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Loïc Foissy. Commutative and non-commutative bialgebras of quasi-posets and applications to Ehrhart polynomials. *Advances in Pure and Applied Mathematics*, 2019, 10 (1), pp.27-63. 10.1515/apam-2016-0051 . hal-01321986v2

**HAL Id: hal-01321986**

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Submitted on 14 Nov 2016

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# Commutative and non-commutative bialgebras of quasi-posets and applications to Ehrhart polynomials

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## Abstract

To any poset or quasi-poset is attached a lattice polytope, whose Ehrhart polynomial we study from a Hopf-algebraic point of view. We use for this two interacting bialgebras on quasi-posets. The Ehrhart polynomial defines a Hopf algebra morphism taking its values in  $\mathbb{Q}[X]$ ; we deduce from the interacting bialgebras an algebraic proof of the duality principle, a generalization and a new proof of a result on B-series due to Whright and Zhao, using a monoid of characters on quasi-posets, and a generalization of Faulhaber's formula.

We also give non-commutative versions of these results: polynomials are replaced by packed words. We obtain in particular a non-commutative duality principle.

**Keywords.** Ehrhart polynomials; Quasi-posets; Characters monoids; Interacting bialgebras

**AMS classification.** 16T30; 06A11

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## Introduction

Let  $P$  be a lattice polytope, that is to say that all its vertices are in  $\mathbb{Z}^n$ . The Ehrhart polynomial  $ehr_P^{cl}(X)$  is the unique polynomial such that, for all  $k \geq 1$ ,  $ehr_P^{cl}(k)$  is the number of points in  $\mathbb{Z}^n \cap kP$ , where  $kP$  is the image of  $P$  by the homothety of center 0 and ratio  $k$ . For example, if  $S$  is the square  $[0, 1]^n$  and  $T$  is the triangle of vertices  $(0, 0)$ ,  $(1, 0)$  and  $(1, 1)$ :

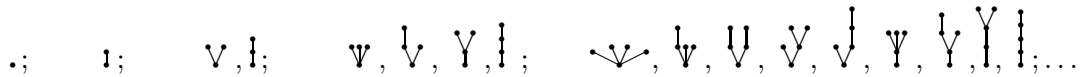
$$ehr_S^{cl}(X) = (X + 1)^2, \quad ehr_T^{cl}(X) = \frac{(X + 1)(X + 2)}{2}.$$

These polynomials satisfy the reciprocity principle: for all  $k \geq 1$ ,  $(-1)^{dim(P)} ehr^{cl}(-k)$  is the number of points of  $\mathbb{Z}^n \cap k\dot{P}$ , where  $\dot{P}$  is the interior of  $P$ . For example:

$$ehr_S^{cl}(-X) = (X - 1)^2, \quad ehr_T^{cl}(-X) = \frac{(X - 1)(X - 2)}{2}.$$

We refer to [2] for general results on Ehrhart polynomials.

It turns out that these polynomials appear in the theory of B-series (B for Butcher [4]), as explained in [3, 6]. We now consider rooted trees:



If  $t$  is a rooted tree, we orient its edges from the root to the leaves. If  $i, j$  are two vertices of  $t$ , we shall write  $i \xrightarrow{t} j$  if there is an edge from  $i$  to  $j$  in  $t$ .

To any rooted tree  $t$ , whose vertices are indexed by  $1 \dots n$ , we associate a lattice polytope  $pol(t)$  in a following way:

$$pol(t) = \left\{ (x_1, \dots, x_n) \in [0, 1]^n \mid \forall 1 \leq i, j \leq n, (i \xrightarrow{t} j) \implies (x_i \leq x_j) \right\}$$

For example, if  $t = \mathbf{1}$ , indexed as  $\mathbf{1}_1^1$ , then  $pol(t) = T$ .

We can consider the Ehrhart polynomial  $ehr_{pol(t)}^{cl}(X)$ , which we shall simply denote by  $ehr_t^{cl}(X)$ : for all  $k \geq 1$ ,

$$ehr_t^{cl}(k) = \# \left\{ (x_1, \dots, x_n) \in \{0, \dots, k\}^n \mid \forall 1 \leq i, j \leq n, (i \xrightarrow{t} j) \implies (x_i \leq x_j) \right\}.$$

Note that  $ehr_t^{cl}$  does not depend on the indexation of the vertices of  $t$ . By the duality principle:

$$(-1)^n ehr_t^{cl}(-k) = \# \left\{ (x_1, \dots, x_n) \in \{1, \dots, k - 1\}^n \mid \forall 1 \leq i, j \leq n, (i \xrightarrow{t} j) \implies (x_i < x_j) \right\}.$$

A B-series is a formal series indexed by rooted trees, of the form:

$$\sum_t a_t \frac{t}{\text{aut}(t)} = a_{\bullet} + a_{\mathbf{1}} \mathbf{1} + a_{\mathbf{V}} \frac{\mathbf{V}}{2} + a_{\mathbf{1}\mathbf{1}} \mathbf{1}\mathbf{1} + \dots,$$

where  $\text{aut}(t)$  is the number of automorphisms of  $t$ . The following B-series is of special importance in numerical analysis:

$$E = \sum_t \frac{1}{t!} \frac{t}{\text{aut}(t)} = \bullet + \frac{1}{2} a_{\mathbf{1}} \mathbf{1} + \frac{1}{3} \frac{\mathbf{V}}{2} + \frac{1}{6} \mathbf{1}\mathbf{1} + \dots,$$

where  $t!$  is the tree factorial (see definition 33). This series is the formal solution of an ordinary differential equation, describing the flow of a vector field. The set of B-series is given a group structure by a substitution operation, which is dually represented by the contraction-extraction coproduct defined in [5]. The inverse of  $E$  is called the backward error analysis:

$$E^{-1} = \sum_t \lambda_t \frac{t}{\text{aut}(t)!}.$$

Wright and Zhao [19] proved that these coefficients  $\lambda_t$  are related to Ehrhart polynomials:

$$\lambda_t = (-1)^{|t|} \frac{d \text{ehr}_t^{\text{cl}}(X)}{dX} \Big|_{X=-1}.$$

We shall in this text study Ehrhart polynomial attached to quasi-posets in a combinatorial Hopf-algebraic way. A quasi-poset  $P$  is a pair  $(A, \leq_P)$ , where  $A$  is a finite set and  $\leq_P$  is a reflexive and transitive relation on  $A$ . The isoclasses of quasi-posets are represented by their Hasse graphs:

$$\mathbf{1}; \quad \bullet; \quad \dots, \mathbf{1}, \bullet_2; \quad \dots, \bullet_1, \dots_2, \mathbf{V}, \mathbf{\Lambda}, \mathbf{1}\mathbf{1}, \mathbf{1}^2, \mathbf{1}_2, \bullet_3, \dots$$

In particular, rooted trees can be seen as quasi-posets. For any quasi-poset  $P = (\{1, \dots, n\}, \leq_P)$ , the polytope associated to  $P$  is:

$$\text{pol}(P) = \{(x_1, \dots, x_n) \in [0, 1]^n \mid \forall 1 \leq i, j \leq n, (i \leq_P j) \implies (x_i \leq x_j)\}.$$

We put  $\text{ehr}_P(X) = \text{ehr}_{\text{pol}(P)}^{\text{cl}}(X-1)$ ; note the translation by  $-1$ , which will give us objects more suitable to our purpose. In other words, for all  $k \geq 1$ :

$$\text{ehr}_P(k) = \#\{(x_1, \dots, x_n) \in \{1, \dots, k\}^n \mid \forall 1 \leq i, j \leq n, (i \leq_P j) \implies (x_i \leq x_j)\}.$$

We also define a polynomial  $\text{ehr}_P^{\text{str}}(X)$  such that for all  $k \geq 1$ :

$$\text{ehr}_P^{\text{str}}(k) = \#\{(x_1, \dots, x_n) \in \{1, \dots, k\}^n \mid \forall 1 \leq i, j \leq n, (i \leq_P j \text{ and not } j \leq_P i) \implies (x_i < x_j)\}.$$

See definition 21 and proposition 22 for more details. These polynomials can be inductively computed, with the help of the minimal elements of  $P$  (proposition 25).

We shall consider two products  $m$  and  $\downarrow$ , and two coproducts  $\Delta$  and  $\delta$  on the space  $\mathcal{H}_{\mathbf{qp}}$  generated by isoclasses of quasi-posets. The coproduct  $\Delta$ , defined in [10, 11] by restriction to open and closed sets of the topologies associated to quasi-posets, makes  $(\mathcal{H}_{\mathbf{qp}}, m, \Delta)$  a graded, connected Hopf algebra, and  $(\mathcal{H}_{\mathbf{qp}}, \downarrow, \Delta)$  an infinitesimal bialgebra; the coproduct  $\delta$ , defined in [8] by an extraction-contraction operation, makes  $(\mathcal{H}_{\mathbf{qp}}, m, \delta)$  a bialgebra. Moreover,  $\delta$  is also a right coaction of  $(\mathcal{H}_{\mathbf{qp}}, m, \delta)$  over  $(\mathcal{H}_{\mathbf{qp}}, m, \Delta)$ , and  $(\mathcal{H}_{\mathbf{qp}}, m, \Delta)$  becomes a Hopf algebra in the category of  $(\mathcal{H}_{\mathbf{qp}}, m, \delta)$ -comodules, which we summarize telling that  $(\mathcal{H}_{\mathbf{qp}}, m, \Delta)$  and  $(\mathcal{H}_{\mathbf{qp}}, m, \delta)$

are two bialgebras in cointeraction (definition 1). For example, the bialgebras  $(\mathbb{K}[X], m, \Delta)$  and  $(\mathbb{K}[X], m, \delta)$  where  $m$  is the usual product of  $\mathbb{K}[X]$  and  $\Delta, \delta$  are the coproducts defined by

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \delta(X) = X \otimes X,$$

are two cointeracting bialgebras.

Ehrhart polynomials  $ehr_P(X)$  and  $ehr_P^{str}(X)$  can now be seen as maps from  $\mathcal{H}_{\mathbf{qp}}$  to  $\mathbb{K}[X]$ , and both are Hopf algebra morphisms from  $(\mathcal{H}_{\mathbf{qp}}, m, \Delta)$  to  $(\mathbb{K}[X], m, \Delta)$  (theorem 23); we shall prove in theorem 37 that  $ehr^{str}$  is the unique morphism from  $\mathcal{H}_{\mathbf{qp}}$  to  $\mathbb{K}[X]$  compatible with both bialgebra structures on  $\mathcal{H}_{\mathbf{qp}}$  and  $\mathbb{K}[X]$ . Using the cointeraction between the two bialgebra structures on  $\mathcal{H}_{\mathbf{qp}}$ , we show that the monoid  $M_{\mathbf{qp}}$  of characters of  $(\mathcal{H}_{\mathbf{qp}}, m, \delta)$  acts on the set  $E_{\mathcal{H}_{\mathbf{qp}} \rightarrow \mathbb{K}[X]}$  of Hopf algebra morphisms from  $(\mathcal{H}_{\mathbf{qp}}, m, \Delta)$  to  $\mathbb{K}[X]$  (lemma 5). Moreover, there exists a particular homogeneous morphism  $\phi_0 \in E_{\mathcal{H}_{\mathbf{qp}} \rightarrow \mathbb{K}[X]}$  such that for all quasi-poset  $P$ :

$$\phi_0(P) = \lambda_P X^{cl(P)} = \frac{\mu_P}{cl(P)!} X^{cl(P)},$$

where  $\mu_P$  is the number of heap-orderings of  $P$  and  $cl(P)$  is the number of equivalence classes of the equivalence associated to the quasi-order of  $P$  (proposition 32). This formula simplifies if  $P$  is a rooted tree: in this case,

$$\phi_0(P) = \frac{1}{P!} X^{|P|}.$$

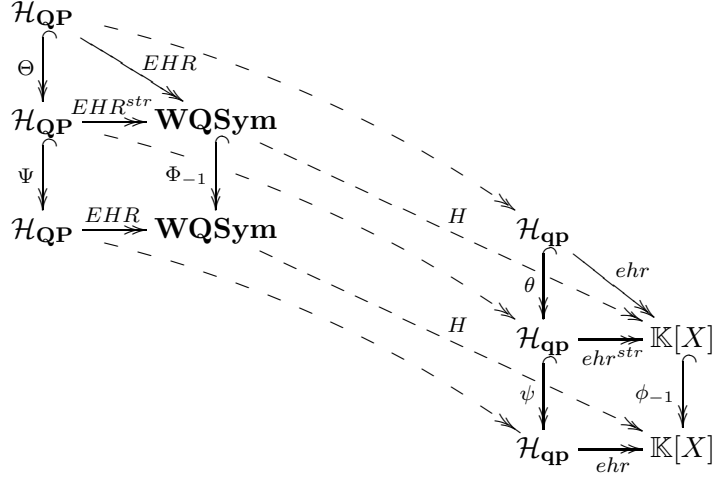
We prove that there exist characters  $\alpha$  and  $\alpha^{str}$  in  $M_{\mathbf{qp}}$ , such that for any quasi-poset  $P$ :

$$ehr_P(X) = \sum_{\sim \triangleleft P} \frac{\mu_{P/\sim}}{cl(\sim)!} \alpha_{P|\sim} X^{cl(\sim)}, \quad ehr_P^{str}(X) = \sum_{\sim \triangleleft P} \frac{\mu_{P/\sim}}{cl(\sim)!} \alpha_{P|\sim}^{str} X^{cl(\sim)},$$

where the sum is over a certain family of equivalence relations  $\sim$  on the set of vertices of  $V$ ,  $P|\sim$  is a restriction operation and  $P/\sim$  is a contraction operation. Applied to corollas, this gives Faulhaber's formula. We prove that  $\alpha^{str}$  is the inverse of the character  $\lambda$  associated to  $\phi_0$  (theorem 37), which is a generalization, as well as a Hopf-algebraic proof, of Wright and Zhao's result. We also give an algebraic proof of the duality principle (theorem 38), and we define a Hopf algebra automorphism  $\theta : (\mathcal{H}_{\mathbf{qp}}, m, \Delta) \rightarrow (\mathcal{H}_{\mathbf{qp}}, m, \Delta)$  with the help of the cointeraction of the two bialgebra structures on  $\mathcal{H}_{\mathbf{qp}}$ , satisfying  $ehr^{str} \circ \theta = ehr$  (proposition 39).

We propose non-commutative versions of these results in the last section of the paper. Here, (isoclasses of) quasi-posets are replaced by quasi-posets on sets  $[n] = \{1, \dots, n\}$ , making a Hopf algebra  $\mathcal{H}_{\mathbf{QP}}$ , in cointeraction with  $(\mathcal{H}_{\mathbf{qp}}, m, \delta)$ , and  $\mathbb{K}[X]$  is replaced by the Hopf algebra of packed words **WQSym** [15]. We define two surjective Hopf algebra morphisms  $EHR$  and  $EHR^{str}$  from  $\mathcal{H}_{\mathbf{QP}}$  to **WQSym** (proposition 41), generalizing  $ehr$  and  $ehr^{str}$ . The automorphism  $\theta$  is generalized as a Hopf algebra automorphism  $\Theta : \mathcal{H}_{\mathbf{QP}} \rightarrow \mathcal{H}_{\mathbf{QP}}$ , such that  $EHR^{str} \circ \Theta = EHR$  (proposition 42), and we formulate a non-commutative duality principle

(theorem 45), and we obtain a commutative diagram of Hopf algebras:



The two triangles reflects the properties of morphisms  $\Theta$  and  $\theta$ , whereas the two squares are the duality principles.

**Acknowledgment.** The research leading these results was partially supported by the French National Research Agency under the reference ANR-12-BS01-0017.

**Notations.** We denote by  $\mathbb{K}$  a commutative field of characteristic zero. All the objects (vector spaces, algebra, and so on) in this text are taken over  $\mathbb{K}$ .

## 1 Bialgebras in cointeraction

We give in this section some general results on bialgebras in cointeractions. They will be used in the sequel for quasi-posets, leading to Ehrhart polynomials. We shall use them on graphs in order to obtain chromatic polynomials in [9].

### 1.1 Definition

**Definition 1** Let  $A$  and  $B$  be two bialgebras. We shall say that  $A$  and  $B$  are in cointeraction if:

- $B$  coacts on  $A$ , via a map  $\rho : \begin{cases} A & \longrightarrow & A \otimes B \\ a & \longrightarrow & \rho(a) = a_1 \otimes a_0. \end{cases}$
- $A$  is a bialgebra in the category of  $B$ -comodules, that is to say:
  - $\rho(1_A) = 1_A \otimes 1_B$ .
  - $m_{2,4}^3 \circ (\rho \otimes \rho) \circ \Delta_A = (\Delta_A \otimes Id) \circ \rho$ , with:

$$m_{2,4}^3 : \begin{cases} A \otimes B \otimes A \otimes B & \longrightarrow & A \otimes A \otimes B \\ a_1 \otimes b_1 \otimes a_2 \otimes b_2 & \longrightarrow & a_1 \otimes a_2 \otimes b_1 b_2. \end{cases}$$

Equivalently, in Sweedler's notations, for all  $a \in A$ :

$$(a^{(1)})_1 \otimes (a^{(2)})_1 \otimes (a^{(1)})_0 (a^{(2)})_0 = (a_1)^{(1)} \otimes (a_1)^{(2)} \otimes a_0.$$

- For all  $a, b \in A$ ,  $\rho(ab) = \rho(a)\rho(b)$ .
- For all  $a \in A$ ,  $(\varepsilon_A \otimes Id) \circ \rho(a) = \varepsilon_A(a)1_B$ .

Examples of bialgebras in interaction can be found in [5] (for rooted trees) and in [14] (for various families of graphs). Another example is given by the algebra  $\mathbb{K}[X]$ , with its usual product  $m$ , and the two coproducts defined by:

$$\Delta(X) = X \otimes 1 + 1 \otimes X, \quad \delta(X) = X \otimes X.$$

The bialgebras  $(\mathbb{K}[X], m, \Delta)$  and  $(\mathbb{K}[X], m, \delta)$  are in cointeraction, via the coaction  $\rho = \delta$ . Identifying  $\mathbb{K}[X] \otimes \mathbb{K}[X]$  and  $\mathbb{K}[X, Y]$ :

$$\Delta(P)(X, Y) = P(X + Y), \quad \delta(P)(X, Y) = P(XY).$$

**Remark.** If  $A$  and  $B$  are in cointeraction, the coaction of  $B$  on  $A$  is an algebra morphism.

**Proposition 2** *Let  $A$  and  $B$  be two bialgebras in cointeraction. We assume that  $A$  is a Hopf algebra, with antipode  $S$ . Then  $S$  is a morphism of  $B$ -comodules, that is to say:*

$$\rho \circ S = (S \otimes Id) \circ \rho$$

**Proof.** We work in the space  $End_{\mathbb{K}}(A, A \otimes B)$ . As  $A \otimes B$  is an algebra and  $A$  is a coalgebra, it is an algebra for the convolution product  $\circledast$ :

$$\forall f, g \in End_{\mathbb{K}}(A, A \otimes B), f \circledast g = m_{A \otimes B} \circ (f \otimes g) \circ \Delta_A.$$

Its unit is:

$$\eta : \begin{cases} A & \longrightarrow & A \otimes B \\ a & \longrightarrow & \varepsilon(a)1_A \otimes 1_B. \end{cases}$$

We consider three elements in this algebra, respectively  $\rho$ ,  $F_1 = (S \otimes Id) \circ \rho$  and  $F_2 = \rho \circ S$ . Firstly:

$$\begin{aligned} (F_1 \circledast \rho)(a) &= S((a^{(1)})_1)(a^{(2)})_1 \otimes (a^{(1)})_0(a^{(2)})_0 \\ &= S((a_1)^{(1)})(a_1)^{(2)} \otimes a_0 \\ &= \varepsilon_A(a_1)1_A \otimes a_0 \\ &= \varepsilon_A(a)1_A \otimes 1_B \\ &= \eta(a). \end{aligned}$$

Secondly:

$$\begin{aligned} (\rho \circledast F_2)(a) &= (a^{(1)})_1 S(a^{(2)})_1 \otimes (a^{(1)})_0 (S(a^{(2)}))_0 \\ &= \varepsilon_A(a)(1_A)_1 \otimes (1_A)_0 \\ &= \varepsilon_A(a)1_A \otimes 1_B \\ &= \eta(a). \end{aligned}$$

We obtain that  $F_1 \circledast \rho = \rho \circledast F_2 = \eta$ , so  $F_1 = F_1 \circledast \eta = F_1 \circledast \rho \circledast F_2 = \eta \circledast F_2 = F_2$ .  $\square$

## 1.2 Monoids actions

**Proposition 3** *Let  $A$  and  $B$  be two bialgebras in cointeraction, through the coaction  $\rho$ . We denote by  $M_A$  and  $M_B$  the monoids of characters of respectively  $A$  and  $B$ . Then  $B$  acts on  $A$  by monoid endomorphisms, via the map:*

$$\leftarrow : \begin{cases} M_A \times M_B & \longrightarrow & M_A \\ (\phi, \lambda) & \longrightarrow & \phi \leftarrow \lambda = (\phi \otimes \lambda) \circ \rho. \end{cases}$$

**Proof.** We denote by  $*$  the convolution product of  $M_B$  and by  $\star$  the convolution product of  $M_A$ . As  $\rho : A \rightarrow A \otimes B$  is an algebra morphism,  $\leftarrow$  is well-defined. Let  $\phi \in M_A$ ,  $\lambda_1, \lambda_2 \in M_B$ .

$$\begin{aligned} (\phi \leftarrow \lambda_1) \leftarrow \lambda_2 &= (\phi \otimes \lambda_1 \otimes \lambda_2) \circ (\rho \otimes Id) \circ \rho \\ &= (\phi \otimes \lambda_1 \otimes \lambda_2) \circ (Id \otimes \Delta_B) \circ \rho \\ &= \phi \leftarrow (\lambda_1 * \lambda_2). \end{aligned}$$

So  $\leftarrow$  is an action. Let  $\phi_1, \phi_2 \in M_A$ ,  $\lambda \in M_B$ . For all  $a \in A$ :

$$\begin{aligned} ((\phi_1 \star \phi_2) \leftarrow \lambda)(a) &= (\phi_1 \otimes \phi_2 \otimes \lambda) \circ (\Delta_A \otimes Id) \circ \rho(a) \\ &= (\phi_1 \otimes \phi_2 \otimes \lambda)((a_0)^{(1)} \otimes (a_0)^{(2)} \otimes a_1) \\ &= (\phi_1 \otimes \phi_2 \otimes \lambda)((a^{(1)})_0 \otimes (a^{(2)})_0 \otimes (a^{(1)})_1 (a^{(2)})_1) \\ &= \phi_1((a^{(1)})_0) \lambda((a^{(1)})_1) \phi_2((a^{(2)})_0) \lambda((a^{(2)})_1) \\ &= (\phi_1 \leftarrow \lambda)(a^{(1)}) (\phi_2 \leftarrow \lambda)(a^{(2)}) \\ &= ((\phi_1 \leftarrow \lambda) \star (\phi_2 \leftarrow \lambda))(a). \end{aligned}$$

