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Global Existence and Uniqueness of a 2D-Transient State of a Coupled Radiative-Conductive Heat Transfer Problem

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Abstract

This paper deals with global existence and uniqueness results for a transient nonlinear radiative-conductive system in two dimensional case. This system describes the heat transfer for a grey, semi-transparent and non-scattering medium with general boundary conditions. We reformulate the full transient state system as a fixed-point problem. The existence and uniqueness proof is based on Schauder fixed point Theorem.

1 Introduction

The aim of this work is to prove the global existence and uniqueness of the solution for a transient combined radiative-conductive system in two dimensional case with general boundary conditions when the initial condition is

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assumed to be nonnegative. The medium is assumed grey, semi-transparent and non-scattering.

Let us consider a bounded, open, connected and convex set $\Omega \subset \mathbb{R}^2$, with C^2 boundary. \mathcal{D} is the unit disk. Let $\boldsymbol{\beta} \in \mathcal{D}$, $\mathbf{x} \in \Omega$, $t \in (0, \tau)$ for $\tau > 0$, $\mathcal{X} = \Omega \times \mathcal{D}$, $\mathcal{Q}_\tau = (0, \tau) \times \Omega$ and $\Sigma_\tau = (0, \tau) \times \partial\Omega$. Let \mathbf{n} be the outward unit normal to the boundary $\partial\Omega$. We denote

$$\partial\Omega_- = \{(x, \boldsymbol{\beta}) \in \partial\Omega \times \mathcal{D} \text{ such that } \boldsymbol{\beta} \cdot \mathbf{n} < 0\}.$$

The full system of a combined nonlinear radiation-conduction heat transfer is written in dimensionless form,

$$I(t, \mathbf{x}, \boldsymbol{\beta}) + \boldsymbol{\beta} \cdot \nabla_{\mathbf{x}} I(t, \mathbf{x}, \boldsymbol{\beta}) = T^4(t, \mathbf{x}) \quad (t, \mathbf{x}, \boldsymbol{\beta}) \in (0, \tau) \times \mathcal{X} \quad (1)$$

$$\partial_t T(t, \mathbf{x}) - \Delta T(t, \mathbf{x}) + 2\pi\theta T^4(t, \mathbf{x}) = \theta G(t, \mathbf{x}) \quad (t, \mathbf{x}) \in \mathcal{Q}_\tau \quad (2)$$

$$a\partial_n T(t, \mathbf{x}) + bT(t, \mathbf{x}) = g(t, \mathbf{x}) \quad (t, \mathbf{x}) \in \Sigma_\tau \quad (3)$$

$$I(t, \mathbf{x}, \boldsymbol{\beta}) = h(t, \mathbf{x}, \boldsymbol{\beta}) \quad (t, \mathbf{x}, \boldsymbol{\beta}) \in (0, \tau) \times \partial\Omega_- \quad (4)$$

$$T(0, \mathbf{x}) = T_0(\mathbf{x}) \quad \mathbf{x} \in \Omega \quad (5)$$

where θ is a positive dimensionless constant, a and b are nonnegative real numbers, T_0 , h and g are smooth and nonnegative initial data. The incident radiation intensity G is given by

$$G(t, \mathbf{x}) = \int_{\mathcal{D}} I(t, \mathbf{x}, \boldsymbol{\beta}) \frac{2}{\sqrt{1 - |\boldsymbol{\beta}|^2}} d\boldsymbol{\beta} \quad (t, \mathbf{x}) \in \mathcal{Q}_\tau. \quad (6)$$

In this paper we assume that the mean radiation intensity of the blackbody verifies the Stefan-Boltzmann law which is proportional to T^4 . The radiative transfer equation (RTE) (1) and the conductive equation (CE) (2) are coupled via the source term $\theta\{G - 2\pi T^4\}$. We use nonhomogeneous Dirichlet boundary conditions for radiation equation and different cases of boundary conditions for CE. For a fuller treatment of the dimensionless form of radiative conductive heat transfer system, we refer the reader to [26].

Radiative-conductive heat transfer problems are the subject of various fields of engineering and science. In the literature, this problem is studied using two different types of model. In the first type, the problem is described using an unique parabolic partial differential equation. In the second type of model, the modeling of the radiation and the conduction is given by a coupled system of partial differential equations where each phenomenon is described by an equation.

There is a huge mathematical theory in the first case, see [5, 6, 7, 8, 31, 45, 3, 4, 9, 34, 35, 36, 37]. For example, the paper [5] is devoted to the study of a nonstationary nonlinear nonlocal initial boundary value problem governing radiative conductive heat transfer in opaque bodies with surfaces whose properties depend on the radiation frequency. This paper is a natural

extension of the work done in [7], where the corresponding stationary problem was treated. In [30], the authors considered the conductive radiative heat transfer in a scattering and absorbing medium bounded by two reflecting and radiating plane surfaces. The existence and uniqueness of a solution of this problem is established using an iterative procedure.

In [36], M. Laitinen and T. Tiihonen studied the well-posedness of a class of models describing heat transfer by conduction and radiation in the stationary case. The employed theory covers different types of grey materials, that is, both semitransparent and opaque bodies as well as isotropic or non-isotropic scattering/reflection provided that the material properties do not depend on the wavelength of the radiation.

In this paper, we consider the second type of model where the phenomenon is expressed as a coupled system of nonlinear partial differential equations in two-dimensional case. In previous works we can find theoretical results of existence and uniqueness in one-dimensional case. Indeed, in the Kelley's paper [28], the authors considered a steady-state combined radiative-conductive heat transfer. In Asllanaaj et al.[12] the authors generalized the Kelley's study and they proved the existence and uniqueness of the 1-D system of coupled radiative conductive in the steady state associated to the nonhomogeneous Dirichlet boundary with the black surfaces. The medium is assumed to be a non-grey anisotropic absorbing, emitting, scattering, with axial symmetry and non homogeneous. They considered a nonlinear conduction equation due to the temperature dependence of the thermal conductivity. However, the approach developed by Asllanaaj et al. [12] is just adaptable to 1D dimensional geometry. In addition, M. M. Porzio and Ó. López Pouso proved in [44] an existence and uniqueness theorem for the non-grey coupled convection-conduction-radiation system associated to the mixed nonhomogenous Dirichlet and homogenous Neumann boundary conditions by means of accretive operators theory. Leaving aside the grey or non-grey character, the main difference between our problem and the one studied in [44] is that we do not include the transient term in the RTE. This is an interesting point because this term is really negligible in a wide range of applications, and also because the techniques used in [44] do not allow disregarding it. Moreover, in our study we discuss different types of boundary conditions.

In this paper we prove the global existence and uniqueness of solutions for the nonlinear radiative conductive system in 2-dimensional case associated to the nonhomogeneous Dirichlet boundary conditions for radiation equation and for different type of conductive boundary conditions. The Schauder fixed point theorem is the principal tool used to solve this problem.

Recently, some attention has been accorded to numerical methods to study the radiative transfer and the nonlinear radiative-conductive heat transfer problem including optimal control problems, for more details see [10, 11, 12, 13, 14, 18, 21, 22, 23, 24, 26, 40, 41, 42, 43, 39, 29, 27, 25].

