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# Typology of Axioms for a Weighted Modal Logic

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## Abstract

In a weighted modal logics framework, this paper studies the definition of weighted extensions for the classical modal axioms. It discusses the notion of relevant weight values, in a specific weighted Kripke semantics and exploiting accessibility relation properties. Different generalisations of the classical axioms are constructed and, from these, a typology of weighted axioms is built, distinguishing between four types, depending on their relations to their classical counterparts and to the, possibly equivalent, frame conditions.

## 1 Introduction

Weighted extensions of modal logics aim at increasing their expressiveness by enriching the two classical modal operators,  $\Box$  and  $\Diamond$ , with integer or real valued degrees. These extensions are based on infinitely many weighted modal operators  $\Box_\alpha$  and  $\Diamond_\alpha$ ,  $\alpha$  denoting the numerical weights. These modalities make it possible to introduce fine distinctions among the pieces of knowledge modeled in the formalism, which can then be used to infer nuanced new knowledge and thus allow, for example, reasoning on partial beliefs.

In this framework, this paper studies weighted extensions of the classical modal axioms: these, which can be seen as defining rules for the combination of the modal operators  $\Box$  and  $\Diamond$ , establish relations between formulae in which they occur once, repeatedly or in combination. For instance the classical axiom (4), written  $\vdash \Box\varphi \rightarrow \Box\Box\varphi$ , states that an implication holds between a single occurrence and repetitions of  $\Box$ . Similarly, axiom (D),  $\vdash \Box\varphi \rightarrow \Diamond\varphi$ , establishes a relation between the two modal operators.

This paper first proposes a semantic interpretation for  $\Box_\alpha$  and  $\Diamond_\alpha$  in the framework of Kripke's semantics, based on a relative counting of accessible validating worlds that relaxes the condition on the universal quantifier defining  $\Box$  in Kripke's semantics. The proposed semantics offers the advantage of being informative enough to serve as a basis for the definition of weighted axioms.

The paper then examines the transposition of these axioms to the case of this weighted modal logic, setting rules for the combination of the weighted modal operators  $\Box_\alpha$  and  $\Diamond_\alpha$ .

Starting with candidate weighted axioms, obtained by replacing each modality of a classical axiom with a weighted one, each with its own weight, the paper discusses how these weights depend on each other. This issue can be illustrated by axiom (D), whose associated weighted candidate takes the form  $\vdash \Box_\alpha\varphi \rightarrow \Diamond_\beta\varphi$ . The question is then to establish a relevant valuation for  $\beta$  depending on  $\alpha$  (or reciprocally).

We propose to address this task from a semantic point of view, interpreting the candidates in the particular weighted Kripke semantic we propose. The approach we apply identifies weight dependencies which hold either in any frame or under specific frame conditions. Moreover, we also study whether the frames in which the obtained axioms hold all satisfy specific conditions. This can be considered as opening the way to the definition of a weighted correspondence theory. Note that the aim here is not to build an axiomatisation of the proposed weighted modal logic semantics, but to study the transposition of classical axioms to the weighted case.

We then establish a typology of weighted modal axioms that distinguishes between four types, depending on their relation to their classic counterparts and to the frame conditions the latter correspond to: type I groups axioms that cannot be relaxed using the degrees of freedom offered by the proposed weights. Type II is made of weighted axioms that preserve the frame conditions of their usual versions. Types III and IV contain the weighted axioms that require a modification of the conditions imposed on the frame, respectively when correspondence cannot be proved or when it can.

The paper is organised as follows: Section 2 presents an informal comparative study of existing weighted modal logics. Section 3 introduces the semantics used to build weighted axioms with the method described in Section 4. Section 5 presents the resulting typology of weighted modal axioms.

## 2 Existing Weighted Modal Logics

After presenting the notations used in this paper, this section briefly describes existing weighted extensions of modal logics, first with the approaches that modify the definition of Kripke frames, integrating weights either in the accessibility relation or in the worlds. It then describes the counting models, that preserve the classical frame definition but alter the quantification used in the modal operator definitions.

## 2.1 Notations

Using the usual notations (eg see [Blackburn *et al.*, 2001]), a frame  $F = \langle W, R \rangle$  is a couple composed of a non-empty set  $W$  of worlds and a binary accessibility relation  $R$  on  $W$ . A model  $\mathcal{M} = \langle F, s \rangle$  is a couple formed by a frame  $F$  and a valuation  $s$  which assigns truth values to each atomic formula in each world in  $W$ .

For a given model  $\mathcal{M}$  and any world  $w$  in  $W$ , we denote by  $R_w$  its set of accessible worlds:

$$R_w = \{w' \in W \mid wRw'\} \quad (1)$$

In addition, considering the usual definition of semantic validity for the symbol  $\models$ , we define, for any formula  $\varphi$ , the set  $R_w(\varphi)$ :

$$R_w(\varphi) = \{w' \in R_w \mid \mathcal{M}, w' \models \varphi\} \quad (2)$$

For any formula  $\varphi$ , the classical interpretations of  $\Box\varphi$  and  $\Diamond\varphi = \neg\Box\neg\varphi$  are respectively based on the universal or existential quantification of accessible worlds which satisfy  $\varphi$ . Using the previous notations, they are written:

$$\mathcal{M}, w \models \Box\varphi \Leftrightarrow \forall w' \in R_w, \mathcal{M}, w' \models \varphi \quad (3)$$

$$\Leftrightarrow R_w(\varphi) = R_w$$

$$\mathcal{M}, w \models \Diamond\varphi \Leftrightarrow \exists w' \in R_w, \mathcal{M}, w' \models \varphi \quad (4)$$

$$\Leftrightarrow R_w(\varphi) \neq \emptyset$$

$$\Leftrightarrow |R_w(\varphi)| > 0$$

## 2.2 Weighted Accessibility Relation

A first category of weighted modal logics extends the classical Kripke model by replacing the accessibility relation  $R$  with a set of indexed relations  $R^\alpha$ , usually with  $\alpha \in [0, 1]$ . They then define weighted modalities  $\Box_\alpha$ , respectively associated with each relation  $R^\alpha$ , in accordance with the definitions given in Eq. (3) and (4). Three approaches can be distinguished depending on the interpretation of the weight, which can belong to different formal frameworks such as probability theory, possibility theory [Zadeh, 1978] or fuzzy set theory [Zadeh, 1965].

