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Andrew Latto, Eric Tenenbaum. Compactly supported wavelets and the numerical solution of Burgers' equation. Comptes rendus de l'Académie des sciences. Série I, Mathématique, 1990, 311 série I, pp.903-909. hal-01321545

HAL Id: hal-01321545

<https://hal.science/hal-01321545>

Submitted on 27 May 2016

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Compactly supported wavelets and the numerical solution of Burgers' equation

Andrew LATTO and Eric TENENBAUM

Abstract — We present numerical solutions of Burgers' equation based on the combination of standard Galerkin techniques and a class of wavelet functions. The compactly supported orthonormal wavelet basis of Daubechies is found to stably represent the solution for small, and even vanishing, viscosity. The oscillations at a shock associated with Gibbs' phenomena are, in comparison with finite difference and spectral approximations, reduced and confined to the vicinity of the shock.

Les ondelettes à support compact et la solution numérique de l'équation de Burgers

Résumé — On présente dans cette Note une méthode de résolution de l'équation de Burgers basée sur la combinaison de techniques classiques de type Galerkin avec des bases fonctionnelles de type ondelettes. Les bases orthonormales d'ondelettes à support compact, dues à I. Daubechies fournissent des solutions stables, même lorsque la viscosité est petite et, éventuellement, nulle. Par rapport à des approximations par différences finies, ou spectrales, les oscillations numériques au voisinage des chocs, associées au phénomène de Gibbs, sont de faible amplitude et restent localisées à proximité des chocs.

Version française abrégée — Les ondelettes récemment découvertes par I. Daubechies constituent une base orthonormale de fonctions à support compact et de carré sommable sur la droite ou le cercle. Ces ondelettes possèdent la propriété qu'un ensemble fini d'entre elles, convenablement construit, engendre un espace fonctionnel contenant tous les polynômes de degré inférieur ou égal à un degré donné. Ces diverses propriétés suggèrent que les ondelettes peuvent être utiles pour la résolution numérique des équations aux dérivées partielles. Dans cette Note, on présente des résultats relatifs à la résolution numérique de l'équation de Burgers

$$u_t - uu_x = \sigma u_{xx}$$

par les méthodes d'approximation utilisant les ondelettes à support compact mentionnées ci-dessus.

La méthodologie utilisée combine des méthodes de Galerkin classiques avec les ondelettes et sera dénommée méthode d'ondelettes-Galerkin. Les bases de Galerkin utilisées dans cette étude sont dérivées d'une fonction génératrice φ , à support compact, satisfaisant

$$\varphi(x) = \sum a_k \varphi(2x - k),$$

les a_k étant ajustés de façon à donner à φ des propriétés convenablement choisies, concernant par exemple sa régularité. Les calculs présentés dans cette Note font appel à une fonction φ telle que six coefficients a_k seulement sont différents de zéro. La base de Galerkin est construite à partir de φ et de ses translatées $\varphi(x - k)$. La discrétisation en temps utilisée est de type semi-implicite, et est donnée par

$$(u_{n+1} - u_n)/\Delta t - u_n u_{n+1, x} = \sigma u_{n+1, xx}$$

avec $u_0(x) = u(x, 0)$ donnée. Dans l'approche de Galerkin, les coefficients du développement de u_{n+1} sont choisis de façon à être l'unique ensemble de coefficients pour lesquels l'équation ci-dessus reste valide par projection sur l'espace engendré par la base de Galerkin considérée.

Note présentée par Roland GLOWINSKI.

La méthode d'ondelettes-Galerkin (OG) peut être utilisée pour calculer de façon stable les solutions de l'équation de Burgers pour de très faibles valeurs du coefficient de viscosité σ , incluant le cas limite $\sigma=0$. Les oscillations numériques associées au phénomène de Gibbs à l'endroit des chocs sont confinées à un petit voisinage de ceux-ci et sont, de plus, d'amplitude bien moindre que ceux produits par des approximations par différences finies, ou spectrales. En fait, même pour les très faibles valeurs du coefficient de viscosité σ , les oscillations se stabilisent et les solutions approchées convergent vers la (non oscillante) bonne solution.

Ce bon comportement a lieu pour des valeurs de σ pour lesquelles les approximations par différences finies et spectrales sont instables. De plus, la composante oscillante de la solution a une modulation régulière ce qui permet de la filtrer pour obtenir une solution correcte.