Asllanaj et al. [13] simulated transient heat transfer by radiation and conduction in two-dimensional complex shaped domains with structured and unstructured triangular meshes working with an absorbing, emitting and non-scattering grey medium.

The plan of this paper is as follows: Section 2, contains the statement of the main result (Theorem 2.1). Section 3 is devoted to its proof based on Schauder fixed point theorem.

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2 Main results

In order to state the main result, we introduce the following notations

$$L^p(\mathbf{Q}_\tau) = L^p(0, \tau; L^p(\Omega)) \text{ for all } p \in [1, \infty)$$

$$W_2^{2,1}(\mathbf{Q}_\tau) := \{\phi \text{ s.t. } \phi, \phi_t, \phi_{x_i}, \phi_{x_i x_j} \in L^2(\mathbf{Q}_\tau)\}.$$

According to the method introduced in [19] to solve the neutron equations, we consider the following space

$$\mathcal{W}^2 = \{v \in L^2(\mathcal{X}); \beta \cdot \nabla_x v \in L^2(\mathcal{X})\}$$

and the following subset of $\partial\Omega \times \mathcal{D}$

$$\partial\Omega_+ = \{(x, \beta) \in \partial\Omega \times \mathcal{D} \text{ and } \beta \cdot \mathbf{n} > 0\}.$$

We denote by

$$L^2 = L^2(\mathcal{X}), \quad L_-^2 = L^2(\partial\Omega_-; |\beta \cdot \mathbf{n}| dx d\beta)$$

and

$$L_+^2 = L^2(\partial\Omega_+; |\beta \cdot \mathbf{n}| dx d\beta),$$

the spaces of square integrable functions in \mathcal{X} , $\partial\Omega_-$ and $\partial\Omega_+$, respectively. Let us denote by \mathcal{W} the following subset of \mathcal{W}^2 :

$$\mathcal{W} = \{T \in \mathcal{W}^2; T|_{\partial\Omega_-} \in L_-^2\}.$$

The space \mathcal{W} is a Hilbert space when is equipped with the scalar product

$$(u, v)_{\mathcal{W}} = \int_{\mathcal{X}} uv dx d\beta + \int_{\mathcal{X}} (\beta \cdot \nabla_x u)(\beta \cdot \nabla_x v) dx d\beta + \int_{\partial\Omega_+} (\beta \cdot \mathbf{n}) uv dx d\beta$$

and the norm

$$\|u\|_{\mathcal{W}}^2 = \|u\|_{L^2}^2 + \|\beta \cdot \nabla_x u\|_{L^2}^2 + \|u\|_{L_+^2}^2.$$

Our result will be obtained under the following assumptions about the initial data

$$\begin{aligned} T_0 &\text{ is nonnegative, belongs to } H^1(\Omega), \\ h &\in L^2(0, \infty; L^2_-) \text{ is nonnegative,} \\ g &\in W_\infty^{2,1}((0, \infty) \times \bar{\Omega}) \cap C^{2,1}((0, \infty) \times \bar{\Omega}) \text{ is nonnegative.} \end{aligned} \quad (7)$$

The main result of this paper is the following Theorem.

Theorem 2.1. *Assume that the data verifies (7). For all $\tau > 0$, the system of equation (1)-(5) has a unique nonnegative solution (T, I) such that $T \in W_2^{2,1}(\mathbf{Q}_\tau)$ and $I \in L^2(0, \tau; \mathbf{W})$. Moreover, there exist $C = C(\Omega, \tau, \theta) > 0$ such that*

$$\|I\|_{L^2(0, \tau; \mathbf{W})} \leq \sqrt{2\pi} \|T\|_{L^8(\mathbf{Q}_\tau)}^4 + \|h\|_{L^2(0, \tau; L^2_-)}, \quad (8)$$

and

$$\|T\|_{W_2^{2,1}(\mathbf{Q}_\tau)} \leq C \left(\|G\|_{L^2(\mathbf{Q}_\tau)} + \|T_0\|_{H^1(\Omega)} + \|g\|_{L^2(0, \tau; H^{\frac{3}{2}}(\Omega))} \right). \quad (9)$$

Remark 2.2. *The Theorem 2.1 shows the existence and uniqueness of the solution for all $\tau > 0$ which implies a global existence and uniqueness of the solution for radiative conductive heat transfer system.*

The next section is devoted to the construction of a completely continuous mapping \mathcal{H} on a suitable set, composed by three continuous maps. Moreover, the Schauder fixed point Theorem is employed to prove the global existence and uniqueness of solution of the nonlinear coupled radiative conductive heat transfer system (1)-(5).

3 Global existence and uniqueness of solution for the coupled system

In this section, we show that the existence of a solution T , and implicitly the existence of a solution I , of the coupled system of equations (1)-(5) is related to the existence of a solution of a fixed point problem. We will apply the Schauder fixed point theorem to a well-chosen map \mathcal{H} . To do so, we must show that this map \mathcal{H} is well defined and completely continuous.

At first, we fixe $\tau > 0$, let M_τ a positive constant only depending on T_0 , g , h and τ , satisfying the following hypotheses

$$\begin{aligned} \star \left\| T_0 \right\|_{L^5(\Omega)}^5 &\leq \pi \theta M_\tau^8, \\ \star \left\| h \right\|_{L^2(0, \tau; L^2_-)} &\leq \sqrt{2\pi} M_\tau^4, \\ \star C_g^* &\leq \frac{M_\tau^8}{4}. \end{aligned} \quad (10)$$

where

$$C_g^* = \frac{4}{50\pi\theta a^2 b} \tau \left| \partial\Omega \right| \left\| g \right\|_{L^\infty((0,\tau) \times \bar{\Omega})} \quad \text{for Robin Boundary conditions } (a > 0, b > 0),$$

$$C_g^* = \tau \left(\frac{12}{75\pi^2\theta^2} \left| \Omega \right| + \frac{\left| \partial\Omega \right|}{4a^2\pi\theta} C(\Omega) \left\| g \right\|_{L^\infty((0,\tau) \times \bar{\Omega})}^5 \right), \quad \text{for}$$

Neumann Boundary conditions $(a > 0, b = 0)$,

$$C_g^* = \frac{2}{b^8} \tau \left| \Omega \right| \left\| g \right\|_{L^\infty(\mathbf{Q}_\tau)}^8 + \frac{7}{192\pi^2\theta^2 b^8} \tau \left| \Omega \right| \left\| (\partial_t + \Delta) g^4 \right\|_{L^\infty(\mathbf{Q}_\tau)}^{8/7} \\ + \frac{8}{25\pi\theta b^8} \left| \Omega \right| \left\| g \right\|_{L^\infty(\mathbf{Q}_\tau)}^5 + \frac{\tau}{b^5\pi\theta} \left| \partial\Omega \right| \left\| g \partial_n g^4 \right\|_{L^\infty((0,\tau) \times \bar{\Omega})},$$

for Dirichlet Boundary conditions $(a = 0, b > 0)$,

and $C(\Omega)$ is a positive constant depending only on Ω .

