On singular Fano varieties with a divisor of Picard number one

Pedro Montero

ON SINGULAR FANO VARIETIES WITH A DIVISOR OF PICARD NUMBER ONE

Pedro MONTERO

Abstract. — In this paper we study the geometry of mildly singular Fano varieties on which there is an effective prime divisor of Picard number one. Afterwards, we address the case of toric varieties. Finally, we treat the lifting of extremal contractions to universal covering spaces in codimension 1.

Contents

1. Introduction

The aim of this article is to study the geometry of mildly singular Fano varieties on which there is a prime divisor of Picard number one. Recall that a Fano variety is a normal complex algebraic variety whose anticanonical divisor has some positive multiple which is an ample Cartier divisor.

A first related result is given by L. Bonavero, F. Campana and J. A. Wiśniewski in the sequel of articles **[Bon02]** and **[BCW02]**, where the authors classified (toric) Fano varieties of dimension $n \geq 3$ on which there is a divisor isomorphic to $\mathbb{P}^{n-1}$ and later used these results to study (toric) complex varieties whose blow-up at a point is Fano. For instance, in the toric case we have the following result.

Theorem 1.1 **([Bon02**, Theorem 2]**).** — *Let X be a smooth toric Fano variety of dimension $n \geq 3$. Then, there exists a toric divisor D of X isomorphic to $\mathbb{P}^{n-1}$ if and only if one of the following situations occurs:*

1. *$X \cong \mathbb{P}^n$ and D is a linear codimension 1 subspace of X.*
2. *$X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n-1}) \cong \mathrm{Bl}_{\mathbb{P}^{n-2}}(\mathbb{P}^n)$ and D is a fiber of the projection on $\mathbb{P}^1$.*
3. *$X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(a))$, where $0 \leq a \leq n-1$, and D is either the divisor $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}})$ or the divisor $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}}(a))$.*
4. *X is isomorphic to the blow-up of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}} \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(a+1))$ along a linear $\mathbb{P}^{n-2}$ contained in the divisor $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}})$, where $0 \leq a \leq n-2$, and D is either the strict transform of the divisor $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}})$ or the strict transform of the divisor $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}}(a+1))$.*

In particular, this classification leads $\rho_X \leq 3$. Some years later, T. Tsukioka in **[Tsu06]** used some arguments from **[And85]** and **[BCW02]** to generalize these results and proved, more generally, that a smooth Fano variety X of dimension $n \geq 3$ containing an effective prime divisor of Picard number one must satisfy $\rho_X \leq 3$.

The bound $\rho_X \leq 3$ was recently proved by G. Della Noce in **[DN14**, Remark 5.5], when X is supposed to be a $\mathbb{Q}$-factorial Gorenstein Fano variety of dimension $n \geq 3$ with canonical singularities, with at most finitely many non-terminal points, and under the more general assumption of the existence of an effective prime divisor $D \subseteq X$ such that the real vector space $\mathrm{N}_1(D, X) := \mathrm{Im}\,(\mathrm{N}_1(D) \to \mathrm{N}_1(X))$ of numerical classes of 1-cycles on X that are equivalent to 1-cycles on D, is one-dimensional.

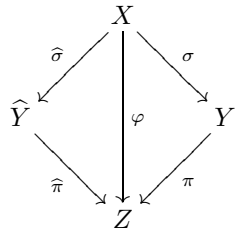
In the smooth case, C. Casagrande and S. Druel provide in **[CD15]** a classification (and examples) of all cases with maximal Picard number $\rho_X = 3$.

Theorem 1.2 **([CD15**, Theorem 3.8]**).** — *Let X be a Fano manifold of dimension $n \geq 3$ and $\rho_X = 3$. Let $D \subseteq X$ be a prime divisor with $\dim_{\mathbb{R}} \mathrm{N}_1(D, X) = 1$. Then X is isomorphic to the blow-up of a Fano manifold $Y \cong \mathbb{P}_Z(\mathcal{O}_Z \oplus \mathcal{O}_Z(a))$ along an irreducible subvariety of dimension $(n-2)$ contained in a section of the $\mathbb{P}^1$-bundle $\pi : Y \to Z$, where Z is a Fano manifold of dimension $(n-1)$ and $\rho_Z = 1$.*

Firstly, we recall in §3 that a mildly singular Fano variety X always has an extremal ray $R \subseteq \overline{\mathrm{NE}}(X)$ whose intersection with a given effective divisor is positive. The rest of §3 is devoted to the study of these extremal contractions in the case that the given divisor has Picard number one. This allows us to prove the following result in §4.

Theorem A. — *Let X be a $\mathbb{Q}$-factorial Gorenstein Fano variety of dimension $n \geq 3$ with canonical singularities and with at most finitely many non-terminal points. Assume that there exists an effective prime divisor $D \subseteq X$ such that $\dim_{\mathbb{R}} \mathrm{N}_1(D, X) = 1$*

and that $\rho_X = 3$. Then, there is a commutative diagram



where σ (resp. $\widehat{\sigma}$) corresponds to a divisorial contraction of an extremal ray $R \subseteq \overline{\mathrm{NE}}(X)$ (resp. $\widehat{R} \subseteq \overline{\mathrm{NE}}(X)$) which is the blow-up in codimension two of an irreducible subvariety of dimension $(n-2)$, and φ is a contraction of fiber type, finite over D, corresponding to the face $R + \widehat{R} \subseteq \overline{\mathrm{NE}}(X)$. Moreover, $D \cdot R > 0$, Y and $\widehat{Y}$ are $\mathbb{Q}$-factorial varieties with canonical singularities and with at most finitely many non-terminal points, Y is Fano and Z is a $\mathbb{Q}$-factorial Fano variety with rational singularities.

The results of S. Cutkosky on the contractions of terminal Gorenstein threefolds **[Cut88]**, together with the previous result imply the following corollary.

Corollary B. — *Let X be a $\mathbb{Q}$-factorial Gorenstein Fano threefold with terminal singularities. Assume that there exists an effective prime divisor $D \subseteq X$ such that $\dim_{\mathbb{R}} \mathrm{N}_1(D, X) = 1$ and that $\rho_X = 3$. Then, X is factorial and it can be realized as the blow-up of a smooth Fano threefold Y along a locally complete intersection curve $C \subseteq Y$. Moreover, Y is isomorphic to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(a))$, where $a \geq 0$.*

In the case $\rho_X = 2$, we obtain in §5 the following extension of **[CD15**, Remark 3.2 and Proposition 3.3] to mildly singular Fano varieties X with $\rho_X = 2$, on which there is an effective prime divisor of Picard number one.

Theorem C. — *Let X be a $\mathbb{Q}$-factorial Gorenstein Fano variety of dimension $n \geq 3$ with canonical singularities and with at most finitely many non-terminal points. Assume that there exists an effective prime divisor $D \subseteq X$ such that $\dim_{\mathbb{R}} \mathrm{N}_1(D, X) = 1$ and that $\rho_X = 2$. There are two possibilities:*

1. *If D is not nef, then there is an extremal contraction sending D to a point.*
2. *If D is nef, then $S = D^{\perp} \cap \mathrm{NE}(X)$ is an extremal ray. One of the following assertions must hold:*
   a) *cont_S is of fiber type onto $\mathbb{P}^1$, and D is a fiber.*
   b) *cont_S is a divisorial contraction sending its exceptional divisor G to a point, and such that $G \cap D = \emptyset$.*
   c) *cont_S is a small contraction and there is a flip $X \dashrightarrow X'$ and a contraction of fiber type $\psi : X' \to Y'$ such that the general fiber is isomorphic to $\mathbb{P}^1$, with anticanonical degree 2. Moreover, ψ is finite over the strict transform of D in X'.*

In order to extend the classification results to higher dimensions, we will restrict ourselves to the case of toric varieties. In that case, the combinatorial description

of the MMP for toric varieties treated in §6, as well as some particular properties of them, will allow us to prove the following result in §7.

Theorem D. — *Let X be a $\mathbb{Q}$-factorial Gorenstein toric Fano variety of dimension $n \geq 3$ with canonical singularities and with at most finitely many non-terminal points. Assume that there exists an effective prime divisor $D \subseteq X$ such that $\dim_{\mathbb{R}} \mathrm{N}_1(D, X) = 1$ and that $\rho_X = 3$. Then, there exist $\mathbb{Q}$-factorial Gorenstein toric Fano varieties Y and Z, with terminal singularities, such that*

1. *$X \cong \overline{\mathrm{Bl}_A(Y)}$, the normalized blow-up of an invariant toric subvariety $A \subseteq Y$ of dimension $(n-2)$; and*
2. *$Y \cong \mathbb{P}_Z(\mathcal{O}_Z \oplus \mathcal{O}_Z(a))$, where $a \geq 0$ and $\mathcal{O}_Z(1)$ is the ample generator of $\mathrm{Pic}(Z)$.*

In the toric setting, we obtain in §8 results that extend L. Bonavero's description of the extremal contractions in the case $\rho_X = 2$ to mildly singular toric Fano varieties. If X is supposed to have isolated canonical singularities then we obtain the following classification.

Theorem E. — *Let X be a $\mathbb{Q}$-factorial Gorenstein toric Fano variety of dimension $n \geq 3$ with isolated canonical singularities. Assume that there exists an effective prime divisor $D \subseteq X$ such that $\dim_{\mathbb{R}} \mathrm{N}_1(D, X) = 1$ and that $\rho_X = 2$. Then, either*

1. *$X \cong \mathbb{P}_Y(\mathcal{O}_Y \oplus \mathcal{O}_Y(a))$ for some toric variety Y. Moreover, Y is a $\mathbb{Q}$-factorial Gorenstein toric Fano variety of dimension $(n-1)$ with terminal singularities and Fano index i_Y, and $0 \leq a \leq i_Y - 1$. In particular, X has only terminal singularities.*
2. *X is isomorphic to the blow-up of a toric variety Y along an invariant subvariety $A \subseteq Y$ of dimension $(n-2)$, contained in the smooth locus of Y. Moreover, Y is isomorphic to either*
   (a) *$\mathbb{P}^n$,*
   (b) *$\mathbb{P}(1^{n-1}, 2, n+1)$ if n is even, or*
   (c) *$\mathbb{P}(1^{n-1}, a, b)$, where $1 \leq a < b \leq n$ are two relatively prime integers such that $a|(n-1+b)$ and $b|(n-1+a)$.*

   In particular, Y is a $\mathbb{Q}$-factorial Gorenstein Fano variety with $\rho_Y = 1$ and it has at most two singular points. Conversely, the blow-up of any of the listed varieties Y along an invariant irreducible subvariety $A \subseteq Y$ of dimension $(n-2)$ and contained in the smooth locus of Y, leads to a toric variety X satisfying the hypothesis.

Moreover, in the case of contractions of fiber type we obtain the following result without the assumption of isolated singularities.

Proposition F. — *Let X be a $\mathbb{Q}$-factorial Gorenstein toric Fano variety of dimension $n \geq 3$ with canonical singularities and with at most finitely many non-terminal points. Assume that there exists an effective prime divisor $D \subseteq X$ such that $\dim_{\mathbb{R}} \mathrm{N}_1(D, X) = 1$ and that $\rho_X = 2$. Let $R \subseteq \overline{\mathrm{NE}}(X)$ be an extremal ray such that $D \cdot R > 0$ and assume that the corresponding extremal contraction $\pi : X \to Y$ is of fiber type. Then, $X \cong \mathbb{P}_Y(\mathcal{O}_Y \oplus \mathcal{O}_Y(a))$. Moreover, Y is a $\mathbb{Q}$-factorial Gorenstein*

Fano variety of dimension $(n-1)$ *with terminal singularities and Fano index* i_Y*, and* $0 \leq a \leq i_Y - 1$*. In particular,* X *has only terminal singularities.*

Finally, §9 is devoted to show that the extremal contractions studied in §8 lift to universal coverings in codimension 1, introduced by W. Buczyńska in [**Buc08**]. See Definition 9.8 for the notion of Poly Weighted Space (PWS), introduced by M. Rossi and L. Terracini in [**RT16a**] and proved to be universal covering spaces in codimension 1 for $\mathbb{Q}$-factorial toric varieties.

In particular, we obtain the following description of divisorial contractions of toric mildly Fano varieties with Picard number two. It should be noticed that even if the combinatorial description of these divisorial contractions is very simple (see Lemma 6.9) and it coincides with the one of the blow-up of a subvariety of dimension $(n-2)$ in the smooth case, it may happen that the morphisms are not globally a blow-up of the coherent sheaf of ideals of a (irreducible and reduced) subvariety but only a blow-up in codimension two if the singularities are not isolated (see Example 8.2).

Proposition G. — *Let* X *be a* $\mathbb{Q}$*-factorial Gorenstein toric Fano variety of dimension* $n \geq 3$ *with canonical singularities and with at most finitely many non-terminal points. Assume that there exists an effective prime divisor* $D \subseteq X$ *such that* $\dim_{\mathbb{R}} \mathrm{N}_1(D, X) = 1$ *and that* $\rho_X = 2$*. Let* $R \subseteq \overline{\mathrm{NE}}(X)$ *be an extremal ray such that* $D \cdot R > 0$ *and let us denote by* $\pi : X \to Y$ *the corresponding extremal contraction. Assume that* π *is birational. Then there exist weights* $\lambda_0, \ldots, \lambda_n \in \mathbb{Z}_{>0}$ *and a cartesian diagram of toric varieties*

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\widehat{\pi}} & \mathbb{P}(\lambda_0, \ldots, \lambda_n) \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow{\pi} & Y \end{array}$$

where vertical arrows denote the corresponding canonical universal coverings in codimension 1, and $\widehat{X}$ *is a Gorenstein Fano PWS with canonical singularities and with at most finitely many non-terminal points such that* $\rho_{\widehat{X}} = 2$*. Moreover,* $\widehat{\pi} : \widehat{X} \to \mathbb{P}(\lambda_0, \ldots, \lambda_n)$ *is a divisorial contraction sending its exceptional divisor* $\widehat{E} \subseteq \widehat{X}$ *onto an invariant subvariety* $\widehat{A} \subseteq \mathbb{P}(\lambda_0, \ldots, \lambda_n)$ *of dimension* $(n-2)$*.*

Acknowledgements. — I would like to thank my supervisors, Stéphane DRUEL and Catriona MACLEAN, for their unstinting help and their valuable guidance.

2. Notation and preliminary results

Through out this article all varieties will be assumed to be reduced and irreducible schemes of finite type over the field $\mathbb{C}$ of complex numbers. Its smooth locus will be denoted by $X_{\mathrm{reg}} \subseteq X$, while $\mathrm{Sing}(X) = X \setminus X_{\mathrm{reg}}$ denotes its singular locus.

We will use the notation and results of the Minimal Model Program (MMP for short) in [**KM98**].

Let X be a normal projective variety, we consider

$$\mathrm{N}_1(X) = \left\{ \sum_{\text{finite}} a_i C_i \mid a_i \in \mathbb{R}, C_i \text{ irreducible curve in } X \right\} / \equiv$$

where $\equiv$ denotes the numerical equivalence. Let $\mathrm{NE}(X) \subseteq \mathrm{N}_1(X)$ be the convex cone generated by the classes of effective 1-cycles, i.e., $1-$cycles with non-negative coefficients, and $\overline{\mathrm{NE}}(X)$ be its topological closure. We denote by $[C]$ the class of C in $\mathrm{N}_1(X)$.