Au vu de ces résultats, la méthode d'ondelettes-Galerkin semble fournir une technique robuste et générale pour *capturer* les chocs, et plus généralement pour résoudre les équations aux dérivées partielles dont les solutions présentent des discontinuités ou de forts gradients.

INTRODUCTION. — The wavelets of Daubechies are a newly-discovered compactly-supported and orthonormal basis for square integrable functions on the line or circle ([1], [2]). These wavelets can be chosen such that a locally finite subset spans a space including all polynomials of any fixed degree [1]. These properties suggest that wavelets might be useful in the numerical solution of nonlinear partial differential equations. In this Note we present results for the numerical evaluation of Burgers' equation using compactly supported wavelets. We combine standard Galerkin methods with these functions to form what we refer to as the wavelet-Galerkin method.

This method can stably represent the solutions to Burgers' equation for small, and even vanishing, viscosity. The oscillations associated with Gibbs' phenomena at a shock are confined to the vicinity of the shock, and are smaller than those that arise in finite difference and spectral approximations. Even for quite small viscosities, the oscillations stabilize and decay to the correct (non-oscillatory) solution. This occurs for values of the viscosity at which standard finite difference methods become unstable, and which standard spectral methods are either inaccurate or unstable. In addition, the oscillatory component of the solution has a regular modulation and can be smoothed to approximate the correct solution.

The wavelet-Galerkin method may provide a uniform, non-problem-specific technique for shock capture and, more generally, for resolving solutions with localized and sharp gradients.

2. BURGERS' EQUATION. — Burgers' equation

$$u_t - uu_x = \sigma u_{xx}$$

describes the evolution of the field $u = u(x, t)$ under nonlinear advection and linear dissipation. When the viscosity σ is null the field will develop a *shock* in a time $t^* = \min(1/u_{0,x})$. For *small* viscosity the solution will be a slightly smoothed version of the inviscid (shock) solution. That is, sharp gradients will develop and slowly dissipate as $t \rightarrow \infty$ and the solution decays to zero. For moderate values of the viscosity the solution decays to zero and gradients do not intensify [3]. Burgers' equation is a useful test case for numerical methods due to its simplicity and predictable dynamics. The challenge is to resolve the sharp gradients/shocks that occur at small and vanishing viscosity and accurately track their evolution. In this Note we assume periodic boundary

conditions and set $u_0 = -\sin x$. A useful property of Burgers' equation is that for certain values of time and viscosity the exact solution can be generated numerically using the Cole-Hopf transformation [3]. This makes it possible to compare the accuracy of different numerical methods by comparing them with the exact solution.

3. WAVELET-GALERKIN METHOD. — In this study we utilize the Daubechies wavelets. They are compactly-supported and form an orthonormal basis for square integrable functions ([1], [2]). The wavelet function can be derived from the scaling function φ and it is this function, together with its dilations and translations, that we use for our numerical analysis. The scaling function φ satisfies the *scaling relation*

$$\varphi(x) = \sum a_k \varphi(2x - k),$$

and can be made to have desired properties by suitable choice of the coefficients $\{a_k\}$. The scaling function φ will have compact support if and only if only finitely many a_k are nonzero. The orthonormality of the set of functions $\varphi(x - k)$ can be obtained from the conditions $\sum a_k a_{k-2m} = 2\delta_{0m}$ and $\sum (-1)^k k^m a_k = 0$. See [4] for a relaxation of these conditions preserving orthonormality. We use the Daubechies scaling function defined by six nonzero coefficients, or D6. It has a continuous first derivative and the vector space spanned by φ and its translates contains all polynomials of degree two. We will numerically solve Burgers' equation by projecting the solution of that PDE onto the space spanned by the integer translates of D6. That is, we assume the solution to be approximated by its projection into the subspace defined by the scaling function and a finite number of its translates

$$u = \sum_{k=-M}^N u_k \varphi(x - k).$$

Substitution of this into Burgers' equation and projection of the result onto the subspace Ω spanned by $\{\varphi(x - k) : k = -M, -M + 1, \dots, N\}$ defines a set of ordinary differential equations for the coefficients $u_k(t)$. The good approximation properties of Ω are a consequence of both the orthogonality and compact-support of the basis elements. In Ω the orthogonal basis is the most compactly-supported basis possible since linear combinations of translates of the scaling function cannot have a smaller

