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Could the Darboux's forces be an alternative to the dark matter/energy?

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Abstract: In this paper we study the possibility that the Darboux-Halphen's forces be a classical alternative to Dark matter and Dark energy hypotheses. We study their two limiting cases, the Newton's and the Hooke's, and we compare our result with the Friedman's equation. At end an important part of this work is to propose an experiment on the Earth, in order to valid or invalid our results.

Keywords: Central force; conic; dark matter; force of gravitation; galaxies; modified gravitation; PACS Number:04.70.Kd

1. Introduction

It is well known that an important problem of the contemporary physics is to describe the celestial mechanics for large distances. For example, it appears that the classical laws of gravitation failed to describe the "flat" curves of rotation of the spirals galaxies without add, inside these galaxies, an important amount of a mysterious matter, called "dark matter" (1-3). Another important problem of the celestial mechanics is to explain the expansion, at an increasing rate, of the universe [4]. Indeed, classical theories of gravitation are always attractive. To solve this contradiction, physicians have to suggest, that another mysterious energy, called this time "dark energy", exists inside the Universe [5, 6].

Problem is that there is no proof of the reality of the "dark matter" and the "dark energy" despite several important experiments on the Earth [7]. Moreover the most accurate study to try to detect them in the space seems also have failed [8].

Consequently, an important contemporary way of research is to modify the classical laws of Gravitation. Several alternative theories are in competition as for example, the $f(R)$ theories of gravity [9], the scalar-tensor gravity theories [10] or the Modified Newtonian dynamics (MOND) theory [11]. But it appears today that no one of these theories does unanimity.

In our case we follow a different way of research. Indeed we studied the different forces which lead to conic trajectories and we looked for one of them –or a family of them- which could describe correctly the astronomical observations we listed before. Indeed our hypothesis is that we have to consider the trajectory to obtain the force (and not the other way around) as it was historically done in the genesis of the theories of gravitation.

By a way of consequence of this initial hypothesis we studied relations between closed trajectories and forces and we learned that important works has been done about this problem at the 19th century. In particular in 1873 the mathematician Bertrand published a theorem [12,13] which continue to attract attention until today (several demonstrations have recently been published [14-18]). In this theorem Bertrand proved that it exists only two central and conservative forces which lead to closed trajectories, the Newton's and the Hooke's. Moreover it is interesting to note that these trajectories are in the two cases conic. Difference is that the Newton's force is directed toward the foci of the conic

and the Hooke's toward its center. It appears also that the work of Bertrand was completed four years later with the important results of the two mathematicians Halphen and Darboux. Indeed they published in 1877 [19] the expression of two families of central forces which lead to conics trajectories. It seemed us very interesting to note that these forces admits two limiting cases, the Hooke's and the Newton's (other demonstrations has been published later [20, 21]).

A consequence of this bibliography was to assume that these forces are perhaps the forces we are looking for. Firstly because they lead to conic trajectories and because one of their limiting case corresponds well to the force of gravitation in the case of small distances (as for example our solar system). Secondly because these forces are central and allow consequently to preserve the Angular momentum of the system (we give its expression in the paper). And thirsty because their second limiting case (the Hooke's) is often evoked has an alternative to the Dark Energy (in the Friedman equation) to describe the expansion of the Universe.

Consequently is seemed us interesting to study these forces and in particular to investigate if the force of Hooke could be a force of gravitation valuable in the case of largest distances. We present in this paper our results about these assumptions and we discuss their validity. At end we purpose a possible experiment inside our solar system to valid or invalid the model we obtained.

2. Family of central forces which lead to conics trajectories: a part of the Darboux's forces

To obtain these forces we used an original method, which is to generalize the Binet's equation. Indeed as distinguished from it we consider not only the radial acceleration but also the tangential acceleration. We present here this method which has been published elsewhere [22, 23].

2.1 Generalization of the Binet's equation

As usual in celestial mechanics we will use the polar system of coordinate $(F; \vec{e}_R; \vec{e}_\theta)$ where F (foci of the conic) is the origin of this system, r is the radial distance to the origin with the relation

$$F\vec{M} = r.\vec{e}_R$$

And the angle θ is measured from the periapsis of the orbit. In this system of coordinate the acceleration is given by the classical relation

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_R + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta$$

But the orbital shape is more concisely described by the reciprocal $u = \frac{1}{r}$ as a function of θ .

And by using the relations

$$\begin{aligned}\dot{r} &= \frac{d}{dt} r = \frac{d}{dt} \frac{1}{u} = -\frac{\dot{u}}{u^2} \\ \dot{u} &= \frac{d}{dt} u = \frac{d\theta}{dt} \frac{du}{d\theta} = \dot{\theta} u' \\ \dot{u}' &= \frac{d}{dt} u' = \frac{d\theta}{dt} \frac{d}{d^2\theta} u = \dot{\theta} u''\end{aligned}$$

We obtain a generalization of the Binet's equation.

$$\vec{a} = \frac{-u''u^2\dot{\theta}^2 - u^3\ddot{\theta}^2 + 2u'^2u\dot{\theta}^2 - u'u^2\ddot{\theta}}{u^4}\vec{e}_R + \frac{-2u'u^2\dot{\theta}^2 + u^3\ddot{\theta}}{u^4}\vec{e}_\theta$$