Let $Z \subseteq X$ be a closed subset and $\iota : Z \to X$ be the natural inclusion, we define

$$\mathrm{N}_1(Z, X) = \iota_* \mathrm{N}_1(Z) \subseteq \mathrm{N}_1(X).$$

For us, a divisor will always be a Weil divisor. Let us denote by K_X the class of a canonical divisor in $\mathrm{Cl}(X)$. A complete normal variety X is said to be a Fano variety if there exists a positive multiple of $-K_X$ which is Cartier and ample. A Fano variety is therefore always projective.

The variety X is said to be Gorenstein if K_X is a Cartier divisor and its singularities are Cohen-Macaulay. The property of being Gorenstein is local and open, so the Gorenstein locus of X is the open subset containing all the Gorenstein points of X (it contains X_{reg}, in particular).

We follow the usual convention, and we say that X is a $\mathbb{Q}$-Gorenstein variety if some positive multiple of K_X is a Cartier divisor; we do not require Cohen-Macaulay singularities. In this case, the Gorenstein index of X is the smallest positive integer $\ell \in \mathbb{Z}_{>0}$ such that ℓK_X is a Cartier divisor.

In the same way as for $1-$cycles, we can define $\mathrm{N}^1(X)$ as the vector space of $\mathbb{Q}$-Cartier divisors with real coefficients, modulo numerical equivalence $\equiv$. We denote by $[D]$ the class of D in $\mathrm{N}^1(X)$.

We have a non-degenerated bilinear form

$$\begin{aligned} \mathrm{N}^1(X) \times \mathrm{N}_1(X) &\longrightarrow \mathbb{R} \\ ([D], [C]) &\longmapsto D \cdot C \end{aligned}$$

given by the intersection product between curves and divisors in X. Then, $\dim_{\mathbb{R}} \mathrm{N}_1(X) = \dim_{\mathbb{R}} \mathrm{N}^1(X) =: \rho_X$ is called the Picard number of X. It is a classical fact that $\dim_{\mathbb{R}} \mathrm{N}^1(X)$ is finite.

An extremal ray R of $\overline{\mathrm{NE}}(X)$ is a 1-dimensional subcone such that if $u, v \in \overline{\mathrm{NE}}(X)$, $u + v \in R$ imply that $u, v \in R$. We denote

$$\mathrm{Locus}(R) = \bigcup_{[C] \in R} C \subseteq X.$$

If D is a $\mathbb{Q}$-Cartier divisor and $R \subseteq \overline{\mathrm{NE}}(X)$ is a extremal ray, then the sign $D \cdot R > 0$ (resp. $=, <$) is well defined. From now on, all the varieties are assumed to be $\mathbb{Q}$-factorial unless explicitly mentioned.

A contraction of X is a projective surjective morphism $\varphi : X \to Y$ with connected fibers, where Y is a normal projective variety. In particular, $\varphi_* \mathcal{O}_X = \mathcal{O}_Y$.

Let us recall the notion of singularities of pairs for $\mathbb{Q}$-factorial varieties (see [**KM98**, §2.3] for details).

Definition 2.1. — Let X be a $\mathbb{Q}$-factorial normal projective variety, and $\Delta = \sum a_i \Delta_i$ an effective $\mathbb{Q}$-divisor on X. Let $f : Y \to X$ be a log-resolution of the pair (X, Δ), i.e., a birational projective morphism f whose exceptional locus $\mathrm{Exc}(f)$ is the union of the effective prime divisors E_i's and such that $\Delta_Y + \sum E_i$ is a simple normal crossing divisor, where Δ_Y is the strict transform of Δ in Y by f. Using numerical equivalence, we have

$$K_Y + \Delta_Y \cong f^*(K_X + \Delta) + \sum_{E_i} a(E_i; X, \Delta) E_i.$$

The numbers $a(E_i; X, \Delta) \in \mathbb{Q}$ are independent of the log-resolution and depend only on the discrete valuation that corresponds to E_i.
Suppose that all $a_i \leq 1$ and that f is a log-resolution of (X, Δ), we say that the pair (X, Δ) is

$$\begin{cases} \text{terminal} & a(E_i; X, \Delta) > 0 \text{ for every exceptional } E_i \\ \text{canonical} \quad \text{if} & a(E_i; X, \Delta) \geq 0 \text{ for every exceptional } E_i \\ \text{klt} & a(E_i; X, \Delta) > -1 \text{ for every exceptional } E_i \text{ and all } a_i < 1 \end{cases}$$

Here klt means "Kawamata log terminal". If the conditions above hold for one log-resolution of (X, Δ), then they hold for every log-resolution of (X, Δ).
We say that X is terminal (canonical,...) or that it has terminal (canonical,...) singularities if $(X, 0)$ is a terminal (canonical,...) pair.

From the Cone Theorem (see [**KM98**, Chapter 3]), if X is klt then for each extremal face $F \subseteq \overline{\mathrm{NE}}(X) \cap \{-K_X > 0\}$ there exists a unique morphism $\mathrm{cont}_F : X \to X_F$ with connected fibers, called the extremal contraction of F, from X onto a normal projective variety X_F such that the irreducible curves contracted by cont_F to points are exactly the curves whose classes in $\mathrm{N}_1(X)$ belongs to F.

We denote by $\mathrm{Exc}(\mathrm{cont}_F)$ the exceptional locus of cont_F, i.e., the subset of X where cont_F is not an isomorphism. There are three possibilities:

- $\dim(X_F) < \dim(X)$; we say that cont_F is of fiber type.
- cont_F is birational and its exceptional locus is an effective divisor E (prime if F is an extremal ray) such that $E \cdot R < 0$; we say that cont_F is a divisorial contraction.
- cont_F is birational and its exceptional locus has codimension > 1 in X; we say that cont_F is a small contraction.

If X is klt and $R \subseteq \overline{\mathrm{NE}}(X)$ is a K_X-negative extremal ray, then there is a short exact sequence

$$0 \longrightarrow \mathrm{Pic}\, X_R \xrightarrow{L \mapsto \mathrm{cont}_R^* L} \mathrm{Pic}\, X \xrightarrow{M \mapsto M \cdot [C]} \mathbb{Z},$$

where $C \subseteq X$ is a curve whose numerical class generates the extremal ray R.

In the case when X is a klt Fano variety we have a more precise description: the Mori cone of X is finite rational polyhedral, generated by classes of rational curves;

in particular $\overline{\mathrm{NE}}(X) = \mathrm{NE}(X)$. Moreover, Kawamata-Viehweg theorem implies that $\mathrm{H}^i(X, \mathcal{O}_X) = \{0\}$ for $i > 0$.

After the works of Birkar, Cascini, Hacon and McKernan, if X is a klt Fano variety, then X is a Mori Dream Space (see [**BCHM10**, Corollary 1.3.2] and [**HK00**]).

For us, a $\mathbb{P}^1-$bundle (or fibration in $\mathbb{P}^1$) is a smooth morphism all of whose fibers are isomorphic to $\mathbb{P}^1$.

Let $\mathcal{E}$ be a vector bundle over a variety X, we denote by $\mathbb{P}_X(\mathcal{E})$ the scheme $\mathbf{Proj}_X\left(\bigoplus_{d\geq 0} S^d(\mathcal{E})\right)$.

Finally, we recall that if $\lambda_0, \ldots, \lambda_n$ are positive integers with $\gcd(\lambda_0, \ldots, \lambda_n) = 1$ then we define the associated Weighted Projective Space (WPS) to be

$$\mathbb{P}(\lambda_0, \ldots, \lambda_n) = \left(\mathbb{C}^{n+1}\backslash\{0\}\right) / \sim,$$

where $\sim$ is the equivalence relation

$$(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n) \Leftrightarrow x_i = \varepsilon^{\lambda_i} y_i, i = 0, \ldots, n \text{ for some } \varepsilon \in \mathbb{C}^*.$$

Moreover, $\mathbb{P}(\lambda_0, \ldots, \lambda_n)$ is a toric variety with Picard number one and torsion-free class group. In general, we say that a $\mathbb{Q}$-factorial complete toric variety is a Poly Weighted Space (PWS) if its class group is torsion-free.

3. Study of the extremal contractions

In this section we study extremal contractions of mildly singular Fano varieties X that admits an effective prime divisor $D \subseteq X$ such that $\dim_{\mathbb{R}} \mathrm{N}_1(D, X) = 1$.

Firstly, notice that klt Fano varieties always have an extremal ray whose intersection with a given effective prime divisor is positive (c.f. [**BCW02**, Lemme 2]).

Lemma 3.1. — *Let X be a $\mathbb{Q}$-factorial klt Fano variety and let $D \subseteq X$ be an effective prime divisor. Then, there exists an extremal ray $R \subseteq \overline{\mathrm{NE}}(X)$ such that $D \cdot R > 0$.*

Proof. — The Cone Theorem [**KM98**, Theorem 3.7] implies that $\overline{\mathrm{NE}}(X) = \mathrm{NE}(X)$ is a rational polyhedral cone generated by a finite number of extremal rays $R_1, \ldots, R_s$. Let $C \subseteq X$ be any curve such that $D \cdot [C] > 0$. Since C is numerically equivalent to a positive sum of extremal curves, $[C] = \sum_{i=1}^{s} a_i [C_i]$ with $a_i \geq 0$ and $[C_i] \in R_i$, we can pick one such that $D \cdot R_i > 0$. □

Secondly, we have that the contraction of an extremal ray whose intersection with an effective prime divisor of Picard number one is positive has at most one-dimensional fibers.

Lemma 3.2. — *Let X be a $\mathbb{Q}$-factorial klt variety and $D \subseteq X$ be an effective prime divisor such that $\dim_{\mathbb{R}} \mathrm{N}_1(D, X) = 1$. Let us suppose that $\rho_X > 1$ and that there exists $R \subseteq \overline{\mathrm{NE}}(X)$ extremal ray such that $D \cdot R > 0$. Then, $R \nsubseteq \mathrm{N}_1(D, X)$. In particular, the extremal contraction cont_R is finite on D and all the fibers of cont_R are at most of dimension 1.*

Proof. — We follow the proof of [**CD15**, Lemma 3.1]. Assume, to the contrary, that $R \subseteq \mathrm{N}_1(D, X)$. The numerical class of every irreducible curve $C \subseteq D$ must belong to R, since $\dim_{\mathbb{R}} \mathrm{N}_1(D, X) = 1$. Thus, cont_R sends D to a point and $D \subseteq \mathrm{Locus}(R)$. If cont_R is birational, then we would have that $-D$ is an effective divisor by the negativity lemma, a contradiction. The contraction cont_R is then of fiber type and $\mathrm{Locus}(R) = X$. As $\rho_X > 1$ and $\mathrm{cont}_R(D)$ is a point, there exists a non-trivial fiber of cont_R disjoint of D, and then $D \cdot R = 0$, a contradiction.

As a consequence we have that all the fibers of cont_R are at most of dimension 1. In fact, $\mathrm{cont}_R|_D$ is a finite morphism: if there exists a curve $C \subseteq D$ contained in a fiber, we would have that $[C] \in R$ and therefore $R \subseteq \mathrm{N}_1(D, X)$. □

We will need the following result.

Proposition 3.3. — *Let X be a $\mathbb{Q}$-factorial klt Fano variety of dimension $n \geq 3$. Let us suppose that $\rho_X > 1$ and that there exists an effective prime divisor such that $\dim_{\mathbb{R}} \mathrm{N}_1(D, X) = 1$. Let $R \subseteq \overline{\mathrm{NE}}(X)$ be an extremal ray such that $D \cdot R > 0$ and let us denote by $\varphi_R : X \to X_R$ the corresponding extremal contraction. Then,*

1. *If φ_R is of fiber type, then X_R is a $\mathbb{Q}$-factorial Fano variety with rational singularities of dimension $(n-1)$ such that $\rho_{X_R} = 1$.*
2. *If φ_R is birational and we suppose that X has Gorenstein canonical singularities with at most finitely many non-terminal points, then φ_R is a divisorial contraction and there exists a closed subset $S \subseteq X_R$ with $\mathrm{codim}_{X_R}(S) \geq 3$ such that $X_R \smallsetminus S \subseteq X_{R,\mathrm{reg}}$, $\mathrm{codim}_X \varphi_R^{-1}(S) \geq 2$, $X \smallsetminus \varphi^{-1}(S) \subseteq X_{\mathrm{reg}}$ and*

$$\varphi_R|_{X \smallsetminus \varphi_R^{-1}(S)} : X \smallsetminus \varphi_R^{-1}(S) \longrightarrow X_R \smallsetminus S$$

is the blow-up of a $(n-2)$-dimensional smooth subvariety in $X_R \smallsetminus S$. Moreover, X_R is a $\mathbb{Q}$-factorial Fano variety with canonical singularities with at most finitely many non-terminal points. In particular,

$$K_X \cdot [F] = E \cdot [F] = -1,$$

for every irreducible curve F such that $[F] \in R$, where $E = \mathrm{Exc}(\varphi_R)$ is the exceptional divisor of φ_R.

Proof. — Let us suppose that $\varphi_R : X \to X_R$ is of fiber type. Then, $\varphi_R|_D : D \to X_R$ is a finite morphism, by Lemma 3.2, and thus $\dim_{\mathbb{C}} X_R = n-1$. Since X_R is a $\mathbb{Q}$-factorial projective variety, the projection formula and the fact that $\dim_{\mathbb{R}} \mathrm{N}_1(D, X) = 1$ imply that $\rho_{X_R} = 1$. In particular, every big divisor on X_R is ample.

Let us prove that X_R is a Fano variety in this case. Since X is Fano klt, there exists an effective $\mathbb{Q}$-divisor Δ_{X_R} on X_R such that (X_R, Δ_{X_R}) is klt and such that $-(K_{X_R} + \Delta_{X_R})$ is ample, by [**PS09**, Lemma 2.8]. In particular, X_R has rational singularities and $-K_{X_R}$ is a big divisor, and therefore ample.

Let us suppose now that $\varphi_R : X \to X_R$ is a birational contraction. Then, it follows from [**DN14**, Theorem 2.2] that φ_R is a divisorial contraction given by the blow-up in codimension two of an irreducible subvariety of dimension $(n-2)$ on X_R, and that X_R is a $\mathbb{Q}$-factorial Fano variety with canonical singularities with at most finitely

many non-terminal points. Finally, the ampleness of the anticanonical divisor $-K_{X_R}$ follows verbatim the proof in the smooth case given in [**CD15**, Lemma 3.1]. □

Remark 3.4. — Let X be a Gorenstein Fano variety of dimension $n \geq 3$ with canonical singularities with at most finitely many non-terminal points, and let $D \subseteq X$ be an effective prime divisor such that $\dim_{\mathbb{R}} \mathrm{N}_1(D, X) = 1$. Then, $\rho_X \leq 3$, by [**DN14**, Remark 5.5].

Remark 3.5. — Let X be a $\mathbb{Q}$-factorial klt variety of dimension $n \geq 3$ and let $\varphi_R : X \to X_R$ be a contraction of fiber type, associated to an extremal ray $R \subseteq \overline{\mathrm{NE}}(X)$, such that all the fibers are one-dimensional. Then [**AW97**, Corollary 1.9] implies that all fibers of φ_R are connected and their irreducible components are smooth rational curves.

Let us denote by $[F_x]$ the numerical class of the funtamental 1-cycle associated to the scheme theoretic fiber $\varphi_R^{-1}(x)$ over the point $x \in X_R$, and by $[\varphi_R^{[-1]}(x)]$ the numerical class of the cycle theoretic fiber over $x \in X_R$ in the sense of J. Kollár [**Kol96**, §I.3].

