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KÜLSHAMMER IDEALS OF ALGEBRAS OF QUATERNION TYPE

ALEXANDER ZIMMERMANN

ABSTRACT. For a symmetric algebra A over a field K of characteristic $p > 0$ Külshammer constructed a descending sequence of ideals of the centre of A . If K is perfect this sequence was shown to be an invariant under derived equivalence and for algebraically closed K under stable equivalence of Morita type. Erdmann classified algebras of tame representation type which may be blocks of group algebras, and Holm classified Erdmann's list up to derived equivalence. In both classifications certain parameters occur in the classification, and it was unclear if different parameters lead to different algebras. Erdmann's algebras fall into three classes, namely of dihedral, semidihedral and of quaternion type. In previous joint work with Holm we used Külshammer ideals to distinguish classes with respect to these parameters in case of algebras of dihedral and semidihedral type. In the present paper we determine the Külshammer ideals for algebras of quaternion type and distinguish again algebras with respect to certain parameters.

INTRODUCTION

Erdmann gave in [4] a list of basic symmetric algebras of tame representation type which include all the algebras which may be Morita equivalent to blocks of finite groups of tame representation type. She obtained these algebras by means of properties of the Auslander-Reiten quiver which are known to hold for blocks of group rings with tame representation type. These algebras are subdivided into three classes, those of dihedral type, of semidihedral type and of quaternion type, corresponding to the possible defect group in case of group algebras, and actually defined by the behaviour of their Auslander-Reiten quiver. Holm refined Erdmann's classification in [7] to a classification up to derived equivalence. However, in [4, 7] the algebras are defined by quivers with relations, and the relations involve certain parameters, corresponding mostly to deformations of the socle of the algebras. It was unclear in some cases if different parameters lead to different derived equivalence classes of algebras. The question of non trivial socle deformations appears to be a very subtle one in this special case, but also in general, and only few progress was made on this question until very recently.

In [19] we showed that a certain sequence of ideals of the centre of a symmetric algebra defined previously by Héthelyi, Horváth, Külshammer and Murray are actually invariant under derived equivalences of this algebra. In joint work [15] with Liu and Zhou we showed that this invariant is also an invariant under stable equivalences of Morita type. We call this sequence of ideals the Külshammer ideals. In joint work [10] with Holm we observed that the Külshammer ideals behave in a very subtle manner with respect to the deformation parameters. Using this observation we showed that some of the parameters are invariants under derived equivalence (and also under stable equivalence of Morita type by [15]) for certain families of algebras of dihedral and of semidihedral type. The present paper is a continuation and completion of [10].

In order to apply the theory of Külshammer ideals we need to use the symmetrising form explicitly, and in our previous work this was done mostly by ad-hoc arguments from a general theory which we developed for Frobenius algebras. In the present paper we extend the theory for Frobenius algebras to the symmetrising form of symmetric algebras by a fairly simple argument. This is the subject of Proposition 3. In this note we compute the Külshammer ideals for algebras of quaternion type and distinguish the algebras with two simple modules and get a complete picture for characteristic different from 2, and an almost complete picture in the case of base fields of characteristic 2. The Külshammer ideal depends in a quite subtle way on the parameters, and we want to stress the

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fact that we need to compute the ideals as ideals, and as in [10] it is not sufficient to consider the dimensions only. We also deal with the case of algebras of quaternion type with three simple modules, where Külshammer ideals distinguish algebras in characteristic 2 with parameter d in case d is a square in K or not. The invariance of Külshammer ideals under derived or Morita equivalence is proved only in case K is perfect, whence when all elements of K are squares. Isomorphic algebras have isomorphic Külshammer ideal structure.

The paper is organised as follows. In Section 1 we recall basic facts about Külshammer ideals and improve the general methods needed to compute the Külshammer ideal structure for symmetric algebras. In particular, we prove Proposition 3 there. In Section 2 we apply the general theory to algebras of quaternion type, and we prove our main result Theorem 8 there.

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1. REVIEW ON KÜLSHAMMER IDEALS AND HOW TO COMPUTE THEM

The aim of this section is to briefly give the necessary background on Külshammer ideals, as introduced by B. Külshammer [13]. Morita invariance of Külshammer ideals (then named generalised Reynolds’ ideals) was shown in [3], [6]. Külshammer ideals were proved to be a derived invariant in [19], were used in [8, 12, 2] to classify weakly symmetric algebras of polynomial growth or domestic type up to derived equivalences, in [10] for a derived equivalence classification of algebras of dihedral or semidihedral type, in [11] for deformed preprojective algebras of type L , and in [17] for the derived equivalence classification of certain special biserial algebras. The concept was generalised to general finite-dimensional algebras in [1], to an invariant of Hochschild (co)homology for symmetric algebras [20] and in [21] for general algebras. Külshammer ideals were shown to be an invariant under stable equivalences of Morita type [15, 14]. An overview is given in [22, 23].

Let K be a field of characteristic $p > 0$. Any finite-dimensional symmetric K -algebra A has an associative, symmetric, non-degenerate K -bilinear form $\langle -, - \rangle : A \times A \rightarrow K$. For any K -linear subspace M of A we denote the orthogonal space by M^\perp with respect to this form. Moreover, let $[A, A]$ be the K -subspace of A generated by all commutators $[a, b] := ab - ba$, where $a, b \in A$. For any $n \geq 0$ set

$$T_n(A) = \left\{ x \in A \mid x^{p^n} \in [A, A] \right\}.$$

Then, by [13], for any $n \geq 0$, the orthogonal space $T_n(A)^\perp$ is an ideal of the center $Z(A)$ of A , called *n-th Külshammer ideal*. These ideals form a descending sequence

$$Z(A) = K(A)^\perp = T_0(A)^\perp \supseteq T_1(A)^\perp \supseteq T_2(A)^\perp \supseteq \dots \supseteq T_n(A)^\perp \supseteq \dots$$

with intersection of all ideals $T_n(A)^\perp$ for $n \in \mathbb{N}$ being the Reynolds’ ideal $R(A) = Z(A) \cap \text{soc}(A)$. In [6] it has been shown that if K is perfect, then the sequence of Külshammer ideals is invariant under Morita equivalences. Later, it was shown that the sequence of Külshammer ideals is invariant under derived and stable equivalences of Morita type. The following theorem recalls part of what is known.

Theorem 1.

- [19, Theorem 1] *Let A and B be finite-dimensional symmetric algebras over a perfect field K of positive characteristic p . If A and B are derived equivalent, then there is an isomorphism $\varphi : Z(A) \rightarrow Z(B)$ between the centers of A and B such that $\varphi(T_n(A)^\perp) = T_n(B)^\perp$ for all positive integers n .*
- (cf e.g. [23, Proposition 6.8.9]) *Let A and B be derived equivalent finite dimensional K -algebras over a field K , which is a splitting field for A and for B . Then the elementary divisors of the Cartan matrices of A and of B coincide. In particular, the determinant of the Cartan matrices coincides.*
- [15, Corollary 6.5] *If A and B are stably equivalent of Morita type, and if K is an algebraically closed field, then $\dim_K(T_n(A)/[A, A]) = \dim_K(T_n(B)/[B, B])$.*

We note that in the proof of [19, Theorem 1] the hypothesis that K is algebraically closed is never used. The assumption on the field K to be perfect is sufficient.

The aim of the present note is to show how these derived invariants can be applied to some subtle questions in the derived equivalence classifications of algebras of quaternion type.

In order to compute the Külshammer ideals we need a symmetrising form. We showed in [11] (see also [23]) that every Frobenius form arises as in the following proposition.

