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WILF CLASSIFICATION OF TRIPLES OF 4-LETTER PATTERNS I

DAVID CALLAN, TOUFIK MANSOUR, AND MARK SHATTUCK

ABSTRACT. This is the first of two papers in which we determine all 242 Wilf classes of triples of 4-letter patterns by showing that there are 32 non-singleton Wilf classes. There are 317 symmetry classes of triples of 4-letter patterns and after computer calculation of initial terms, the problem reduces to showing that counting sequences that appear to be the same (agree in the first 16 terms) are in fact identical. This amounts to counting avoiders for 107 representative triples. The insertion encoding algorithm (INSENC) applies to many of them and some others have been previously counted. Thus there remain 36 triples. In this paper, we find the generating function for the first 18 of these triples and in a second paper, we treat the other 18. The generating function turns out to be algebraic in each case. Our methods are both combinatorial and analytic, including decompositions by left-right maxima and by initial letters. Sometimes this leads to an algebraic equation for the generating function, sometimes to a functional equation or a multi-index recurrence that succumbs to the kernel method. A bijection is used in one of the cases (Case 50).

Keywords: pattern avoidance, Wilf-equivalence, kernel method, insertion encoding algorithm

1. INTRODUCTION

In recent decades pattern avoidance has received a lot of attention. It has a prehistory in the work of MacMahon [10] and Knuth [6] but the paper that really sparked the current interest is by Simion and Schmidt [12]. They thoroughly analyzed 3-letter patterns, including a bijection between 123- and 132-avoiding permutations, thereby explaining the first (nontrivial) instance of what is, in modern terminology, a Wilf class. Since then the problem has been addressed on several other discrete structures, such as compositions, k -ary words, and set partitions; see, e.g., the texts [4, 11] and references contained therein.

Permutations avoiding a single 4-letter pattern have been well studied (see, e.g., [14, 15, 17]). There are 56 symmetry classes of pairs of 4-letter patterns, for all but 8 of which the avoiders have been enumerated [1]. Le [9] established that these 56 symmetry classes form 38 distinct Wilf classes. Vatter [16] showed that of these 38, 12 can be enumerated with regular insertion encodings (the INSENC algorithm). Some of these generating functions were computed by hand by Kremer and Shiu [7].

Much less is known about larger sets/longer patterns. Here, we consider the 317 symmetry classes of triples of 4-letter patterns and determine their Wilf classes. First, we used the software of Kuszmaul [8] to compute the initial terms $\{|S_n(T)|\}_{n=1}^{16}$ for a choice of T in each of the 317 symmetry classes. These results are available for reference in Table 2 in the Appendix of the arXiv full length version [3] of the paper. There are 242 distinct 16-term sequences among the 317, giving a lower bound of 242 on the number of Wilf classes. We will show that whenever two sequences in Table 2 [3] agree in the first 16 terms, they are in fact identical, and so 242 is also an upper

bound. To do so, we find the generating function for every triple whose 16-term counting sequence is repeated in Table 2 [3]. Thirty-eight of them can be found by INSENC; some others have already been counted and are referenced in Table 1 of the arXiv version [3].

There remain 36 triples to enumerate with 15 distinct counting sequences (cases). In this paper, we treat the first 9 cases (numbered 50, 55, 166, 171, 174, 177, 191, 196, 201, where the numbering is taken from the full table of 242 counting sequences [3]). These 9 cases cover 18 of the 36 triples. Table 1 is a compendium of the results. The second paper will treat the remaining 18 triples.

Summarizing, there are 242 Wilf classes of triples of 4-letter patterns, of which 210 are singleton (trivial) and 32 contain more than one symmetry class (nontrivial). Enumeration of the trivial Wilf classes will be treated in a forthcoming paper.

Theorem 1 (Main Theorem). *There are exactly 242 Wilf classes (32 nontrivial Wilf classes) of triples of 4-letter patterns in permutations.*

Table 1: Nontrivial Wilf classes of three 4-letter patterns up to Case 201, numbering taken from Table 2 [3].

| Start of Table | | | |
|----------------|--|--|--|
| No. | T | $\sum_{n>0} S_n(T) x^n$ | Reference |
| 6 | {1432,2134,3412}, {1234,1432,3412} | $\frac{1-6x+16x^2-22x^3+21x^4-8x^5+2x^6}{(1-x)^7}$ | INSENC |
| 50 | {2143,3412,2341}, {2143,2341,4231} {3412,1432,1243} | $\frac{1-6x+13x^2-11x^3+5x^4}{(1-x)^2(1-2x)(1-3x+x^2)}$ | Thm. 3, 4, 5 |
| 55 | {1342,3124,4213}, {1234,2143,4123} {1324,2143,2341} | $\frac{1-6x+12x^2-8x^3+3x^4-x^5}{(1-x)(1-3x+x^2)^2}$ | INSENC Thm. 8 |
| 56 | {1342,2143,4123}, {1432,3412,4123} | $\frac{1-5x+8x^2-5x^3+3x^4}{(1-x)^2(1-4x+3x^2-x^3)}$ | INSENC |
| 78 | {1324,1342,4312}, {1234,2413,3412} | $\frac{1-8x+26x^2-42x^3+36x^4-14x^5}{(1-x)^3(1-2x)^3}$ | INSENC |
| 94 | {1432,2134,4132}, {1234,1432,4132} | $\frac{1-8x+25x^2-37x^3+27x^4-9x^5}{(1-3x+x^2)(1-x)^2(1-2x)^2}$ | INSENC |
| 108 | {1432,3124,4123}, {1324,1432,4123} | $\frac{1-7x+17x^2-15x^3+4x^4-2x^5}{(1-2x)(1-3x+x^2)^2}$ | INSENC |
| 112 | {1243,1432,2134}, {1432,3142,4123} | $\frac{1-4x+4x^2-3x^3+x^4}{(1-x+x^2)(1-4x+2x^2)}$ | INSENC |
| 126 | {2431,4213,1324}, {3142,4123,1234} | $\frac{1-7x+18x^2-20x^3+11x^4-2x^5}{(1-x)^2(1-3x+x^2)^2}$ | INSENC |
| 127 | {1342,2413,4312}, {1243,2341,2413} | $\frac{1-8x+24x^2-33x^3+22x^4-7x^5}{(1-x)^3(1-3x)(1-3x+x^2)}$ | INSENC |
| 129 | {1432,2413,3124}, {1432,2143,3124} | $\frac{(1-x)^2(1-3x+x^2)}{1-6x+12x^2-11x^3+3x^4-x^5}$ | INSENC |
| 157 | {1324,1342,4213}, {1432,3124,4132} | $\frac{(1-x)(1-6x+11x^2-4x^3)}{(1-2x)(1-3x+x^2)^2}$ | INSENC |
| 166 | {3412,3142,1243}, {3412,3142,1324} | $\frac{1-9x+30x^2-44x^3+27x^4-7x^5}{(1-3x)(1-x)(1-3x+x^2)^2}$ | Thm. 10, 12 |
| 170 | {3142,4231,4321}, {1234,1243,2413} | $\frac{(1-3x+x^2)^2}{(1-x)(1-2x)(1-4x+2x^2)}$ | INSENC |
| 171 | {3124,1342,4123}, {1342,1324,4123} | $\frac{1-4x+5x^2-x^3+(1-4x+3x^2-x^3)\sqrt{1-4x}}{(1-x)(1-3x+x^2)(1-2x+\sqrt{1-4x})}$ | Thm. 13, 17 |
| 173 | {1342,1423,4213}, {1423,2341,2431} | $\frac{1-5x+8x^2-7x^3+2x^4}{1-6x+12x^2-13x^3+6x^4-x^5}$ | INSENC |
| 174 | {1432,3412,3421}, {1342,3412,4312} {2134,2341,2413}, {1342,3142,4312} {2143,2314,2341}, {2143,2314,2431} | $\frac{1-6x+10x^2-3x^3+x^4}{(1-3x+x^2)(1-4x+2x^2)}$ | INSENC Thm. 18, INSENC Thm. 20, 21 |
| 177 | {2143,2341,2413}, {2143,2341,3241} | $\frac{1-4x+3x^2-x^3}{1-5x+6x^2-3x^3}$ | Thm. 22, 23 |
| 191 | {1342,2134,2413}, {1324,1432,3124} {1423,2134,2413}, {3142,4132,4321} | $\frac{(1-x)(1-2x)(1-3x)}{1-7x+16x^2-14x^3+3x^4}$ | Thm. 24, INSENC INSENC |

| Continuation of Table 1 | | | |
|-------------------------|--|---|-----------------------|
| No. | T | $\sum_{n>0} S_n(T) x^n$ | Reference |
| 196 | $\{1342, 3142, 4213\}, \{1324, 1432, 2134\}$ $\{2143, 2431, 3241\}, \{1342, 3142, 4123\}$ | $\frac{1-5x+7x^2-4x^3}{1-6x+11x^2-9x^3+2x^4}$ | INSENC Thm. 25, 27 |
| 201 | $\{3142, 1324, 1243\}, \{1342, 1423, 2314\}$ | $\frac{1-3x+x^2}{1-x}C(x)^3$ | Thm. 28, 31 |
| End of Table | | | |

2. PRELIMINARIES AND NOTATION

We say a permutation is *standard* if its support set is an initial segment of the positive integers, and for a permutation π whose support is any set of positive integers, $\text{St}(\pi)$ denotes the standard permutation obtained by replacing the smallest entry of π by 1, the next smallest by 2, and so on. As usual, a standard permutation π *avoids* a standard permutation τ if there is no subsequence ρ of π for which $\text{St}(\rho) = \tau$. In this context, τ is called a *pattern*, and for a list T of patterns, $S_n(T)$ denotes the set of permutations of $[n] = \{1, 2, \dots, n\}$ that avoid all the patterns in T .

A permutation has an obvious representation as a matrix diagram,

| | | |
|---|---|---|
| • | | |
| | | • |
| | • | |

matrix diagram of the permutation 312

and it will often be convenient to use such diagrams where shaded areas always indicate regions that contain no entries (blank regions may generally contain entries but in a few cases, as noted and clear from the context, they don't).

The eight symmetries of a square, generated by rotation and reflection, partition patterns and sets of patterns into symmetry classes on each of which the counting sequence for avoiders is obviously constant. Thus if π avoids τ then, for example, π^{-1} avoids τ^{-1} since inversion corresponds to flipping the matrix diagram across a diagonal. It sometimes happens (and remarkably often) that different symmetry classes have the same counting sequence, and all symmetry classes with a given counting sequence form a *Wilf class*. Thus Wilf classes correspond to counting sequences.

Throughout, $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ denotes the generating function for the Catalan numbers $C_n := \frac{1}{n+1} \binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}$. As is well known [2], $C(x)$ is the generating function for $(|S_n(\pi)|)_{n \geq 0}$ where π is any one of the six 3-letter patterns. Occasionally, we need the generating function for avoiders of a 3-letter and a 4-letter pattern; see [2] for a comprehensive list.

A permutation π expressed as $\pi = i_1\pi^{(1)}i_2\pi^{(2)} \dots i_m\pi^{(m)}$ where $i_1 < i_2 < \dots < i_m$ and $i_j > \max(\pi^{(j)})$ for $1 \leq j \leq m$ is said to have m *left-right maxima* (at i_1, i_2, \dots, i_m). Given nonempty sets of numbers S and T , we will write $S < T$ to mean $\max(S) < \min(T)$ (with the inequality vacuously holding if S or T is empty). In this context, we will often denote singleton sets simply by the element in question. Also, for a number k , $S - k$ means the set $\{s - k : s \in S\}$.

Our approach is ultimately recursive. In each case, we examine the structure of an avoider, usually by splitting the class of avoiders under consideration into subclasses according to a judicious choice of parameters which may involve, for example, left-right maxima, initial letters, positions of given letters, and whether resulting subpermutations are empty or not. The choice is made so that each

member of a subclass can be decomposed into independent parts. The generating function for the subclass (a summand of the full generating function) is then the product of the generating functions for the parts, and we speak of the “contribution” of the various parts to the generating function for that subclass. From the structure, we are able to find an equation for the generating function $F_T(x) := \sum_{n \geq 0} |S_n(T)|x^n$, where T is the triple under consideration. This equation is often algebraic and, if linear or quadratic, as it is in all cases treated here, easy to solve explicitly once found. It also often comes in the form of a functional equation requiring the *kernel method* (see, e.g., [5] for an exposition). For one of the triples in Case 50, an explicit bijection is found which establishes the result. In every case, the generating function turns out to be algebraic.

Furthermore, in several cases, especially those where recurrences are made use of, we have in fact counted members of the avoidance class in question according to the distribution of one or more statistics, specific to the class, and have assumed particular values of the parameters to obtain the avoidance result. In some of these cases, to aid in solving the recurrence, certain auxiliary arrays related to the statistic are introduced. This leads to systems of linear functional equations to which we apply the kernel method, adapted for a system. See, for example, the proof below of the second symmetry class in Case 171. Also, in instances where the kernel method is used, it is usually possible (if desired) to solve the functional equation in its full generality yielding a polynomial generalization of the avoidance result. In other cases, one may extend the result by counting members of the class in question having a fixed number of left-right maxima. We refer the reader to the discussion following the proof of the first triple in Case 50 below.

We now proceed to the proofs for the 9 cases listed in the Introduction.

3. PROOFS

3.1. **Case 50.** The three representative triples T are:

{2143, 2341, 3412} (Theorem 3)

{2143, 2341, 4231} (Theorem 4)

{2143, 2341, 3421} (Theorem 5)

In order to deal with this case, we define the following two generating functions for each triple T : $H_T(x)$ is the generating function for T -avoiders with first letter $n - 1$ and $J_T(x)$ is the generating function for T -avoiders with second letter n .

Lemma 2. *For each triple T in Case 50, $H_T = J_T$.*

Proof. For each pattern in Case 50, its matrix diagram is invariant under the involution “flip in the diagonal line $y = -x$ ”. Consequently, the set of T -avoiders is invariant under this flip. But the flip interchanges the permutations whose first letter is $n - 1$ and the permutations whose second letter is n . \square

Theorem 3. *Let $T = \{2143, 2341, 3412\}$. Then*

$$F_T(x) = \frac{1 - 6x + 13x^2 - 11x^3 + 5x^4}{(1 - x)^2(1 - 2x)(1 - 3x + x^2)}.$$

Proof. Let $G_m(x)$ be the generating function for T -avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. For $G_m(x)$ with $m \geq 2$, we first need an equation for $J_T(x)$. Consider a permutation $\pi = in\pi' \in S_n(T)$ counted by J_T . Clearly, the contribution for the case $i = n-1$ is given by $\frac{x^2}{1-x}$. If $i = 1 \neq n-1$, the contribution is $x^2(F_T(x) - 1)$. Otherwise, $n \geq 4$ and $1 < i < n-1$ and π can be written as $in\beta'(n-1)\beta''$ with at least one of β', β'' nonempty. Consider three cases: (1) β' is empty, (2) β'' is empty, (3) neither of β', β'' is empty. In each of cases 1 and 2, the map “delete $n-1$ and standardize” is a bijection to the one-size-smaller permutations counted by J_T that do not start with a 1. Hence, in each of these cases, the contribution is $x(J_T(x) - x^2F_T(x))$. In case 3, $\pi \setminus \{n-1\}$ must have the form $n-2 \ n \ n-3 \ n-4 \ \dots \ 2 \ 1$ with $n-4$ positions available for $n-1$, namely, immediately before $1, 2, \dots, n-4$. Hence the contribution in this case is $x^5/(1-x)^2$.