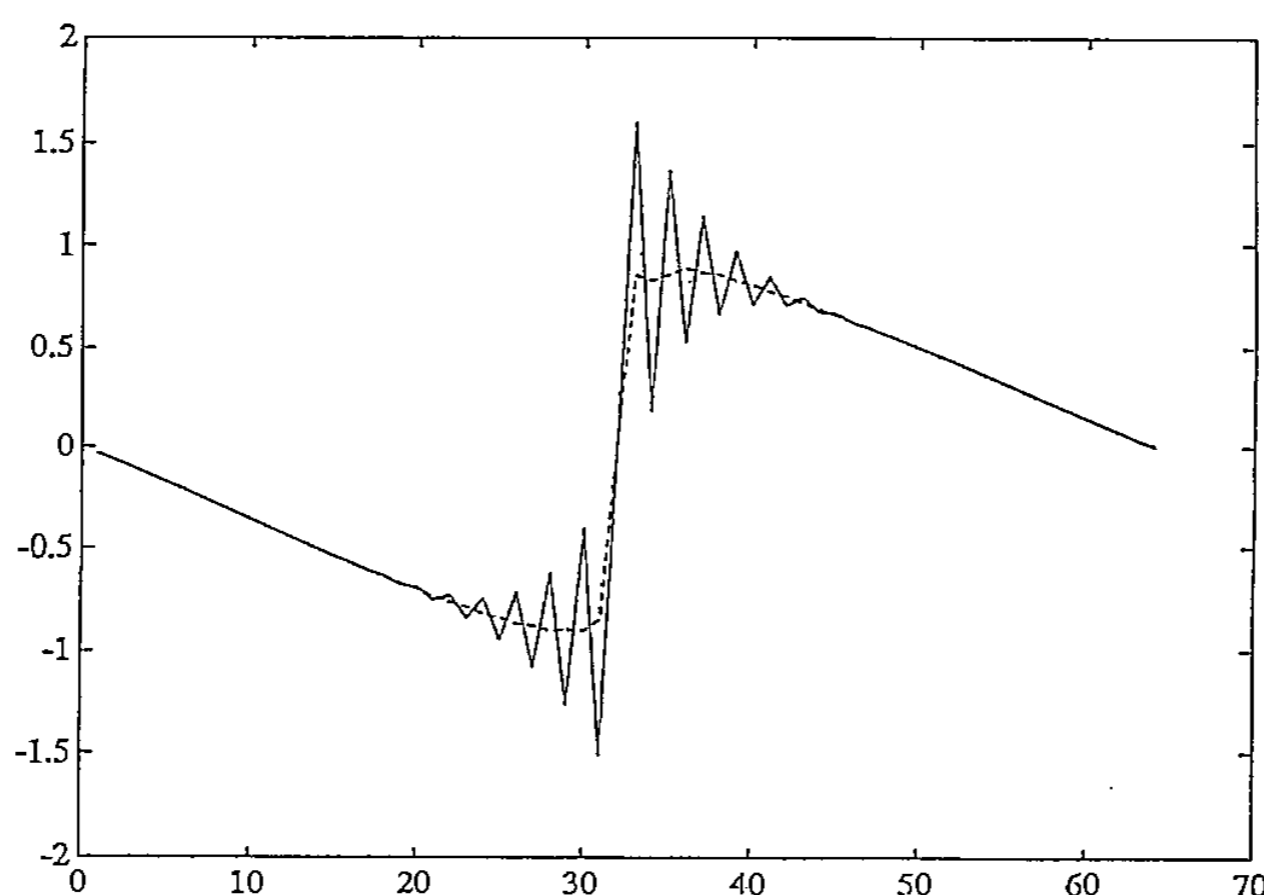


Fig. 1. — Smoothed and unsmoothed WG at time 18.

Fig. 1. — Solution OG filtrée et non filtrée au temps 18.