In the probabilistic case [Shirazi and Amir, 2007], the interpretation given to the accessibility weights relies on the conditional probability of transition from one world to another. Combinations of weights are, therefore, led in the usual probabilistic way.

In the fuzzy case [Bou *et al.*, 2009], the relation weights represent the strength of the relation, expressing that a world is *more or less* accessible: they describe an imprecision on the accessibility.

Since these fuzzy weighted relations correspond to  $\alpha$ -cuts of fuzzy relations, they satisfy a nesting property such that  $\forall \alpha, \beta \in [0, 1]$ , if  $\alpha \geq \beta$  then  $w_1 R^\alpha w_2 \Rightarrow w_1 R^\beta w_2$ . This in turn implies relations between modalities, expressed as a decreasing graduality property:

$$\forall \alpha, \beta \in [0, 1], \text{ if } \alpha \geq \beta \text{ then } \models \Box_\alpha \varphi \rightarrow \Box_\beta \varphi \quad (5)$$

The fuzzy interpretation thus leads to a multi-modal logic with dependent –or at least comparable– modalities.

In the possibilistic case [Fariñas del Cerro and Herzig, 1991], the relation weights represent the uncertainty on the

accessibility between worlds: they allow to express doubts regarding the very existence of a link between worlds, where the fuzzy model delivers information about its intensity. The possibilistic approach leads to multiple independent modalities.

## 2.3 Weighted Worlds

A second category of weighted modal logics considers that weights apply to worlds and not to the relation. Consequently, the weights have a global effect: they are set, regardless of the reference world and its accessible successors. Conversely, weighted relations exhibit a local effect, since weights are specific to each pair of worlds.

[Boutillier, 1994] enriches classical Kripke frames with a distribution of qualitative possibilities [Zadeh, 1978] over  $W$ , denoted  $\pi$ : worlds are considered as more or less possible.  $\pi$  is used to define the accessibility relation as:

$$R_w = \{w' \in W \mid \pi(w) \leq \pi(w')\}$$

The  $\Box$  and  $\Diamond$  semantics are then defined in the classical way, cf Eq. (3) and (4), using this relation. As a consequence, a formula  $\Box\varphi$  holds in  $w$  if and only if  $\varphi$  is satisfied in all worlds that are at least as possible as  $w$ . Note that  $\Box$  and  $\Diamond$  remain unweighted: this integration of weights actually does not lead to weighted modalities.

Also, the accessibility relation induced by  $\pi$  is necessarily antisymmetric, transitive and reflexive, restricting the expressivity of the ensuing modalities.

The distribution of possibilities  $\pi$  can also be generalised to formulae, defining  $\Pi(\varphi) = \max_{w \in W} \{\pi(w) \mid \mathcal{M}, w \models \varphi\}$  [Dubois *et al.*, 2012]. This model allows to build a generalised possibilistic logic, interpreted in an epistemic framework.

[Laverny and Lang, 2004] similarly enrich the classical Kripke model with weights on the worlds, where these weights represent some semantic property of the world independently of any formal paradigm: to each world is associated a so-called exceptionality degree that represents how different –or unrepresentative– the world is. An exceptionality degree is then assigned to each formula by:

$$except(\varphi) = \min_{w \in W} \{except(w) \mid \mathcal{M}, w \models \varphi\}$$

The proposed definition for the induced weighted modality does not preserve the classical definition of Eq. (3) but states:

$$\mathcal{M}, w \models \Box_\alpha \varphi \Leftrightarrow except(\neg\varphi) \geq \alpha$$

This definition means that the more exceptional a contradiction, the higher the weight.

Two properties of this exceptionality based definition of weighted modalities stand out: first, the validity of a modal formula is global and does not depend on the reference world where it is interpreted. Indeed,  $\mathcal{M}, w \models \Box_\alpha \varphi \Leftrightarrow \mathcal{M} \models \Box_\alpha \varphi$ . Second, due to the inequality in their definition, a dependence between modalities can be observed: the decreasing graduality property given in Eq. (5) also applies for this model.

## 2.4 Counting Approach

The counting approach [Fine, 1972; Fattorosi-Barnaba and Cerrato, 1988; Caro, 1988; van der Hoek and Meyer, 1992]

does not modify Kripke definitions of frames to integrate weights, neither on worlds nor on the relation, but modifies the modality definition, using a counting approach. Contrary to all previously discussed approaches, the weights considered here are integers and are, as a consequence, denoted  $n$ .

The counting approach modifies the quantification constraints on accessible validating worlds in Eq. (3) and (4). Indeed, the interpretation of  $\diamond_n$  is based on a hardening of the existential quantifier of Eq. (4): it no longer requires that at least one accessible world satisfies the formula but that at least  $n$  do. Formally, the counting approach defines  $\diamond_n$  and, by duality,  $\square_n$ , as,  $\forall n \in \mathbb{N}$ :

$$\mathcal{M}, w \models \diamond_n \varphi \Leftrightarrow |R_w(\varphi)| \geq n \quad (6)$$

$$\mathcal{M}, w \models \square_n \varphi \Leftrightarrow |R_w(\neg\varphi)| < n \quad (7)$$

The  $\square_n$  modality is weighted by the number of invalidating accessible worlds:  $n$  can be interpreted as a measure of contradiction.

Whereas this definition relies on absolute counting, majority logic [Pacuit and Salame, 2006] considers a specific case of relative counting: it introduces a modal operator expressing that a formula is true in more than half of the accessible worlds. It addresses the issue of its semantics in the case of infinite sets of worlds  $W$ .