Adding all the contributions, and solving for $J_T(x)$, yields

$$(1) \quad J_T(x) = x^2F_T(x) + \frac{x^3(1-x+x^2)}{(1-x)^2(1-2x)}.$$

Now let $m \geq 2$ and let us write an equation for $G_m(x)$. Let $\pi = i_1\pi^{(1)}i_2\pi^{(2)} \dots i_m\pi^{(m)} \in S_n(T)$ with exactly m left-right maxima. By considering the cases where $\pi^{(1)}$ is not empty or where $\pi^{(1)}$ is empty and $\pi^{(2)}$ either has a letter smaller than i_1 or it doesn't (see the next figure), we obtain

$x^{m-2} \left(H_T(x) - \frac{x^2}{1-x} \right) \quad x^{m-2} (J_T(x) - x^2F_T(x)) \quad xG_{m-1}(x)$

$$(2) \quad G_m(x) = x^{m-2} \left(H_T(x) - \frac{x^2}{1-x} \right) + x^{m-2} (J_T(x) - x^2F_T(x)) + xG_{m-1}(x)$$

for $m \geq 2$. By summing (2) over $m \geq 2$ and using the expressions for $G_0(x)$ and $G_1(x)$, we obtain

$$(3) \quad F_T(x) = 1 + \frac{1}{(1-x)^2} (H_T(x) - x^2/(1-x) + J_T(x) + x(1-2x)F_T(x)).$$

Eliminating $J_T = H_T$ from (1) and (3) gives the desired expression for F_T . \square

It is possible to generalize the preceding result as follows. Define the generating function $G(x, q) = G_T(x, q) = \sum_{m \geq 0} G_m(x)q^m$, where $T = \{2143, 2341, 3412\}$. Multiplying both sides of (2) by q^m , and summing over $m \geq 2$, implies

$$G(x, q) - 1 - xqF_T(x) = \frac{q^2}{1-xq} \left(2J_T(x) - x^2F_T(x) - \frac{x^2}{1-x} \right) + xq(G(x, q) - 1).$$

Solving for $G(x, q)$ then yields

$$\begin{aligned} G(x, q) &= 1 + \frac{q}{(1-xq)^2} \left(xF_T(x) + \frac{2x^3q(1-x+x^2)}{(1-x)^2(1-2x)} - \frac{x^2q}{1-x} \right) \\ &= 1 + \frac{xq(1-6x+13x^2-11x^3+5x^4-xq(1-8x+20x^2-19x^3+10x^4-2x^5))}{(1-x)^2(1-xq)^2(1-2x)(1-3x+x^2)}. \end{aligned}$$

Taking $q = 1$ in the last formula, and simplifying, recovers Theorem 3:

$$G_T(x, 1) = F_T(x) = \frac{1-6x+13x^2-11x^3+5x^4}{(1-x)^2(1-2x)(1-3x+x^2)}.$$

Extracting the coefficient of q^m from $G(x, q)$ implies that the generating function $G_m(x)$ for the number of T -avoiders having exactly m left-right maxima is given by

$$G_m(x) = \frac{x^m(1-8x+20x^2-19x^3+10x^4-2x^5) + mx^{m+1}(2-7x+8x^2-5x^3+2x^4)}{(1-x)^2(1-2x)(1-3x+x^2)}, \quad m \geq 1.$$

Note that taking $m = 1$ in the last equation gives back the obvious formula $G_1(x) = xF_T(x)$.

Remark: A comparable formula for $G_T(x, q)$ may be obtained in a similar fashion for other T in subsequent cases where left-right maxima are made use of in counting the avoiders in question.

We now turn our attention to the case when $T = \{2143, 2341, 4231\}$.

Theorem 4. *Let $T = \{2143, 2341, 4231\}$. Then*

$$F_T(x) = \frac{1-6x+13x^2-11x^3+5x^4}{(1-x)^2(1-2x)(1-3x+x^2)}.$$

Proof. The proof follows along the lines of Theorem 3. Let $G_m(x)$ be the generating function for T -avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xK$, where

$$K = \sum_{n \geq 0} |S_n(231, 2143)|x^n = \frac{1-2x}{1-3x+x^2}$$

(see [13, Seq. A001519]).

To write an equation for $H_T(x)$, we consider $\pi = (n-1)\pi'n\pi'' \in S_n(T)$. If $\pi'\pi''$ is empty, then the contribution is x^2 . Otherwise, we consider the following two cases:

- the letter $n-2$ belongs to π' , which implies that $\pi = (n-1)\beta'(n-2)\beta''n\pi''$. If $\beta''\pi''$ is empty, we get a contribution of x^3K by deleting the letters $n-1, n-2, n$ from π . If $\beta''\pi''$ is nonempty, then $\beta' < \beta''\pi''$ and we get a contribution of $\frac{x}{1-x}(H_T(x) - x^2)$.
- the letter $n-2$ belongs to π'' , which implies $\pi = (n-1)\pi'n\beta'(n-2)\beta''$. If β'' is not empty, then $\pi'n\beta' = 12 \cdots jn(j+1)(j+2) \cdots i'$ where β'' is a permutation of $i'+1, \dots, n-3$, so we have a contribution of $\frac{x^3}{(1-x)^2}(K-1)$. If β'' is empty, then π' is an increasing subsequence, say $\pi' = j_1j_2 \cdots j_d$ with $j_1 < j_2 < \cdots < j_d$. If $d = 0$, then the contribution is x^3K , otherwise π can be written as $\pi = (n-1)j_1j_2 \cdots j_dn\beta'(n-2)$ with $d \geq 1$. Since π avoids 2341 and 4231, we have $\pi = (n-1)12 \cdots (d-1)j_dn\beta^{(1)}\beta^{(2)}(n-2)$, where $j_d\beta^{(1)}\beta^{(2)}$ is a permutation of $d, d+1, \dots, n-3$ and $\beta^{(1)} < j_d < \beta^{(2)}$. By considering whether $\beta^{(1)}$ is empty or not, we get $x^{d+3}K + \frac{x^{d+3}}{1-x}(K-1)$. By summing over $d \geq 1$, it follows that the contribution in this case is given by $\frac{x^4}{1-x}K + \frac{x^4}{(1-x)^2}(K-1)$.

Thus, $H_T(x) = x^2 + x^3K + \frac{x}{1-x}(H_T(x) - x^2) + \frac{x^3}{(1-x)^2}(K - 1) + x^3K + \frac{x^4}{1-x}K + \frac{x^4}{(1-x)^2}(K - 1)$, which implies

$$H_T(x) = \frac{(1 - 3x + 3x^2 + x^3)x^2}{(1 - 2x)(1 - 3x + x^2)}.$$

Now, we are ready to write an equation for $G_m(x)$, where $m \geq 2$. Using the same decomposition as in the proof of Theorem 3, we obtain

$$\begin{aligned} G_m(x) &= x^{m-2}(H_T(x) - x^2K) + x^{m-2}(J_T(x) - x^2K) + xG_{m-1}(x) \\ &= 2x^{m-2}(H_T(x) - x^2K) + xG_{m-1}(x). \end{aligned}$$

Summing over $m \geq 2$, we find $\sum_{m \geq 2} G_m(x) = (2H_T(x) - x^2(1+x)K)/(1-x)^2$. Using the expressions for $G_0(x)$ and $G_1(x)$, it is seen that $F_T(x) = \sum_{m \geq 0} G_m(x)$ simplifies to the desired expression. \square

Theorem 5. *There is a bijection between the set of {2143, 2341, 3412}-avoiders and the set of {2143, 2341, 3421}-avoiders.*

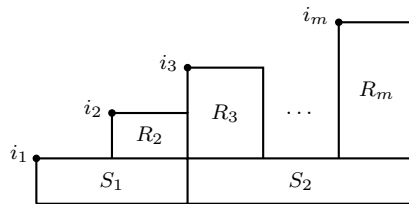
Proof. Let A_n and B_n denote the subsets of S_n whose members avoid the patterns in the sets {2143, 2341, 3412} and {2143, 2341, 3421}, respectively. We will define a bijection f from A_n to B_n as follows. If $n \leq 3$, we may clearly take f to be the identity, so assume $n \geq 4$. Given $\pi = \pi_1\pi_2 \cdots \pi_n \in A_n$, let $a = \pi_1$ denote the first letter of π . If $a = 1$ so that π has the form $1\pi'$, define f recursively by $f(\pi) = 1 \oplus f(\pi' - 1)$. (Here, \oplus is the direct sum, thus $132 \oplus 123 = 132456$.) Henceforth, assume $a > 1$. We consider cases according to descending values of the position j of n in π .

If $j = n$ so that π has the form $\pi'n$, define f , again recursively, by $f(\pi) = f(\pi')n$.

If $3 \leq j \leq n - 1$, then π has the form $a\beta'n\beta''$ with β' and β'' nonempty. We claim (i) $a = n - 1$, (ii) β' avoids 231, and (iii) β'' is decreasing. To establish the claims, we need a preliminary result.

Lemma 6. *If $\pi \in A_n$ has ≥ 3 left-right maxima and 1 is not its first letter, then n is its last letter.*

Proof. Suppose $i_1 = a, i_2, \dots, i_m = n$ are the left-right maxima of π with $m \geq 3$, so that all other entries of π lie in the rectangles R_2, R_3, \dots, R_m and S_1, S_2 shown in the figure.



An entry $x \in S_2$ implies $i_1i_2i_3x$ is a forbidden 2341. Hence $S_2 = \emptyset$ and so $1 \in S_1$. Now $x \in R_m$ implies i_1i_mx is a forbidden 2143. Hence, $R_m = \emptyset$ and $i_m = n$ is the last letter in π . \square

Corollary 7. *If $\pi \in A_n$ and 1 is not its first letter and n is not its last letter, then π has at most two left-right maxima.* \square

Now, if claim (i) fails, $a \leq n - 2$. Since n is not the last letter of π , Corollary 7 implies that a and n are the only two left-right maxima of π . Consequently, $n - 1$ occurs after n and, since $j \geq 3$, there is a letter $x < a$ before n . But then $axn(n - 1)$ is a forbidden 2143.

If (ii) fails and cdb is a 231 in β' , take any $x \in \beta''$. If $x > c$, then $cbnx$ is a forbidden 2143, while if $x < c$, then $cdnx$ is a forbidden 2341.

If (iii) fails, there are letters $x < y$ in β'' . But then $(n - 1)nxy$ is a forbidden 3412.

It follows from (ii) that the initial segment $a\beta'n$ of π avoids 3421, since 3421 contains the pattern 231. Define $f(\pi) = a\beta'nr(\beta'')$, where $r(\beta)$ denotes the reverse of a permutation β . It is clear that $f(\pi)$ also avoids 3421 and so is in B_n .

If $j = 2$ so that π begins $an \cdots$, we introduce the condition

$$(4) \quad a + 1 \neq \pi_n \text{ and the successor and predecessor of } a + 1 \text{ in } \pi \text{ are both } < a,$$

and consider two cases.

- Condition (4) holds. Here, $2 \leq a \leq n - 2$ and π must have the form $a\alpha'\beta'(a + 1)\beta''\alpha''$ with $\beta' = (a - 1)(a - 2) \cdots (b + 1)$ and $\beta'' = b(b - 1) \cdots 1$ for some $b \in [a - 2]$, where α' starts with n and is decreasing, and α'' is increasing, possibly empty. Conversely, one may verify that a permutation with a decomposition of this form belongs to A_n . Define $f(\pi) = a\alpha'r(\beta'')(a + 1)r(\beta')\alpha''$.
- Condition (4) fails. Here, π must have the form $a\alpha'\beta\alpha''$ with $\beta = (a - 1)(a - 2) \cdots 1$ (hence, nonempty), where α' starts with n and is decreasing, and α'' is increasing, possibly empty. Define $f(\pi) = a\alpha'r(\beta)\alpha''$.

Finally, if $j = 1$ so that $\pi = n\pi'$, define f recursively by $f(\pi) = nf(\pi')$.

The mapping f preserves left-right maxima and their positions. The reader may check that f is reversible and is a bijection from A_n to B_n . \square

3.2. Case 55.

Theorem 8. *Let $T = \{1324, 2143, 2341\}$. Then*

$$F_T(x) = \frac{1 - 6x + 12x^2 - 8x^3 + 3x^4 - x^5}{(1 - x)(1 - 3x + x^2)^2}.$$

Proof. Let $a_T(n; i_1, i_2, \dots, i_s)$ be the number of permutations $i_1 i_2 \cdots i_s \pi \in S_n(T)$. The initial conditions $a_T(n; n) = a_T(n; n - 1) = a_T(n - 1)$ and $a_T(n; 1) = |S_{n-1}(213, 2341)|$ easily follow from the definitions. It is well known that $|S_{n-1}(213, 2341)| = F_{2n-3}$ (F_n is the n th Fibonacci number defined by $F_n = F_{n-1} + F_{n-2}$, with $F_0 = 0$ and $F_1 = 1$).

Now let $2 \leq i \leq n - 2$ and let us focus on the second letter of π . Clearly, $a_T(n; i, n) = a_T(n - 1; i)$, and also $a_T(n; i, j) = 0$ for all $j = i + 2, i + 3, \dots, n - 1$. Note that any permutation $\pi = i(i + 1)\pi' \in S_n(T)$ can be written as $\pi = i(i + 1)\pi''(i + 2)(i + 3) \cdots n$, so $a_T(n; i, i + 1) = |S_{i-1}(132, 2341)|$ and it is also known that $|S_{i-1}(132, 2341)| = F_{2i-3}$. A permutation $\pi = ij\pi' \in S_n(T)$ with $n - 2 \geq i > j \geq 1$ satisfies $\pi = ij\pi'(j + 2) \cdots (i - 1)(i + 1)(i + 2) \cdots n$, where π' is a permutation of $1, 2, \dots, j - 1, j + 1$ that avoids 132 and 2341, and so $a_T(n; i, j) = |S_j(132, 2341)| = F_{2j-1}$. Hence

$$a_T(n; i) = F_1 + F_3 + F_5 + \cdots + F_{2i-3} + F_{2i-3} + a_T(n - 1; i),$$

which, by the fact that $F_1 + F_3 + F_5 + \dots + F_{2i-3} = F_{2i-2}$, implies for $2 \leq i \leq n-2$ that

$$a_T(n; i) - a_T(n-1; i) = F_{2i-1}.$$

By summing both sides of the last equation over $i = 2, 3, \dots, n-2$ and using the initial conditions, we obtain for $n \geq 3$,

$$\begin{aligned} a_T(n) - a_T(n-1) &= 2a_T(n-1) - a_T(n-2) + F_{2n-3} - F_{2n-5} + \sum_{i=2}^{n-2} F_{2i-1} \\ &= 2a_T(n-1) - a_T(n-2) + F_{2n-3} - F_{2n-5} + F_{2n-4} - 1 \\ &= 2a_T(n-1) - a_T(n-2) + 2F_{2n-4} - 1. \end{aligned}$$

It is routine to solve this difference equation for the generating function $F_T(x)$ using

$$\sum_{n \geq 0} F_{2n-1} x^n = \frac{1-2x}{1-3x+x^2}.$$

□

3.3. Case 166. The two representative triples T are:

$\{1243, 3142, 3412\}$ (Theorem 10)

$\{1324, 3142, 3412\}$ (Theorem 12)

3.3.1. $\mathbf{T} = \{1243, 3142, 3412\}$. Let $b_n(i) = |S_T(n; i, n)|$ and $b_n(i, j) = |S_T(n; i, n, j)|$ so that $b_n(i)$ and $b_n(i, j)$ count T -avoiders for which the second letter is n by first letter, i , and third letter, j . We first obtain a recurrence for $b_n(i, j)$.

