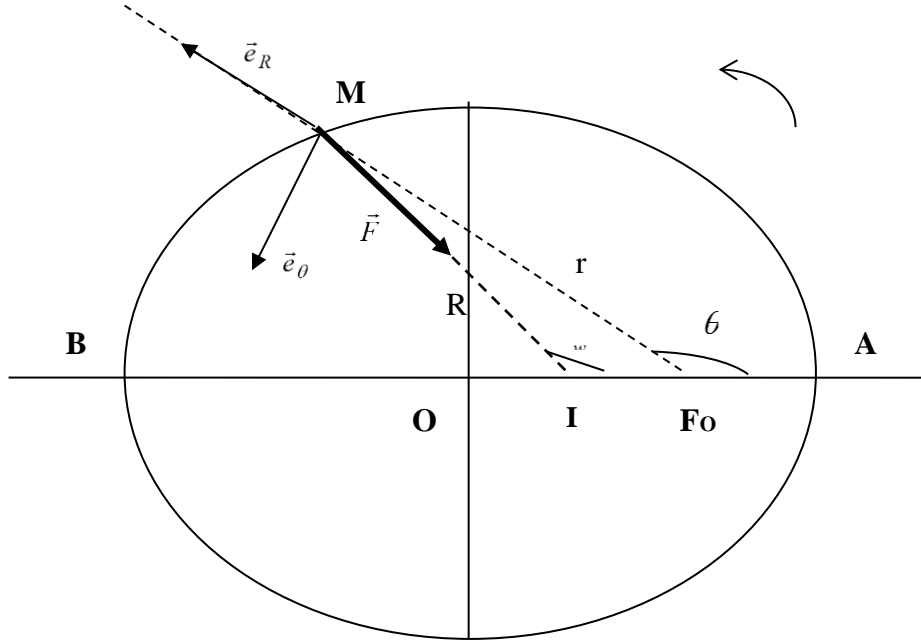


Figure 1. A part of the Darboux's forces

The force admits in thus a symmetry around the periapsis. This indicates that

$$F(w) = F(-w) \quad (2)$$

Force is described by an even function. We deduce that

$$b_1 = 0 \quad (3)$$

And Force given by (1) becomes

$$F = \frac{\mu}{(1 - a_1 R \cos \omega)^3} R \quad (4)$$

We re-write again this relationship with a simpler form. We call F_0 the foci of the conic and O its center. e is the eccentricity and a the semi major axis. We introduce a constant C_1 defined by

$$0 \leq C_1 \leq 1 \quad (5)$$

And we write

$$\begin{aligned} OI &= aeC_1 \\ IF_0 &= ae(1 - C_1) \end{aligned} \quad (6)$$

Consequently we obtain the relationship

$$R \cos \omega = ae(1 - C_1) + r \cos \theta \quad (7)$$

Using the relation specific to the conic

$$r = F_0 M = \frac{a(1 - e^2)}{1 + e \cos \theta} \quad (8)$$

We obtain

$$R \cos \omega = \frac{a - r - ae^2 C_1}{e} \quad (9)$$

And the force becomes

$$F = \frac{\mu}{\left(1 - a_1 \frac{a - r - ae^2 C_1}{e}\right)^3} R \quad (10)$$

We have noticed that we can obtain a simpler expression of this force. Indeed if we introduce a second constant A_F given by

$$A_F = \frac{\mu a^2 C_1}{a_1} e(1 + C_1(1 - e^2))^2 \quad (11)$$

With

$$a_1 = \frac{eC_1}{a(1 + C_1(1 - e^2))} \quad (12)$$

$$\mu = \frac{A_F}{a^3(1 + C_1(1 - e^2))^3}$$

(10) becomes simply

$$F = \frac{A_F}{(C_1(r - a) + a)^3} R \quad (13)$$

With the radius given by

$$R = \sqrt{r^2 + a(1 - C_1)(a(2 - e^2) - 2r)} \quad (14)$$

We can immediately of our limiting case: it is obtained choosing $C_1 = 1$. Indeed in this case $R = r$ and magnitude of the force becomes the inverse square law given by

$$F = \frac{A_F}{r^2} \quad (15)$$

Considering the case where this force is the Newton's we obtain

$$A_F = GM_G m_G \quad (16)$$

Where G is the universal constant of gravity, M_G the gravitational mass of the center of force and m_G the gravitational mass of the body which orbits around it.

Note that if we choose $C_1 = 0$ we obtain $R = \sqrt{r^2 + a(a(2 - e^2) - 2r)} = OM$ and force becomes the Hooke's given by

$$F = \frac{A_F}{a^3} OM \quad (17)$$

3. Determination of the acceleration

We present now a method to obtain the accelerations which lead to conic trajectories. To obtain these accelerations we generalize the Binet's equation. We present here this original method (at our knowledge) which has been published elsewhere [18].