By noticing that this equation can be written

$$\bar{a} = \left[\frac{-u''u^2\dot{\theta}^2 - u^3\dot{\theta}^2}{u^4} + \frac{u'2u'u^2\dot{\theta}^2 - u^3\ddot{\theta}}{u^4} \right] \bar{e}_R + \frac{-2u'u^2\dot{\theta}^2 + u^3\ddot{\theta}}{u^4} \bar{e}_\theta$$

We introduce two functions given by

$$Y(u) = \frac{-u''u^2\dot{\theta}^2 - u^3\dot{\theta}^2}{u^4}$$

And

$$Z(u) = \frac{-2u'u^2\dot{\theta}^2 + u^3\ddot{\theta}}{u^4}$$

And the acceleration becomes

$$\bar{a} = \left[Y(u) - \frac{u'}{u} Z(u) \right] \bar{e}_R + Z(u) \bar{e}_\theta$$

We can now write the system of equation

$$\begin{cases} a_R = Y(u) - \frac{u'}{u} Z(u) \\ a_\theta = Z(u) \end{cases}$$

We introduce a new function $f(u)$ definite by

$$Y(u) = -A.f(u)$$

Where A is constant. To obtain $r(\theta)$ as a conic, we have to solve a differential equation as

$$u'' + u = B$$

Where B is a second constant. Consequently we have now to introduce a relation between $Y(u)$ and $\dot{\theta}$. This relation is

$$\dot{\theta} = Cu\sqrt{f(u)}$$

Where C is a constant of the motion. Indeed with this relation we obtain

$$u''\dot{\theta}^2 - u\dot{\theta}^2 = -Au^2f(u)$$

And

$$u'' + u = B = \frac{A}{C^2}$$

This differential equation leads now to the classical solution

$$r(\theta) = \frac{p}{1 + e\cos\theta}$$

The parameter p of the conic is

$$p = a(1 - e^2) = \frac{C^2}{A}$$

Where e is the eccentricity and a the semi major axis. Thus we obtain

$$C = \sqrt{A}\sqrt{a(1 - e^2)}$$

We have now to determine the tangential component of the acceleration and by using

$$\ddot{\theta} = \frac{C.\dot{r}}{r^2\sqrt{f}} \left[\frac{1}{2} f' r - f \right]$$

Where

$$f' = \frac{d}{dr} f(r)$$

We obtain

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{2} \frac{Cf'\dot{r}}{\sqrt{f}} + C \frac{\dot{r}}{r} \sqrt{f}$$

Consequently our family of force (per unit mass) is with respect for the Newton's law of dynamics

$$\vec{F} = -Af.\vec{e}_R + \dot{r}^2 \left[\frac{1}{2} \frac{f'}{f} + \frac{1}{r} \right] \vec{e}_R + C\dot{r}\sqrt{f} \left[\frac{1}{2} \frac{f'}{f} + \frac{1}{r} \right] \vec{e}_\theta$$

Or, more simply

$$\vec{F} = -Af.\vec{e}_R + \dot{r}^2 \left[\frac{1}{2} \frac{f'}{f} + \frac{1}{r} \right] \vec{e}_R + r \frac{dr}{d\theta} \dot{\theta}^2 \left[\frac{1}{2} \frac{f'}{f} + \frac{1}{r} \right] \vec{e}_\theta \quad (1)$$

Where A and C are two constants. Their physical dimensions depend on the choice of $f(r)$.

2.2. Determination of the central forces

To obtain our family of force we have now to add a condition to the previously relation: this condition is that the force has to be central. But the center of this force can't be wherever inside (or outside) the ellipse (as in the case of the Darboux's force). Indeed if we want that our force becomes gradually a force of Hooke we have to consider that the center of force is located between the foci and the center of the ellipse.

Consequently to determine the family of central forces we introduce a point, called I which is located on the right (AB). The distance FI is called Δ (Figure 1).

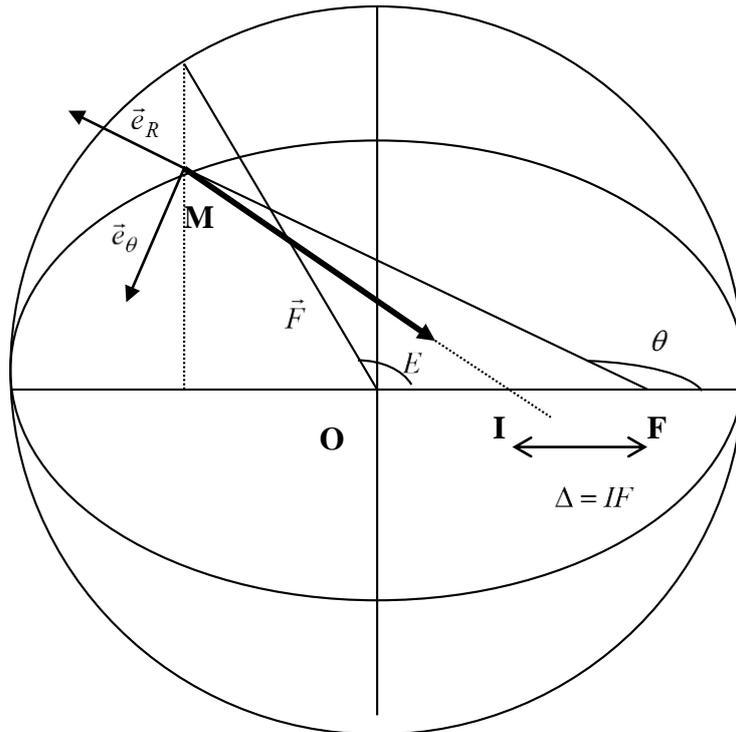


Figure 1. Representation of the central force

If the force is directed to I then the vector product

$$\vec{IM} * \vec{F} = \vec{0}$$

By noticing that, in our system of coordinate $F\vec{e}_R\vec{e}_\theta$ the vector \vec{IM} is given by

$$\vec{IM} \begin{vmatrix} \Delta \cos\theta + r \\ -\Delta \sin\theta \end{vmatrix}$$