Contrary to the approaches described in the previous subsections 2.1 and 2.2, which rely on a semantic definition, the counting approach has also been axiomatised, in both the absolute and relative cases [Caro, 1988; Pacuit and Salame, 2006]: the models propose manipulation rules for the weighted modalities.

### 3 Proposed Semantics

This section describes the semantics we propose for a weighted modal logic. It relies on a relative counting approach: despite its limitation to finite sets of worlds  $W$ , the normalisation constraint it imposes offers the benefits of rich information that allow to establish weighted extensions of the modal axioms, as discussed in Sections 4 and 5.

Syntactically, for  $p \in \mathbb{P}$  denoting a set of propositional variables and  $\alpha \in [0, 1]$  a numerical coefficient, we consider the set of all well-formed formulae according to the language

$$F := p \mid \neg F \mid F \wedge F \mid F \vee F \mid F \rightarrow F \mid \square_\alpha F \mid \diamond_\alpha F$$

#### 3.1 Definition

The semantics we propose follows the same principle as the relative counting approach described in Section 2.4, viz. based on counting proportions of validating worlds to relax the universal and harden the existential quantification constraints of Eq. (3) and (4).

It is defined when  $W$  is finite, in a frequentist interpretation, as a normalised cardinality. This proportion has the added benefit of making the modality weight independent of frame connectivity: the evaluation of the truth value of a formula  $\square_\alpha \varphi$  in a world  $w$  is not obfuscated by the number  $|R_w|$  of accessible worlds  $w$  has.

Formally, the proposed weighted modality  $\square_\alpha$  is defined as,  $\forall \alpha \in [0, 1]$ :

$$\begin{cases} \mathcal{M}, w \models \square_\alpha \varphi & \Leftrightarrow \frac{|R_w(\varphi)|}{|R_w|} \geq \alpha & \text{if } R_w \neq \emptyset \\ \mathcal{M}, w \models \square_\alpha \varphi & & \text{otherwise} \end{cases} \quad (8)$$

This definition thus relaxes the universal quantifier in Eq. (3), only requiring that a proportion of the accessible worlds satisfy the formula  $\varphi$ , instead of all of them.

By duality, the relation  $\diamond_\alpha$  is defined as,  $\forall \alpha \in [0, 1]$

$$\begin{cases} \mathcal{M}, w \models \diamond_\alpha \varphi & \Leftrightarrow \frac{|R_w(\varphi)|}{|R_w|} > 1 - \alpha & \text{if } R_w \neq \emptyset \\ \mathcal{M}, w \not\models \diamond_\alpha \varphi & & \text{otherwise} \end{cases} \quad (9)$$

The modality  $\diamond_\alpha$  requires that at least a proportion  $1 - \alpha$  of accessible worlds satisfy  $\varphi$ , instead of at least one accessible world: similar to the counting approach of Section 2.4, it thus hardens the existential quantifier, requiring more than just one accessible validating world. Note that, consequently, the higher the  $\alpha$ , the less demanding the condition. Also, because  $\diamond_\alpha \varphi = \neg \square_\alpha \neg \varphi$ , the loose inequality in Eq. (8) becomes a strict one for  $\diamond$ , in Eq. (9).

#### 3.2 Properties

This section establishes and discusses some properties satisfied by the proposed weighted modal operators.

##### Boundary Cases

As stated in the following proposition, the boundary case  $\alpha = 1$  corresponds to the classical modalities, whereas  $\alpha = 0$  is a tautology for  $\square$  and a contradiction for  $\diamond$ :

##### Proposition 1.

$$\begin{array}{ll} \square_1 \varphi = \square \varphi & \models \square_0 \varphi \\ \diamond_1 \varphi = \diamond \varphi & \models \neg \diamond_0 \varphi \end{array}$$

The proofs of this proposition follow directly from the definitions given in Eq. (8) and (9) and are, thus, omitted.

As a consequence, the case  $\alpha = 0$  can be considered as trivial and uninformative and it should, generally, be ignored. However, in the case where it is the only value for which a weighted formula holds, it expresses rich knowledge: considering  $\square_0$  for instance, for  $w \in W$  such that  $R_w \neq \emptyset$  and  $|R_w(\varphi)|/|R_w| = 0$ ,  $\mathcal{M}, w \models \square_1 \neg \varphi$ .

##### Decreasing Graduality

Due to the transitivity of the inequality relation on which the proposed semantics relies, the decreasing graduality property is satisfied:

**Proposition 2.** *The definition of  $\square_\alpha$  given in Eq. (8) satisfies the graduality property defined in Eq. (5).*

The proof follows directly from the definitions given in Eq. (8) and Eq. (5) and is, therefore, also omitted.

Proposition 2 implies that, up to a maximal degree, a formula holds for all lower weights. Notice that this property provides another justification for the uninformativeness of the  $\square_0$  modality underlined above. More generally, as a result, the most informative weight for the  $\square_\alpha$  modality

is the maximal admissible value, since all others can be inferred from it. This property will be crucial for establishing weighted extensions of modal axioms, as discussed in Section 4.

By duality, similar results hold for the  $\diamond_\alpha$  modality, with an increasing graduality property: for  $\diamond_\alpha$ , the most informative weight is the minimal admissible value.

### Relations between $\Box_\alpha$ and $\diamond_\alpha$

Let us underline that the preserved duality constraint, according to which  $\Box_\alpha\varphi = \neg\diamond_\alpha\neg\varphi$ , does not guarantee the equivalence between  $\Box_\alpha$  and  $\diamond_{1-\alpha}$ . Indeed, due to the fact that the  $\Box_\alpha$  definition relies on a non-strict inequality whereas  $\diamond_\alpha$  relies on a strict one, it can be shown that one implication holds but the other does not: (the case  $\alpha = 0$  is covered by Proposition 1).