3.1. Generalization of the Binet's equation

As usual in celestial mechanics we will use the polar system of coordinate $(F_O; \vec{e}_R; \vec{e}_\theta)$ where F_O (foci of the conic) is the origin of this system, r is the radial distance to the origin with the relation

$$F_O \vec{M} = r \vec{e}_R \quad (18)$$

And the angle θ is measured from the periapsis of the orbit. In this system of coordinate the acceleration is given by the classical relation

$$\vec{a}_c = (\ddot{r} - r\dot{\theta}^2)\vec{e}_R + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta \quad (19)$$

But the orbital shape is more concisely described by the reciprocal $u = \frac{1}{r}$ as a function of θ . By using the relations

$$\begin{aligned} \dot{r} &= \frac{d}{dt} r = \frac{d}{dt} \frac{1}{u} = -\frac{\dot{u}}{u^2} \\ \dot{u} &= \frac{d}{dt} u = \frac{d\theta}{dt} \frac{du}{d\theta} = \dot{\theta} u' \\ \dot{r}' &= -\frac{\dot{\theta} u'}{u^2} \\ \dot{u}' &= \frac{d}{dt} u' = \frac{d\theta}{dt} \frac{d}{d\theta} u' = \dot{\theta} u'' \\ \ddot{r} &= -\frac{(\ddot{\theta} u' + \dot{\theta} \dot{u}') u^2 - 2\dot{u} u \dot{\theta} u'}{u^4} = -\frac{(\ddot{\theta} u' + \dot{\theta}^2 u'') u^2 - 2u \dot{\theta}^2 u'^2}{u^4} \end{aligned} \quad (20)$$

We obtain a generalization of the Binet's equation.

$$\vec{a}_c = \frac{-u'' u^2 \dot{\theta}^2 - u^3 \dot{\theta}^2 + 2u'^2 u \dot{\theta}^2 - u' u^2 \ddot{\theta}}{u^4} \vec{e}_R + \frac{-2u' u^2 \dot{\theta}^2 + u^3 \ddot{\theta}}{u^4} \vec{e}_\theta \quad (21)$$

Noting that this equation can be written

$$\vec{a}_c = \left[\frac{-u'' u^2 \dot{\theta}^2 - u^3 \dot{\theta}^2}{u^4} + \frac{u' 2u' u^2 \dot{\theta}^2 - u^3 \ddot{\theta}}{u^4} \right] \vec{e}_R + \frac{-2u' u^2 \dot{\theta}^2 + u^3 \ddot{\theta}}{u^4} \vec{e}_\theta \quad (22)$$

We introduce two functions given by

$$Y(u) = \frac{-u'' u^2 \dot{\theta}^2 - u^3 \dot{\theta}^2}{u^4} \quad (23)$$

And

$$Z(u) = \frac{-2u' u^2 \dot{\theta}^2 + u^3 \ddot{\theta}}{u^4} \quad (24)$$

And the acceleration becomes

$$\vec{a}_c = \left[Y(u) - \frac{u'}{u} Z(u) \right] \vec{e}_R + Z(u) \vec{e}_\theta \quad (25)$$

We can now write the system of equation

$$\begin{cases} a_{CR} = Y(u) - \frac{u'}{u} Z(u) \\ a_{C\theta} = Z(u) \end{cases} \quad (26)$$

We introduce a new function $f(u)$ definite by

$$Y(u) = -A_C f(u) \quad (27)$$

Where A_C is constant. To obtain $r(\theta)$ as a conic, we have to solve a differential equation as

$$u'' + u = B \quad (28)$$

Where B is a second constant. Consequently we have now to introduce a relation between $Y(u)$ and $\dot{\theta}$. This relation is

$$\dot{\theta} = Cu\sqrt{f(u)} \quad (29)$$

Where C is a constant of the motion. Indeed with this relation we obtain

$$u'' \dot{\theta}^2 - u \dot{\theta}^2 = -A_C u^2 f(u) \quad (30)$$

And

$$u'' + u = B = \frac{A_C}{C^2} \quad (31)$$

This differential equation leads now to the classical solution

$$r(\theta) = \frac{p}{1 + e \cos \theta} \quad (32)$$

The parameter p of the conic is

$$p = a(1 - e^2) = \frac{C^2}{A_C} \quad (33)$$

Where e is the eccentricity and a the semi major axis. Thus we obtain

$$C = \sqrt{A_C} \sqrt{a(1 - e^2)} \quad (34)$$

We have now to determine the tangential component of the acceleration and by using

$$\ddot{\theta} = \frac{C \dot{r}}{r^2 \sqrt{f}} \left[\frac{1}{2} f' r - f \right] \quad (35)$$

Where

$$f' = \frac{d}{dr} f(r) \quad (36)$$

We obtain

$$a_{C\theta} = r \ddot{\theta} + 2 \dot{r} \dot{\theta} = \frac{1}{2} \frac{C f' \dot{r}}{\sqrt{f}} + C \frac{\dot{r}}{r} \sqrt{f} \quad (37)$$

Consequently our family of acceleration is given by

$$\vec{a}_C = -A_C f \cdot \vec{e}_R + \dot{r}^2 \left[\frac{1}{2} \frac{f'}{f} + \frac{1}{r} \right] \vec{e}_R + C \dot{r} \sqrt{f} \left[\frac{1}{2} \frac{f'}{f} + \frac{1}{r} \right] \vec{e}_\theta \quad (38)$$

Where A_C and C are two constants. Their physical dimensions depend on the choice of $f(r)$. For example consider that this function is given by

$$f(r) = \frac{1}{r^2} \quad (39)$$

Acceleration becomes

$$\vec{a}_c = -A_c \frac{1}{r^2} \vec{e}_R \quad (40)$$

Unities of constant of rotation A_c are $m^{-3}s^{-2}$.