Proposition 2. [11] *Let $A = KQ/I$ be a Frobenius algebra given by the quiver Q and ideal of relations I , and fix an K -basis \mathcal{B} of A consisting of pairwise distinct non-zero paths of the quiver Q . Assume that \mathcal{B} contains a basis of $\text{soc}(A)$. Then the following statements hold:*

- (1) *Define an K -linear mapping ψ on the basis elements by*

$$\psi(b) = \begin{cases} 1 & \text{if } b \in \text{soc}(A) \\ 0 & \text{otherwise} \end{cases}$$

for $b \in \mathcal{B}$. Then an associative non-degenerate K -bilinear form $\langle -, - \rangle$ for A is given by $\langle x, y \rangle := \psi(xy)$.

- (2) *Any Frobenius form arises this way for some choice of a basis \mathcal{B} .*

If A is a basic symmetric algebra over an algebraically closed field K , then $A = KQ/I$ and we want to determine those bases \mathcal{B}_s of $\text{soc}(A)$ which yield a symmetric form. This was still an open problem in previous papers dealing with Külshammer ideals (cf [11, Remark 2.9], [10, Remark 3.2]). The following proposition solves the problem and sheds new light on the structure of the symmetrising form of a symmetric basic algebra.

Proposition 3. *If A is a basic symmetric algebra over an algebraically closed field K . If \mathcal{B}_s is a K -basis of $\text{soc}(A)$ such that $b - b' \in [A, A]$ for all $b, b' \in \mathcal{B}_s$. Then we may put $\psi(b) = 1$ for all $b \in \mathcal{B}_s$ and obtain a symmetrising form on A using Proposition 2.*

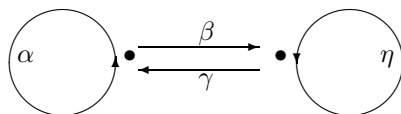
Proof. Let ψ be a map leading to a symmetric Frobenius form. Such a form must exist, and is given by a choice of a K -basis of $\text{soc}(A)$ by Proposition 2 item (2). We may suppose that for each primitive idempotent e of A , $\text{soc}(A)e$ is generated as K -vector space by a single path p_e of maximal length. Suppose that for another primitive idempotent f of A we get $p_e - p_f \in [A, A]$, then we need to have $\psi(p_e) = \psi(p_f)$. Indeed, $[A, A]^\perp = Z(A)$ by [13] (see also [23, Lemma 2.9.10]), and hence $\langle (p_e - p_f), ? \rangle$ is a linear form (with argument denoted “?”) with kernel containing $Z(A)$. On the other hand, p_e and p_f are both maximal paths, and so $p_e x = p_f y = x p_e = y p_f = 0$ for all arrows x, y in Q . Hence, $\langle (p_e - p_f), x \rangle$ is zero for all $x \in \text{rad}(A)$. Therefore, we only need to evaluate $\langle (p_e - p_f), ? \rangle$ on $x = 1 \in Z(A)$, which gives $\psi(p_e) = \psi(p_f)$. ■

Remark 4. Proposition 3 gives the general situation for indecomposable basic symmetric algebras A . Indeed, using [23, Proposition 2.7.4] it is not hard to see that we can always find a basis \mathcal{B}_s of $\text{soc}(A)$ such that the difference of two elements of \mathcal{B}_s is in the commutator subspace.

2. ALGEBRAS OF QUATERNION TYPE

2.1. Two simple modules. Erdmann gave a classification of algebras which could appear as blocks of tame representation type. These algebras fall in three classes, the algebras of dihedral, the algebras of semidihedral and the algebras of quaternion type. Erdmann’s classification was up to Morita equivalence. Holm [7, Appendix B] gave a classification up to derived equivalence and obtained for non local algebras of quaternion type two families of algebras, one with two simple modules, one with three simple modules. Each of the families has a quiver, and the relations depend on a number of parameters.

The quiver for the algebra with two simples is the following.



Let $k \geq 1, s \geq 3$ integers and $a \in K^\times, c \in K$. Then we get an algebra $Q(2\mathfrak{B})_1^{k,s}(a, c)$ with relations $\beta\eta = (\alpha\beta\gamma)^{k-1}\alpha\beta, \eta\gamma = (\gamma\alpha\beta)^{k-1}\gamma\alpha, \alpha^2 = a \cdot (\beta\gamma\alpha)^{k-1}\beta\gamma + c \cdot (\beta\gamma\alpha)^k, \gamma\beta = \eta^{s-1}, \alpha^2\beta = 0, \gamma\alpha^2 = 0$

Remark 5. The centre of this algebra is of dimension $k+s+2$ and the Cartan matrix is $\begin{pmatrix} 4k & 2k \\ 2k & k+s \end{pmatrix}$ with determinant $4ks$. Hence, using Theorem 1, if $D^b(Q(2\mathfrak{B})_1^{k,s}(a, c)) \simeq D^b(Q(2\mathfrak{B})_1^{k',s'}(a', c'))$, then $4ks = 4k's'$ and $k+s+2 = k'+s'+2$. Therefore $(k+s)^2 = (k'+s')^2$ and $(k-s)^2 = (k'-s')^2$, which implies $k = k'$ and $s = s'$, or $k = s'$ and $k' = s$.

Lemma 6. Let K be a field, and let $A^{k,s}(a, c) := Q(2\mathfrak{B})_1^{k,s}(a, c)$. Then, $Z(A^{k,s}(a, c))$ has a K -basis formed by the disjoint union

$\{\eta - (\alpha\beta\gamma)^{k-1}\alpha\} \dot{\cup} \{\eta^t \mid 2 \leq t \leq s\} \dot{\cup} \{(\alpha\beta\gamma)^u + (\beta\gamma\alpha)^u + (\gamma\alpha\beta)^u \mid 1 \leq u \leq k-1\} \dot{\cup} \{1, (\alpha\beta\gamma)^k, \alpha^2\}$
and is isomorphic, as commutative K -algebra, with

$$Z(A^{k,s}(a, c)) \simeq K[U, Y, S, T]/(Y^{s+1}, U^k - Y^s - 2T, S^2, T^2, SY, SU, ST, UY, UT, YT)$$

where

$$\begin{aligned} U &:= (\alpha\beta\gamma) + (\beta\gamma\alpha) + (\gamma\alpha\beta) \\ Y &:= \eta - (\alpha\beta\gamma)^{k-1}\alpha \\ S &:= \alpha^2 \\ T &:= (\alpha\beta\gamma)^k \end{aligned}$$

Proof. First, $\eta - (\alpha\beta\gamma)^{k-1}\alpha$ commutes trivially with η , since $\eta\alpha = 0 = \alpha\eta$. Now

$$\begin{aligned} \alpha(\eta - (\alpha\beta\gamma)^{k-1}\alpha) - (\eta - (\alpha\beta\gamma)^{k-1}\alpha)\alpha &= \alpha^2(\beta\gamma\alpha)^{k-1} - (\alpha\beta\gamma)^{k-1}\alpha^2 \\ &= (a(\beta\gamma\alpha)^{k-1}\beta\gamma + c(\beta\gamma\alpha)^k)(\beta\gamma\alpha)^{k-1} \\ &\quad - (\beta\gamma\alpha)^{k-1}(a(\beta\gamma\alpha)^{k-1}\beta\gamma + c(\beta\gamma\alpha)^k) \\ &= a((\beta\gamma\alpha)^{k-1}\beta\gamma(\beta\gamma\alpha)^{k-1} - (\beta\gamma\alpha)^{2k-2}\beta\gamma) \\ &= a(\beta\gamma\alpha)^{k-1}(\beta\gamma(\beta\gamma\alpha)^{k-1} - (\beta\gamma\alpha)^{k-1}\beta\gamma) \end{aligned}$$

This is trivially 0 if $k = 1$, and if $k > 1$, then

$$(\beta\gamma\alpha)^{2k-2}\beta\gamma = \beta(\gamma\alpha\beta)^{2k-2}\gamma = \beta(\gamma\alpha\beta)^k(\gamma\alpha\beta)^{k-2}\gamma = \beta\eta^s(\gamma\alpha\beta)^{k-2}\gamma = 0$$