We obtain

$$\Delta \sin\theta \cdot F_R + (\Delta \cos\theta + r) \cdot F_\theta = 0$$

With the relations specific to the conics

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \quad \text{and} \quad \dot{r} = \frac{eC\sin\theta}{a(1-e^2)} r \sqrt{f} \quad (2)$$

We obtain

$$\Delta = \frac{ae(2f + rf')}{f'(r-a) + 2f}$$

To obtain a central force we are looking for the family of functions $f(r)$ which leads to Δ as a constant. Consequently we write the equation

$$\frac{d}{dr} \Delta = 0$$

This relation leads to

$$\frac{ea^2(2ff'' - 3f'^2)}{(f'(r+a) - 2f)^2} = 0$$

And the solving is

$$f(r) = \frac{1}{(C_1 r + C_2)^2}$$

Where C_1 and C_2 are constant. The position of I is given by

$$\Delta = \frac{aC_2 e}{C_2 + aC_1} \quad (3)$$

And the force becomes

$$\vec{F} = A \frac{ar(C_2 - rC_1) - C_2(a^2(1-e^2) + r^2)}{ar(C_1 r + C_2)^3} \vec{e}_R + A \frac{eC_2}{(C_1 r + C_2)^3} \sin\theta \cdot \vec{e}_\theta \quad (4)$$

Consequently we have determined a family of central forces which lead to conic trajectories. These forces are depending on two constants C_1 and C_2 are directed toward the point I . The distance IF is given by

$$IF = \frac{aC_2 e}{C_2 + aC_1}$$

And the distance OI by

$$OI = OF - IF = ae - \frac{aC_2 e}{C_2 + aC_1} = \frac{a^2 C_1 e}{C_2 + aC_1}$$

Note we obtain a simple ratio

$$\frac{OI}{IF} = \frac{aC_1}{C_2}$$

To simplify the expression we call R the distance IM . This one is given by

$$R = IM = \sqrt{(\Delta + r \cos \theta)^2 + (r \sin \theta)^2} = \sqrt{\Delta^2 + r^2 + 2r \cdot \cos \theta}$$

And by using

$$r = \frac{a(1 - e^2)}{1 + e \cos \theta}$$

We obtain

$$R = \frac{\sqrt{2aC_1C_2(-ar + r^2 + a^2(1 - e^2)) + C_2^2(2a^2 - e^2a^2 + r^2 - 2ar) + C_1^2(a^2r^2)}}{(C_1a + C_2)}$$

We determine now the magnitude of our force. This one is given by

$$F = A \sqrt{\left[\frac{ar(C_2 - rC_1) - C_2(a^2(1 - e^2) + r^2)}{ar(C_1r + C_2)^3} \right]^2 + \left[\frac{eC_2}{(C_1r + C_2)^3} \sin \theta \right]^2}$$

And by using

$$\sin \theta = \frac{\sqrt{e^2r^2 - [a(1 - e^2) - r]^2}}{er}$$

We obtain

$$F = \frac{A}{a} \frac{\sqrt{2aC_1C_2(-ar + r^2 + a^2(1 - e^2)) + C_2^2(2a^2 - e^2a^2 + r^2 - 2ar) + C_1^2(a^2r^2)}}{(C_1r + C_2)^3}$$

Consequently we can write

$$F = \frac{A}{a} \frac{C_1a + C_2}{(C_1r + C_2)^3} R \quad (5)$$

2.3. Limiting cases

We can distinguish two limiting cases: Firstly if $C_2 = 0$: the force given by (4) becomes

$$\vec{F} = -\frac{A}{C_1^2 r^2} \vec{e}_R$$

And we obtain

$$\Delta = 0$$

Consequently by choosing $C_1 = 1$ this force is the Newton's.

Secondly If $C_1 = 0$ the force becomes

$$\vec{F} = A \frac{ar - a^2(1 - e^2) + r^2}{arC_2^2} \vec{e}_R + \frac{Ae}{C_2^2} \sin \theta \cdot \vec{e}_\theta$$

Its magnitude is given by

$$\|\vec{F}\| = \frac{A}{aC_2^2} \sqrt{a^2(1-e^2) + r^2 + a^2 - 2ar}$$

By noticing that the distance OM , where M is the point-particle and O the center of the conic, is given by

$$OM = \sqrt{a^2(1-e^2) + r^2 + a^2 - 2ar}$$

We obtain

$$\|\vec{F}\| = \frac{A}{aC_2^2} OM \quad (6)$$

The center of the force is located at O (center of the conic) because

$$\Delta = FI = ae = FO$$

Consequently this force is the Hooke.