#### Proposition 3.

$$\begin{aligned} \forall \alpha \in (0, 1] \quad & \models \diamond_\alpha\varphi \rightarrow \Box_{1-\alpha}\varphi \\ & \not\models \Box_\alpha\varphi \rightarrow \diamond_{1-\alpha}\varphi \end{aligned}$$

*Proof.* Let  $\mathcal{M} = \langle\langle W, R \rangle, s\rangle$  be any model and  $w \in W$ . It holds that

$$\begin{aligned} \mathcal{M}, w \models \diamond_\alpha\varphi & \Leftrightarrow R_w \neq \emptyset \text{ and } \frac{|R_w(\varphi)|}{|R_w|} > 1 - \alpha \\ & \Rightarrow R_w \neq \emptyset \text{ and } \frac{|R_w(\varphi)|}{|R_w|} \geq 1 - \alpha \\ & \Rightarrow \mathcal{M}, w \models \Box_{1-\alpha}\varphi \end{aligned}$$

The fact that the second implication  $\Box_\alpha\varphi \rightarrow \diamond_{1-\alpha}\varphi$  is not a tautology can be proved using a counterexample, such as the frame in Fig. 1:  $w \models \Box_{2/3}\varphi$  but  $w \not\models \diamond_{1/3}\varphi$ , as  $|R_w(\varphi)|/|R_w| = 1/3$  does not satisfy a strict inequality. ■

Another relation establishes an equivalence between the classical  $\diamond$  and a weighted  $\Box_\alpha$ :

**Proposition 4.** For any model  $\mathcal{M} = \langle\langle W, R \rangle, s\rangle$  and any  $w \in W$ ,

$$\mathcal{M}, w \models \diamond_1\varphi \Leftrightarrow R_w \neq \emptyset \text{ and } \mathcal{M}, w \models \Box_{\frac{1}{|R_w|}}\varphi$$

*Proof.* Let  $\mathcal{M} = \langle\langle W, R \rangle, s\rangle$  be any model and  $w \in W$ . It holds that

$$\begin{aligned} \mathcal{M}, w \models \diamond_1\varphi & \Leftrightarrow \exists w' \in R_w, \mathcal{M}, w' \models \varphi \\ & \Leftrightarrow R_w \neq \emptyset \text{ and } \frac{|R_w(\varphi)|}{|R_w|} \geq \frac{1}{|R_w|} \end{aligned}$$

## 4 Principles for Building Weighted Extensions of Modal Axioms

Axioms in classical modal logic [Blackburn *et al.*, 2001] can be seen as defining rules for the combination of the modal operators  $\Box$  and  $\diamond$ , establishing relations between formulae in which they occur once, repeatedly or in combination.

We propose to study their weighted transposition, defined as the formulae obtained when replacing each modality of a classical axiom with a weighted one, each with its own weight. Thus, the classical axiom (4), written  $\vdash \Box\varphi \rightarrow \Box\Box\varphi$ , leads to a weighted extension noted  $\vdash \Box_\alpha\varphi \rightarrow \Box_\beta\Box_\gamma\varphi$ .

More precisely, we propose to examine how these weights depend on each other, in a semantic approach based on the interpretation of weighted modal logic presented in the previous section: the method we consider consists in identifying weight dependence which holds either in any frame or under specific frame conditions. Moreover, we study whether the frames in which the obtained axioms hold satisfy specific conditions. This section presents the principles used to set the values for the introduced weights.

### 4.1 Inequality Constraints on Candidate Weights

When interpreted as elements of an inference system, in order to allow rich inferences, axioms that take the form of implications should have premises that are easy to satisfy and informative conclusions.

This informal principle gives hints regarding relevant weight values exploiting the axiom structure, more precisely the position of the considered modal operator  $\Box_\alpha$ , in combination with the crucial decreasing graduality property: when  $\Box_\alpha$  is in the conclusion of the implication,  $\alpha$  should be maximal. Indeed, all lower values can be inferred from it and the most informative case is the highest value.

Conversely, if  $\Box_\alpha$  is in the premise of the implication,  $\alpha$  should be minimal: it indicates the lowest value that still allows to infer the conclusion, using modus ponens. Indeed, any proved formula of the form  $\Box_\beta\varphi$  with greater  $\beta$  induces the required  $\Box_\alpha\varphi$ , triggering the axiom inference.

By duality, for the  $\diamond_\alpha$  operator that satisfies an increasing graduality property, the converse definition of relevant values applies:  $\alpha$  should be minimal for  $\diamond_\alpha$  in the conclusion and maximal in the premise.

As a consequence, weighted extensions of classical modal axioms can be qualified as enriched, relaxed or loosened variants of their non-weighted counterparts, depending on the position of the weighted modalities and the weight values.

Indeed, if the weighted axiom is established for  $\Box_\alpha$  with a high value for  $\alpha$  in the conclusion, the induced axiom can be considered as enriched: it allows inference of informative elements. Note that this configuration is interesting only if there is a weighted modality in the premise: otherwise, the classical axiom allows to conclude with the  $\Box_1$  modality, and thus all  $\Box_\alpha$  by decreasing graduality.

When the weighted axiom contains  $\Box_\alpha$  in its premise, it can be considered as a relaxation of the classical version: it allows to infer a conclusion even if the strongest hypothesis is not satisfied.

Finally, there can be more complex variations leading to a weighted axiom that can only be considered as a loosening of the classical version, as discussed in section 5.