Since X is $\mathbb{Q}$-Gorenstein, the intersection number $K_X \cdot [\varphi_R^{[-1]}(x)]$ is independent of the closed point $x \in X_R$, by [**Kol96**, Proposition I.3.12]. In our context (terminal Gorenstein threefolds or toric varieties), $\varphi_R : X \to X_R$ will be additionally flat and hence the cycle theoretic fiber will coincide with the fundamental class of the scheme theoretic fiber (see [**Kol96**, Definition I.3.9 and Corollary I.3.15]).

Under the flatness hypothesis, if we suppose moreover that K_X is a Cartier divisor then we will have that $-K_X \cdot [F_x] = 2$ for all $x \in X_R$, by generic smoothness. We have therefore three possibilities in that case:

1. F_x is an irreducible and generically reduced rational curve such that $(F_x)_{\mathrm{red}} \cong \mathbb{P}^1$ and that $-K_X \cdot [F_x] = 2$.
2. $[F_x] = 2[C]$ as 1-cycles, where C is an irreducible and generically reduced rational curve such that $C_{\mathrm{red}} \cong \mathbb{P}^1$ and that $-K_X \cdot [C] = 1$.
3. $F_x = C \cup C'$, with $C \neq C'$ irreducible and generically reduced rational curves such that $C_{\mathrm{red}} \cong C'_{\mathrm{red}} \cong \mathbb{P}^1$ and that $-K_X \cdot [C] = -K_X \cdot [C'] = 1$.

Let us finish the section with two results concerning birational extremal contractions.

Theorem 3.6 **([DN14**, Theorem 3.1]**).** — *Let X be a $\mathbb{Q}$-factorial Fano variety with canonical singularities. Then, for any prime divisor $D \subseteq X$, there exists a finite sequence (called a special Mori program for the divisor $-D$)*

$$X = X_0 \overset{\sigma_0}{\dashrightarrow} X_1 \dashrightarrow \cdots \dashrightarrow X_{k-1} \overset{\sigma_{k-1}}{\dashrightarrow} X_k \overset{\psi}{\to} Y$$

such that, if $D_i \subseteq X_i$ is the transform of D for $i = 1, \ldots, k$ and $D_0 := D$, the following hold:

1. *$X_1, \ldots, X_k$ and Y are $\mathbb{Q}$-factorial projective varieties and $X_1, \ldots, X_k$ have canonical singularities.*
2. *for every $i = 0, \ldots, k$, there exists an extremal ray Q_i of X_i with $D_i \cdot Q_i > 0$ and $-K_{X_i} \cdot Q_i > 0$ such that:*

(a) *for* $i = 1, \dots, k-1$, $\mathrm{Locus}(Q_i) \subsetneq X_i$, *and* σ_i *is either the contraction of* Q_i *(if* Q_i *is divisorial), or its flip (if* Q_i *is small);*
(b) *the morphism* $\psi : X_k \to Y$ *is the contraction of* Q_k *and* ψ *is a fiber type contraction.*

The following result is a particular case of [**DN14**, Lemma 3.3]. We include the statement with our notation for completeness.

Lemma 3.7. — *Let* X_0 *be a* $\mathbb{Q}$*-factorial Gorenstein Fano variety of dimension* $n \geq 3$ *with canonical singularities and with at most finitely many non-terminal points. Let* $D_0 \subseteq X_0$ *be an effective prime divisor and let us suppose that there is a diagram*

$$X_0 \overset{\sigma_0}{\dashrightarrow} X_1 \overset{\sigma_1}{\dashrightarrow} X_2,$$

where σ_i *is the birational map associated to the contraction of an extremal ray* $Q_i \subseteq \overline{\mathrm{NE}}(X_i)$ *such that* $D_i \cdot Q_i > 0$*, for* $i = 0, 1, 2$*; as in Theorem 3.6. If* $Q_i \not\subseteq \mathrm{N}_1(D_i, X_i)$ *for* $i = 0, 1$*, then both* σ_0 *and* σ_1 *are divisorial contractions,* $\mathrm{Exc}(\sigma_i)$ *is contained in the Gorenstein locus of* X_i *and* $\mathrm{Exc}(\sigma_0)$ *is disjoint from the transform of* $\mathrm{Exc}(\sigma_1)$ *in* X_0*.*

4. The extremal case $\rho_X = 3$

In this section we study the extremal contractions of mildly singular Fano varieties X on which there is an effective prime divisor of Picard number one and such that $\rho_X = 3$. As we pointed out in Remark 3.4, this is the largest possible Picard number for such varieties. Compare with the smooth case [**CD15**, Lemma 3.1 and Theorem 3.8].

Proof of Theorem A. — Since X is a $\mathbb{Q}$-factorial klt Fano variety, there is an extremal ray $R \subseteq \overline{\mathrm{NE}}(X)$ such that $D \cdot R > 0$, by Lemma 3.1. We denote by $\sigma := \varphi_R : X \to Y$ the associated extremal contraction. We note that Proposition 3.3 implies that σ is a divisorial contraction sending the effective prime divisor $E = \mathrm{Exc}(\sigma)$ onto a subvariety $A \subseteq Y$ of dimension $(n-2)$. Moreover, Y is a Fano variety with canonical singularities and with at most finitely many non-terminal points, such that if we denote by D_Y the image of D by σ, then we have that $A \subseteq D_Y$ and that $\dim_{\mathbb{R}} \mathrm{N}_1(D, X) = 1$.

Since Y is $\mathbb{Q}$-factorial klt Fano variety, there is an extremal ray $Q \subseteq \overline{\mathrm{NE}}(Y)$ such that $D_Y \cdot Q > 0$, by Lemma 3.1. We denote by $\pi := \varphi_Q : Y \to Z$ the associated extremal contraction. Let us prove that π is of fiber type. Assume, to the contrary, that π is a birational contraction. Hence, Lemma 3.7 implies that both π and $\varphi = \pi \circ \sigma$ are divisorial contractions, and that the exceptional locus $\mathrm{Exc}(\varphi)$ consists of two disjoint effective prime divisors. Since $D_Y \cdot Q > 0$ we have that $\mathrm{Exc}(\pi) \cdot [C] > 0$ for every irreducible curve $C \subseteq D_Y$. In particular, $\mathrm{Exc}(\pi) \cap A \neq \emptyset$, as $\dim_{\mathbb{C}} A = n-2 \geq 1$, contradicting the fact that the exceptional divisors are disjoint.

Let $\widehat{R} \subseteq \overline{\mathrm{NE}}(X)$ be the extremal ray such that $\mathrm{cont}_{R+\widehat{R}} = \varphi$. Then, φ can be factorized as $\varphi = \widehat{\pi} \circ \widehat{\sigma}$, where $\widehat{\sigma} := \mathrm{cont}_{\widehat{R}} : X \to \widehat{Y}$ and $\widehat{\pi} := \mathrm{cont}_{\widehat{\pi}(R)} : \widehat{Y} \to Z$. Since φ has fibers of dimension 1, both contractions must have fibers of dimension at

most 1. Notice that the general fiber of φ is not contracted by $\widehat{\sigma}$, hence $\widehat{\sigma}$ must be divisorial and $\widehat{\pi}$ a contraction of fiber type, by the same arguments as above. □

The results of S. Cutkosky on the contractions of terminal Gorenstein threefolds and Theorem A above lead to Corollary B.

Proof of Corollary B. — By Theorem A, there is a diagram

$$X \xrightarrow{\sigma} Y \xrightarrow{\pi} Z \ , \qquad X \xrightarrow{\varphi} Z \ ,$$

where $\sigma : X \to Y$ is a divisorial contraction sending a prime divisor $E = \mathrm{Exc}(\sigma) \subseteq X$ onto a curve $C \subseteq Y$, and π and φ are both extremal contractions of fiber type whose fibers are of dimension 1. All these varieties are $\mathbb{Q}$-factorial Fano varieties, and Y has terminal singularities. Moreover, X is factorial by [**Cut88**, Lemma 2].

By [**Cut88**, Lemma 3 and Theorem 4], $C \subseteq Y$ is an irreducible reduced curve which is a locally complete intersection, Y is a factorial threefold which is smooth near the curve C, and $\sigma : X \to Y$ is the blow-up of the ideal sheaf $\mathcal{I}_C$. In particular, Y is a Gorenstein Fano threefold with terminal singularities and therefore [**Cut88**, Theorem 7] implies that Z is a smooth del Pezzo surface and $\pi : Y \to Z$ is a (possibly singular) conic bundle over Z. We note that $\rho_Z = 1$ and hence $Z \cong \mathbb{P}^2$.

Let $H = \sigma(C) \subseteq Z$. Since π is finite on $D_Y = \sigma(D)$ and $C \subseteq D_Y$, we have that H is an effective prime divisor on $\mathbb{P}^2$, which is therefore ample. Let us denote by S_π the locus of points of $\mathbb{P}^2$ over which π is not a smooth morphism. By [**Gro71**, Proposition II.1.1], is a closed subset of $\mathbb{P}^2$. Then, S_π has pure codimension 1 on $\mathbb{P}^2$ or $S_\pi = \emptyset$, by [**ArRM**, Theorem 3].

Let us suppose that S_π is not empty. If we take $z \in H \cap S_\pi$ and we denote by $F_z \subseteq Y$ its fiber by π, then we have that $F_z \cap C \neq \emptyset$ (as $z \in H$) and that the 1-cycle on Y associated to F_z is of the form $[F_z] = [C] + [C']$, where C and C' are (possibly equal) irreducible and generically reduced rational curves such that $C_{\mathrm{red}} \cong C'_{\mathrm{red}} \cong \mathbb{P}^1$ (as $z \in S_\pi$), by Remark 3.5. Thus, if we denote by $\tilde{F}_z \subseteq X$ the strict transform of F_z on X by σ, we will have $-K_X \cdot [\tilde{F}_z] \geq 3$, contradicting the fact that the anticanonical degree of every fiber of $\varphi = \pi \circ \sigma$ is 2 (see Remark 3.5). We conclude in this way that $\pi : Y \to \mathbb{P}^2$ is a $\mathbb{P}^1$-bundle, and then Y is an smooth threefold by [**ArRM**, Theorem 5].

Finally, let us notice that if D_Y is not nef then there is a birational contraction $Y \to Y_0$ sending D_Y to a point, by [**CD15**, Remark 3.2], and hence [**CD15**, Lemma 3.9] implies that $Y \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(a))$, with $a \geq 0$. On the other hand, if D_Y is nef then we apply [**CD15**, Proposition 3.3] to conclude that either there is a divisorial contraction $Y \to Y_0$ sending an effective prime divisor $G_Y \subseteq Y$ to a point, or there is a contraction of fiber type $Y \to \mathbb{P}^1$; small contractions are excluded since X is a Fano variety, by the same argument (3.3.8) in [**CD15**]. In the first case [**CD15**, Lemma 3.9] allows us to conclude, while in the second case we have that $Y \cong \mathbb{P}^2 \times \mathbb{P}^1$, by [**Cas09**, Lemma 4.9]. □

5. The case $\rho_X = 2$

In the case $\rho_X = 2$ we can describe the extremal contraction associated to the other extremal ray in the Mori cone of X (compare with [**CD15**, Proposition 3.3]). We will need the following result of G. V. Ravindra and V. Srinivas (see [**RS06**]).

Theorem 5.1. — *Let X be a complex projective variety, smooth in codimension 1 and let $\mathcal{L}$ be a ample line bundle over X. If $V \subseteq \mathrm{H}^0(X, \mathcal{L})$ is a linear subspace which gives a base point free linear system $|V|$ on X, then there is a dense Zariski open set of divisors $E \in |V|$ such that the restriction map*

$$\mathrm{Cl}(X) \to \mathrm{Cl}(E)$$

is an isomorphism, if $\dim_{\mathbb{C}} X \geq 4$, and is injective, with finitely generated cokernel, if $\dim_{\mathbb{C}} X = 3$.

Corollary 5.2. — *Let X be a $\mathbb{Q}$-factorial Fano variety of dimension $n \geq 3$ with klt singularities. If $\rho_X > 1$ and D is an effective prime divisor such that $\dim_{\mathbb{R}} \mathrm{N}_1(D, X) = 1$, then D is not an ample $\mathbb{Q}$-divisor.*

Proof. — Assume, to the contrary, that D is an ample $\mathbb{Q}$-divisor. Let $m \in \mathbb{Z}_{>0}$ such that mD is a very ample Cartier divisor on X and use the complete linear system $|mD|$ to embed $X \hookrightarrow \mathbb{P}(V)$. Let us define the projective incidence variety

$$\mathfrak{D} = \{(x, [E]) \in X \times \mathbb{P}(V) \mid x \in E\} \subseteq X \times \mathbb{P}(V),$$

and let $\pi : \mathfrak{D} \to \mathbb{P}(V)$ be the second projection.

Let $\mathcal{H}$ be the relative Hilbert scheme of curves associated to the morphism $\pi : \mathfrak{D} \to \mathbb{P}(V)$, which is a projective scheme with countably many irreducible components. Let us denote by $Z_i \subseteq \mathbb{P}(V)$ the image of the components of $\mathcal{H}$ that do not dominate $\mathbb{P}(V)$. They are closed subsets of $\mathbb{P}(V)$.

Thus, if we take $[E] \in \mathbb{P}(V) - \cup Z_i$ (a very general point on $\mathbb{P}(V)$), then for every curve C_E on E there is a dominant component of $\mathcal{H}$ such that C_E is one the curves parametrized by this component. Since the image of this dominant component is in fact the whole projective space $\mathbb{P}(V)$, then we have that there is a curve C_D on D which is also parametrized for this component. In particular, C_E and C_D are numerically equivalent. But, since there is only one curve on D up to numerical equivalence, we will have $\dim_{\mathbb{R}} \mathrm{N}_1(E, X) = 1$.

On the other hand, Theorem 5.1 implies that we can also suppose that this very general divisor $E \in |D|$ is chosen in such a way the restriction

$$\mathrm{Cl}(X) \to \mathrm{Cl}(E)$$

is injective. Since $\rho_X > 1$ and $\dim_{\mathbb{R}} \mathrm{N}_1(E, X) = 1$, we have that the inclusion $\mathrm{N}_1(E) \to \mathrm{N}_1(X)$ is not surjective. Therefore, the induced map on the dual spaces obtained by restriction $\mathrm{N}^1(X) \to \mathrm{N}^1(E)$ is not injective.

Since X is a $\mathbb{Q}$-factorial klt Fano variety, we have that numerical and linear equivalence coincide, by [**AD14**, Lemma 2.5]. Hence, we have the following diagram

$$\begin{array}{ccccc} \mathrm{Cl}(X)\otimes_{\mathbb{Z}}\mathbb{R} & \xleftarrow{\cong} & \mathrm{Pic}(X)\otimes_{\mathbb{Z}}\mathbb{R} & \xrightarrow{\cong} & \mathrm{N}^1(X) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Cl}(E)\otimes_{\mathbb{Z}}\mathbb{R} & \longleftarrow & \mathrm{Pic}(E)\otimes_{\mathbb{Z}}\mathbb{R} & \longrightarrow & \mathrm{N}^1(E) \end{array}$$

Therefore, $\mathrm{N}^1(X) \to \mathrm{N}^1(E)$ is not injective if and only if $\mathrm{Pic}(E)\otimes_{\mathbb{Z}}\mathbb{R} \to \mathrm{N}^1(E)$ is not injective.