2.4. Are these forces the Halfphen-Darboux forces?

We can answer to this question considering the results of Darboux and Halphen. The second family of their forces [19] is given by

$$F = \frac{\mu}{R^2 \left(\frac{1}{R} - a_1 \cos w - b_1 \sin w \right)^3}$$

Where a_1 , b_1 , μ are three constants and w the angle of revolution. To verify if the forces are equal we write the relation

$$F = \frac{\mu}{R^2 \left(\frac{1}{R} + a_1 \cos w + b_1 \sin w \right)^3} = \frac{A}{a} \frac{C_1 a + C_2}{(C_1 r + C_2)^3} R$$

This relation becomes

$$\frac{\mu}{(1 + a_1 R \cos w + b_1 R \sin w)^3} R = \frac{A}{a} \frac{C_1 a + C_2}{(C_1 r + C_2)^3} R \quad (7)$$

In our case the angle w is given by (Figure 1)

$$w = \hat{MIF}$$

And we obtain

$$R \cos w = r \cos \theta + \frac{aeC_2}{C_2 + aC_1}$$

And by using the relation (2)

$$r = \frac{ae^2 C_2}{C_2 + aC_1} + a(1-e^2) - eR \cos w$$

(7) becomes

$$\frac{\mu}{(1 + a_1 R \cos w + b_1 R \sin w)^3} = \frac{A}{a} \frac{(C_1 a + C_2)^4}{(C_2^2 + a^2(1-e^2)C_1^2 + 2aC_1C_2 - eRC_1(C_2 + aC_1)\cos w)^3}$$

Thus

$$\frac{\mu}{(1 + a_1 R \cos w + b_1 R \sin w)^3} = \frac{A}{a} \frac{\frac{(C_1 a + C_2)^4}{(C_2^2 + a^2(1 - e^2)C_1^2 + 2aC_1C_2)^3}}{\left(1 - \frac{eC_1(C_2 + aC_1)}{C_2^2 + a^2(1 - e^2)C_1^2 + 2aC_1C_2} R \cos w\right)^3}$$

By analogy it appears that the Darboux's coefficient are

$$a_1 = \frac{-eC_1(C_2 + aC_1)}{aC_1C_2 + a^2(1 - e^2)C_1^2 + C_2^2}$$

$$b_1 = 0$$

$$\mu = \frac{A}{a} \frac{(C_1 a + C_2)^4}{(C_2^2 + a^2(1 - e^2)C_1^2 + 2aC_1C_2)^3}$$

And the force can be written

$$F = \frac{A}{a} \frac{\frac{(C_1 a + C_2)}{(C_2^2 + a^2(1 - e^2)C_1^2 + 2aC_1C_2)^3}}{\left(1 + \frac{eC_1}{aC_1C_2 + a^2(1 - e^2)C_1^2 + C_2^2} R \cos w\right)^3} R$$

Consequently the forces we obtained are well a part of the Darboux's forces. The center of these forces is located between the foci of the conic and its origin. Therefore the expressions of the coefficient are different and simpler in our case. In particular by using them we can obtain simple expressions of the angular momentum and of the equation of time.

2.5. Determination of the angular momentum

We determine it as usual at the center of the force (I). In order to do it we begin to determine the angular momentum at F , foci of the conic. We use the relation

$$\vec{L}_F = r \cdot \vec{e}_R * (\dot{r} \cdot \vec{e}_R + r \dot{\theta} \cdot \vec{e}_\theta) = r^2 \dot{\theta} \cdot \vec{e}_Z$$

And by using

$$\dot{\theta} = \frac{C}{r} \sqrt{f(r)} = \frac{C}{r} \frac{1}{C_1 r + C_2} \quad (8)$$

we obtain

$$\vec{L}_F = \frac{Cr}{C_1 r + C_2} \cdot \vec{e}_Z$$

The angular momentum at I is given by

$$\vec{L}_I = \vec{L}_F + I \vec{F} * (\dot{r} \cdot \vec{e}_R + r \dot{\theta} \cdot \vec{e}_\theta)$$

By noticing that

$$I \vec{F} = \Delta (\cos \theta \cdot \vec{e}_R - \sin \theta \cdot \vec{e}_\theta)$$

\vec{L}_I becomes

$$\vec{L}_I = \vec{L}_F + \Delta [r \dot{\theta} \cos \theta + \dot{r} \sin \theta] \vec{e}_Z$$

And

$$\vec{L}_I = \frac{Cr}{C_1 r + C_2} + \frac{aC_2 e}{C_2 + aC_1} [r \dot{\theta} \cos \theta + \dot{r} \sin \theta] \vec{e}_Z$$

By using the relations specific to the conics (2) we obtain

$$\vec{L}_l = \frac{aC}{C_2 + aC_1} \vec{e}_z \quad (m^2 s^{-1})$$

Consequently, the angular momentum is constant for all the forces and the trajectory is plane.