So  $\leftarrow$  is an action by monoid endomorphisms. □

**Example.** We take  $A = (\mathbb{K}[X], m, \Delta)$ ,  $B = (\mathbb{K}[X], m, \delta)$  and  $\rho = \delta$ . We consider the map:

$$ev : \begin{cases} \mathbb{K} & \rightarrow & \mathbb{K}[X]^* \\ \lambda & \rightarrow & \begin{cases} \mathbb{K}[X] & \rightarrow & \mathbb{K} \\ P(X) & \rightarrow & ev_\lambda(P) = P(\lambda). \end{cases} \end{cases}$$

Then  $ev$  is an isomorphism from  $(\mathbb{K}, +)$  to  $(M_A, \star)$  and from  $(\mathbb{K}, \cdot)$  to  $(M_B, *)$ . Moreover, for all  $\lambda, \mu \in \mathbb{K}$ :

$$ev_\lambda \leftarrow ev_\mu = ev_{\lambda\mu}.$$

**Proposition 4** *Let  $A$  and  $B$  be two bialgebras in cointeraction, through the coaction  $\rho$ .*

1. *Let  $H$  be a bialgebra. We denote by  $M_B$  the monoid of characters of  $B$  and by  $E_{A \rightarrow H}$  the set of bialgebra morphisms from  $A$  to  $H$ . Then  $M_B$  acts on  $E_{A \rightarrow H}$  via the map:*

$$\leftarrow : \begin{cases} E_{A \rightarrow H} \times M_B & \rightarrow & E_{A \rightarrow H} \\ (\phi, \lambda) & \rightarrow & \phi \leftarrow \lambda = (\phi \otimes \lambda) \circ \rho \end{cases}$$

2. *Let  $H_1$  and  $H_2$  be two bialgebras and let  $\theta : H_1 \rightarrow H_2$  be a bialgebra morphism. For all  $\phi \in E_{A \leftarrow H_1}$ , for all  $\lambda \in M_B$ , in  $E_{A \leftarrow H_2}$ :*

$$\theta \circ (\phi \leftarrow \lambda) = (\theta \circ \phi) \leftarrow \lambda.$$

3. *if  $\lambda, \mu \in M_B$ , in  $E_{A \rightarrow A}$ :*

$$(Id \leftarrow \lambda) \circ (Id \leftarrow \mu) = Id \leftarrow (\lambda * \mu).$$

*The following map is an injective monoid morphism:*

$$\begin{cases} (M_B, *) & \rightarrow & (E_{A \rightarrow A}, \circ) \\ \lambda & \rightarrow & Id \leftarrow \lambda. \end{cases}$$



**Proof.** 1. For all  $\phi \in E_{A \leftarrow B}$ ,  $\lambda \in M_B$ ,  $\phi \leftarrow \lambda : A \rightarrow H \otimes \mathbb{K} = H$ . As  $\phi$ ,  $\lambda$  and  $\rho$  are algebra morphisms, by composition  $\phi \leftarrow \lambda$  is an algebra morphism. Let  $a \in A$ .

$$\begin{aligned}
\Delta_H(\phi \leftarrow \lambda(a)) &= \Delta_H(\phi(a_0)\lambda(a_1)) \\
&= \lambda(a_1)\Delta_H \circ \phi(a_1) \\
&= \lambda(a_1)\phi(a_0)^{(1)} \otimes \phi(a_0)^{(2)} \\
&= \lambda(a_1)\phi((a_0)^{(1)}) \otimes \phi((a_0)^{(2)}) \\
&= \lambda((a^{(1)})_1(a^{(2)})_1)\phi((a^{(1)})_0) \otimes \phi((a^{(2)})_0) \\
&= \lambda((a^{(1)})_1)\lambda((a^{(2)})_1)\phi((a^{(1)})_0) \otimes \phi((a^{(2)})_0) \\
&= \phi((a^{(1)})_0)\lambda((a^{(1)})_1) \otimes \phi((a^{(2)})_0)\lambda((a^{(2)})_1) \\
&= \phi \leftarrow \lambda(a^{(1)}) \otimes \phi \leftarrow \lambda(a^{(2)}) \\
&= ((\phi \leftarrow \lambda) \otimes (\phi \leftarrow \lambda)) \circ \Delta_A(a).
\end{aligned}$$

So  $\phi \leftarrow \lambda \in E_{A \rightarrow H}$ .

Let  $\phi \in E_{A \rightarrow H}$ . For all  $a \in A$ ,  $\phi \leftarrow \eta \circ \varepsilon(a) = \phi(a_0)\varepsilon(a_1) = \phi(a)$ . Let  $\lambda, \mu \in M_B$ .

$$(\phi \leftarrow \lambda) \leftarrow \mu = (\phi \otimes \lambda \otimes \mu) \circ (\rho \otimes Id) \circ \rho = (\phi \otimes \lambda \otimes \mu) \circ (Id \otimes \Delta_B) \circ \rho = \phi \leftarrow (\lambda * \mu).$$

So  $\leftarrow$  is indeed an action of  $M_B$  on  $E_{A \rightarrow H}$ .

2. Let  $a \in H$ .

$$(\theta \circ \phi) \leftarrow \lambda(a) = \theta \circ \phi(a_1)\lambda(a_0) = \theta(\phi(a_1)\lambda(a_0)) = \theta(\phi \leftarrow \lambda(a)) = \theta \circ (\phi \leftarrow \lambda)(a).$$

So  $(\theta \circ \phi) \leftarrow \lambda = \theta \circ (\phi \leftarrow \lambda)$ .

3. Consequently, if  $\lambda, \mu \in M_B$ , in  $E_{A \rightarrow A}$ :

$$(Id \leftarrow \lambda) \circ (Id \leftarrow \lambda) = (Id \leftarrow \lambda) \leftarrow \mu = Id \leftarrow (\lambda * \mu).$$

If  $Id \leftarrow \lambda = Id$ , then, composing by  $\varepsilon'$ , we obtain  $\varepsilon' * \lambda = \varepsilon'$ , so  $\lambda = \varepsilon'$ . □

**Example.** We take  $A = (\mathbb{K}[X], m, \Delta)$ ,  $B = (\mathbb{K}[X], m, \delta)$  and  $\rho = \delta$ . In  $E_{A \rightarrow A}$ , for any  $\lambda \in \mathbb{K}$ :

$$Id \leftarrow ev_\lambda(X) = ev_\lambda(X)X = \lambda X,$$

so for any  $P \in \mathbb{K}[X]$ ,  $(Id \leftarrow ev_\lambda)(P) = P(\lambda X)$ . In this case, the monoids  $(M_B, *)$  and  $(E_{A \rightarrow A}, \circ)$  are isomorphic.

### 1.3 Polynomial morphisms

In this section, we deal with a family  $(A, m, \Delta, \delta)$  such that:

1.  $(A, m, \Delta)$  is a graded, connected Hopf algebra. As a graded algebra, it is isomorphic to the symmetric algebra  $S(V)$ , where  $V$  is a graded subspace of  $A$ .
2.  $(A, m, \delta)$  is a bialgebra.
3.  $(A, m, \Delta)$  and  $(A, m, \delta)$  are in cointeraction, through the coaction  $\delta$ .
4.  $V_1 = A_1$  has a basis  $(g_i)_{i \in I}$  such that:

$$\forall i \in I, \delta(g_i) = g_i \otimes g_i.$$

We shall denote by  $J$  the set of sequences  $\alpha = (\alpha_i)_{i \in I}$  with a finite support. For all  $\alpha \in J$ , we put  $g_\alpha = \prod_{i \in I} g_i^{\alpha_i}$ . These are group-like elements of  $(A, m, \delta)$ .

5. For all  $n \geq 2$ ,  $V_n$  can be decomposed as:

$$V_n = \bigoplus_{i \in I, \alpha \in J} V_n(g_i, g_\alpha),$$

such that for all  $x \in V_n(g_i, g_\alpha)$ :

$$\delta(x) - g_i \otimes x - x \otimes g_\alpha \in S(V_1 \oplus \dots \oplus V_{n-1})^{\otimes 2}.$$

The counit of  $(A, m, \Delta)$  will be denoted by  $\varepsilon$  and the counit of  $(A, m, \delta)$  by  $\varepsilon'$ . We denote by  $M_B$  the monoid of characters of  $(A, m, \delta)$ .

**Remark.** If  $x \in V_n(g_i, g_\alpha)$ , as  $\varepsilon'(g_i) = \varepsilon'(g_\alpha) = 1$ , necessarily,  $\varepsilon'(x) = 0$  and:

$$\delta(x) - g_i \otimes x - x \otimes g_\alpha \in \text{Ker}(\varepsilon')^{\otimes 2}.$$

**Lemma 5** *Let  $\lambda \in M_B$ . It has an inverse in  $M_B$  if, and only if, for all  $i \in I$ ,  $\lambda(g_i) \neq 0$ .*

**Proof.**  $\implies$ . Let  $\mu$  be the inverse of  $\lambda$  in  $M_B$ . For all  $i \in I$ ,  $\lambda * \mu(g_i) = \lambda(g_i)\mu(g_i) = \varepsilon'(g_i) = 1$ , so  $\lambda(g_i) \neq 0$ .

$\impliedby$ . We define two characters  $\mu, \nu \in M_B$  by inductively defining  $\mu_n = \mu|_{V_n}$  and  $\nu_n = \nu|_{V_n}$ . For  $n = 1$ , we put  $\mu_1(g_i) = \nu_1(g_i) = \lambda(g_i)^{-1}$ . Let us assume that  $g_1, \dots, g_{n-1}$  are already defined, with  $n \geq 2$ . If  $x \in V_n(g_i, g_\alpha)$ , we put:

$$\delta(x) - g_i \otimes x - x \otimes g_\alpha = \sum x'_k \otimes x''_k \in S(V_1 \oplus \dots \oplus V_{n-1})^{\otimes 2}.$$

Hence, for all  $k$ ,  $\mu(x'_k)$  and  $\nu(x''_k)$  are defined. We put:

$$\begin{aligned} \mu_n(x) &= \prod_{i \in I} \frac{1}{\lambda(g_i)^{\alpha_i}} \left( \varepsilon'(x) - \mu(g_i)\lambda(x) - \sum \mu(x'_k)\lambda(x''_k) \right), \\ \nu_n(x) &= \frac{1}{\lambda(g_i)} \left( \varepsilon'(x) - \lambda(x)\nu(g_\alpha) - \sum \lambda(x'_k)\nu(x''_k) \right). \end{aligned}$$

Consequently,  $\mu, \nu \in M_B$  and for all  $x \in V$ ,  $\mu * \lambda(x) = \lambda * \nu(x) = \varepsilon'(x)$ , so  $\mu * \lambda = \lambda * \nu = \varepsilon'$ , and finally  $\mu = \mu * (\lambda * \nu) = (\mu * \lambda) * \nu = \nu$ , so  $\lambda$  is invertible in  $M_B$ .  $\square$

**Lemma 6** *Let  $C$  be a graded, connected Hopf algebra. We denote by  $C_+ = \text{Ker}(\varepsilon_C)$  its augmentation ideal. Let  $\lambda : C_+ \rightarrow \mathbb{K}$  be any linear map. There exists a unique coalgebra morphism  $\phi : C \rightarrow \mathbb{K}[X]$  such that:*

$$\forall x \in C_+, \frac{d\phi(x)}{dX}(0) = \lambda(x).$$

Moreover:

1.  $\phi$  is homogeneous if, and only if, for all  $n \geq 2$ ,  $\lambda(C_n) = (0)$ .
2.  $\phi$  is a Hopf algebra morphism if, and only if, for all  $x, y \in C_+$ ,  $\lambda(xy) = 0$ .

**Proof.** Let  $\pi : \mathbb{K}[X] \rightarrow \text{Vect}(X)$  be the canonical projection. For any  $P \in \mathbb{K}[X]$ :

$$\pi(P) = \frac{dP}{dX}(0)X.$$

*Existence.* We define  $\phi|_{C_n}$  by induction on  $n$ . For  $n = 0$ , we put  $\phi(1) = 1$ . Let us assume that  $\phi|_{C_0 \oplus \dots \oplus C_{n-1}}$  is defined such that for all  $x \in C_0 \oplus \dots \oplus C_{n-1}$ ,

$$\Delta \circ \phi(x) = (\phi \otimes \phi) \circ \Delta(x), \quad \pi \circ \phi(x) = \lambda(x)X.$$

Let  $x \in C_n$ . As  $\tilde{\Delta}(x) \in (C_1 \oplus \dots \oplus C_{n-1})^{\otimes 2}$ , we can consider the element

$$y = (\phi \otimes \phi) \circ \tilde{\Delta}(x) \in \mathbb{K}[X]_+^{\otimes 2}.$$

We put:

$$y = \sum_{i,j \geq 1} a_{i,j} \frac{X^i}{i!} \otimes \frac{X^j}{j!}.$$

Moreover:

$$\begin{aligned} \tilde{\Delta} \otimes Id(y) &= ((\tilde{\Delta} \circ \phi) \otimes \phi) \circ \tilde{\Delta}(x) \\ &= (\phi \otimes \phi \otimes \phi) \circ (\tilde{\Delta} \otimes Id) \circ \tilde{\Delta}(x) \\ &= (\phi \otimes \phi \otimes \phi) \circ (Id \otimes \tilde{\Delta}) \circ \tilde{\Delta}(x) \\ &= (\phi \otimes (\tilde{\Delta} \circ \phi)) \circ \tilde{\Delta}(x) \\ &= (Id \otimes \tilde{\Delta})(y). \end{aligned}$$

Hence:

$$\sum_{i,j,k \geq 1} a_{i+j,k} \frac{X^i}{i!} \otimes \frac{X^j}{j!} \otimes \frac{X^k}{k!} = \sum_{i,j,k \geq 1} a_{ij+k} \frac{X^i}{i!} \otimes \frac{X^j}{j!} \otimes \frac{X^k}{k!}.$$

For all  $i, j, k \geq 1$ ,  $a_{i+j,k} = a_{i,j+k}$ , so there exist scalars  $a_n$  such that for all  $i, j \geq 1$ ,  $a_{i,j} = a_{i+j}$ . We obtain that:

$$y = \sum_{n \geq 2} a_n \left( \sum_{i,j \geq 1, i+j=n} \frac{X^i}{i!} \otimes \frac{X^j}{j!} \right) = \sum_{n \geq 2} a_n \tilde{\Delta} \left( \frac{X^n}{n!} \right).$$

We then put  $\phi(x) = \sum_{n \geq 2} a_n \frac{X^n}{n!} + \lambda(x)X$ .

We obtain in this way a coalgebra morphism such that  $\pi \circ \phi(x) = \lambda(x)X$  for all  $x \in C$ .

*Unicity.* Let  $\phi, \psi : C \rightarrow \mathbb{K}[X]$  be coalgebra morphisms such that  $\pi \circ \phi = \pi \circ \psi$ . Let us prove that  $\phi(x) = \psi(x)$  for all  $x \in A_n$ ,  $n \geq 0$ , by induction on  $n$ . For  $n = 0$ , as the unique group-like element of  $\mathbb{K}[X]$  is 1,  $\phi(1) = \psi(1) = 1$ . Let us assume the result at all ranks  $k < n$ . Then, if  $x \in A_n$ :

$$\tilde{\Delta} \circ \phi(x) = (\phi \otimes \phi) \circ \tilde{\Delta}(x) = (\psi \otimes \psi) \circ \tilde{\Delta}(x) = \tilde{\Delta} \circ \psi(x),$$

so  $\phi(x) - \psi(x) \in Ker(\tilde{\Delta}) = Vect(X)$ . Hence,  $0 = \pi \circ \phi(x) - \pi \circ \psi(x) = \phi(x) - \psi(x)$ .

1.  $\implies$ . If  $\phi$  is homogeneous, then, for all  $n \geq 2$ ,  $\phi(C_n) \subseteq Vect(X^n)$ , so:

$$\pi \circ \phi(C_n) = \lambda(C_n)X = (0).$$

1.  $\Leftarrow$ . Let us go back to the construction of  $\phi$  is the *Existence* part. If  $n \geq 1$ ,  $x \in C_1$ , then  $\phi(x) = \lambda(x)X$  is homogeneous of degree 1. If  $n \geq 2$ , then, by homogeneity,  $a_{i,j} = 0$  if  $i + j \neq n$ , so  $a_k = 0$  if  $n \neq k$ , and  $\phi(x) = a_n \frac{X^n}{n!}$ :  $\phi$  is homogeneous.

2.  $\implies$ . Let  $x, y \in C_+$ . Then  $\phi(x), \phi(y) \in \mathbb{K}[X]_+ = X\mathbb{K}[X]$ , so  $\phi(xy) = \phi(x)\phi(y) \in X^2\mathbb{K}[X]$  and  $\pi \circ \phi(xy) = \lambda(xy)X = 0$ .

2.  $\Leftarrow$ . Let us consider  $\phi_1 = m \circ (\phi \otimes \phi)$  and  $\phi_2 = \phi \circ m$ . By composition, they are both coalgebra morphisms from the graded bialgebra  $C \otimes C$  to  $\mathbb{K}[X]$ . Moreover, if  $x, y \in C_+$ :

$$\begin{aligned} \pi \circ \phi_1(1 \otimes y) &= \pi \circ \phi(y), & \pi \circ \phi_2(1 \otimes y) &= \pi \circ \phi(y), \\ \pi \circ \phi_1(x \otimes 1) &= \pi \circ \phi(x), & \pi \circ \phi_2(x \otimes 1) &= \pi \circ \phi(x), \\ \pi \circ \phi_1(x \otimes y) &= \pi(\phi(x)\phi(y)) = 0, & \pi \circ \phi_2(x \otimes y) &= \lambda(xy)X = 0. \end{aligned}$$

So, for all  $z \in (C \otimes C)_+ = (\mathbb{K}1 \otimes C_+) \oplus (C_+ \otimes \mathbb{K}1) \oplus (C_+ \otimes C_+)$ ,  $\pi \circ \phi_1(z) = \pi \circ \phi_2(z)$ . By the *Unicity* part,  $\phi_1 = \phi_2$ , so  $\phi$  is an algebra morphism.  $\square$

**Remark.** If  $x \in V_1$ ,  $\phi(x) = \lambda(x)X$ .

**Theorem 7** *Under the hypotheses 1–5, there exists a unique homogeneous Hopf algebra morphism  $\phi_0 : (A, m, \Delta) \rightarrow (\mathbb{K}[X], m, \Delta)$  such that:*

$$\forall x \in A_1, \phi_0(x) = \varepsilon'(x)X.$$

Moreover, there exists a unique character  $\lambda_0 \in M_B$ , invertible in  $M_B$ , such that:

$$\forall n \geq 0, \forall x \in A_n, \phi_0(x) = \lambda_0(x)X^n.$$

**Proof. Existence.** Let  $\lambda \in A^*$  such that  $\lambda(x) = \varepsilon'(x)$  if  $x \in A_1$  and  $\lambda(x) = 0$  if  $x \in A_n$ ,  $n \geq 2$ . We denote by  $\phi_0$  the unique coalgebra morphism such that  $\frac{d\phi_0(x)}{dX}(0) = \lambda(x)$  for all  $x \in A_+$ . By the first point of lemma 6,  $\phi_0$  is homogeneous. As  $A_+^2 \subseteq A_{\geq 2}$ , by the second point of lemma 6,  $\phi_0$  is an algebra morphism.

*Unicity.* If  $\phi$  is such a morphism, by the first point of lemma 6, for all  $x \in A_n$ ,  $n \geq 2$ ,  $\pi \circ \phi(x) = 0$ ; hence, for all  $x \in A_+$ ,  $\pi \circ \phi_0 = \pi \circ \phi$ . By the unicity in lemma 6,  $\phi = \phi_0$ .

By homogeneity of  $\phi_0$ , for all  $x \in A_n$ , there exists a unique scalar  $\lambda_0(x) \in \mathbb{K}$  such that:

$$\phi_0(x) = \lambda_0(x)X^n.$$

If  $x \in A_m$ ,  $y \in A_n$ ,  $xy \in A_{m+n}$  and then:

$$\phi(xy) = \lambda_0(xy)X^{n+m} = \phi(x)\phi(y) = \lambda_0(x)\lambda_0(y)X^{n+m},$$

so  $\lambda_0 \in M_B$ . For all  $i \in I$ ,  $\phi(g_i) = \varepsilon'(g_i)X = X$ , so  $\lambda_0(g_i) = 1$ . By lemma 5,  $\lambda_0$  is an invertible element of  $M_B$ .  $\square$

**Theorem 8** *Under the hypotheses 1–5, the following map is a bijection:*

$$\theta : \begin{cases} M_B & \longrightarrow E_{A \rightarrow \mathbb{K}[X]} \\ \lambda & \longrightarrow \phi_0 \leftarrow \lambda. \end{cases}$$

Moreover, if  $\phi = \theta(\lambda)$ , with  $\lambda \in M_B$ , then for all  $x \in V$ ,

$$\lambda(x) = \frac{d\phi(x)}{dX}(0).$$

**Proof.** Let  $\lambda \in M_B$ , and  $\phi = \phi_0 \leftarrow \lambda$ . For all  $i \in I$ ,  $\phi(g_i) = X\lambda(g_i)$ , so  $\frac{d\phi(g_i)}{dX}(0) = \lambda(g_i)$ . If  $n \geq 2$  and  $x \in V_n(g_i, g_\alpha)$ , we put:

$$\delta(x) = g_i \otimes x + \sum x'_k \otimes x''_k,$$

with for all  $k$ ,  $x'_k$  homogeneous of degree  $\geq 2$  or homogeneous of degree 1, with  $\varepsilon'(x'_k) = 0$ . For all  $k$ ,  $\pi \circ \phi_0(x'_k) = 0$ . We obtain:

$$\pi \circ \phi(x) = \pi \left( \lambda(x)X + \sum \phi_0(x'_k)\lambda(x''_k) \right) = \lambda(x)X.$$

By the unicity in lemma 6,  $\phi$  is injective. If  $\psi \in E_{A \rightarrow \mathbb{K}[X]}$ , we define  $\lambda \in M_B$  by:

$$\forall x \in V, \lambda(x) = \frac{d\psi(x)}{dX}(0).$$

We put  $\phi = \phi_0 \leftarrow \lambda$ . Then for all  $x \in V$ ,  $\frac{d\phi(x)}{dX}(0) = \frac{d\psi(x)}{dX}(0) = \lambda(x)$ . If  $x \in A_+^2$ , by the second point of lemma 6,  $\frac{d\phi(x)}{dX}(0) = \frac{d\psi(x)}{dX}(0) = 0$ . Finally, as  $V$  generates  $A$ ,  $A_+ = V + A_+^2$ , and for all  $x \in A_+$ ,  $\frac{d\phi(x)}{dX}(0) = \frac{d\psi(x)}{dX}(0)$ . By the unicity in lemma 6,  $\phi = \psi$ , so  $\theta$  is surjective.  $\square$

**Corollary 9** *Under the hypotheses 1–5, for any  $\mu \in M_B$ , there exists a unique Hopf algebra morphism  $\phi : A \rightarrow \mathbb{K}[X]$ , such that:*

$$\forall x \in A, \phi(x)(1) = \mu(1).$$

*This morphism is:  $\phi_0 \leftarrow (\lambda_0^{*-1} * \mu)$ .*

**Proof.** Let  $\phi$  be a Hopf algebra morphism from  $A$  to  $\mathbb{K}[X]$ . By theorem 8, there exists  $\lambda \in M_B$  such that  $\phi = \phi_0 \leftarrow \lambda$ . Let  $x \in A$ . We write  $x = \sum x'_k \otimes x''_k$ , with, for all  $k$ ,  $x'_k$  homogeneous of degree  $n_k$ . Then:

$$\phi(x)(1) = \left( \sum \lambda_0(x'_k)X^{n_k}\lambda(x''_k) \right)_{|X=1} = \sum \lambda_0(x'_k)\lambda(x''_k) = \lambda_0 * \lambda(x).$$

So  $\phi$  satisfies the required conditions if, and only if,  $\lambda_0 * \lambda = \mu$ , if, and only if,  $\lambda = \lambda_0^{*-1} * \mu$ , as  $\lambda_0$  is invertible. So such a  $\phi$  exists and is unique.  $\square$

**Corollary 10** *Under the hypotheses 1–5, there exists a unique morphism  $\phi_1 : A \rightarrow \mathbb{K}[X]$  such that:*

1.  $\phi_1$  is a Hopf algebra morphism from  $(A, m, \Delta)$  to  $(\mathbb{K}[X], m, \Delta)$ .
2.  $\phi_1$  is a bialgebra morphism from  $(A, m, \delta)$  to  $(\mathbb{K}[X], m, \delta)$ .