3.2. Central acceleration

To obtain our accelerations we have now to add a condition about their origin. . We consider that center of acceleration is located between the foci and the origin of the conic.

We call again Δ the distance F_0I . If the force is directed to I then the vector product

$$\vec{IM} * \vec{a}_c = \vec{0} \quad (41)$$

Noting that, in our system of coordinate $F\vec{e}_R\vec{e}_\theta$ the vector \vec{IM} is given by

$$\vec{IM} \begin{vmatrix} \Delta \cos \theta + r \\ -\Delta \sin \theta \end{vmatrix} \quad (42)$$

We obtain

$$\Delta \sin \theta a_{cR} + (\Delta \cos \theta + r) a_{c\theta} = 0 \quad (43)$$

With relation (8) and (29)

$$\dot{r} = \frac{eC \sin \theta}{a(1-e^2)} r \sqrt{f(r)} \quad (44)$$

And using relations (38) and (44) result is

$$\Delta = \frac{ae(rf' + 2f)}{f'(r-a) + 2f} \quad (45)$$

To obtain a central acceleration we are looking for the family of functions $f(r)$ which leads to Δ as a constant. Consequently we write the equation

$$\frac{d}{dr} \Delta = 0 \quad (46)$$

Solving is

$$f(r) = \frac{1}{(C'_1 r + C'_2)^2} \quad (47)$$

Where C'_1 and C'_2 are two constant (notations of X' will be used in all the paper for accelerations, X for forces). Note that Δ is simply given by

$$\Delta = \frac{eaC'_2}{aC'_1 + C'_2} \quad (48)$$

We can now obtain the mathematical expression of our force. Introducing relation (47) in equations (38) (and using (44)) acceleration becomes

$$\vec{a}_c = A_c \frac{ar(C'_2 - rC'_1) - C'_2(a^2(1 - e^2) + r^2)}{ar(C'_1 r + C'_2)^3} \vec{e}_R + A_c \frac{eC'_2}{(C'_1 r + C'_2)^3} \text{Sin} \theta \vec{e}_\theta \quad (49)$$

Its magnitude is

$$a_c = \frac{A_c}{a} \frac{\sqrt{2aC'_1 C'_2 (-ar + r^2 + a^2(1 - e^2) + C'_2{}^2(2a^2 - e^2a^2 + r^2 - 2ar) + C'_1{}^2 a^2 r^2)}}{(C'_1 r + C'_2)^3} \quad (50)$$

Noting that the distance $IM = R$ is given by

$$R = \sqrt{(\Delta + r \cos \theta)^2 + (r \sin \theta)^2} = \sqrt{\Delta^2 + r^2 + 2r \cos \theta} \quad (51)$$

After simplification

$$R = \frac{\sqrt{2aC'_1 C'_2 (-ar + r^2 + a^2(1 - e^2) + C'_2{}^2(2a^2 - e^2a^2 + r^2 - 2ar) + C'_1{}^2 a^2 r^2}}{C'_1 r + C'_2} \quad (52)$$

We obtain expression of the central acceleration

$$a_c = \frac{A_c}{a} \frac{C'_1 a + C'_2}{(C'_1 r + C'_2)^3} R \quad (53)$$

We recognize our two limiting cases: if $C'_2 = 0$ acceleration is Newtonian. If $C'_1 = 0$ it is harmonic.

If we consider now that this acceleration is a gravitational acceleration (Newtonian case) we have to choose

$$A_c = GM_G \quad (54)$$

Where G is the constant of gravity and M_G the gravitational mass of the center of force.

3.3. Relation between force and acceleration

Force and acceleration are linked by the Newton's second law of motion given by

$$\vec{F} = \frac{d}{dt} m_I \vec{V} \quad (55)$$

Where m_I is the inertial mass and \vec{V} the speed. If we consider the general case relation (3) becomes

$$\vec{F} = \dot{m}_I \vec{V} + m_I \vec{a}_c \quad (56)$$

We consider that the inertial mass is constant on all the trajectory. Consequently (56) becomes

$$\vec{F} = m_I \vec{a}_c \quad (57)$$

Using (53) magnitude of the force is thus given by

$$F = \frac{A_c}{a} m_I \frac{C'_1 a + C'_2}{(C'_1 r + C'_2)^3} R \quad (58)$$

We write the equality between the two expressions of the force given by (13) and (58)

$$F = \frac{A_F}{(C_1(r - a) + a)^3} R = \frac{A_c}{a} m_I \frac{C'_1 a + C'_2}{(C'_1 r + C'_2)^3} R \quad (59)$$

Consequently

$$\frac{A_F}{(C_1(r-a)+a)^3} = \frac{A_C}{a} m_I \frac{C'_1 a + C'_2}{(C'_1 r + C'_2)^3} \quad (60)$$

We write also the equality of $\Delta = IF_0$ given by (6) and (48)

$$\Delta = ea(1-C_1) = \frac{eaC'_2}{aC'_1+C'_2} \quad (61)$$

And we obtain relationship

$$\frac{C'_2}{a(1-C'_1)} = \frac{C'_1}{C_1} \quad (62)$$

We introduce (62) inside equation (60)

$$\frac{A_F}{(C_1(r-a)+a)^3} = A_C m_I \frac{C_1^2}{C_1'^2 (C_1(r-a)+a)^3} \quad (63)$$