2.6. Equation of time

We can also give the equation of time of the motion. As usual, we write this equation by using the eccentric anomaly E (Figure 1). Indeed, (in the case of an ellipse) we have the relation

$$\dot{\theta} = \dot{E} \frac{\sqrt{1-e^2}}{1-e\cos E}$$

By using equation (8)

$$\dot{E} \frac{\sqrt{1-e^2}}{1-e\cos E} = \frac{C}{r} \frac{1}{C_1 r + C_2}$$

And by using

$$r = a(1 - e\cos E)$$

We obtain

$$\dot{E} [C_1 a(1 - e\cos E) + C_2] = \frac{C}{a\sqrt{1-e^2}} = \sqrt{\frac{A}{a}}$$

By an integration

$$C_1 a(E - e\sin E) + C_2 E = \sqrt{\frac{A}{a}} t$$

Consequently if trajectory is bounded we obtain

$$E(C_1 a + C_2) - eC_1 a \sin E = \sqrt{\frac{A}{a}} t$$

$$(C_1 a + C_2) \left[E - e \frac{C_1 a}{C_1 a + C_2} \sin E \right] = \sqrt{\frac{A}{a}} t$$

And

$$E - e \frac{C_1 a}{C_1 a + C_2} \sin E = \frac{1}{(C_1 a + C_2)} \sqrt{\frac{A}{a}} t = mt$$

Thus the mean motion is given by

$$m = \frac{1}{(C_1 a + C_2)} \sqrt{\frac{A}{a}}$$

If the force is the Newton's we obtain the equation of time of Kepler

$$E - e \sin E = \sqrt{\frac{A}{a^3}} t$$

And if the force is the Hooke's we obtain simply

$$E = \sqrt{\frac{A}{aC_2^2}} t$$

2.7. Conclusion of this mathematical study

In this part we determined a family of central forces which lead to conic trajectories. The forces we obtained are identical to the Darboux's forces when the center of these forces is located between the foci and the origin of the conic.

It appears that with respect for the Bertrand's theorem these forces aren't conservative except for the two limiting case (the Newton's and the Hooke's). In the following of this paper we will use them to try to build our theory of gravitation.

3. A force of gravitation valuable for large distances?

To verify our initial assumption we have to determine possible expressions of the coefficients C_1 and C_2 . To determine a possible expression of them we study as usual limiting cases. In the Newton's case the expression of these coefficients is simple and given by

$$\begin{aligned} C_1 &= 1 \\ C_2 &= 0 \end{aligned}$$

Therefore in the second limiting case (the Hooke's) these coefficient have to be different. We will now study if a possible expression of them could describe the curve of rotation of the galaxies.

3.1. Curve of rotation of the galaxies: determination of C_2

In the case of the force of Hooke the coefficient $C_1 = 0$. Moreover if our assumption is possible we should obtain a flat curve of rotation by a correct choice of the constant factor C_2 . In order to determine it we study as usual the circular motion. Indeed, this motion describes with a good approximation the motion of the stars around the center of galaxies. In this particular motion, the eccentricity of the conic is given by $e = 0$ and $a = r = R$. Consequently, the force given by relation (5) becomes simply

$$\vec{F} = -\frac{A}{aC_2^2} R\vec{e}_R = -\frac{A}{C_2^2} \vec{e}_R$$

The curve of rotation is obtained by written the equality of the acceleration and the force

$$-\frac{V^2}{r} = -\frac{A}{C_2^2}$$

Where V is the speed of the point-particle. We obtain

$$V = \frac{\sqrt{Ar}}{C_2}$$

And consequently

$$V = \frac{\sqrt{Aa}}{C_2}$$

In order to obtain a flat curve of rotation, and with respect for the physical dimension of C_2 (m) we see that this factor has to be linked to the semi-major axis, as

$$C_2 = \sqrt{r_0 a} \quad (m)$$

Where r_0 is constant (m). Moreover, we can determine this constant, because we have to respect the Tully Fisher law [25]. Indeed, this empirical relation suggests a correspondence between the visible mass of galaxies and the velocity as

$$V^4_{\infty} \propto (M)$$

Where M is the total mass of the galaxy and V_{∞} the constant speed of the stars. A is proportional to the mass of the galaxy and given by

$$A = GM$$

Where G is the universal constant of gravity. Thus

$$V_{\infty} = \sqrt{\frac{A}{r_0}} \propto A^{1/4}$$

And consequently

$$r_0 \propto A^{1/2}$$

This condition, with respect for physical dimensions, leads to write the relation

$$r_0 = \sqrt{a_0 A}$$

Where a_0 is an acceleration. This acceleration should be constant in the entire universe, and can be linked with the Milgrom's acceleration used in MOND [11]. With this relation V is constant for all the stars at the periphery of one galaxy and is in agreement with the Tully-Fisher law. This speed is given by

$$V = [GMa_0]^{1/4}$$

This expression is equal to the expression which was obtained in MOND and which has been tested with success [11]. We see thus that by using this constant acceleration, which has been introduced in another theory, The Darboux's forces could describe the curve of rotation of galaxies without dark matter hypothesis. Note that unlike MOND this model is in agreement with the corpus of classical physics. Indeed we used in this part a force which is well-known since centuries.

By using relation (4) and with

$$C_1 = 0$$

The force can be written

$$\vec{F} = -\frac{A}{ar_0} \left[\frac{a}{r} (1-e^2) + \frac{r}{a} - 1 \right] \vec{e}_R + \frac{A}{ar_0} e \sin \theta \cdot \vec{e}_{\theta}$$

We can now use this expression to study the parabolic motion.

3.3. Consequence on the parabolic motion: dynamics of the Universe

We see here the predictions of this model about the dynamics of the universe. To do it we study the limiting case of parabolic motion. The eccentricity is given by

$$e = 1$$

And in the simplest case the angular θ is determined by

$$\theta = \pi$$

Consequently our force becomes simply

$$\vec{F} = -\frac{A}{ar_0} \left[\frac{r}{a} - 1 \right] \vec{e}_R$$

We can see that the sign of this force can be positive or negative, and is depending on the ratio $\frac{r}{a}$.