Hence

$$\begin{aligned} a(\beta\gamma\alpha)^{k-1}(\beta\gamma(\beta\gamma\alpha)^{k-1} - (\beta\gamma\alpha)^{k-1}\beta\gamma) &= a(\beta\gamma\alpha)^{k-1}\beta\gamma(\beta\gamma\alpha)(\beta\gamma\alpha)^{k-2} \\ &= a(\beta\gamma\alpha)^{k-1}\beta\eta^{s-1}(\gamma\alpha)(\beta\gamma\alpha)^{k-2} \\ &= a(\beta\gamma\alpha)^{k-1}\beta\eta^{s-2}(\gamma\alpha\beta)^{k-1}\gamma\alpha^2(\beta\gamma\alpha)^{k-2} \\ &= 0 \end{aligned}$$

since $\gamma\alpha^2 = 0$. The relations $\beta\eta = (\alpha\beta\gamma)^{k-1}\alpha\beta$ and $\eta\gamma = (\gamma\alpha\beta)^{k-1}\gamma\alpha = \gamma(\alpha\beta\gamma)^{k-1}\alpha$ show that $\eta - (\alpha\beta\gamma)^{k-1}\alpha$ commutes with β , and with γ . Now, if $k > 1$, then $(\eta - (\alpha\beta\gamma)^{k-1}\alpha)^2 = \eta^2$, and if $k = 1$, then $(\eta - (\alpha\beta\gamma)^{k-1}\alpha)^2 = \eta^2 + \alpha^2$. Since $\alpha^2\beta = \gamma\alpha^2 = 0$, it is clear that α^2 is central. Hence η^t is central for each $t \geq 2$. Now,

$$\alpha\beta\gamma\beta = \alpha\beta\eta^{s-1} = \alpha(\alpha\beta\gamma)^{k-1}\alpha\beta\eta^{s-2} = 0$$

since $\alpha^2\beta = 0$. Likewise $\gamma\beta\gamma\alpha = 0$. Hence

$$\beta U = \beta((\alpha\beta\gamma) + (\beta\gamma\alpha) + (\gamma\alpha\beta)) = \beta\gamma\alpha\beta = ((\alpha\beta\gamma) + (\beta\gamma\alpha) + (\gamma\alpha\beta))\beta = U\beta$$

and

$$\gamma U = \gamma((\alpha\beta\gamma) + (\beta\gamma\alpha) + (\gamma\alpha\beta)) = \gamma\alpha\beta\gamma = \gamma((\alpha\beta\gamma) + (\beta\gamma\alpha) + (\gamma\alpha\beta)) = U\gamma.$$

Now,

$$\begin{aligned}
\eta U &= \eta((\alpha\beta\gamma) + (\beta\gamma\alpha) + (\gamma\alpha\beta)) \\
&= \eta(\gamma\alpha\beta) = (\gamma\alpha\beta)^{k-1}\gamma\alpha^2\beta \\
&= 0 \\
&= \gamma\alpha(\alpha\beta\gamma)^{k-1}\alpha\beta \\
&= \gamma\alpha\beta\eta \\
&= ((\alpha\beta\gamma) + (\beta\gamma\alpha) + (\gamma\alpha\beta))\eta \\
&= U\eta
\end{aligned}$$

and

$$\alpha U = \alpha((\alpha\beta\gamma) + (\beta\gamma\alpha) + (\gamma\alpha\beta)) = \alpha\beta\gamma\alpha = ((\alpha\beta\gamma) + (\beta\gamma\alpha) + (\gamma\alpha\beta))\alpha = U\alpha$$

Hence U is central, and then we only need to compute U^u to get the result. Finally, socle elements in basic algebras are always central, and 1 is central of course. We know by [7] that the centre is $2 + k + s$ -dimensional, and obtain therefore the result. ■

Remark 7. Skowroński and Erdmann show in [16, Lemma 5.7] that if K is an algebraically closed field, then $Q(2\mathfrak{B})_1^{k,s}(a, c) \simeq Q(2\mathfrak{B})_1^{k,s}(1, c')$ for some $c' \in K$ and if K is of characteristic different from 2, then $Q(2\mathfrak{B})_1^{k,s}(a, c) \simeq Q(2\mathfrak{B})_1^{k,s}(a, 0)$. We can examine their computations again to get slightly better results.

Suppose that K admits any k -th root, i.e. for all $x \in K$ there is $y \in K$ with $y^k = x$. We want to simplify the parameters a, c . Replace α by $x_\alpha\alpha$, β by $x_\beta\beta$, γ by $x_\gamma\gamma$ and η by $x_\eta\eta$ for non zero scalars $x_\alpha, x_\beta, x_\gamma, x_\eta$. Then the relations above are equivalent to

$$\begin{aligned}
x_\eta\beta\eta &= x_\alpha^k(x_\beta x_\gamma)^{k-1}(\alpha\beta\gamma)^{k-1}\alpha\beta, \\
x_\eta\eta\gamma &= x_\alpha^k(x_\beta x_\gamma)^{k-1}(\gamma\alpha\beta)^{k-1}\gamma\alpha, \\
x_\alpha^2\alpha^2 &= a \cdot x_\alpha^{k-1}(x_\beta x_\gamma)^k(\beta\gamma\alpha)^{k-1}\beta\gamma + c \cdot x_\alpha^k(x_\beta x_\gamma)^k(\beta\gamma\alpha)^k, \\
x_\gamma x_\beta \gamma \beta &= x_\eta^{s-1}\eta^{s-1}, \\
\alpha^2\beta &= 0, \\
\gamma\alpha^2 &= 0
\end{aligned}$$

We first choose x_β such that put $x_\beta x_\gamma = x_\eta^{s-1}$ to get the system

$$\begin{aligned}
\beta\eta &= x_\alpha^k x_\eta^{(k-1)(s-1)-1}(\alpha\beta\gamma)^{k-1}\alpha\beta, \\
\eta\gamma &= x_\alpha^k x_\eta^{(k-1)(s-1)-1}(\gamma\alpha\beta)^{k-1}\gamma\alpha, \\
\alpha^2 &= a \cdot x_\alpha^{k-3} x_\eta^{k(s-1)}(\beta\gamma\alpha)^{k-1}\beta\gamma + c \cdot x_\alpha^{k-2} x_\eta^{k(s-1)}(\beta\gamma\alpha)^k, \\
\gamma\beta &= \eta^{s-1}, \\
\alpha^2\beta &= 0, \\
\gamma\alpha^2 &= 0
\end{aligned}$$

Then we put $x_\alpha = x_\eta^{-\frac{(k-1)(s-1)-1}{k}}$ and obtain the system

$$\begin{aligned}
\beta\eta &= (\alpha\beta\gamma)^{k-1}\alpha\beta, \\
\eta\gamma &= (\gamma\alpha\beta)^{k-1}\gamma\alpha, \\
\alpha^2 &= a \cdot x_\eta^{-\frac{(k-1)(s-1)-1}{k} \cdot (k-3) + k(s-1)}(\beta\gamma\alpha)^{k-1}\beta\gamma + c \cdot x_\eta^{-\frac{(k-1)(s-1)-1}{k} \cdot (k-2) + k(s-1)}(\beta\gamma\alpha)^k, \\
\gamma\beta &= \eta^{s-1}, \\
\alpha^2\beta &= 0, \\
\gamma\alpha^2 &= 0
\end{aligned}$$

Now, $-\frac{(k-1)(s-1)-1}{k} \cdot (k-3) + k(s-1) = 0$ implies $k = 3$ and $s = 1$, which are excluded parameters, the algebra being defined only for $s \geq 3$. This number can be simplified and we therefore define $u(k, s) := (k-3) + 4(k-1)(s-1)$. Hence if there is an element $y_a \in K$ such that $y_a^{u(k,s)} = a^{k^2}$,

then we may choose such a parameter x_η , such that we may assume $a = 1$. This holds in particular if K is algebraically closed. We obtain this way $A(a, 0) \simeq A(1, 0)$ if K is sufficiently big, i.e. there is an element y_a satisfying $y_a^{u(k,s)} = a^{k^2}$. Moreover, $\frac{(k-1)(s-1)-1}{k}(k-2) + k(s-1) = 0$ implies $k = 2$ and $s = 1$. These are non admissible parameters, the algebra being defined only for $s \geq 3$. This last expression can be simplified and we define $v(k, s)$ such that $v(k, s) := (k-2) + (3k-2)(s-1)$. If there is $y_c \in K$ such that $y_c^{v(k,s)} = c^{k^2}$, then we can therefore choose an appropriate x_η such that we can assume $c = 1$ if $c \neq 0$. Again, this is trivially true if K is algebraically closed.