Let us consider a line bundle $\mathcal{L}$ on E such that $\mathcal{L} \equiv 0$. Notice that E has klt singularities since it is a general member of the ample linear system $|mD|$, by [**KM98**, Lemma 5.17]. In particular, E has rational singularities and thus if we consider a resolution of singularities $\varepsilon : \widetilde{E} \to E$, then the Leray spectral sequence leads to $h^i(\widetilde{E}, \mathcal{O}_{\widetilde{E}}) = h^i(E, \mathcal{O}_E)$ and $h^i(E, \mathcal{L}) = h^i(\widetilde{E}, \varepsilon^*\mathcal{L})$ for all $i \geq 0$. On the other hand, the short exact sequence of sheaves

$$0 \to \mathcal{O}_X(-E) \to \mathcal{O}_X \to \mathcal{O}_E \to 0$$

and the vanishing $h^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$, give us $h^1(E, \mathcal{O}_E) = h^2(X, \mathcal{O}_X(-E))$. Since X is a normal variety with Cohen-Macaulay singularities, Serre's duality implies

$$\mathrm{H}^2(X, \mathcal{O}_X(-E)) \cong \mathrm{H}^2(X, \mathcal{O}_X(-mD)) \cong \mathrm{H}^{n-2}(X, \mathcal{O}_X(K_X + mD))^\vee.$$

So, by taking m large enough at the beginning if necessary, we can suppose that $h^1(E, \mathcal{O}_E) = 0$ by the Kawamata-Viehweg vanishing theorem.

We get that $h^1(\widetilde{E}, \mathcal{O}_{\widetilde{E}}) = 0$ and hence an inclusion $\mathrm{Pic}(\widetilde{E}) \hookrightarrow \mathrm{H}^2(\widetilde{E}, \mathbb{Z})$. Clearly $\varepsilon^*\mathcal{L} \equiv 0$ and thus $\varepsilon^*\mathcal{L} \cong \mathcal{O}_{\widetilde{E}}$. By the projection formula, $\mathcal{L} \cong \mathcal{O}_E$, a contradiction. □

We end this section by proving Theorem C.

Proof of Theorem C. — If D is not nef, then there exists an extremal ray $R \subseteq \mathrm{NE}(X)$ such that $D \cdot R < 0$, and therefore $\mathrm{Exc}(\mathrm{cont}_R) \subseteq D$. Since $\dim_{\mathbb{R}} \mathrm{N}_1(D, X) = 1$ we must have that $\mathrm{Exc}(\mathrm{cont}_R) = D$ and that $\mathrm{cont}_R(D)$ is a point.

If D is nef, then it is not ample by Corollary 5.2, and thus $S = D^\perp \cap \mathrm{NE}(X)$ is an extremal ray, as $\rho_X = 2$. If we denote $G = \mathrm{Locus}(S) \subsetneq X$, the proof follows almost verbatim the proof in the smooth case given in [**CD15**, Proposition 3.3], up to modify the argument (3.3.6) in [**CD15**] by using Lemma 3.7 to get a contradiction if we suppose that ψ is a birational contraction. □

6. The toric MMP

We will analyze now the toric case. We may refer the reader to [**CLS**] for the general theory of toric varieties and to [**Mat02**] for details of the toric MMP. We will keep the same notation as [**CLS**].

Let $N \cong \mathbb{Z}^n$ be a lattice, $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ its dual lattice and let $N_{\mathbb{R}}$ (resp. $N_{\mathbb{Q}}$) and $M_{\mathbb{R}}$ (resp. $M_{\mathbb{Q}}$) be their real scalar (resp. rational scalar) extensions. Let us denote by $\langle \cdot, \cdot \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$ the natural $\mathbb{R}-$bilinear pairing.

Let $\Delta_X \subseteq N_{\mathbb{R}} \cong \mathbb{R}^n$ be a fan. As we will see, most of the properties of our interest in the context of the MMP of the associated toric variety $X = X(\Delta_X)$, can be translated into combinatorial properties of the fan Δ_X.

Sometimes we will write N_X instead of N in order to emphasize the dependence of $X(\Delta_X)$ on the lattice where primitive generators of Δ_X belong.

Let $\Delta_X(k)$ be the set of k dimensional cones in Δ_X. In the same way, if $\sigma \in \Delta_X$ is a cone, we will denote by $\sigma(k)$ the set of its $k-$dimensional faces. Usually, we will not distinguish between $1-$dimensional cones $\rho \in \Delta_X(1)$ (or $1-$dimensional faces $\rho \in \sigma(1)$) and the primitive vector $u_\rho \in N$ generating them.

If $\sigma \in \Delta_X(k)$ we will denote by U_σ the associated affine toric variety, and by $V(\sigma) \subseteq X(\Delta_X)$ the closed invariant subvariety of codimension k. In particular, each $\rho \in \Delta_X(1)$ corresponds to an invariant Weil divisor $V(\rho)$ on X; such a cone is called a *ray*. Similarly, each cone of codimension 1 $\omega \in \Delta_X(n-1)$ corresponds to an invariant rational curve on X; such a cone is called a *wall*.

It is a classical fact that every Weil divisor on a toric variety X is linearly equivalent to an invariant toric divisor (see [**CLS**, Theorem 4.1.3]). The same holds for effective curves on complete toric varieties.

Theorem 6.1 **([Rei83**, Proposition 1.6]**).** — *Let $X = X(\Delta_X)$ be a complete toric variety of dimension n. Then*

$$\mathrm{NE}(X) = \sum_{\omega \in \Delta_X(n-1)} \mathbb{R}_{\geq 0}[V(\omega)].$$

In particular, $\mathrm{NE}(X)$ *is a closed rational polyhedral cone and it is strictly convex if and only if X is a projective variety.*

All affine toric varieties associated to strongly convex rational polyhedral cones are normal (see [**CLS**, Theorem 1.3.5]). Thus, a toric variety X associated to a fan Δ_X is also normal. Moreover, we can check if a toric variety associated to a fan Δ_X is smooth or not by looking at all the cones $\sigma \in \Delta_X$. In fact, if we say that a cone $\sigma \in \Delta_X$ is smooth if and only if the associated affine toric variety U_σ is smooth, then we have the following result.

Proposition 6.2 **([CLS**, Proposition 11.1.2 and Proposition 11.1.8**])**

Let X be the toric variety associated to the fan Δ_X. Then,

$$\mathrm{Sing}(X) = \bigcup_{\sigma \text{ not smooth}} V(\sigma)$$

and

$$X_{\mathrm{reg}} = \bigcup_{\sigma \text{ smooth}} U_\sigma.$$

Moreover, given a $d-$dimensional simplicial cone $\sigma \subseteq N_{\mathbb{R}}$ with generators $u_1, \ldots, u_d \in N$, let $N_\sigma = \mathrm{Span}(\sigma) \cap N$ and define the multiplicity of σ by

$$\mathrm{mult}(\sigma) = [N_\sigma : \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_d].$$

Then,

1. *σ is smooth if and only if* $\mathrm{mult}(\sigma) = 1$.

2. *Let $e_1, \dots, e_d$ be a basis of N_σ and write $u_i = \sum_{i=1}^{d} a_{ij} e_j$. Then,*
$$\operatorname{mult}(\sigma) = |\det(a_{ij})|\,.$$
3. *If $\tau \preceq \sigma$ is a face of σ, then*
$$\operatorname{mult}(\sigma) = \operatorname{mult}(\tau)[N_\sigma : N_\tau + \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_d].$$
In particular, $\operatorname{mult}(\tau) \leq \operatorname{mult}(\sigma)$ *whenever* $\tau \preceq \sigma$.

Remark 6.3. — Let X be a $n-$dimensional toric variety associated to a simplicial fan Δ_X, i.e., a fan whose cones are all simplicial, on which there is a cone σ of full dimension n. If X is smooth in codimension k, namely the closed invariant subset $\operatorname{Sing}(X) \subseteq X$ is such that $\operatorname{codim}_X \operatorname{Sing}(X) \geq k+1$, then we can choose a basis of $\mathbb{Z}^n \cong N$ in such a way the first k generators of the cone σ corresponds to the first k elements of the canonical basis of $\mathbb{Z}^n$.

In general, most of the interesting kind of singularities can also be characterized by looking at the (maximal) cones belonging to the fan.

Theorem 6.4. — *Let σ be a strongly convex rational polyhedral cone and let U_σ be the corresponding affine toric variety, then the following hold.*

1. *U_σ is Cohen-Macaulay.*
2. *U_σ is $\mathbb{Q}$-factorial if and only if σ is simplicial.*
3. *U_σ is $\mathbb{Q}$-Gorenstein if and only if there exists $m_\sigma \in M_\mathbb{Q}$ such that $\langle m_\sigma, \rho\rangle = 1$, for every ray $\rho \in \sigma(1)$. In this case, the Gorenstein index of U_σ is the smallest positive integer $\ell \in \mathbb{Z}_{>0}$ such that $\ell m_\sigma \in M$.*
4. *If U_σ is $\mathbb{Q}$-Gorenstein then U_σ has klt singularities.*
5. *If U_σ is $\mathbb{Q}$-Gorenstein then U_σ has terminal singularities of Gorenstein index ℓ if and only if exists $m_\sigma \in M$ such that*
$$\langle m_\sigma, u_\rho\rangle = \ell \text{ for all } u_\rho \in \operatorname{Gen}(\sigma) \text{ and}$$
$$\langle m_\sigma, u_\rho\rangle > \ell \text{ for all } u_\rho \in \sigma \cap N \setminus (\{0\} \cup \operatorname{Gen}(\sigma)).$$
The element m_σ is uniquely determined whenever σ is of maximal dimension in the fan.
6. *If U_σ is $\mathbb{Q}$-Gorenstein then U_σ has canonical singularities of Gorenstein index ℓ if and only if exists $m_\sigma \in M$ such that*
$$\langle m_\sigma, u_\rho\rangle = \ell \text{ for all } u_\rho \in \operatorname{Gen}(\sigma) \text{ and}$$
$$\langle m_\sigma, u_\rho\rangle \geq \ell \text{ for all } u_\rho \in \sigma \cap N \setminus (\{0\} \cup \operatorname{Gen}(\sigma)).$$
The element m_σ is uniquely determined whenever σ is of maximal dimension in the fan.
7. *If U_σ is Gorenstein then U_σ has canonical singularities.*

Here, $\operatorname{Gen}(\sigma)$ *denotes the set of primitive vectors u_ρ generating all the rays $\rho = \{\lambda u_\rho \mid \lambda \geq 0\} \in \sigma(1)$*

Proof. — We may refer the reader to the survey [**Dai02**] for proofs or references to proofs. □

Remark 6.5. — By Theorem 6.4 above, if X is a $\mathbb{Q}$-factorial toric variety with canonical singularities then we have the decomposition

$$\mathrm{Sing}(X) = \bigcup_{\substack{\sigma \text{ canonical} \\ \text{non-terminal}}} V(\sigma) \cup \bigcup_{\substack{\sigma \text{ terminal} \\ \text{non-smooth}}} V(\sigma).$$

Therefore, if X is a $\mathbb{Q}$-factorial toric variety with canonical singularities and with at most finitely many non-terminal points, then the (finite) set of canonical points is made up by some invariant points $V(\sigma)$, where σ are of maximal dimension in the fan.

Let us introduce now the necessary elements to run the Toric MMP.

Definition 6.6. — Let $U \subseteq N_{\mathbb{R}}$ be a rational vector subspace; a collection of cones Δ^* is a degenerate fan with vertex U if it satisfies the usual conditions of a fan with strict convexity of cones replaced by

$$\forall \sigma \in \Delta^*,\ \sigma \cap -\sigma = U.$$

This coincides with the usual notion of a fan $\Delta = \Delta^*/U$ in the quotient space $N_{\mathbb{R}}/U$.

In our setting, we will always deal with $\mathbb{Q}$-factorial complete toric varieties, i.e., toric varieties having a simplicial fan whose support is the whole space $N_{\mathbb{R}}$, by Theorem 6.4 above and [**CLS**, Theorem 3.4.1]. In the toric case, every extremal ray $R \subseteq \overline{\mathrm{NE}}(X)$ of such a variety will correspond to a invariant curve C_ω such that $R = \mathbb{R}_{\geq 0}[C_\omega]$ or, equivalently, to a wall $\omega \in \Delta_X(n-1)$.

Let us suppose that $\omega = \mathrm{cone}(u_1, \ldots, u_{n-1})$, where u_i are primitive vectors. Since Δ_X is a simplicial fan, ω separates two maximal cones $\sigma = \mathrm{cone}(u_1, \ldots, u_{n-1}, u_n)$ and $\sigma' = \mathrm{cone}(u_1, \ldots, u_{n-1}, u_{n+1})$, where u_n and u_{n+1} are primitive on rays on opposite sides of ω. The $n+1$ vectors $u_1, \ldots, u_{n+1}$ are linearly dependent. Hence, they satisfy a so called *wall relation*:

$$b_n u_n + \sum_{i=1}^{n-1} b_i u_i + b_{n+1} u_{n+1} = 0,$$

where $b_n, b_{n+1} \in \mathbb{Z}_{>0}$ and $b_i \in \mathbb{Z}$ for $i = 1, \ldots, n-1$. By reordering if necessary, we can assume that

$$\begin{aligned} b_i < 0 \quad &\text{for} \quad 1 \leq i \leq \alpha \\ b_i = 0 \quad &\text{for} \quad \alpha + 1 \leq i \leq \beta \\ b_i > 0 \quad &\text{for} \quad \beta + 1 \leq i \leq n+1. \end{aligned}$$

Let us introduce the notation

$$\Delta(\omega) = \sigma + \sigma' = \mathrm{cone}(u_1, \ldots, u_{n+1})$$

and

$$U(\omega) = \mathrm{cone}(u_1, \ldots, u_\alpha, u_{\beta+1}, \ldots, u_{n+1}).$$

This wall relation and the signs of the coefficients involved allow us to describe the nature of the associated contraction.

Theorem 6.7 **([Rei83**, Theorem 2.4 and Corollary 2.5])

Let X be a $\mathbb{Q}$-factorial complete toric variety of dimension n associated to the fan $\Delta_X \subseteq N_{\mathbb{R}}$, and suppose that $R \subseteq \overline{\mathrm{NE}}(X)$ is an extremal ray of X. Let us remove from $\Delta_X(n-1)$ all the walls ω associated to curves from R and for each such ω replace the two adjacent maximal cones σ and σ' from $\Delta_X(n)$ by the cone $\Delta(\omega)$. Then, α, β and $U_R = U(\omega)$ are independent of ω and, by taking respectively their faces in $\Delta_X(i)$, where $i \leq n-2$, we get a complete fan $\Delta_R^* \subseteq N_{\mathbb{R}}$, degenerate with vertex U_R if $\alpha = 0$, non-degenerate if $\alpha > 0$. Moreover, if $\alpha = 0$ then $\Delta_R := \Delta_R^*/U_R$ is a complete simplicial fan. If $\alpha = 1$, then $\Delta_R := \Delta_R^*$ is simplicial.

Furthermore, in this case the induced morphism of toric varieties $\varphi_R : X = X(\Delta_X) \to X_R = X(\Delta_R)$ is the contraction of the extremal ray R in the sense of Mori theory. Moreover, the exceptional locus of φ_R corresponds to the irreducible closed invariant subvariety $A \subseteq X$ associated to the cone $\mathrm{cone}(u_1, \ldots, u_\alpha) \in \Delta_X(\alpha)$, which is contracted onto the irreducible closed invariant subvariety $B \subseteq X_R$ corresponding to the cone $U_R \in \Delta_X^*(n-\beta)$ (if $\alpha = 0$ then both are equal to the whole variety X and X_R, respectively). Therefore, $\dim_{\mathbb{C}} A = n - \alpha$, $\dim_{\mathbb{C}} B = \beta$, and $\varphi_R|_A : A \to B$ is a flat morphism, all whose fibers are (possibly non-reduced) projective toric varieties of Picard number 1 and dimension $n - \alpha - \beta$.