Indeed, if $r < a$ the force is repulsive. However, it also appears that this force can be attractive if $r > a$.

With this result we can try to build a simple model about the dynamics of the Universe. We consider, for example, the interaction of two clusters of galaxies between themselves. We approximate their motion with a parabolic motion, and the force of gravitation is the force we present here.

If the clusters move away from one another it is possible that this force is repulsive. Consequently, their relative speed will increase, at least for a time. It appears that if our idea is correct this kind of interactions is actually majority inside the Universe. However when the distance between the clusters progresses this force becomes a day attractive. Consequently, we can describe the actual expansion of the Universe at an increasing rate, and simultaneously build a model of the expansion of the Universe compatible with the classical idea of "Big Bang / Big Crunch". This simple model doesn't need "dark energy" hypothesis, because the Universe is considerate as a kind of harmonic oscillator.

Indeed for a closed homogeneous and isotropic universe, we can compare it to a kind of spring. The expansion of the universe is accelerated because this "spring" is today compressed. This point indicates that the free end of the spring hasn't reached its "relaxed" position. After this point the force becomes attractive and the rate of the expansion decreases.

We will now briefly compare this model with the Friedmann's equations.

3.4. Comparison with the Friedmann's equations

The first equation of Friedmann is used for modeling an isotropic and homogeneous universe and is derivated from the Einstein's field equations [25]. In this equation Einstein has inserted a factor called the cosmological constant. By taking care of this factor the first Friedmann's equation becomes

$$\dot{a}_1^2 - \frac{8\pi G}{3} \rho(t) a_1^2 - \frac{\Lambda c^2}{3} a_1^2 = kc^2$$

Where a_1 is the scale factor, G the Newton's gravitational constant, Λ the cosmological constant, c the speed of light in vaccum, $\rho(t)$ the density of mass of the universe and $\frac{k}{a_1^2}$ the spatial curvature in

any time slice of the Universe. By considering that $\rho(t) \propto a_1^{-3}$ [26] the equation of Friedmann becomes

$$\dot{a}_1^2 - \frac{m}{a_1} - \frac{\Lambda c^2}{3} a_1^2 = kc^2 \quad (9)$$

Where m is constant.

The first term of this equation can be compared with a Kinetic energy. The second term with a Newtonian potential, the third with an isotropic harmonic oscillator potential.

The right-hand side of this equation is constant. Consequently this equation can be understood as the conservation of the Mechanical energy.

If we have a look on equation (9) we can deduce that the force we present here doesn't lead to a comparable equation. Indeed we can't write a potential which is only the addition of a Newtonian and harmonic oscillator potential, because in this case the trajectory won't be a conic.

Therefore if $a_1 \rightarrow \infty$ we can consider that the Newtonian potential disappears in equation (9). It appears that in this case this equation possesses an interesting analogy with our work. Indeed, in the case of parabolic motion our force is given by

$$\vec{F} = -\frac{A}{ar_0} \left[\frac{r}{a} - 1 \right] \vec{e}_R$$

In this limiting case the force is conservative. Consequently we can determine a potential which is only depending on r . This one is given by

$$V_p(r) = -\int -\frac{A}{ar_0} \left[\frac{r}{a} - 1 \right] dr = \frac{1}{2} \frac{A}{a^2 r_0} r^2 - \frac{A}{ar_0} r + V_{p1}$$

Where V_1 is constant. By writing as usual that the potential has to be null when $r = a$ we obtain

$$V_{p1} = \frac{1}{2} \frac{A}{r_0}$$

And the potential becomes

$$V_p(r) = \frac{1}{2} \frac{A}{ar_0} [r - a]^2$$

This force is located at O and the distance $OM = R$ is given by using the relation

$$R = \sqrt{a^2(1 - e^2) + r^2 + a^2 - 2ar}$$

With $e = 1$ and consequently

$$R = r - a$$

Consequently the potential (per mass unity) can be written

$$V_p(R) = \frac{1}{2} \frac{A}{ar_0} R^2$$

The Kinetic energy (per mass unity) is given by

$$E_C = \frac{1}{2} V^2 = \frac{1}{2} (\dot{r}^2 + (r\dot{\theta})^2)$$

By using equations (2) and (8) we obtain

$$E_C = \frac{1}{2} A \frac{(2a - r)r}{a^2 r_0} = \frac{1}{2} \frac{A}{a^2 r_0} (a^2 - R^2)$$

Thus the Mechanical energy is

$$E_M = E_C + V_p(R) = \frac{1}{2} \frac{A}{r_0}$$

And we obtain

$$\frac{1}{2} \dot{r}^2 + \frac{1}{2} \frac{A}{ar_0} R^2 = \frac{1}{2} \frac{A}{r_0} \quad (10)$$

We can now compare equations (9) and (10). It appears that we can write two proportionality relations

$$-\frac{\Lambda c^2}{3} a_1^2 \propto \frac{1}{2} \frac{A}{ar_0} R^2$$

And

$$kc^2 \propto \frac{1}{2} \frac{A}{r_0}$$

Where A is linked to the total mass of the Universe by

$$A = GM_U$$

An r_0 by

$$r_0 = \sqrt{\frac{GM_U}{a_0}}$$

Consequently the previously relation can be written

$$-\frac{\Lambda c^2}{3} a_1^2 \propto \frac{1}{2} \frac{\sqrt{GM_U a_0}}{a} (r-a)^2$$

Where a is the semi-major axis of the parabola and

$$kc^2 \propto \frac{1}{2} \sqrt{GM_U a_0}$$

It seems thus that if this analogy is correct the cosmological constant should be negative. Moreover this cosmological constant should be depending on the Millgrom's acceleration.