As a consequence, combining our computation and the result [16, Lemma 5.7], if K is algebraically closed of characteristic different from 2, then $Q(2\mathfrak{B})_1^{k,s}(a, c) \simeq Q(2\mathfrak{B})_1^{k,s}(1, 0)$.

Theorem 8. *Let $A(a, c)^{k,s}$ be the algebra $Q(2\mathfrak{B})_1^{k,s}(a, c)$ over a field K of characteristic p . Let $a, c \in K \setminus \{0\}$. We get the following cases.*

(1) *Suppose $p = 2$.*

(a) *If $k = 1$, then*

(i) *if s is even or if a is not a square in K , then*

$$\dim_K(T_1^\perp(A(a, c)^{k,s})) = \dim_K(T_1^\perp(A(a, 0)^{k,s}))$$

(ii) *if s is odd and a is a square in K , then*

$$\dim_K(T_1^\perp(A(a, c)^{k,s})) = \dim_K(T_1^\perp(A(a, 0)^{k,s})) - 1.$$

(b) *If $k > 1$ is odd, then*

(i) *if s is even and if c is a square in K , then*

$$\dim_K(T_1^\perp(A(a, c)^{k,s})) = \dim_K(T_1^\perp(A(a, 0)^{k,s}))$$

(ii) *if s is odd or if c is not a square in K , then*

$$\dim_K(T_1^\perp(A(a, c)^{k,s})) = \dim_K(T_1^\perp(A(a, 0)^{k,s})) - 1.$$

(c) *If k is even, then*

(i) *if c is a square in K , then*

$$\dim_K(T_1^\perp(A(a, c)^{k,s})) = \dim_K(T_1^\perp(A(a, 0)^{k,s}))$$

(ii) *if c is not a square in K , then*

$$\dim_K(T_1^\perp(A(a, c)^{k,s})) = \dim_K(T_1^\perp(A(a, 0)^{k,s})) - 1.$$

(2) *Suppose K is algebraically closed of characteristic $p = 2$.*

(a) *Then $A(a, 0)^{k,s} \simeq A(1, 0)^{k,s}$ and*

(i) *if k and s are even, then*

$$Z(A(a, 0)^{k,s})/T_1^\perp(A(a, 0)^{k,s}) \simeq K[U, Y, S]/(U^{k/2} - Y^{s/2}, S^2, UY, US, YS)$$

(ii) *if $k > 1$ or s is odd, then*

$$Z(A(a, 0)^{k,s})/T_1^\perp(A(a, 0)^{k,s}) \simeq K[U, Y, S]/(U^{\lceil k/2 \rceil}, Y^{\lceil s/2 \rceil}, S^2, UY, US, YS)$$

(b) *If $c \neq 0$, then*

(i) *if k and s are even, then*

$$Z(A(a, c)^{k,s})/T_1^\perp(A(a, c)^{k,s}) \simeq K[U, Y]/(U^{k/2} - Y^{s/2}, UY)$$

(ii) *if $k > 1$ or s is odd, then*

$$Z(A(a, c)^{k,s})/T_1^\perp(A(a, c)^{k,s}) \simeq K[U, Y]/(U^{\lceil (k+1)/2 \rceil}, Y^{\lceil (s+1)/2 \rceil}, UY)$$

(c) *If $k = 1$, then*

(i) *if (s is odd and $c = 0$) or (s is even and $c \neq 0$),*

$$Z(A(a, c)^{1,s})/T_1^\perp(A(a, c)^{1,s}) \simeq K[Y, S]/(Y^{\lceil (s+1)/2 \rceil}, S^2, YS)$$

(ii) *if (s is odd and $c \neq 0$) or (s is even and $c = 0$),*

$$Z(A(a, c)^{1,s})/T_1^\perp(A(a, c)^{1,s}) \simeq K[Y]/(Y^{\lceil s/2 \rceil})$$

(3) *If $p \neq 2$, and if K is algebraically closed, then $A(a, c)^{k,s} \simeq A(1, 0)^{k,s}$.*

- (4) If $p > 3$ or $n \geq 2$, then the dimension of the Külshammer ideal $T_n(A(a, c)^{k, s})^\perp$ does not depend on the parameters a, c .

Suppose that K is algebraically closed. Then $A(a, 0) \simeq A(1, 0)$ and if $c \neq 0$ then $A(a, c) \simeq A(a', 1) \simeq A(1, c')$ for some $a', c' \in K^\times$.

Remark 9. The hypothesis that K is algebraically closed is too strong. A more precise, somewhat technical statement is given at the end of Remark 7.

Remark 10. The case $p = 3$ is special if K is not algebraically closed. Then the first Külshammer ideal detects if the parameter a is a third power in case k or s is divisible by 3. This shows that the isomorphism $A(a, 0) \simeq A(1, 0)$ for algebraically closed fields really uses at least perfect base fields.

Proof of Theorem 8. The isomorphisms $A(a, 0) \simeq A(1, 0)$ and if $c \neq 0$ then $A(1, c') \simeq A(a, c) \simeq A(a', 1)$ for some $a', c' \in K^\times$ follow from Remark 7. We first determine a basis of $Q(\mathfrak{B})_1^{k, s}(a, c)$.

$$\mathcal{B}_1 := \{\alpha(\beta\gamma\alpha)^n, (\beta\gamma\alpha)^n\beta\gamma, (\beta\gamma\alpha)^m, e_1(\alpha\beta\gamma)^\ell \mid 0 \leq \ell \leq k, 1 \leq m \leq k-1, 0 \leq n \leq k-1\}$$

$$\mathcal{B}_2 := \{e_2\eta^t, (\gamma\alpha\beta)^m \mid 0 \leq t \leq s, 1 \leq m \leq k-1\}$$

$$\mathcal{B}_3 := \{(\beta\gamma\alpha)^n\beta, \alpha(\beta\gamma\alpha)^n\beta \mid 0 \leq n \leq k-1\}$$

$$\mathcal{B}_4 := \{(\gamma\alpha\beta)^n\gamma, (\gamma\alpha\beta)^n\gamma\alpha \mid 0 \leq n \leq k-1\}$$

which is a generating set, using the known Cartan matrix.

As a next step we need to compute the commutator space. Clearly, non closed paths are commutators, since if $e_i p e_j \neq 0$ for some path p and $e_i \neq e_j$, then $p = e_i p - p e_i$. Hence $\mathcal{B}_3 \cup \mathcal{B}_4 \subseteq [A(a, c), A(a, c)]$. Moreover,

$$\begin{aligned} \alpha(\beta\gamma\alpha)^n &= \alpha(\beta\gamma\alpha)^n - (\beta\gamma\alpha)^n\alpha \in [A(a, c), A(a, c)] \quad \forall n \geq 1 \\ (\beta\gamma\alpha)^m &= (\alpha\beta\gamma)^m = (\gamma\alpha\beta)^m \in A(a, c)/[A(a, c), A(a, c)] \quad \forall m \geq 0 \\ (\beta\gamma\alpha)^m\beta\gamma &= \gamma(\beta\gamma\alpha)^m\beta = 0 \in A(a, c)/[A(a, c), A(a, c)] \quad \forall m \geq 1 \end{aligned}$$

and hence

$$\mathcal{B}_{comm} := \{\alpha, (\alpha\beta\gamma)^m, \eta^t, e_1, e_2 \mid 1 \leq t \leq s-1, 1 \leq m \leq k\}$$

is a generating set of $A(a, c)/[A(a, c), A(a, c)]$. Since the dimension of the centre of $A(a, c)$ equals the dimension of $A(a, c)/[A(a, c), A(a, c)]$, the algebra $A(a, c)$ being symmetric, and both are of dimension $2 + k + s$, by [7], we get that \mathcal{B}_{comm} is actually a basis of $A(a, c)/[A(a, c), A(a, c)]$.