In general, if $\omega \in \Delta_X(n-1)$ is any wall of Δ_X (not necessarily corresponding to an extremal ray), we will also have a wall relation allowing us to compute the intersection number of the curve C_ω with every invariant divisor D_ρ, $\rho \in \Delta_X(1)$.

Proposition 6.8 **([CLS**, Proposition 6.4.4]). — *Let Δ_X be a simplicial fan in $N_{\mathbb{R}} \cong \mathbb{R}^n$ and $\omega = \mathrm{cone}(u_1, \ldots, u_{n-1}) \in \Delta_X(n-1)$ be a wall, separating the two maximal cones $\sigma = \mathrm{cone}(u_1, \ldots, u_{n-1}, u_n)$ and $\sigma' = \mathrm{cone}(u_1, \ldots, u_{n-1}, u_{n+1})$, satisfying the wall relation*

$$b_n u_n + \sum_{i=1}^{n-1} b_i u_i + b_{n+1} u_{n+1} = 0,$$

where $b_n, b_{n+1} \in \mathbb{Z}_{>0}$ and $b_i \in \mathbb{Z}$ for $i = 1, \ldots, n-1$. Then, if we denote by $V(u_\rho) := V(\rho)$ the invariant Weil divisor associated to the primitive vector $u_\rho \in N$ corresponding to $\rho \in \Delta_X(1)$ and by C the invariant curve associated to the wall ω, we have that

1. *$V(u) \cdot [C] = 0$ for all $u_\rho \notin \{u_1, \ldots, u_n, u_{n+1}\}$.*
2. *$V(u_n) \cdot [C] = \dfrac{\mathrm{mult}(\omega)}{\mathrm{mult}(\sigma)}$ and $V(u_{n+1}) \cdot [C] = \dfrac{\mathrm{mult}(\omega)}{\mathrm{mult}(\sigma')}$.*
3. *$V(u_i) \cdot [C] = \dfrac{b_i \,\mathrm{mult}(\omega)}{b_n \,\mathrm{mult}(\sigma)} = \dfrac{b_i \,\mathrm{mult}(\omega)}{b_{n+1} \,\mathrm{mult}(\sigma')}$ for $i \in \{2, \ldots, n-1\}$.*

In the setting of toric varieties, we will be interested in analyzing the extremal contractions appearing in §3 in terms of the description given by Theorem 6.7. Let us begin by the birational case.

Lemma 6.9. — *Let $X = X(\Delta_X)$ be a $\mathbb{Q}$-factorial Gorenstein toric variety of dimension $n \geq 3$. Let $\varphi_R : X \to X_R$ be a divisorial contraction such that*

1. *$\mathrm{Exc}(\varphi_R) = E$ is an invariant prime divisor on X.*

2. $A = \varphi_R(E)$ *is an invariant subvariety of codimension two.*
3. $E \cdot [F] = K_X \cdot [F] = -1$ *for every non-trivial fiber* F *of* φ_R.

Let us suppose that $\varphi_R : X \to X_R$ *is defined by the contraction of the wall* $\omega = \mathrm{cone}(u_1, \ldots, u_{n-1})$ *that separates the maximal cones* $\sigma = \mathrm{cone}(u_1, \ldots, u_{n-1}, u_n)$ *and* $\sigma' = \mathrm{cone}(u_1, \ldots, u_{n-1}, u_{n+1})$. *Then, up to reordering if necessary, the wall relation satisfied by these cones is of the form*

$$u_n + u_{n+1} = u_1.$$

Proof. — By Theorem 6.7, up to reordering if necessary, we can suppose that $\varphi_R : X \to X_R$ is defined by the relation

$$\alpha u_n + \lambda u_1 + \beta u_{n+1} = 0, \tag{1}$$

where $\alpha, \beta \in \mathbb{Z}_{>0}$, $\lambda \in \mathbb{Z}_{<0}$ and $E = V(u_1)$. Let us denote by $C = V(\omega)$ the invariant curve associated to the wall ω.

By Proposition 6.8, $V(u) \cdot [C] = 0$ for $u \notin \{u_1, \ldots, u_{n+1}\}$. Moreover, the wall relation gives us $V(u_i) \cdot [C] = 0$ for $i \in \{2, \ldots, n-1\}$ and

$$V(u_1) \cdot [C] = \frac{\lambda}{\alpha} V(u_n) \cdot [C] = \frac{\lambda}{\beta} V(u_{n+1}) \cdot [C].$$

By hypothesis, $V(u_1) \cdot [C] = -1$ and thus $V(u_n) \cdot [C] = -\frac{\alpha}{\lambda}$ and $V(u_{n+1}) \cdot [C] = -\frac{\beta}{\lambda}$.

It is well know that for a toric variety $X = X(\Delta_X)$, $K_X = -\sum_{\rho \in \Delta_X(1)} V(\rho)$ is a invariant canonical divisor on X (see [**CLS**, Theorem 8.2.3]), and thus the condition $-K_X \cdot [C] = 1$ can be translated into

$$-1 + \left(-\frac{\alpha}{\lambda}\right) + \left(-\frac{\beta}{\lambda}\right) = 1 \Leftrightarrow \alpha + \beta = -2\lambda. \tag{2}$$

On the other hand, we should notice that we can suppose that $\gcd(\lambda, \alpha) = \gcd(\lambda, \beta) = \gcd(\alpha, \beta) = 1$. In fact, if $\gcd(\alpha, \beta) = d > 1$, then the equation (1) implies $d | \lambda$, as u_1 is a primitive vector. The same argument applies to the other two pairs.

By assumption, $K_X = -\sum_{\rho \in \Delta_X(1)} V(\rho)$ is a Cartier divisor, i.e., for each maximal cone $\sigma \in \Delta_X(n)$, there is $m_\sigma \in M$ with $\langle m_\sigma, u_\rho \rangle = 1$ for all $\rho \in \sigma(1)$. In our setting, this condition applied to the two maximal cones σ and σ' tells us that there exists two elements $m, m' \in M$ such that

$$\begin{aligned} \langle m, u_i \rangle &= 1 \qquad \text{for} \qquad i \in \{1, \ldots, n\}, \\ \langle m', u_i \rangle &= 1 \qquad \text{for} \qquad i \in \{1, \ldots, n-1, n+1\}. \end{aligned}$$

From the equation (1) we obtain

$$\alpha + \lambda + \beta \langle m, u_{n+1} \rangle = 0, \text{ and} \tag{3}$$

$$\alpha \langle m', u_n \rangle + \lambda + \beta = 0 \tag{4}$$

By using the equation (2) and (3) we obtain that $\lambda = \beta(\langle m, u_{n+1}\rangle - 1)$ and thus $\beta = 1$, as $\beta \in \mathbb{Z}_{>0}$ and $\gcd(\lambda, \beta) = 1$. In the same way, by using the equation (2) and (4), we deduce that $\alpha = 1$ and hence $\lambda = -1$. Finally, we get the relation

$$u_n + u_{n+1} = u_1,$$

and $\langle m, u_{n+1}\rangle = \langle m', u_n\rangle = 0$. □

Let us consider now the case when $\varphi_R : X \to X_R$ is a contraction of fiber type.

Remark 6.10. — Let $X = X(\Delta_X)$ be a $\mathbb{Q}$-factorial toric variety of dimension $n \geq 3$ and let $\varphi_R : X \to X_R$ be an extremal contraction such that $\dim_{\mathbb{C}} X_R = n - 1$. Then, φ_R is a flat morphism whose generic fiber is isomorphic to $\mathbb{P}^1$. Let us denote by S_{φ_R} the locus of points of X_R over which φ_R is not smooth. Then, [**ArRM**, Theorem 3] implies that either S_{φ_R} is empty or of pure codimension 1 in X_R, as $\mathbb{Q}$-factorial toric varieties have locally quotient singularities at every point (see [**CLS**, Theorem 11.4.8]).

As a consequence we have the following description for contractions of fiber type.

Lemma 6.11. — *Let X be a $\mathbb{Q}$-factorial projective toric variety of dimension $n \geq 3$. Let us suppose that X is smooth in codimension two and that there is an extremal contraction $\varphi_R : X \to X_R$ of fiber type onto a $\mathbb{Q}$-factorial projective toric variety of dimension $(n-1)$. Then φ_R is a $\mathbb{P}^1-$bundle. Moreover, there is a split vector bundle $\mathcal{E}$ of rank 2 on X_R and an isomorphism $X \cong \mathbb{P}_{X_R}(\mathcal{E})$ over X_R.*

Proof. — Since $\operatorname{codim}_X \operatorname{Sing}(X) \geq 3$ and $\operatorname{codim}_{X_R} \operatorname{Sing}(X_R) \geq 2$, we can take $n-2$ general hyperplane sections in X_R in order to obtain a smooth surface $S \subseteq X$ such that $B := \varphi_R(S)$ is a smooth curve. Hence $\varphi_R|_S : S \to B$ will be a morphism from a smooth projective surface onto a smooth projective curve such that its generic fiber is isomorphic to $\mathbb{P}^1$. The Tsen's theorem implies that $\varphi_R|_S$ admits a section and therefore the fibers of $\varphi_R|_S$ are generically reduced. Thus, outside a codimension two closed subset of X_R, the fibers of φ_R are generically reduced. Then, φ_R is a smooth morphism outside a codimension two closed subset of X_R. We conclude by the Remark 6.10 that φ_R is a smooth morphism. Finally, since $\varphi_R : X \to X_R$ is a toric morphism between toric varieties of relative dimension 1, it admits two disjoint invariant sections $s_0 : X_R \to X$ and $s_\infty : X_R \to X$ passing through the two invariant points of all the fibers of φ. Hence, [**ArRM**, Remark 8] implies that there exists a rank 2 split vector bundle $\mathcal{E}$ such that $X \cong \mathbb{P}_{X_R}(\mathcal{E})$. □

7. The extremal case $\rho_X = 3$ for toric varieties

We are now able to prove the structure theorem for toric varieties with Picard number 3.

Proof of Theorem D. — By Theorem A, we obtain a diagram

$$X \xrightarrow{\sigma} Y \xrightarrow{\pi} Z\,, \qquad X \xrightarrow{\varphi} Z$$

where $\sigma : X \to Y$ is a divisorial contraction sending a toric prime divisor E onto an invariant subvariety A of codimension two on Y, and π and φ are both extremal contractions of fiber type whose fibers are of dimension 1. All these varieties are $\mathbb{Q}$-factorial toric Fano varieties.

By Lemma 6.11, $\pi : Y \to Z$ is a $\mathbb{P}^1-$bundle isomorphic to the projectivization of the rank 2 split vector bundle $\mathcal{E} = \mathcal{L}' \oplus \mathcal{L}$. As $\mathbb{P}_Z(\mathcal{E}) \cong \mathbb{P}_Z(\mathcal{E} \otimes \mathcal{M})$ for any line bundle $\mathcal{M} \in \mathrm{Pic}(Z)$, we can suppose that $\mathcal{L}' \cong \mathcal{O}_Z$. On the other hand, since $\mathrm{Pic}(Z)$ is isomorphic to $\mathbb{Z}$, we can consider an ample generator $\mathcal{O}_Z(1)$ of $\mathrm{Pic}(Z)$ and an integer $a \in \mathbb{Z}$ such that $\mathcal{L} \cong \mathcal{O}_Z(a)$. Up to tensor by $\mathcal{L}^\vee$, we can always suppose that $a \geq 0$. In particular, both Y and Z must have at most terminal singularities, since Y has a most a finite number of canonical singularities and π is locally trivial.

It should be noticed that in this case $A \subseteq Y$ is contained in one of the two disjoint invariant sections associated to the $\mathbb{P}^1-$bundle $\pi : Y \to Z$. In fact, by Theorem 6.7 (taking $\alpha = 0$ and $\beta = n-1$) we have that $\pi : Y \to Z$ is defined by a wall relation of the form

$$b_n u_n + \sum_{i=1}^{n-1} 0 \cdot u_i + b_{n+1} u_{n+1} = 0,$$

where $\omega = \mathrm{cone}(u_1, \ldots, u_{n-1})$ is a wall generating the extremal ray associated to this contraction, which separates the two maximal cones $\sigma = \mathrm{cone}(u_1, \ldots, u_{n-1}, u_n)$ and $\sigma' = \mathrm{cone}(u_1, \ldots, u_{n-1}, u_{n+1})$.

Thus $U(\omega) = \mathrm{cone}(u_n, u_{n+1}) = \mathrm{Span}(u_n) = \mathrm{Span}(u_{n+1}) \cong \mathbb{R}$ and $\pi : Y \to Z$ is induced by the quotient $N_{\mathbb{R}} \to N_{\mathbb{R}}/U(\omega)$. Since $A \subseteq D_Y = \sigma(D)$ and $\pi|_{D_Y}$ is a finite morphism, A is sent onto an invariant subvariety of dimension $n-2$, a divisor on Z. Thus, $A \subseteq D(u_n) := D_0$ or $A \subseteq D(u_{n+1}) := D_\infty$, otherwise it will be sent onto a subvariety of dimension $n-3$.

If we suppose that $A \subseteq D_\infty$, where $D_\infty \cong Z \subseteq Y$ is one of these disjoint invariant sections, then for any invariant affine open subset $U \subseteq Z$ such that $\pi^{-1}(U) \cong U \times \mathbb{P}^1$ we will have

$$\pi^{-1}(U) \setminus (\pi^{-1}(U) \cap D_\infty) \cong U \times \mathbb{A}^1.$$

The open set $U \times \mathbb{A}^1$ is therefore isomorphic to an open set on Y contained in the locus where $\sigma^{-1} : Y \dashrightarrow X$ is an isomorphism. In particular, $U \times \mathbb{A}^1$ is an affine Gorenstein toric variety and thus U is also Gorenstein. We conclude in this way that both Y and Z are Gorenstein varieties.

As a consequence of the formula $K_X = \sigma^*(K_Y) + E$ we have that $E = \mathrm{Exc}(\sigma)$ is a Cartier divisor. Let us prove that $X \to Y$ verifies the universal property of the blow-up. The short exact sequence of sheaves

$$0 \to \mathcal{O}_X(-E) \to \mathcal{O}_X \to \mathcal{O}_E \to 0$$

gives

$$0 \to \sigma_* \mathcal{O}_X(-E) \to \sigma_* \mathcal{O}_X \to (\sigma|_E)_* \mathcal{O}_E \to \mathrm{R}^1 \sigma_* \mathcal{O}_X(-E) \to \cdots,$$

where $\sigma_* \mathcal{O}_X = \mathcal{O}_Y$ since $\sigma : X \to Y$ is a contraction.

Notice that $\sigma|_E : E \to A$ is a $\mathbb{P}^1-$bundle. In fact, since K_X is a Cartier divisor and $-K_X \cdot [F] = 1$ for any non-trivial fiber F of σ, it follows that the scheme theoretic fiber

F is an irreducible and generically reduced rational curve on X. Then, by [**Kol96**, Theorem II.2.8], $\sigma|_E : E \to A$ is a $\mathbb{P}^1-$bundle and thus $(\sigma|_E)_*\mathcal{O}_E = \mathcal{O}_A$.