Note that when the second letter is n , the conditions $i = 1$ and $j \in [2, n-2]$ force the last entry to be $n-1$ or 2 for else a forbidden pattern will be present. Deleting this entry gives contributions of $b_{n-1}(1, j)$, $b_{n-1}(1, j-1)$ to $b_n(1, j)$ according as the last entry is $n-1$ or 2 . Clearly, $b_n(1, n-1) = b_{n-1}(1)$. Hence, for $i = 1$, we have the following table of recursive values for $b_n(1, j)$:

| | | |
|-------------|-----------------------------------|--------------|
| j | $\in [2, n-2]$ | $n-1$ |
| $b_n(1, j)$ | $b_{n-1}(1, j) + b_{n-1}(1, j-1)$ | $b_{n-1}(1)$ |

Similarly, the conditions $2 \leq i \leq n-2$ and $j \in [i+1, n-2]$ force the last entry to be $n-1$ or 1 and deleting it gives contributions of $b_{n-1}(i, j)$, $b_{n-1}(i-1, j-1)$ according as the last entry is $n-1$ or 1 . Hence, for $2 \leq i \leq n-2$, we have the table (the straightforward verification of entries other than $j \in [i+1, n-2]$ is left to the reader):

| | | | | |
|-------------|----------------|----------------|-------------------------------------|--------------|
| j | $\in [1, i-2]$ | $i-1$ | $\in [i+1, n-2]$ | $n-1$ |
| $b_n(i, j)$ | 0 | $b_{n-1}(i-1)$ | $b_{n-1}(i, j) + b_{n-1}(i-1, j-1)$ | $b_{n-1}(i)$ |

Lastly, for $i = n-1$, $b_n(n-1) = 1$ because $\pi = (n-1)n\pi' \in S_n(T)$ implies $\pi = (n-1)n(n-2)(n-3)\dots 1$.

The table for $i = 1$ yields $b_n(1) = \sum_{j=2}^{n-1} b_n(1, j) = \sum_{j=2}^{n-2} (b_{n-1}(1, j) + b_{n-1}(1, j-1)) + b_n(1, n-1) = 3b_{n-1}(1) - b_{n-2}(1)$ for $n \geq 4$. Together with the initial conditions $b_2 = b_3 = 1$, this recurrence implies that $b_n(1) = F_{2n-5}$ for $n \geq 2$, where F_n is the n th Fibonacci number (defined by $F_n = F_{n-1} + F_{n-2}$ for all n , with $F_0 = 0$ and $F_1 = 1$).

Lemma 9. Let $b_n = \sum_{i=1}^{n-1} |S_T(n; i, n)|$. Then the generating function for the sequence b_n is given by

$$B(x) = \sum_{n \geq 2} b_n x^n = \frac{x^2(1-5x+7x^2-x^3)}{(1-3x+x^2)(1-3x)(1-x)}.$$

Proof. Clearly, $b_n = \sum_{i=1}^{n-1} b_n(i)$. Using the preceding results, we have

$$\begin{aligned} b_n &= 2b_{n-1} + \sum_{i=2}^{n-3} \sum_{j=i+1}^{n-2} b_n(i, j) + \sum_{j=2}^{n-2} b_n(1, j) \\ &= 2b_{n-1} + \sum_{i=1}^{n-4} \sum_{j=i+1}^{n-3} b_{n-1}(i, j) + \sum_{i=2}^{n-3} \sum_{j=i+1}^{n-2} b_{n-1}(i, j) + \sum_{j=2}^{n-2} b_n(1, j) \\ &= 2b_{n-1} + b_{n-1} - 2b_{n-2} + b_{n-1} - b_{n-2} + F_{2n-7} - F_{2n-7} \\ &= 4b_{n-1} - 3b_{n-2} + F_{2n-8}, \end{aligned}$$

with $b_1 = 0$, $b_2 = 1$ and $b_3 = 2$. Hence, by using the fact that $\sum_{n \geq 4} F_{2n-8} x^n = \frac{x^5}{1-3x+x^2}$, we obtain $B(x) = \frac{x^2(1-5x+7x^2-x^3)}{(1-3x+x^2)(1-3x)(1-x)}$. \square

Theorem 10. We have

$$F_T(x) = \frac{1-9x+30x^2-44x^3+27x^4-7x^5}{(1-3x)(1-x)(1-3x+x^2)^2}.$$

Proof. Let $G_m(x)$ be the generating function for T -avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. Now let us write an equation for $G_m(x)$ with $m = 2$. Let $\pi = i\pi'n\pi'' \in S_n(T)$ with two left-right maxima. If $\pi = i(i-1)\cdots i'n\pi''$, then the contribution is $B(x)/(1-x)$ (see Lemma 9). Otherwise, $\pi = i\pi'n\pi''$ where π' is a permutation on $\{i', i'+1, \dots, i-1\}$ that avoids T and has at least one ascent. Since π avoids 1243, $\pi'' = (i'-1)\cdots 21$. Thus, the contribution is given by $\frac{x^2}{1-x}(F_T(x) - 1/(1-x))$. Hence,

$$G_2(x) = \frac{1}{1-x}B(x) + \frac{x^2}{1-x} \left(F_T(x) - \frac{1}{1-x} \right).$$

Now let us write an equation for $G_m(x)$ with $m \geq 3$. Let $\pi = i_1\pi^{(1)} \cdots i_m\pi^{(m)} \in S_n(T)$ with exactly m left-right maxima, i_1, i_2, \dots, i_m . Since π avoids 1243, we have that $\pi^{(2)}, \dots, \pi^{(m)}$ are all $< i_2$. If $\pi^{(1)}$ has an ascent, then $\pi^{(1)} > \pi^{(2)} > \dots > \pi^{(m)}$ and $\pi^{(1)}$ avoids T and $\pi^{(2)} \cdots \pi^{(m)} = (i'-1)\cdots 1$ with i' the minimal letter of $\pi^{(1)}$. Thus, the contribution is given by $\frac{x^m}{(1-x)^{m-1}}(F_T(x) - 1/(1-x))$.

From now, we can assume that $\pi^{(1)} = (i-1)\cdots(i'+1)i'$. Let $s \in \{2, 3, \dots, m\}$ be the minimal number such that $\pi^{(s)}$ contains a letter from the set $[i'-1]$. We have the following cases:

- $s = 2$: Here $\pi^{(2)}\pi^{(3)} \cdots \pi^{(m)} = (i'-1)\cdots 21$, where $\pi^{(2)}$ is the subsequence of all letter of $\pi^{(2)}$ that are smaller than i' . Hence, by the definition of $B(x)$ (see Lemma 9), we get a contribution of $\frac{x^{m-2}}{(1-x)^{m-1}}(B(x) - x^2K(x))$, where $K(x) = \sum_{n \geq 0} |S_n(132, 3412)|x^n = \frac{1-2x}{1-3x+x^2}$ (see [13, Seq. A001519]).

- $s = 3, 4, \dots, m-1$: Here $\pi^{(2)} > \pi^{(3)} > \dots > \pi^{(s-1)} > i_1 > \pi^{(s)} > \pi^{(s+1)} > \dots > \pi^{(m)}$ and $\pi^{(s)} \dots \pi^{(m)} = (i'_1 - 1) \dots 21$ and $\pi^{(3)} \dots \pi^{(s-1)} = (i'_1 - 1) \dots (i_1 + 2)(i_1 + 1)$, where i'_1 is the minimal letter of $\pi^{(2)}$. Moreover, $\pi^{(2)}$ avoids 132 and 3412. Thus, the contribution is given by $\frac{x^{m+1}}{(1-x)^m} K(x)$.
- $s = m$. Here $i'_1 = 1$ and $\pi^{(2)} > \pi^{(3)} > \dots > \pi^{(m)} > i_1$ and $\pi^{(3)} \dots \pi^{(s-1)} = (i'_1 - 1) \dots (i_1 + 2)(i_1 + 1)$ and $\pi^{(2)}$ avoids 132 and 3412, where i'_1 is the minimal letter of $\pi^{(2)}$. Thus, the contribution is given by $\frac{x^m}{(1-x)^m} K(x)$.

By the preceding cases, we obtain for $m \geq 3$,

$$G_m(x) = \frac{x^m}{(1-x)^{m-1}} \left(F_T(x) - \frac{1}{1-x} \right) + \frac{x^{m-2}}{(1-x)^{m-1}} (B(x) - x^2 K(x)) \\ + (m-3) \frac{x^{m+1}}{(1-x)^m} K(x) + \frac{x^m}{(1-x)^m} K(x).$$

Since $F_T(x) = \sum_{m \geq 0} G_m(x)$, we have

$$F_T(x) = 1 + xF_T(x) + \frac{1}{1-x} B(x) + \sum_{m \geq 2} \frac{x^m}{(1-x)^{m-1}} \left(F_T(x) - \frac{1}{1-x} \right) \\ + \sum_{m \geq 3} \frac{x^{m-2}}{(1-x)^{m-1}} (B(x) - x^2 K(x)) + \sum_{m \geq 4} (m-3) \frac{x^{m+1}}{(1-x)^m} K(x) \\ + \sum_{m \geq 3} \frac{x^m}{(1-x)^m} K(x).$$

After several algebraic operations, and solving for $F_T(x)$, we complete the proof. \square

3.3.2. $\mathbf{T} = \{1324, 3142, 3412\}$. Let $b_n(i, j) = |S_T(n; i, n, j)|$. Then, in analogy with the previous subsection,

$$b_n(i, j) = F_{2n-2j-3}, \quad 2 \leq i+1 < j \leq n-2,$$

with $b_n(i+1, i) = b_n(i, i+1) = b_n(i, n-1) = b_{n-1}(i)$ and $b_n(n-2, n-1) = 1$.

Lemma 11. Let $b_n(i) = |S_T(n; i, n)|$, $b_n = \sum_{i=1}^{n-1} b_n(i)$, and $B(x) = \sum_{n \geq 2} b_n x^n$. Then

$$B(x) = \frac{x^2(1-5x+7x^2-x^3)}{(1-3x+x^2)(1-3x)(1-x)}.$$

Proof. From the preceding results, we have

$$b_n = 3b_{n-1} - 1 + \sum_{j=1}^{n-4} F_{2j-1}(n-3-j),$$

with $b_2 = 1$. Thus,

$$b_n - b_{n-1} = 3b_{n-1} - 3b_{n-2} + F_1 + F_3 + \dots + F_{2n-9},$$

which, by the fact that $F_1 + F_3 + \dots + F_{2n-11} = F_{2n-8}$, implies

$$b_n = 4b_{n-1} - 3b_{n-2} + F_{2n-8},$$

with $b_2 = 1$ and $b_3 = 2$. This is the same recurrence as in Lemma 9. \square

Theorem 12. *We have*

$$F_T(x) = \frac{1 - 9x + 30x^2 - 44x^3 + 27x^4 - 7x^5}{(1 - 3x)(1 - x)(1 - 3x + x^2)^2}.$$

Proof. Let $G_m(x)$ be the generating function for T -avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. By using similar arguments as in the proof of Theorem 10, we obtain that (see Lemma 11) $G_2(x) = K(x)B(x)$.

Now let us write an equation for $G_m(x)$ with $m \geq 3$. Let $\pi = i_1\pi^{(1)} \dots i_m\pi^{(m)} \in S_n(T)$ with exactly m left-right maxima. Since π avoids 1324, we have that $i_1 > \pi^{(1)} > \dots > \pi^{(m-1)}$. Let $s \in \{2, 3, \dots, m-1\}$ (maybe s does not exist) be the minimal number such that $\pi^{(s)} \neq \emptyset$. We have the following cases:

- $s = 2, 3, \dots, m-1$: Here $\pi^{(2)} = \pi^{(3)} = \dots = \pi^{(s-1)} = \emptyset$ and there is no letter in $\pi^{(m)}$ between i_{s-1} and i_s . Thus, the contribution is given by $\frac{x^s}{1-x}G_{m+1-s}(x)$.
- s does not exist. Here $\pi^{(2)} = \pi^{(3)} = \dots = \pi^{(m-1)} = \emptyset$. So there two cases, either $\pi^{(m)}$ contains a letter from the set $\{i_1 + 1, \dots, i_{m-1} - 1\}$ or not. If yes, then $\pi^{(m)}$ can be written as $\pi^{(m)} = \pi'(i_{m-1} - 1) \dots (i_{m-2} + 2)(i_{m-2} + 1) \dots (i_1 - 1) \dots 21$ with π' and $\pi^{(1)}$ avoiding 132 and 3412. Thus, the contribution is given by $\frac{x^m}{1-x}(1/(1-x)^{m-2} - 1)K(x)^2$. Otherwise, $\pi^{(m)}$ does not contain any letter from the set $\{i_1 + 1, \dots, i_{m-1} - 1\}$, which yields a contribution of $x^{m-2}G_2(x)$.

Combining the previous cases, we obtain

$$G_m(x) = x^{m-2}G_2(x) + \frac{x^m}{1-x}(1/(1-x)^{m-2} - 1)K(x)^2 + \sum_{s=2}^{m-1} \frac{x^s}{1-x}G_{m+1-s}(x).$$

Since $F_T(x) = \sum_{m \geq 0} G_m(x)$, we have

$$\begin{aligned} F_T(x) &= 1 + xF_T(x) + \frac{1}{1-x}K(x)B(x) + \sum_{m \geq 3} \frac{x^m}{1-x}(1/(1-x)^{m-2} - 1)K(x)^2 \\ &\quad + \frac{x^2}{(1-x)^2}(F_T(x) - 1 - xF_T(x)). \end{aligned}$$

Solving for $F_T(x)$ and using Lemma 11, we complete the proof. \square

3.4. Case 171. The two representative triples T are:

$\{1423, 2314, 2341\}$ (Theorem 13)

$\{1324, 1342, 4123\}$ (Theorem 17)

3.4.1. $\mathbf{T} = \{1423, 2314, 2341\}$.