We only need to work in $A(a, c)/[A(a, c), A(a, c)]$, and therefore we need to consider linear combinations of paths in \mathfrak{B}_{comm} only. We can omit idempotents, since computing modulo the radical these idempotents remain idempotents, and are hence never nilpotent modulo commutators. Hence we only need to consider linear combinations of elements in the set

$$\{\eta^t, \alpha, (\beta\gamma\alpha)^m \mid 1 \leq t \leq s-1, 1 \leq m \leq k\}.$$

We deal with the case $p = 2$. Let hence $p = 2$. In the commutator quotient squaring is semilinear (cf e.g. [13], [23, Lemma 2.9.3]).

If $k > 1$ is odd, then

$$\begin{aligned} 0 &= \left(\sum_{t=1}^{s-1} x_t \eta^t + u\alpha + \sum_{m=1}^k y_m (\beta\gamma\alpha)^m \right)^2 \\ &= \sum_{t=1}^{s-1} x_t^2 \eta^{2t} + u^2 c (\beta\gamma\alpha)^k + \sum_{m=1}^k y_m^2 (\beta\gamma\alpha)^{2m} \\ &= \sum_{1 \leq t \leq s/2} x_t^2 \eta^{2t} + u^2 c (\beta\gamma\alpha)^k + \sum_{m=1}^{(k-1)/2} y_m^2 (\beta\gamma\alpha)^{2m} \end{aligned}$$

which implies $x_t = 0$ for all $t \leq \frac{s}{2}$, $y_m = 0$ for all $m \leq \frac{k-1}{2}$. If $c = 0$ then there is no other constraint. Suppose $c \neq 0$. Then, if s is odd we get $u = 0$. If s is even, then $x_{s/2}^2 + cu^2 = 0$ which has a solution if and only if c is a square in K . Hence, computing in $A(a, c)/[A(a, c), A(a, c)]$ we get

$$T_1(A(a, c)) = \begin{cases} \langle \alpha, \eta^t, (\beta\gamma\alpha)^m \mid t > \frac{s}{2}, m > \frac{k-1}{2} \rangle_K & \text{if } c = 0 \\ \langle \eta^t, (\beta\gamma\alpha)^m \mid t > \frac{s}{2}, m > \frac{k-1}{2} \rangle_K & \text{if } s \text{ is odd and } c \neq 0 \\ \langle \eta^t, (\beta\gamma\alpha)^m \mid t > \frac{s}{2}, m > \frac{k-1}{2} \rangle_K & \text{if } s \text{ is even and } 0 \neq c \notin K^2 \\ \langle \eta^t, (\beta\gamma\alpha)^m, \eta^{s/2} + d\alpha \mid t > \frac{s}{2}, m > \frac{k-1}{2} \rangle_K & \text{if } s \text{ is even and } 0 \neq c = d^2 \end{cases}$$

If k is even, then

$$\begin{aligned} 0 &= \left(\sum_{t=1}^{s-1} x_t \eta^t + u\alpha + \sum_{m=1}^k y_m (\beta\gamma\alpha)^m \right)^2 \\ &= \sum_{t=1}^{s-1} x_t^2 \eta^{2t} + u^2 c (\beta\gamma\alpha)^k + \sum_{m=1}^k y_m^2 (\beta\gamma\alpha)^{2m} \\ &= \sum_{1 \leq t \leq (s-1)/2} x_t^2 \eta^{2t} + u^2 c (\beta\gamma\alpha)^k + \sum_{m=1}^{k/2} y_m^2 (\beta\gamma\alpha)^{2m} \end{aligned}$$

which implies $y_m = 0$ for $1 \leq m < k/2$ and $x_t = 0$ for $t \leq \frac{s-1}{2}$.

If $c = 0$, then $x_{s/2} + y_{k/2} = 0$ in case s is even, and $y_{k/2} = 0$ in case s is odd.

Suppose $c \neq 0$. If s is odd, then $y_{k/2}^2 + cu^2 = 0$, and if s is even, then $y_{k/2}^2 + x_{s/2}^2 + cu^2 = 0$. Again $y_{k/2}^2 + cu^2 = 0$ and $y_{k/2}^2 + x_{s/2}^2 + cu^2 = 0$ has non zero solutions if and only if c is a square.

Computing in $A(a, c)/[A(a, c), A(a, c)]$ we get

$$T_1(A(a, c)) = \begin{cases} \langle \alpha, \eta^t, (\beta\gamma\alpha)^m \mid t > \frac{s-1}{2}, m > \frac{k}{2} \rangle_K & \text{if } s \text{ is odd and } c = 0 \\ \langle \alpha, \eta^{s/2} + (\beta\gamma\alpha)^{k/2}, \eta^t, (\beta\gamma\alpha)^m \mid t > \frac{s}{2}, m > \frac{k}{2} \rangle_K & \text{if } s \text{ is even and } c = 0 \\ \langle \eta^t, (\beta\gamma\alpha)^m \mid t > \frac{s-1}{2}, m > \frac{k}{2} \rangle_K & \text{if } s \text{ is odd and } 0 \neq c \notin K^2 \\ \langle \eta^t, (\beta\gamma\alpha)^m, (\beta\gamma\alpha)^{k/2} + d\alpha \mid t > \frac{s-1}{2}, m > \frac{k}{2} \rangle_K & \text{if } s \text{ is odd and } 0 \neq c = d^2 \\ \langle \eta^t, (\beta\gamma\alpha)^m, (\beta\gamma\alpha)^{k/2} + \eta^{s/2} \mid t > \frac{s}{2}, m > \frac{k}{2} \rangle_K & \text{if } s \text{ is even and } 0 \neq c \notin K^2 \\ \langle \eta^t, (\beta\gamma\alpha)^m, (\beta\gamma\alpha)^{k/2} + \eta^{s/2}, \eta^{s/2} + d\alpha \mid t > \frac{s}{2}, m > \frac{k}{2} \rangle_K & \text{if } s \text{ is even and } 0 \neq c = d^2 \end{cases}$$

If $k = 1$, then, since $\beta\gamma = \gamma\beta = \eta^{s-1}$ in the commutator quotient,

$$0 = \left(\sum_{t=1}^{s-1} x_t \eta^t + u\alpha + y_1 (\beta\gamma\alpha) \right)^2 = \sum_{t=1}^{s-1} x_t^2 \eta^{2t} + u^2 a \eta^{s-1} + u^2 c (\beta\gamma\alpha)$$

which implies $x_t = 0$ for $1 \leq t \leq \frac{s-2}{2}$.

If $c = 0$, then $x_{s/2} = 0$ in case s is even, and $x_{(s-1)/2}^2 + au^2 = 0$ in case s is odd. This last equation has non zero solutions if and only if $a \in K^2$.