Moreover,  $\phi_1 = \phi_0 \leftarrow \lambda_0^{*-1}$  and, for all  $x \in A$ ,  $\phi_1(x)(1) = \varepsilon'(x)$ .

**Proof.** *Unicity.* If such a  $\phi_1$  exists, then for all  $x \in A$ ,  $\phi_1(x)(1) = \varepsilon' \circ \phi_1(x) = \varepsilon'(x)$ . By corollary 9,  $\phi_1 = \phi_0 \leftarrow (\lambda_0^{*-1} * \varepsilon') = \phi_0 \leftarrow \lambda_0^{*-1}$ .

*Existence.* Let  $\phi_1$  be the unique Hopf algebra morphism in  $E_{A \rightarrow \mathbb{K}[X]}$  such that for all  $x \in A$ ,  $\phi_1(x)(1) = \varepsilon'(x)$ . Recall that we identify  $\mathbb{K}[X] \otimes \mathbb{K}[X]$  with  $\mathbb{K}[X, Y]$ ; for all  $P \in \mathbb{K}[X]$ :

$$\Delta(P)(X, Y) = P(X + Y), \quad \delta(P)(X, Y) = P(XY).$$

Let us fix  $x \in A$ . Let us prove that for all  $k, l \in \mathbb{N}^*$ ,  $\delta \circ \phi_1(x)(k, l) = (\phi_1 \otimes \phi_1) \circ \Delta(x)(k, l)$  by induction on  $k$ . First, observe that  $\delta \circ \phi_1(x)(k, l) = \phi_1(x)(kl)$ . If  $k = 1$ , then:

$$(\phi_1 \otimes \phi_1) \circ \Delta(x)(1, l) = \phi_1((\varepsilon' \otimes Id) \circ \Delta(x))(l) = \phi_1(x)(l).$$

Let us assume the result at rank  $k$ . Then:

$$\begin{aligned}
(\phi_1 \otimes \phi_1) \circ \Delta(x)(k+1, l) &= (\Delta \otimes Id) \circ (\phi_1 \otimes \phi_1) \circ \delta(x)(k, 1, l) \\
&= (\phi_1 \otimes \phi_1 \otimes \phi_1) \circ (\Delta \otimes Id) \circ \delta(x)(k, 1, l) \\
&= (\phi_1 \otimes \phi_1 \otimes \phi_1) \circ m_{2,4}^3 \circ (\delta \otimes \delta) \circ \Delta(x)(k, 1, l) \\
&= (\phi_1 \otimes \phi_1 \otimes \phi_1 \otimes \phi_1) \circ (\delta \otimes \delta) \circ \Delta(x)(k, l, 1, l) \\
&= (\phi_1 \otimes \phi_1) \circ \Delta(x)(kl, l) \\
&= \Delta \circ \phi_1(x)(kl, l) \\
&= \phi_1(x)(kl + l) \\
&= \phi_1(x)((k+1)l).
\end{aligned}$$

So the result is true for all  $k, l \geq 1$ . Hence,  $(\phi_1 \otimes \phi_1) \circ \delta(x) = \delta \circ \phi_1(x)$ :  $\phi_1$  is a bialgebra morphism from  $(A, m, \delta)$  to  $(\mathbb{K}[X], m, \delta)$ .  $\square$

## 2 Examples from quasi-posets

### 2.1 Definition

**Definition 11** 1. Let  $A$  be a set finite set. A quasi-order on  $A$  is a transitive, reflexive relation  $\leq$  on  $A$ . If  $\leq$  is a quasi-order on  $A$ , we shall say that  $(A, \leq)$  is a quasi-poset. If  $P$  is a quasi-poset:

(a) Its isoclass is denoted by  $[P]$ .

(b)  $\sim_P$  is defined by:

$$\forall a, b \in A, a \sim_P b \text{ if } (a \leq b \text{ and } b \leq a).$$

It is an equivalence on  $A$ .

(c)  $\bar{A} = A / \sim_P$  is given an order by:

$$\forall a, b \in A, \bar{a} \leq \bar{b} \text{ if } a \leq b.$$

The poset  $(\bar{A}, \leq)$  is denoted by  $\bar{P}$ .

(d) The cardinality of  $\bar{P}$  is denoted by  $cl(P)$ .

2. Let  $n \in \mathbb{N}$ .

(a) The set of quasi-posets which underlying set is  $[n] = \{1, \dots, n\}$  is denoted by  $\mathbf{QP}(n)$ .

(b) The set of posets which underlying set is  $[n]$  is denoted by  $\mathbf{P}(n)$ .

(c) The set of isoclasses of quasi-posets of cardinality  $n$  is denoted by  $\mathbf{qp}(n)$ .

(d) The set of isoclasses of quasi-posets of cardinality  $n$  is denoted by  $\mathbf{p}(n)$ .

We put:

$$\begin{aligned}
\mathbf{QP} &= \bigsqcup_{n \geq 0} \mathbf{QP}(n), & \mathbf{P} &= \bigsqcup_{n \geq 0} \mathbf{P}(n), & \mathbf{qp} &= \bigsqcup_{n \geq 0} \mathbf{qp}(n), & \mathbf{p} &= \bigsqcup_{n \geq 0} \mathbf{p}(n), \\
\mathcal{H}_{\mathbf{QP}} &= \text{Vect}(\mathbf{QP}), & \mathcal{H}_{\mathbf{P}} &= \text{Vect}(\mathbf{P}), & \mathcal{H}_{\mathbf{qp}} &= \text{Vect}(\mathbf{qp}) & \mathcal{H}_{\mathbf{p}} &= \text{Vect}(\mathbf{p}).
\end{aligned}$$

As posets are quasi-posets, there are canonical injections from  $\mathcal{H}_{\mathbf{P}}$  into  $\mathcal{H}_{\mathbf{QP}}$  and from  $\mathcal{H}_{\mathbf{p}}$  into  $\mathcal{H}_{\mathbf{qp}}$ . Moreover, the map  $P \rightarrow \bar{P}$  induces surjective maps from  $\mathcal{H}_{\mathbf{QP}}$  to  $\mathcal{H}_{\mathbf{P}}$  and from

$\mathcal{H}_{\mathbf{QP}}$  to  $\mathcal{H}_{\mathbf{P}}$ , both denoted by  $\xi$ . The map  $P \rightarrow [P]$  induces maps  $\sqcup : \mathcal{H}_{\mathbf{QP}} \rightarrow \mathcal{H}_{\mathbf{QP}}$  and  $\sqcup : \mathcal{H}_{\mathbf{P}} \rightarrow \mathcal{H}_{\mathbf{P}}$ . The following diagram commutes:

$$\begin{array}{ccc}
\mathcal{H}_{\mathbf{QP}} & \xrightarrow{\sqcup} & \mathcal{H}_{\mathbf{QP}} \\
\xrightarrow{\xi} & & \xrightarrow{\xi} \\
& \searrow & \swarrow \\
& \mathcal{H}_{\mathbf{P}} & \xrightarrow{\sqcup} & \mathcal{H}_{\mathbf{P}} \\
& \swarrow & & \swarrow \\
\mathcal{H}_{\mathbf{P}} & \xrightarrow{\sqcup} & \mathcal{H}_{\mathbf{P}} \\
& \nearrow & \nearrow \\
& \mathcal{H}_{\mathbf{P}} & \xrightarrow{\sqcup} & \mathcal{H}_{\mathbf{P}} \\
& \nearrow & \nearrow \\
& \mathcal{H}_{\mathbf{P}} & \xrightarrow{\sqcup} & \mathcal{H}_{\mathbf{P}}
\end{array} \tag{1}$$

We shall represent any element  $P$  of  $\mathbf{QP}$  by the Hasse graph of  $\overline{P}$ , indicating on the vertices the elements of the corresponding equivalence class. For example, the elements of  $\mathbf{QP}(n)$ ,  $n \leq 3$ , are:

$$\begin{aligned}
& 1; \quad \cdot_1; \quad \cdot_1 \cdot_2, \mathbf{1}_1^2, \mathbf{1}_2^1, \cdot_1 \cdot_2; \quad \cdot_1 \cdot_2 \cdot_3, \cdot_1 \mathbf{1}_2^3, \cdot_1 \mathbf{1}_3^2, \cdot_2 \mathbf{1}_1^3, \cdot_2 \mathbf{1}_3^1, \cdot_3 \mathbf{1}_1^2, \cdot_3 \mathbf{1}_2^1, \cdot_1 \cdot_2 \cdot_3, \cdot_2 \cdot_1 \cdot_3, \cdot_3 \cdot_1 \cdot_2, \\
& {}^2\mathbf{V}_1^3, {}^1\mathbf{V}_2^3, {}^1\mathbf{V}_3^2, {}^2\mathbf{\Lambda}_3^1, {}^1\mathbf{\Lambda}_3^2, {}^1\mathbf{\Lambda}_3^3, \mathbf{1}_1^3, \mathbf{1}_2^3, \mathbf{1}_3^1, \mathbf{1}_3^2, \mathbf{1}_3^3, \mathbf{1}_1^2, \mathbf{1}_2^1, \mathbf{1}_2^2, \mathbf{1}_1^3, \mathbf{1}_2^3, \mathbf{1}_3^1, \mathbf{1}_3^2, \mathbf{1}_3^3, \mathbf{1}_1^2, \mathbf{1}_2^1, \mathbf{1}_2^2, \mathbf{1}_1^3, \mathbf{1}_2^3, \mathbf{1}_3^1, \mathbf{1}_3^2, \mathbf{1}_3^3, \cdot_1 \cdot_2 \cdot_3.
\end{aligned}$$

We shall represent any element  $P \in \mathbf{qp}$  by the Hasse graph of  $\overline{P}$ , indicating on the vertices the cardinality of the corresponding equivalence class, if this cardinality is not equal to 1. For example, the elements of  $\mathbf{qp}(n)$ ,  $n \leq 3$ , are:

$$1; \quad \cdot; \quad \dots, \mathbf{1}, \cdot_2; \quad \dots, \cdot \mathbf{1}, \cdot_2; \quad \mathbf{V}, \mathbf{\Lambda}, \mathbf{1}, \mathbf{1}^2, \mathbf{1}_2, \cdot_3.$$

## 2.2 First coproduct

By Alexandroff's theorem [1, 18], finite quasi-posets are in bijection with finite topological spaces. Let us recall the definition of the topology attached to a quasi-poset.

**Definition 12** 1. Let  $P = (A, \leq)$  be a quasi-poset. An open set of  $P$  is a subset  $O$  of  $A$  such that:

$$\forall i, j \in A, (i \in O \text{ and } i \leq j) \implies (j \in O).$$

The set of open sets of  $P$  (the topology associated to  $P$ ) is denoted by  $\text{top}(P)$ .

2. Let  $P = (A, \leq)$  be a quasi-poset and  $B \subseteq A$ . We denote by  $P|_B$  the quasi-poset  $(B, \leq|_B)$ .
3. Let  $P = (A, \leq_P)$  be a quasi-poset. We assume that  $A$  is also given a total order  $\leq$ : for example,  $A$  is a subset of  $\mathbb{N}$ . If the cardinality of  $A$  is  $n$ , there exists a unique increasing bijection  $f$  from  $[n]$ , with its usual order, to  $(A, \leq)$ . We denote by  $\text{Std}(P)$  the quasi-poset, element of  $\mathbf{QP}(n)$ , defined by:

$$\forall i, j \in [n], i \leq_{\text{Std}(P)} j \iff f(i) \leq_P f(j).$$

**Proposition 13** 1. We define a product  $m$  on  $\mathcal{H}_{\mathbf{QP}}$  in the following way: if  $P \in \mathbf{QP}(k)$ ,  $Q \in \mathbf{QP}(l)$ , then  $PQ = m(P, Q) \in \mathbf{QP}(k+l)$  and

$$\begin{aligned}
\forall i, j \in [k+l], i \leq_{PQ} &\iff (1 \leq i, j \leq k \text{ and } i \leq_P j) \\
&\text{or } (k+1 \leq i, j \leq k+l \text{ and } i-k \leq_Q j-k).
\end{aligned}$$

2. We define a second product  $\downarrow$  on  $\mathcal{H}_{\mathbf{QP}}$  in the following way: if  $P \in \mathbf{QP}(k)$ ,  $Q \in \mathbf{QP}(l)$ , then  $PQ = m(P, Q) \in \mathbf{QP}(k+l)$  and

$$\begin{aligned}
\forall i, j \in [k+l], i \leq_{PQ} &\iff (1 \leq i, j \leq k \text{ and } i \leq_P j) \\
&\text{or } (k+1 \leq i, j \leq k+l \text{ and } i-k \leq_Q j-k) \\
&\text{or } (1 \leq i \leq k < j \leq k+l).
\end{aligned}$$

3. We define a coproduct  $\Delta$  on  $\mathcal{H}_{\mathbf{QP}}$  in the following way:

$$\forall P \in \mathbf{QP}(n), \Delta(P) = \sum_{O \in \text{top}(P)} \text{Std}(P|_{[n] \setminus O}) \otimes \text{Std}(P|_O).$$

Then  $(\mathcal{H}_{\mathbf{QP}}, m, \Delta)$  is a non-commutative, non-cocommutative Hopf algebra, and  $(\mathcal{H}_{\mathbf{QP}}, \downarrow, \Delta)$  is an infinitesimal bialgebra.

**Proof.** See [10, 11]. □

**Examples.** If  $\{a, b\} = \{1, 2\}$  and  $\{i, j, k\} = \{1, 2, 3\}$ :

$$\begin{aligned} \Delta(\bullet_1) &= \bullet_1 \otimes 1 + 1 \otimes \bullet_1, \\ \Delta(\mathbf{1}_a^b) &= \mathbf{1}_a^b \otimes 1 + 1 \otimes \mathbf{1}_a^b + \bullet_a \otimes \bullet_b, \\ \Delta({}^j\mathbf{V}_i^k) &= {}^j\mathbf{V}_i^k \otimes 1 + 1 \otimes {}^j\mathbf{V}_i^k + \mathbf{1}_i^j \otimes \bullet_k + \mathbf{1}_i^k \otimes \bullet_j + \bullet_i \otimes \bullet_{j \bullet k}, \\ \Delta({}_j\mathbf{A}_k^i) &= {}_j\mathbf{A}_k^i \otimes 1 + 1 \otimes {}_j\mathbf{A}_k^i + \bullet_j \otimes \mathbf{1}_k^i + \bullet_k \otimes \mathbf{1}_j^i + \bullet_{j \bullet k} \otimes \bullet_i, \\ \Delta(\mathbf{1}_i^j) &= \mathbf{1}_i^j \otimes 1 + 1 \otimes \mathbf{1}_i^j + \bullet_i \otimes \mathbf{1}_j^i + \mathbf{1}_i^j \otimes \bullet_k. \end{aligned}$$

**Remark.** This Hopf algebraic structure is compatible with the morphisms of (1), that is to say:

1.  $\mathcal{H}_{\mathbf{P}}$  is a Hopf subalgebra of  $\mathcal{H}_{\mathbf{QP}}$ .

2. observe that:

- If  $(P_1, P_2)$  and  $(Q_1, Q_2)$  are pairs of isomorphic quasi-posets, then  $P_1 Q_1$  and  $P_2 Q_2$  are isomorphic.
- If  $P_1$  and  $P_2$  are isomorphic quasi-posets of  $\mathbf{QP}(n)$ , and if  $\phi : [n] \rightarrow [n]$  is an isomorphism from  $P_1$  to  $P_2$ , then the topology associated to  $P_2$  is the image by  $\phi$  of the topology associated to  $P_1$  and for any subset  $I$  of  $P_1$ ,  $\phi|_I$  is an isomorphism from  $(P_1)|_I$  to  $(P_2)|_{\phi(I)}$ .

Consequently, the surjective map  $\llbracket \cdot \rrbracket : \mathcal{H}_{\mathbf{QP}} \rightarrow \mathcal{H}_{\mathbf{qp}}$  is compatible with the product and the coproduct:  $\mathcal{H}_{\mathbf{qp}}$  inherits a Hopf algebra structure. Its product is the disjoint union of quasi-posets. For any quasi-poset  $P = (A, \leq_P)$ :

$$\Delta(\llbracket P \rrbracket) = \sum_{O \in \text{top}(P)} \llbracket P|_{A \setminus O} \rrbracket \otimes \llbracket P|_O \rrbracket.$$

3.  $\mathcal{H}_{\mathbf{p}}$  is a Hopf subalgebra of  $\mathcal{H}_{\mathbf{qp}}$ .

4. All the morphisms in (1) are Hopf algebra morphisms.

**Definition 14** 1. We shall say that a finite quasi-poset  $P = (A, \leq_P)$  is connected if its associated topology is connected.

2. For any finite quasi-poset  $P$ , we denote by  $cc(P)$  the number of connected components of its associated topology.

It is well-known that  $P$  is connected if, and only if, the Hasse graph of  $\overline{P}$  is connected. Any quasi-poset  $P$  can be decomposed as the disjoint union of its connected components; in an algebraic setting,  $\mathcal{H}_{\mathbf{qp}}$  is generated as a polynomial algebra by the connected quasi-posets. This is not true in  $\mathcal{H}_{\mathbf{QP}}$ : for example,  $\mathbf{1}_1^2$  is both not connected and indecomposable in  $\mathcal{H}_{\mathbf{QP}}$ .



### 2.3 Second coproduct

**Definition 15** Let  $P = (A, \leq_P)$  be a quasi-poset and let  $\sim$  be an equivalence on  $A$ .

1. We define a second quasi-order  $\leq_{P|\sim}$  on  $A$  by the relation:

$$\forall x, y \in A, x \leq_{P|\sim} y \text{ if } (x \leq_P y \text{ and } x \sim y).$$

2. We define a third quasi-order  $\leq_{P/\sim}$  on  $A$  as the transitive closure of the relation  $\mathcal{R}$  defined by:

$$\forall x, y \in A, x \mathcal{R} y \text{ if } (x \leq_P y \text{ or } x \sim y).$$

3. We shall say that  $\sim$  is  $P$ -compatible and we shall denote  $\sim \triangleleft P$  if the two following conditions are satisfied:

- The restriction of  $P$  to any equivalence class of  $\sim$  is connected.
- The equivalences  $\sim_{P/\sim}$  and  $\sim$  are equal. In other words:

$$\forall x, y \in A, (x \leq_{P/\sim} y \text{ and } y \leq_{P/\sim} x) \implies x \sim y;$$

note that the converse assertion trivially holds.

**Remarks.**

1.  $P|\sim$  is the disjoint union of the restriction of  $\leq_P$  to the equivalence classes of  $\sim$ .

2. Let  $x, y \in P$ . Then  $x \leq_{P/\sim} y$  if there exist  $x_1, x'_1, \dots, x_k, x'_k \in A$  such that:

$$x \leq_P x_1 \sim x'_1 \leq_P \dots \leq_P x_k \sim x'_k \leq_P y.$$

3. If  $\sim \triangleleft P$ , then:

- (a) The equivalence classes of  $\sim_{P/\sim}$  are the equivalence classes of  $\sim$  and are included in a connected component of  $P$ . This implies that the connected components of  $P/\sim$  are the connected components of  $P$ . Consequently:

$$cl(P/\sim) = cl(\sim), \quad cc(P/\sim) = cc(P), \quad (2)$$

where  $cl(\sim)$  is the number of equivalence classes of  $\sim$ .

- (b) If  $x \sim_P y$  and  $x \sim y$ , then  $x \sim_{P|\sim} y$ : the equivalence classes of  $\sim_{P|\sim}$  are the equivalence classes of  $\sim_P$ ; the connected components of  $P|\sim$  are the equivalence classes of  $\sim$ . Consequently:

$$cl(P|\sim) = cl(P), \quad cc(P|\sim) = cl(\sim). \quad (3)$$

**Definition 16** Let  $P \in \mathbf{QP}$ . We shall say that  $P$  is discrete if  $[\overline{P}] = \cdot^{cl(P)}$ .

In other words,  $P$  is discrete if, and only if,  $\sim_P = \leq_P$ .

**Definition 17** We define a second coproduct  $\delta$  on  $\mathcal{H}_{\mathbf{QP}}$  in the following way: for all  $P \in \mathbf{QP}$ ,

$$\delta(P) = \sum_{\sim \triangleleft P} (P/\sim) \otimes (P|\sim).$$

Then  $(\mathcal{H}_{\mathbf{QP}}, m, \delta)$  is a bialgebra. Its counit  $\varepsilon'$  is given by:

$$\forall P \in \mathbf{QP}, \varepsilon'(P) = \begin{cases} 1 & \text{if } P \text{ is discrete,} \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** Firstly, let us prove the compatibility of  $\delta$  and  $m$ . Let  $P = (A, \leq_P)$  and  $Q = (B, \leq_Q)$  be two elements of  $\mathbf{QP}$ . Let  $\sim$  be an equivalence relation on  $P$ . We denote by  $\sim'$  and  $\sim''$  the restriction of  $\sim$  to  $P$  and  $Q$ . Then:

- If  $\sim \triangleleft PQ$ , then as the equivalence classes of  $\sim$  are connected, they are included in  $A$  or in  $B$ . Consequently, if  $x \in A$  and  $y \in B$ ,  $x$  and  $y$  are not equivalent for  $\sim$ . Moreover,  $\sim' \triangleleft P$  and  $\sim'' \triangleleft Q$ , and:

$$PQ| \sim = (P| \sim')(Q| \sim''), \quad PQ/ \sim = (P/ \sim')(Q/ \sim'').$$

- Conversely, if  $\sim' \triangleleft P, \sim'' \triangleleft Q$  and for all  $x \in A, y \in B$ ,  $x$  and  $y$  are not  $\sim$ -equivalent, then  $\sim \triangleleft PQ$ .