Theorem 13. *Let $T = \{1423, 2314, 2341\}$. Then*

$$F_T(x) = \frac{1 - 4x + 5x^2 - x^3 + (1 - 4x + 3x^2 - x^3)\sqrt{1 - 4x}}{(1 - x)(1 - 3x + x^2)(1 - 2x + \sqrt{1 - 4x})}.$$

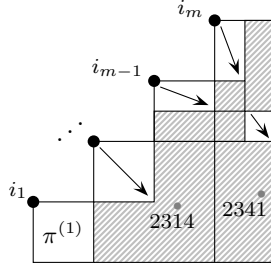
Proof. Let $G_m(x)$ be the generating function for T -avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. Next, we consider $m \geq 3$. If $\pi = i_1\pi^{(1)} \cdots i_m\pi^{(m)}$ avoids T then

$$\pi^{(1)} < \pi^{(2)} < \cdots < \pi^{(m-2)} < \pi^{(m-1)}i_m\pi^{(m)}$$

and $\pi^{(1)}$ avoids 231 and 1423 (due to the presence of i_m), $\pi^{(j)} = (i_{j+1} - 1) \cdots (i_j + 2)(i_j + 1)$ for $j = 2, 3, \dots, m-2$, and

$$i_{m-1}\pi^{(m-1)}i_m\pi^{(m)} = i_{m-1}(i_{m-1} - 1) \cdots \ell i_m(i_m - 1) \cdots (i_{m-1} + 1) \ell(\ell - 1) \cdots (i_{m-2} + 1).$$

These results are explained in the next figure, where entries are decreasing as indicated by arrows to avoid 1423, and other shaded regions are empty to avoid the indicated pattern.



There are m left-right maxima and m regions containing arrows (to be filled with an arbitrary number of “balls”), and the generating function $K := \sum_{n \geq 0} |S_n(231, 1423)|x^n$ for $\{231, 1423\}$ -avoiders is $\frac{1-2x}{1-3x+x^2}$ [13, Seq. A001519]. Hence, for $m \geq 3$,

$$G_m(x) = \frac{x^m}{(1-x)^m} K.$$

Now we find an explicit formula for $G_2(x)$. In order to do that, we define the following notation. Let $g_k(x)$ denote the generating function for T -avoiders in S_n with two left-right maxima and leftmost letter $n-1-k$, $0 \leq k \leq n-2$. Let $g'_k(x)$ to be the generating function for T -avoiders with two left-right maxima $(n-1-k)n$ in the two leftmost positions.

First, we find an equation for $g_k(x)$. Let $\pi = (n-1-k)\pi'\pi'' \in S_n(T)$ with two left-right maxima and leftmost letter $n-1-k$. If π' is empty, then the contribution is $g'_k(x)$. Otherwise, since π' avoids 2314, π' has the form $\beta'd\beta''$ where $\beta' < \beta'' < d$. If β' is empty, then the contribution is $x \sum_{j \geq 0} g_{k+j}(x)$, upon considering deletion of $n-1-k$. If β' is not empty, then π has the form

$$(n-1-k)\beta'd(d-1) \cdots d'n(n-1) \cdots (n-k)(n-2-k) \cdots (d+1)(d'-1) \cdots (d''+1),$$

for some d' , where d'' is the largest letter of β' . The contributions are x^{k+2} for $\{n-k-1, n-k, \dots, n\}$, x for d , $1/(1-x)^3$ for the 3 decreasing sequences, and $K-1$ for β' , hence $\frac{x^{k+3}(K-1)}{(1-x)^3}$ altogether. By combining all the contributions, we obtain

$$g_k(x) = g'_k(x) + x \sum_{j \geq 0} g_{k+j}(x) + \frac{x^{k+3}(K-1)}{(1-x)^3}$$

for $k \geq 1$, with initial condition $g_0(x) = x(F_T(x) - 1)$ (delete $n-1$, which can play no role in a forbidden pattern).

Define the generating function $G(x, u) = \sum_{k \geq 0} g_k(x)u^k$. Note that $G(x, 1) = G_2(x)$ since $G_2(x) = \sum_{k \geq 0} g_k(x)$. The preceding recurrence for $g_k(x)$ can now be written as

(5)

$$G(x, u) = G'(x, u) + x(F_T(x) - 1) - x^2 F_T(x) + \frac{xu}{1-u}(G(x, 1) - G(x, u)) + \frac{x^4 u(K-1)}{(1-x)^3(1-xu)},$$

where $G'(x, u) = \sum_{k \geq 0} g'_k(x)u^k$.

Next, let us write an equation for $g'_k(x)$. So suppose $\pi = (n-1-k)n\pi' \in S_n(T)$ has two left-right maxima. Clearly, $g'_0(x) = x^2 F_T(x)$ (delete the first two letters, $n-1$ and n). For $k \geq 1$, π can be written as $\pi = (n-1-k)n\beta'(n-1)\beta''$. If β' is empty, then the contribution is given by $xg'_{k-1}(x)$. Otherwise, similar to the g_k case, β' has the form $\gamma'd\gamma''$ where $\gamma' < \gamma'' < d$, and by considering whether γ' is empty or not, we obtain the contribution $x^2 \sum_{j \geq 0} g_{k-1+j}(x) + \frac{x^{k+3}(K-1)}{(1-x)^3}$. Combining all the previous cases yields

$$g'_k(x) = xg'_{k-1}(x) + x^2 \sum_{j \geq 0} g_{k-1+j}(x) + \frac{x^{k+3}(K-1)}{(1-x)^3}$$

for $k \geq 1$, with $g'_0(x) = x^2 F_T(x)$. Multiply by u^k and sum over $k \geq 0$ to obtain

$$(6) \quad G'(x, u) = xuG'(x, u) + x^2 F_T(x) + \frac{x^2 u}{1-u}(G(x, 1) - uG(x, u)) + \frac{x^4 u}{(1-x)^3(1-xu)}(K-1).$$

Solving (6) for $G'(x, u)$, and substituting into (5), yields

$$\begin{aligned} \frac{1-u+xu^2}{1-u}G(x, u) &= \frac{x((1-x)^3 + xu(2-xu)(x^2 - (1-x)^3))}{(1-x)^3(1-xu)} \\ &\quad - x(1-xu+x^2u)F_T(x) - \frac{x^4 u(2-xu)K}{(1-x)^3(1-xu)} - \frac{xu(1+x-xu)}{1-u}G(x, 1). \end{aligned}$$

To solve the preceding functional equation, we apply the kernel method and take $u = C(x)$, which cancels out the $G(x, u)$ term. A calculation, using the identity $xC(x)^2 = C(x) - 1$ to simplify the result (best done by computer), now yields

$$G(x, 1) = \frac{x(-1 + (2-x)C(x))}{1 + xC(x)}F_T(x) - \frac{x(1-5x+9x^2-4x^3+x^4-x^2C(x))}{(1-x)^2(1-3x+x^2)(1+x(1-C(x)))}.$$

Hence, since $F_T(x) = \sum_{m \geq 0} G_m(x)$ and $G_2(x) = G(x, 1)$, we obtain

$$F_T(x) = 1 + xF_T(x) + G(x, 1) + \frac{x^3 K}{(1-x)^2(1-2x)},$$

which leads to

$$F_T(x) = \frac{1-4x+5x^2-x^3+(1-4x+3x^2-x^3)\sqrt{1-4x}}{(1-x)(1-3x+x^2)(1-2x+\sqrt{1-4x})},$$

as required. \square

3.4.2. $\mathbf{T} = \{1324, 1342, 4123\}$. For this case, we define $a(n; i_1, i_2, \dots, i_k)$ for $n \geq k$ to be the number of T -avoiding permutations of length n whose first k letters are i_1, i_2, \dots, i_k . Let $a(n) = \sum_{i=1}^n a(n; i)$ for $n \geq 1$ and $\mathcal{T}_{i,j}$ be the subset of permutations enumerated by $a(n; i, j)$. It is convenient to consider separately the cases when either the second or third letter equals n . To this end, let $e(n; i) = a(n; i, n)$ for $1 \leq i \leq n-2$ (with $e(n; n-1)$ defined to be zero) and $f(n; i, j) = a(n; i, j, n)$ for $4 \leq i \leq n-1$ and $1 \leq j \leq i-3$. The arrays $a(n; i, j)$, $e(n; i)$ and $f(n; i, j)$ are determined recursively as follows.

Lemma 14. *We have*

$$(7) \quad a(n; i, i-2) = a(n-2; i-2) + e(n-1; i-2) + \sum_{j=1}^{i-3} a(n-1; i-1, j), \quad 3 \leq i \leq n,$$

$$(8) \quad a(n; i, j) = f(n; i, j) + \sum_{\ell=1}^{j-1} a(n-1; i-1, \ell), \quad 4 \leq i \leq n-1 \quad \text{and} \quad 1 \leq j \leq i-3,$$

$$(9) \quad a(n; n, j) = \sum_{\ell=1}^j a(n-1; n-1, \ell), \quad 1 \leq j \leq n-3,$$

$$(10) \quad e(n; i) = e(n-1; i) + \sum_{j=1}^i a(n-1; n-1, j), \quad 1 \leq i \leq n-3,$$

and

$$(11) \quad f(n; i, j) = f(n-1; i, j) + \sum_{\ell=1}^{j-1} a(n-2; n-2, \ell), \quad 4 \leq i \leq n-2 \quad \text{and} \quad 1 \leq j \leq i-3,$$

with $e(n; n-2) = C_{n-2}$ for $n \geq 3$ and $f(n; n-1, j) = a(n-1; n-1, j)$ for $1 \leq j \leq n-4$. Furthermore, we have $a(n; i, j) = 0$ if $n \geq 4$ and $1 \leq i < j-1 < n-1$, $a(n; i, i+1) = a(n-1; i)$ if $1 \leq i \leq n-1$, and $a(n; i, i-1) = a(n-1; i-1)$ if $2 \leq i \leq n$.

Proof. The formulas for $a(n; i, i+1)$ and $a(n; i, i-1)$, and for $a(n; i, j)$ when $i < j-1 < n-1$, follow from the definitions. In the remaining cases, let x denote the third letter of a T -avoiding permutation. For (7), note that members of $\mathcal{T}_{i, i-2}$ when $i < n$ must have $x = i-1$, $x = n$ or $x < i-2$, lest there be an occurrence of 1324 or 1342. This is seen to give $a(n-2; i-2)$, $e(n-1; i-2)$ and $\sum_{j=1}^{i-3} a(n-1; i-1, j)$ possibilities, respectively, which implies (7). Observe that (7) also holds when $i = n$ since $e(n-1; n-2) = 0$, by definition. For (8), note that members of $\mathcal{T}_{i, j}$ where $i < n$ and $j \leq i-3$ must have $x = n$ or $x < j$ (as $x = j+1$ is not permitted due to 4123 and $j+2 \leq x \leq n-1$ is not due to 1324, 1342). In the second case, the letter j becomes extraneous and thus may be deleted since i, x imposes a stricter requirement on later letters than does j, x (with $x < j$ making j redundant with respect to 1324, 1342). Relation (8) then follows from the definitions. For (9), note that members of $\mathcal{T}_{n, j}$ where $j \leq n-3$ must have $x = n-1$ or $x < j$ in order to avoid 4123, which accounts for the $\ell = j$ term and the remaining terms, respectively, in the sum on the right-hand side.

For (10), note that members of $\mathcal{T}_{i, n}$ where $i \leq n-3$ must have $x = n-1$, $x < i$ or $x = i+1$. The letter n may be deleted in the first case, while the i may be deleted in the latter two (as n, x imposes a stricter requirement on subsequent letters than i, x). Thus, there are $e(n-1; i)$,

$\sum_{j=1}^{i-1} a(n-1; n-1, j)$ and $a(n-1; n-1, i)$ possibilities, respectively, which implies (10). That $e(n; n-2) = C_{n-2}$ follows from the fact that members of $\mathcal{T}_{n-2, n}$ are synonymous with 123-avoiding permutations of length $n-2$ which are well known to be enumerated by C_{n-2} (note that $n-2$ is redundant due to n). Finally, to show (11), note that permutations counted by $f(n; i, j)$ must have fourth letter y equal $n-1$ or less than j . If $y = n-1$, then the letter n may be deleted and thus there are $f(n-1; i, j)$ possibilities, by definition. If $y < j$, then n, y imposes a stricter requirement on the remaining letters with respect to 4123 than does i, y or i, j , with the i and j also redundant with respect to 1324 or 1342 due to y . Thus, both i and j may be deleted in this case, yielding $\sum_{\ell=1}^{j-1} a(n-2; n-2, \ell)$ possibilities, which implies (11). That $f(n; n-1, j) = a(n-1; n-1, j)$ holds for $1 \leq j \leq n-4$ since the letter n may be deleted in this case, which completes the proof. \square

In order to solve the recurrences of the prior lemma, we introduce the following functions: $a_n(u) = \sum_{i=1}^n a(n; i)u^i$ for $n \geq 1$, $b_{n,i}(v) = \sum_{j=1}^{i-2} a(n; i, j)v^j$ for $3 \leq i \leq n$, $b_n(u, v) = \sum_{i=1}^{n-1} b_{n,i}(v)u^i$ for $n \geq 3$,

$$c_{n,i}(v) = a(n-1; i) + a(n-1; i-1) + b_{n,i}(v), \quad 1 \leq i \leq n-1,$$

$c_n(u, v) = \sum_{i=1}^{n-1} c_{n,i}(v)u^i$ for $n \geq 2$, $e_n(u) = \sum_{i=1}^{n-2} e(n; i)u^i$ for $n \geq 3$, $f_{n,i}(v) = \sum_{j=1}^{i-3} f(n; i, j)v^j$ for $4 \leq i \leq n-1$, and $f_n(u, v) = \sum_{i=4}^{n-1} f_{n,i}(v)u^i$ for $n \geq 5$.