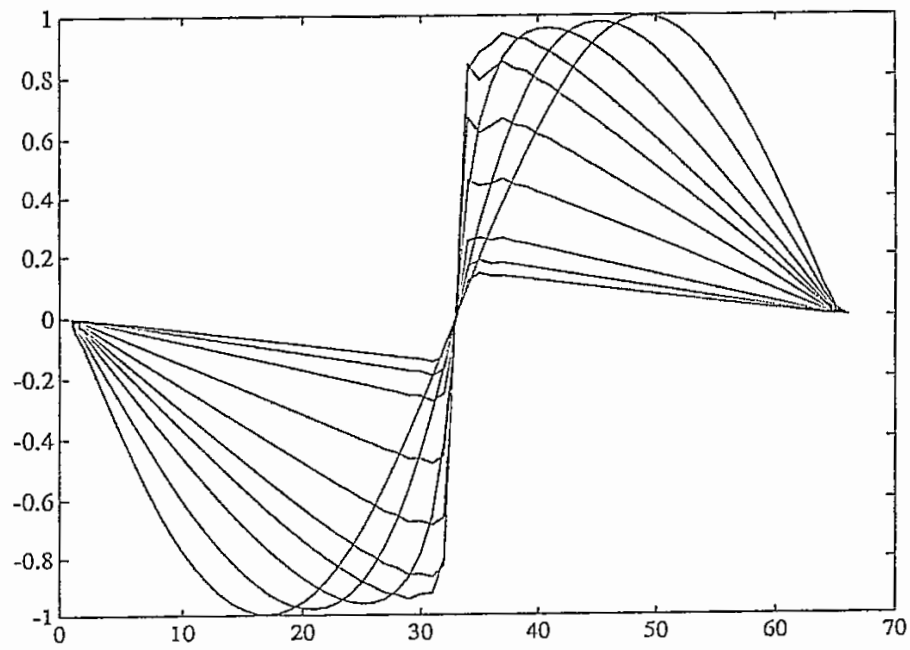


Fig. 2. — Progression of various times from 1 to 200 of smoothed WG.
 Fig. 2. — Progression entre les temps 1 et 200 de la solution OG filtrée.

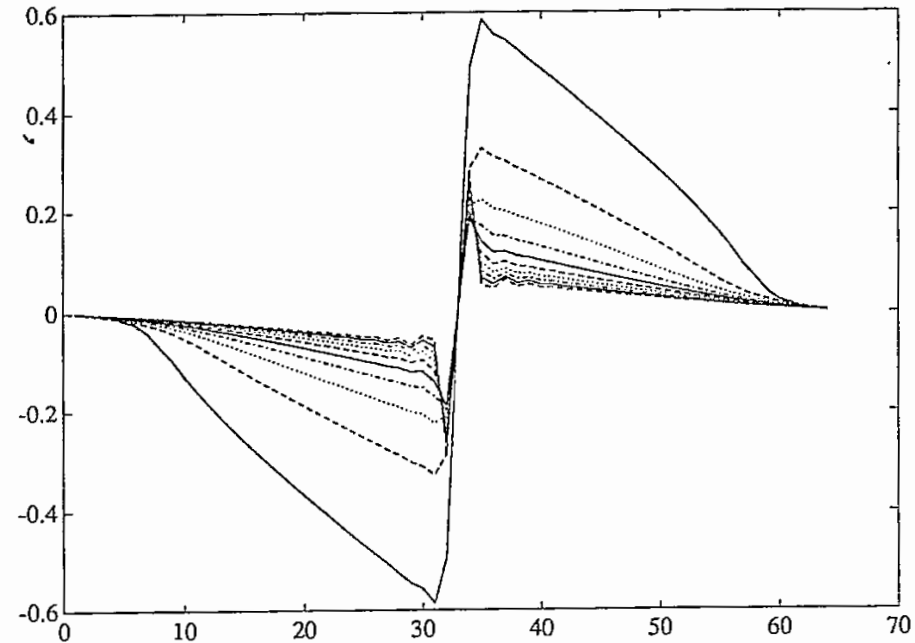


Fig. 4. — Smoothed spectral at times 40 to 400 by 40.
 Fig. 4. — Solution spectrale filtrée aux temps 40 à 400 par 40.