Note that it is always possible to choose a large enough constant M_τ satisfying the assumptions (10) for any given value of $\tau > 0$.

We introduce the following sets

$$E_1^\tau = \{T \in L^8(\mathbf{Q}_\tau); \quad \left\| T \right\|_{L^8(\mathbf{Q}_\tau)} \leq M_\tau\}, \\ E_2^\tau = \{I \in L^2(0, \tau; L^2(\mathcal{X})); \quad \left\| I \right\|_{L^2(0,\tau; L^2(\mathcal{X}))}^2 \leq 2\pi M_\tau^8\}, \\ E_3^\tau = \{G \in L^2(\mathbf{Q}_\tau); \quad \left\| G \right\|_{L^2(\mathbf{Q}_\tau)}^2 \leq 4\pi^2 M_\tau^8\}.$$

The map $\mathcal{H} : E_1^\tau \longrightarrow E_1^\tau$ is a composition of three maps

$$\mathcal{H} = \mathcal{H}_3 \circ \mathcal{H}_2 \circ \mathcal{H}_1.$$

The map $\mathcal{H}_1 : E_1^\tau \longrightarrow E_2^\tau$ is defined as follows, for $T \in E_1^\tau$, $\mathcal{H}_1(T) \in E_2^\tau$ is the solution of the radiative transfer equation(1)-(5). On the other hand, the map $\mathcal{H}_2 : E_2^\tau \longrightarrow E_3^\tau$ is defined in the following way, for $I \in E_2^\tau$, $\mathcal{H}_2(I) = G \in E_3^\tau$ where G is given by (6). Finally, the map $\mathcal{H}_3 : E_3^\tau \longrightarrow E_1^\tau$ is defined as follows, for $G \in E_3^\tau$, $\mathcal{H}_3(G) \in E_1^\tau$ is the solution of CE (1)-(5).

To study \mathcal{H} , we will be studying in great detail the maps \mathcal{H}_1 , \mathcal{H}_2 and \mathcal{H}_3 .

3.1 The maps \mathcal{H}_1 and \mathcal{H}_2

Now, we focus on the maps \mathcal{H}_1 and \mathcal{H}_2 , we give some properties of the solution of the RTE (1) using nonhomogeneous radiative Dirichlet boundary conditions.

We start by recalling the Green's formula, see [16]:

$$\int_{\mathcal{X}} (\boldsymbol{\beta} \cdot \nabla_x u) v dx d\boldsymbol{\beta} + \int_{\mathcal{X}} (\boldsymbol{\beta} \cdot \nabla_x v) u dx d\boldsymbol{\beta} = \int_{\partial\Omega \times \mathcal{D}} (\boldsymbol{\beta} \cdot \mathbf{n}) u v dx d\boldsymbol{\beta}, \quad (11)$$

for all $(u, v) \in \mathcal{W} \times \mathcal{W}$.

Theorem 3.1. *Let us consider $T \in E_1^\tau$. Under the assumptions (7), (10), the problem (1), (4) has a unique nonnegative solution $\mathcal{H}_1(T) \in L^2(0, \tau; \mathcal{W})$. Moreover, \mathcal{H}_1 is a well-posed and continuous map from E_1^τ to E_2^τ .*

Proof. Let $T \in E_1^\tau$, $t \in [0, \tau]$, we have $T^4(t) \in L^2(\Omega)$. Using a result about the existence and uniqueness of the solution of the transport equation, see [19], the boundary value problem (1), (4) has a unique solution $I(t) \in L^2(\mathcal{X})$.

In addition, using the linearity of (1), the solution I of the problem (1)-(5) is given by $I = I_0 + w$ where I_0 is a solution of (1), (4) for $h \equiv 0$ and w is a solution of (1), (4) without the second member T^4 .

We start by the homogeneous problem

$$\begin{aligned} \boldsymbol{\beta} \cdot \nabla_x I_0(t, \mathbf{x}, \boldsymbol{\beta}) + I_0(t, \mathbf{x}, \boldsymbol{\beta}) &= T^4(t, \mathbf{x}) \quad \forall (t, \mathbf{x}, \boldsymbol{\beta}) \in (0, \tau) \times \mathcal{X} & (12) \\ I_0(t, \mathbf{x}, \boldsymbol{\beta}) &= 0 \quad \forall (t, \mathbf{x}, \boldsymbol{\beta}) \in \partial\Omega_-. & (13) \end{aligned}$$

If we multiply the equation (12) by I_0 and we integrate in space, we obtain

$$\int_{\mathcal{X}} I_0^2(t) dx d\boldsymbol{\beta} + \int_{\mathcal{X}} (\boldsymbol{\beta} \cdot \nabla_x I_0)(t) I_0(t) dx d\boldsymbol{\beta} = \int_{\mathcal{X}} T^4(t) I_0(t) dx d\boldsymbol{\beta}. \quad (14)$$

Using the Cauchy-Schwarz inequality, we obtain the following bound of the second member of (14). Thus

$$\left| \int_{\mathcal{X}} T^4(t) I_0(t) dx d\boldsymbol{\beta} \right| \leq \|T^4(t)\|_{L^2} \|I_0(t)\|_{L^2},$$

since T is independent of the direction $\boldsymbol{\beta}$ and $mes(\mathcal{D}) = 2\pi$, we deduce:

$$\left| \int_{\mathcal{X}} T^4(t) I_0(t) dx d\boldsymbol{\beta} \right| \leq \sqrt{2\pi} \|T^4(t)\|_{L^2(\Omega)} \|I_0(t)\|_{L^2}.$$

In order to involve the boundary value, we use Green's formula (11) and (13) to get

$$\begin{aligned} 2 \int_{\mathcal{X}} I_0(t) (\boldsymbol{\beta} \cdot \nabla_x I_0)(t) dx d\boldsymbol{\beta} &= \int_{\partial\Omega \times \mathcal{D}} (\boldsymbol{\beta} \cdot \mathbf{n}) I_0^2(t) d\Gamma d\boldsymbol{\beta} \\ &= \int_{\partial\Omega_+} (\boldsymbol{\beta} \cdot \mathbf{n}) I_0^2(t) d\Gamma d\boldsymbol{\beta}. \end{aligned} \quad (15)$$

Using the definition of $\partial\Omega_+$, we can conclude that the right hand side term is nonnegative. Finally, we have the following inequality

$$\|I_0(t)\|_{L^2}^2 + \frac{1}{2} \|I_0(t)\|_{L_+^2} \leq \sqrt{2\pi} \|T^4(t)\|_{L^2(\Omega)} \|I_0(t)\|_{L^2}.$$