But as we wrote it previously it seems that it is only an analogy. Therefore we can perhaps consider that the mathematical similarity between the two equations is an argument for our model. Indeed the introduction of the cosmological constant is often evoked as a solution to understand the expansion of the universe.

4. A way to build a general force of gravitation?

We saw that the force of Hooke could describe the "flat" curve of rotation of the galaxies without dark matter. Moreover, it could also perhaps explain the expansion of the Universe without dark energy. However, it is certain that this force can't be the dominant force in the case of small distances, as solar systems for example. Consequently, the nature of force has to change gradually when the acceleration decreases.

To try to describe this evolution, we use our previously work about the central forces. We saw that the forces given by the relation

$$\vec{F} = A \frac{ar(C_2 - rC_1) - C_2(a^2(1 - e^2) + r^2)}{ar(C_1r + C_2)^3} \vec{e}_R + A \frac{eC_2}{(C_1r + C_2)^3} \text{Sin}\theta \vec{e}_\theta$$

Are central and directed toward a point I definite by

$$FI = \frac{aC_2e}{C_2 + aC_1}$$

We can determine the constants for our limiting cases: if $r \ll r_0$ the force is the Newton's and consequently $C_2 = 0$. If $r \gg r_0$ the force is the Hooke's consequently $C_2 = \sqrt{r_0 a}$ and $C_1 = 0$. We can list our results in the table 1.

		C_1	C_2
Small distances	$a \ll r_0$	1	0
Large distances	$a \gg r_0$	0	$\sqrt{r_0 a}$
General case		$g\left(\frac{a}{r_0}\right)$	$h(a, r_0)$

Tab 1. Coefficients of the central force

In this table the function $g\left(\frac{a}{r_0}\right)$ is defined by the following relation

$$0 \leq g\left(\frac{a}{r_0}\right) \leq 1$$

And the function $h(a, r_0)$ could be

$$h(r_0, a) = \sqrt{r_0 a} \left[1 - \left[g\left(\frac{a}{r_0}\right) \right]^m \right]^n$$

For example

$$h(r_0, a) = \sqrt{r_0 a} \left[1 - g\left(\frac{a}{r_0}\right) \right]$$

Or perhaps

$$h(r_0, a) = \sqrt{r_0 a} \sqrt{1 - \left[g\left(\frac{a}{r_0}\right) \right]^2}$$

5. Possible tests inside the solar system?

Our model indicates that in our solar system, the force of gravitation shouldn't be exactly the Newton's.. Naturally, the modification should be a tiny amount of the force of gravitation, should be more important at the periphery of the system than at its center. Moreover this part won't modify the trajectories of celestial bodies, because these trajectories are always conic, but should modify the mean motion.

We can use this reasoning to purpose an experiment inside our solar system or perhaps at the surface of the Earth. Indeed if trajectories are not modified, it isn't the case of the equation of time. Consequently it appears that a classical experiment, the study of the free fall motion, could help to valid or invalid our model. We will now try to prove it by using our equations about the conics.

5.1. Equation of the free fall motion

We saw in part 2. that the radial acceleration is given by

$$a_R = -Af + \dot{r}^2 \left[\frac{1}{2} \frac{f'}{f} + \frac{1}{r} \right]$$

And, by using the relations specifics to the conics

$$r = \frac{a(1-e^2)}{1+e\cos\theta} \quad \text{and} \quad \dot{r} = \frac{dr}{dt} = \frac{d\theta}{dt} \frac{dr}{d\theta} = \dot{\theta} \frac{dr}{d\theta}$$

We obtain, after simplification

$$a_R = -\frac{1}{2} \frac{A}{ar} (2f(r)(a^2(1-e^2) + r^2 - ar) + f'(r)(ra^2(1-e^2) + r^3 - 2r^2a))$$

And, in the case of free fall motion ($e=1$) the radial acceleration becomes

$$\ddot{r} = -\frac{1}{2} \frac{A}{a} (2f(r)(r-a) + f'(r)(r^2 - 2ra))$$

If we consider that the Newton's force is the correct force of gravitation, valid in all the cases, then the radial acceleration has to be given by

$$\ddot{r} = -\frac{A}{r^2}$$

Consequently we can write a differential equation

$$-\frac{A}{r^2} = -\frac{1}{2} \frac{A}{a} (2f(r)(r-a) + f'(r)(r^2 - 2ra))$$

And after simplification

$$-\frac{1}{r^2} = -\frac{1}{2a} (2f(r)(r-a) + f'(r)(r^2 - 2ra))$$

The solving of this equation is

$$f(r) = \frac{1}{r^2} \left[\frac{-2a + C_1 r}{-2a + r} \right]$$

Where C_1 is constant. With this relation and by using (3) the total speed becomes

$$V = \sqrt{\frac{A(2a - C_1 r)}{ar}}$$

However this speed has to be equal to zero if

$$r = 2a$$

(Case of the free fall motion without initial speed). Thus we have to write that $C_1 = 1$ and

$$f(r) = \frac{1}{r^2}$$

With this expression of $f(r)$ the force is the Newton's. Consequently, we can't obtain a force different from this force which leads to the same acceleration, in the case of the free fall motion. This point indicates that if a force of gravitation different from the Newton's is valuable for large distances we should detect an anomaly around this law. As we wrote it previously, this anomaly should be a tiny proportion of it but it is possible to think that with a contemporary precision, we could detect it. Note that this test is only a modern version of the experiments of Galilee. Moreover note that it should detect all variations around the Newton's law and not only our model. At end it appears that this kind of anomaly has been suspected in the past [27].