On the other hand, the Cartier divisor $-(K_X + E)$ is σ-ample and therefore $\mathrm{R}^i\,\sigma_*\mathcal{O}_X(-E) = 0$ for $i > 0$, by [**AW97**, Vanishing Theorem 1.1]. Hence, the above long exact sequence becomes

$$0 \to \sigma_*\mathcal{O}_X(-E) \to \mathcal{O}_Y \to \mathcal{O}_A \to 0,$$

and thus $\mathcal{I}_A \cong \sigma_*\mathcal{O}_X(-E)$.

Let us follow [**AW93**] and notice that $\sigma : X \to Y$ is a local contraction supported by the Cartier divisor $K_X - E$. Let F be any non-trivial fiber of σ. Then, by [**AW93**, Theorem 5.1], the evaluation morphism

$$\sigma^*\sigma_*\mathcal{O}_X(-E) \to \mathcal{O}_X(-E)$$

is surjective at every point of F. On the other hand, $\sigma_*\mathcal{O}_X(-E) \cong \mathcal{I}_A$ and $\sigma^{-1}\mathcal{I}_A \cdot \mathcal{O}_X$ is defined to be the image of $\sigma^*\mathcal{I}_A \to \mathcal{O}_X(-E)$. Thus, $\sigma^{-1}\mathcal{I}_A \cong \mathcal{O}_X(-E)$ is an invertible sheaf.

Then, by the universal property of the normalized blow-up, σ factorizes as

$$X \xrightarrow{\tau} \overline{\mathrm{Bl}_A(Y)} \xrightarrow{\nu\circ\varepsilon} Y\ ,$$

where $\varepsilon : \mathrm{Bl}_A(Y) \to Y$ is the blow-up of the coherent sheaf of ideals $\mathcal{I}_A$ and $\nu : \overline{\mathrm{Bl}_A(Y)} \to \mathrm{Bl}_A(Y)$ its normalization.

Since σ contracts only the irreducible divisor E, τ contracts no divisor. If τ is not finite, it is a small contraction sending a curve $C \subseteq E$ to a point. The rigidity lemma [**KM98**, Lemma 1.6] applied to the $\mathbb{P}^1$-bundle $E \to A$ and the morphism $\tau(E) \to A$ implies that τ contracts the divisor E, a contradiction. Hence τ is a finite and birational morphism onto a normal variety, and therefore τ is an isomorphism by Zariski's Main Theorem. □

Remark 7.1. — By [**CLS**, Proposition 11.4.22], if $X = X(\Delta_X)$ is a $\mathbb{Q}$-factorial toric variety with Gorenstein terminal singularities, then $\mathrm{codim}_X \mathrm{Sing}(X) \geq 4$. Therefore, if we start with a toric variety X of dimension 3 or 4 satisfying the hypothesis of Theorem D, we will obtain that Z is a smooth toric variety, implying that Y and X must be both smooth toric varieties too.

8. The case $\rho_X = 2$ for toric varieties

In this section we study the extremal contractions described in §3 for toric toric varieties with Picard number 2. Let us begin with the proof of Proposition F.

Proof of Proposition F. — The isomorphism $X \cong \mathbb{P}_Y(\mathcal{O}_Y \oplus \mathcal{O}_Y(a))$, with $a \geq 0$, follows from Lemma 6.11 and the fact that $\rho_Y = 1$. On the other hand, the condition $0 \leq a \leq i_Y$ is in fact equivalent to the condition of X being Fano. To see this, let us recall that the adjunction formula for projectivized vector bundles [**BS95**, §1.1.7] gives

$$K_X = \pi^*\mathcal{O}_Y(a - i_Y) - 2\xi,$$

where ξ is the tautological divisor on $\mathbb{P}_Y(\mathcal{O}_Y \oplus \mathcal{O}_Y(a))$ and i_Y is defined in such a way $\mathcal{O}_Y(-K_Y) \cong \mathcal{O}_Y(i_Y)$. Now, we notice that if F is any fiber π, then

$$-K_X \cdot [F] = 2\xi \cdot [F] = 2 > 0.$$

On the other hand, if $C \subseteq Y$ is an irreducible reduced curve and $C_X \subseteq X$ is the image of C by the section associated to the quotient $\mathcal{O}_Y \oplus \mathcal{O}_Y(a) \to \mathcal{O}_Y$, then

$$-K_X \cdot [C_X] = \deg(C) \cdot (i_Y - a).$$

Hence, X is Fano if and only if $0 \leq a \leq i_Y - 1$, since the numerical classes of these curves above generates the Mori cone of X. □

Let us prove now Theorem E.

Proof of Theorem E. — Let $R \subseteq \overline{\mathrm{NE}}(X)$ be an extremal ray such that $D \cdot R > 0$ and let $\pi : X \to Y$ be the corresponding extremal contraction. If π is of fiber type then Proposition F leads us to the first case.

Let us suppose that $\pi : X \to Y$ is a birational contraction. Since Y is a complete and simplicial toric variety of dimension n and Picard number one, the fan of Y contains exactly $n+1$ rays. Let us denote by $u_1, \ldots, u_{n+1} \in N$ the primitive lattice vectors generating these rays.

By Proposition 3.3, $\pi : X \to Y$ is a divisorial contraction sending an irreducible invariant divisor $E = V(u_E)$ onto a codimension two subvariety $A \subseteq Y$, and Y is a $\mathbb{Q}$-factorial toric Fano variety with isolated canonical singularities. Moreover, $E \cdot [F] = K_X \cdot [F] = -1$ for every non-trivial fiber F of π.

Notice that the fan of X contains exactly $n+2$ rays, generated by the primitive lattice vectors $u_1, \ldots, u_{n+1}, u_E \in N$. Now, Lemma 6.9 implies that, up to reordering if necessary, π is induced by the wall relation

$$u_1 + (-1)u_E + u_2 = 0.$$

There are exactly $n-1$ walls satisfying this relation (corresponding to the fibers over the $n-1$ invariants points of A). Namely, the walls

$$\omega_i = \mathrm{cone}(u_E, u_3, \ldots, \widehat{u_i}, \ldots, u_{n+1}) \quad \text{with } i \in \{3, \ldots, n+1\},$$

separating the two maximal cones $\sigma_i = \mathrm{cone}(u_E, u_3, \ldots, \widehat{u_i}, \ldots, u_{n+1}, u_1)$ and $\sigma_i' = \mathrm{cone}(u_E, u_3, \ldots, \widehat{u_i}, \ldots, u_{n+1}, u_2)$.

From this relation and Proposition 6.8, we can compute:

$$V(u_E) \cdot [C_{\omega_i}] = -\frac{\mathrm{mult}(\omega_i)}{\mathrm{mult}(\sigma_i)} = -\frac{\mathrm{mult}(\omega_i)}{\mathrm{mult}(\sigma_i')} = -1,$$

and

$$-K_X \cdot [C_{\omega_i}] = \frac{\mathrm{mult}(\omega_i)}{\mathrm{mult}(\sigma_i)} = \frac{\mathrm{mult}(\omega_i)}{\mathrm{mult}(\sigma_i')} = 1.$$

Therefore, $\mathrm{mult}(\omega_i) = \mathrm{mult}(\sigma_i) = \mathrm{mult}(\sigma_i')$.

On the other hand, Remark 6.3 implies that $\mathrm{mult}(\omega_i) = 1$ since X has isolated singularities. Hence, both σ_i and σ_i' are smooth cones.

We get that $\tau_i = \mathrm{cone}(u_1, u_2, u_3, \ldots, \widehat{u_i}, \ldots, u_{n+1}) \in \Delta_Y(n)$ is a smooth cone for $i \in \{3, \ldots, n+1\}$. Thus, $A = V(u_1, u_2)$ is contained in the smooth locus of Y. The

isomorphism $X \cong \mathrm{Bl}_A(Y)$ follows from the description of the blow-up of a smooth toric variety along an irreducible invariant (smooth) subvariety (see [**CLS**, Definition 3.3.17]).

Since Y has $n-1$ smooth maximal cones and a complete simplicial toric variety of dimension n and Picard number one has exactly $n+1$ maximal cones (corresponding to $n+1$ invariant points), Y has at most 2 singular points and they are outside $A \subseteq Y$. In particular, Y is a $\mathbb{Q}$-factorial Gorenstein toric Fano variety with $\rho_Y = 1$.

Let us prove that Y is isomorphic to one of the listed varieties. Since the fan of Y contains a smooth maximal cone, we can suppose that the vectors $u_1, \ldots, u_n$ correspond to the first n elements of the canonical basis of $\mathbb{Z}^n$, by Remark 6.3.

Let us write $u_{n+1} = (-a_1, \ldots, -a_n)$, with $a_i \in \mathbb{Z}_{>0}$ for $i \in \{1, \ldots, n\}$. For each $i \in \{3, \ldots, n+1\}$ we have that $\mathrm{mult}(\sigma_i) = 1$. Therefore, Proposition 6.2 leads $\mathrm{mult}(\sigma_i) = |\det(e_1, e_1 + e_2, e_3, \ldots, \widehat{e_i}, \ldots, e_n, u_{n+1})| = a_i = 1$, for $i \in \{3, \ldots, n\}$. Hence, we can write $u_{n+1} = (-a, -b, -1, \ldots, -1)$, with $a, b \in \mathbb{Z}_{>0}$. It should be noticed that Y has isolated singularities if and only if $\gcd(a, b) = 1$.

Thus, $Y \cong \mathbb{P}(1^{n-1}, a, b)$ with $a, b \in \mathbb{Z}_{>0}$ relatively prime integers. Now, Y is a Gorenstein Weighted Projective Space if and only if $a|(n-1+a+b)$ and $b|(n-1+a+b)$, by [**CK99**, Lemma 3.5.6]. Equivalently, $a|(n-1+b)$ and $b|(n-1+a)$. If $a = b$ the only possibility is $a = b = 1$, leading to (a): $Y \cong \mathbb{P}^n$.

Let us suppose that $1 \leq a < b$ and notice that $ab|(n-1+a+b)$ since $\gcd(a, b) = 1$. On the other hand,

$$\frac{n-1+a+b}{ab} = 1 \Leftrightarrow \frac{n-1+a+b}{ab} < 2$$
$$\Leftrightarrow (2b-1)\left(a - \frac{1}{2}\right) - n + \frac{1}{2} > 0.$$

Since $a \geq 1$, this condition is fulfilled when $(2b-1) \cdot \frac{1}{2} - n + \frac{1}{2} > 0 \Leftrightarrow b > n$.

Therefore, $b \geq n+1$ implies that $n-1+a+b = ab$ or, equivalently, $n = (a-1)(b-1)$. This leads to $a = 2$, $b = n+1$ and hence to (b): $Y \cong \mathbb{P}(1^{n-1}, 2, n+1)$ and n must be even. Finally, if $1 \leq a < b \leq n$ we get the last case (c): $Y \cong \mathbb{P}(1^{n-1}, a, b)$.

Conversely, given one of these listed varieties Y with their fans as above and considering X to be the blow-up of Y along $A = V(e_1, e_2)$, we obtain a projective toric variety satisfying the hypothesis.

In fact, since A is contained in the smooth locus of Y we obtain that X is a $\mathbb{Q}$-factorial Gorenstein toric variety with isolated canonical singularities. In order to prove that X is Fano we need to analyze the second extremal contraction that corresponds to the wall relation on X given by

$$bu_E + (a-b)e_1 + e_3 + \cdots + e_n + u_{n+1} = 0.$$

In any of the three listed cases we will obtain

$$-K_X \cdot [C_\omega] = \frac{n-1+a}{b} \in \mathbb{Z}_{>0},$$

proving that X is a Fano variety, by the Cone Theorem and Kleiman's criterion of ampleness. □

As a consequence we obtain the following list of possible admissible weights $(a,b) \in \mathbb{Z}^2_{>0}$ that corresponds to varieties $Y \cong \mathbb{P}(1^{n-1},a,b)$ as in Theorem E.2. Compare with **[Kas13]** and **[Mir85]**.

TABLE 1. Admissible $Y \cong \mathbb{P}(1^{n-1},a,b)$ as in Theorem E.2

n	Weights $(a,b) \in \mathbb{Z}^2_{>0}$
3	$(1,1),(1,3)$
4	$(1,1),(1,2),(1,4),(2,5)$
5	$(1,1),(1,5)$
6	$(1,1),(1,3),(1,6),(2,7),(3,4)$
7	$(1,1),(1,7)$
8	$(1,1),(1,2),(1,4),(1,8),(2,3),(2,9),(3,5)$
9	$(1,1),(1,3),(1,9)$
10	$(1,1),(1,2),(1,5),(1,10),(2,11)$

Remark 8.1. — From the wall relation

$$bu_E + (a-b)e_1 + e_3 + \cdots + e_n + u_{n+1} = 0$$

we can deduce the nature of the second extremal contraction $\varphi : X \to W$. In the smooth case it is a contraction of fiber type onto $W \cong \mathbb{P}^1$. In the singular case, it is a divisorial contraction sending its exceptional locus onto a point. Moreover, $W \cong \mathbb{P}(1^{n-1},a,b-a)$ since

$$au_E + (b-a)e_2 + e_3 + \cdots + e_n + u_{n+1} = 0.$$

We conclude with an example showing that is the hypothesis of isolated singularities in Theorem E.2 cannot be dropped. We exhibit an example of a $\mathbb{Q}$-factorial Gorenstein toric Fano fivefold X with terminal singularities, whose singular locus is one-dimensional and that admits a birational extremal contraction $\pi : X \to Y$ which is not a blow-up, but only a blow-up in codimension two, and where Y is a non-Gorenstein $\mathbb{Q}$-factorial toric Fano fivefold.

Example 8.2. — Let us consider the fan $\Delta_X \subseteq \mathbb{R}^5$ generated by the vectors

$$e_1, e_2, e_3, e_4, e_5, u_6, u_E,$$

where $\{e_1,\ldots,e_5\}$ is the canonical basis of $\mathbb{R}^5$, $u_6 = (-1,-1,-1,-2,-3)$ and $u_E = (-1,-1,-1,-2,-2)$.

It can be checked by using **[Macaulay2]** that X is a $\mathbb{Q}$-factorial Gorenstein Fano fivefold. Moreover, it can be checked by hand that its singular locus is one-dimensional, consisting only of terminal points, and given by

$$\mathrm{Sing}(X) = V(e_1,e_2,e_3,u_E) \cup V(e_1,e_2,e_3,e_4,u_6).$$

The wall relation $e_5 + (-1)u_E + u_6 = 0$ determines an extremal contraction $\pi : X \to Y$ sending the Weil divisor $V(u_E) \subseteq X$ (which is not Cartier) onto $A = V(e_5, u_6) \subseteq Y$. Finally, the relation

$$e_1 + e_2 + e_3 + 2e_4 + 3e_5 + u_6 = 0$$

implies that $Y \cong \mathbb{P}(1^4, 2, 3)$, which is not Gorenstein, by [**CK99**, Lemma 3.5.6].

9. Toric universal coverings in codimension 1

The aim of this section is to describe the contraction of extremal rays appearing in the previous sections by using universal coverings in codimension 1.

Let us recall some of the results and definitions introduced in [**Buc08**]. We will focus on the case of complex normal varieties.

Definition 9.1. — Let X be a complex normal algebraic variety. A *covering in codimension 1* is a finite surjective morphism $\varphi : Y \to X$ which is unramified in codimension 1. Namely, there exists a subvariety $V \subseteq X$ such that

1. $\operatorname{codim}_X(V) \geq 2$.
2. $\varphi|_{\varphi^{-1}(X \setminus V)} : \varphi^{-1}(X \setminus V) \to X \setminus V$ is a topological covering.