By the definitions, we have

$$(12) \quad a_n(u) = c_n(u, 1) + e_n(u) + C_{n-1}u^n, \quad n \geq 1.$$

Assume $b_{n,1}(v) = b_{n,2}(v) = 0$. By (7) and (8), we have for $3 \leq i \leq n-1$,

$$\begin{aligned} b_{n,i}(v) &= (a(n-2; i-2) + b_{n-1, i-1}(1) + e(n-1; i-2))v^{i-2} + f_{n,i}(v) \\ &\quad + \sum_{j=1}^{i-3} v^j \sum_{\ell=1}^{j-1} a(n-1; i-1, \ell) \\ &= (a(n-2; i-2) + e(n-1; i-2))v^{i-2} + f_{n,i}(v) + \frac{v}{1-v}(b_{n-1, i-1}(v) - v^{i-2}b_{n-1, i-1}(1)). \end{aligned}$$

Multiplying both sides of the last equation by u^i , and summing over $3 \leq i \leq n-1$, yields

$$(13) \quad \begin{aligned} b_n(u, v) &= u^2(a_{n-2}(uv) + e_{n-1}(uv) - C_{n-3}(uv)^{n-2}) + f_n(u, v) \\ &\quad + \frac{u}{1-v}(vb_{n-1}(u, v) - b_{n-1}(uv, 1)), \quad n \geq 3. \end{aligned}$$

By (7) and (9), we get

$$(14) \quad \begin{aligned} b_{n,n}(v) &= (a(n-2; n-2) + b_{n-1, n-1}(1))v^{n-2} + \frac{1}{1-v}(b_{n-1, n-1}(v) - v^{n-2}b_{n-1, n-1}(1)) \\ &= C_{n-3}v^{n-2} + \frac{1}{1-v}(b_{n-1, n-1}(v) - v^{n-1}b_{n-1, n-1}(1)), \quad n \geq 3. \end{aligned}$$

By the definitions, we have

$$(15) \quad \begin{aligned} c_n(u, v) &= a_{n-1}(u) + u(a_{n-1}(u) - a(n-1; n-1)u^{n-1}) + b_n(u, v) \\ &= (u+1)a_{n-1}(u) - C_{n-2}u^n + b_n(u, v), \quad n \geq 2. \end{aligned}$$

Multiplying both sides of (10) by u^i , and summing over $1 \leq i \leq n-3$, gives

$$\begin{aligned} e_n(u) &= e(n; n-2)u^{n-2} + e_{n-1}(u) + \sum_{j=1}^{n-3} a(n-1; n-1, j) \left(\frac{u^j - u^{n-2}}{1-u} \right) \\ (16) \quad &= C_{n-2}u^{n-2} + e_{n-1}(u) + \frac{1}{1-u} (b_{n-1, n-1}(u) - u^{n-2}b_{n-1, n-1}(1)), \quad n \geq 3. \end{aligned}$$

By (11), we have

$$f_{n,i}(v) = f_{n-1,i}(v) + \frac{1}{1-v} \sum_{\ell=1}^{i-3} a(n-2; n-2, \ell) (v^{\ell+1} - v^{i-2}), \quad 4 \leq i \leq n-2,$$

with

$$f_{n, n-1}(v) = \sum_{j=1}^{n-4} f(n; n-1, j)v^j = \sum_{j=1}^{n-4} a(n-1; n-1, j)v^j = b_{n-1, n-1}(v) - C_{n-3}v^{n-3}, \quad n \geq 5.$$

We then get

$$\begin{aligned} f_n(u, v) - f_{n, n-1}(v)u^{n-1} &= f_{n-1}(u, v) + \frac{1}{1-v} \sum_{\ell=1}^{n-4} a(n-2; n-2, \ell) v^{\ell+1} \sum_{i=\ell+3}^{n-2} u^i \\ &\quad - \frac{1}{1-v} \sum_{\ell=1}^{n-4} a(n-2; n-2, \ell) \sum_{i=\ell+3}^{n-2} u^i v^{i-2} \\ &= f_{n-1}(u, v) + \frac{uv}{(1-u)(1-v)} (u^2 b_{n-2, n-2}(uv) - u^{n-2} b_{n-2, n-2}(v)) \\ (17) \quad &\quad - \frac{u}{v(1-uv)(1-v)} (u^2 v^2 b_{n-2, n-2}(uv) - (uv)^{n-2} b_{n-2, n-2}(1)), \quad n \geq 5. \end{aligned}$$

Let $C(x) = \sum_{n \geq 0} C_n x^n$. Define the generating functions $a(x; u) = \sum_{n \geq 1} a_n(u) x^n$, $b(x; u, v) = \sum_{n \geq 3} b_n(u, v) x^n$, $c(x; u, v) = \sum_{n \geq 2} c_n(u, v) x^n$, $d(x; v) = \sum_{n \geq 3} b_{n,n}(v) x^n$, $e(x; u) = \sum_{n \geq 3} e_n(u) x^n$, and $f(x; u, v) = \sum_{n \geq 5} f_n(u, v) x^n$. By (15) at $v=1$, we have

$$(18) \quad c(x; u, 1) = x(u+1)a(x; u) + b(x; u, 1) - x^2 u^2 C(xu).$$

Rewriting recurrences (12)–(14), (16), and (17) in terms of generating functions, and applying (18) to the relation obtained from (12), yields the following system of functional equations.

Lemma 15. *We have*

$$(19) \quad (1 - x(u+1))a(x; u) = xu(1-xu)C(xu) + b(x; u, 1) + e(x; u),$$

$$\begin{aligned} \left(1 - \frac{xuv}{1-v}\right) b(x; u, v) &= f(x; u, v) + xu^2 e(x; uv) - x^3 u^3 v C(xuv) - \frac{xu}{1-v} b(x; uv, 1) \\ (20) \quad &\quad + x^2 u^2 a(x; uv), \end{aligned}$$

$$(21) \quad \left(1 - \frac{x}{1-v}\right) d(x; v) = x^3 v C(xv) - \frac{x}{1-v} d(xv; 1),$$

$$(22) \quad (1-x)e(x; u) = x^2(C(xu) - 1) + \frac{x}{u(1-u)} (ud(x; u) - d(xu; 1)),$$

and

$$(23) \quad \begin{aligned} (1-x)f(x; u, v) &= xd(xu; v) - x^3u^2(C(xuv) - 1) + \frac{x^2uv}{(1-u)(1-v)}(u^2d(x; uv) - d(xu; v)) \\ &\quad - \frac{x^2u}{v(1-uv)(1-v)}(u^2v^2d(x; uv) - d(xuv; 1)). \end{aligned}$$

Note that the last three equations in the prior lemma are independent of the first two.

Lemma 16. *We have*

$$(24) \quad d(x; v) = \frac{x^2v(1 - (1-x)C(xv))}{1-x-v},$$

$$(25) \quad e(x; u) = \frac{x^2u(1 - (1-x)C(xu))}{(1-x)(1-x-u)},$$

and

$$(26) \quad f(x; 1/(1-x), 1-x) = \frac{1-6x+9x^2-2x^3-(1-3x)(1-x)\sqrt{1-4x}}{2x(1-x)^2}.$$

Proof. Replacing x with x/v in (21) gives

$$(27) \quad \left(1 - \frac{x}{v(1-v)}\right) d(x/v; v) = \frac{x^3}{v^2}C(x) - \frac{x}{v(1-v)}d(x; 1).$$

Applying the kernel method to (27), and taking $v = \frac{1+\sqrt{1-4x}}{2} = \frac{1}{C(x)}$, yields

$$d(x; 1) = x^2 \left(1 - \frac{1}{C(x)}\right) C^2(x) = x(C(x) - 1) - x^2C(x) = (x - x^2)C(x) - x$$

and thus

$$\begin{aligned} d(x; v) &= \frac{x^3v(1-v)C(xv)}{1-x-v} - \frac{x}{1-x-v}(xv(1-xv)C(xv) - xv) \\ &= \frac{x^2v(1 - (1-x)C(xv))}{1-x-v}. \end{aligned}$$

Formula (25) now follows from (24) and (22). By taking $u \rightarrow 1/v$ in (23), we obtain

$$\begin{aligned} &(1-x)f(x; 1/v, v) \\ &= xd(x/v; v) - x^3/v^2(C(x) - 1) + \frac{x^2}{(1-1/v)(1-v)}(d(x; 1)/v^2 - d(x/v; v)) \\ &\quad - \frac{x^2}{v^2(1-v)} \lim_{u \rightarrow 1/v} \frac{u^2v^2d(x; uv) - d(xuv; 1)}{1-uv} \\ &= xd(x/v; v) - x^3/v^2(C(x) - 1) + \frac{x^2}{(1-1/v)(1-v)}(d(x; 1)/v^2 - d(x/v; v)) \\ &\quad + \frac{x^2(2d(x; 1) + \frac{d}{dw}d(x; w)|_{w=1} - x\frac{d}{dx}d(x; 1))}{v^2(1-v)}. \end{aligned}$$

Substituting $v = 1-x$ in the last expression, and using (24), yields (26). \square

We can now determine the generating function $F_T(x)$.

Theorem 17. *Let $T = \{1324, 1342, 4123\}$. Then*

$$F_T(x) = \frac{1 - 3x + x^2 - x^3 - (1-x)^3\sqrt{1-4x}}{2x(1-4x+4x^2-x^3)}.$$

Proof. In the notation above, we seek to determine $1 + a(x; 1)$. By (20) with $u = 1/(1-x)$ and $v = 1-x$ and by (19) with $u = 1$, we have

$$\begin{aligned} f(x; 1/(1-x), 1-x) &= -\frac{x}{(1-x)^2}e(x; 1) + \frac{x^3}{(1-x)^2}C(x) + \frac{1}{1-x}b(x; 1, 1) - \frac{x^2}{(1-x)^2}a(x; 1), \\ a(x; 1) &= \frac{x(1-x)}{1-2x}C(x) + \frac{1}{1-2x}b(x; 1, 1) + \frac{1}{1-2x}e(x; 1). \end{aligned}$$

Substituting the expressions for $e(x; 1)$ and $f(x; 1/(1-x), x)$ from the prior lemma, and then solving the system that results for $a(x; 1)$ and $b(x; 1, 1)$, we obtain

$$\begin{aligned} a(x; 1) &= \frac{1 - 5x + 9x^2 - 9x^3 + 2x^4 - (1-x)^3\sqrt{1-4x}}{2x(1-4x+4x^2-x^3)}, \\ b(x; 1, 1) &= \frac{1 - 8x + 22x^2 - 23x^3 + 5x^4 - (1-6x+12x^2-7x^3+x^4)\sqrt{1-4x}}{2x(1-3x+x^2)}. \end{aligned}$$

Hence,

$$1 + a(x; 1) = \frac{1 - 3x + x^2 - x^3 - (1-x)^3\sqrt{1-4x}}{2x(1-4x+4x^2-x^3)},$$

as desired. \square

3.5. Case 174. The three representative triples T are:

$\{2134, 2341, 2413\}$ (Theorem 18)

$\{2143, 2314, 2341\}$ (Theorem 20)

$\{2143, 2314, 2431\}$ (Theorem 21)

Theorem 18. *Let $T = \{2134, 2341, 2413\}$. Then*

$$F_T(x) = \frac{1 - 6x + 10x^2 - 3x^3 + x^4}{(1-3x+x^2)(1-4x+2x^2)}.$$

Proof. Let $G_m(x)$ be the generating function for T -avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. Now let us write an equation for $G_m(x)$ with $m \geq 2$.

For $m = 2$, let $\pi = i\pi'n\pi'' \in S_n(T)$ with two left-right maxima, i and n , and consider cases on i . If $i = n-1$, then $\pi \rightarrow \pi'(n-1)\pi''$ is a bijection to nonempty T -avoiders of length $n-1$, giving a contribution of $x(F_T(x)-1)$. If $2 \leq i \leq n-2$ and π'' has the form $(n-1)(n-2)\cdots(i+1)\pi'''$, then $\pi \rightarrow \pi'i\pi'''$ is a bijection to T -avoiders of length i , giving a contribution of $x^2/(1-x)(F_T(x)-1-x)$.

Lemma 19. *If $\pi = i\pi'n\pi'' \in S_n(T)$ has two left-right maxima, $i \leq n-2$ and π'' does not have the form $(n-1)(n-2)\cdots(i+1)\pi'''$, then $\pi' = \emptyset$ and $i = 1$.*

Proof. Note first that all letters of $[i+1, n-1]$ must occur prior to any letters of $[i-1]$ within π'' in order to avoid 2413. By hypothesis, there exist $a, b \in \pi''$ such that $i < a < b$ with a occurring before b . If $x \in \pi'$, then $ixab$ is a 2134. Hence, $\pi' = \emptyset$. If $i > 1$, then 1 occurs (i) before a or (ii) after b . If (i), $i1ab$ is a 2134; if (ii), $iab1$ is a 2341, both forbidden. Hence, $i = 1$. \square

By the lemma, the only remaining case is $\pi = 1n\pi''$ with $n \geq 3$ (since $n = 2$ falls under the case $i = n - 1$). Here, $\pi \rightarrow \pi''$ is a bijection to T -avoiders of length $n - 2$, giving a contribution of $x^2(F_T(x) - 1)$. Summing all contributions, we find

$$G_2(x) = \frac{x}{1-x}(F_T(x) - 1) + x^2 \left(F_T(x) - \frac{1}{1-x} \right).$$

For $m \geq 3$, let $\pi = i_1\pi^{(1)} \cdots i_m\pi^{(m)} \in S_n(T)$ with m left-right maxima. Then $\pi^{(1)} = \pi^{(2)} = \cdots = \pi^{(m-2)} = \emptyset$ for else a 2134 is present; also $\pi^{(m-1)} > i_{m-3}$ (with $i_0 = 0$) and $\pi^{(m)} > i_{m-2}$ or a 2341 is present. Consequently, if there is no letter between i_{m-3} and i_{m-2} , then π has the form $12 \cdots (m-2)\pi'$ where π' is a permutation of $m-1, m-2, \dots, n$ with two left-right maxima that avoids T , giving a contribution of $x^{m-2}G_2(x)$. On the other hand, if there is a letter between i_{m-3} and i_{m-2} , the reader may verify that π must have the form

$$\pi = 12 \cdots (m-3)i_{m-2}i_{m-1}(i_{m-1}-1) \cdots (i_{m-2}+1)\pi'i_m(i_m-1) \cdots (i_{m-1}+1),$$

where π' is a nonempty permutation of $i_{m-3}+1, \dots, i_{m-2}-1$ that avoids 213 and 2341. Here, the contribution is $\frac{x^m}{(1-x)^2}(K-1)$, where $K = \sum_{n \geq 0} |S_n(213, 2341)|x^n = \frac{1-2x}{1-3x+x^2}$ (see [13, Seq. A001519]). Hence, for $m \geq 3$,

$$G_m(x) = x^{m-2}G_2(x) + \frac{x^m}{(1-x)^2}(K-1).$$

Since $F_T(x) = \sum_{m \geq 0} G_m(x)$, we have

$$F_T(x) = 1 + xF_T(x) - \frac{x}{(1-x)^2}((x^2 - x - 1)F_T(x) + x + 1) + \frac{(K-1)x^3}{(1-x)^3},$$

with solution the stated $F_T(x)$. \square

Theorem 20. *Let $T = \{2143, 2314, 2341\}$. Then*

$$F_T(x) = \frac{1 - 6x + 10x^2 - 3x^3 + x^4}{(1 - 3x + x^2)(1 - 4x + 2x^2)}.$$

Proof. Let $G_m(x)$ be the generating function for T -avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. Now let us write an equation for $G_m(x)$ with $m \geq 3$. Suppose $\pi = i_1\pi^{(1)} \cdots i_m\pi^{(m)} \in S_n(T)$ with $m \geq 3$ left-right maxima i_1, i_2, \dots, i_m . Then, because π avoids 2314 and 2341,

$$\pi^{(1)} < i_1 < \pi^{(2)} < i_2 < \cdots < \pi^{(m-2)} < i_{m-2} < i_{m-1}\pi^{(m-1)}i_m\pi^{(m)}.$$

If $\pi^{(1)} = \cdots = \pi^{(j-1)} = \emptyset$ and $\pi^{(j)} \neq \emptyset$ with $j = 1, 2, \dots, m-2$, then we have that $\pi^{(j+1)} = \cdots = \pi^{(m)} = \emptyset$ (π avoids 2143). So the contribution for each $j = 1, 2, \dots, m-2$ is $x^m(K-1)$, where $K = \sum_{n \geq 0} |S_n(231, 2143)|x^n = \frac{1-2x}{1-3x+x^2}$ (see [13, Seq. A001519]). If $\pi^{(1)} = \cdots = \pi^{(m-2)} = \emptyset$, then the contribution is given by $x^{m-2}G_2(x)$. Thus,

$$G_m(x) = (m-2)x^m(K-1) + x^{m-2}G_2(x), \quad m \geq 3.$$

It remains to find a formula for $G_2(x)$. So suppose $\pi = i\pi'n\pi''$ and consider whether π' is empty or not. If $\pi' = \emptyset$, then $\pi \rightarrow i\pi''$ is a bijection to nonempty T -avoiders, giving a contribution of $x(F_T(x) - 1)$. If $\pi' \neq \emptyset$, say $x \in \pi'$, then $i = n - 1$ because $i < n - 1$ implies $ixn(n - 1)$ is a 2143. So π has first letter $n - 1$ (contributes $x(F_T(x) - 1)$) and second letter $\neq n$ (hence, subtract $x^2F_T(x)$) for a net contribution of $x(F_T(x) - 1 - xF_T(x))$. Thus,

$$G_2(x) = x(F_T(x) - 1) + x(F_T(x) - 1 - xF_T(x)).$$

Since $F_T(x) = \sum_{m \geq 0} G_m(x)$, we have

$$F_T(x) = 1 + xF_T(x) + \frac{x}{1-x}(2F_T(x) - 2 - xF_T(x)) + \frac{x^3(K-1)}{(1-x)^2},$$

with solution the stated $F_T(x)$. □

Theorem 21. *Let $T = \{2143, 2314, 2431\}$. Then*

$$F_T(x) = \frac{1 - 6x + 10x^2 - 3x^3 + x^4}{(1 - 3x + x^2)(1 - 4x + 2x^2)}.$$

Proof. Let $G_m(x)$ be the generating function for T -avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. Now let us write an equation for $G_m(x)$ with $m \geq 2$. Suppose $\pi = i_1\pi^{(1)} \dots i_m\pi^{(m)} \in S_n(T)$ with $m \geq 2$ left-right maxima i_1, i_2, \dots, i_m . Then, because π avoids 2314,

$$\pi^{(1)} < i_1 < \pi^{(2)} < i_2 < \dots < \pi^{(m-2)} < i_{m-2} < \pi^{(m-1)} < i_{m-1}.$$

If $\pi^{(1)}, \dots, \pi^{(m)}$ are all empty, the contribution is x^m . Now suppose the π 's are not all empty and j is minimal such that $\pi^{(j)} \neq \emptyset$.