Hence:

$$\begin{aligned} \delta(PQ) &= \sum_{\sim \triangleleft PQ} (PQ/ \sim) \otimes (PQ| \sim) \\ &= \sum_{\sim' \triangleleft P, \sim'' \triangleleft Q} (P/ \sim')(Q/ \sim'') \otimes (P| \sim')(Q| \sim'') \\ &= \delta(P)\delta(Q). \end{aligned}$$

Let us now prove the coassociativity of  $\delta$ . Let  $P \in \mathbf{QP}$ .

*First step.* We put:

$$A = \{(r, r') \mid r \triangleleft P, r' \triangleleft P/r\}, \quad B = \{(s, s') \mid s \triangleleft P, s' \triangleleft P|s\}.$$

We consider the maps:

$$F : \begin{cases} A & \longrightarrow B \\ (r, r') & \longrightarrow (r', r), \end{cases} \quad G : \begin{cases} B & \longrightarrow A \\ (s, s') & \longrightarrow (s', s). \end{cases}$$

$F$  is well-defined: we put  $(s, s') = (r', r)$ . The equivalence classes of  $s$  are the equivalence classes of  $r'$ , so are  $P$ -connected. If  $x \sim_{P/s} y$ , there exist  $x_1, x'_1, \dots, x_k, s'_k$  and  $y_1, y'_1, \dots, y_l, y_l$  such that:

$$x \leq_P x_1 r' x'_1 \leq_P \dots \leq_P x_k r' x'_k \leq_P y, \quad y \leq_P y_1 r' y'_1 \leq_P \dots \leq_P y_l r' y'_l \leq_P x.$$

Hence:

$$x \leq_{P/r} x_1 r' x'_1 \leq_{P/r} \dots \leq_{P/r} x_k r' x'_k \leq_{P/r} y, \quad y \leq_{P/r} y_1 r' y'_1 \leq_{P/r} \dots \leq_{P/r} y_l r' y'_l \leq_{P/r} x.$$

So  $x \sim_{P/r} y$ . As  $r' \triangleleft P/r, x \sim_P y: s \triangleleft P$ .

Let us assume that  $xs'y$ . Then  $xry$ , so, as  $r \triangleleft y$ , there exists a path from  $x$  to  $y$  in the Hasse graph of  $P$ , made of vertices all  $r$ -equivalent to  $x$  and  $y$ . If  $x'$  and  $y'$  are two elements of this path, Then  $x'r'y'$ , so  $x' \leq_{G/r} y'$  and finally  $x' \leq_{(P/r)/r'} y'$ . As  $r' \triangleleft P/r, x'r'y'$ , so  $xsy$ . So the elements of this path are all  $P|s$ -equivalent: the equivalence classes of  $s'$  are  $P|s$ -connected.

Let us assume that  $x \sim_{(P|s)/s'} y$ . There exist  $x_1, x'_1, \dots, x_k, x'_k$  and  $y_1, y'_1, \dots, y_l, y'_l$  such that:

$$x \leq_{P|r'} x_1 r x'_1 \leq_{P|r'} \dots \leq_{P|r'} x_k r x'_k \leq_{P|r'} y, \quad y \leq_{P|r'} y_1 r y'_1 \leq_{P|r'} \dots \leq_{P|r'} y_l r y'_l \leq_{P|r'} x.$$

Then:

$$x \leq_P x_1 r x'_1 \leq_P \dots \leq_P x_k r x'_k \leq_P y, \quad y \leq_P y_1 r y'_1 \leq_P \dots \leq_P y_l r y'_l \leq_P x,$$

So  $x \leq_{P/r} y$  and  $y \leq_{P/r} x$ . As  $r \triangleleft P, xry$ , so  $xs'y$ : we obtain that  $s' \triangleleft P|s$ .

$G$  is well-defined: let  $(s, s') \in B$  and let us put  $G(s, s') = (r, r')$ . The equivalence classes of  $r$  are  $P|s$ -connected, so are  $P$ -connected. Let us assume that  $x \sim_{P/r} y$ . There exist  $x_1, x'_1, \dots, x_k, x'_k$  and  $y_1, y'_1, \dots, y_l, y'_l$  such that:

$$x \leq_P x_1 s' x'_1 \leq_P \dots \leq_P x_k s' x'_k \leq_P y, \quad y \leq_P y_1 s' y'_1 \leq_P \dots \leq_P y_l s' y'_l \leq_P x.$$

As the equivalence classes of  $s'$  are  $P|s$ -connected, all this elements are in the same connected component of  $P|s$ , so are  $s$ -equivalent:

$$x \leq_{P|s} x_1 s' x'_1 \leq_{P|s} \dots \leq_{P|s} x_k s' x'_k \leq_{P|s} y, \quad y \leq_{P|s} y_1 s' y'_1 \leq_{P|s} \dots \leq_{P|s} y_l s' y'_l \leq_{P|s} x.$$

Hence,  $x \sim_{(P|s)/s'} y$ , so as  $s' \triangleleft P | s$ ,  $x s' y$ , so  $x r y$ :  $r \triangleleft P$ .

The equivalence classes of  $r'$  are the equivalence classes of  $s$ , so are  $P$ -connected and therefore  $P/r$ -connected. Let us assume that  $x \sim_{(P/r)/r'} y$ . Note that if  $x' s' y'$ , then  $x'$  and  $y'$  are in the same connected component of  $P|s$ , so  $x' s y'$ . By the definition of  $\leq_{P/s'}$  as a transitive closure, using this observation, we obtain:

$$x \leq_P x_1 s x'_1 \leq_P \dots \leq_P x_k s x'_k \leq_P y, \quad y \leq_P y_1 s y'_1 \leq_P \dots \leq_P y_l s y'_l \leq_P x.$$

So  $x \sim_{P/s} y$ . As  $s \triangleleft P$ ,  $x s y$ , so  $x r' y$ :  $r' \triangleleft P/r$ .

Clearly,  $F$  and  $G$  are inverse bijections.

*Second step.* Let  $(r, r') \in A$  and let  $F(r, r') = (s, s')$ . Note that if  $x r y$ , then  $x / \sim_{P/r} y$ , so  $x / \sim_{(P/r)/r'} y$ , so  $x r' y$  as  $r' \triangleleft P/r$ . Then:

$$\begin{aligned} \leq_{(P/r)/r'} &= \text{transitive closure of } ((x r' y) \text{ or } (x \leq_{P/r} y)) \\ &= \text{transitive closure of } ((x r' y) \text{ or } (x \leq_P y) \text{ or } (x \leq_r y)) \\ &= \text{transitive closure of } ((x r' y) \text{ or } (x \leq_P y)) \\ &= \text{transitive closure of } ((x s y) \text{ or } (x \leq_P y)) \\ &= \leq_{P/s}. \end{aligned}$$

So  $P/s = (P/r)/r'$ .

$$\begin{aligned} \leq_{(P|s)/s'} &= \text{transitive closure of } ((x s' y) \text{ or } (x \leq_{P|s} y)) \\ &= \text{transitive closure of } ((x r y) \text{ or } (x \leq_{P|r'} y)) \\ &= \text{transitive closure of } ((x r y) \text{ or } ((x \leq_P y) \text{ and } (x r' y))) \\ &= \text{transitive closure of } (((x r y) \text{ or } (x \leq_P y)) \text{ and } ((s r y) \text{ or } (x r' y))) \\ &= \text{transitive closure of } ((x \leq_{P/r} y) \text{ and } (s r' y)) \\ &= \leq_{(P/r)|r'}. \end{aligned}$$

So  $(P|s)/s' = (P/r)|r'$ . For all  $x, y$ :

$$\begin{aligned} x \leq_{(P|s)/s'} y &\iff (x \leq_{P|s} y) \text{ and } (x s' y) \\ &\iff (x \leq_P y) \text{ and } x s y \text{ and } (x s' y) \\ &\iff (x \leq_P y) \text{ and } x r' y \text{ and } (x r y) \\ &\iff (x \leq_P y) \text{ and } (x r y) \\ &\iff x \leq_{P|r} y. \end{aligned}$$

So  $(P|s)|s' = P|r$ . Finally:

$$\begin{aligned} (\delta \otimes Id) \circ \delta(P) &= \sum_{(r, r') \in A} (P/r)/r' \otimes (P/r)|r' \otimes P|r \\ &= \sum_{(s, s') \in B} P/s \otimes (P|s)/s' \otimes (P|s)|s' \\ &= (Id \otimes \delta) \circ \delta(P). \end{aligned}$$

So  $\mathcal{H}_{\mathbf{QP}}$  is a bialgebra.

Let  $P$  be a quasi-poset. If  $P$  is discrete, then  $\delta(P) = P \otimes P$ , so  $(\varepsilon' \otimes Id) \circ \delta(P) = (Id \otimes \varepsilon') \circ \Delta(P) = P$ . If  $P$  is not discrete, there are three types of relations  $\sim \triangleleft P$ :

1. The equivalence classes of  $\sim$  are the connected components of  $P$ : in this case,  $P| \sim = P$  and  $P/ \sim = P_1$  is discrete.
2.  $\sim = \sim_P$ : in this case,  $P/ \sim = P$  and  $P| \sim = P_2$  is discrete.
3.  $\sim$  is not one of two preceding relations: in this case, nor  $P/ \sim$ , nor  $P| \sim$  is discrete.

So:

$$\delta(P) - P_1 \otimes P - P \otimes P_2 \in Ker(\varepsilon') \otimes Ker(\varepsilon'),$$

which implies that  $(\varepsilon' \otimes Id) \circ \delta(P) = (Id \otimes \varepsilon') \circ \Delta(P) = P$ . □

**Examples.** If  $\{a, b\} = \{1, 2\}$  and  $\{i, j, k\} = \{1, 2, 3\}$ :

$$\begin{aligned} \delta(\cdot_1) &= \cdot_1 \otimes \cdot_1, \\ \delta(\mathbf{1}_a^b) &= \mathbf{1}_a^b \otimes \cdot_{a \cdot b} + \cdot_{a, b} \otimes \mathbf{1}_a^b, \\ \delta({}^j \mathbf{V}_i^k) &= {}^j \mathbf{V}_i^k \otimes \cdot_{i \cdot j \cdot k} + \mathbf{1}_{i, j}^k \otimes \mathbf{1}_i^j \cdot_k + \mathbf{1}_{i, k}^j \otimes \mathbf{1}_i^k \cdot_j + \cdot_{i, j, k} \otimes {}^j \mathbf{V}_i^k, \\ \delta({}_j \mathbf{\Lambda}_k^i) &= {}_j \mathbf{\Lambda}_k^i \otimes \cdot_{i \cdot j \cdot k} + \mathbf{1}_k^{i, j} \otimes \mathbf{1}_j^i \cdot_k + \mathbf{1}_j^{i, k} \otimes \mathbf{1}_k^i \cdot_j + \cdot_{i, j, k} \otimes {}_j \mathbf{\Lambda}_k^i, \\ \delta(\mathbf{!}_i^k) &= \mathbf{!}_i^k \otimes \cdot_{i \cdot j \cdot k} + \mathbf{1}_{i, j}^k \otimes \mathbf{!}_i^j \cdot_k + \mathbf{!}_i^{j, k} \otimes \cdot_i \mathbf{!}_j^k + \cdot_{i, j, k} \otimes \mathbf{!}_i^k. \end{aligned}$$

**Remarks.**

1.  $\delta$  is the internal coproduct of [8].
2.  $(\mathcal{H}_{\mathbf{QP}}, m, \delta)$  is not a Hopf algebra: for all  $n \geq 1$ ,  $\delta(\cdot_n) = \cdot_n \otimes \cdot_n$ , and  $\cdot_n$  has no inverse in  $\mathcal{H}_{\mathbf{QP}}$ .
3. This coproduct is also compatible with the map  $\llbracket \cdot \rrbracket$ , so we obtain a bialgebra structure on  $\mathcal{H}_{\mathbf{qp}}$  with the coproduct defined by:

$$\delta(\llbracket P \rrbracket) = \sum_{\sim \triangleleft P} \llbracket P/ \sim \rrbracket \otimes \llbracket P| \sim \rrbracket.$$

4.  $\mathcal{H}_{\mathbf{P}}$  and  $\mathcal{H}_{\mathbf{p}}$  are not stable under  $\delta$ , as if  $P$  is a poset and  $\sim \triangleleft P$ ,  $P/ \sim$  is not necessarily a poset (although  $P| \sim$  is). However, there is a way to define a coproduct  $\bar{\delta} = (\xi \otimes Id) \circ \delta$  on  $\mathcal{H}_{\mathbf{p}}$ :

$$\forall P \in \mathbf{P}(n), \bar{\delta}(\llbracket P \rrbracket) = \sum_{\sim \triangleleft P} = \overline{\llbracket P/ \sim \rrbracket} \otimes \llbracket P| \sim \rrbracket.$$

$(\mathcal{H}_{\mathbf{p}}, m, \bar{\delta})$  is a quotient of  $(\mathcal{H}_{\mathbf{qp}}, m, \delta)$  through the map  $\xi$ .

## 2.4 Cointeractions

**Theorem 18** *We consider the map:*

$$\rho = (Id \otimes \llbracket \cdot \rrbracket) \circ \delta : \begin{cases} \mathcal{H}_{\mathbf{QP}} & \longrightarrow \mathcal{H}_{\mathbf{QP}} \otimes \mathcal{H}_{\mathbf{qp}} \\ P \in \mathbf{QP} & \longrightarrow \sum_{\sim \triangleleft P} (P/ \sim) \otimes \llbracket P| \sim \rrbracket. \end{cases}$$

*The bialgebras  $(\mathcal{H}_{\mathbf{QP}}, m, \Delta)$  and  $(\mathcal{H}_{\mathbf{qp}}, m, \delta)$  are in cointeraction via  $\rho$ .*

**Proof.** By composition,  $\rho$  is an algebra morphism. Let us take  $P \in \mathbf{QP}(n)$ . We put:

$$A = \{(r, O) \mid r \triangleleft P, O \in \text{top}(P/r)\}, \quad B = \{(O, s, s') \mid O \in \text{top}(P), s \triangleleft P_{[n] \setminus O}, s' \triangleleft P_{|O}\}.$$

*First step.* We define a map  $F : A \rightarrow B$ , sending  $(r, O)$  to  $(O, s, s')$ , by:

- $xy$  if  $xry$  and  $x, y$  are in the same connected component of  $P_{|[n] \setminus O}$ .
- $xs'y$  if  $xry$  and  $x, y$  are in the same connected component of  $P_{|O}$ .

Let us prove that  $F$  is well-defined. Let us take  $x, y \in [n]$ , with  $x \in O$  and  $x \leq_P y$ . Then  $x \leq_{P/r} y$ . as  $O$  is an open set of  $P/r$ ,  $y \in O$ :  $O$  is an open set of  $P$ . By definition, the equivalence classes of  $s'$  are the intersection of the equivalence classes of  $r$  and of the connected components of  $O$ . As  $O$  is a union of equivalence classes of  $r$ , they are  $P_{|O}$ -connected. If  $x \sim_{P_{|O}/s'} y$ , then  $x \sim_{P/r} y$  and  $x$  and  $y$  are in the same connected component of  $O$ . As  $r \triangleleft r$ ,  $xry$ , so  $xs'y$ :  $s' \triangleleft P_{|O}$ . Similarly,  $s \triangleleft P_{|[n] \setminus O}$ .

*Second step.* We define a map  $G : B \rightarrow A$ , sending  $(O, s, s')$  to  $(O, r, r')$ , by:

$$xry \text{ if } (x, y \notin O \text{ and } xsy) \text{ or } (x, y \in O \text{ and } xs'y).$$

Let us prove that  $G$  is well-defined. Let  $x, y \in [N]$ , with  $x \in O$  and  $x \leq_{P/r} y$ . There exists  $x_1, x'_1, \dots, x_k, x'_k$  such that:

$$x \leq_P x_1 r x'_1 \leq_P \dots \leq_P x_k r x'_k \leq_P y.$$

As  $O$  is an open set of  $P$ ,  $x_1 \in O$ ; by definition of  $r$ ,  $x'_1 \in O$ . Iterating, we obtain that  $x_2, x'_2, \dots, x_k, x'_k, y \in O$ . So  $O$  is open in  $P/r$ .

Let us assume that  $xry$ . Then  $x, y \in O$  or  $x, y \notin O$ . As  $s \triangleleft P_{|[n] \setminus O}$  and  $P_{|O}$ , there exists a path from  $x$  to  $y$  in the Hasse graph of  $P$  formed by elements  $s$ - or  $s'$ -equivalent to  $x$  and  $y$ , so the equivalence classes of  $r$  are  $P$ -connected.

Let us assume that  $x \sim_{P/r} y$ . here exists  $x_1, x'_1, \dots, x_k, s'_k$  and  $y_1, y'_1, \dots, y_l, y_l$  such that:

$$x \leq_P x_1 r x'_1 \leq_P \dots \leq_P x_k r x'_k \leq_P y, \quad y \leq_P y_1 r y'_1 \leq_P \dots \leq_P y_l r y'_l \leq_P x.$$

If  $x, y \in O$ , then all these elements are in  $O$ , so  $x \sim_{P_{|O}/s'} y$ , and then  $xs'y$ , so  $xry$ . If  $x, y \notin O$ , as  $O$  is an open set, none of these elements is in  $O$ , so  $x \sim_{P_{|[n] \setminus O}/s} y$ , so  $xsy$  and finally  $xry$ :  $r \triangleleft P$ .

*Third step.* Let  $(r, O) \in A$ . We put  $F(r, O) = (O, s, s')$  and  $G(O, s, s') = (\tilde{r}, O)$ . If  $xry$ , as  $O$  is an open set of  $P/r$ , both  $x$  and  $y$  are in  $O$  or both are not in  $O$ . Hence,  $xsy$  or  $xs'y$ , so  $x\tilde{r}y$ .

If  $x\tilde{r}y$ , then  $xsy$  or  $xs'y$ , so  $xry$ :  $\tilde{r} = r$  and  $G \circ F = Id_A$ .

Let  $(O, s, s') \in B$ . We put  $G(O, s, s') = (r, O)$  and  $F(r, O) = (O, \tilde{s}, \tilde{s}')$ . If  $xsy$ , then  $x$  and  $y$  are in the same connected component of  $[n] \setminus O$  as  $s \triangleleft P_{|[n] \setminus O}$  and  $xry$ , so  $x\tilde{s}y$ . If  $x\tilde{s}y$ , then  $xry$ , so  $xsy$ : we obtain that  $\tilde{s} = s$ . Similarly,  $\tilde{s}' = s'$ , which proves that  $F \circ G = Id_B$ .

We proved that  $F$  and  $G$  are inverse bijections. Let  $(r, O) \in A$  and  $(O, s, s') = F(O, r)$ .

$$\begin{aligned} \leq_{(P/r)_{|[n] \setminus O}} &= \text{transitive closure of } (xry \text{ and } x \leq_P y) \text{ restricted to } [n] \setminus O \\ &= \text{transitive closure of } (xry \text{ and } x \leq_{P_{|[n] \setminus O}} y) \\ &= \text{transitive closure of } (xsy \text{ and } x \leq_{P_{|[n] \setminus O}} y) \\ &= \leq_{P_{|[n] \setminus [n]}/s}. \end{aligned}$$

So  $(P/r)_{|[n] \setminus O} = P_{|[n] \setminus O}/s$ . Similarly,  $(P/r)_{|O} = P_{|O}/s'$ .

Let us now consider  $P|_R$ . Its connected components are the equivalence classes of  $r$ , that is to say the equivalence classes of  $s$  and  $s'$ ; for any such equivalence class  $I$ ,  $(P|_R)|_I = P|_I$ . So  $P|_R$  is the disjoint union of  $(P|_{[n]\setminus O})|_s$  and  $(P|_O)|_{s'}$ , and therefore is isomorphic to  $Std(P|_{[n]\setminus O})|_s Std((P|_O)|_{s'})$ , but not equal, because of the reindexation induced by the standardization. Hence,  $[P|_R] = [(P|_{[n]\setminus O})|_s] [(P|_O)|_{s'}]$ .

Finally:

$$\begin{aligned} (\Delta \otimes Id) \circ \rho(P) &= \sum_{(r,O) \in A} (G/r)|_{[n]\setminus O} \otimes (G/r)|_O \otimes [G|_r] \\ &= \sum_{(O,s,s') \in B} (P|_{[n]\setminus O})|_s \otimes (P|_O)|_{s'} \otimes [(P|_{[n]\setminus O})|_s] [(P|_O)|_{s'}] \\ &= m_{2,4}^3 \circ (\rho \otimes \rho) \circ \Delta(P). \end{aligned}$$

Moreover,  $(\varepsilon \otimes Id) \circ \rho(P) = \delta_{P,1} 1 \otimes 1 = \varepsilon(P) 1 \otimes 1$ .  $\square$

**Remark.** As noticed in [8],  $(\mathcal{H}_{\mathbf{QP}}, m, \Delta)$  and  $(\mathcal{H}_{\mathbf{QP}}, m, \delta)$  are not in cointeraction through  $\delta$ .

Taking the quotient through  $[\ ]$ :

**Corollary 19** *The bialgebras  $(\mathcal{H}_{\mathbf{qp}}, m, \Delta)$  and  $(\mathcal{H}_{\mathbf{qp}}, m, \delta)$  are in cointeraction via  $\delta$ . Moreover,  $\mathcal{H}_{\mathbf{qp}}$  is given a graduation by:*

$$\forall n \geq 0, (\mathcal{H}_{\mathbf{qp}})_n = Vect(P \in \mathbf{qp} \mid cl(P) = n).$$

With this graduation, hypotheses 1–5 of section 1.3 are satisfied.