Moreover, a *universal covering in codimension 1* is a covering in codimension 1 which is universal in the sense that for any covering in codimension 1 $f : Z \to X$ there exists a (not necessarily unique) covering in codimension 1, $g : Y \to Z$, such that $\varphi = f \circ g$.

Proposition 9.2 **([Buc08**, Corollary 3.10 and Remark 3.14**])**

A covering in codimension 1, $\varphi : Y \to X$, is universal if and only if $\pi_1(Y_{\mathrm{reg}})$ is trivial.

Proposition 9.3. — *Let X and Y be normal projective varieties and $\varphi : Y \to X$ be a covering in codimension 1. Then,*

1. *If K_X is a Cartier divisor, then K_Y is a Cartier divisor.*
2. *If X is a Fano variety, then Y is a Fano variety.*
3. *If X has terminal (resp. canonical) singularities, then Y also has terminal (resp. canonical) singularities.*

Proof. — As $\varphi : Y \to X$ is unramified in codimension 1, there is no ramification divisor and hence $\varphi^* K_X = K_Y$, implying (1). The point (2) follows from [**EGAII**, Proposition 5.1.12], while (3) follows from [**Kol97**, Proposition 3.16]. □

In the case of toric varieties we can describe the ramification divisor of a toric finite surjective morphism.

Lemma 9.4 **([AP13**, Lemma 3.3**]).** — *Let $\varphi : Y \to X$ be a finite morphism of toric varieties corresponding to the map of fans $\Phi : (N_Y, \Delta_Y) \to (N_X, \Delta_X)$ given by the inclusion of lattices $N_Y \subseteq N_X$ of finite index, so that $N_X \otimes_{\mathbb{Z}} \mathbb{R} = N_Y \otimes_{\mathbb{Z}} \mathbb{R}$ and $\Delta_X = \Delta_Y$. Then,*

1. *φ is equivariant with respect to the homomorphism of tori $T_Y \to T_X$.*
2. *φ is an abelian cover with Galois group $G = \ker(T_Y \to T_X) \cong N_X / N_Y$.*

3. *The ramification divisor* $\mathrm{Ram}(\varphi)$ *is supported on the torus invariant divisors* $V(\rho)$*, with multiplicities* $d_\rho \geq 1$ *defined by the condition that the integral generator of* $N_Y \cap \mathbb{R}_{\geq 0} u_\rho$ *is* $d_\rho u_\rho$*, for every ray* $\rho = \mathbb{R}_{\geq 0} u_\rho \in \Delta_X(1)$.

Moreover, we have the following theorem characterizing the fundamental group of the smooth locus for $\mathbb{Q}$-factorial toric varieties.

Theorem 9.5 **([Buc08**, Corollary 3.10 and Theorem 4.8] **and [RT16a**, Theorem 2.4]**)**

Let X *be a* $\mathbb{Q}$*-factorial toric variety defined by the fan* $\Delta_X \subseteq N_{\mathbb{R}}$ *and let* $N_{\Delta_X(1)} \subseteq N$ *be the sublattice of* N *generated by the primitive lattice generators* $u_\rho \in N$ *of all the rays* $\rho \in \Delta_X(1)$*. Then,*

$$\pi_1(X_{\mathrm{reg}}) \cong N/N_{\Delta_X(1)} \cong \mathrm{Tors}(\mathrm{Cl}(X)).$$

Example 9.6. — Let $X = X(\Delta_X)$ be a complete $\mathbb{Q}$-factorial toric variety of dimension n such that $\rho_X = 1$. Then, we will say that X is a Fake Weighted Projective Space.

This name comes from the following observation: the fan Δ_X has cone generators $u_0, \ldots, u_n \in \Delta_X(1)$, and the maximal cones of Δ_X are generated by the n-element subsets of $\{u_0, \ldots, u_n\} \subseteq N \cong \mathbb{Z}^n$. As they are linearly dependent,

$$\sum_{i=0}^{n} \lambda_i u_i = 0,$$

for some $\lambda_0, \ldots, \lambda_n \in \mathbb{Z}_{\geq 1}$. Therefore, $\pi_1(X_{\mathrm{reg}}) = \{0\}$ if and only if the primitive lattice vectors $u_0, \ldots, u_n \in N$ generate the lattice N. If it is the case we will have that $X \cong \mathbb{P}(\lambda_0, \ldots, \lambda_n)$, by **[CLS**, Example 5.1.14].

From Lemma 9.4 and Theorem 9.5 we can deduce the following structure theorem for $\mathbb{Q}$-factorial toric complete varieties of Picard number one.

Theorem 9.7 **([Buc08**, Theorem 6.4]**).** — *Let* X *be a Fake Weighted Projective Space of dimension* n*. There exists a unique universal covering in codimension 1* $\varphi : \mathbb{P}(\lambda_0, \ldots, \lambda_n) \to X$*, canonically identifying* X *as a finite geometric quotient of* $\mathbb{P}(\lambda_0, \ldots, \lambda_n)$ *by the torus-equivariant action of* $\pi_1(X_{\mathrm{reg}}) \cong \mathrm{Tors}(\mathrm{Cl}(X))$.

Following **[RT16a]** and **[RT16b]**, it is natural to consider $\mathbb{Q}$-factorial complete toric varieties with torsion-free class group as analogs of Weighted Projective Spaces.

Definition 9.8. — Let $X = X(\Delta_X)$ be a $\mathbb{Q}$-factorial complete toric variety of dimension n. We define the *canonical universal covering in codimension 1 of* X to be the covering in codimension 1 $\pi_X : \widehat{X} \to X$ corresponding to the map of fans

$$\Pi_X : (N_{\Delta_X(1)}, \Delta_X) \to (N_X, \Delta_X).$$

Moreover, we say that X is a Poly Weighted Space (PWS) if

$$\pi_1(X_{\mathrm{reg}}) \cong N/N_{\Delta_X(1)} \cong \mathrm{Tors}(\mathrm{Cl}(X)) \cong \{0\}.$$

After the recent works of M. Rossi and L. Terracini, there is an explicit combinatorial construction (via Gale duality) of the canonical universal covering in codimension 1 of any $\mathbb{Q}$-factorial complete toric variety. This extends Theorem 9.7 to higher class group rank varieties (see [**RT16b**, Theorem 2.2] for details).

The remaining of the section will be devoted to study contractions $X \to Y$ of extremal rays as in the previous section via universal coverings in codimension 1, and without the assumption of isolated singularities in the divisorial case.

The case of extremal contraction of fiber type was studied by Y. Kawamata in [**Kaw06**, Lemma 4.1].

Let X be a $\mathbb{Q}$-factorial projective toric variety and let $R = \mathbb{R}_{\geq 0}[C_\omega] \subseteq \overline{\mathrm{NE}}(X)$ be an extremal ray defining a contraction of fiber type $\varphi_R : X \to X_R$, where $C_\omega = V(u_1, \ldots, u_{n-1})$ is an invariant curve contracted by φ_R. Then, the wall $\omega = \mathrm{cone}(u_1, \ldots, u_{n-1})$ separates two maximal cones $\sigma = \mathrm{cone}(u_1, \ldots, u_{n-1}, u_n)$ and $\sigma' = \mathrm{cone}(u_1, \ldots, u_{n-1}, u_{n+1})$.

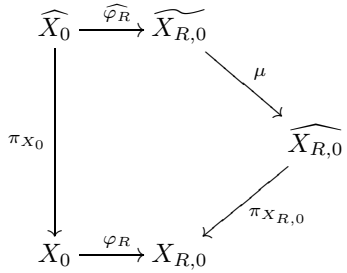
Following the same notation as in Theorem 6.7, we have that (up to reordering, if necessary) the contraction of R is defined by the projection

$$N := N_X \xrightarrow{\Phi} N_{X_R} := N_X / \left(\mathrm{Span}(u_{\beta+1}, \ldots, u_{n+1}) \cap N_X\right).$$

Write $\Phi(u_i) = d_i \overline{u}_i$ for primitive vectors $\overline{u}_i$ in N_{X_R} and positive integers d_i, where $1 \leq i \leq \beta$. Then, this $\overline{u}_i$ define a β-dimensional cone $\sigma_0 \in \Delta_{X_R}$ which is of maximal dimension by Theorem 6.7 and hence it corresponds to an invariant open affine subset $X_{R,0}$ of X_R.

In this setting, we have that the contraction of fiber type $\varphi_R : X \to X_R$ becomes (locally) trivial over the invariant open affine subset $X_{R,0} \subseteq X_R$ after a finite morphism of toric varieties (possibly ramified in codimension 1).

Lemma 9.9 **([Kaw06**, Lemma 4.1]**).** — *Let $X_{R,0} \subseteq X_R$ be an invariant open affine subset and let $X_0 = \varphi_R^{-1}(X_{R,0}) \subseteq X$. Then, there is a commutative diagram of toric morphisms*



that satisfies the following conditions:

(a) *π_{X_0} and $\pi_{X_{R,0}}$ are the corresponding canonical universal coverings in codimension 1.*

(b) *μ is a finite surjective morphism such that $\pi_{X_{R,0}} \circ \mu$ has ramification order d_i over $V(\overline{u}_i) \subseteq X_{R,0}$.*

(c) *$\widehat{\varphi_R}$ is a trivial fibration, whose fiber is a WPS.*

Remark 9.10. — If $\varphi_R : X \to X_R$ is a toric extremal contraction of fiber type, then φ_R is not necessarily locally trivial, as there may be some multiple fibers. However, we have the following combinatorial criterion to decide whether is locally trivial over an invariant open affine subset of X_R (see [**CaDR08**, Remark 3.3 and Remark 3.8]).

Let $\omega = \mathrm{cone}(u_1, \ldots, u_{n-1}) \in \Delta_X(n-1)$ be a wall defining the contraction and $U(\omega) = U_R$ as in Theorem 6.7. Then,

$$\Delta_F := \{\sigma \in \Delta_X \mid \sigma \subseteq U_R\}$$

can be seen as a fan in the real vector space U_R, defining a toric variety F. Moreover, we have that $\varphi^{-1}(x)_{\mathrm{red}} \cong F$ for all $x \in X_R$.

In this setting, invariant fibers of φ_R are given by $V(\tau)$ with $\tau \in \Delta_X$ such that $\dim \tau = \dim X_R$ and $\tau \cap U_R = \{0\}$. Let $p = V(\sigma_0)$ be a fixed point of the torus action and $V(\tau)$ the fiber of φ_R over p. Then, the following are equivalent:

1. The scheme theoretic fiber of φ_R over p is reduced.
2. $\varphi_R^{-1}(U_{\sigma_0}) \cong U_{\sigma_0} \times F$.
3. $N = (U_R \cap N) \oplus (\mathrm{Span}(\tau) \cap N)$.

Proposition 9.11. — *Let X be a $\mathbb{Q}$-factorial Gorenstein toric Fano variety of dimension $n \geq 3$ with canonical singularities and with at most finitely many non-terminal points. Assume that there exists an effective prime divisor $D \subseteq X$ such that $\dim_{\mathbb{R}} \mathrm{N}_1(D, X) = 1$ and that $\rho_X = 2$. Let $R \subseteq \overline{\mathrm{NE}}(X)$ be an extremal ray such that $D \cdot R > 0$ and let us denote by $\pi : X \to Y$ the corresponding extremal contraction. Assume that π is of fiber type. Then there exist weights $\lambda_0, \ldots, \lambda_{n-1} \in \mathbb{Z}_{>0}$ and a cartesian diagram of toric varieties*

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\widehat{\pi}} & \mathbb{P}(\lambda_0, \ldots, \lambda_{n-1}) \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow{\pi} & Y \end{array}$$

where vertical arrows denote the corresponding universal covering in codimension 1, and $\widehat{X}$ is a Gorenstein Fano PWS with terminal singularities such that $\rho_{\widehat{X}} = 2$. Moreover, $\widehat{\pi} : \widehat{X} \to \mathbb{P}(\lambda_0, \ldots, \lambda_{n-1})$ leads to an isomorphism

$$\widehat{X} \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}(\lambda_0,\ldots,\lambda_{n-1})} \oplus \mathcal{O}_{\mathbb{P}(\lambda_0,\ldots,\lambda_{n-1})}(a)).$$

Proof. — By Proposition F, $\pi : X \to Y$ is a $\mathbb{P}^1-$bundle and X is isomorphic to $\mathbb{P}_Y(\mathcal{O}_Y \oplus \mathcal{O}_Y(a))$, where Y is a $\mathbb{Q}$-factorial Gorenstein Fano variety with terminal singularities. In particular, if U_{σ_0} is an invariant open affine subset of Y with fixed point $p = V(\sigma_0)$ and $V(\tau)$ is the fiber of π over p, then π is a locally trivial fibration over U_{σ_0} and hence

$$N = (U_R \cap N) \oplus (\mathrm{Span}(\tau) \cap N),$$

by Remark 9.10 above. Thus, the contraction is defined by the projection

$$N_X \xrightarrow{\mathrm{pr}_2} \mathrm{Span}(\tau) \cap N_X \cong N_Y.$$

This implies that all the ramification orders d_i appearing in Lemma 9.9 are equal to 1.

This holds for every invariant open affine open subset of Y and hence it follows that the induced morphism between the universal coverings in codimension 1, $\widehat{\pi} : \widehat{X} \to \widehat{Y}$, is a locally trivial fibration all whose fibers are isomorphic to $\mathbb{P}^1$. Moreover, this commutative diagram is cartesian in the category of schemes by [**Mol16**, Lemma 2.2.7].

Since $\rho_Y = 1$, it follows that $\widehat{Y} \cong \mathbb{P}(\lambda_0, \ldots, \lambda_{n-1})$ for some weights $\lambda_0, \ldots, \lambda_{n-1} \in \mathbb{Z}_{>0}$, by Example 9.6.

Finally, we have that both $\widehat{X}$ and $\widehat{Y}$ are Fano Gorenstein varieties with terminal singularities, by Proposition 9.3. □

Example 9.12. — Let X as in Proposition 9.11 and suppose that $\mathrm{Tors}(\mathrm{Cl}(X)) \cong \{0\}$, i.e., that $X \cong \widehat{X}$. The extremal contraction of fiber type

$$X \to \mathbb{P}(\lambda_0, \ldots, \lambda_{n-1})$$

leads to an isomorphism $X \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}(\lambda_0,\ldots,\lambda_{n-1})} \oplus \mathcal{O}_{\mathbb{P}(\lambda_0,\ldots,\lambda_{n-1})}(a))$. Then,

(a) X is Gorenstein $\Leftrightarrow \mathbb{P}(\lambda_0, \ldots, \lambda_{n-1})$ is Gorenstein $\Leftrightarrow \lambda_i | h$ for every $i \in \{0, \ldots, n-1\}$, by [**CK99**, Lemma 3.5.6].

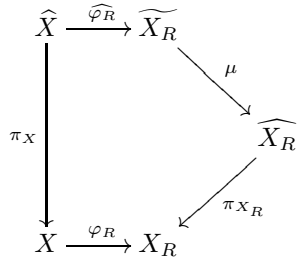
(b) X is terminal $\Leftrightarrow \mathbb{P}(\lambda_0, \ldots, \lambda_{n-1})$ is terminal $\Leftrightarrow \sum_{i=0}^{n} \{\lambda_i \kappa / h\} \in \{2, \ldots, n-1\}$ for each $\kappa \in \{2, \ldots, h-2\}$, by [**Kas13**, Proposition 2.3].