First, suppose $j \in [m - 1]$. Then $\pi^{(1)} = \dots = \pi^{(j-1)} = \emptyset$ by supposition and $\pi^{(j+1)} = \dots = \pi^{(m-1)} = \emptyset$ (to avoid 2143). Furthermore, $\pi_m > i_{j-1}$ (to avoid 2431) and $\pi_m < i_j$ (to avoid 2143). Hence, $\pi = 12 \dots (j - 1)i_j\pi^{(j)}(i_j + 1)(i_j + 2) \dots n\pi^{(m)}$. So “delete $12 \dots (j - 1)$ and $(i_j + 1)(i_j + 2) \dots (n - 1)$ and standardize” is a bijection to T -avoiders with second largest letter in first position and largest letter not in second position, giving a contribution of $x^{m-1}(F_T(x) - 1 - xF_T(x))$ as in Theorem 20, for each $j \in [m - 1]$.

Next, suppose $j = m$ so that $\pi = i_1i_2 \dots i_m\pi^{(m)}$ with $\pi^{(m)} \neq \emptyset$. Then, because π avoids 2431, π has the form $i_1i_2 \dots i_m\beta^{(1)}\beta^{(2)} \dots \beta^{(m)}$ with $\beta^{(1)} < i_1 < \beta^{(2)} < i_2 < \dots < \beta^{(m)} < i_m$. Let ℓ be the minimal index such that $\beta^{(\ell)}$ is not empty, and say $x \in \beta^{(\ell)}$. Then $\beta^{(j)} = \emptyset$ for $j \geq \ell + 2$ because if $y \in \beta^{(j)}$ with $j \geq \ell + 2$, then $i_{j-2}i_{j-1}xy$ is a 2314. Furthermore, $\beta^{(\ell+1)}$ is increasing, because $z > y$ in $\beta^{(\ell+1)}$ implies $i_\ell xzy$ is a 2143.

If $\beta^{(\ell+1)} = \emptyset$, we get a contribution of $x^m(F_T(x) - 1)$ for each $\ell \in [m]$.

If $\beta^{(\ell+1)} \neq \emptyset$, then $\beta^{(\ell)}$ must avoid 231. So “delete the initial m letters (= the left-right maxima) and standardize” is a bijection to pairs $(\gamma^{(\ell)}, \gamma^{(\ell+1)})$ with $\gamma^{(\ell)}$ a nonempty $\{231, 2143\}$ -avoider and $\gamma^{(\ell+1)}$ an initial segment of the positive integers. Thus we get, for each $\ell \in [m - 1]$, a contribution of $x^m \frac{x}{1-x}(K - 1)$, where $K = \sum_{n \geq 0} |S_n(231, 2143)|x^n = \frac{1-2x}{1-3x+x^2}$ (see [13, Seq. A001519]). Summing all contributions, we have for $m \geq 2$,

$$G_m(x) = x^m + (m - 1)x^{m-1}(F_T(x) - 1 - xF_T(x)) + mx^m(F_T(x) - 1) + (m - 1) \left(\frac{x^{m+1}}{1-x}(K - 1) \right).$$

Since $F := F_T(x) = \sum_{m \geq 0} G_m(x)$, we find

$$F = 1 + xF + \frac{x}{(1-x)^2}(F-1-xF) + \frac{(2-x)x^2}{(1-x)^2}(F-1) + \frac{x^2}{1-x} + \frac{x^3}{(1-x)^3}(K-1),$$

with solution the stated $F_T(x)$. \square

3.6. Case 177. The two representative triples T are:

$\{2143, 2341, 2413\}$ (Theorem 22)

$\{2143, 2341, 3241\}$ (Theorem 23)

Theorem 22. *Let $T = \{2143, 2341, 2413\}$. Then*

$$F_T(x) = \frac{1 - 4x + 3x^2 - x^3}{1 - 5x + 6x^2 - 3x^3}.$$

Proof. Let $G_m(x)$ be the generating function for T -avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. Now let us write an equation for $G_m(x)$ with $m \geq 2$. So suppose $\pi = i_1\pi^{(1)}i_2\pi^{(2)} \cdots i_m\pi^{(m)} \in S_n(T)$ with $m \geq 2$ left-right maxima. We consider the following three cases:

- $\pi^{(1)} \neq \emptyset$. Here, the only letters occurring after i_2 that are $> i_1$ are i_3, \dots, i_m (to avoid 2143) and no letter occurring after i_3 is $< i_1$ (to avoid 2341). So π has the form $i_1\pi^{(1)}(i_1+1)\gamma^{(2)}(i_1+2) \cdots n$ with $\gamma^{(2)} < i_1$. Thus, $(i_1+2) \cdots n$ contributes x^{m-2} and deleting these letters is a bijection to T -avoiders of length r for some $r \geq 2$ with first letter $r-1$ and second letter $\neq r$, contributing $x(F_T-1) - x^2F_T$.
- $\pi^{(1)} = \emptyset$ and $\pi^{(2)}$ has a letter $a < i_1$. Here, no non-left-right max letter occurring after i_3 (if present) is $> i_1$ (2143) and, again, no letter occurring after i_3 is $< i_1$ (2341). Also, in π_2 , all letters $> i_1$ occur before all letters $< i_1$ (2413). So π has the form $i_1i_2\gamma^{(2)}\gamma^{(1)}(i_2+1) \cdots n$ with $\gamma^{(1)} < i_1 < \gamma^{(2)} < i_2$ and, furthermore, $\gamma^{(2)}$ is decreasing because $b < c$ in $\gamma^{(2)}$ implies i_1bca is a 2341. So $\pi = i_1i_2(i_2-1) \cdots (i_1+1)\gamma^{(1)}(i_2+1) \cdots n$ with $\gamma^{(1)}$ a T -avoider of length $\in [n-m]$, giving a contribution of $\frac{x^m}{1-x}(F_T(x)-1)$.
- $\pi^{(1)} = \emptyset$ and $\pi^{(2)} > i_1$. This condition implies $i_1 = 1$ (obvious if $m = 2$ and because $i_1i_2i_31$ would be a 2341 if $m \geq 3$), giving a contribution of $xG_{m-1}(x)$.

Summing the contributions, we have for $m \geq 2$,

$$G_m(x) = x^{m-1}(F_T(x) - 1 - xF_T(x)) + \frac{x^m}{1-x}(F_T(x) - 1) + xG_{m-1}(x).$$

Since $F_T(x) = \sum_{m \geq 0} G_m(x)$, we find that

$$F_T(x) = 1 + xF_T(x) + \frac{x}{1-x}(F_T(x) - 1 - xF_T(x)) + \frac{x^2}{(1-x)^2}(F_T(x) - 1) + x(F_T(x) - 1),$$

which, by solving for $F_T(x)$, completes the proof. \square

Theorem 23. *Let $T = \{2143, 2341, 3241\}$. Then*

$$F_T(x) = \frac{1 - 4x + 3x^2 - x^3}{1 - 5x + 6x^2 - 3x^3}.$$

Proof. Let $G_m(x)$ be the generating function for T -avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. Now let us write an equation for $G_m(x)$ with $m \geq 2$. So suppose $\pi = i_1\pi^{(1)}i_2\pi^{(2)} \cdots i_m\pi^{(m)} \in S_n(T)$ with $m \geq 2$ left-right maxima. We consider the following three cases:

- $\pi^{(1)} \neq \emptyset$. Since T contains 2143 and 2341, π has the form $i_1\pi^{(1)}(i_1+1)\gamma^{(2)}(i_1+2) \cdots n$ with $\gamma^{(2)} < i_1$, as in Theorem 22. Furthermore, $\pi^{(1)} < \gamma^{(2)}$ for else i_1, i_1+1 are the 3 and 4 of a 3241. Hence, if $\pi^{(1)}$ is increasing, then $\pi^{(1)} = 12 \cdots i$ for some $i \geq 1$ and $\pi \rightarrow \text{St}(\gamma^{(2)})$ is a bijection to T -avoiders, giving a contribution of $\frac{x^{m+1}}{1-x}F_T(x)$. On the other hand, if $\pi^{(1)}$ is not increasing, then $\gamma^{(2)} = \emptyset$ because $b > a$ in $\pi^{(1)}$ and $c \in \gamma^{(2)}$ implies bai_2c is a 2143. So, $\pi \rightarrow \pi^{(1)}$ is a bijection to non-identity T -avoiders, giving a contribution of $x^m(F_T(x) - \frac{1}{1-x})$.
- $\pi^{(1)} = \emptyset$ and $\pi^{(2)}$ has a letter smaller than i_1 . Here π has the form $i_1i_2\pi^{(2)}(i_2+1) \cdots n$ (to avoid 2143, 2341) with $i_1 \neq 1$, and $\pi \rightarrow i_1\pi^{(2)}$ is a bijection to T -avoiders of length ≥ 2 with first letter $\neq 1$, giving a contribution of $x^{m-1}(F_T(x) - 1 - xF_T(x))$.
- $\pi^{(1)} = \emptyset$ and $\pi^{(2)} > i_1$. As in Theorem 22, $i_1 = 1$ and the contribution is $xG_{m-1}(x)$.

Summing the contributions, we have for $m \geq 2$,

$$G_m(x) = \frac{x^{m+1}}{1-x}F_T(x) + x^m \left(F_T(x) - \frac{1}{1-x} \right) + x^{m-1}(F_T(x) - 1 - xF_T(x)) + xG_{m-1}(x).$$

Since $F_T(x) = \sum_{m \geq 0} G_m(x)$, we find that

$$\begin{aligned} F_T(x) &= 1 + xF_T(x) + \frac{x^3}{(1-x)^2}F_T(x) + \frac{x^2}{1-x} \left(F_T(x) - \frac{1}{1-x} \right) \\ &\quad + \frac{x}{1-x}(F_T(x) - 1 - xF_T(x)) + x(F_T(x) - 1), \end{aligned}$$

which, by solving for $F_T(x)$, completes the proof. \square

3.7. Case 191.

Theorem 24. *Let $T = \{1342, 2134, 2413\}$. Then*

$$F_T(x) = \frac{(1-x)(1-2x)(1-3x)}{1-7x+16x^2-14x^3+3x^4}.$$

Proof. Let $G_m(x)$ be the generating function for T -avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. Now let us write an equation for $G_m(x)$ with $m \geq 2$.

For $m = 2$, let $\pi = i\pi'n\pi'' \in S_n(T)$ with two left-right maxima. The entries after n and $> i$ all precede entries $< i$ or else i, n are the “2,4” of a 2413. Hence, $\pi = i\pi'n\beta'\beta''$ with $\beta' > i > \beta''$.

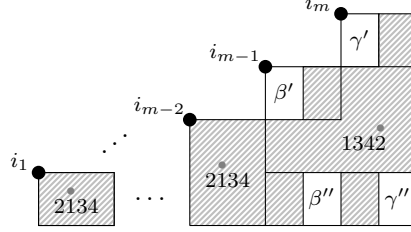
If $\pi' = \emptyset$ so that $\pi = in\beta'\beta''$, then β' avoids 231 or i would start a 1342. So β' avoids 2134 and 231 (which subsumes 2413), and β'' avoids T . The generating function $K(x)$ for $\{231, 2134\}$ -avoiders is $K(x) = 1 + \frac{x(1-3x+3x^2)}{(1-x)(1-2x)^2}$ [13, Seq. A005183]. Thus the contribution is $x^2K(x)F_T(x)$.

If $\pi' \neq \emptyset$, then β' is decreasing (or $i \max(\pi')$ would start a 2134) and $\text{St}(\pi'n\beta'')$ is a T -avoider that does not start with its max. Thus, by deleting i , we have a contribution of $\frac{x}{1-x}(F_T(x) - 1 - xF_T(x))$.

Hence,

$$G_2(x) = x^2K(x)F_T(x) + \frac{x}{1-x}(F_T(x) - 1 - xF_T(x)).$$

Now, suppose $m \geq 3$ and π avoids T with m left-right maxima $i_1 < i_2 < \dots < i_m$. Then π has the form shown in the figure below with shaded regions empty to avoid a pattern involving the gray bullet as indicated, and entries in β' preceding entries in β'' to avoid 2413, and similarly for γ', γ'' .



We consider 4 cases according as β', β'' are empty or not.

If β', β'' are both empty, then γ' avoids 231 (else i_{m-1} is the “1” of a 1342) and γ'' avoids T , giving a contribution of $x^m K(x) F_T(x)$.

If $\beta' \neq \emptyset, \beta'' = \emptyset$, then β' avoids 213 due to i_m (2134) and avoids 231 due to i_{m-2} (1342). The generating function $L(x)$ for $\{213, 231\}$ -avoiders is $L(x) = \frac{1-x}{1-2x}$ [12]. Also, γ' is decreasing (2134), and γ'' avoids T . By deleting the left-right maxima and γ' , the contribution is $\frac{x^m}{1-x}(L(x) - 1)F_T(x)$.

If $\beta' = \emptyset, \beta'' \neq \emptyset$, then γ' is decreasing once again and $\text{St}(\beta'' i_m \gamma'')$ avoids T and does not start with its max. Deleting i_1, \dots, i_{m-1} and γ' , the contribution is $\frac{x^{m-1}}{1-x}(F_T(x) - 1 - xF_T(x))$.