Suppose $c \neq 0$. If s is even, then $x_{s/2}^2 + cu^2 = 0$. This has non zero solutions if and only if $c \in K^2$. If s is odd, then $cu^2 = 0$ and $x_{(s-1)/2}^2 + au^2 = 0$. Hence s odd implies $u = 0 = x_{(s-1)/2}$. Computing again in $A(a, c)/[A(a, c), A(a, c)]$,

$$T_1(A(a, c)) = \begin{cases} \langle \eta^{(s-1)/2} + b\alpha, \eta^t, (\beta\gamma\alpha) \mid t > \frac{s-1}{2} \rangle_K & \text{if } s \text{ is odd, } a = b^2 \text{ and } c = 0 \\ \langle \eta^t, (\beta\gamma\alpha) \mid t > \frac{s-1}{2} \rangle_K & \text{if } s \text{ is odd, } a \notin K^2 \text{ and } c = 0 \\ \langle \eta^t, (\beta\gamma\alpha) \mid t > \frac{s}{2} \rangle_K & \text{if } s \text{ is even and } c = 0 \\ \langle \eta^t, (\beta\gamma\alpha) \mid t > \frac{s-1}{2} \rangle_K & \text{if } s \text{ is odd and } c \neq 0 \\ \langle \eta^t, (\beta\gamma\alpha) \mid t > \frac{s}{2} \rangle_K & \text{if } s \text{ is even and } 0 \neq c \notin K^2 \\ \langle \eta^t, (\beta\gamma\alpha), \eta^{s/2} + d\alpha \mid t > \frac{s}{2} \rangle_K & \text{if } s \text{ is even and } 0 \neq c = d^2 \end{cases}$$

It is easy to see that computing $T_n(A(a, c)/[A(a, c), A(a, c)])$ for $n \geq 2$ (and any $p \geq 2$ in this case) yields expressions which are independent of a, c .

In order to compute the Külshammer ideal $T_1(A(a, c))^\perp$ we need to give the symmetrising form of $A(a, c)$. In order to do so, we use Proposition 2 and Proposition 3, and therefore we will assume that

K is an algebraically closed field, even though it might not be strictly necessary for all arguments. We define a trace map

$$A(a, c) \xrightarrow{\psi} K$$

by

$$\psi(\eta^s) = (\alpha\beta\gamma)^k = 1, \text{ and } \psi(x) = 0 \text{ if } x \notin \text{soc}(A(a, c)).$$

The fact that ψ defines a symmetric form since $\eta^s = \eta\gamma\beta = (\gamma\alpha\beta)^k$ and hence

$$\eta^s - (\alpha\beta\gamma)^k = (\gamma\alpha\beta)^k - (\alpha\beta\gamma)^k = [\gamma, \alpha\beta(\gamma\alpha\beta)^{k-1}]$$

is a commutator.

Recall from Lemma 6 that $\dim_K(Z(A(a, c))) = k + s + 2$. We proceed case by case.

$k > 1$ **odd** and $c = 0$: Recall that in this case

$$T_1(A(a, c))/[A(a, c), A(a, c)] = \left\langle \alpha, \eta^t, (\beta\gamma\alpha)^m \mid t > \frac{s}{2}; m > \frac{k-1}{2} \right\rangle_K$$

Hence

$$\begin{aligned} \dim_K(T_1(A(a, 0))/[A(a, 0), A(a, 0)]) &= \begin{cases} 1 + \frac{s}{2} + \frac{k+1}{2} - 1 & \text{if } s \text{ is even} \\ 1 + \frac{s+1}{2} + \frac{k+1}{2} - 1 & \text{if } s \text{ is odd} \end{cases} \\ &= \begin{cases} \frac{s}{2} + \frac{k+1}{2} & \text{if } s \text{ is even} \\ \frac{s+1}{2} + \frac{k+1}{2} & \text{if } s \text{ is odd} \end{cases} \end{aligned}$$

observing that $(\beta\gamma\alpha)^k - \eta^s \in [A(a, c), A(a, c)]$. Therefore

$$\begin{aligned} \dim_K(T_1(A(a, 0))^\perp) &= k + s + 2 - \begin{cases} \frac{s}{2} + \frac{k+1}{2} & \text{if } s \text{ is even} \\ \frac{s+1}{2} + \frac{k+1}{2} & \text{if } s \text{ is odd} \end{cases} \\ &= \begin{cases} \frac{s}{2} + 1 + \frac{k+1}{2} & \text{if } s \text{ is even} \\ \frac{s+1}{2} + \frac{k+1}{2} & \text{if } s \text{ is odd} \end{cases} \end{aligned}$$

But, in case s is even,

$$\left\{ \eta^t, (\alpha\beta\gamma)^u + (\beta\gamma\alpha)^u + (\gamma\alpha\beta)^u, (\alpha\beta\gamma)^k \mid u \geq \frac{k+1}{2}, t \geq \frac{s}{2} \right\} \subseteq T_1(A(a, c))^\perp,$$

and in case s is odd,

$$\left\{ \eta^t, (\alpha\beta\gamma)^u + (\beta\gamma\alpha)^u + (\gamma\alpha\beta)^u, (\alpha\beta\gamma)^k \mid u \geq \frac{k+1}{2}, t \geq \frac{s+1}{2} \right\} \subseteq T_1(A(a, c))^\perp.$$

This is a basis of a subspace of the centre of the dimension as required, and hence the set above set is a basis of $T_1(A(a, c))^\perp$. Hence, with these parameters, if s is even then

$$Z(A(a, c))/T_1^\perp(A(a, c)) \simeq K[U, Y, S]/(Y^{s/2}, U^{(k+1)/2}, S^2, YS, US, UY)$$

and if s is odd, then

$$Z(A(a, c))/T_1^\perp(A(a, c)) \simeq K[U, Y, S]/(Y^{(s+1)/2}, U^{(k+1)/2}, S^2, YS, US, UY).$$

$k > 1$ **odd**, $c \neq 0$, and s is **odd**: Recall

$$T_1(A(a, c))/[A(a, c), A(a, c)] = \left\langle \eta^t, (\beta\gamma\alpha)^m \mid t > \frac{s}{2}; m > \frac{k-1}{2} \right\rangle_K.$$

In this case we get $\alpha^2 \in T_1(A(a, c))^\perp$, and using the preceding discussion we get that

$$\left\{ \alpha^2, (\beta\gamma\alpha)^k, \eta^t, (\alpha\beta\gamma)^u + (\beta\gamma\alpha)^u + (\gamma\alpha\beta)^u \mid u \geq \frac{k+1}{2}, t \geq \frac{s+1}{2} \right\}$$

is a K -basis of $T_1(A(a, c))^\perp$. Hence in this case

$$Z(A(a, c))/T_1^\perp(A(a, c)) \simeq K[U, Y]/(Y^{(s+1)/2}, U^{(k+1)/2}, UY)$$

$k > 1$ **odd**, $d^2 = c \neq 0$, and s is **even**. Recall

$$T_1(A(a, c))/[A(a, c), A(a, c)] = \left\langle d\alpha + \eta^{s/2}, \eta^t, (\beta\gamma\alpha)^m \mid t > \frac{s}{2}; m > \frac{k-1}{2} \right\rangle_K.$$

Then

$$\left(\eta^{s/2} + d\alpha\right) \cdot \left(\eta^{s/2} - \frac{d}{ca}\alpha^2\right) = \eta^s + \frac{d^2}{ca}\alpha^3 = \eta^s + \frac{c}{ca}\alpha^3 = \eta^s + \frac{1}{a} \cdot a(\alpha\beta\gamma)^k$$

and this maps to 0 by ψ . Hence,

$$\left\{ d\alpha^2 + \eta^{s/2}, (\alpha\beta\gamma)^k, \eta^t, (\alpha\beta\gamma)^u + (\beta\gamma\alpha)^u + (\gamma\alpha\beta)^u \mid u \geq \frac{k-1}{2}, t \geq \frac{s}{2} + 1 \right\}$$

is a K -basis of $T_1(A(a, c))^\perp$. Therefore in this case

$$Z(A(a, c))/T_1^\perp(A(a, c)) \simeq K[U, Y]/(Y^{\frac{s}{2}+1}, U^{(k+1)/2}, UY)$$

k even and $c = 0$ and s is odd. Recall

$$T_1(A(a, c))/[A(a, c), A(a, c)] = \left\langle d\alpha + (\beta\gamma\alpha)^{k/2}, \eta^t, (\beta\gamma\alpha)^m \mid t > \frac{s-1}{2}; m > \frac{k}{2} \right\rangle_K.$$