(c) X is Fano $\Leftrightarrow 0 \leq a \leq i_{\mathbb{P}(\lambda_0,\ldots,\lambda_{n-1})} - 1 = h - 1$, by Proposition F and the formula [**Mor75**, Proposition 2.3] for the canonical divisor of $\mathbb{P}(\lambda_0, \ldots, \lambda_{n-1})$.

Here, $h = \sum_{i=0}^{n-1} \lambda_i$ and $\{x\}$ denotes the fractional part of $x \in \mathbb{R}$.

The case of divisorial extremal contractions follows in a similar way.

Lemma 9.13. — *Let X be a $\mathbb{Q}$-factorial projective toric variety and $R \subseteq \overline{\mathrm{NE}}(X)$ an extremal ray defining a divisorial contraction $\varphi_R : X \to X_R$. Then, there is a commutative diagram of toric morphisms*



that satisfies the following conditions:

(a) *π_X and π_{X_R} are the corresponding canonical universal coverings in codimension 1.*

(b) *μ is a finite surjective morphism unramified in codimension 1 and given by the inclusion of lattices $N_{\Delta_{X_R}(1)} \subseteq N_{\Delta_X(1)}$ of index $d_E \geq 1$, defined by the condition*

that the integral generator of $\mathbb{R}_{\geq 0}u_E \cap N_{\Delta_{X_R}(1)}$ *is* $d_E u_E$*, where* $V(u_E)$ *is the exceptional divisor of* φ_R*.*

Moreover,

(c) $\widehat{\varphi_R} : \widehat{X} \to \widetilde{X_R}$ *is a divisorial contraction with* $(\pi_X)_* \operatorname{Exc}(\widehat{\varphi_R}) = \operatorname{Exc}(\varphi_R)$ *and* $(\pi_{X_R} \circ \mu)_* \widehat{\varphi_R}(\operatorname{Exc}(\widehat{\varphi_R})) = \varphi_R(\operatorname{Exc}(\varphi_R))$*.*

(d) *If* X *(resp.* X_R*) is a Fano variety then* $\widehat{X}$ *(resp.* $\widehat{X_R}$ *and* $\widetilde{X_R}$*) is.*

(e) *If* X *(resp.* X_R*) has Gorenstein singularities then* $\widehat{X}$ *(resp.* $\widehat{X_R}$ *and* $\widetilde{X_R}$*) does.*

(f) *If* X *has terminal (resp. canonical) singularities, then all varieties in the diagram have terminal (resp. canonical) singularities.*

Proof. — Let us suppose that $\varphi_R : X \to X_R$ is given by the contraction of the wall $\omega = \operatorname{cone}(u_1, \ldots, u_{n-1})$ separating the maximal cones $\sigma = \operatorname{cone}(u_1, \ldots, u_{n-1}, u_n)$ and $\sigma' = \operatorname{cone}(u_1, \ldots, u_{n-1}, u_{n+1})$. Then, the wall relation satisfied by these cones (defining the contraction) is given by

$$b_u u_n + \sum_{i=1}^{n-1} b_i u_i + b_{n+1} u_{n+1} = 0,$$

where $b_n, b_{n+1} \in \mathbb{Z}_{>0}$ and $b_i \in \mathbb{Z}$.

Since φ_R is a divisorial contraction we can suppose that (up to reordering, if necessary) $b_1 < 0$ and $b_2, \ldots, b_{n-1} \geq 0$, by Theorem 6.7. Thus, $E = \operatorname{Exc}(\varphi_R) = V(u_1)$ and the contraction corresponds to the stellar subdivision of the cone

$$\sigma = \operatorname{cone}(u_2, \ldots, u_n, u_{n+1}) \in \Delta_{X_R}(n)$$

with respect to the primitive lattice vector u_1 satisfying the wall relation above.

The canonical universal covering in codimension 1 of X (resp. X_R) is given by the fan Δ_X (resp. Δ_{X_R}) but seen in the sublattice $N_{\Delta_X(1)}$ (resp. $N_{\Delta_{X_R}(1)}$) of N.

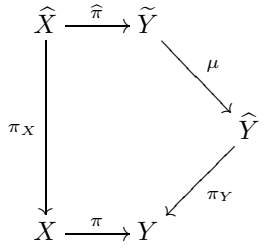
Clearly we have the inclusion of lattices $N_{\Delta_{X_R}(1)} \subseteq N_{\Delta_X(1)}$, which is of finite index since $(-b_1)u_1 \in N_{\Delta_{X_R}(1)}$, by the wall relation above. Hence, we obtain an induced finite surjective morphism of toric varieties $\mu : \widetilde{X_R} \to \widehat{X_R}$ that is unramified in codimension 1, by Lemma 9.4.

Now, the fan of $\widehat{X}$ is obtained by the stellar subdivision of the fan of $\widetilde{X_R}$ with respect to the primitive vector u_1 satisfying the wall relation above, obtaining the desired commutative diagram that satisfies (a), (b) and (c) by construction.

Finally, the last three assertions follows from Proposition 9.3 together with [**KM98**, Corollary 3.43]. □

We can now prove Proposition G.

Proof of Proposition G. — By Lemma 9.13 there is a commutative diagram of toric morphisms



where π_X and π_Y are the corresponding canonical universal coverings in codimension 1 and $\mu : \widetilde{Y} \to \widehat{Y}$ is a finite surjective morphism defined by the inclusion of lattices $N_{\Delta_Y(1)} \subseteq N_{\Delta_X(1)}$.

It should be noticed that under the hypothesis of the statement, Lemma 6.9 implies that $N_{\Delta_Y(1)} = N_{\Delta_X(1)}$ and hence $\widetilde{Y} \cong \widehat{Y}$. Moreover, this commutative diagram is cartesian in the category of schemes by [**Mol16**, Lemma 2.2.7].

Finally, we have that $\widehat{Y} \cong \mathbb{P}(\lambda_0, \ldots, \lambda_n)$ for some weights $\lambda_0, \ldots, \lambda_n \in \mathbb{Z}_{>0}$, by Example 9.6. The result follows now directly from Lemma 9.13. □

Example 9.14. — Let X as in Proposition G and suppose that $\mathrm{Tors}(\mathrm{Cl}(X)) \cong \{0\}$, i.e., that $X \cong \widehat{X}$. The extremal divisorial contraction

$$\pi : X \to \mathbb{P}(\lambda_0, \ldots, \lambda_n)$$

determines the shape of the fan of X in terms of the fan of $\mathbb{P}(\lambda_0, \ldots, \lambda_n)$ (which is well known):

The fan of $\mathbb{P}(\lambda_0, \ldots, \lambda_n)$ is given by $n+1$ lattice primitive vectors $u_0, \ldots, u_n$ that generates the lattice N and that satisfy the relation

$$\sum_{i=0}^{n} \lambda_i u_i = 0.$$

Let us suppose that π contracts $E = V(u_E) \subseteq X$ onto the invariant subvariety $A = V(u_i, u_j) \subseteq \mathbb{P}(\lambda_0, \ldots, \lambda_n)$, of codimension two. Then, $u_E = u_i + u_j$, by Lemma 6.9.

In particular, the same computation used to prove [**CK99**, Lemma 3.5.6] shows that X is a Fano Gorenstein variety if and only if $\lambda_i | h$, $\lambda_j | h$, $\lambda_k | (h - \lambda_i)$ and $\lambda_k | (h - \lambda_j)$ for every $k \neq i, j$, where $h = \sum_{i=0}^{n} \lambda_i$.

However, to the best of the author's knowledge, the characterization such X having terminal singularities is more subtle. Indeed, if X is a $\mathbb{Q}$-factorial Fano Gorenstein toric variety then it corresponds to a simplicial reflexive lattice polytope $P \subseteq N_{\mathbb{R}}$ (see [**Bat94**] for details). In [**Nil05**, Corollary 3.7], B. Nill characterizes all polytopes among these ones that correspond to varieties with only terminal singularities, but it does not seem easy to translate this characterization into a function of the weights $\lambda_0, \ldots, \lambda_n$.

Finally, it should be noticed that if one of the weights is equal to 1, say $\lambda_0 = 1$, then we have a coordinate-wise description of the primitive vectors defining the fan

of $\mathbb{P}(1, \lambda_1, \ldots, \lambda_n)$. Namely, the canonical basis of $\mathbb{Z}^n$ together with the vector $(-\lambda_1, \ldots, -\lambda_n) \in \mathbb{Z}^n$. In this case, we can explicitly compute the Cartier data $\{m_\sigma\}_{\sigma \in \Delta_X(n)} \subseteq M$ of K_X, allowing us to decide whether the singularities of X are terminal or not.

In particular, we compute that X has only Gorenstein terminal singularities if all the integers

$$\frac{h}{\lambda_i}, \frac{h}{\lambda_j}, \frac{h-\lambda_i}{\lambda_k}, \frac{h-\lambda_j}{\lambda_k}$$

considered before, are equal or greater than 3. The variety defined in Example 8.2 satisfy this condition.

References

[AP13] V. Alexeev and R. Pardini, *On the existence of ramified abelian covers*, Rend. Sem. Mat. Univ. Politec. Torino, 71, 3-4 (2013), pages 307-315.

[And85] T. Ando, *On extremal rays of the higher dimensional varieties*, Invent. Math. 81, 1985, pages 347-357.

[AW93] M. Andreatta and J. A. Wiśniewski, *A note on nonvanishing and applications*, Duke Math. J., 72 (1993), no. 3, pages 739-755.

[AW97] M. Andreatta and J. A. Wiśniewski, *A view on contractions of higher-dimensional varieties*, Algebraic geometry-Santa Cruz 1995, Proc. Sympos. Pure Math., 62, Part 1, Amer. Math. Soc., Providence, RI, 1997, pages 153-183.

[AD14] C. Araujo and S. Druel, *On codimension 1 del Pezzo foliations on varieties with mild singularities*, Mathematische Annalen, 360 (3-4), 2014, pages 769-798.

[ArRM] C. Araujo and J. Ramón-Marí, *Flat deformations of* $\mathbb{P}^n$, Bulletin of the Brazilian Mathematical Society, New Series, 2014, Volume 45, Issue 3, pages 371-383.

[Bat94] V.V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, J. Algebr. Geom. 3, 1994, pages 493-535.

[BS95] M. C. Beltrametti and A. J. Sommese, *The adjunction theory of complex projective varieties*, Walter de Gruyter, 16, 1995.

[BCHM10] C. Birkar, P. Cascini, C. D. Hacon and J. McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc., 2010 (23), pages 405-468.

[Bon02] L. Bonavero, *Toric varieties whose blow-up at a point is Fano*, Tôhoku Math. J. (2) 54, 2002, pages 593-597.

[BCW02] L. Bonavero, F. Campana and J. A. Wiśniewski, *Variétés projectives complexes dont l'éclatée en un point est de Fano*, C. R. Math. Acad. Sci. Paris 334, 2002, pages 463-468.

[Buc08] W. Buczyńska, *Fake weighted projective spaces*, Electronic preprint `arXiv:0805.1211`, 2008.

[Cas09] C. Casagrande, *On Fano manifolds with a birational contraction sending a divisor to a curve*, The Michigan Mathematical Journal, 58 (3), 2009, pages 783-805.

[CaDR08] C. Casagrande and S. Di Rocco, *Projective Q-factorial toric varieties covered by lines*, Communications in Contemporary Mathematics 10, 2008, pages 363-389.

[CD15] C. Casagrande and S. Druel, *Locally unsplit families of rational curves of large anticanonical degree on Fano manifolds*, Internat. Math. Res. Notices, 2015 (21), pages 10756-10800.

[CK99] D. Cox and S. Katz, *Mirror symmetry and algebraic geometry*, American Mathematical Soc., No. 68, 1999.

[CLS] D. Cox, J. Little and H. Schenck, *Toric varieties*, GTM 124, AMS, 2011.

[Cut88] S. Cutkosky, *Elementary contractions of Gorenstein threefolds*, Mathematische Annalen, 1988 (3), vol. 280, pages 521-525.

[Dai02] D. I. Dais, *Resolving 3-dimensional toric singularities*, Geometry of toric varieties, Sémin. Cong., 6, pages 155-186, Soc. Math. France, Paris, 2002.

[DN14] G. Della Noce, *On the Picard Number of Singular Fano Varieties*, Int. Math. Res. Notices, 2014 (4), pages 955-990.

[Macaulay2] Daniel R. Grayson and Michael E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at `http://www.math.uiuc.edu/Macaulay2/`.

[Gro71] A. Grothendieck, *Séminaire de Géométrie Algébrique du Bois Marie 1 (1960-1961): Revêtements étales et groupe fondamental*, Springer-Verlag, Berlin, 1971, Lecture Notes in Mathematics, Vol. 224.

[EGAII] A. Grothendieck, *Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphisms*, IHES Publ. Math., n. 8, 1961.

[HK00] Y. Hu and S. Keel, *Mori dream spaces and GIT*, Michigan Mathematical Journal, 48 (2000), pages 331-348.

[Kas13] A. M. Kasprzyk, *Classifying terminal weighted projective space*, Electronic preprint `arXiv:1304.3029`, 2013.

[Kaw06] Y. Kawamata, *Derived categories of toric varieties*, Michigan Math. J. 54 (3), 2006, pages 517-535.

[Kol97] J. Kollár, *Singularities of Pairs*, In Algebraic Geometry, Santa Cruz 1995, volume 62 of Proc. Symp. Pure Math., Amer. Math. Soc., 1997.

[Kol96] J. Kollár, *Rational curves on algebraic varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 32, Springer-Verlag, Berlin, 1996.

[KM98] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, 1998.

[Mat02] K. Matsuki, *Introduction to the Mori Program*, Universitext (1979), Springer, 2002.

[Mir85] R. Miranda, *Gorenstein toric threefolds with isolated singularities and cyclic divisor class group*, Rocky Mountain J. Math, 1985, Volume 15 (1), pages 39-46.

[Mol16] S. Molcho, *Universal Weak Semistable Reduction*, Electronic preprint `arXiv:1601.00302`, 2016.

[Mor75] S. Mori, *On a Generalization of Complete Intersections*, J. Math. Kyoto Univ., 15, 1975, pages 619-646.

[Nil05] B. Nill, *Gorenstein toric Fano varieties*, Manuscripta Math. 116, no. 2, 2005, pages 183-210.

[PS09] Yu. G. Prokhorov and V. V. Shokurov, *Towards the second main theorem on complements*, J. Algebraic Geom, 18 (1), 2009, pages 151-199.

[RS06] G. V. Ravindra and V. Srinivas, *The Grothendieck-Lefschetz theorem for normal projective varieties*, J. Algebraic Geom, 2006 (15), pages 563-590.

[Rei83] M. Reid, *Decomposition of toric morphisms*, Arithmetic and geometry, vol. II, Progr. Math. 36, pages 395-418, Birhäuser, Boston, 1983.

[RT16a] M. Rossi and L. Terracini, *$\mathbb{Z}$-linear Gale duality and poly weighted spaces (PWS)*, Linear Algebra and Its Applications, 495, 2016, pages 256-288.

[RT16b] M. Rossi and L. Terracini, *A $\mathbb{Q}$-factorial complete toric variety is a quotient of a poly weighted space*, to appear in Annali di Matematica Pura e Applicata.

[Tsu06] T. Tsukioka, *Classification of Fano manifolds containing a negative divisor isomorphic to projective space*, Geom. Dedicata 123, 2006, pages 179-186.

Pedro MONTERO

Univ. Grenoble Alpes, Institut Fourier, F-38000 Grenoble, France.

Pedro MONTERO • *E-mail :* pedro.montero@univ-grenoble-alpes.fr