We obtain an interesting simplification

$$A_F = A_C m_I \frac{C_1^2}{C_1'^2} \quad (64)$$

3.4. Ratio gravitational/inertial masses

To determine our constants we consider the particular case where force and acceleration are gravitational and directed toward the foci of the conic, i.e. that they are Newtonian. In this case we know that

$$\begin{aligned} C_1 &= C'_1 = 1 \\ A_F &= GM_G m_G \\ A_C &= GM_G \end{aligned} \quad (65)$$

Thus (64) becomes

$$GM_G m_G = GM_G m_I \frac{C_1^2}{C_1'^2} \quad (66)$$

We introduce the ratio

$$\eta = \frac{m_G}{m_I} \quad (67)$$

and we obtain

$$\eta = \frac{C_1^2}{C_1'^2} \quad (68)$$

If potential is Newtonian $\eta = 1$. If it is harmonic it is undetermined because the two coefficients tend to 0.

4. Theory of gravitation

It is well known that the speed at the periphery of one spiral galaxy is approximately constant. Astronomers and theorists noticed [19, 20] that this speed is well described by the relation

$$V_{\infty} = [GM_V a_0]^{1/4} \quad (69)$$

Where M_V is the visible mass of the galaxy and a_0 the Millgrom's acceleration [19] given by

$$a_0 \approx 1,2 \cdot 10^{-10} \text{ms}^{-2} \quad (70)$$

M_V is called « visible mass » because, to determine it astronomers analyzed the light which is coming from these stars: they have thus obtained the visible part of the quantity of matter, i.e. a part of its inertial mass. (Gravitational mass is deduced by its gravitational effects). To simplify our study we have assumed that the visible mass is equal to the totality of the inertial mass of the galaxy. Consequently we have re-written (69)

$$V_{\infty} = [GM_I a_0]^{1/4} \quad (71)$$

Where M_I is the total inertial mass of the galaxy. We introduce a constant

$$r_0 = \sqrt{\frac{GM_I}{a_0}} \quad (72)$$

Which is thus a constant of the galaxy. We have assumed that our two limiting cases are defined by:

- $r_0 \gg a$ the force is the Newton's
- $r_0 \ll a$ it is the Hooke's.

We have considered that this condition was satisfied at the periphery of the galaxies. Moreover we have approximated the trajectories of stars to circles as is customary [20, 21]. The acceleration given by equation (53) is

$$a_c = \frac{GM_G}{a} \frac{C'_1 a + C'_2}{(C'_1 r + C'_2)^3} R \quad (73)$$

Becomes, at the periphery of galaxies, and for circular motion ($C'_1 = 0$ and $r = R = a$).

$$a_c = \frac{GM_G}{C'_2{}^2} \quad (74)$$

We write the equality of accelerations for circular motion

$$a_c = \frac{GM_G}{C'_2{}^2} = \frac{V_{\infty}^2}{a} \quad (75)$$

And using (71) and (72) we obtain

$$\frac{GM_G}{C'_2{}^2} = \frac{GM_I}{ar_0} \quad (76)$$

Consequently

$$C'_2 = \sqrt{\eta} \sqrt{ar_0} \quad (77)$$

Following problem was thus to obtain the mathematical form of C'_2 . We have assumed that this form was

$$C'_2 = d(1 - C'_1) \quad (78)$$

Where d is an unknown distance. (Note that the boundaries conditions of acceleration are respected). To determine it we have written that in our limiting case we had

$$C'_2 = d \quad (79)$$

Consequently using (62) and (68)

$$d = \sqrt{\eta} \sqrt{ar_0} = \frac{a(1-C_1)}{C'_2} \sqrt{ar_0} = \frac{a}{d} \sqrt{ar_0} \quad (80)$$

And we have obtained

$$d = [a^3 r_0]^{1/4} \quad (81)$$

And using (20)

$$C'_2 = [a^3 r_0]^{1/4} (1-C'_1) \quad (82)$$

Then we have written the relation given by (13)

$$\frac{C_1}{C'_1} = \frac{a(1-C_1)}{C'_2} \quad (83)$$

We have thus obtained

$$\frac{C_1}{C'_1} = \frac{a}{[a^3 r_0]^{1/4}} \left[\frac{1-C_1}{1-C'_1} \right] = \left[\frac{a}{r_0} \right]^{1/4} \left[\frac{1-C_1}{1-C'_1} \right] \quad (84)$$

Which have led to

$$C'_1 = \frac{C_1 r_0^{1/4}}{C_1 (r_0^{1/4} - a^{1/4}) + a^{1/4}} \quad (85)$$

We have met here a problem to determine the coefficient. Indeed we had only one equation for two unknowns. To reduce the choice we used boundaries conditions. Indeed we know that C_1, C'_1 and η are three continuous functions defined by

$$\begin{aligned} 0 &\leq C_1 \leq 1 \\ 0 &\leq C'_1 \leq 1 \\ 1 &\leq \eta \leq \sqrt{\frac{a}{r_0}} \end{aligned} \quad (86)$$

We know also that

$$\begin{array}{llll} \text{If} & a \ll r_0 & \text{then} & \begin{array}{l} C_1 \rightarrow 1 \\ C'_1 \rightarrow 1 \\ \eta \rightarrow 1 \end{array} \end{array} \quad (87)$$