**Proof.** Here,  $V$  is the space generated by the set of connected quasi-posets; the basis  $(g_i)_{i \in I}$  of group-like elements of  $(\mathcal{H}_{\mathbf{qp}})_1$  is  $(\bullet_n)_{n \geq 1}$ .  $\square$

We denote by  $M_{\mathbf{qp}}$  the monoid of characters of  $(\mathcal{H}_{\mathbf{qp}}, m, \delta)$ . Its product is denoted by  $*$ . Using proposition 4 on  $\mathcal{H}_{\mathbf{QP}}$ :

**Corollary 20** *Let  $\lambda \in M_{\mathbf{qp}}$ . The following map is a Hopf algebra endomorphism:*

$$\phi_\lambda : \begin{cases} (\mathcal{H}_{\mathbf{QP}}, m, \Delta) & \longrightarrow (\mathcal{H}_{\mathbf{QP}}, m, \Delta) \\ P \in \mathbf{QP} & \longrightarrow \sum_{\sim \triangleleft P} \lambda_{[P|\sim]} P / \sim. \end{cases}$$

It is bijective if, and only if, for all  $n \geq 1$ ,  $\lambda_{\bullet_n} \neq 0$ . If this holds,  $\phi_\lambda^{-1} = \phi_{\lambda^{-1}}$ .

**Proof.**  $\phi_\lambda = Id \leftarrow \lambda$ , so is an element of  $E_{\mathcal{H}_{\mathbf{QP}} \rightarrow \mathcal{H}_{\mathbf{QP}}}$ .

$\implies$ . For all  $n \geq 0$ ,  $\phi_\lambda(\bullet_n) = \lambda_{\bullet_n} \bullet_n$ . As  $\phi_\lambda$  is injective,  $\lambda_{\bullet_n} \neq 0$ .

$\impliedby$ . By proposition 5,  $\lambda$  is invertible in  $M_{\mathbf{qp}}$ : let us denote its inverse by  $\mu$ . Then, by proposition 4:

$$\phi_\lambda \circ \phi_\mu = Id \leftarrow (\lambda * \mu) = Id \leftarrow \varepsilon' = Id.$$

Similarly,  $\phi_\mu \circ \phi_\lambda = Id$ .  $\square$

### 3 Ehrhart polynomials

**Notations.** For all  $k \geq 0$ , we denote by  $H_k$  the  $k$ -th Hilbert polynomial:

$$H_k(X) = \frac{X(X-1)\dots(X-k+1)}{k!}.$$

### 3.1 Definition

**Definition 21** Let  $P \in \mathbf{QP}(n)$  and  $k \geq 1$ . We put:

$$\begin{aligned} L_P(k) &= \{f : [n] \rightarrow [k] \mid \forall i, j \in [n], i \leq_P j \implies f(i) \leq f(j)\}, \\ L_P^{str}(k) &= \{f \in L_P(k) \mid \forall i, j \in [n], (i \leq_P j \text{ and } f(i) = f(j)) \implies i \sim_P j\}, \\ W_P(k) &= \{w \in L_P(k) \mid w([n]) = [k]\}, \\ W_P^{str}(k) &= \{w \in L_P^{str}(k) \mid w([n]) = [k]\}. \end{aligned}$$

By convention:

$$L_P(0) = L_P^{str}(0) = W_P(0) = W_P^{str}(0) = \begin{cases} \emptyset & \text{if } P \neq 1, \\ \{1\} & \text{if } P = 1. \end{cases}$$

We also put:

$$L_P = \bigcup_{k \geq 0} L_P(k), \quad L_P^{str} = \bigcup_{k \geq 0} L_P^{str}(k), \quad W_P = \bigsqcup_{k \geq 0} W_P(k), \quad W_P^{str} = \bigsqcup_{k \geq 0} W_P^{str}(k).$$

Note that the elements of  $W_P$  and  $W_P^{str}$  are packed words (see definition 40).

**Proposition 22** Let  $P \in \mathbf{QP}$ . There exist unique polynomials  $ehr_P$  and  $ehr_P^{str} \in \mathbb{Q}[X]$ , such that for  $k \geq 0$ :

$$ehr_P(k) = \#L_P(k), \quad ehr_P^{str}(k) = \#L_P^{str}(k).$$

**Proof.** This is obvious if  $P = 1$ , with  $ehr_1(X) = ehr_1^{str}(X) = 1$ . Let us assume that  $P \in \mathbf{QP}(n)$ ,  $n \geq 1$ . Note that if  $i > n$ ,  $W_P(i) = 0$ . For all  $k \geq 1$ :

$$\#L_P(k) = \sum_{i=1}^k \#W_P(i) \binom{k}{i} = \sum_{i=1}^k \#W_P(i) H_i(k) = \sum_{i=1}^n \#W_P(i) H_i(k).$$

So:

$$ehr_P(X) = \sum_{i=1}^n \#W_P(i) H_i(X).$$

Moreover, if  $k = 0$ :

$$ehr_P(0) = \sum_{i=1}^n \#W_P(i) H_i(0) = \#L_P(0).$$

In the same way:

$$ehr_P^{str}(X) = \sum_{i=1}^n \#W_P^{str}(i) H_i(X).$$

These are indeed elements of  $\mathbb{Q}[X]$ . □

#### Remarks.

1. Let  $P, Q \in \mathbf{QP}(n)$ .

- If they are isomorphic, then  $ehr_P(k) = ehr_Q(k)$  for all  $k \geq 1$ , so  $ehr_P = ehr_Q$ .
- If  $w \in L_P$ , for all  $x, y \in P$  such that  $x \sim_P y$ , then  $w(x) \leq w(y)$  and  $w(y) \leq w(x)$ , so  $w(x) = w(y)$ :  $w$  goes through the quotient by  $\sim_P$ . We obtain in this way a bijection from  $L_P(k)$  to  $L_{\overline{P}}(k)$  for all  $k$ , so  $ehr_P = ehr_{\overline{P}}$ . Similarly,  $ehr_P^{str} = ehr_{\overline{P}}^{str}$ .

Hence, we obtain maps, all denoted by  $ehr$  and  $ehr^{str}$ , such the following diagrams commute:

$$\begin{array}{ccc} \mathcal{H}_{\mathbf{QP}} & \xrightarrow{\sqcup} & \mathcal{H}_{\mathbf{QP}} & \xrightarrow{\xi} & \mathcal{H}_{\mathbf{P}} \\ & \searrow & \downarrow & \swarrow & \\ & ehr & \mathbb{K}[X] & ehr & \end{array} \qquad \begin{array}{ccc} \mathcal{H}_{\mathbf{QP}} & \xrightarrow{\sqcup} & \mathcal{H}_{\mathbf{QP}} & \xrightarrow{\xi} & \mathcal{H}_{\mathbf{P}} \\ & \searrow & \downarrow & \swarrow & \\ & ehr^{str} & \mathbb{K}[X] & ehr^{str} & \end{array}$$

2. Let  $P \in \mathbf{P}(n)$ . The classical definition of the Ehrhart polynomial  $ehr^{cl}(t)$  is the number of of integral points of  $tPol(P)$ , where  $Pol(P)$  is the polytope associated to  $P$ . Hence,  $ehr^{cl}(X) = ehr(X + 1)$ .

**Theorem 23** *The morphisms  $ehr, ehr^{str} : \mathcal{H}_{\mathbf{QP}}, \mathcal{H}_{\mathbf{QP}}, \mathcal{H}_{\mathbf{P}} \rightarrow \mathbb{K}[X]$  are Hopf algebra morphisms.*

**Proof.** It is enough to prove it for  $ehr, ehr^{str} : \mathcal{H}_{\mathbf{P}} \rightarrow \mathbb{K}[X]$ .

*First step.* Let  $P \in \mathbf{P}(n)$ . Let us prove that for all  $k, l \geq 0$ :

$$ehr_P(k + l) = \sum_{O \in Top(P)} ehr_{P_{[n] \setminus O}}(k) ehr_{P_O}(l), \quad ehr_P^{str}(k + l) = \sum_{O \in Top(P)} ehr_{P_{[n] \setminus O}}^{str}(k) ehr_{P_O}^{str}(l).$$

Let  $f \in L_P(k + l)$ . We put  $O = f^{-1}(\{k + 1, \dots, k + l\})$ . If  $x \in O$  and  $x \leq_P y$ , then  $f(x) \leq f(y)$ , so  $y \in O$ :  $O$  is an open set of  $P$ . By restriction, the following maps are elements of respectively  $L_{P_{[n] \setminus O}}(k)$  and  $L_{P_O}(l)$ :

$$f_1 : \begin{cases} [n] \setminus O & \longrightarrow [k] \\ x & \longrightarrow f(x), \end{cases} \qquad f_2 : \begin{cases} O & \longrightarrow [l] \\ x & \longrightarrow f(x) - k. \end{cases}$$

This defines a map:

$$\nu : \begin{cases} L_P(k + l) & \longrightarrow \bigsqcup_{O \in Top(P)} L_{P_{[n] \setminus O}}(k) \times L_{P_O}(l) \\ f & \longrightarrow (f_1, f_2). \end{cases}$$

This map is clearly injective; moreover:

$$\nu(L_P^{str}(k + l)) \subseteq \bigsqcup_{O \in Top(P)} L_{P_{[n] \setminus O}}^{str}(k) \times L_{P_O}^{str}(l).$$

Let us prove that  $f$  is surjective. Let  $(f_1, f_2) \in L_{P_{[n] \setminus O}}(k) \otimes L_{P_O}(l)$ , with  $O \in Top(P)$ . We define a map  $f : P \rightarrow [k + l]$  by:

$$f(x) = \begin{cases} f_1(x) & \text{if } x \notin O, \\ f_2(x) + k & \text{if } x \in O. \end{cases}$$

Let  $x \leq_P y$ . As  $O$  is an open set of  $P$ , three cases are possible:

- $x, y \notin O$ : then  $f_1(x) \leq f_1(y)$ , so  $f(x) \leq f(y)$ .
- $x, y \in O$ : then  $f_2(x) \leq f_2(y)$ , so  $f(x) \leq f(y)$ .
- $x \notin O, y \in O$ : then  $f(x) \leq k < f(y)$ .



So  $f \in L_P(k+l)$ , and  $v(f) = (f_1, f_2)$ :  $v$  is surjective, and finally bijective. Moreover, if  $f_1 \in L_{[n]\setminus O}^{str}(k)$  and  $f_2 \in L_{P|_O}^{str}(l)$ , then  $f = v^{-1}(f_1, f_2) \in L_P^{str}(k+l)$ . Finally:

$$\begin{aligned} f(L_P(k+l)) &= \bigsqcup_{O \in Top(P)} L_{P|_{[n]\setminus O}}(k) \times L_{P|_O}(l), \\ f(L_P^{str}(k+l)) &= \bigsqcup_{O \in Top(P)} L_{P|_{[n]\setminus O}}^{str}(k) \times L_{P|_O}^{str}(l). \end{aligned}$$

Taking the cardinals, we obtain the announced result.

*Second step.* Let  $P \in \mathbf{P}(m)$ ,  $Q \in \mathbf{P}(n)$ , and  $f : [m+n] \rightarrow [k]$ . Then  $f \in L_{PQ}(k)$  if, and only if,  $f|_{[m]} \in L_P(k)$  and  $Std(f|_{[m+n]\setminus[m]}) \in L_Q(k)$ . So  $ehr_{PQ}(k) = ehr_P(k)ehr_Q(k)$ , and then  $ehr_{PQ}(X) = ehr_P(X)ehr_Q(X)$ :  $ehr$  is an algebra morphism.

Let  $P$  be a finite poset, and  $k, l \geq 0$ . By the first step:

$$\begin{aligned} (ehr \otimes ehr) \circ \Delta(P)(k, l) &= \sum_{O \in Top(P)} ehr_{P|_{[m]\setminus O}}(k)ehr_{P|_O}(l) \\ &= ehr_P(k+l) \\ &= \Delta \circ ehr(P)(k, l). \end{aligned}$$

As this is true for all  $k, l \geq 1$ ,  $(ehr \otimes ehr) \circ \Delta(P) = \Delta \circ ehr(P)$ . Moreover:

$$\varepsilon \circ ehr(P) = ehr_P(0) = \begin{cases} 1 & \text{if } P = 1, \\ 0 & \text{otherwise,} \end{cases}$$

so  $\varepsilon \circ ehr = \varepsilon$ .

The proof is similar for  $ehr^{str}$ . □

### 3.2 Recursive computation of $ehr$ and $ehr^{str}$

Let us recall this classical result:

**Lemma 24** *We consider the following maps:*

$$L : \begin{cases} \mathbb{K}[X] & \longrightarrow \mathbb{K}[X] \\ H_k(X) & \longrightarrow H_{k+1}(X). \end{cases}$$

The map  $L$  is injective, and  $L(\mathbb{K}[X]) = \mathbb{K}[X]_+$ . Moreover, for all  $P \in \mathbb{K}[X]$ , for all  $n \geq 0$ :

$$L(P)(n+1) = P(0) + \dots + P(n).$$

**Proof.** Let us consider  $P = H_k(X)$ . For all  $n \geq 0$ :

$$\begin{aligned} H_k(0) + \dots + H_k(n) &= \binom{0}{k} + \dots + \binom{n}{k} \\ &= \binom{k}{k} + \dots + \binom{n}{k} \\ &= \binom{n+1}{k+1} \\ &= H_{k+1}(n+1) \\ &= L(H_k)(n+1). \end{aligned}$$

By linearity, for any  $P \in \mathbb{K}[X]$ ,  $L(P)(n+1) = P(0) + \dots + P(n)$  for all  $n \geq 1$ . □

**Proposition 25** Let  $P \in \mathbf{P}(n)$ .

$$\begin{aligned} ehr_P(X) &= L \left( \sum_{\emptyset \neq O \in Top(P)} ehr_{P_{|[n] \setminus O}}(X) \right), \\ ehr_P^{str}(X) &= L \left( \sum_{\emptyset \neq O \in Top(P), \text{ discrete}} ehr_{P_{|[n] \setminus O}}^{str}(X) \right). \end{aligned}$$

**Proof.** Let  $n \geq 1$ . As  $L_Q(1)$  is reduced to a singleton for all finite poset  $Q$ :

$$\begin{aligned} ehr_P(n+1) &= \sum_{O \in Top(P)} ehr_{P_{|[n] \setminus O}}(n) ehr_{P_O}(1) \\ &= \sum_{\emptyset \neq O \in Top(P)} ehr_{P_{|[n] \setminus O}}(n) + ehr_P(n). \end{aligned}$$

We put:

$$Q(X) = \sum_{\emptyset \neq O \in Top(P)} ehr_{P_{|[n] \setminus O}}(X).$$

In particular:

$$Q(0) = \sum_{\emptyset \neq O \in Top(P)} ehr_{P_{|[n] \setminus O}}(0) = ehr_{\emptyset}(0) + 0 = 1 = ehr_P(1).$$

Then:

$$\begin{aligned} ehr_P(n+1) &= Q(n) + ehr_P(n) \\ &= Q(n) + Q(n-1) + ehr_P(n-1) \\ &\vdots \\ &= Q(n) + Q(n-1) + \dots + Q(1) + ehr_P(1) \\ &= Q(n) + \dots + Q(1) + Q(0) \\ &= L(Q)(n+1). \end{aligned}$$

So  $ehr_P = L(Q)$ .

For  $ehr_P^{str}$ , observe that  $ehr_Q^{str}(1) = 1$  if  $Q$  is discrete, and 0 otherwise, which implies:

$$ehr_P^{str}(n+1) = \sum_{\emptyset \neq O \in Top(P), \text{ discrete}} ehr_{P_{|[n] \setminus O}}^{str}(n) + ehr_P^{str}(n).$$

The end of the proof is similar. □

### Examples.

$$\begin{aligned}
ehr_{\bullet}(X) &= H_1(X) = X, \\
ehr_{\mathbf{1}}(X) &= H_1(X) + H_2(X) = \frac{X(X+1)}{2}, \\
ehr_{\mathbf{V}}(X) &= ehr_{\mathbf{\wedge}}(X) = H_1(X) + 3H_2(X) + 2H_3(X) = \frac{X(X+1)(2X+1)}{6}, \\
ehr_{\mathbf{\dagger}}(X) &= H_1(X) + 2H_2(X) + H_3(X) = \frac{X(X+1)(X+2)}{6}; \\
\\
ehr_{\bullet}^{str}(X) &= H_1(X) = X, \\
ehr_{\mathbf{\dagger}}^{str}(X) &= H_2(X) = \frac{X(X-1)}{2}, \\
ehr_{\mathbf{V}}^{str}(X) &= ehr_{\mathbf{\wedge}}^{str}(X) = H_2(X) + 2H_3(X) = \frac{X(X-1)(2X-1)}{6}, \\
ehr_{\mathbf{\dagger}}^{str}(X) &= H_3(X) = \frac{X(X-1)(X-2)}{6}.
\end{aligned}$$

### 3.3 Characterization of quasi-posets by packed words

**Lemma 26** *Let  $P \in \mathbf{QP}(n)$  and let  $I_1, \dots, I_k$  be distinct minimal classes of the poset  $\overline{P}$ ; let  $w' \in W_{P_{[n] \setminus (I_1 \sqcup \dots \sqcup I_k)}}^{str}$ . The following map belongs to  $W_P^{str}$ :*

$$w : \begin{cases} [n] & \longrightarrow \mathbb{N}^* \\ x \in I_p, 1 \leq p \leq k & \longrightarrow p \\ x \notin I_1 \sqcup \dots \sqcup I_k & \longrightarrow w'(x). \end{cases}$$

**Proof.** Let us assume that  $i \leq_P j$ .

- If  $i \in I_p$ , as  $I_p$  is a minimal class of  $\overline{P}$ ,  $j \in I_p$  or  $j \notin I_1 \sqcup \dots \sqcup I_k$ . In the first case,  $w(i) = w(j)$ ; in the second case,  $w(i) \leq k < w(j)$ . If moreover  $w(i) = w(j)$ , then necessarily  $j \in I_p$ , so  $i \sim_P j$ .
- If  $i \notin I_1 \sqcup \dots \sqcup I_k$ , as  $i \leq_P j$ ,  $j \notin I_1 \sqcup \dots \sqcup I_k$ , so  $i \leq_{P_{[n] \setminus (I_1 \sqcup \dots \sqcup I_k)}} j$  and  $w'(i) \leq w'(j)$ , so  $w'(i) \leq w'(j)$ . If moreover  $w(i) = w(j)$ , then  $w'(i) = w'(j)$ , so  $i \sim_{P_{[n] \setminus (I_1 \sqcup \dots \sqcup I_k)}} j$  and finally  $i \sim_P j$ .

As a conclusion,  $w \in W_P^{str}$ . □

**Remark.** This lemma implies that  $W_P^{str}$  is non-empty for any non-empty quasi-poset  $P$ .

**Proposition 27** *Let  $P = ([n], \leq_P)$  be a quasi-poset and let  $i, j \in [n]$ . The following assertions are equivalent:*

1.  $i \leq_P j$ .
2. For all  $w \in L_P$ ,  $w(i) \leq w(j)$ .
3. For all  $w \in L_P^{str}$ ,  $w(i) \leq w(j)$ .
4. For all  $w \in W_P$ ,  $w(i) \leq w(j)$ .
5. For all  $w \in W_P^{str}$ ,  $w(i) \leq w(j)$ .

**Proof.** Obviously:

$$\begin{array}{ccccc} 1. & \Longrightarrow & 2. & \Longrightarrow & 3. \\ & & \Downarrow & & \Downarrow \\ & & 4. & \Longrightarrow & 5. \end{array}$$

It is enough to prove that  $5. \implies 1.$  We proceed by induction on  $n$ . If  $n = 1$ , there is nothing to prove. Let us assume the result at all ranks  $< n$ . Let  $i, j \in [n]$ , such that we do not have  $i \leq_P j$ . Let us prove that there exists  $w \in W_P^{str}$ , such that  $w(i) > w(j)$ . There exists a minimal element  $k \in [n]$ , such that  $k \leq_P j$ ; let  $I$  be the class of  $k$  in  $\bar{P}$ . By hypothesis on  $i$ ,  $i$  and  $k$  are not equivalent for  $\sim_P$ , so  $i \notin I$ . If  $j \in I$ , let us choose an element  $w' \in W_{P_{[n] \setminus I}}^{str}$ ; if  $j \notin I$ , then by the induction hypothesis, there exists  $w' \in W_{P_{[n] \setminus I}}^{str}$ , such that  $w'(i) > w'(j)$ . By lemma 26, the following map is an element of  $W_P^{str}$ :

$$w : \begin{cases} [n] & \longrightarrow \mathbb{N} \\ x \in I & \longrightarrow 1 \\ x \notin I & \longrightarrow w'(x) + 1 \end{cases}$$

If  $j \in I$ , then  $w(j) = 1 < w(i)$ ; if  $j \notin I$ ,  $w(i) = w'(i) + 1 > w'(j) + 1 = w(j)$ . In both cases,  $w(i) > w(j)$ .  $\square$

### 3.4 Link with linear extensions

Let  $P \in \mathbf{QP}(n)$ . Linear extensions, as defined in [10], are elements of  $W_P^{str}$ : they are the elements  $f \in W_P^{str}$  such that

$$\forall i, j \in [n], f(i) = f(j) \iff i \sim_P j.$$

It may happens that not all elements of  $W_P^{str}$  are linear extensions. For example, if  $P = {}^2\mathbf{V}_1^3$ ,  $W_P^{str}(3) = \{(123), (132), (122)\}$ , and  $(122)$  is not a linear extension of  $P$ . The set of linear extensions of  $P$  will be denoted by  $E_P$ .

**Definition 28** Let  $w$  and  $w'$  be two packed words of the same length  $n$ . We shall say that  $w \leq w'$  if:

$$\forall i, j \in [n], w(i) < w(j) \implies w'(i) < w'(j).$$

**Proposition 29** Let  $P \in \mathbf{QP}(n)$ . Then:

$$W_P = \bigcup_{w \in E_P} \{w' \mid w' \leq w\}.$$

This union may be not disjoint. Moreover, the maximal elements of  $W_P$  for the order of definition 28 are the elements of  $E_P$ .

**Proof.**  $\subseteq$ . Let  $w \in W_P$ . For all  $1 \leq p \leq \max(w)$ , we put  $I_p = w^{-1}(p)$ . Let  $f_p$  be a linear extension of  $P_{|I_p}$ . Let us consider:

$$f : \begin{cases} [n] & \longrightarrow \mathbb{N} \\ i & \longrightarrow \max(f_1) + \dots + \max(f_{p-1}) + f_p(i) \text{ if } i \in I_p. \end{cases}$$

By construction, if  $w(i) < w(j)$ , then  $f(i) < f(j)$ :  $w \leq f$ . Let us prove that  $f \in E_P$ .

If  $i \leq_P j$ , then as  $w \in W_P$ ,  $w(i) \leq w(j)$ . If  $w(i) = w(j) = p$ , then  $i \leq_{P_{|I_p}} j$ , so  $f_p(i) \leq f_p(j)$ , and  $f(i) \leq f(j)$ . If  $w(i) < w(j)$ , then  $f(i) < f(j)$ .

If  $f(i) = f(j)$ , then  $w(i) = w(j) = p$ , and  $f_p(i) = f_p(j)$ . As  $f_p \in E_{P_{|I_p}}$ ,  $i \sim_{P_{|I_p}} j$ , so  $i \sim_P j$ .