Then the discussion of the case $k > 1$ odd and $c = 0$ shows that

$$\left\{ \eta^t, (\alpha\beta\gamma)^k, (\alpha\beta\gamma)^u + (\beta\gamma\alpha)^u + (\gamma\alpha\beta)^u \mid u \geq \frac{k}{2}, t \geq \frac{(s+1)}{2} \right\}$$

is a K -basis of $T_1(A(a, c))^\perp$. Hence in this case

$$Z(A(a, c))/T_1^\perp(A(a, c)) \simeq K[U, Y, S]/(Y^{(s+1)/2}, U^{k/2}, S^2, YS, US, UY)$$

k even and $c = 0$ and s is even. Recall

$$T_1(A(a, c))/[A(a, c), A(a, c)] = \left\langle \alpha, \eta^{s/2} + (\beta\gamma\alpha)^{k/2}, \eta^t, (\beta\gamma\alpha)^m \mid t > \frac{s}{2}; m > \frac{k}{2} \right\rangle_K.$$

Then

$$\left\{ (\alpha\beta\gamma)^k, \eta^t, (\alpha\beta\gamma)^u + (\beta\gamma\alpha)^u + (\gamma\alpha\beta)^u, \eta^{s/2} + (\beta\gamma\alpha)^{k/2} \mid u \geq \frac{k}{2} + 1, t \geq \frac{s}{2} + 1 \right\}$$

is a K -basis of $T_1(A(a, c))^\perp$. Hence in this case

$$Z(A(a, c))/T_1^\perp(A(a, c)) \simeq K[U, Y, S]/(Y^{s/2} - U^{k/2}, S^2, YS, US, UY)$$

k even, $0 \neq c = d^2$, s odd: Recall

$$T_1(A(a, c))/[A(a, c), A(a, c)] = \left\langle d\alpha + (\beta\gamma\alpha)^{k/2}, \eta^t, (\beta\gamma\alpha)^m \mid t > \frac{s-1}{2}; m > \frac{k}{2} \right\rangle_K.$$

Since $\dim(Z(A)) = k + s + 2$, and since

$$\dim(T_1(A(a, c)))/[A(a, c), A(a, c)] = \frac{k}{2} + \frac{s+1}{2},$$

we get

$$\dim(T_1(A(a, c))^\perp) = 2 + k + s - \frac{k}{2} - \frac{s+1}{2} = \frac{k}{2} + \frac{s+1}{2}.$$

Moreover,

$$\left(\frac{d}{c}\alpha^2 + U^{k/2}\right) \cdot \left(d\alpha + (\beta\gamma\alpha)^{k/2}\right) = \frac{d^2}{c}\alpha^3 + dU^{k/2}\alpha + \frac{d}{c}(\beta\gamma\alpha)^{k/2} + (\beta\gamma\alpha)^k$$

and this maps to 0 by the map ψ . Therefore

$$\left\{ (\beta\gamma\alpha)^k, \frac{d}{c}\alpha^2 + (\alpha\beta\gamma)^{k/2} + (\beta\gamma\alpha)^{k/2} + (\gamma\alpha\beta)^{k/2}, \eta^t, (\alpha\beta\gamma)^u + (\beta\gamma\alpha)^u + (\gamma\alpha\beta)^u \mid u \geq \frac{k}{2} + 1, t \geq \frac{s+1}{2} \right\}$$

is a K -basis of $T_1(A(a, c))^\perp$, and therefore

$$Z(A(a, c))/T_1(A(a, c))^\perp \simeq K[U, Y]/(U^{k/2}, Y^{(s+1)/2}, UY).$$

k even, $c = d^2 \neq 0$, and s even: Recall

$$T_1(A(a, c))/[A(a, c), A(a, c)] = \left\langle d\alpha + \eta^{s/2}, (\beta\gamma\alpha)^{k/2} + \eta^{s/2}, \eta^t, (\beta\gamma\alpha)^m \mid t > \frac{s}{2}; m > \frac{k}{2} \right\rangle_K.$$

Then

$$\left\{ \alpha^2 + d\eta^{s/2}, (\beta\gamma\alpha)^k, \eta^t, (\alpha\beta\gamma)^u + (\beta\gamma\alpha)^u + (\gamma\alpha\beta)^u, \eta^{s/2} + (\beta\gamma\alpha)^{k/2} \mid u \geq \frac{k}{2} + 1, t \geq \frac{s}{2} + 1 \right\}$$

is a K -basis of $T_1(A(a, c))^\perp$. Hence in this case

$$Z(A(a, c))/T_1^\perp(A(a, c)) \simeq K[U, Y]/(Y^{s/2} - U^{k/2}, UY)$$

If $k = 1$ and $c = 0$ and s odd: Since $\dim(Z(A(a, c))) = 3 + s$, and since

$$\dim(T_1(A(a, c))/[A(a, c), A(a, c)]) = 3 + \frac{s-1}{2},$$

we obtain $\dim(T_1(A(a, c))^\perp) = \frac{s+1}{2}$. Observe that $\eta^s = (\alpha\beta\gamma) + (\beta\gamma\alpha) + (\gamma\alpha\beta) = U$. Then we get

$$\left\{ \beta\gamma\alpha, \eta^t, \mid t \geq \frac{s+1}{2} \right\}$$

is a K -basis of $T_1(A(a, c))^\perp$. Therefore

$$Z(A(a, c))/T_1^\perp(A(a, c)) \simeq K[Y, S]/(Y^{(s+1)/2}, S^2, YS)$$

If $k = 1$ and $c = 0$ and s even: Since $\dim(Z(A(a, c))) = 3 + s$, and since

$$\dim(T_1(A(a, c))/[A(a, c), A(a, c)]) = 1 + \frac{s}{2},$$

we obtain $\dim(T_1(A(a, c))^\perp) = 2 + \frac{s}{2}$. Hence

$$\left\{ \alpha^2, \beta\gamma\alpha, \eta^t, \mid t \geq \frac{s}{2} \right\}$$

is a K -basis of $T_1(A(a, c))^\perp$ and

$$Z(A(a, c))/T_1^\perp(A(a, c)) \simeq K[Y]/Y^{s/2}.$$

If $k = 1$ and $c \neq 0$ and s odd: Since $\dim(Z(A(a, c))) = 3 + s$, and since

$$\dim(T_1(A(a, c))/[A(a, c), A(a, c)]) = 2 + \frac{s-1}{2},$$

we obtain $\dim(T_1(A(a, c))^\perp) = 1 + \frac{s+1}{2}$. Hence

$$\left\{ \alpha^2, \beta\gamma\alpha, \eta^t, \mid t \geq \frac{s+1}{2} \right\}$$

is a K -basis of $T_1(A(a, c))^\perp$ and

$$Z(A(a, c))/T_1^\perp(A(a, c)) \simeq K[Y]/Y^{(s+1)/2}.$$

If $k = 1$ and $c \neq 0$ and s even: Since $\dim(Z(A(a, c))) = 3 + s$, and since

$$\dim(T_1(A(a, c))/[A(a, c), A(a, c)]) = 2 + \frac{s}{2},$$

we obtain $\dim(T_1(A(a, c))^\perp) = 1 + \frac{s}{2}$. Hence

$$\left\{ \beta\gamma\alpha, \eta^t, \mid t \geq 1 + \frac{s}{2} \right\}$$

is a K -basis of $T_1(A(a, c))^\perp$ and

$$Z(A(a, c))/T_1^\perp(A(a, c)) \simeq K[Y, S]/(Y^{(s+2)/2}, S^2, YS).$$

■

Corollary 11. *Let K be an algebraically closed field of characteristic $p \in \mathbb{N} \cup \{\infty\}$, and let a, a', c be non-zero elements in K , and let $c', c'' \in K$.*