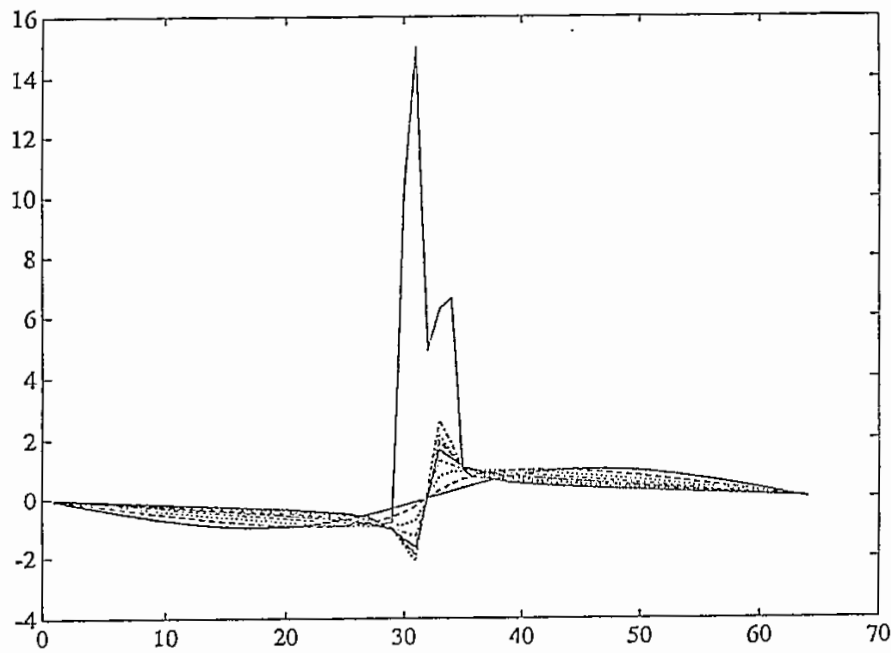


Fig. 3. — Smoothed finite difference times 3 to 43 by 5.
 Fig. 3. — Solution par différences finies filtrée aux temps 3 à 43 par 5.

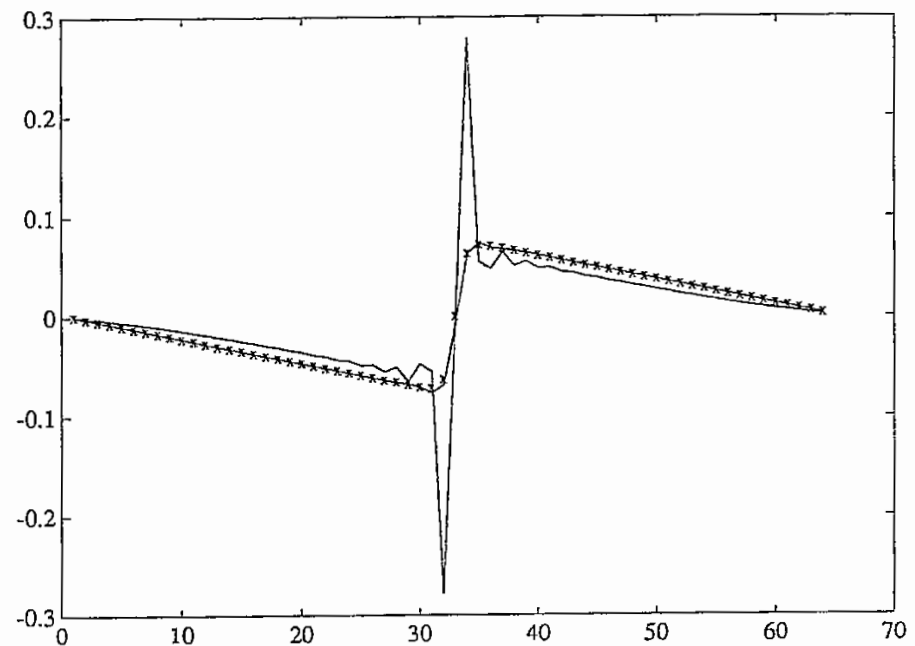


Fig. 5. — Exact: x, smoothed spectral: —, unsmoothed WG: --, time = 400.
 Fig. 5. — Solution exacte : x, spectrale filtrée : —, OG non filtrée : --, temps = 400.

support than the scaling function itself. The orthogonality of the basis diagonalizes the u_t term, preserving the finite bandwidth of the coefficient matrix. The asymmetry of the scaling function does not introduce a significant asymmetry in the numerical solution. Use of the mirror image of the scaling function does not change, to a tolerance, the numerical solution.

In this Note we utilize the *semi-implicit* time differencing scheme for solving Burgers' equation. That is, we solve for u_{n+1} from

$$(u_{n+1} - u_n)/\Delta t - u_n u_{n+1, x} = \sigma u_{n+1, xx},$$

where $u_0 = u(x, 0)$. In the Galerkin approach, the coefficients in the expression of u_{n+1} are chosen to be the unique set of coefficients that make the above equation hold when both sides are projected back to the space spanned by ϕ and its translates.