### 4.2 Using Frame Conditions

A second tool to establish weight dependence for weighted extensions of modal axioms is provided by the frame conditions associated to classical modal axioms in correspondence theory [Van Benthem, 1984]. Indeed, the semantic counterparts of modal axioms comes with specific classes of frames, constrained by conditions on the accessibility relation which is, for instance, required to be reflexive or symmetric. Table 1 lists the definition of the most frequent relation properties.

serial	$\forall u, \exists v, uRv$
reflexive	$\forall u, uRu$
symmetric	$\forall u, v, uRv \Rightarrow vRu$
shift-reflexive	$\forall u, v, uRv \Rightarrow vRv$
transitive	$\forall u, v, w (uRv \wedge vRw) \Rightarrow uRw$
euclidean	$\forall u, v, w (uRv \wedge uRw) \Rightarrow vRw$
dense	$\forall u, \forall v uRv \Rightarrow \exists w, (uRw \wedge wRv)$

Table 1: Most common properties of a relation  $R$  defined on  $W \times W$ , where  $u, v, w \in W$ .

Therefore, when interpreting the weighted extension of a classical modal axiom from a semantic point of view, we only consider frames satisfying the corresponding conditions, to examine if specific relations are imposed on the weight values under these assumptions. This principle also guarantees compatibility with the boundary case where all introduced weights equal 1.

$\Box_\alpha$ , which presents a more expressive interpretation than  $\Box$ , is also less informative. Indeed, knowing that a proportion of accessible worlds is a model for a given formula does not give information on this formula's evaluation in all considered worlds:  $\Box_\alpha$  gives a global indication and leads to uncertainty for any precisely specified world. As a consequence, it is expected that establishing weighted extensions of the axioms may require to impose more constraining frame conditions. More precisely, it can be the case that the obtained axiom does not exclude the case where the conclusion is of the form  $\Box_0$ . As discussed in Section 3.2, this is uninformative. The approach we propose thus consists in looking for conditions that exclude such counter-example frames, hardening the classic condition.

Finally, when a relevant weighted axiom has been established, under possibly hardened frame conditions, we study whether a converse proposition holds, i.e. whether the frames in which the obtained weighted axiom holds necessarily satisfy the considered condition. This can be considered as opening the way to the definition of a weighted correspondence theory.

## 5 Typology of Weighted Axioms

This section presents the obtained results, i.e. the weighted extensions of the classical modal axioms when applying the principles presented in the previous section. Regarding the semantic interpretation, we consider the definition presented in Section 3, whose constraints allow to establish weight dependence.

The results, listed in Table 2, are organised in a 4-type typology, whose definition is given in the first subsection. The following subsections then successively detail the 4 axiom types.

### 5.1 Four Types of Weighted Axiom

Four types of axioms, whose content is described below have been identified:

- (I) Unweighted axioms
- (II) Weighted axioms with classical correspondence
- (III) Weighted axioms without correspondence
- (IV) Weighted axioms with enriched correspondence

(K $_\alpha$ )	$\Box_\alpha(\varphi \rightarrow \psi) \rightarrow (\Box_\beta\varphi \rightarrow \Box_{\max(0, \alpha + \beta - 1)}\psi)$	(II)
(CD $_\alpha$ )	$\Diamond_\alpha\varphi \rightarrow \Box_{1-\alpha}\varphi$	(IV)
(D $_\alpha$ )	$\Box_\alpha\varphi \rightarrow \Diamond_{1-\alpha+\varepsilon}\varphi$	serial (II)
(M $_\alpha$ )	$\Box_1\varphi \rightarrow \varphi$	reflexive (I)
(B $_\alpha$ )	$\varphi \rightarrow \Box_1\Diamond_1\varphi$	symmetric (I)
( $\Box$ M $_\alpha$ )	$\Box_1(\Box_1\varphi \rightarrow \varphi)$	shift-reflexive (I)
(4 $_\alpha$ )	$\Box_\alpha\varphi \rightarrow \Box_1\Box_\alpha\varphi$	transitive & euclidean (III)
(5 $_\alpha$ )	$\Diamond_\alpha\varphi \rightarrow \Box_1\Diamond_{\alpha+\varepsilon}\varphi$	transitive & euclidean (III)
(C4 $_\alpha$ )	$\Box_\alpha\Box_1\varphi \rightarrow \Box_\alpha\varphi$	dense & shift-reflexive (IV)
(C4 $_\alpha$ )	$\Box_1\Box_\beta\varphi \rightarrow \Box_\beta\varphi$	dense & transitive (IV)
(C4 $_\alpha$ )	$\Box_1\Box_\beta\varphi \rightarrow \Diamond_1\varphi$	dense & transitive (IV)
(C4 $_\alpha$ )	$\Box_\alpha\Box_\beta\varphi \rightarrow \Box_\beta\varphi$	transitive & euclidean (IV)

Table 2: Obtained weighted axioms with associated (not necessarily corresponding) frame conditions and type, as defined in Section 5.1.  $\alpha, \beta$  are real numbers in  $[0, 1]$  and  $\varepsilon \in (0, \alpha]$ .

These types depend on the relation between the weighted axioms and their classical counterparts and the frame conditions the latter correspond to: type I groups axioms that cannot be relaxed using the degrees of freedom offered by the weights. Type II is composed of weighted axioms that preserve the frame conditions of their usual versions. Types III and IV contain the weighted axioms that require a modification of the conditions imposed on the frame, respectively when correspondence cannot be proved or when it can.

Note that a given classical axiom can have several weighted extensions, depending on the considered frame conditions.

The following subsections detail each type in turn, each only describes one example.

### 5.2 Type I: Unweighted Axioms

The first type groups axioms for which the only possible weighting is the usual boundary case where the weights equal 1: they cannot be weakened and do not benefit from the weighting relaxation.

This is, for instance, the case of axiom (M), whose general weighted form is  $\vdash \Box_\alpha\varphi \rightarrow \varphi$ . By compatibility with the classic case this formula must be true within any frame with a reflexive relation. However, in the case where the maximal admissible weight is  $\alpha < 1$ , the reflexivity constraint cannot guarantee the reference world is not the (or one of the) worlds where  $\varphi$  does not hold. Other additional constraints (such as transitivity, symmetry or euclideanity) would not give information about  $w$  itself. It can be shown easily, by construction, that:

**Theorem 1.**  $\forall \alpha \in [0, 1)$ , there exists a model  $\mathcal{M} = \langle \langle W, R \rangle, s \rangle$  with reflexive  $R$  and  $w \in W$  such that  $\mathcal{M}, w \models \Box_\alpha\varphi$  but  $\mathcal{M}, w \not\models \varphi$ .