$\supseteq$ . Let  $w \in E_P$  and  $w' \leq w$ . If  $i \leq_P j$ , then  $w(i) \leq w(j)$  as  $w$  is a linear extension of  $P$ . As  $w' \leq w$ ,  $w'(i) \leq w'(j)$ , so  $w' \in W_P$ .

Let  $w$  be a maximal element of  $W_P$ . There exists a linear extension  $w'$  of  $P$ , such that  $w \leq w'$ . As  $w$  is maximal,  $w = w'$  is a linear extension of  $P$ . Conversely, if  $w$  is a linear extension of  $P$  and  $w \leq w'$  in  $W_P$ , then as  $w$  is a linear extension of  $P$ ,  $\max(w) = cl(P)$ . Moreover, as  $w \leq w'$ ,  $\max(w) \leq \max(w')$ . As  $w' \in W_P$ ,  $\max(w') \leq cl(P)$ , which implies that  $\max(w) = \max(w') = cl(P)$ , and finally  $w = w'$ :  $w$  is a maximal element of  $W_P$ .  $\square$

**Example.** For  $P = {}^2\mathbf{V}_1^3$ :

$$\begin{aligned} E_P &= \{(123), (132)\}; \\ W_P &= \{(123), (122), (112), (111)\} \cup \{(132), (122), (121), (111)\} \\ &= \{(123), (132), (122), (112), (121), (111)\}. \end{aligned}$$

Note that the two components of  $W_P$  are not disjoint.

**Remark.** A similar result is proved in [10] for  $T$ -partitions of a quasi-poset, generalizing Stanley's result [17] for  $P$ -partitions of posets; nevertheless, this is different here, as the union is not a disjoint one.

## 4 Characters associated to $ehr$ and $ehr^{str}$

Recall that  $(M_{\mathbf{qp}}, *)$  the monoid of characters of  $(\mathcal{H}_{\mathbf{qp}}, m, \delta)$ .

By theorems 7 and 23:

**Proposition 30** 1. *There exists a unique homogeneous Hopf algebra morphism  $\phi_0$  from  $(\mathcal{H}_{\mathbf{qp}}, m, \Delta)$  to  $(\mathbb{K}[X], m, \Delta)$  such that:*

$$\forall n \geq 1, \phi_0(\bullet_n) = X.$$

*There exists a unique character  $\lambda \in M_{\mathbf{qp}}$  such that for all  $P \in \mathbf{qp}$ ,*

$$\phi_0(P) = \lambda_P X^{cl(P)}.$$

2. *There exist unique characters  $\alpha, \alpha^{str} \in M_{\mathbf{qp}}$ , such that:*

$$ehr = \phi_0 \leftarrow \alpha, \quad ehr^{str} = \phi_0 \leftarrow \alpha^{str}.$$

**Remarks.**

1. Let  $P \in \mathbf{qp}$ . Then  $ehr_P = ehr_{\overline{P}}$  and  $ehr_P^{str} = ehr_{\overline{P}}^{str}$ , so  $\alpha_P = \alpha_{\overline{P}}$  and  $\alpha_P^{str} = \alpha_{\overline{P}}^{str}$ .
2. Still by theorem 7, there exists a unique homogeneous Hopf algebra morphism  $\phi'_0$  from  $\mathcal{H}_{\mathbf{p}}$  to  $\mathbb{K}[X]$  such that  $\phi'_0(\bullet) = X$ . By unicity,  $\phi_0 = \phi'_0 \circ \Xi$ , so for any  $P \in \mathbf{qp}$ ,  $\lambda_P = \lambda_{\overline{P}}$ .

### 4.1 The character $\lambda$

**Lemma 31** *For all  $P \in \mathbf{P}(n)$ ,  $n \geq 0$ :*

$$\lambda_{[P]} = \begin{cases} 1 & \text{if } P = 1, \\ \frac{1}{n} \sum_{M \in \max(P)} \lambda_{[P]_{[n] \setminus \{M\}}} & \text{otherwise.} \end{cases} = \frac{1}{n} \sum_{m \in \min(P)} \lambda_{[P]_{[n] \setminus \{m\}}}$$

**Proof.** Let  $P \in \mathbf{P}(n)$ , with  $n \geq 0$ . Recall that  $\pi : \mathbb{K}[X] \rightarrow \text{Vect}(X)$  is the canonical projection.

$$\begin{aligned}
(Id \otimes \pi) \circ \Delta \circ \phi_0([P]) &= \lambda_{[P]}(Id \otimes \pi) \circ \Delta(X^n) \\
&= \lambda_{[P]} n X^{n-1}; \\
= (Id \otimes \pi) \circ (\phi_0 \otimes \phi_0) \circ \Delta([P]) &= \sum_{O \in \text{Top}(P)} \lambda_{[P|_O]} \lambda_{[P_{[n] \setminus O}]} X^{|[n] \setminus O|} \pi(X^{|O|}) \\
&= \sum_{O \in \text{Top}(P), |O|=1} \lambda_{[P|_O]} \lambda_{[P_{[n] \setminus O}]} X^{n-1} \\
&= \sum_{M \in \text{max}(P)} \lambda_{[P_{[n] \setminus \{M\}}]} X^{n-1}.
\end{aligned}$$

This implies the first equality. The second is proved by considering  $(\pi \otimes Id) \circ \Delta \circ \phi_0([P])$ .  $\square$

**Remarks.**

1. This lemma allows to inductively compute  $\lambda_P$ . This gives:

$P$	$\cdot$	$\dagger$	$\vee$	$\wedge$	$\ddagger$	$\Psi$	$\mathfrak{A}$	$\Downarrow$	$\Uparrow$	$\Upsilon$	$\mathfrak{A}$	$\ddagger$	$\mathbb{N}$	$\mathfrak{M}$	$\diamond$
$\lambda_P$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{5}{24}$	$\frac{1}{6}$	$\frac{1}{12}$

2. If  $P = (P, \leq)$  is a finite poset, we denote by  $P^{op}$  the opposite poset  $(P, \geq)$ . It is not difficult to deduce from this lemma that  $\lambda_P = \lambda_{P^{op}}$ .

**Proposition 32** *Let  $P \in \mathbf{P}(n)$ . The number of elements of  $W_P(n)$  of  $P$  is denoted by  $\mu_P$ : in other words,  $\mu_P$  is the number of bijections  $f$  from  $[n]$  to  $[n]$  such that for all  $x, y \in [n]$ ,*

$$x \leq_P y \implies f(x) \leq f(y).$$

*These bijections are called heap-orderings of  $P$ . Then, for any finite poset  $P$ ,  $\lambda_P = \frac{\mu_P}{n!}$ .*

**Proof.** Let us fix a non-empty finite poset  $P \in \mathbf{P}(n)$ . The set of heap-orderings of  $P$  is  $HO(P) = W_P(n)$ . We consider the map:

$$\begin{cases} HO(P) & \longrightarrow \bigsqcup_{M \in \text{max}(P)} HO(P \setminus \{M\}) \\ f & \longrightarrow f_{[n-1]} \in HO(P \setminus \{f^{-1}(n)\}). \end{cases}$$

It is not difficult to prove that this is a bijection. So:

$$\mu_P = \sum_{M \in \text{max}(P)} \mu_{P \setminus \{M\}}; \quad \frac{\mu_P}{n!} = \frac{1}{n} \sum_{M \in \text{max}(P)} \frac{\mu_{P \setminus \{M\}}}{|P \setminus \{M\}|!}.$$

An easy induction on  $|P|$  then proves that  $\lambda_P = \frac{\mu_P}{n!}$  for all  $P$ .  $\square$

This formula can be simplified for rooted forests.

**Definition 33** *Let  $P$  be a non-empty finite poset.*

1. We put:

$$P! = \prod_{i \in V(P)} \#\{j \in V(P) \mid i \leq_P j\}.$$

By convention,  $1! = 1$ .

2. We shall say that  $P$  is a rooted forest if  $P$  does not contain any subposet isomorphic to  $\Lambda$ .

For example, here are isoclasses of rooted forests of cardinality  $\leq 4$ :

$$1; \quad .; \quad \cdot; \dots; \quad \vee, \uparrow, \dots; \quad \Psi, \downarrow, \Upsilon, \updownarrow, \vee \cdot, \updownarrow, \updownarrow, \dots, \dots$$

**Examples.**

$P$	$\cdot$	$\uparrow$	$\vee$	$\Lambda$	$\updownarrow$	$\Psi$	$\aleph$	$\downarrow$	$\uparrow$	$\Upsilon$	$\aleph$	$\updownarrow$	$\mathbb{N}$	$\aleph$	$\diamond$
$P!$	1	2	3	4	6	4	8	8	12	12	18	24	6	9	16

**Proposition 34** For all finite poset  $P$ ,  $\lambda_P \geq \frac{1}{P!}$ , with equality if, and only if,  $P$  is a rooted forest.

**Proof.** We proceed by induction on  $n = |P|$ . It is obvious if  $n = 0$ . Let us assume the result at all ranks  $< n$ .

$$\begin{aligned} \lambda_P &= \frac{1}{|P|} \sum_{m \in \min(P)} \lambda_{P \setminus \{m\}} \\ &\geq \frac{1}{|P|} \sum_{m \in \min(P)} \prod_{i \in V(P), i \neq m} \frac{1}{\#\{j \in V(P) \mid j \neq m, i \leq_P j\}} \\ &= \frac{1}{|P|} \sum_{m \in \min(P)} \prod_{i \in V(P), i \neq m} \frac{1}{\#\{j \in V(P) \mid i \leq_P j\}} \\ &= \frac{1}{P!} \frac{1}{|P|} \sum_{m \in \min(P)} \#\{j \in V(P) \mid m \leq_P j\}. \end{aligned}$$

For any  $j \in A$ , there exists  $m \in \min(P)$  such that  $m \leq_P j$ , so:

$$\sum_{m \in \min(P)} \#\{j \in V(P) \mid m \leq_P j\} \geq |P|.$$

Consequently,  $\lambda_P \geq \frac{1}{P!}$ .

Let us assume that this is an equality. Then:

$$\sum_{m \in \min(P)} \#\{j \in V(P) \mid m \leq_P j\} = |P|.$$

Consequently, for all  $j \in \min(P)$ , there exists a unique  $m \in \min(P)$  such that  $m \leq_P j$ . Moreover, for all  $m \in \min(P)$ ,  $\lambda_{P \setminus \{m\}} = \frac{1}{P \setminus \{m\}!}$ . By the induction hypothesis,  $P \setminus \{m\}$  is a rooted forest; this implies that  $P$  is also a rooted forest.

Let us assume that  $P$  is a rooted forest. For any  $j \in V(P)$ , there exists a unique  $m \in \min(P)$  such that  $m \leq_P j$ , so:

$$\sum_{m \in \min(P)} \#\{j \in V(P) \mid m \leq_P j\} = |P|.$$

Moreover, for all  $m \in \min(P)$ ,  $P \setminus \{m\}$  is also a rooted forest. By the induction hypothesis,  $\lambda_{P \setminus \{m\}} = \frac{1}{P \setminus \{m\}!}$ . Hence,  $\lambda_P = \frac{1}{P!}$ .  $\square$

## 4.2 The character $\alpha^{str}$

Let us now apply theorem 8 to  $ehr$  and  $ehr^{str}$ :

**Theorem 35** *For all finite connected quasi-poset  $P$ , we have:*

$$\alpha_P = \frac{d \, ehr_P}{dX}(0), \quad \alpha_P^{str} = \frac{d \, ehr_P^{str}}{dX}(0).$$

For any quasi-poset  $P$ :

$$ehr_P(X) = \sum_{\sim \triangleleft P} \frac{\mu_{P/\sim}}{cl(\sim)!} \alpha_{P|\sim} X^{cl(\sim)}, \quad ehr_P^{str}(X) = \sum_{\sim \triangleleft P} \frac{\mu_{P/\sim}}{cl(\sim)!} \alpha_{P|\sim}^{str} X^{cl(\sim)},$$

where  $cl(\sim)$  is the number of equivalence classes of  $\sim$ .

Let us give a few values of  $\alpha$ :

$P$	$\cdot$	$\mathfrak{I}$	$\mathfrak{V}$	$\mathfrak{A}$	$\mathfrak{I}$	$\mathfrak{V}$	$\mathfrak{A}$	$\mathfrak{V}$	$\mathfrak{I}$	$\mathfrak{Y}$	$\mathfrak{A}$	$\mathfrak{I}$	$\mathfrak{N}$	$\mathfrak{M}$	$\mathfrak{D}$
$\alpha_P$	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	0	0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{6}$

**Lemma 36** *Let  $P \in \mathbf{QP}$ , not discrete. Then:*

$$(-1)^{cl(P)} ehr_P(-1) = ehr_P^{str}(1) = \begin{cases} 1 & \text{if } P \text{ is discrete,} \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** If  $P$  is discrete, then  $ehr_P(X) = ehr_P^{str}(X) = X^{cl(P)}$  and the result is obvious. Let us assume that  $P$  is not discrete. There exists a unique map  $f$  from  $[n]$  to  $[1]$ ; as  $P$  is not discrete,  $f \notin L_P^{str}(1)$ , so  $ehr_P^{str}(1) = 0$ . We now proceed with  $ehr_P(-1)$ .

*First step.* Let us prove that  $L(H_k(-X)) = -H_{k+1}(-X)$  for all  $k \geq 0$ . For all  $l, n \geq 0$ :

$$H_l(-n) = (-1)^l \binom{n+l-1}{l}.$$

For all  $k, n \geq 0$ :

$$\begin{aligned} L(H_k(-X))(n+1) &= H_k(0) + \dots + H_k(-n) \\ &= (-1)^k \sum_{i=0}^n \binom{i+k-1}{k} \\ &= (-1)^k \sum_{j=k}^{n+k-1} \binom{j}{k} \\ &= (-1)^k \binom{n+k}{k+1} \\ &= -H_{k+1}(-(n+1)). \end{aligned}$$

*Second step.* Let us prove that  $L((X+1)\mathbb{K}[X]) \subseteq (X+1)\mathbb{K}[X]$ . For all  $k \geq 2$ , let us put  $H_k(-X) = X(X+1)L_k(X)$ ;  $(L_k(X))_{k \geq 2}$  is a basis of  $\mathbb{K}[X]$ , which implies that  $(H_k(-X))_{k \geq 2}$  is a basis of  $X(X+1)\mathbb{K}[X]$ , and that  $(X+1) \sqcup (H_k(-X))_{k \geq 2}$  is a basis of  $(X+1)\mathbb{K}[X]$ . First:

$$L(X+1) = L(H_1(X) + H_0(X)) = H_2(X) + H_1(X) = \frac{X(X-1)}{2} + X = \frac{X(X+1)}{2} \in (X+1)\mathbb{K}[X];$$



if  $k \geq 2$ , by the first step,  $L(H_k(-X)) = -H_{k+1}(-X) \in (X+1)\mathbb{K}[X]$ .

*Last step.* We can replace  $P$  by  $\overline{P}$ , and we now assume that  $P \in \mathbf{P}(n)$ . There is nothing to prove if  $n = 0, 1$ . Let us assume the result at all rank  $< n$ . Then, by the second step and the induction hypothesis:

$$\begin{aligned}
ehr_{[P]}(-1) &= L \left( \sum_{\emptyset \neq O \in Top(P)} ehr_{[P|_{[n] \setminus O}]}(X) \right)_{|X=-1} \\
&= L \left( \sum_{\substack{\emptyset \neq O \in Top(P) \\ P|_{[n] \setminus O} \text{ discrete}}} ehr_{[P|_{[n] \setminus O}]}(X) \right)_{|X=-1} \\
&= L \left( \sum_{[n] \neq J \subseteq min(P)} ehr_{[P|_J]}(X) \right)_{|X=-1} \\
&= L \left( \sum_{J \subseteq min(P)} ehr_{[P|_J]} J(X) \right)_{|X=-1} \\
&= L \left( \sum_{J \subseteq min(P)} X^{|J|} \right)_{|X=-1} \\
&= L \left( \underbrace{(1+X)^{|min(P)|}}_{\in (X+1)\mathbb{K}[X]} \right)_{|X=-1} \\
&= 0.
\end{aligned}$$

For the fourth equality, note that  $P$  is not discrete, so  $min(P) \neq P$ . □

In other words, for any  $P \in \mathbf{qp}$ ,  $(-1)^{cl(P)} ehr_P(-1) = ehr_P^{str}(1) = \varepsilon'(P)$ . By corollary 10:

**Theorem 37** *ehr<sup>str</sup> is the unique morphism from  $\mathcal{H}_{\mathbf{qp}}$  to  $\mathbb{K}[X]$  such that:*

1. *ehr<sup>str</sup> is a Hopf algebra morphism from  $(\mathcal{H}_{\mathbf{qp}}, m, \Delta)$  to  $(\mathbb{K}[X], m, \Delta)$ .*
2. *ehr<sup>str</sup> is a bialgebra morphism from  $(\mathcal{H}_{\mathbf{qp}}, m, \delta)$  to  $(\mathbb{K}[X], m, \delta)$ .*

Moreover, the character  $\alpha^{str}$  is the inverse of  $\lambda$  in  $M_b$ .

### 4.3 The character $\alpha$ and the duality principle

**Theorem 38** 1. *(Duality principle). For any quasi-poset  $P$ :*

$$ehr_P^{str}(X) = (-1)^{cl(P)} ehr_P(-X).$$

2. *For any quasi-poset  $P$ ,  $\alpha_P = (-1)^{cl(P)+cc(P)} \alpha_P^{str}$ .*

3.  *$\alpha$  is invertible in  $M_{\mathbf{qp}}$ . We denote by  $\beta$  its inverse. For any quasi-poset  $P$ :*

$$\beta_P = (-1)^{cl(P)+cc(P)} \lambda_P = (-1)^{cl(P)+cc(P)} \frac{\mu_P}{cl(P)!}.$$

**Proof.** 1. We consider the morphism:

$$\phi : \begin{cases} \mathcal{H}_{\mathbf{qp}} & \longrightarrow \mathbb{K}[X] \\ P \in \mathbf{qp} & \longrightarrow (-1)^{cl(P)} ehr_P(-X). \end{cases}$$

We put:

$$\theta_1 : \begin{cases} \mathcal{H}_{\mathbf{qp}} & \longrightarrow \mathcal{H}_{\mathbf{qp}} \\ P \in \mathbf{qp} & \longrightarrow (-1)^{cl(P)} P, \end{cases} \quad \theta_2 : \begin{cases} \mathbb{K}[X] & \longrightarrow \mathbb{K}[X] \\ P(X) & \longrightarrow P(-X). \end{cases}$$

Both are Hopf algebra morphisms, and  $\phi = \theta_2 \circ ehr \circ \phi_1$ , so  $\phi$  is a Hopf algebra morphism. If  $P$  is a non discrete quasi-poset, then  $\phi(P)(1) = (-1)^{cl(P)} ehr_P(-1) = 0 = \varepsilon'(x)$ . If  $P$  is a discrete quasi-poset, then  $ehr_P(X) = X^{cl(P)}$ , so  $\phi(P)(1) = 1 = \varepsilon'(x)$ . By corollary 10,  $\phi = \phi_1 = ehr^{str}$ .

2. and 3. Immediate consequences of the first point.  $\square$

**Proposition 39** *The following map is a Hopf algebra automorphism:*

$$\theta : \begin{cases} (\mathcal{H}_{\mathbf{qp}}, m, \Delta) & \longrightarrow (\mathcal{H}_{\mathbf{qp}}, m, \Delta) \\ P & \longrightarrow \sum_{\sim \triangleleft P} P / \sim. \end{cases}$$

*Its inverse is:*

$$\theta^{-1} : \begin{cases} (\mathcal{H}_{\mathbf{qp}}, m, \Delta) & \longrightarrow (\mathcal{H}_{\mathbf{qp}}, m, \Delta) \\ P & \longrightarrow \sum_{\sim \triangleleft P} (-1)^{cl(\sim)+cl(P)} P / \sim. \end{cases}$$

*Moreover:*

$$ehr^{str} \circ \theta = ehr, \quad ehr \circ \theta^{-1} = ehr^{str}.$$

**Proof.** Let  $\iota$  be the character of  $\mathcal{H}_{\mathbf{qp}}$  which sends any quasi-poset to 1. By corollary 20,  $\theta$  is an automorphism and  $\theta^{-1} = \phi_{\iota^* - 1}$ . For any quasi-poset  $P$ :

$$\iota(1) = 1 = ehr_P(1) = \sum_{\sim \triangleleft P} \frac{\mu_{P/\sim}}{cl(P/\sim)!} \alpha_{P/\sim} = \sum_{\sim \triangleleft P} \lambda_{P/\sim} \alpha_{P/\sim} = \lambda * \alpha(P),$$

so  $\iota = \lambda * \alpha$ ; hence, its inverse is  $\beta * \alpha^{str}$ , and for any quasi-poset  $P$ , as  $cl(P/\sim) = cl(\sim)$  and  $cc(P/\sim) = cc(P)$  for any  $\sim \triangleleft P$ :

$$\begin{aligned} \beta * \alpha^{str}(P) &= \sum_{\sim \triangleleft P} (-1)^{cl(\sim)+cc(P)} \frac{\mu_{P/\sim}}{cl(P/\sim)!} \alpha_{P/\sim}^{str} \\ &= (-1)^{cc(P)} ehr_P^{str}(-1) \\ &= (-1)^{cc(P)+cl(P)} ehr^{str}(1) \\ &= (-1)^{cc(P)+cl(P)}. \end{aligned}$$

Hence:

$$\theta^{-1}(P) = Id \leftarrow (\beta * \alpha^{str})(P) = \sum_{\sim \triangleleft P} (-1)^{cc(P/\sim)+cl(P/\sim)} P / \sim = \sum_{\sim \triangleleft P} (-1)^{cl(\sim)+cl(P)} P / \sim.$$

*Moreover:*

$$\begin{aligned} ehr^{str} \circ \theta &= (\phi_0 \leftarrow \alpha^{str}) \circ (Id \leftarrow \iota) \\ &= ((\phi_0 \leftarrow \alpha^{str}) \circ Id) \leftarrow \iota \\ &= (\phi_0 \leftarrow \alpha^{str}) \leftarrow \iota \\ &= \phi_0 \leftarrow (\alpha^{str} * \iota) \\ &= \phi_0 \leftarrow (\alpha^{str} * \lambda * \alpha) \\ &= \phi_0 \leftarrow \alpha \\ &= ehr. \end{aligned}$$

$\square$

#### 4.4 A link with Bernoulli numbers

For any  $k \in \mathbb{N}$ , let  $c_k$  be the corolla quasi-poset with  $k$  leaves:  $c_k = ([k+1], \leq_{c_k})$ , with  $1 \leq_{c_k} 2, \dots, k+1$ :

$$c_0 = \bullet_1, \quad c_1 = \mathbf{1}_1^2, \quad c_2 = {}^2\mathbf{V}_1^3, \quad c_3 = {}^2\mathbf{V}_1^4 \dots$$

By proposition 34,  $\lambda_{c_k} = \frac{1}{k+1}$ . Moreover:

$$\begin{aligned} L_{c_k} &= \{f : [k+1] \longrightarrow \mathbb{N}^* \mid f(1) \leq f(2), \dots, f(k+1)\}, \\ L_{c_k}^{str} &= \{f : [k+1] \longrightarrow \mathbb{N}^* \mid f(1) < f(2), \dots, f(k+1)\}, \end{aligned}$$

so, for all  $n \geq 1$ :

$$Ehr_{c_k}^{str}(n) = (n-1)^k + \dots + 1^k = S_k(n),$$

where  $S_k(X)$  is the unique polynomial such that for all  $n \geq 1$ ,  $S_k(n) = 1^k + \dots + (n-1)^k$ . As a consequence,  $\alpha_{c_k}^{str}$  is equal to the  $k$ -th Bernoulli number  $b_k$ .