Thus

$$\|I_0(t)\|_{L^2} \leq \sqrt{2\pi} \|T^4(t)\|_{L^2(\Omega)}. \quad (16)$$

If we multiply (12) by $I_0 + \boldsymbol{\beta} \cdot \nabla_x I_0$ and we integrate in space, we get

$$\begin{aligned} \int_{\mathcal{X}} (I_0 + \boldsymbol{\beta} \cdot \nabla_x I_0)(t) I_0(t) dx d\boldsymbol{\beta} + \int_{\mathcal{X}} (\boldsymbol{\beta} \cdot \nabla_x I_0)(t) (I_0 + \boldsymbol{\beta} \cdot \nabla_x I_0)(t) dx d\boldsymbol{\beta} \\ = \int_{\mathcal{X}} T^4(t) (I_0 + \boldsymbol{\beta} \cdot \nabla_x I_0)(t) dx d\boldsymbol{\beta}. \end{aligned} \quad (17)$$

Using the Cauchy-Schwarz inequality, we verify that the second member of (17) is bounded,

$$\left| \int_{\mathcal{X}} T^4(t) \boldsymbol{\beta} \cdot \nabla_x I_0(t) dx d\boldsymbol{\beta} \right| \leq \|T^4(t)\|_{L^2} \|\boldsymbol{\beta} \cdot \nabla_x I_0(t)\|_{L^2}.$$

We have also

$$\begin{aligned} \int_{\mathcal{X}} I_0^2(t) dx d\boldsymbol{\beta} + 2 \int_{\mathcal{X}} (\boldsymbol{\beta} \cdot \nabla_x I_0)(t) I_0(t) dx d\boldsymbol{\beta} + \int_{\mathcal{X}} (\boldsymbol{\beta} \cdot \nabla_x I_0)^2(t) dx d\boldsymbol{\beta} \\ = \|I_0(t)\|_{L^2}^2 + \|\boldsymbol{\beta} \cdot \nabla_x I_0(t)\|_{L^2}^2 + 2 \int_{\mathcal{X}} (\boldsymbol{\beta} \cdot \nabla_x I_0)(t) I_0(t) dx d\boldsymbol{\beta}. \end{aligned}$$

Using (15), it follows that

$$\begin{aligned} \|I_0(t)\|_{L^2}^2 + \|\boldsymbol{\beta} \cdot \nabla_x I_0(t)\|_{L^2}^2 + \|I_0(t)\|_{L^2_+}^2 \leq \|T^4(t)\|_{L^2} \|\boldsymbol{\beta} \cdot \nabla_x I_0(t)\|_{L^2} \\ + \|T^4(t)\|_{L^2} \|I_0(t)\|_{L^2}, \end{aligned}$$

consequently

$$\|I_0(t)\|_{\mathcal{W}}^2 = \|I_0(t)\|_{L^2}^2 + \|\boldsymbol{\beta} \cdot \nabla_x I_0(t)\|_{L^2}^2 + \|I_0(t)\|_{L^2_+}^2 \leq \|T^4(t)\|_{L^2} \|I_0(t)\|_{\mathcal{W}}.$$

In this way

$$\|I_0(t)\|_{\mathcal{W}} \leq \sqrt{2\pi} \|T^4(t)\|_{L^2(\Omega)}. \quad (18)$$

Now, we study the nonhomogeneous boundary value problem:

$$\boldsymbol{\beta} \cdot \nabla_x w(t, \mathbf{x}, \boldsymbol{\beta}) + w(t, \mathbf{x}, \boldsymbol{\beta}) = 0 \quad \forall (t, \mathbf{x}, \boldsymbol{\beta}) \in (0, \tau) \times \mathcal{X} \quad (19)$$

$$w(t, \mathbf{x}, \boldsymbol{\beta}) = h(t, \mathbf{x}, \boldsymbol{\beta}) \quad \forall (t, \mathbf{x}, \boldsymbol{\beta}) \in (0, \tau) \times \partial\Omega_-. \quad (20)$$

Multiplying by w and integrating in \mathcal{X} , we find that

$$\int_{\mathcal{X}} w^2(t) dx d\boldsymbol{\beta} + \int_{\mathcal{X}} (\boldsymbol{\beta} \cdot \nabla_x w(t)) w(t) dx d\boldsymbol{\beta} = 0.$$

Using Green's formula (11), we thus get

$$\|w(t)\|_{L^2}^2 + \frac{1}{2} \|w(t)\|_{L^2_+}^2 - \frac{1}{2} \|h(t)\|_{L^2_-}^2 = 0$$

and then

$$\|w(t)\|_{L^2} \leq \frac{1}{\sqrt{2}} \|h(t)\|_{L^2_-}. \quad (21)$$

If we multiply (19) by $\beta \cdot \nabla_x w$ and we integrate in \mathcal{X} , we obtain

$$\int_{\mathcal{X}} w(\beta \cdot \nabla_x w(t)) dx d\beta + \int_{\mathcal{X}} (\beta \cdot \nabla_x w(t))^2 dx d\beta = 0$$

then

$$\|\beta \cdot \nabla_x w(t)\|_{L^2}^2 + \frac{1}{2} \|w(t)\|_{L^2_+}^2 = \frac{1}{2} \|h(t)\|_{L^2_-}^2.$$

From the above it follows that

$$\|w(t)\|_{L^2}^2 \leq \frac{1}{2} \|h(t)\|_{L^2_-}^2$$

then

$$\|w(t)\|_{L^2}^2 + \|\beta \cdot \nabla_x w(t)\|_{L^2}^2 + \frac{1}{2} \|w(t)\|_{L^2_+}^2 \leq \|h(t)\|_{L^2_-}^2.$$

Hence

$$\|w(t)\|_{\mathcal{W}} \leq \|h(t)\|_{L^2_-}. \quad (22)$$

Since $I = I_0 + w$, the estimates (16) and (21) imply

$$\|I(t)\|_{L^2} \leq \sqrt{2\pi} \|T^4(t)\|_{L^2(\Omega)} + \frac{1}{\sqrt{2}} \|h(t)\|_{L^2_-},$$

Finally, in a similar way, according to (18) and (22), we obtain

$$\|I(t)\|_{\mathcal{W}} \leq \sqrt{2\pi} \|T^4(t)\|_{L^2(\Omega)} + \|h(t)\|_{L^2_-}.$$

If we integrate in time between 0 and τ , we obtain

$$\begin{aligned} \|I\|_{L^2(0,\tau;L^2)} &\leq \sqrt{2\pi} \|T\|_{L^8(\mathcal{Q}_\tau)}^4 + \frac{1}{\sqrt{2}} \|h\|_{L^2(0,\tau;L^2_-)} \\ \|I\|_{L^2(0,\tau;\mathcal{W})} &\leq \sqrt{2\pi} \|T\|_{L^8(\mathcal{Q}_\tau)}^4 + \|h\|_{L^2(0,\tau;L^2_-)}, \end{aligned}$$

thus

$$\|I\|_{L^2(0,\tau;L^2)} \leq \sqrt{2\pi} \|T\|_{L^8(\mathcal{Q}_\tau)}^4 + \frac{1}{\sqrt{2}} \|h\|_{L^2(0,\infty;L^2_-)} \quad (23)$$

Then, from (7),(10) we deduce

$$\|I\|_{L^2(\mathcal{Q}_\tau)} \leq 2\sqrt{2\pi} M_\tau^4.$$

Consequently, $I \in E_2^\tau$ and then \mathbb{H}_1 is a well-posed map.