To apply the Galerkin method we require the definite integrals of products of scaling functions and their derivatives. The direct numerical evaluation of these terms can have inaccuracies associated with the oscillations of the derivatives. We have therefore developed exact methods for the evaluation of these terms. These methods will be described elsewhere.

4. DESCRIPTION OF RESULTS. — We present numerical results for viscosity $\sigma = .032$, period 64 and Reynolds number 2000. The initial data for figures 1 through 6 is $u_0 = -\sin((2\pi/64)x)$. The wavelet-Galerkin (WG) solutions are compared to finite difference and spectral solutions. The same semi-implicit time differencing scheme is used in all cases. As the shock develops, the wavelet-Galerkin solution develops oscillations. However, a simple three-point averaging of this solution of the form $\tilde{u}_k = (u_{k-1} + 2u_k + u_{k+1})/4$ produces a smooth and accurate approximation to the solution. Note that this smoothing is used only to remove the oscillations and does not produce an artificial viscosity; each time step is calculated from the unsmoothed version of the previous time step. The same smoothing is used for each method compared.

Figure 1 compares the smoothed and unsmoothed versions of the function produced by the wavelet-Galerkin technique after 18 time steps, around the time of shock formation.

Figure 2 shows that the smoothed wavelet-Galerkin technique displays the qualitatively correct behavior over time: a shock that quickly builds to near singularity, then slowly dissipates.

Figure 3 shows the behavior of the smoothed finite difference technique over time. While the results are quite good up until shock formation, the magnitude of u near the shock then grows without bound, and eventually wild oscillations develop.

Figure 4 shows the evolution of the solution by the smoothed spectral method over time. While the results are accurate during shock formation, the results become quite inaccurate in the vicinity of the shock as the shock dissipates. For smaller values of the viscosity the spectral method is unstable. See also Figure 5.

The wavelet-Galerkin technique is stable. Figure 5 compares the unsmoothed wavelet-Galerkin solution, the smoothed spectral solution, and the exact solution for large time. Note that for large time, the viscosity term causes the numerically-induced oscillations in the wavelet-Galerkin solution to die out, resulting in a qualitatively accurate solution even without smoothing. In contrast, the smoothed spectral solution at this time value is highly inaccurate, with an error near the shock of the same size as the exact solution. This error is caused by the ambient oscillations in the smoothed solution in the neighborhood of the shock (second-order Gibbs oscillations).

Figure 6 shows that up until the time of shock formation, the wavelet-Galerkin method produces reasonable results even for the inviscid case. This technique appears to be stable for large time even in the inviscid case.

To compare the exact, smoothed and unsmoothed solutions at smaller times, we consider a solution with an initial condition $u_0 = -\sin((12\pi/128)x)$ and viscosity of 0.064 on a period of 128.

Figure 7 shows the exact and smoothed solutions at a time soon after the formation of the shocks.

Figure 8 shows the exact, smoothed and unsmoothed solutions. With reference to the units from the previous figure, this figure is from 20 to 45.

Figure 9 shows the exact and unsmoothed solutions after the oscillations shown in figure 8 have dissipated.

In conclusion, we note the following results.

1. The wavelet-Galerkin (WG) method appears to be stable for all viscosities, including zero.
2. The WG converges to close to the exact solution for large time even for quite small viscosities. This is a consequence of the stability and of the reduction of Gibbs' phenomena.
3. The three term smoothing of oscillating WG closely approximates the exact solution for all times for quite small viscosities.
4. The solution remains stable (bounded) even in the inviscid case, since the Gibbs' phenomenon does not propagate throughout the solution.

The wavelet-Galerkin solution appears to handle error in a way that the information content of the solution is not destroyed and, in a sense, can be recovered from the approximation. We will explore the possibilities and their implications more fully in subsequent publications. In this paper we compare similar implementations of the wavelet-Galerkin, finite-difference and spectral methods. We will also consider comparisons with optimized versions of these methods using geometrical constraints (variation diminishing algorithms) and methods that filter (smooth) the solution during the course of the calculation in a subsequent publication.

The authors would like to thank Roland Glowinski, Wayne Lawton, Howard Resnikoff, and John Weiss for their valuable suggestions and assistance with this work. This work was supported in part by the Defense Advanced Research Projects Agency, DARPA Order No. 7092 under AFOSR contract F4960-89-C-0125.

Note remise le 26 mars 1990, acceptée après révision le 5 septembre 1990.