And

$$\begin{array}{llll} \text{If} & a \gg r_0 & \text{then} & \begin{array}{l} C_1 \rightarrow 0 \\ C'_1 \rightarrow 0 \\ \eta \rightarrow \sqrt{\frac{a}{r_0}} \end{array} \end{array} \quad (88)$$

Moreover when $a=r_0$ we assumed force is composed at an equal part of the Newton's and the Hooke's. For this reason we assume that in this case we have

$$\Delta = \frac{1}{2} e r_0 \quad (89)$$

This means that point l is located at the same distance between O and F. we deduce that at this point

$$\begin{aligned} C_1 &= \frac{1}{2} \\ C_2 &= \frac{1}{2} r_0 \end{aligned} \quad (90)$$

We have tested several functions in order to see if they could respect our conditions. It appears that we obtained a family of simple and proximate relations which allows to respect all its conditions and could be consequently correct. It is the reason for what we suggest

$$C_1 = \frac{r_0^k}{r_0^k + a^k} \quad (91)$$

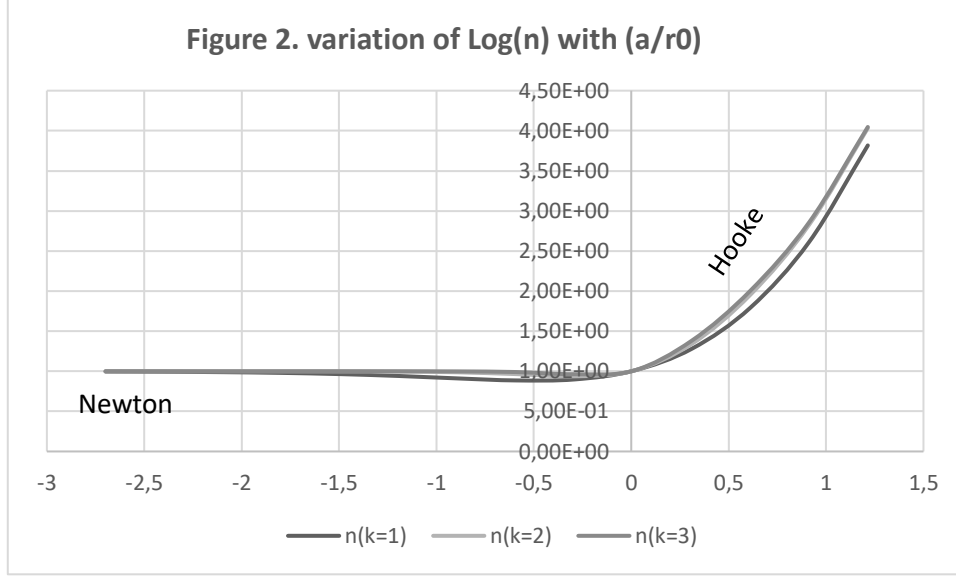
Where k is a positive and natural integer. Introducing ratio

$$x = \frac{a}{r_0} \quad (92)$$

We have obtained relationships

$$\left\{ \begin{aligned} C_1 &= \frac{1}{1+x^k} \\ C_2 &= a \frac{x^k}{1+x^k} \\ C'_1 &= \frac{1}{1+x^{k+1/4}} \\ C'_2 &= [a^3 r_0]^{1/4} \frac{x^{k+1/4}}{1+x^{k+1/4}} \\ \eta &= \left[\frac{1+x^{k+1/4}}{1+x^k} \right]^2 \end{aligned} \right. \quad (93)$$

Note that coefficients have the same mathematical form and that all our limiting conditions are respected. In the following of the paper we will use these coefficient to do numerical simulation. First of them is about η (Figure 2)



The increasing of gravitational mass is corresponding in our model to the “missing mass”, i.e. the “Dark Matter”. Indeed its amount in Universe has been determined assuming equivalence principle was respected (i.e. that inertial mass and gravitational are equivalent). Note that this kind of result has already been suggested in another theories (for example in reference [21]).

5. Curve of rotation of galaxies: comparison with MOND

To obtain this curve we consider as usual the circular motion. Using (53) and (93) we write the equality of the acceleration

$$\frac{V^2}{r} = \frac{GM_G}{(C_1 (r - [a^3 r_0]^{1/4}) + [a^3 r_0]^{1/4})^2} \quad (94)$$

We have the relationships $r = a = R$. Speed is thus given by

$$V = \frac{\sqrt{GM_G}}{a^{1/4}} \frac{1}{C_1 (a^{1/4} - r_0^{1/4}) + r_0^{1/4}} \quad (95)$$

And using relationship (68) and (85) we obtain

$$V = \sqrt{\frac{GM_G}{a} \frac{C_1}{C_1}} = \sqrt{\frac{GM_I}{a} \frac{C_1}{C_1}} \sqrt{\eta} = \sqrt{\frac{GM_I}{a}} \eta = \sqrt{\frac{GM_I}{a}} \frac{[1 + x^{k+1/4}]^2}{[1 + x^k]^2} \quad (96)$$

With $x = a/r_0$. Note that in this relation we can distinguish our two limiting cases:

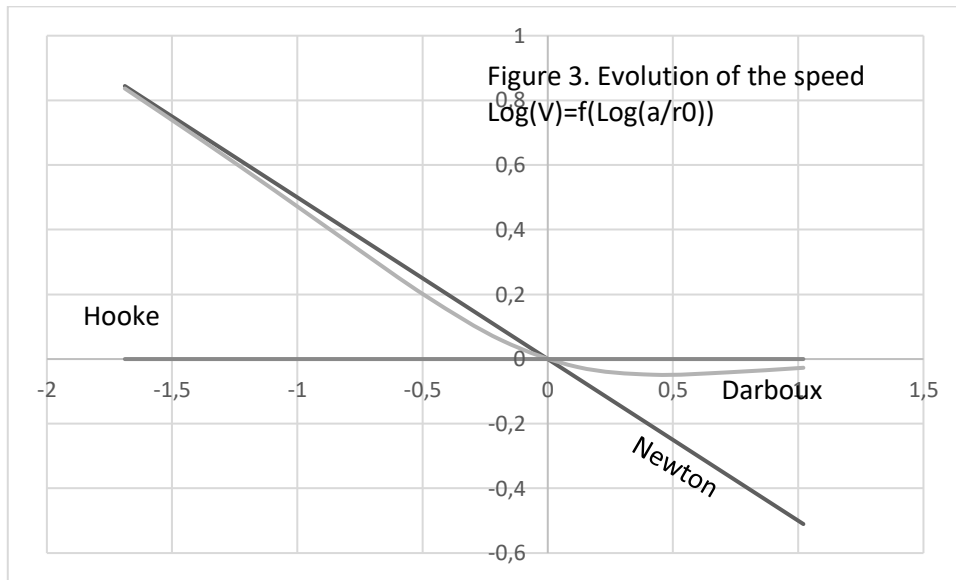
If $r_0 \gg a$ the speed becomes the Newton's given by

$$V = \sqrt{\frac{GM_G}{a}} \quad (97)$$

If $r_0 \ll a$ this speed becomes

$$V = \sqrt{\frac{GM_I}{r_0^2}} = [GM_I a_0]^{1/4} \quad (98)$$

We present the graph we obtained for $k = 2$ (Figure 3).



For comparison with our solar system Earth is located at $\text{Log}(a/r_0) \approx -5$ and pluton at $\text{Log}(a/r_0) \approx -3$:

Difference between the Newton's and our model are tiny. But we can expect modification of the dynamics at the extreme periphery of the solar system.

It is also interesting to compare the speed we have obtained with the speed due to MOND theory. Indeed it is well known that this theory allows to obtain very good fit of curve of rotation of spiral galaxies. If functions are proximate we will deduce that the model could likely be correct to describe these curves.

MOND theory assumes that “the Newtonian acceleration g_N produced by the visible matter is linked to the true acceleration g by means of an interpolating function μ » [20] given by

$$\mu\left(\frac{g}{a_0}\right)g = g_N \quad (99)$$

Where

$$\mu(y) = 1 \text{ if } y \gg 1 \quad (100)$$

$$\mu(y) = y \text{ if } y \ll 1 \quad (101)$$

Force and acceleration are linked by

$$\vec{F} = m_i \mu \left(\frac{g}{a_0} \right) \vec{g} \quad (102)$$

Where F is the Newton's force. In his theory inertial and gravitational mass are equal and (25) becomes in the case of circular motion

$$\frac{GM_G}{a^2} = \mu \left(\frac{g}{a_0} \right) \frac{V^2}{a} \quad (103)$$

Several possible expressions of μ has been tested with success. The most popular choice was the "standard" μ function [20]

$$\mu_1 = \frac{y}{\sqrt{1+y^2}} \quad (104)$$

And the "simple" μ function [13]

$$\mu_2 = \frac{y}{1+y} \quad (105)$$

Note that these two functions allows to obtain excellent fit for $a_0 \approx 1,2 \cdot 10^{-10} \text{ms}^{-2}$. Using acceleration given by

$$g = \frac{V^2}{a} \quad (106)$$

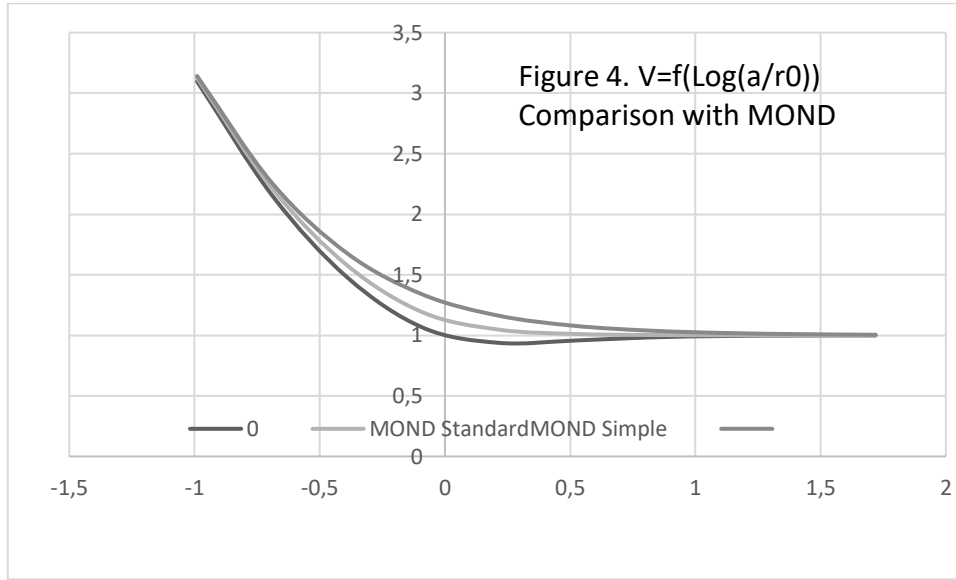
And introducing this relation into (103) we obtain two expressions of the speed

$$V_1 = \sqrt{\frac{GM_G}{ar_0} \left[\frac{r_0^2 + \sqrt{r_0^4 + 4a^4}}{2} \right]^{1/4}} = \sqrt{\frac{GM_G}{a} \left[\frac{1 + \sqrt{1 + 4x^4}}{2} \right]^{1/4}} \quad (107)$$