Using the positivity of h and the maximum principle [1], this implies that the solution I of (1), (4) is nonnegative.

Now, we show the continuity of the map \mathcal{H}_1 . We consider I_1, I_2 two solutions of (1) associated to T_1, T_2 , respectively. Let $t \in [0, \tau]$, we have

$$\|I_1(t) - I_2(t)\|_{L^2(\mathcal{X})} \leq \sqrt{2\pi} \|T_1^4(t) - T_2^4(t)\|_{L^2(\Omega)}. \quad (24)$$

Using the generalized Hölder's inequality, we have the following inequality

$$\|T_1^4 - T_2^4\|_{L^2(\mathcal{Q}_\tau)}^2 \leq \|T_1 - T_2\|_{L^8(\mathcal{Q}_\tau)}^2 \|T_1 + T_2\|_{L^8(\mathcal{Q}_\tau)}^2 \|T_1^2 + T_2^2\|_{L^4(\mathcal{Q}_\tau)}^2. \quad (25)$$

Hence

$$\begin{aligned} \|T_1^2 + T_2^2\|_{L^4(\mathcal{Q}_\tau)}^4 &\leq \|T_1\|_{L^8(\mathcal{Q}_\tau)}^8 + 4\|T_1\|_{L^8(\mathcal{Q}_\tau)}^6 \|T_2\|_{L^8(\mathcal{Q}_\tau)}^2 + \|T_2\|_{L^8(\mathcal{Q}_\tau)}^8 \\ &\quad + 6\|T_1\|_{L^8(\mathcal{Q}_\tau)}^4 \|T_2\|_{L^8(\mathcal{Q}_\tau)}^4 + 4\|T_1\|_{L^8(\mathcal{Q}_\tau)}^2 \|T_2\|_{L^8(\mathcal{Q}_\tau)}^6. \end{aligned}$$

Since $T_1, T_2 \in E_1^\tau$, then we deduce that

$$\|T_1^2 + T_2^2\|_{L^4(\mathcal{Q}_\tau)} \leq 2M_\tau^2. \quad (26)$$

On the other hand, we have

$$\|T_1 + T_2\|_{L^8(\mathcal{Q}_\tau)} \leq 2M_\tau, \quad (27)$$

it follows that

$$\|T_1^4 - T_2^4\|_{L^2(\mathcal{Q}_\tau)} \leq 4M_\tau^3 \|T_1 - T_2\|_{L^8(\mathcal{Q}_\tau)},$$

From (24), we deduce that

$$\|I_1 - I_2\|_{L^2(0, \tau; L^2(\mathcal{X}))} \leq 4\sqrt{2\pi} M_\tau^3 \|T_1 - T_2\|_{L^8(\mathcal{Q}_\tau)}. \quad (28)$$

The last inequality shows the continuity of \mathcal{H}_1 . \square

Now, we give some properties of the map \mathcal{H}_2 .

Proposition 3.2. *Under the hypotheses (7), (10), \mathcal{H}_2 is a well posed and continuous map from E_2^τ to E_3^τ . Moreover, for all I solution of the problem (1), (4), $G = \mathcal{H}_2(I)$ is a nonnegative.*

Proof. Let us consider $I \in E_2^\tau$ and $G = \mathcal{H}_2(I)$, then we have

$$\|G\|_{L^2(\mathcal{Q}_\tau)} \leq \sqrt{2\pi} \|I\|_{L^2(0, \tau; L^2(\mathcal{X}))}. \quad (29)$$

Hence, for all $I \in E_2^\tau$, $G = \mathcal{H}_2(I)$ belongs to E_3^τ . Since I is nonnegative then G is nonnegative. Therefore \mathcal{H}_2 is a well posed map. Since \mathcal{H}_2 is a linear function, from the inequality (29), \mathcal{H}_2 is a continuous map. \square

3.2 The map \mathcal{H}_3

In this subsection we introduce some properties of the map \mathcal{H}_3 .

Proposition 3.3. *Let $\tau > 0$, $G \in E_3^T$. Under the assumptions (7), (10), the problem (2),(3),(5) has a nonnegative solution $T \in W_2^{2,1}(\mathbf{Q}_\tau)$.*

Proof. Let $\tau > 0$. For $T_0 \in H^1(\Omega)$, $G \in L^2(\Omega)$ the proof of the existence and uniqueness of the solution of the problem (2),(3),(5), see [2, 32].

Now, in order to prove the non-negativity of the solution of (2),(3),(5), let us consider F defined in $(0, \tau) \times \Omega \times \mathbb{R}$ by

$$F(t, \mathbf{x}, y) = \theta (G(t, \mathbf{x}) - 2\pi y^4).$$

The equation (2) can be rewritten

$$\begin{cases} \partial_t T(t, \mathbf{x}) - \Delta T(t, \mathbf{x}) = F(t, \mathbf{x}, T(t, \mathbf{x})) & \text{for } (t, \mathbf{x}) \in]0, \tau] \times \Omega \\ a\partial_n T(t, \mathbf{x}) + bT(t, \mathbf{x}) = g(t, \mathbf{x}) & \text{for } (t, \mathbf{x}) \in]0, \tau] \times \partial\Omega \\ T(0, \mathbf{x}) = T_0(\mathbf{x}) & \text{for } \mathbf{x} \in \Omega. \end{cases} \quad (30)$$

Now, we define \bar{F} in $(0, \tau) \times \Omega \times \mathbb{R}$ by

$$\bar{F}(t, \mathbf{x}, y) = \begin{cases} \theta (G(t, \mathbf{x}) - 2\pi y^4) & \text{if } y \geq 0 \\ \theta G(t, \mathbf{x}) & \text{if } y < 0. \end{cases}$$

Let us consider \bar{T} the solution of the following system

$$\begin{cases} \partial_t \bar{T}(t, \mathbf{x}) - \Delta \bar{T}(t, \mathbf{x}) = \bar{F}(t, \mathbf{x}, \bar{T}(t, \mathbf{x})) & \text{for } (t, \mathbf{x}) \in]0, \tau] \times \Omega \\ a\partial_n \bar{T}(t, \mathbf{x}) + b\bar{T}(t, \mathbf{x}) = g(t, \mathbf{x}) & \text{for } (t, \mathbf{x}) \in]0, \tau] \times \partial\Omega \\ \bar{T}(0, \mathbf{x}) = T_0(\mathbf{x}) & \text{for } \mathbf{x} \in \Omega \end{cases} \quad (31)$$

Our goal is to prove that the solution \bar{T} of this equation remains nonnegative over the time. Indeed, in this case \bar{F} and F coincide, therefore we have by the uniqueness of the solution $T = \bar{T}$ which is nonnegative.

We set $\bar{T}^+ = \max(\bar{T}, 0)$ and $\bar{T}^- = \max(-\bar{T}, 0)$, such that $\bar{T} = \bar{T}^+ - \bar{T}^-$.