*Proof.* Let  $\alpha \in [0, 1)$ . Finding a counter-example is sufficient to prove this theorem. Let  $F = \langle W, R \rangle$  be a frame containing  $n$  worlds, where  $n$  is such that  $(n-1)/n \geq \alpha$  and  $R$  is reflexive, let  $w \in W$  be such that  $R_w = W$ . Let  $s$  be the valuation such that

- (i)  $x \models \varphi$  for all  $x \in W \setminus \{w\}$
- (ii)  $w \models \neg\varphi$

It holds that  $w \models \Box_\alpha\varphi$  but  $w \not\models \varphi$ . ■

Similar considerations can be applied to the classical axioms (B) and ( $\Box M$ ), whose weighted extensions equal their classic counterparts, as presented in Table 2.

### 5.3 Type II: Weighted Axioms with Classic Correspondence

Type II axioms offer a relaxed version of their classic counterparts and can be established under the same frame conditions. Moreover, the classically associated relation constraint is preserved and sufficient to have relevant values. This example applies to axiom (K), as stated by the following theorem:

**Theorem 2** ( $K_\alpha$ ).  $\forall \alpha, \beta \in [0, 1]$

$$\models \Box_\alpha(\varphi \rightarrow \psi) \rightarrow (\Box_\beta\varphi \rightarrow \Box_\gamma\psi)$$

where  $\gamma = \max(0, \alpha + \beta - 1)$

*Proof.* Let  $F = \langle W, R \rangle$  be a frame and  $w \in W$ . If  $R_w = \emptyset$ ,  $w$  trivially satisfies all three modal formulae and thus the implication. If  $|R_w| > 0$ , the proof consists in applying the modus ponens in accessible worlds where both  $\varphi \rightarrow \psi$  and  $\varphi$  are satisfied:  $R_w(\varphi \rightarrow \psi) \cap R_w(\varphi) \subseteq R_w(\psi)$ . Now by definition of the cardinal of set intersection:

$$|R_w(\varphi \rightarrow \psi) \cap R_w(\varphi)| = |R_w(\varphi \rightarrow \psi)| + |R_w(\varphi)| - |R_w(\varphi \rightarrow \psi) \cup R_w(\varphi)|$$

As  $|R_w| \geq |R_w(\varphi \rightarrow \psi) \cup R_w(\varphi)|$ , it holds that:

$$\begin{aligned} |R_w(\psi)| &\geq |R_w(\varphi \rightarrow \psi) \cap R_w(\varphi)| \\ &\geq |R_w(\varphi \rightarrow \psi)| + |R_w(\varphi)| - |R_w| \end{aligned}$$

Thus:  $\frac{|R_w(\psi)|}{|R_w|} \geq \alpha + \beta - 1$ .  $\blacksquare$

Similarly, as indicated in Table 2, a weighted extension of axiom (D) is established for any serial frame, and reciprocally. It states that  $\vdash \Box_\alpha\varphi \rightarrow \Diamond_{1-\alpha+\varepsilon}\varphi$  for all  $\alpha \in [0, 1]$  and  $\varepsilon \in (0, \alpha]$ . The weighted variant ( $D_\alpha$ ) completes the properties stated in Prop. 3 that relates the two weighted modal operators.

### 5.4 Type III: Weighted Axioms without Correspondence

This section establishes axioms where the classical frame conditions are not sufficient to establish weighted variants and proposes the addition of relevant requirements.

It can be illustrated with axiom (4), written  $\vdash \Box\varphi \rightarrow \Box\Box\varphi$  and classically associated with transitivity. A weighted variant is of the form  $\vdash \Box_\alpha\varphi \rightarrow \Box_\beta\Box_\gamma\varphi$  and the issue is to determine the appropriate values for  $\beta$  and  $\gamma$  for a given  $\alpha$ .

Now the sole condition that  $R$  is transitive does not allow to establish such a result:

**Theorem 3.**  $\forall \alpha \in [0, 1]$ , there exists a model  $\mathcal{M} = \langle F, s \rangle$  with  $R$  transitive and  $w \in W$  such that  $\mathcal{M}, w \models \Box_\alpha\varphi$  and  $\mathcal{M}, w \not\models \Box_1\Box_1\neg\varphi$ .

*Proof.* The proof consists in building such a model  $\mathcal{M}$ . For a given  $\alpha < 1$ , let  $m, q \in \mathbb{N}^*$  such that  $\alpha \leq m/(m+q)$ . Let  $W$  be a set of  $1 + q + m$  worlds,  $w \in W$ ,  $R$  the binary relation between worlds and  $s$  the valuation defined such that

$$(i) R_w = W \setminus \{w\}$$

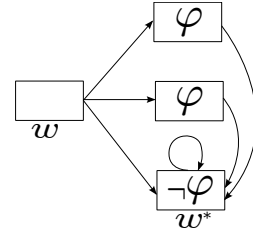


Figure 1: Counter-example model proving Th. 3 for  $\alpha = \frac{2}{3}$

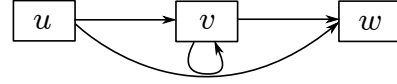


Figure 2: Frame showing the converse of Th. 4 does not hold

$$(ii) |R_w(\varphi)| = m$$

$$(iii) |R_w(\neg\varphi)| = q$$

$$(iv) \forall x \in R_w, \text{ let } R_x = \{w_n\} \text{ for one } w_n \in R_w(\neg\varphi)$$

By definition,  $R$  is transitive. Denoting  $F = \langle W, R \rangle$  and  $\mathcal{M} = \langle F, s \rangle$ , it holds that  $\mathcal{M}, w \models \Box_{m/(m+q)}\varphi$ , therefore, using the graduality property,  $\mathcal{M}, w \models \Box_\alpha\varphi$ .