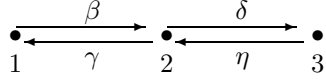
- If $p \neq 2$, then $Q(2\mathfrak{B})_1^{k,s}(a, c) \simeq Q(2\mathfrak{B})_1^{k,s}(1, 0)$.
- If $p = 2$, then $D^b(Q(2\mathfrak{B})_1^{k,s}(a, c)) \not\simeq D^b(Q(2\mathfrak{B})_1^{k,s}(a', 0))$.
- For any characteristic of K we get

$$\left(D^b(Q(2\mathfrak{B})_1^{k,s}(a, c'')) \simeq D^b(Q(2\mathfrak{B})_1^{k',s'}(a', c')) \right) \Rightarrow ((k = k' \text{ and } s = s') \text{ or } (k = s' \text{ and } s = k')).$$

Proof. The first statement is an immediate consequence of Theorem 8 item (2) and [16, Lemma 5.7]. The second statement follows from Theorem 8 item (2)(a), (2)(b) and the main result in [19]. Indeed, the isomorphism type of the centre modulo the first Külshammer ideal differs in case $c = 0$ and $c \neq 0$. The third statement follows from Remark 5. ■

2.2. Three simple modules. For three simple modules there are two families, $Q(3\mathcal{K})^{a,b,c}$ and $Q(3\mathcal{A})_1^{2,2}(d)$. According to [7] the derived classification of the case $Q(3\mathcal{K})^{a,b,c}$ is complete, whereas the classification for the case $Q(3\mathcal{A})_1^{2,2}(d)$ is complete up to the scalar $d \in K \setminus \{0, 1\}$.

The quiver $3\mathcal{A}$ is

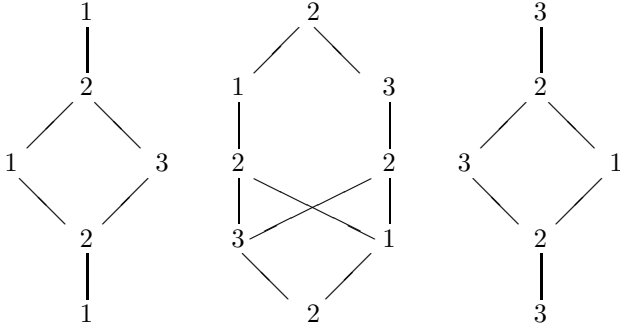


$B(d) := Q(3\mathcal{A})_1^{2,2}(d)$ is the quiver algebra modulo the relations

$$\beta\delta\eta = \beta\gamma\beta, \delta\eta\gamma = \gamma\beta\gamma, \eta\gamma\beta = d \cdot \eta\delta\eta, \gamma\beta\delta = d \cdot \delta\eta\delta, \beta\delta\eta\delta = 0, \eta\gamma\beta\gamma = 0$$

for $d \in K \setminus \{0, 1\}$.

Following [4] the Cartan matrix of $B(d)$ is $\begin{pmatrix} 4 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix}$ and the centre is 6-dimensional. The Loewy series of the projective indecomposable modules are given below.



We obtain a basis of the socle of $B(d)$ by

$$\{s_1 := \beta\delta\eta\gamma, s_2 := \eta\gamma\beta\delta, s_3 := \gamma\beta\delta\eta\}.$$

The closed paths of the algebra are

$$\{e_0, e_1, e_2, \beta\gamma, \gamma\beta, \delta\eta, \eta\delta, \beta\delta\eta\gamma, \eta\gamma\beta\delta, \gamma\beta\delta\eta\}.$$

The centre is formed by linear combinations of closed paths and has a basis

$$\{1, \beta\gamma + \gamma\beta + \frac{1}{d}\eta\delta, \beta\gamma + \delta\eta + \eta\delta, \beta\delta\eta\gamma, \eta\gamma\beta\delta, \gamma\beta\delta\eta\}$$

as is easily verified. Non closed paths are clearly commutators. Obviously

$$\beta\delta\eta\gamma \equiv \eta\gamma\beta\delta \equiv \gamma\beta\delta\eta \pmod{[B(d), B(d)]}.$$

Moreover,

$$\beta\gamma - \gamma\beta \in [B(d), B(d)] \text{ and } \delta\eta - \eta\delta \in [B(d), B(d)].$$

Since the dimension of the centre of $B(d)$, we get a basis of $B(d)/[B(d), B(d)]$ by

$$\{e_0, e_1, e_2, \beta\gamma, \delta\eta, \beta\delta\eta\gamma\}.$$

We now suppose that the characteristic p of K is $p = 2$. If x is a square in K , then denote $y = \sqrt{x}$ if $y^2 = x$. We compute

$$(\beta\gamma)^2 = \beta\gamma\beta\gamma = \beta\delta\eta\gamma \text{ and } (\delta\eta)^2 = \delta\eta\delta\eta = \frac{1}{d}\delta\eta\gamma\beta.$$

If d is a square in K , then

$$((\gamma\beta) + \sqrt{d}(\eta\delta))^2 = \gamma\eta\gamma\beta + d\eta\delta\eta\delta = \delta\eta\gamma\beta + \eta\gamma\beta\delta \in [B(d), B(d)]$$

so that $T_1(B(d))/[B(d), B(d)]$ is 1-dimensional. If d is not a square, then $T_1(B(d)) = [B(d), B(d)]$.

Let us consider the centre. Denote $\beta\gamma + \gamma\beta + \frac{1}{d}\eta\delta = x$ and $\beta\gamma + \delta\eta + \eta\delta = y$. Then we get

$$\begin{aligned} x^2 &= (\beta\gamma + \gamma\beta + \frac{1}{d}\eta\delta)^2 = \beta\delta\eta\gamma + \frac{1}{d}\delta\eta\gamma\beta + \frac{1}{d}\eta\gamma\beta\delta = s_1 + \frac{1}{d}s_2 + \frac{1}{d}s_2 \\ y^2 &= (\beta\gamma + \delta\eta + \eta\delta)^2 = \beta\delta\eta\gamma + \delta\eta\gamma\beta + \frac{1}{d^3}\eta\gamma\beta\delta = s_1 + s_2 + \frac{1}{d^3}s_3 \\ xy &= (\beta\gamma + \delta\eta + \eta\delta) \cdot (\beta\gamma + \gamma\beta + \frac{1}{d}\eta\delta) = \beta\delta\eta\gamma + \delta\eta\gamma\beta + \frac{1}{d^2}\eta\gamma\beta\delta = s_1 + s_2 + \frac{1}{d^2}s_3 \end{aligned}$$

The coefficient matrix above has determinant $\frac{(d-1)^2}{d^4}$ and since $d \neq 1$, the elements x^2, y^2, xy are linearly independent, and hence $Z(B(d)) \simeq K[x, y]/(x^3, y^3, x^2y, xy^2)$. Moreover, choose the Frobenius form given by Proposition 3, and an elementary computation gives that $T_1^\perp(B(d))$ has a basis $\{x, s_1, s_2, s_3\}$, and therefore $Z(B(d))/T_1^\perp(B(d)) \simeq K[y]/y^2$, independently of d .

Theorem 12. *Let K be a field of characteristic 2, and let $B(d)$ be the algebra $Q(3\mathcal{A})_1^{2,2}(d)$. Then $\dim_K(T_1^\perp(B(d))/R(B(d))) = 1$ if d is a square in K , and $\dim_K(T_1^\perp(B(d))/R(B(d))) = 0$ if d is not a square in K .*

Proof: is done above. ■

Remark 13. If K is perfect of characteristic 2, then all elements of K are squares, by definition. However, Theorem 1 needs that K is perfect for derived equivalences and K is even algebraically closed for stable equivalences of Morita type. The hypothesis that K is perfect is really essential for the proofs. Hence, the case where the Külshammer ideal distinguishes the algebras in Theorem 12 does not allow to distinguish the derived equivalence classes. If K is not perfect we do not know if these algebras are derived equivalent or not.

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