Multiplying the equation (31) by $(-\bar{T}^-)$ and integrating over Ω , we obtain

$$-\int_{\Omega} \partial_t \bar{T}(t, \mathbf{x}) \bar{T}^-(t, \mathbf{x}) d\mathbf{x} + \int_{\Omega} \Delta \bar{T}(t, \mathbf{x}) \bar{T}^-(t, \mathbf{x}) d\mathbf{x} = -\int_{\Omega} \bar{F}(t, \mathbf{x}, \bar{T}) \bar{T}^-(t, \mathbf{x}) d\mathbf{x}.$$

Now, we have

$$-\int_{\Omega} \partial_t \bar{T}(t, \mathbf{x}) \bar{T}^-(t, \mathbf{x}) d\mathbf{x} = \frac{1}{2} \partial_t \int_{\Omega} (\bar{T}^-(t, \mathbf{x}))^2 d\mathbf{x}, \quad (32)$$

$$\begin{aligned} -\int_{\Omega} \bar{F}(t, \mathbf{x}, \bar{T}) \bar{T}^-(t, \mathbf{x}) d\mathbf{x} &= -\int_{\{\bar{T} < 0\}} \bar{F}(t, \mathbf{x}, \bar{T}) \bar{T}^-(t, \mathbf{x}) d\mathbf{x} \\ &= -\theta \int_{\{\bar{T} < 0\}} G(t, \mathbf{x}) \bar{T}^-(t, \mathbf{x}) d\mathbf{x} \leq 0, \end{aligned} \quad (33)$$

and

$$\int_{\Omega} \Delta \bar{T}(t, \mathbf{x}) \bar{T}^{-}(t, \mathbf{x}) d\mathbf{x} = \int_{\Omega} (\nabla \bar{T}^{-}(t, \mathbf{x}))^2 d\mathbf{x} + \int_{\partial\Omega} \partial_n \bar{T}(t, \mathbf{x}) \bar{T}^{-}(t, \mathbf{x}) d\Gamma.$$

If $a > 0$ (Robin or Neumann boundary conditions), then

$$\begin{aligned} \int_{\partial\Omega} \partial_n \bar{T}(t, \mathbf{x}) \bar{T}^{-}(t, \mathbf{x}) d\Gamma &= -\frac{b}{a} \int_{\partial\Omega} \bar{T}(t, \mathbf{x}) \bar{T}^{-}(t, \mathbf{x}) d\Gamma \\ &\quad + \frac{1}{a} \int_{\partial\Omega} g(t, \mathbf{x}) \bar{T}^{-}(t, \mathbf{x}) d\Gamma \\ &= \frac{b}{a} \int_{\partial\Omega} (\bar{T}^{-}(t, \mathbf{x}))^2 d\Gamma \\ &\quad + \frac{1}{a} \int_{\partial\Omega} g(t, \mathbf{x}) \bar{T}^{-}(t, \mathbf{x}) d\Gamma \geq 0. \end{aligned} \quad (34)$$

Now, if we have $a = 0$ (thus $b > 0$), since $\bar{T}^{-} = 0$ on $\partial\Omega$ then

$$\int_{\partial\Omega} \partial_n \bar{T}(t, \mathbf{x}) \bar{T}^{-}(t, \mathbf{x}) d\Gamma = 0. \quad (35)$$

In the both cases, we have

$$\int_{\Omega} \Delta \bar{T}(t, \mathbf{x}) \bar{T}^{-}(t, \mathbf{x}) d\mathbf{x} \geq 0. \quad (36)$$

Consequently, (32), (33) and (36) imply

$$\frac{1}{2} \partial_t \int_{\Omega} (\bar{T}^{-}(t, \mathbf{x}))^2 d\mathbf{x} \leq 0. \quad (37)$$

As T_0 is nonnegative, we deduce from (37) that $\bar{T}^{-} \equiv 0$. It follows that \bar{T} and consequently T are nonnegative in $(0, \tau) \times \Omega$.

In the following, we prove that $T \in W_2^{2,1}(\mathbf{Q}_{\tau})$. For it, let us introduce z the solution of the parabolic problem

$$\begin{cases} \partial_t z(t, \mathbf{x}) - \Delta z(t, \mathbf{x}) = \theta G(t, \mathbf{x}) & \text{for } (t, \mathbf{x}) \in]0, \tau] \times \Omega \\ a \partial_n z(t, \mathbf{x}) + bz(t, \mathbf{x}) = g(t, \mathbf{x}) & \text{for } (t, \mathbf{x}) \in]0, \tau] \times \partial\Omega \\ z(t, \mathbf{x}) = T_0 & \text{for } \mathbf{x} \in \Omega \end{cases} \quad (38)$$

Since $G \in L^2(\mathbf{Q}_{\tau})$, $T_0 \in H^1(\Omega)$ and thanks to a result on parabolic regularity, see[32], then $z \in W_2^{2,1}(\mathbf{Q}_{\tau})$ and there exists a constant $\tilde{C} > 0$ such that

$$\|z\|_{W_2^{2,1}(\mathbf{Q}_{\tau})} \leq \tilde{C} \left(\|G\|_{L^2(\mathbf{Q}_{\tau})} + \|T_0\|_{H^1(\Omega)} + \|g\|_{L^2(0, \tau; H^{\frac{3}{2}}(\Omega))} \right). \quad (39)$$

For more details, we refer the reader to [20, p.197]. Then, using the Sobolev embedding we deduce that T is a subsolution of (30), then using the maximum principle, we have that $T \leq z$. Consequently, T belongs to $L^8(\mathbf{Q}_{\tau})$.

Now, by the bootstrap argument we obtain T belongs to $W_2^{2,1}(\mathbf{Q}_\tau)$. Indeed, since G and T^4 belong to $L^2(\mathbf{Q}_\tau)$, we have $\theta G - 4\pi\theta T^4$ belongs to $L^2(\mathbf{Q}_\tau)$. Consequently, using the same result on parabolic regularity we obtain $T \in W_2^{2,1}(\mathbf{Q}_\tau)$. \square

Theorem 3.4. *Under the hypotheses of Proposition 3.3, $\mathcal{H}_3(E_3^\tau) \subseteq E_1^\tau$.*

Proof. Let $G \in E_3^\tau$. We have already proved that T belongs to $L^8(\mathbf{Q}_\tau)$. However, to prove that $T = \mathcal{H}_3(G) \in E_1^\tau$, we need a more precise control of $\|T\|_{L^8(\mathbf{Q}_\tau)}$.