Moreover, as  $\forall u \in R_w, \mathcal{M}, u \models \Box_1\neg\varphi$ , it holds that  $\mathcal{M}, w \not\models \Box_1\Box_1\neg\varphi$ : there is no  $\beta > 0$  such that  $\mathcal{M}, w \models \Box_\beta\Box_\gamma\varphi$  for  $\gamma > 0$ .  $\blacksquare$

Such a counter-example model is illustrated in Figure 1 for  $\alpha = 2/3$ , with  $m = 2$  and  $q = 1$ .

Therefore, transitivity is not a sufficient condition to have guarantees on the values of  $\beta$  and  $\gamma$ . It is thus necessary to harden the frame conditions by adding another constraint, euclideanity in Th. 4 below. Indeed, it can prevent the existence of sinkhole worlds, the ones in  $R_w(\neg\varphi)$  in the previous proof. Transitivity is kept to preserve the compatibility with the classic case obtained when the weights equal 1, leading to the theorem:

**Theorem 4** ( $4_\alpha$ ).  $\forall F = \langle W, R \rangle$ ,

$R$  is transitive and euclidean

$$\Rightarrow \forall \alpha \in [0, 1], F \models \Box_\alpha\varphi \rightarrow \Box_1\Box_\alpha\varphi$$

*Proof.* The proof relies on the fact that, for a transitive and euclidean relation,  $\forall w' \in R_w, R_{w'} = R_w$ . As a consequence, for all valuations  $s$  and for all  $\alpha \in [0, 1]$ , if  $w \models \Box_\alpha\varphi$ , then all accessible worlds  $w' \in R_w$  also satisfy  $w' \models \Box_\alpha\varphi$ , that is  $w \models \Box_1\Box_\alpha\varphi$ .  $\blacksquare$

However, the converse does not hold: Fig. 2 shows a counter-example with a frame  $F = \langle W, R \rangle$ , with  $W = \{u, v, w\}$  such that  $F \models \Box_\alpha\varphi \rightarrow \Box_1\Box_\alpha\varphi$ , for all  $\alpha \in [0, 1]$ , for all valuations  $s$  and for all worlds  $w \in W$ , but  $R$  is not euclidean.

The axiom established in Theorem 4 is powerful as the first modality in its conclusion is weighted by the maximal possible degree and the second one precisely by the degree  $\alpha$  appearing in the premise of the implication. Therefore, a

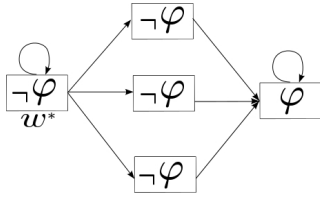


Figure 3: Counter-example model proving Th. 5 for  $\alpha = \frac{3}{4}$

weighted axiom with greater degree cannot be considered, meaning this axiom cannot be “improved”.

The same kind of result, shown in Table 2 but not detailed here, can be proved for the weighed extension of the classic (5) axiom: it possesses the same structure as axiom (4) with a single  $\diamond_\alpha$  operator in its premise and the combination of two modal operators in its conclusion.

### 5.5 Type IV: Weighted Axioms with Enriched Correspondence

Weighted axioms of type IV are defined as extensions for which additional frame conditions must be considered. The difference with type III comes from the fact that, in their case, correspondence can be proved.

We illustrate this category with the case of axiom  $(C4_\alpha)$ : its classic counterpart states  $\vdash \Box\Box\varphi \rightarrow \Box\varphi$  and is associated to the density frame condition. The general weighted version takes the form  $\Box_\alpha\Box_\beta\varphi \rightarrow \Box_\gamma\varphi$  but, as stated in the following theorem, density alone is not sufficient to guarantee such a property: for any  $\alpha$  and  $\beta$  value, a model can be built for which  $\gamma = 0$ .

**Theorem 5.**  $\forall \alpha \in [0, 1], \forall \beta \in [0, 1]$ , there is a model  $\mathcal{M} = \langle F, s \rangle$  with  $R$  dense and  $w \in W$  such that  $\mathcal{M}, w \models \Box_\alpha\Box_\beta\varphi$  and  $\mathcal{M}, w \models \Box_1\neg\varphi$

*Proof.* Again, the proof consists in building such a model  $\mathcal{M} = \langle F, s \rangle$ . For a given  $\alpha \in [0, 1]$  and  $\beta \in [0, 1]$ , let  $n \in \mathbb{N}$  be such that  $n \geq \alpha/(1 - \alpha)$ . Let  $W$  be a set of  $n + 2$  worlds,  $w^*$  and  $w'$  two distinct worlds from  $W$  and  $R$  and  $s$  such that

- (i)  $\mathcal{M}, w' \models \varphi$
- (ii)  $\forall w \in W \setminus \{w'\}, \mathcal{M}, w \models \neg\varphi$
- (iii)  $R_{w^*} = W \setminus \{w'\}$
- (iv)  $\forall w \in W \setminus \{w^*\}, R_w = \{w'\}$

By construction,  $R$  is dense.

Then  $\forall w \in W \setminus \{w^*\}, \mathcal{M}, w \models \Box_1\varphi$ , which implies by decreasing graduality,  $\mathcal{M}, w \models \Box_\beta\varphi$ . Therefore

$$\begin{aligned} |R_{w^*}(\Box_\beta\varphi)| = n &\Rightarrow \frac{|R_{w^*}(\Box_\beta\varphi)|}{|R_{w^*}|} = \frac{n}{n+1} \geq \alpha \\ &\Rightarrow \mathcal{M}, w^* \models \Box_\alpha\Box_\beta\varphi \end{aligned}$$

But  $\mathcal{M}, w^* \models \Box_1\neg\varphi$  so  $\nexists \gamma > 0$  such that  $\mathcal{M}, w^* \models \Box_\gamma\varphi$ . ■

Such a counter-example frame is illustrated on Fig. 3 for  $\alpha = 0.75$ .