If β', β'' are both nonempty, then β' avoids 213 and 231, γ' is decreasing, and $\text{St}(\beta'' i_m \gamma'')$ avoids T and does not start with its max. Again deleting i_1, \dots, i_{m-1} and γ' , the contribution is $\frac{x^{m-1}}{1-x}(L(x) - 1)(F_T(x) - 1 - xF_T(x))$.

Hence, for $m \geq 3$,

$$G_m(x) = x^m K(x) F_T(x) + \frac{x^m}{1-x}(L(x) - 1)F_T(x) + \frac{x^{m-1}}{1-x}L(x)(F_T(x) - 1 - xF_T(x)).$$

Since $F_T(x) = \sum_{m \geq 0} G_m(x)$, we obtain

$$\begin{aligned} F_T(x) &= 1 + xF_T(x) + x^2 K(x) F_T(x) + \frac{x}{1-x}(F_T(x) - 1 - xF_T(x)) \\ &\quad + \frac{x^3}{1-x}K(x)F_T(x) + \frac{x^3}{(1-x)^2}(L(x) - 1)F_T(x) + \frac{x^2}{(1-x)^2}L(x)(F_T(x) - 1 - xF_T(x)). \end{aligned}$$

By solving for $F_T(x)$, we complete the proof. \square

3.8. Case 196. The two representative triples T are:

$\{2143, 2431, 3241\}$ (Theorem 25)

$\{2413, 2431, 3214\}$ (Theorem 27)

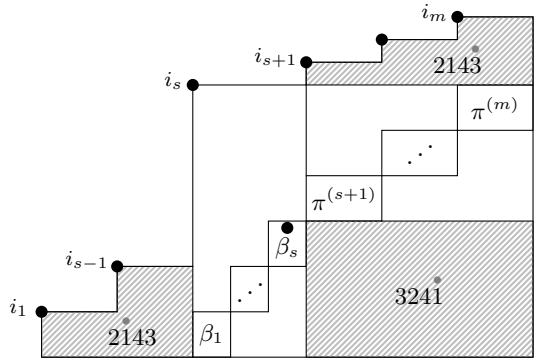
Theorem 25. Let $T = \{2143, 2431, 3241\}$. Then

$$F_T(x) = \frac{1 - 5x + 7x^2 - 4x^3}{1 - 6x + 11x^2 - 9x^3 + 2x^4}.$$

Proof. Let $G_m(x)$ be the generating function for T -avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. For $m \geq 2$, we need a simple lemma. Let $S_{n,m}$ denote the set of all permutations $\pi = i_1\pi^{(1)}i_2\pi^{(2)} \dots i_m\pi^{(m)} \in S_n$ with m left-right maxima i_1, i_2, \dots, i_m and $R_{n,m}$ the subset that satisfy the condition (*) $\pi^{(1)} = \emptyset$ and $\pi^{(j)} < i_{j-1}$ for all $j = 2, 3, \dots, m$. Let $S_{n,m}(T)$ and $R_{n,m}(T)$ have the obvious meaning.

Lemma 26. *For $m \geq 2$, the map $\phi : i_1\pi^{(1)}i_2\pi^{(2)} \dots i_m\pi^{(m)} \rightarrow i_1\pi^{(2)}i_2\pi^{(3)} \dots i_{m-1}\pi^{(m)}$ is a bijection from $R_{n,m}$ to $S_{n-1,m-1}$. Furthermore, the restriction of ϕ to $R_{n,m}(T)$ is a bijection to $S_{n-1,m-1}(T)$. \square*

Now suppose $\pi \in S_{n,m}(T)$ with $m \geq 2$ and consider cases according as condition (*) holds or not. If (*) holds, so that $\pi \in R_{n,m}(T)$, then the contribution is $xG_{m-1}(x)$, by the lemma. If (*) does not hold, then there exists an index $s \in [m]$ such that $\pi^{(s)}$ contains a letter between i_{s-1} and i_s . This imposes restrictions on π as illustrated:



where dark bullets indicate mandatory entries, shaded regions are empty to avoid the pattern involving a light bullet as indicated, blank regions are empty, and the β 's and π 's are in the displayed order to avoid 2431.

Thus, the contribution is $x^m \sum_{s=1}^m L_{m,s;s}(x)$, where $L_{m,s;d}(x)$ is the generating function for such avoiders π (as illustrated) with $\beta_1 = \dots = \beta_{s-d} = \emptyset$, $1 \leq d \leq s$, when the left-right maxima are understood to make no contribution, i.e., are weighted with 1 rather than x . Note the latter condition on the β 's is vacuous—no restriction—when $d = s$. We need to introduce d because we can get a recurrence for $L_{m,s;d}$ in terms of $L_{m,s;d-1}$ that will yield $L_{m,s;s}$. To do so, let $2 \leq d \leq s$ and consider whether β_{s-d+1} is empty or not. If $\beta_{s-d+1} = \emptyset$, clearly the contribution is $L_{m,s;d-1}(x)$. If $\beta_{s-d+1} \neq \emptyset$ then, to avoid 2143, β_i is increasing (could be empty) for $s-d+2 \leq i \leq s-1$ while β_s is increasing and nonempty. Moreover, also to avoid 2143 (but utilizing different letters), $\pi^{(j)} = \emptyset$ for all $j = s+1, s+2, \dots, m$. There are $d-1$ β 's required to be increasing and so the contribution is $(F_T(x) - 1) \frac{x}{(1-x)^{d-1}}$.

Adding the two contributions, we have

$$(28) \quad L_{m,s;d}(x) = L_{m,s;d-1}(x) + (F_T(x) - 1) \frac{x}{(1-x)^{d-1}}.$$

To complete the recurrence, we need an expression for $L_{m,s;1}(x)$. Here, $\beta_1, \dots, \beta_{s-1}$ are all empty. Set $r = m - s$, $M_r = L_{m,s;1}(x)$ and relabel β 's and π 's so that the boxes not required to be empty

for M_r contain $\beta_0 \neq \emptyset, \pi^{(1)}, \dots, \pi^{(r)}$. Now consider variable r . Clearly, $M_0 = F_T(x) - 1$ and we obtain a recurrence for M_r , $r \geq 1$, conditioning on the first nonempty $\pi^{(j)}$. If all π 's are empty, the contribution is $F_T(x) - 1$. Otherwise, let $j \in [r]$ be minimal with $\pi^{(j)} \neq \emptyset$. Then β_0 is increasing ($b > a$ in β_0 would make ba the 21 of a 2143) and the contribution is $\frac{x}{1-x} M_{r-j}$ (because $\pi^{(j)}$ can play the role of β_0). So $M_r = F_T(x) - 1 + \frac{x}{1-x} \sum_{j=1}^r M_{r-j}$ for $r \geq 1$, with $M_0 = F_T(x) - 1$. This recurrence has solution $M_r = \frac{F_T(x)-1}{(1-x)^r}$. So $L_{m,s;1}(x) = \frac{F_T(x)-1}{(1-x)^{m-s}}$, the initial condition for recurrence (28), with solution

$$L_{m,s;d}(x) = (F_T(x) - 1) \left(\frac{1}{(1-x)^{d-1}} + \frac{1}{(1-x)^{m-s}} - 1 \right).$$

Hence,

$$\begin{aligned} G_m(x) &= xG_{m-1}(x) + x^m \sum_{s=1}^m L_{m,s;s}(x) \\ &= xG_{m-1}(x) + x^m \sum_{s=1}^m (F_T(x) - 1) \left(\frac{1}{(1-x)^{s-1}} + \frac{1}{(1-x)^{m-s}} - 1 \right), \end{aligned}$$

which implies

$$G_m(x) = xG_{m-1}(x) + x^m \frac{(F_T(x) - 1)(2(1-x)^{1-m} - mx - 2(1-x))}{x}.$$

By summing over all $m \geq 2$ and using the initial condition $G_0(x) = 1$ and $G_1(x) = xF_T(x)$, we obtain

$$F_T(x) - 1 - xF_T(x) = x(F_T(x) - 1) + \frac{(2x^2 - 3x + 2)(F_T(x) - 1)x^2}{(1-2x)(1-x)^2},$$

with solution the desired F_T . □

Theorem 27. *Let $T = \{2413, 2431, 3214\}$. Then*

$$F_T(x) = \frac{1 - 5x + 7x^2 - 4x^3}{1 - 6x + 11x^2 - 9x^3 + 2x^4}.$$

Proof. Let $a_n = |S_n(T)|$ and let $a_n(i_1, i_2, \dots, i_s)$ denote the number of permutations $i_1 i_2 \dots i_s \pi \in S_n(T)$. We will obtain expressions for $a_n(i, j)$ and $a_n(i)$ and deduce a recurrence for a_n . Clearly, $a_n(1) = a_n(n) = a_{n-1}$. For $2 \leq i \leq n-1$, we have the following expressions for $a_n(i, j)$:

$$a_n(i, j) = \begin{cases} a_{n-1}(i-1) & \text{if } j = 1, \\ a_{j-1} & \text{if } 2 \leq j < i, \\ a_{n-1}(i) & \text{if } j = i+1, \\ 0 & \text{if } j \geq i+2. \end{cases}$$

For the first item, “delete 1 and standardize” is a bijection from T -avoiders that begin $i1$ to one-size-smaller T -avoiders that begin $i-1$. For the second item, n occurs before 1 (3214) and so $\pi = ij\pi'n\pi''1\pi'''$. Also, $\pi' > j$ (3214), $\pi'' < j$ (2431), $\pi''' < j$ (2413), and π' is increasing (2431). These results imply that $\pi = ij(j+1) \dots \widehat{i} \dots n\beta$ (where \widehat{i} indicates that i is missing) with $\beta \in S_{j-1}(T)$. The easy proofs of the last two items are left to the reader.

Since $a_n = \sum_{i=1}^n a_n(i) = a_n(1) + \sum_{i=2}^{n-1} \sum_{j=1}^n a_n(i, j) + a_n(n)$, the preceding results yield

$$a_n = \sum_{i=1}^{n-3} (n-2-i)a_i + 2 \sum_{i=1}^{n-1} a_{n-1}(i) + 2(a_{n-1} - a_{n-2})$$

for $n \geq 3$, which implies

$$a_n = 4a_{n-1} - 2a_{n-2} + \sum_{i=1}^{n-3} (n-2-i)a_i,$$

with $a_0 = a_1 = 1$ and $a_2 = 2$. Since $F_T(x) = \sum_{n \geq 0} a_n x^n$, the recurrence for a_n translates to

$$F_T(x) - 1 - x - 2x^2 = 4x(F_T(x) - 1 - x) - 2x^2(F_T(x) - 1) + \frac{x^3}{(1-x)^2}(F_T(x) - 1),$$

with solution the desired F_T . □

3.9. Case 201. The two representative triples T are:

$\{1243, 1324, 3142\}$ (Theorem 28)

$\{1342, 1423, 2314\}$ (Theorem 31)

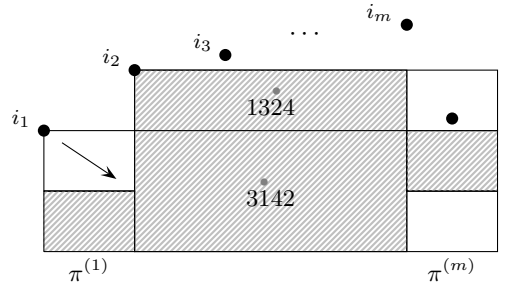
3.9.1. $T = \{1243, 1324, 3142\}$.

Theorem 28. *Let $T = \{1243, 1324, 3142\}$. Then*

$$F_T(x) = \frac{1 - 3x + x^2}{1 - x} C^3(x).$$

Proof. Let $G_m(x)$ be the generating function for T -avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. Now suppose $m \geq 2$ and $\pi = i_1\pi^{(1)} \cdots i_m\pi^{(m)}$ avoids T . Then $\{i_2, i_3, \dots, i_m = n\}$ are consecutive integers (a gap would give a 1324 or a 1243). We consider two cases.

- $i_1 = n - m + 1$, its maximum possible value. Here, $\pi^{(1)} > \pi^{(2)} > \dots > \pi^{(m)}$ (to avoid 3142). Also, for $j \in [m - 1]$, $\pi^{(j)}$ avoids 132 (or i_{j+1} is the “4” of a 1324), and $\pi^{(m)}$ avoids T . Hence, the contribution is $x^m C(x)^{m-1} F_T(x)$.
- $i_1 < n - m + 1$. Here, i_1 and i_2 are not consecutive and π has the form



where the top shaded rectangle is empty (1324) and hence $\pi^{(m)}$ contains i_1+1 ; the rectangle below it is empty (3142); $\pi^{(1)}$ is decreasing (or $i_m(i_1+1)$ would terminate a 1243) and $i_1\pi^{(1)}$ has no gaps (else there exist $a < b < i_1$ with $a \in \pi^{(1)}$ and $b \in \pi^{(m)}$ and then i_1ai_mb is a 3142). Also, $i_1\pi^{(m)}$ avoids T and does not start with its maximal letter. Hence, the contribution is $\frac{x^{m-1}}{1-x}(F_T(x) - 1 - xF_T(x))$.

Combining the preceding cases gives $G_m(x)$ for $m \geq 2$, and by summing over all $m \geq 0$, we obtain

$$F_T(x) = 1 + \sum_{m \geq 1} x^m C(x)^{m-1} F_T(x) + \sum_{m \geq 2} \frac{x^{m-1}}{1-x} (F_T(x) - 1 - xF_T(x)),$$

which implies

$$F_T(x) = 1 + xC(x)F_T(x) + \frac{x}{(1-x)^2} (F_T(x) - 1 - xF_T(x)),$$

with solution $F_T(x) = \frac{1-3x+x^2}{(1-x)(1-2x+(x^2-x)C(x))}$, equivalent to the stated expression. \square

3.9.2. $\mathbf{T} = \{1342, 1423, 2314\}$. To enumerate the members of $S_n(T)$, we consider the relative positions of the letters n and $n-1$ within a permutation. More precisely, given $1 \leq i, j \leq n$ with $i \neq j$, let $a(n; i, j)$ denote the number of permutations $\pi = \pi_1\pi_2 \cdots \pi_n \in S_n(T)$ such that $\pi_i = n$ and $\pi_j = n-1$. If $n \geq 2$ and $1 \leq i \leq n$, then let $a(n; i) = \sum_{j=1}^n a(n; i, j)$, with $a(1; 1) = 1$. The array $a(n; i, j)$ is determined by the following recurrence relations.

Lemma 29. *If $n \geq 3$, then*

$$(29) \quad a(n; i, i-1) = a(n-1; i-1, i-2) + \sum_{j=2}^{i-2} a(n-1; i-1, j), \quad 3 \leq i \leq n,$$

and

$$(30) \quad a(n; i, j) = a(n-1; i, j) + a(n-1; i-1, j-1) + \sum_{k=j+1}^{n-1} a(n-1; j-1, k), \quad 2 \leq i \leq j-2.$$

Furthermore, we have $a(n; i, 1) = a(n-1; i-1)$ for $2 \leq i \leq n$, $a(n; 1, j) = a(n-1; j-1)$ for $2 \leq j \leq n$, $a(n; i, j) = a(n-1; i-1, j)$ for $2 \leq j \leq i-2$, and $a(n; i, i+1) = a(n-1; i)$ for $1 \leq i \leq n-1$.