As $T \in W_2^{2,1}(\mathbf{Q}_\tau)$ then T^4 belongs to $L^2(0, \tau; H^1(\Omega))$. Thus, we can multiply the equation (2) by T^4 and we integrate over Ω , we obtain for all $t \in (0, \tau)$

$$\begin{aligned} \frac{1}{5} \frac{d}{dt} \|T(t)\|_{L^5(\Omega)}^5 + 4 \int_{\Omega} (\nabla T(t, \mathbf{x}))^2 T^3(t, \mathbf{x}) d\mathbf{x} - \int_{\partial\Omega} \partial_n T(t, \mathbf{x}) T^4(t, \mathbf{x}) d\Gamma \\ + 2\pi\theta \int_{\Omega} T^8(t, \mathbf{x}) d\mathbf{x} = \theta \int_{\Omega} G(t, \mathbf{x}) T^4(t, \mathbf{x}) d\mathbf{x}. \end{aligned} \quad (40)$$

Using the Young's inequality, we get

$$\begin{aligned} \frac{1}{5} \frac{d}{dt} \|T(t)\|_{L^5(\Omega)}^5 + \frac{16}{25} \int_{\Omega} (\nabla T^{\frac{5}{2}}(t, \mathbf{x}))^2 d\mathbf{x} + 2\pi\theta \int_{\Omega} T^8(t, \mathbf{x}) d\mathbf{x} \\ \leq \frac{\theta}{8\pi} \int_{\Omega} G^2(t, \mathbf{x}) d\mathbf{x} + \int_{\partial\Omega} \partial_n T(t, \mathbf{x}) T^4(t, \mathbf{x}) d\Gamma. \end{aligned} \quad (41)$$

For each type of boundary conditions, the treatment of the boundary terms will be different. For this way we start by the simplest case Robin boundary conditions ($a > 0, b > 0$).

Using Young's inequality, choosing $\epsilon_1 = \frac{5b}{4}$, we have

$$\begin{aligned} \int_{\partial\Omega} \partial_n T(t) T^4(t, \mathbf{x}) d\Gamma &= -\frac{b}{a} \int_{\partial\Omega} T^5(t, \mathbf{x}) d\Gamma + \frac{1}{a} \int_{\partial\Omega} g T^4(t, \mathbf{x}) d\Gamma \\ &\leq -\frac{b}{a} \int_{\partial\Omega} T^5(t, \mathbf{x}) d\Gamma + \frac{4\epsilon_1}{5a} \int_{\partial\Omega} T^5(t, \mathbf{x}) d\Gamma \\ &\quad + \frac{1}{5a\epsilon_1} \int_{\partial\Omega} g^5(t, \mathbf{x}) d\Gamma \\ &\leq \frac{4}{25ab} \int_{\partial\Omega} g^5(t, \mathbf{x}) d\Gamma. \end{aligned}$$

Thus from (41), it follows that

$$\begin{aligned} \frac{1}{5} \frac{d}{dt} \|T(t)\|_{L^5(\Omega)}^5 + 2\pi\theta \int_{\Omega} T^8(t, \mathbf{x}) d\mathbf{x} &\leq \frac{\theta}{8\pi} \int_{\Omega} G^2(t, \mathbf{x}) d\mathbf{x} \\ &\quad + \frac{4}{25ab} \int_{\partial\Omega} g^5(t, \mathbf{x}) d\Gamma. \end{aligned}$$

We integrate in time between 0 and τ , we obtain

$$\|T\|_{L^8(\mathbf{Q}_\tau)}^8 \leq \frac{1}{16\pi^2} \|G\|_{L^2(\mathbf{Q}_\tau)}^2 + \frac{4}{50\pi\theta a^2 b} \tau \left| \partial\Omega \right| \|g\|_{L^\infty((0,\tau) \times \bar{\Omega})} + \frac{1}{2\pi\theta} \|T_0\|_{L^5(\Omega)}^5.$$

Since $G \in E_3^\tau$, g and T_0 satisfy (10), it follows that

$$\|T\|_{L^8(\mathbf{Q}_\tau)}^8 \leq M_\tau^8. \quad (42)$$

Hence, we have $T \in E_1$. Then the map \mathcal{H}_3 is well-posed.

Now, we consider the Neumann boundary conditions ($a > 0, b = 0$). Let consider the boundary term of (41)

$$\begin{aligned} \int_{\partial\Omega} \partial_n T(t, \mathbf{x}) T^4(t, \mathbf{x}) d\Gamma &= \frac{1}{a} \int_{\partial\Omega} g(t, \mathbf{x}) T^4(t, \mathbf{x}) d\Gamma \\ &\leq \frac{1}{5a\epsilon} \int_{\partial\Omega} g^5(t, \mathbf{x}) d\Gamma + \frac{4\epsilon}{5a} \int_{\partial\Omega} T^5(t, \mathbf{x}) d\Gamma \\ &\leq \frac{1}{5a\epsilon} \int_{\partial\Omega} g^5(t, \mathbf{x}) d\Gamma + \frac{4\epsilon}{5a} \int_{\partial\Omega} \left(T^{\frac{5}{2}}(t, \mathbf{x}) \right)^2 d\Gamma. \end{aligned}$$

Then there exists $C(\Omega) > 0$, see[15], such that

$$\int_{\partial\Omega} \partial_n T(t, \mathbf{x}) T^4(t, \mathbf{x}) d\Gamma \leq \frac{1}{5a\epsilon} \int_{\partial\Omega} g^5(t, \mathbf{x}) d\Gamma + \frac{4\epsilon}{5a} C(\Omega) \|T^{\frac{5}{2}}(t)\|_{H^1(\Omega)}^2. \quad (43)$$

Choosing $\epsilon = \frac{4a}{5C(\Omega)}$, substituting (43) into (41), for all $t \in (0, \tau)$

$$\begin{aligned} \frac{d}{dt} \|T(t)\|_{L^5(\Omega)}^5 + 10\pi\theta \int_{\Omega} T^8(t, \mathbf{x}) d\mathbf{x} &\leq \frac{5\theta}{8\pi} \int_{\Omega} G^2(t, \mathbf{x}) d\mathbf{x} \\ &+ \frac{16}{5} \|T(t)\|_{L^5(\Omega)}^5 + \frac{5C(\Omega)}{4a^2} \int_{\partial\Omega} g^5(t, \mathbf{x}) d\Gamma, \end{aligned} \quad (44)$$

Using the Young inequality we obtain

$$\|T(t)\|_{L^5(\Omega)}^5 \leq \frac{5\epsilon}{8} \|T(t)\|_{L^8(\Omega)}^8 + \frac{3}{8\epsilon} |\Omega|. \quad (45)$$

Integrating (44) in time between 0 and τ and using (45) we obtain

$$\begin{aligned} \|T(\tau)\|_{L^5(\Omega)}^5 + 10\pi\theta \|T\|_{L^8(\mathbf{Q}_\tau)}^8 &\leq \frac{5\theta}{8\pi} \|G\|_{L^2(\mathbf{Q}_\tau)}^2 + 2\epsilon \|T\|_{L^8(\mathbf{Q}_\tau)}^8 \\ &+ \tau \frac{6}{5\epsilon} |\Omega| + \frac{5C(\Omega)}{4a^2} \int_{\Sigma_\tau} g^5(s, \mathbf{x}) d\Gamma ds \\ &+ \|T_0\|_{L^5(\Omega)}^5. \end{aligned}$$