And

$$V_2 = \sqrt{\frac{GM_G}{ar_0} \left[\frac{r_0^2 + \sqrt{r_0^2 + 4a^2}}{2} \right]^{1/2}} = \sqrt{\frac{GM_G}{a} \left[\frac{1 + \sqrt{1 + 4x^2}}{2} \right]^{1/2}} \quad (108)$$

We plot the functions on figure 4.



5.1. Comparison with “Mass Discrepancy Acceleration Relation”

Recently an empirical correlation (called MDAR”) between radial acceleration and distribution of baryon in a wide range of galaxies has been published [22]. We can thus compare our model with this result.

Correlation is given by

$$g_{OBS} = \frac{g_{BAR}}{1 - \exp\left(-\sqrt{\frac{g_{BAR}}{g_0}}\right)} \quad (109)$$

Where g_{OBS} is the radial observed acceleration, g_{BAR} the acceleration due to visible baryonic matter and g_0 a constant acceleration proximate to the Millgram's. In our case radial acceleration is given (for circular motion) by

$$a_C = \frac{GM_G}{(C'_1 r + C'_2)^2} \quad (110)$$

With coefficients given by (93) and by using

$$g_{BAR} = \frac{GM_G}{r^2} \quad (111)$$

$$g_0 = \frac{GM_{G=l}}{r_0^2}$$

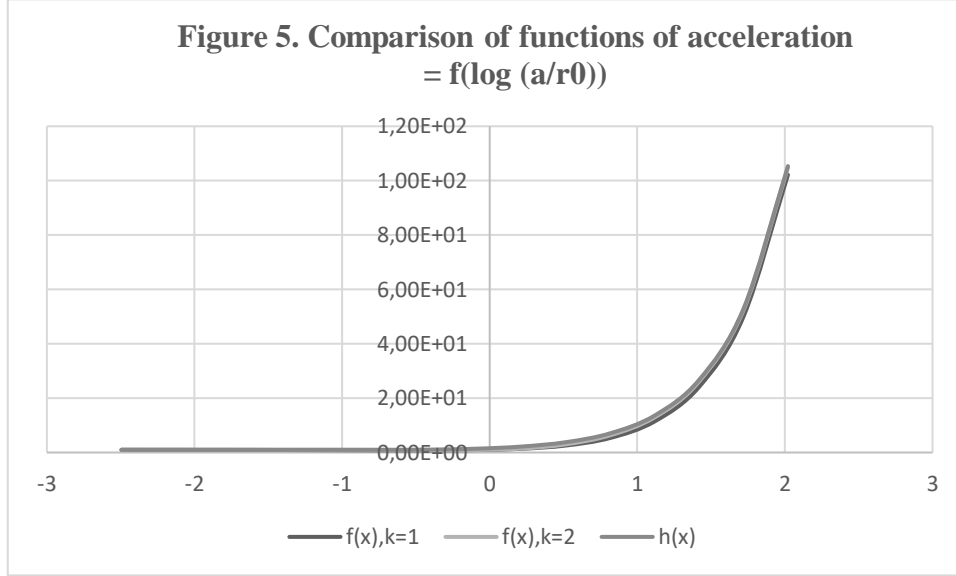
We obtain

$$a_C = g_{BAR} \eta^2 = g_{BAR} \left[\frac{1 + x^{k+1/4}}{1 + x^k} \right]^4 = g_{BAR} f(x) \quad (112)$$

With this notation MDAR becomes

$$g_{OBS} = g_{BAR} \frac{1}{1 - \exp(-\frac{1}{x})} = g_{BAR} \cdot h(x) \quad (113)$$

It seems thus interesting to compare the functions $f(x)$ and $h(x)$. We plot the functions on figure 5.



Curves are proximate and this result seems naturally an encouragement for this model. Indeed the empirical relation has been tested on more 100 galaxies. Moreover note that we have used several values of k . It seems that the simple value $k=2$ is the best.

6. Predictions. Possible test inside our solar system?

Our model doesn't predict modification of the trajectories of planets. But it predicts that the center of mass isn't located exactly at the center of force. In particular we can consider that the center of the Sun isn't located at the foci of the conic. We did simulations but in the case of planets this effect should be undetectable. However it isn't perhaps the case for comets: indeed if we use relation

$$\Delta = a(1 - C_1) \quad (114)$$

And

$$r_0 = \sqrt{\frac{GM_{SUN}}{a_0}} \quad (115)$$

With

$$C_1 = \frac{r_0^2}{a^2 + r_0^2} \quad (116)$$

We have obtained for the comet of Halley a non-negligible value, given by

$$\Delta = 1.74 \cdot 10^7 \text{ m} \quad (117)$$

Perhaps it is possible to test this result.