5.2. Our prediction

If our model is correct we should obtain a tiny difference by comparison with the Newton's acceleration. Indeed a radial acceleration given by the equation (4) for $e = 1$. This equation becomes

$$\ddot{r} = A \frac{a(C_2 - rC_1) - C_2 r}{a(C_1 r + C_2)^3}$$

Therefore if we consider that $C_2 \ll r$ the acceleration becomes

$$\ddot{r} = -\frac{A}{r^2} \left[\frac{1}{C_1^2} + \frac{C_2}{a} \right]$$

The acceleration is proportional to $\frac{1}{r^2}$ and consequently should be really proximate to the Newton's acceleration. A test should be to change a : in this case, with several experiments the constant of proportionality should be a little bit different.

6. Discussion

6.1. Physical interpretation

In this paper, we studied the possibility that the force of Hooke could perhaps be the force of gravitation valuable for large distances, by choosing correct coefficients. In particular the coefficient C_2 has to be linked to the semi major axis of the trajectory. This point seems surprising because we generally consider that the force and the trajectory has to be independent (as in the case of the Newton's force). Consequently this fact seems a problem for this model. However we can try to discuss this assumption with a physical point of view.

Firstly, we can say that the force of Hooke exist already in the gravitation. It is the force which interacts with a point-particle inside a sphere where the density of mass is uniform. This is the consequence of the theorem of Gauss. For example, in a sphere the acceleration is given by

$$\ddot{r} = -\frac{GM_{\text{int}}}{r^2} \cdot \bar{e}_R$$

Where M_{int} is the total mass contained inside the sphere of radius r . If the point particle is dropped out from a distance a of the center and if the mass density is uniform we obtain

$$\ddot{r} = -\frac{C}{a^3} r \cdot \bar{e}_R$$

Where C is constant. We can compare this expression with the acceleration we obtained in a comparable motion with our force

$$\ddot{r} = -\frac{A}{r_0 a^2} r \cdot \bar{e}_R$$

We see that the expressions are proximate. In the two cases the force is depending on the semi major axis. Moreover with this analogy we can suggest a physical interpretation of our force: progressively, when the distance to the center of force increases, the point particle which orbits around the center of mass "considers" that this mass is progressively «diluted» inside a closed volume. This volume is depending on the semi major axis of its trajectory and on the constant r_0 . At the end of this evolution the density of mass inside this closed volume is uniform and the force becomes entirely the Hooke's.

Moreover note that this kind of interpretation was already used in another action-at-a-distance, for example to build the model of the atom of Thomson at the 19th century. Consequently our conclusion is that we can't absolutely exclude the possibility that force of Hooke could be a force of gravitation valuable for large distances.

6.2. The problem of the experiment

A second weak point of this model is the lake of experiments. Indeed this model was entirely built from this assumption: "the trajectory of a point-particle which orbits around a center of mass is always conic". This assumption led us to study the central forces which allow to obtain conic trajectories and to investigate the forces of Darboux.

Note that certain of our predictions seem correct: for example the curve of rotation of the galaxies is “flat” and respects the Tully Fisher law. Moreover, our model could describe the actual expansion of the Universe. Therefore it is also true that we haven't an experimental proof to valid it.

For example we can't predict a modification of the trajectories of the celestial bodies inside the solar system. Indeed with respect for our initial assumption these trajectories are always conic. The only difference we expect is a small modification of their mean motion at the periphery of the solar system. That is the reason why in the part 5 of this paper we purposed an experiment about the free motion. Moreover it is possible that a day the motion of a spacecraft on a hyperbolic trajectory helps to valid or invalids our model. In the past several anomalies has been suspected but because of several perturbations we can't take care these kinds of observations [27].

Therefore we can argue that this model isn't the only one in this case. Indeed the experiments when the acceleration is so small seem very difficult to implement [28]. We can also argue that the majority theory (the dark matter hypothesis) seems have no more experimental results to present.

7. Conclusion: Could the Darboux's force be an alternative to dark matter/ energy?

In our point of view this study show that it seems possible. Indeed we showed that by using these forces we can describe the flat curve of rotation of the galaxies, and explain the actual expansion of the Universe (because these forces can be repulsive in the case of parabolic-hyperbolic motion). We tried to present these assumptions in the part 3 of this paper.

Consequently this work leads us to think that it is perhaps possible to use these forces to build a classical and alternative force of gravitation. In this paper we didn't present this force because we had only a look on its possibility. In particular several theorist problems hasn't been evoked, as for example the two body problem or the principle of equivalence. Therefore we present possible tests inside our solar system to valid or invalid this assumption.

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