Proof. Throughout, let $\pi = \pi_1\pi_2 \cdots \pi_n \in S_n(T)$ be of the form enumerated in the case under consideration. The formulas for $a(n; i, 1)$ and $a(n; 1, j)$ follow from the fact that an initial letter $n-1$ or n within a member of $S_n(T)$ may be safely deleted. To determine $a(n; i, j)$ where $2 \leq j \leq i-2$, first note that within $\pi \in S_n(T)$ in this case, the letter $n-2$ cannot go to the left of $n-1$ (for if it did, then there would be an occurrence of 2314 of the form $(n-2)(n-1)xn$ for some $x < n-2$). Furthermore, the letter $n-2$ cannot go to the right of n , for otherwise there would be an occurrence of 1342 of the form $x(n-1)n(n-2)$ for some $x < n-2$ (since $j \geq 2$ implies $n-1$ is not the first letter). Thus, $n-2$ must go between $n-1$ and n in this case. Note also that $\min\{\pi_{j+1}, \pi_{j+2}, \dots, \pi_{i-1}\} > \max\{\pi_1, \pi_2, \dots, \pi_{j-1}\}$ so as to avoid an occurrence of 2314 (of the form $x(n-1)yn$). Thus, $j \geq 2$ implies the section $\pi_{j+1}\pi_{j+2} \cdots \pi_{i-1}$ is decreasing in order to avoid 1423, whence $\pi_{j+1} = n-2$. It follows that the letter $n-2$ may be deleted, which implies $a(n; i, j) = a(n-1; i-1, j)$ if $1 < j < i-1$. Next, observe that $a(n; i, i+1) = a(n-1; i)$ since the

letter $n - 1$ is extraneous in this case and may be deleted (as none of the patterns in T contain “4” directly followed by “3”).

We now show (29). Note that the letter $n - 2$ in this case must occur to the left of $n - 1$, for otherwise there would be a 1342. If $\pi_1 = n - 2$, there are $a(n - 1; i - 1, i - 2)$ possibilities as the letter $n - 2$ may be deleted since it cannot play the role of a “2” within a 2314. So suppose $\pi_j = n - 2$ for some $2 \leq j \leq i - 2$. Then we must have $\min\{\pi_{j+1}, \pi_{j+2}, \dots, \pi_{i-2}\} > \max\{\pi_1, \pi_2, \dots, \pi_{j-1}\}$ in order to avoid 2314, with $\min\{\pi_1, \pi_2, \dots, \pi_{j-1}\} > \max\{\pi_{i+1}, \pi_{i+2}, \dots, \pi_n\}$ to avoid 1342. Since all of the same restrictions on π are seen to apply if we delete n , it follows that there are $\sum_{j=2}^{i-2} a(n - 1; i - 1, j)$ possibilities if $n - 2$ does not start a permutation. Combining this case with the previous implies formula (29).

Finally, to show (30), it is convenient to write $\pi \in S_n(T)$ enumerated by $a(n; i, j)$ when $1 < i < j - 1$ as $\pi = w^{(1)}w^{(2)} \dots w^{(r)}$, where $w^{(i)}$ for $i < r$ denotes the sequence of letters of π between the i -th and the $(i + 1)$ -st left-right minimum, including the former but excluding the latter (with $w^{(r)}$ comprising all letters to the right of and including the rightmost left-right minimum). Observe that n must be the final letter of some $w^{(\ell)}$. For if not, then $j > i + 1$ implies that there would be an occurrence of 1423 of the form $xny(n - 1)$, where x is a left-right minimum and y is not. Then $n - 2$ must be the first letter of π or go to the right of n , for otherwise, π would contain an occurrence of 2314 of the form $x(n - 2)y(n - 1)$, where x is the first letter and y is the successor of n (and hence a left-right minimum). If $n - 2$ is the first letter, then it is seen to be extraneous (since $n - 1$ occurs to the right of n within π) and thus may be deleted, yielding $a(n - 1; i - 1, j - 1)$ possibilities. If $n - 2$ occurs to the right of n , then it must also occur to the right of $n - 1$ in order to avoid 1423. If $\pi_{j+1} = n - 2$, then it is seen that $n - 2$ may be deleted as there can be no possible occurrence of a pattern in T involving both $n - 2$ and $n - 1$ in this case, whence there are $a(n - 1; i, j)$ possibilities. On the other hand, if $\pi_k = n - 2$ for some $k > j + 1$, then the letter $n - 1$, like n , must be the last letter of some $w^{(\ell)}$ in order to avoid 1423.

We claim that the letter n may be deleted in this case. First note that $i > 1$ implies n belongs to the leftmost $w^{(\ell)}$ such that $w^{(\ell)}$ is not of length one (for otherwise, there would be an occurrence of 2314, with n playing the role of the “4”). If s denotes the index of this $w^{(\ell)}$, then $w^{(s)}$ must be of length two, for if not and $w^{(s)}$ contained a third letter, then π would contain 2314, as witnessed by the subsequence $xyz(n - 1)$, where x and z are the first letters of $w^{(s)}$ and $w^{(s+1)}$ and y is the second letter of $w^{(s)}$. It follows that the letters to the left of n within π form a decreasing sequence. By similar reasoning, the letters between n and $n - 1$ are decreasing since $\pi_k = n - 2$ for some $k > j + 1$. Since $\min\{\pi_1, \pi_2, \dots, \pi_{i-1}\} > \max\{\pi_{i+1}, \pi_{i+2}, \dots, \pi_{j-1}\}$ in order to avoid 1423, it follows that the letters to the left of $n - 1$ excluding n form a decreasing sequence. From this, it is seen that the letter n may be deleted, which gives $\sum_{k=j+1}^{n-1} a(n - 1; j - 1, k)$ additional possibilities. Combining this with the previous cases implies (30) and completes the proof. \square

Define the functions $b_{n,i}(v) = \sum_{j=i+2}^n a(n; i, j)v^j$ for $1 \leq i \leq n - 2$ and $c_{n,i}(v) = \sum_{j=2}^{i-1} a(n; i, j)v^j$ for $3 \leq i \leq n$. Then recurrences (30) and (29) imply

$$(31) \quad b_{n,i}(v) = b_{n-1,i}(v) + vb_{n-1,i-1}(v) + \sum_{j=i+2}^n b_{n-1,j-1}(1)v^j, \quad 2 \leq i \leq n - 2,$$

and

$$(32) \quad c_{n,i}(v) = c_{n-1,i-1}(v) + c_{n-1,i-1}(1)v^{i-1} + a(n - 1; i - 1, i - 2)v^{i-1}, \quad 3 \leq i \leq n.$$

Let $a_n(u) = \sum_{i=1}^n a(n; i)u^i$ for $n \geq 1$, $b_n(u, v) = \sum_{i=2}^{n-2} b_{n,i}(v)u^i$ for $n \geq 4$, $c_n(u, v) = \sum_{i=3}^n c_{n,i}(v)u^i$ for $n \geq 3$, and $d_n(u) = \sum_{i=2}^n a(n; i, i-1)u^i$ for $n \geq 2$. Let $a(n) = a_n(1)$ for $n \geq 1$, with $a(0) = 1$.

By Lemma 29, we have

$$\sum_{i=2}^{n-1} a(n; i, i+1)u^i = \sum_{i=2}^{n-1} a(n-1; i)u^i = a_{n-1}(u) - a(n-2)u, \quad n \geq 2,$$

and

$$\sum_{i=2}^n a(n; i, 1)u^i = \sum_{i=2}^n a(n-1; i-1)u^i = ua_{n-1}(u), \quad n \geq 2.$$

Thus, by the definitions, we have

$$(33) \quad a_n(u) = u(a(n-1) - a(n-2)) + (1+u)a_{n-1}(u) + b_n(u, 1) + c_n(u, 1), \quad n \geq 2,$$

with $a_1(u) = u$, upon considering separately the cases of $a(n; i, j)$ when $i = 1$, $j = 1$ or both $i, j > 1$.

Note that by the definitions,

$$b_{n,1}(v) = \sum_{j=3}^n a(n; 1, j)v^j = \sum_{j=3}^n a(n-1; j-1)v^j = v(a_{n-1}(v) - a(n-2)v), \quad n \geq 3.$$

Multiplying both sides of (31) by u^i , and summing over $2 \leq i \leq n-2$, then yields

$$(34) \quad \begin{aligned} b_n(u, v) &= b_{n-1}(u, v) + uv(b_{n-1}(u, v) + ub_{n-1,1}(v)) + \sum_{j=3}^n b_{n-1, j-1}(1)v^j \sum_{i=2}^{j-2} u^i \\ &= (1+uv)b_{n-1}(u, v) + u^2v^2(a_{n-2}(v) - a(n-3)v) \\ &\quad + \frac{v}{1-u}(u^2b_{n-1}(v, 1) - b_{n-1}(uv, 1)), \quad n \geq 4. \end{aligned}$$

Multiplying both sides of (32) by u^i , and summing over $3 \leq i \leq n$, gives

$$(35) \quad c_n(u, v) = u(c_{n-1}(u, v) + c_{n-1}(uv, 1) + d_{n-1}(uv)), \quad n \geq 3.$$

Finally, using recurrence (29) and noting $a(n; 2, 1) = a(n-2)$, we get

$$(36) \quad \begin{aligned} d_n(u) &= a(n-2)u^2 + ud_{n-1}(u) + \sum_{i=3}^n c_{n-1, i-1}(1)u^i \\ &= a(n-2)u^2 + ud_{n-1}(u) + uc_{n-1}(u, 1), \quad n \geq 2. \end{aligned}$$

Define the generating functions $a(x; u) = \sum_{n \geq 1} a_n(u)x^n$, $b(x; u, v) = \sum_{n \geq 4} b_n(u, v)x^n$, $c(x; u, v) = \sum_{n \geq 3} c_n(u, v)x^n$ and $d(x; u) = \sum_{n \geq 2} d_n(u)x^n$. Rewriting recurrence (33)–(36) in terms of generating functions yields the following system of functional equations.

Lemma 30. *We have*

$$(37) \quad (1-x-xu)a(x; u) = xu(1-x)(1+a(x; 1)) + b(x; u, 1) + c(x; u, 1),$$

$$(38) \quad (1-x-xuv)b(x; u, v) = (xuv)^2(a(x; v) - xva(x; 1) - xv) + \frac{xv}{1-u}(u^2b(x; v, 1) - b(x; uv, 1)),$$

$$(39) \quad (1-xu)c(x; u, v) = xu(c(x; uv, 1) + d(x; uv)),$$

and

$$(40) \quad (1 - xu)d(x; u) = x^2u^2(1 + a(x; 1)) + xuc(x; u, 1).$$

We can now determine the generating function $F_T(x)$.

Theorem 31. *Let $T = \{1342, 1423, 2314\}$. Then*

$$F_T(x) = \frac{1 - 3x + x^2}{1 - x} C^3(x).$$

Proof. In our present notation, we seek $1 + a(x; 1)$. By taking $u = v = 1$ in (37), (39) and (40), and then solving the resulting system for $b(x; 1, 1)$, $c(x; 1, 1)$ and $d(x; 1)$, we obtain

$$\begin{aligned} b(x; 1, 1) &= \frac{(1 - x)(1 - 5x + 6x^2 - x^3)a(x; 1) - x(1 - 4x + 5x^2 - x^3)}{1 - 3x + x^2}, \\ c(x; 1, 1) &= \frac{x^3(1 + a(x; 1))}{1 - 3x + x^2}, \\ d(x; 1) &= \frac{x^2(1 - 2x)(1 + a(x; 1))}{1 - 3x + x^2}. \end{aligned}$$

Hence, equation (38) with $v = 1$ can be written as

$$\begin{aligned} \left(1 + \frac{xu^2}{1 - u}\right) b(x; u, 1) &= (xu)^2((1 - x)a(x; 1) - x) \\ &+ \frac{xu^2}{1 - u} \frac{(1 - x)(1 - 5x + 6x^2 - x^3)a(x; 1) - x(1 - 4x + 5x^2 - x^3)}{1 - 3x + x^2}. \end{aligned}$$

Applying the kernel method to this last equation, it is seen that taking $u = C(x)$ cancels out the left-hand side. This gives, after several algebraic operations, the formula

$$1 + a(x; 1) = \frac{2(1 - 3x + x^2)}{(1 - x)(1 - 3x) + (1 - x)^2\sqrt{1 - 4x}} = \frac{1 - 3x + x^2}{1 - x} C^3(x),$$

as desired. \square

Remark: From the formula for $a(x; 1)$, one can now determine $b(x; 1, 1)$, as well as $c(x; u, 1)$ and $d(x; u, 1)$, by (39) and (40). This in turn allows one to find $b(x; u, 1)$, by (38) at $v = 1$. By (37), one then obtains a formula for $1 + a(x; u)$ which generalizes $F_T(x)$ (reducing to it when $u = 1$).

REFERENCES

- [1] Wikipedia, Permutation pattern, https://en.wikipedia.org/wiki/Permutation_pattern
- [2] Wikipedia, Enumerations of specific permutation classes, https://en.wikipedia.org/wiki/Enumerations_of_specific_permutation_classes
- [3] D. Callan, T. Mansour and M. Shattuck, Wilf classification of triples of 4-letter patterns, preprint, <http://arxiv.org/abs/1605.04969>.
- [4] S. Heubach and T. Mansour, *Combinatorics of Compositions and Words*, CRC Press, Boca Raton, FL, 2009.
- [5] Q. Hou and T. Mansour, Kernel method and linear recurrence system, *J. Computat. Appl. Math.* **261:1** (2008), 227–242.
- [6] D. E. Knuth, *The Art of Computer Programming*, 3rd edition, Addison Wesley, Reading, MA, 1997.
- [7] D. Kremer and W. C. Shiu, Finite transition matrices for permutations avoiding pairs of length four patterns, *Discrete Math.* **268:1-3** (2003), 171–183.
- [8] W. Kuszmaul, Fast algorithms for finding pattern avoiders and counting pattern occurrences in permutations, preprint, arXiv:1509.08216, 2015.

- [9] I. Le, Wilf classes of pairs of permutations of length 4, *Electron. J. Combin.* **12** (2005), #R25.
- [10] P. A. MacMahon, *Combinatory Analysis*, 2 vols, Cambridge University Press, 1915-16.
- [11] T. Mansour, *Combinatorics of Set Partitions*, CRC Press, Boca Raton, FL, 2012.
- [12] R. Simion and F. W. Schmidt, Restricted permutations, *European J. Combin.* **6** (1985), 383–406.
- [13] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, <http://oeis.org>.
- [14] Z. E. Stankova, Forbidden subsequences, *Discrete Math.* **132** (1994), 291–316.
- [15] Z. Stankova, Classification of forbidden subsequences of length four, *European J. Combin.* **17** (1996), 501–517.
- [16] V. Vatter, Finding regular insertion encodings for permutation classes, *J. Symbolic Comput.* **47:3** (2012), 259–265.
- [17] J. West, Generating trees and the Catalan and Schröder numbers, *Discrete Math.* **146** (1995), 247–262.

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