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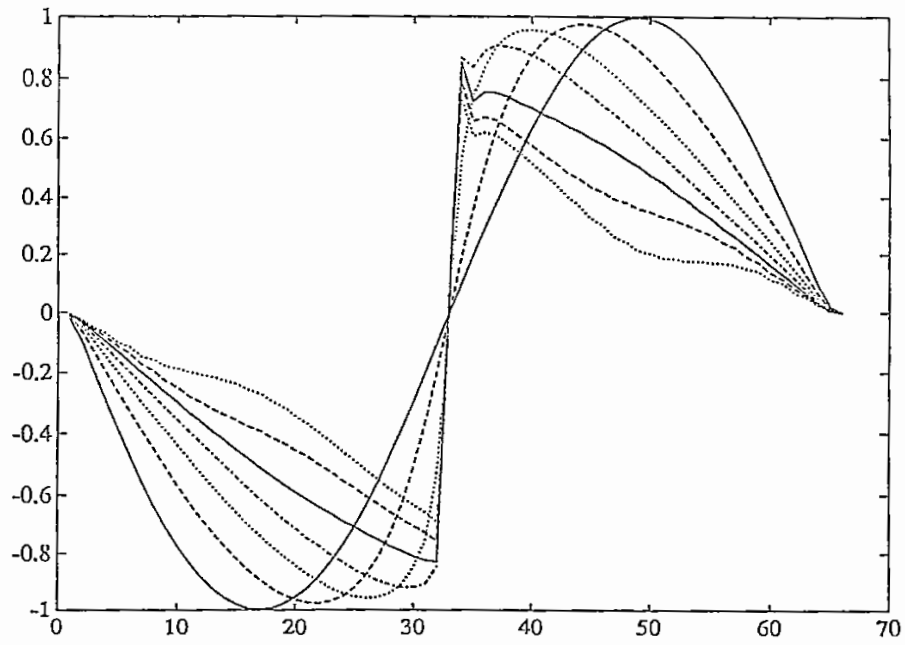


Fig. 6. — Smoothed WG with viscosity times 1 to 35 by 5.
 Fig. 6. — Solution OG filtrée avec viscosité aux temps 1 à 35.

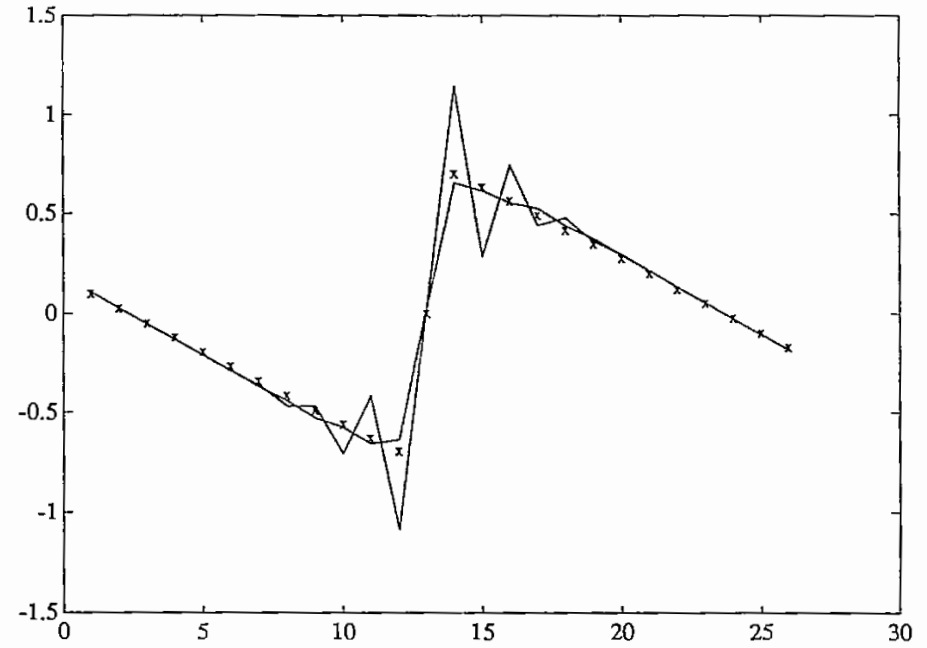


Fig. 8. — Exact: x, smoothed WG: —, unsmoothed WG: --, time=5.
 Fig. 8. — Solution exacte : x, OG filtrée : —, OG non filtrée : --, temps=5.