Another test could be to try to detect perturbations around the Newton's law in the case of parabolic or hyperbolic motion, for example with a spacecraft. Indeed in the case of free fall motion, the acceleration (we have obtained it with a series when $r_0 \rightarrow \infty$) should be

$$\bar{a}_C = \left(-\frac{GM_G}{r^2} + \frac{GM_G}{r^3} \left[\frac{a}{r_0} \right]^2 \right) [2a - r] \bar{e}_R \quad (118)$$

Which is consequently not absolutely equal to the Newton's.

At end we think to another test around the equivalence principle. Indeed model predicts that it isn't absolutely respected. For example if two test-body are dropped out from the same distance from the center of force the semi-major axis of their trajectory will be equal, but the ratio gravitational/inertial masses given by

$$\eta = \frac{m_G}{m_I} = \frac{1}{\sqrt{r_0}} \left[\frac{a^{9/4} + r_0^{9/4}}{a^2 + r_0^2} \right]^2 \quad (119)$$

Will be a little bit different because

$$r_0 = \sqrt{\frac{G(M_G + m_G)}{a_0}} \quad (120)$$

This constant isn't exactly the same for the two test-body. However generally difference should be a negligible amount, because masses of test-body are small in comparison with the mass of the Earth. For example mass of test bodies inside satellite Microscope [23] are around 0.5 and 1.5 Kg and Earth around 10^{24} Kg... Difference of r_0 and consequently of r should be in these conditions really small and likely undetectable. But perhaps we can imagine another experiment to test this principle, based on our relation, where this time semi-major axis of the conic is different.

Note that in a general way in this domain of the Physics we are faced to the lake of experimental results, not only in this model. For example it is the case of the dominant approach (Dark Matter/ Energy approach) and of the majority of alternative theories. Reasons is that tests are very difficult to implement when acceleration is so small [24].

7. Discussion

7.1. About the harmonic potential and the variation of ratio gravitational/inertial masses

In this paper, we studied the possibility that the force of Hooke could perhaps be the force of gravitation valuable for large distances, by choosing correct coefficients. We think consequently it makes senses to ask ourselves if this idea can have a physical reality. It appears that we can find several argument in favor of this hypothesis.

Firstly, we can notice that the force of Hooke exist already in the gravitation. It is the force which interacts with a point-particle inside a sphere where the density of mass is uniform. This is the consequence of the theorem of Gauss. For example, in a sphere the force is given by

$$\vec{F} = -\frac{GM_{G\text{int}}}{r^2} m_I \cdot \vec{e}_R \quad (121)$$

Where $M_{G\text{int}}$ is the total gravitational mass contained inside the sphere of radius r . If the point particle is dropped out from a distance a from the center and if the mass density is uniform we obtain

$$\vec{F} = -\frac{GM_G}{a^3} m_I \cdot r \cdot \vec{e}_R \quad (122)$$

Where C is constant. We can compare this expression with force we obtained in a comparable motion

$$\vec{F} = -\frac{GM_G}{a^3} m_I \cdot r \cdot \vec{e}_R \quad (123)$$

We see that the expressions are proximate. Using this analogy we can suggest a physical interpretation of our force: progressively, when the distance to the center of force increases, the point particle which orbits around the center of mass “considers” that this mass is progressively «diluted» inside a closed volume. This volume is depending on the semi major axis of its trajectory. At the end of this evolution the force becomes entirely the Hooke's. During the same time the ratio gravitational / inertial masses increases. This ratio is equal to 1 when potential is Newtonian, and increases when the gravitational mass seems “diluted”: The two modifications are linked in the model, and replace simultaneous the concepts of “dark energy” and of “dark matter”.

7.2. Strengths and weakness of the model

We can list several strengths of the model we presented. In a first time it is interesting to note that it is in agreement with corpus of classical Physics. In particular, energy and angular momentum are conserved. Note that it is an important difference with other theories, in particular with MOND.

A second interesting point is that model seems describe correctly dynamics of Galaxies and Universe without Dark matter or Dark Energy. Curve of rotation of galaxies are well flat and actual expansion of Universe can be explained. It seems thus that this model present less difficulties than other which failed often to describe simultaneous these two facts.

But naturally model have weak points. In particular the force we used (the force of Darboux) isn't conservative and is depending on the trajectory. Another weak point is that model is non-relativistic, i.e. could describe the reality only when speeds and densities of mass are small. Consequently an important test for the model should be to see if it can become the limiting case of a relativistic theory (it naturally is doesn't exist).

8. Conclusion

In this paper we present a classical model as an alternative to dark matter and dark energy. We used the forces of Darboux to see if they could be transitional forces between the Newton's and the Hooke's. Conclusion is that these forces should lead to curve of rotation of galaxies comparable to MOND theory, and that they could perhaps describe the actual dynamics of the Universe if equivalence principle is violated (except in the Newton's case). Moreover model respect conservation of Energy and of Angular Momentum, and seems in agreement with corpus of classical physics. We

don't expect modification on advance on perihelion of planets inside our solar system but a modification of the mean motion at the extreme periphery of it. Principal difficulty of the model is that the force isn't conservative and consequently is depending on the trajectory.

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