Let  $\sim \triangleleft c_k$ . As the equivalence classes of  $\sim$  are connected:

- The equivalence class of the minimal element 1 of  $c_k$  contains  $i$  leaves,  $0 \leq i \leq k$ .
- The other equivalence classes are formed by a unique leaf.

Hence:

$$\delta([c_k]) = \sum_{i=0}^k \binom{k}{i} [c'_{i,k-i}] \otimes [c_i] \bullet^{k-i},$$

where  $c'_{i,k-i}$  is the quasi poset on  $[k+1]$  such that:

$$1 \sim_{c'_{i,k-i}} \dots \sim_{c'_{i,k-i}} i+1 \leq_{c'_{i,k-i}} i+2, \dots, k+1.$$

Hence, by theorem 35:

$$\begin{aligned} S_k(X) &= \sum_{i=0}^k \binom{k}{i} \lambda_{c'_{i,k-i}} b_i X^{k-i+1} \\ &= \sum_{i=0}^k \binom{k}{i} \lambda_{c'_{i,k-i}} b_i X^{k-i+1} \\ &= \sum_{i=0}^k \binom{k}{i} \lambda_{c_{k-i}} b_i X^{k-i+1} \\ &= \sum_{i=0}^k \binom{k}{i} \frac{b_i}{k-i+1} X^{k-i+1}. \end{aligned}$$

We recover in this way Faulhaber's formula. For all  $n \geq 1$ ,  $ehr_{c_k}(n) = n^k + \dots + 1^k$ , and the duality principle gives, for all  $n \geq 1$ :

$$(-1)^{k+1} S_k(-n) = 1^k + \dots + n^k = S_k(n) + n^k.$$

## 5 Noncommutative version

### 5.1 Reminders on packed words

Let us recall the construction of the Hopf algebra of packed words  $\mathbf{WQSym}$  [15, 16].

**Definition 40** *Let  $w = x_1 \dots x_n$  be a word whose letters are positive integers.*

1. *We shall say that  $w$  is a packed word if there exists an integer  $k$  such that  $\{x_1, \dots, x_n\} = [k]$ . The set of packed words of length  $n$  is denoted by  $\mathbf{PW}(n)$ ; the set of all packed words is denoted by  $\mathbf{PW}$ .*
2. *There exists a unique increasing bijection  $f : \{x_1, \dots, x_n\} \rightarrow [k]$  for a well-chosen  $k$ . We denote by  $\text{Pack}(w)$  the packed word  $f(x_1) \dots f(x_n)$ . Note that  $w$  is packed if, and only if,  $w = \text{Pack}(w)$ .*
3. *Let  $I \subseteq \mathbb{N}$ . Let  $i_1 < \dots < i_p$  be the indices  $i$  such that  $x_i \in I$ . We denote by  $w_I$  the word  $x_{i_1} \dots x_{i_p}$ .*

As a vector space,  $\mathbf{WQSym}$  is generated by the set  $\mathbf{PW}$ . The product is given by:

$$\forall u \in \mathbf{PW}(k), \forall v \in \mathbf{PW}(l), u.v = \sum_{\substack{w=x_1 \dots x_{k+l} \in \mathbf{PW}(k+l), \\ \text{Pack}(x_1 \dots x_k)=u, \\ \text{Pack}(x_{k+1} \dots x_{k+l})=v}} w.$$

The unit is the empty word 1. The coproduct is given by:

$$\forall w \in \mathbf{PW}, \Delta(w) = \sum_{k=0}^{\max(w)} w_{\{1, \dots, k\}} \otimes \text{Pack}(w_{\{k+1, \dots, \max(w)\}}).$$

For example:

$$\begin{aligned} (11).(11) &= (1111) + (1122) + (2211), \\ (11).(12) &= (1112) + (1123) + (2212) + (2213) + (3312), \\ (11).(21) &= (1121) + (1132) + (2231) + (3321), \\ (12).(11) &= (1211) + (1222) + (1233) + (1322) + (2311), \\ (12).(12) &= (1212) + (1213) + (1223) + (1234) + (1323) + (1324) \\ &\quad + (1423) + (2312) + (2313) + (2314) + (2413) + (3412), \\ (12).(21) &= (1221) + (1231) + (1232) + (1243) + (1332) + (1342) \\ &\quad + (1432) + (2321) + (2331) + (2341) + (2431) + (3421), \end{aligned}$$

$$\begin{aligned} \Delta(111) &= (111) \otimes 1 + (111) \otimes 1, \\ \Delta(212) &= (212) \otimes 1 + (1) \otimes (11) + 1 \otimes (212), \\ \Delta(312) &= (312) \otimes 1 + (1) \otimes (21) + (12) \otimes (1) + 1 \otimes (312). \end{aligned}$$

### 5.2 Hopf algebra morphisms in $\mathbf{WQSym}$

**Proposition 41** *The two following maps are surjective Hopf algebra morphisms:*

$$\begin{aligned} EHR : & \begin{cases} (\mathcal{H}_{\mathbf{QP}}, m, \Delta) & \longrightarrow & \mathbf{WQSym} \\ P & \longrightarrow & \sum_{w \in W_P} w, \end{cases} \\ EHR^{str} : & \begin{cases} (\mathcal{H}_{\mathbf{QP}}, m, \Delta) & \longrightarrow & \mathbf{WQSym} \\ P & \longrightarrow & \sum_{w \in W_P^{str}} w. \end{cases} \end{aligned}$$

**Proof.** Let  $P \in \mathbf{QP}(k)$ ,  $Q \in \mathbf{QP}(l)$ , and  $w$  be a packed word of length  $k + l$ . Then:

- $w \in W_{PQ}$  if, and only if,  $Pack(w_1 \dots w_k) \in W_P$  and  $Pack(w_{k+1} \dots w_{k+l}) \in W_Q$ .
- $w \in W_{PQ}^{str}$  if, and only if,  $Pack(w_1 \dots w_k) \in W_P^{str}$  and  $Pack(w_{k+1} \dots w_{k+l}) \in W_Q^{str}$ .

This implies that :

$$EHR(PQ) = EHR(P)EHR(Q), \quad EHR^{str}(PQ) = EHR^{str}(P)EHR^{str}(Q).$$

Let  $P \in \mathbf{QP}(n)$ . We consider the two sets:

$$\begin{aligned} A &= \{(w, k) \mid w \in W_P, 0 \leq k \leq \max(w)\}, \\ B &= \{(O, w_1, w_2) \mid O \in Top(P), w_1 \in W_{Pack(P_{[n] \setminus O})}, w_2 \in W_{Pack(P|_O)}\}. \end{aligned}$$

We define a bijection between  $A$  and  $B$  by  $F(w, k) = (O, w_1, w_2)$ , where:

- $O = w^{-1}(\{k + 1, \dots, \max(w)\})$ .
- $w_1 = Pack(w_{\{1, \dots, k\}})$ .
- $w_2 = Pack(w_{\{k+1, \dots, \max(w)\}})$ .

Then:

$$\begin{aligned} \Delta \circ EHR(P) &= \sum_{(w,k) \in A} Pack(w_{\{1, \dots, k\}}) \otimes Pack(w_{\{k+1, \dots, \max(w)\}}) \\ &= \sum_{(O, w_1, w_2) \in B} w_1 \otimes w_2 \\ &= (EHR \otimes EHR) \circ \Delta(P). \end{aligned}$$

So  $EHR$  is a Hopf algebra morphism. In the same way,  $EHR^{str}$  is a Hopf algebra morphism.

Let  $w$  be a packed word of length  $n$ . We define a quasi-poset structure on  $[n]$  by  $i \leq_P j$  if, and only if,  $w_i \leq w_j$ . Then  $W_P^{str} = \{w\}$ , so  $EHR^{str}(P) = w$ :  $EHR^{str}$  is surjective. If  $w' \in W_P$ , then  $\max(w') \leq \max(w)$  with equality if, and only if,  $w = w'$ . Hence:

$$EHR(P) = w + \text{words } w' \text{ with } \max(w') < \max(w).$$

By a triangular argument,  $EHR$  is surjective. □

**Examples.**

$$\begin{array}{ll} EHR(\bullet_1) = (1), & EHR^{str}(\bullet_1) = (1), \\ EHR(\mathfrak{1}_1^2) = (12) + (11), & EHR^{str}(\mathfrak{1}_1^2) = (12), \\ EHR(\mathfrak{1}_2^1) = (21) + (11), & EHR^{str}(\mathfrak{1}_2^1) = (21), \\ EHR(\bullet_{1 \bullet 2}) = (12) + (21) + (11), & EHR^{str}(\bullet_{1 \bullet 2}) = (12) + (21) + (11), \\ EHR(\bullet_{\bullet 1, 2}) = (11), & EHR^{str}(\bullet_{\bullet 1, 2}) = (11). \end{array}$$

**Remark.** The Hopf algebra  $\mathbf{WQSym}$  has a polynomial representation [16]. Let  $X = \{x_1, x_2, \dots\}$  be an infinite, totally ordered alphabet; then, for any packed word  $w$ :

$$P_w(X) = \sum_{w' \in X^*, Pack(w')=w} w'.$$

With this polynomial representation, for any  $P \in \mathbf{QP}(n)$ :

$$EHR(P) = \sum_{f \in L_P} x_{f(1)} \dots x_{f(n)}, \quad EHR^{str}(P) = \sum_{f \in L_P^{str}} x_{f(1)} \dots x_{f(n)}.$$

**Proposition 42** *The following map is a Hopf algebra automorphism:*

$$\Theta : \begin{cases} (\mathcal{H}_{\mathbf{QP}}, m, \Delta) & \longrightarrow (\mathcal{H}_{\mathbf{QP}}, m, \Delta) \\ P & \longrightarrow \sum_{\sim \triangleleft P} P / \sim . \end{cases}$$

*Its inverse is:*

$$\Theta^{-1} : \begin{cases} (\mathcal{H}_{\mathbf{QP}}, m, \Delta) & \longrightarrow (\mathcal{H}_{\mathbf{QP}}, m, \Delta) \\ P & \longrightarrow \sum_{\sim \triangleleft P} (-1)^{cl(P)+cl(\sim)} P / \sim . \end{cases}$$

*Moreover,  $EHR^{str} \circ \Theta = EHR$  and  $EHR \circ \Theta^{-1} = EHR^{str}$ .*

**Proof.** By corollary 20,  $\Theta = \phi_\iota$  is a Hopf algebra automorphism, where  $\iota$  is defined in the proof of proposition 39. Its inverse is  $\phi_{\iota^{-1}}$ .

Let us prove that:

$$W_G = \sum_{\sim \triangleleft P} W_{G/\sim}^{str}.$$

Let  $w \in W_G$ ; we define an equivalence  $\sim_w$  by  $x \sim_w y$  if  $w(x) = w(y)$  and  $x$  and  $y$  are in the same connected component of  $w^{-1}(w(x))$ . By definition, the equivalence classes of  $\sim_w$  are connected. If  $x \sim_{P/\sim_w} y$ , there exists  $x_1, x'_1, \dots, x_k, x'_k, y_1, y'_1, \dots, y_l, y'_l$  such that:

$$\begin{aligned} x \leq_P x_1 \sim_w x'_1 \leq_P \dots \leq_P x_k \sim_w x'_k \leq_P y, \\ y \leq_P y_1 \sim_w y'_1 \leq_P \dots \leq_P y_l \sim_w y'_l \leq_P x. \end{aligned}$$

As  $w \in W_P$ ,  $w(x) \leq w(x_1) = w(x'_1) \leq \dots \leq w(x_k) = w(x'_k) \leq w(y)$ ; by symmetry,  $w(x) = w(x_1) = \dots = w(x'_k) = w(y) = i$ . Moreover, as the equivalence classes of  $\sim_w$  are connected,  $x$  and  $y$  are in the same connected component of  $w^{-1}(i)$ , so  $x \sim_w y$ :  $\sim_w \triangleleft P$ .

If  $x \leq_P y$  or  $x \sim_w y$ , then  $w(x) \leq w(y)$ . By transitive closure, if  $x \leq_{P/\sim_w} y$ , then  $w(x) \leq w(y)$ , so  $w \in W_{P/\sim_w}$ . Moreover, if  $w(i) \neq w(j)$ , we do not have  $x \sim_w y$ , so  $w \in W_{P/\sim_w}^{str}$ .

Let us assume that  $\sim \triangleleft P$  and let  $w \in W_{P/\sim}^{str}$ . If  $x \leq_P y$ , then  $x \leq_{P/\sim} y$ , so  $w(x) \leq w(y)$ :  $W_{P/\sim}^{str} \subseteq W_P$ .

Let us assume that  $w \in W_{P/\sim}^{str}$ , with  $\sim \triangleleft P$ . If  $x \sim y$ , then  $w(x) = w(y) = i$  and  $x$  and  $y$  are in the same connected component of  $P/\sim$ , so are in the the same connected component of  $w^{-1}(i)$ :  $x \sim_w y$ . If  $x \sim_w y$ , then  $w(x) = w(y) = i$  and there exists  $x_1, x'_1, \dots, x_k, x'_k$  with  $w(x_1) = w(x'_1) = \dots = w(x_k) = w(x'_k) = i$  such that:

$$x \leq_P x_1 \geq_P x'_1 \leq_P \dots \geq_P x'_k \leq_P y.$$

As  $w \in W_{P/\sim}^{str}$ ,  $x \sim_{P/\sim} x_1$ ,  $x_1 \sim_{P/\sim} x'_1, \dots, x'_k \sim_{P/\sim} y$ . So  $x \sim_{P/\sim} y$ ; as  $\sim \triangleleft P$ ,  $x \sim y$ . Finally,  $\sim = \sim_w$ .

We obtain that:

$$EHR(P) = \sum_{w \in W_P} w = \sum_{\sim \triangleleft P} \sum_{w \in W_{P/\sim}^{str}} w = \sum_{\sim \triangleleft P} EHR^{str}(P/\sim) = EHR^{str}(\Theta(P)).$$

So  $Ehr^{str} \circ \Theta = EHR$ . □

**Proposition 43** *Let us consider the following map:*

$$H : \begin{cases} \mathbf{WQSym} & \longrightarrow \mathbb{K}[X] \\ w \in \mathbf{PW} & \longrightarrow H_{\max(w)}(X). \end{cases}$$

This is a surjective Hopf algebra morphism, making the following diagram commuting:

$$\begin{array}{ccccc}
& & \mathcal{H}_{\mathbf{QP}} & & \\
& \swarrow \sqcup & \downarrow \theta & \searrow EHR & \\
\mathcal{H}_{\mathbf{QP}} & & \mathcal{H}_{\mathbf{QP}} & \xrightarrow{EHR^{str}} & \mathbf{WQSym} \\
\downarrow \theta & \swarrow \sqcup & \downarrow \theta & \searrow H & \\
\mathcal{H}_{\mathbf{QP}} & \xrightarrow{ehr^{str}} & \mathbb{K}[X] & & 
\end{array}$$

**Proof.** Let  $P \in \mathbf{QP}$ . Then:

$$ehr(\lfloor P \rfloor) = \sharp W_P(k) H_k(X) = \sum_{w \in W_P} H_{\max(w)}(X) = \sum_{w \in W_P} H(w) = H \circ EHR(P).$$

So  $ehr \circ \sqcup = H \circ EHR$ . Similarly,  $ehr^{str} \circ \sqcup = H \circ EHR^{str}$ .

Let us prove that  $H$  is a Hopf algebra morphism. Let  $w_1, w_2 \in \mathbf{WQSym}$ . There exist  $x_1, x_2 \in \mathcal{H}_{\mathbf{QP}}$ , such that  $w_1 = EHR(x_1)$  and  $w_2 = EHR(x_2)$ . Then:

$$\begin{aligned}
H(w_1 w_2) &= H(EHR(x_1) EHR(x_2)) \\
&= H \circ EHR(x_1 x_2) \\
&= ehr(\lfloor x_1 x_2 \rfloor) \\
&= ehr(\lfloor x_1 \rfloor) ehr(\lfloor x_2 \rfloor) \\
&= H \circ EHR(x_1) H \circ EHR(x_2) \\
&= H(w_1) H(w_2).
\end{aligned}$$

Let  $w \in \mathbf{WQSym}$ . There exists  $x \in \mathcal{H}_{\mathbf{QP}}$  such that  $w = EHR(x)$ .

$$\begin{aligned}
\Delta \circ H(w) &= \Delta \circ H \circ EHR(x) \\
&= (H \otimes H) \circ (EHR \otimes EHR) \circ \Delta(x) \\
&= (H \otimes H) \circ \Delta \circ EHR(x) \\
&= (H \otimes H) \circ \Delta(w).
\end{aligned}$$

So  $H$  is a Hopf algebra morphism. □

### 5.3 The non-commutative duality principle

**Lemma 44** *The following map is an involution and a Hopf algebra automorphism:*

$$\Phi_{-1} : \begin{cases} \mathbf{WQSym} & \longrightarrow & \mathbf{WQSym} \\ w & \longrightarrow & (-1)^{\max(w)} \sum_{\sigma : [\max(w)] \rightarrow [l], \text{ non-decreasing}} \sigma \circ w. \end{cases}$$

**Proof.** Using the surjective morphisms  $EHR^{str}$  and  $ehr^{str}$ , taking the quotients of the cointeracting bialgebras  $(\mathcal{H}_{\mathbf{QP}}, m, \Delta)$  and  $(\mathcal{H}_{\mathbf{QP}}, m, \delta)$ , we obtain that  $(\mathbf{WQSym}, m, \Delta)$  and  $(\mathbb{K}[X], m, \delta)$  are cointeracting bialgebras, with the coaction defined by:

$$\rho = (Id \otimes H) \circ \delta : \mathbf{WQSym} \longrightarrow \mathbf{WQSym} \otimes \mathbb{K}[X]$$

For any packed word  $w$ :

$$\rho(w) = \sum_{\sigma : [k] \rightarrow [l], \text{ non-decreasing}} \sigma \circ w \otimes H_{\max(\text{Pack}(w_{|(\sigma \circ w)^{-1}(1)}))}(X) \dots H_{\max(\text{Pack}(w_{|(\sigma \circ w)^{-1}(l)}))}(X).$$

Using proposition 4, for any  $\lambda \in \mathbb{K}$ , considering the character:

$$ev_\lambda : \begin{cases} \mathbb{K}[X] & \longrightarrow \mathbb{K} \\ P & \longrightarrow P(\lambda), \end{cases}$$

we obtain an endomorphism  $\Phi_\lambda$  of  $(\mathbf{WQSym}, m, \Delta)$  defined by  $\Phi_\lambda = Id \leftarrow ev_\lambda$ . if  $\lambda \neq 0$ ,  $\Phi_\lambda$  is invertible, of inverse  $\Phi_{\lambda^{-1}}$ . For any packed word  $w$ , denoting by  $k$  its maximum:

$$\Phi_\lambda(w) = \sum_{\sigma : [k] \rightarrow [l], \text{ non-decreasing}} H_{\max(\text{Pack}(w_{|(\sigma \circ w)^{-1}(1)}))}(\lambda) \cdots H_{\max(\text{Pack}(w_{|(\sigma \circ w)^{-1}(l)}))}(\lambda) \sigma \circ w.$$

In particular, for  $\lambda = -1$ , for any  $p \in \mathbb{N}$ :

$$H_p(-1) = \frac{(-1)(-2) \cdots (-k)}{k!} = (-1)^k.$$

Hence:

$$\begin{aligned} \Phi_{-1}(w) &= \sum_{\sigma : [k] \rightarrow [l], \text{ non-decreasing}} (-1)^{\max(\text{Pack}(w_{|(\sigma \circ w)^{-1}(1)})) + \cdots + \max(\text{Pack}(w_{|(\sigma \circ w)^{-1}(l)}))} \sigma \circ w \\ &= (-1)^k \sum_{\sigma : [k] \rightarrow [l], \text{ non-decreasing}} \sigma \circ w. \end{aligned}$$

Indeed, if  $x \in (\sigma \circ w)^{-1}(p)$  and  $y \in (\sigma \circ w)^{-1}(q)$ , with  $p < q$ , then  $\sigma \circ w(x) < \sigma \circ w(y)$ ; as  $\sigma$  is non-decreasing,  $x < y$ . So there exists  $n_1 < n_2 < \cdots < n_l = k$  such that for all  $p$ , the values taken by  $w$  on  $(\sigma \circ w)^{-1}(p)$  are  $n_{p-1} + 1, \dots, n_p$ . Hence, the values taken by  $\text{Pack}(w_{|(\sigma \circ w)^{-1}(p)})$  are  $1, \dots, n_p - n_{p-1}$ , so:

$$\max(\text{Pack}(w_{|(\sigma \circ w)^{-1}(1)})) + \cdots + \max(\text{Pack}(w_{|(\sigma \circ w)^{-1}(l)})) = n_1 + n_2 - n_1 + \cdots + n_l - n_{l-1} = n_l = k.$$

In particular,  $\Phi_{-1}$  is an involution and a Hopf algebra automorphism of  $(\mathbf{WQSym}, m, \Delta)$ .  $\square$

**Theorem 45 (Non commutative duality principle)** *For any quasi-poset  $P \in \mathbf{QP}$ :*

$$EHR(P) = (-1)^{cl(P)} \Phi_{-1} \circ ERH^{str}(P), \quad EHR^{str}(P) = (-1)^{cl(P)} \Phi_{-1} \circ ERH(P).$$

**Proof.** We shall use the following involution and Hopf algebra automorphism:

$$\Psi : \begin{cases} \mathcal{H}_{\mathbf{QP}} & \longrightarrow \mathcal{H}_{\mathbf{QP}} \\ p \in \mathbf{QP} & \longrightarrow (-1)^{cl(P)} P. \end{cases}$$

Recall that the character  $\iota$  of  $\mathcal{H}_{\mathbf{QP}}$  sends any  $P \in \mathbf{QP}$  to 1. By the duality principle:

$$\iota \circ \Psi(P) = (-1)^{cl(P)} = (-1)^{cl(P)} ehr(P)(1) = ehr^{str}(-1) = ev_{-1} \circ ehr^{str}(P).$$

So  $\iota \circ \Psi = ev_{-1} \circ ehr^{str}$ .