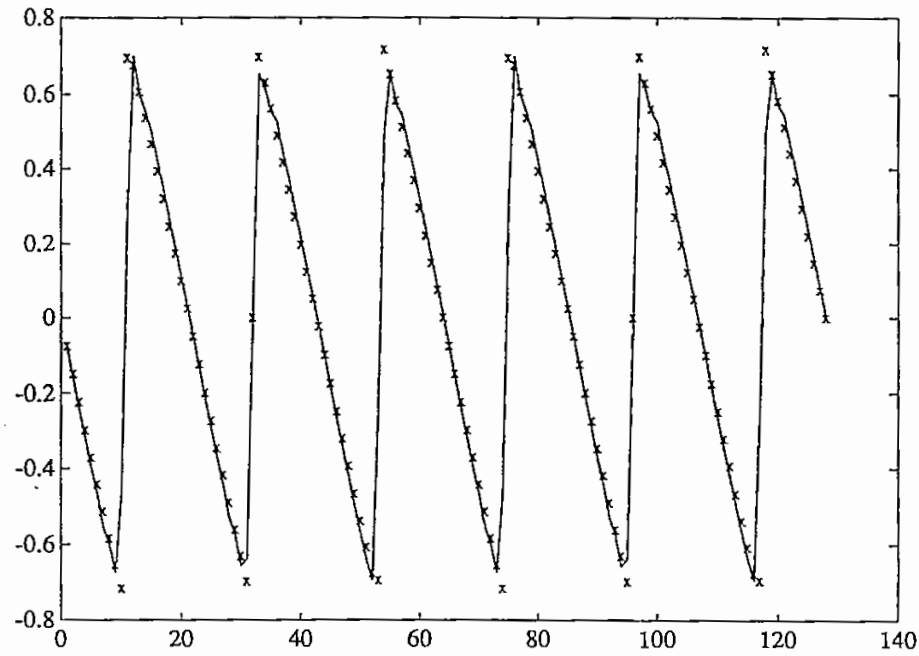


Fig. 7. — An exact: x and a smoothed WG at time 5.
 Fig. 7. — Solution exacte : x et OG filtrée au temps 5.

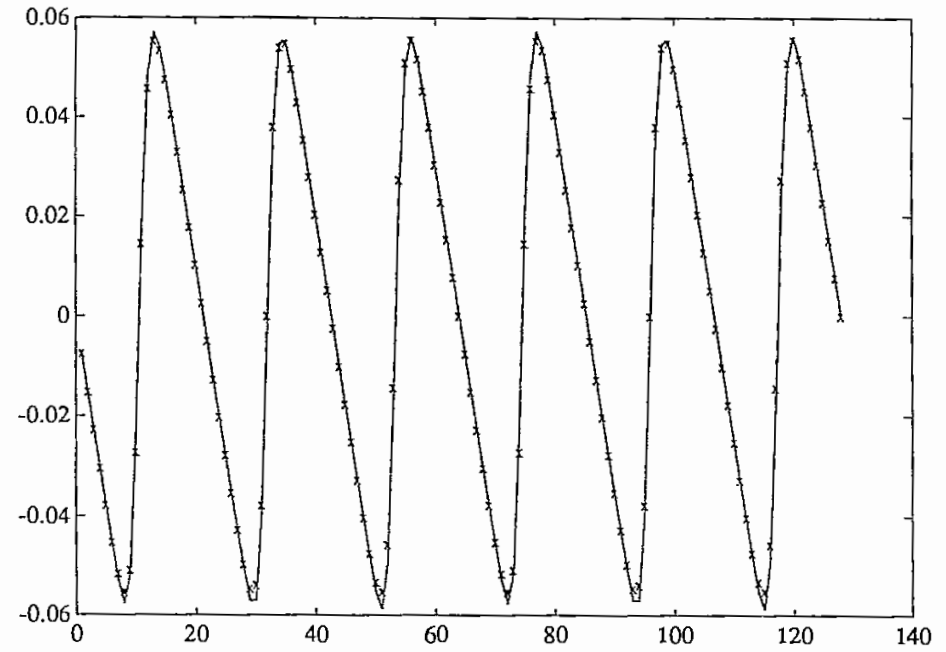


Fig. 9. — Exact: x and unsmoothed WG at time 64.
 Fig. 9. — Solution exacte : x et OG non filtrée au temps 64.