As a consequence, the only way to guarantee a strictly positive value of  $\gamma$  is to add assumptions on the relation properties. As listed in Table 2 four distinct sets of constraints can be added to the accessibility relation, leading to four weighted extensions of  $(C4_\alpha)$ . They differ by the informativeness of their conclusion and the correlated level of constraint their premise imposes.

We give the proof for the strongest version of  $(C4_\alpha)$ . Note that the classical properties is preserved by euclideanity which implies that density holds.

**Theorem 6**  $(C4_\alpha)$ .  $\forall F = \langle W, R \rangle$ ,  
 $R$  is transitive and euclidean  
 $\Rightarrow \forall \alpha \in (0, 1], \forall \beta \in [0, 1], F \models \Box_\alpha\Box_\beta\varphi \rightarrow \Box_\beta\varphi$

*Proof.* Let  $\langle W, R \rangle$  be such that  $R$  is transitive and euclidean. Let a world  $w \in W$  and  $\alpha, \beta \in (0, 1]$  (if  $\beta = 0$  then  $\Box_\beta\varphi$  is true, and thus the implication is).

If  $w \not\models \Box_\alpha\Box_\beta\varphi$ , then  $w \models \Box_\alpha\Box_\beta\varphi \rightarrow \Box_\beta\varphi$ .

If  $w \models \Box_\alpha\Box_\beta\varphi$ , then a proportion  $\alpha > 0$  of worlds accessible from  $w$  satisfy  $\Box_\beta\varphi$ , let  $u$  be such a world. It holds that:

$$\frac{|R_u(\varphi)|}{|R_u|} \geq \beta$$

Now, as  $R$  is transitive and euclidean, it holds that  $\forall w' \in R_w, R_{w'} = R_w$ . In particular,  $R_u = R_w$ . We thus have :

$$\frac{|R_w(\varphi)|}{|R_w|} \geq \beta$$

Therefore,  $w \models \Box_\beta\varphi$ . ■

The converse can be proved by contraposition: if the relation is not transitive and euclidean, then axiom  $(C4_\alpha)$  does not hold. For this proof, we have to build a Kripke frame whose relation is not transitive or not euclidean, and we need to propose values for  $\alpha$  and  $\beta$ , and a valuation such that the axiom does not hold in one world of the frame.

The principle of this proof is illustrated by its first part: we show that for any frame  $\langle W, R \rangle$ , if the relation  $R$  is not transitive, then there exist  $\alpha, \beta$  and a valuation such that  $\exists w \in W$  such that  $w \not\models (C4_\alpha)$ . Formally:

**Theorem 7.**  $\forall F = \langle W, R \rangle$ ,  
 $R$  is not transitive  
 $\Rightarrow \exists \alpha, \beta \in [0, 1], \exists s, \exists w \in W$ ,  
 $\langle \langle W, R \rangle, s \rangle, w \not\models \Box_\alpha\Box_\beta\varphi \rightarrow \Box_\beta\varphi$

*Proof.* Let  $W$  be a finite set of worlds and  $R$  a non-transitive relation: there exists  $u, v, w \in W$  such that  $uRv \wedge vRw \wedge \neg uRw$ . We set  $\alpha = \frac{1}{|R_u|}$  and  $\beta = \frac{1}{|R_v|}$ . Let  $s$  be the valuation such that

- (i)  $w \models \varphi$
- (ii)  $\forall x \in W \setminus \{w\}, x \not\models \varphi$ .

Figure 4 illustrates an example of a such model. It holds that:

- $u \models \Box_1\neg\varphi$  because the only world satisfying  $\varphi$  cannot be accessible from  $u$ :  $w \notin R_u$ . Therefore  $u \not\models \Box_\beta\varphi$
- $v \models \Box_\beta\varphi$  because  $w \in R_v$  and  $w \models \varphi$



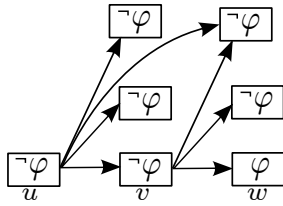


Figure 4: Kripke model where  $R$  is non-transitive

- $u \models \Box_\alpha \Box_\beta \varphi$  because  $v \in R_u$  and  $v \models \Box_\beta \varphi$

Therefore  $\exists \alpha, \beta$  such that  $u \models \Box_\alpha \Box_\beta \varphi$  but  $u \not\models \Box_\beta \varphi$ , which implies  $u \not\models (C4_\alpha)$ . ■

Following the same principle, it can be shown that if the relation is not euclidean then there exists a model which does not satisfy  $(C4_\alpha)$ .

Note that there exists other weighted versions of  $(C4)$ , listed in Table 2. Indeed, with frame conditions weaker than transitivity and euclideanity, relevant values hold for the weights.

## 6 Conclusion and Future Works

This paper studied rules for the combination of weighted modal operators, through the extension of classical axioms. In doing so, it offered a typology of weighted axioms with respect to their relation to their classical counterparts and to the frame conditions the latter correspond to. It discussed the expressiveness increase allowed by the weighting of axioms and how the hardened relation properties allow to balance the induced lack of informations. Thus, some examples were proposed to illustrate most of the issues of weighted axioms.

Amongst the frame conditions considered for establishing the weighted modal axioms, only binary classical relation properties were studied. It would be interesting to consider relaxed versions, in the spirit of some  $\alpha$ -symmetry, to examine what other extended versions of the axioms can be established.

Future works also aim at specifying the proposed weighted modal logic to the doxastic framework, so as to study a belief-based adaptation. From a semantic point of view, the interpretation of the weights as belief degrees will be studied; from an axiomatic point of view, the weighted axioms of the modal logic KD45 and their properties will be considered from the set of established axioms.

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