Let  $P \in \mathbf{QP}$ . Recalling that if  $\sim \triangleleft P$ ,  $cl(P| \sim) = cl(P)$ :

$$\delta \circ \Psi(P) = (-1)^{cl(P)} \sum_{\sim \triangleleft P} P/ \sim \otimes P| \sim = \sum_{\sim \triangleleft P} P/ \sim \otimes (-1)^{cl(P| \sim)} P| \sim = (Id \otimes \Psi) \circ \delta(P).$$

So  $\delta \circ \Psi = (Id \otimes \Psi) \circ \delta$ . Hence, for any  $x \in \mathcal{H}_{\mathbf{QP}}$ :

$$\begin{aligned} EHR \circ \Psi(x) &= EHR^{str} \circ (Id \leftarrow \iota) \circ \Psi(x) \\ &= EHR^{str}(\Psi(x)_0) \iota \circ \Psi(x)_1 \\ &= EHR^{str}(x_0) \iota \circ \Psi(x_1) \\ &= EHR^{str}(x_0) ev_{-1} \circ ehr^{str}(x_1) \\ &= EHR^{str}(x^{(1)}) ev_{-1} \circ EHR^{str}(x^{(2)}) \\ &= EHR \leftarrow ev_{-1}(x) \\ &= (Id \leftarrow ev_{-1}) \circ EHR^{str}(x) \\ &= \Phi_{-1} \circ EHR^{str}(x), \end{aligned}$$



where we denote  $\delta(x) = x^{(1)} \otimes x^{(2)}$  and  $\rho(x) = x_0 \otimes x_1$ . As  $\Phi_{-1}$  and  $\Psi$  are involutions,  $EHR^{str} \circ \Psi = \Phi_{-1} \circ EHR$ .  $\square$

In  $E_{\mathbb{K}[X] \rightarrow \mathbb{K}[X]}$ , putting  $\phi_\lambda = Id \leftarrow ev_\lambda$ , for any  $P \in \mathbb{K}[X]$ ,  $\phi_\lambda(P) = P(\lambda X)$ . Moreover, as  $H$  is compatible with the coactions:

$$H \circ \Phi_\lambda = H \circ (Id \leftarrow ev_\lambda) = H \leftarrow ev_\lambda = (Id \leftarrow ev_\lambda) \circ H = \phi_\lambda \circ H,$$

so:

$$ehr \circ \Psi = H \circ EHR \circ \Psi = H \circ \Phi_{-1} \circ EHR^{str} = \phi_{-1} \circ H \circ EHR^{str} = \phi_{-1} \circ ehr^{str}.$$

In other words, for any  $P \in \mathbf{QP}$ ,  $(-1)^{cl(P)} ehr_P(X) = ehr_P^{str}(-X)$ : we recover the duality principle.

We obtain the commutative diagram of Hopf algebra morphisms:

$$\begin{array}{ccccc}
\mathcal{H}_{\mathbf{QP}} & & & & \\
\downarrow \ominus & \searrow EHR & & & \\
\mathcal{H}_{\mathbf{QP}} & \xrightarrow{EHR^{str}} & \mathbf{WQSym} & \sqcup & \\
\downarrow \Psi & & \downarrow \Phi_{-1} & & \\
\mathcal{H}_{\mathbf{QP}} & \xrightarrow{EHR} & \mathbf{WQSym} & \sqcup & \\
& & & \sqcup & \\
& & & & \mathcal{H}_{\mathbf{qp}} \\
& & & \searrow H & \downarrow \theta \\
& & & & \mathcal{H}_{\mathbf{qp}} \xrightarrow{ehr} \mathbb{K}[X] \\
& & & \searrow H & \downarrow \psi \\
& & & & \mathcal{H}_{\mathbf{qp}} \xrightarrow{ehr^{str}} \mathbb{K}[X] \\
& & & & \downarrow \phi_{-1} \\
& & & & \mathcal{H}_{\mathbf{qp}} \xrightarrow{chr} \mathbb{K}[X]
\end{array}$$

#### 5.4 Compatibility with the other product and coproduct

**Theorem 46** We define a second coproduct  $\delta$  on  $\mathbf{WQSym}$ :

$$\forall w \in \mathbf{PW}, \delta(w) = \sum_{(\sigma, \tau) \in A_w} (\sigma \circ w) \otimes (\tau \circ w),$$

where  $A_w$  is the set of pairs of packed words  $(\sigma, \tau)$  of length  $\max(w)$  such that:

- $\sigma$  is non-decreasing.
- If  $1 \leq i < j \leq \max(w)$  and  $\sigma(i) = \sigma(j)$ , then  $\tau(i) < \tau(j)$ .

Then  $(\mathbf{WQSym}, m, \delta)$  is a bialgebra and  $EHR^{str}$  is a bialgebra morphism from  $(\mathcal{H}_{\mathbf{QP}}, m, \delta)$  to  $(\mathbf{WQSym}, m, \delta)$ .

**Proof.** Let us prove that  $\delta \circ EHR^{str} = (EHR^{str} \otimes EHR^{str}) \circ \delta$ . Let  $P \in \mathbf{QP}$ . We consider the two following sets:

$$\begin{aligned}
A &= \{(\sim, w_1, w_2) \mid \sim \triangleleft P, w_1 \in W_{P/\sim}^{str}, w_2 \in W_{P|\sim}^{str}\}, \\
B &= \{(w, \sigma, \tau) \mid w \in W_P^{str}, (\sigma, \tau) \in A_w\}.
\end{aligned}$$

Let  $(\sim, w_1, w_2) \in A$ . We put  $I_p = w_1^{-1}(p)$  for all  $1 \leq p \leq \max(w_1)$ , and  $w_2^{(p)}$  the standardization of the restriction of  $w_2$  to  $I_p$ . We define  $w$  by:

$$w(i) = w_2^{(p)}(i) + \max w_1^{(2)} + \dots + \max w_{p-1}^{(2)} \text{ if } i \in I_p.$$

Let us prove that  $w \in W_P^{str}$ . If  $x \leq_P y$ , then  $x \leq_{P/\sim} y$ , so  $p = w_1(x) \leq w_2(y) = q$ .

- If  $p < q$ , then  $w(x) < w(y)$ .
- If  $p = q$ , then  $w_1(x) = w_2(y)$  and, as  $x \leq_P y$ ,  $x$  and  $y$  are in the same connected component of  $w^{-1}(p)$ . So  $x \sim_{w_1} y$ , that is to say  $x \sim y$  as  $w_1 \in W_{P/\sim}^{str}$ , and  $x \leq_{P|\sim} y$ , which implies that  $w_2(x) \leq w_2(y)$  and finally  $w(x) \leq w(y)$ .

Let us assume that moreover  $w(x) = w(y)$ . Then  $p = q$  and necessarily,  $w_2(x) = w_2(y)$ . As  $w_2 \in W_{P|\sim}^{str}$ ,  $x \sim_{P|\sim} y$ , so  $x \sim_P y$ .

If  $w(x) = w(y)$ , then by definition of  $w$ ,  $w_1(x) = w_1(y)$ . So there exists a unique  $\sigma : [\max(w)] \rightarrow [\max(w_1)]$ , such that  $w_1 = \sigma \circ w$ . If  $w(x) < w(y)$ , then, by construction of  $w$ ,  $w_1(x) \leq w_1(y)$ :  $\sigma$  is non-decreasing.

There exists a unique  $\tau : [\max(w)] \rightarrow [\max(w_2)]$ , such that  $w_2 = \tau \circ \sigma$ . As  $Pack(w|_{I_p}) = Pack((w_2)|_{I_p})$  for all  $p$ ,  $\tau$  is increasing on  $I_p$ .

To any  $(\sim, w_1, w_2) \in A$ , we associate  $(w, \sigma, \tau) = F(\sim, w_1, w_2) \in B$ , such that  $w_1 = \sigma \circ w$ ,  $w_2 = \tau \circ \sigma$ , and  $\sim = \sim_{\sigma \circ \tau}$ . This defines a map  $F : A \rightarrow B$ .

Let  $(w, \sigma, \tau) \in B$ . We put  $G(w, \sigma, \tau) = (\sim, \sigma, \tau) = (\sim_{\sigma \circ w}, \sigma \circ w, \tau \circ w)$ . If  $x \leq_P y$ , then  $w(x) \leq w(y)$ , so  $w_1(x) = \sigma \circ w(x) \leq \sigma \circ w(y) = w_1(y)$ . If moreover  $w_1(x) = w_1(y)$ , then as  $x \leq_P y$ ,  $x$  and  $y$  are in the same connected component of  $w_1^{-1}(w_1(x))$ , so  $x \sim_{w_1} y$ :  $w_1 \in W_{P/\sim}^{str}$ .

If  $x \leq_{P|\sim} y$ , then  $x \sim_{w_1} y$  and  $x \leq_P y$ , so  $w_1(x) = w_1(y)$  and  $w(x) \leq w(y)$ . By hypothesis on  $\tau$ ,  $\tau \circ w(x) \leq \tau \circ w(y)$ , so  $w_2(x) \leq w_2(y)$ . If moreover  $w_2(x) = w_2(y)$ , by hypothesis on  $\tau$ ,  $w(x) = w(y)$ . As  $w \in W_P^{str}$ ,  $x \sim_P y$ , so  $x \sim_{P|\sim} y$ :  $w_2 \in W_{P|\sim}^{str}$ .

We defined in this way a map  $G : B \rightarrow A$ . If  $(\sim, w_1, w_2) \in A$ :

$$G \circ F(\sim, w_1, w_2) = G(w, \sigma, \tau) = (\sim_{\sigma \circ w}, \sigma \circ w, \tau \circ w) = (\sim_{w_1}, w_1, w_2) = (\sim, w_1, w_2).$$

So  $G \circ F = Id_A$ . If  $(w, \sigma, \tau) \in B$ :

$$F \circ G(w, \sigma, \tau) = F(\sim_{\sigma \circ w}, \sigma \circ w, \tau \circ w) = (w, \sigma, \tau).$$

So  $F \circ G = Id_B$ :  $F$  and  $G$  are inverse bijections.

We obtain:

$$\begin{aligned} (EHR^{str} \otimes EHR^{str}) \circ \delta(P) &= \sum_{(\sim, w_1, w_2) \in A} w_1 \otimes w_2 \\ &= \sum_{(w, \sigma, \tau) \in B} \sigma \circ w \otimes \tau \circ w \\ &= \sum_{w \in W_P^{str}} \delta(w) \\ &= \delta \circ EHR^{str}(P). \end{aligned}$$

So  $EHR^{str}$  is compatible with  $\delta$ .

As  $EHR^{str}$  is compatible with the product  $m$  and the coproduct  $\delta$ ,  $Ker(EHR^{str})$  is a biideal of  $(\mathcal{H}_{\mathbf{QP}}, m, \delta)$ , and  $(\mathbf{WQSym}, m, \delta)$  is identified with the quotient  $\mathcal{H}_{\mathbf{QP}}/Ker(EHR^{str})$ , so is a bialgebra.  $\square$

**Examples.**

$$\begin{aligned} \delta(11) &= (11) \otimes (11), \\ \delta(12) &= (12) \otimes ((11) + (12) + (21)) + (11) \otimes (12), \\ \delta(21) &= (21) \otimes ((11) + (12) + (21)) + (11) \otimes (21). \end{aligned}$$

This coproduct  $\delta$  on **WQSym** is the internal coproduct of [16], dual to the product of the Solomon-Tits algebra.

**Remarks.**

1. The count of  $(\mathbf{WQSym}, m, \delta)$  is given by:

$$\varepsilon_B(w) = \begin{cases} 1 & \text{if } w = (1 \dots 1), \\ 0 & \text{otherwise.} \end{cases}$$

2. There is no coproduct  $\delta'$  on **WQSym** such that  $(EHR \otimes EHR) \circ \delta = \delta' \circ EHR$ . Indeed, if  $\delta'$  is any coproduct on **WQSym**, for  $x = \mathbf{1}_1^2 + \mathbf{1}_2^1 - \bullet_{1,2} - \bullet_{1,2}$ :

$$\delta' \circ EHR(x) = \delta'(0) = 0,$$

but:

$$\begin{aligned} & (EHR \otimes EHR) \circ \delta(x) \\ &= (EHR \otimes EHR)((\mathbf{1}_1^2 + \mathbf{1}_2^1 - \bullet_{1,2}) \otimes \bullet_{1,2} + \bullet_{1,2} \otimes (\mathbf{1}_1^2 + \mathbf{1}_2^1 - \bullet_{1,2} - \bullet_{1,2})) \\ &= (11) \otimes (11). \end{aligned}$$

**Proposition 47**  $H : (\mathbf{WQSym}, m, \delta) \longrightarrow (\mathbb{K}[X], m, \delta)$  is a bialgebra morphism.

**Proof.** Let  $w$  be a packed word. We denote  $k = \max(w)$ . Let  $a, b \in \mathbb{N}$ .

$$\begin{aligned} (H \otimes H) \circ \delta(w)(a, b) &= \sum_{(\sigma, \tau) \in A_w} H_{\max(\sigma \circ w)}(a) H_{\max(\tau \circ w)}(b) \\ &= \sum_{\sigma : [k] \rightarrow [l], \text{ non-decreasing}} \binom{a}{l} \binom{b}{|\sigma^{-1}(1)|} \cdots \binom{b}{|\sigma^{-1}(l)|} \\ &= \sum_{\substack{1 \leq l \leq k, \\ i_1 + \dots + i_l = k}} \binom{a}{l} \binom{b}{i_1} \cdots \binom{b}{i_l} \\ &= \binom{ab}{k} \\ &= H_k(ab) \\ &= \delta(H(w))(a, b). \end{aligned}$$

As this is true for any  $a, b \in \mathbb{N}$ ,  $(H \otimes H) \circ \delta(w) = \delta \circ H$ . □

**Definition 48** Let  $w = w_1 \dots w_k$  and  $w' = w'_1 \dots w'_l$  be two packed words. We put:

$$\begin{aligned} w \downarrow w' &= w_1 \dots w_k (w'_1 + \max(w)) \dots (w'_l + \max(w)), \\ w \circledast w' &= w_1 \dots w_k (w'_1 + \max(w) - 1) \dots (w'_l + \max(w) - 1), \\ w \downarrow\! \! \! \downarrow w' &= w \downarrow w' + w \circledast w'. \end{aligned}$$

These three products are extended to **WQSym** by bilinearity.

**Proposition 49** For all  $x, y \in \mathcal{H}_{\mathbf{QP}}$ :

$$EHR^{str}(x \downarrow y) = EHR^{str}(x) \downarrow EHR^{str}(y), \quad EHR(x \downarrow y) = EHR(x) \downarrow\! \! \! \downarrow EHR(y).$$

**Proof.** Let  $P \in \mathbf{QP}(k)$  and  $Q \in \mathbf{QP}(l)$ . If  $w = w_1 \dots w_{k+l}$  is a packed word of length  $k+l$ :

$$\begin{aligned} w \in W_{P \downarrow Q}^{str} &\iff w_1 \dots w_k \in L_P^{str}, w_{k+1} \dots w_{k+l} \in L_Q^{str}, w_1, \dots, w_k < w_{k+1}, \dots, w_{k+l} \\ &\iff w = w_P \downarrow w_Q, \text{ with } w_P \in W_P^{str}, w_Q \in W_Q^{str}. \end{aligned}$$

So  $W_{P \downarrow Q}^{str} = W_P^{str} \downarrow W_Q^{str}$ , and:

$$EHR^{str}(P \downarrow Q) = \sum_{w_P \in W_P^{str}, w_Q \in W_Q^{str}} w_P \downarrow w_Q = EHR^{str}(P) \downarrow EHR^{str}(Q).$$

If  $w = w_1 \dots w_{k+l}$  is a packed word of length  $k+l$ :

$$\begin{aligned} w \in W_{P \downarrow Q} &\iff w_1 \dots w_k \in L_P, w_{k+1} \dots w_{k+l} \in L_Q, w_1, \dots, w_k \leq w_{k+1}, \dots, w_{k+l} \\ &\iff w = (w_P \downarrow w_Q, \text{ with } w_P \in W_P, w_Q \in W_Q) \\ &\text{or } w = (w_P \otimes w_Q, \text{ with } w_P \in W_P, w_Q \in W_Q). \end{aligned}$$

Note that these two conditions are incompatible:

- in the first case,  $\max(w_1 \dots w_k) = \min(w_{k+1} \dots w_{k+l}) - 1$ ;
- in the second case,  $\max(w_1 \dots w_k) = \min(w_{k+1} \dots w_{k+l})$ .

So  $W_{P \downarrow Q} = (W_P \downarrow W_Q) \sqcup (W_P \otimes W_Q)$ , and:

$$\begin{aligned} EHR(P \downarrow Q) &= \sum_{w_P \in W_P, w_Q \in W_Q} w_P \downarrow w_Q + w_P \otimes w_Q \\ &= EHR(P) \downarrow EHR(Q) + EHR(P) \otimes EHR(Q), \end{aligned}$$

so  $EHR(P \downarrow Q) = EHR(P) \not\downarrow EHR(Q)$ . □

**Remark.** As a consequence,  $(\mathbf{WQSym}, \downarrow, \Delta)$  and  $(\mathbf{WQSym}, \not\downarrow, \Delta)$  are infinitesimal bialgebras [12], as  $(\mathcal{H}_{\mathbf{QP}}, \downarrow, \Delta)$  is [11, 10].

**Corollary 50** For all  $x, y \in \mathbf{WQSym}$ :

$$\Phi_{-1}(x \downarrow y) = \Phi_{-1}(x) \not\downarrow \Phi_{-1}(y) \qquad \Phi_{-1}(x \not\downarrow y) = \Phi_{-1}(x) \downarrow \Phi_{-1}(y).$$

**Proof.** If  $P, Q \in \mathbf{QP}$ , then  $cl(P \downarrow Q) = cl(P) + cl(Q)$ , so:

$$\Psi(P \downarrow Q) = (-1)^{cl(P)+cl(Q)} P \downarrow Q = \Psi(P) \downarrow \Psi(Q).$$

Let  $x, y \in \mathbf{WQSym}$ . There exist  $x', y' \in \mathcal{H}_{\mathbf{QP}}$ , such that  $EHR^{str}(x') = x$  and  $EHR^{str}(y') = y$ . Hence, using the non-commutative duality principle:

$$\begin{aligned} \Phi_{-1}(x \downarrow y) &= \Phi_{-1}(EHR^{str}(x') \downarrow EHR^{str}(y')) \\ &= \Phi_{-1} \circ EHR^{str}(x' \downarrow y') \\ &= \Phi_{-1} \circ EHR^{str} \circ \Psi(x') \downarrow \Psi(y') \\ &= EHR(\Psi(x') \downarrow \Psi(y')) \\ &= EHR \circ \Psi(x') \not\downarrow EHR \circ \Psi(y') \\ &= \Phi_{-1}(\Phi_{-1} \circ EHR \circ \Psi(x')) \not\downarrow \Phi_{-1}(\Phi_{-1} \circ EHR \circ \Psi(y')) \\ &= \Phi_{-1}(EHR^{str}(x')) \not\downarrow \Phi_{-1}(EHR^{str}(y')) \\ &= \Phi_{-1}(x) \not\downarrow \Phi_{-1}(y). \end{aligned}$$

As  $\Phi_{-1}$  is an involution, we obtain the second point. □

## 5.5 Restriction to posets

In [10], the image of the restriction to  $\mathcal{H}_{\mathbf{P}}$  of the map from  $\mathcal{H}_{\mathbf{QP}}$  to  $\mathbf{WQSymb}$  defined by  $T$ -partitions is a Hopf subalgebra, isomorphic to the Hopf algebra of permutations  $\mathbf{FQSymb}$  [13, 7]. This is not the case here:

**Proposition 51**  $EHR(\mathcal{H}_{\mathbf{P}}) = EHR^{str}(\mathcal{H}_{\mathbf{P}}) = \mathbf{WQSymb}$ .

**Proof.** Let  $w$  be a packed word of length  $n$ . We define a poset  $P$  on  $[n]$  by:

$$\forall i, j \in [n], i \leq_P j \text{ if } (i = j) \text{ or } (w(i) < w(j)).$$

Note that if  $i \leq_P j$ , then  $w(i) \leq w(j)$ . If  $i \leq_P j$  and  $j \leq_P k$ , then:

- if  $i = j$  or  $j = k$ , then obviously  $i \leq_P k$ .
- Otherwise,  $w(i) < w(j)$  and  $w(j) < w(k)$ , so  $w(i) < w(k)$  and  $i \leq_P k$ .

Let us assume that  $i \leq_P j$  and  $j \leq_P i$ . Then  $w(i) \leq w(j)$  and  $w(j) \leq w(i)$ , so  $w(i) = w(j)$ . As  $i \leq_P j$ ,  $i = j$ . So  $P$  is indeed a poset, and we observed that  $w \in W_P$ .

Let  $w'$  be a packed word of length  $n$ . Let us prove that  $w' \in W_P^{str}$  if, and only if,  $w \leq w'$ , where  $\leq$  is the order on packed words defined in definition 28.

$\implies$ . Let us assume that  $w' \in W_P^{str}$ . If  $w(i) < w(j)$ , then  $i \leq_P j$ , so  $w'(i) \leq w'(j)$ . Moreover, if  $w'(i) = w'(j)$ , then  $i \leq_P j$ , so  $i = j$  as  $P$  is a poset, and finally  $w(i) = w(j)$ : contradiction. So  $w'(i) < w'(j)$ , we shows that  $w \leq w'$ .

$\impliedby$ . Let us assume that  $w' \leq w$ . If  $i \leq_P j$ , then  $i = j$  or  $w(i) < w(j)$ , so  $w'(i) = w'(j)$  or  $w'(i) < w'(j)$ . If, moreover,  $w'(i) = w'(j)$ , then  $i = j$ ; so  $w' \in W_P^{str}$ .

We obtain an element  $P \in \mathcal{H}_{\mathbf{P}}$  such that:

$$EHR^{str}(P) = \sum_{w \leq w'} w'.$$

As this holds for any  $w$ , by a triangularity argument,  $EHR^{str}(\mathcal{H}_{\mathbf{P}}) = \mathbf{WQSymb}$ . By the non-commutative duality principle:

$$EHR(\mathcal{H}_{\mathbf{P}}) = \Phi_{-1} \circ EHR^{str} \circ \Psi(\mathcal{H}_{\mathbf{P}}) = \Phi_{-1} \circ EHR^{str}(\mathcal{H}_{\mathbf{P}}) = \Phi_{-1}(\mathbf{WQSymb}) = \mathbf{WQSymb},$$

as  $\Phi_{-1}$  is an automorphism of  $\mathbf{WQSymb}$ . □

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