Taking $\epsilon = \frac{5\pi\theta}{2}$, we obtain

$$\begin{aligned} \|T\|_{L^8(\mathbf{Q}_\tau)}^8 &\leq \frac{1}{8\pi^2} \|G\|_{L^2(\mathbf{Q}_\tau)}^2 + \tau \frac{12}{75\pi^2\theta^2} |\Omega| + \frac{C(\Omega)}{4a^2\pi\theta} \int_{\Sigma_\tau} g^5(s, \mathbf{x}) d\Gamma ds \\ &\quad + \frac{1}{5\pi\theta} \|T_0\|_{L^5(\Omega)}^5, \end{aligned}$$

then

$$\begin{aligned} \|T\|_{L^8(\mathbf{Q}_\tau)}^8 &\leq \frac{1}{8\pi^2} \|G\|_{L^2(\mathbf{Q}_\tau)}^2 + \tau \left(\frac{12}{75\pi^2\theta^2} |\Omega| + \frac{|\partial\Omega|}{4a^2\pi\theta} C(\Omega) \|g\|_{L^\infty((0,\tau)\times\bar{\Omega})}^5 \right) \\ &\quad + \frac{1}{5\pi\theta} \|T_0\|_{L^5(\Omega)}^5. \end{aligned}$$

Since $G \in E_3^\tau$, g and T_0 satisfy (10), it follows that

$$\|T\|_{L^8(\mathbf{Q}_\tau)}^8 \leq M_\tau^8. \quad (46)$$

Hence, we have $T \in E_1$. Then the map \mathcal{H}_3 is well-posed.

Finally, we consider the case of Dirichlet boundary conditions ($a = 0, b > 0$). This type of boundary conditions request a different analytical tools.

To bound the last term on the right hand side of (41), we multiply the equation (2) by g^4 (given in (7)) and we integrate over \mathbf{Q}_τ , we get

$$\begin{aligned} \int_0^\tau \int_\Omega \left[\partial_t T(s, \mathbf{x}) - \Delta T(s, \mathbf{x}) + 2\pi\theta T^4(s, \mathbf{x}) \right] g^4(s, \mathbf{x}) d\mathbf{x} ds \\ = \int_0^\tau \int_\Omega G(s, \mathbf{x}) g^4(s, \mathbf{x}) d\mathbf{x} ds \end{aligned}$$

Therefore, we deduce from Green's Formula that

$$\begin{aligned} \int_\Omega \left[T(\tau, \mathbf{x}) g^4(\tau, \mathbf{x}) - T(0, \mathbf{x}) g^4(0, \mathbf{x}) \right] d\mathbf{x} - \int_\Omega \int_0^\tau T(s, \mathbf{x}) \partial_t g^4(s, \mathbf{x}) ds d\mathbf{x} \\ - \int_{\mathbf{Q}_\tau} T(s, \mathbf{x}) \Delta(g^4)(s, \mathbf{x}) d\mathbf{x} ds + \frac{1}{b} \int_{\Sigma_\tau} g(s, \mathbf{x}) \partial_n g^4(s, \mathbf{x}) d\mathbf{x} ds \\ - \int_{\Sigma_\tau} \partial_n T(s, \mathbf{x}) g^4(s, \mathbf{x}) d\mathbf{x} ds + 2\pi\theta \int_{\mathbf{Q}_\tau} T^4(s, \mathbf{x}) g^4(s, \mathbf{x}) d\mathbf{x} ds \\ = \int_{\mathbf{Q}_\tau} G(s, \mathbf{x}) g^4(s, \mathbf{x}) d\mathbf{x} ds. \end{aligned} \quad (47)$$

Using the positivity of G and T_0 , (47) becomes

$$\begin{aligned} \int_{\Sigma_\tau} \partial_n T(s, \mathbf{x}) g^4(s, \mathbf{x}) d\mathbf{x} ds \leq 2\pi\theta \int_{\mathbf{Q}_\tau} T^4(s, \mathbf{x}) g^4(s, \mathbf{x}) d\mathbf{x} ds + \int_\Omega T(\tau, \mathbf{x}) g^4(\tau, \mathbf{x}) d\mathbf{x} \\ - \int_{\mathbf{Q}_\tau} T(s, \mathbf{x}) (\partial_t + \Delta) g^4(s, \mathbf{x}) d\mathbf{x} ds \\ + \frac{1}{b} \int_{\Sigma_\tau} g(s, \mathbf{x}) \partial_n g^4(s, \mathbf{x}) d\mathbf{x} ds \end{aligned}$$

Then, Young's inequality implies that there exists $\epsilon_1 > 0$, $\epsilon_2 > 0$, $\epsilon_3 > 0$

$$\begin{aligned} \int_{\Sigma_\tau} \partial_n T(s, \mathbf{x}) g^4(s, \mathbf{x}) d\mathbf{x} ds &\leq \frac{2\pi\theta}{\epsilon_1} \|T\|_{L^8(\mathbf{Q}_\tau)}^8 + 2\pi\theta\tau\epsilon_1 |\Omega| \|g\|_{L^\infty(\mathbf{Q}_\tau)}^8 + \frac{1}{8\epsilon_2} \|T\|_{L^8(\mathbf{Q}_\tau)}^8 \\ &\quad + \frac{7\epsilon_2}{8} \tau |\Omega| \|(\partial_t + \Delta) g^4\|_{L^\infty(\mathbf{Q}_\tau)}^{8/7} + \frac{1}{5\epsilon_3} \|T(\tau)\|_{L^5(\Omega)}^5 \\ &\quad + \frac{4}{5}\epsilon_3 |\Omega| \|g\|_{L^\infty(\mathbf{Q}_\tau)}^5 + \frac{\tau}{b} |\partial\Omega| \|g\partial_n g^4\|_{L^\infty((0,\tau)\times\bar{\Omega})}. \end{aligned} \quad (48)$$

For the Dirichlet boundary conditions, the inequality (41) becomes

$$\begin{aligned} \frac{1}{5} \frac{d}{dt} \|T(t)\|_{L^5(\Omega)}^5 + \frac{16}{25} \int_{\Omega} (\nabla T^{\frac{5}{2}}(t, \mathbf{x}))^2 d\mathbf{x} + 2\pi\theta \int_{\Omega} T^8(t, \mathbf{x}) d\mathbf{x} \\ \leq \frac{\theta}{8\pi} \int_{\Omega} G^2(t, \mathbf{x}) d\mathbf{x} + \frac{1}{b^4} \int_{\partial\Omega} \partial_n T(t, \mathbf{x}) g^4(t, \mathbf{x}) d\Gamma. \end{aligned} \quad (49)$$

Integrating (49) in time between 0 and τ and using (48) we obtain

$$\begin{aligned} \|T(\tau)\|_{L^5(\Omega)}^5 + 10\pi\theta \|T\|_{L^8(\mathbf{Q}_\tau)}^8 &\leq \frac{5\theta}{8\pi} \|G\|_{L^2(\mathbf{Q}_\tau)}^2 + \frac{10\pi\theta}{b^4\epsilon_1} \|T\|_{L^8(\mathbf{Q}_\tau)}^8 \\ &\quad + \frac{10}{b^4} \pi\theta\tau\epsilon_1 |\Omega| \|g\|_{L^\infty(\mathbf{Q}_\tau)}^8 + \frac{5}{8b^4\epsilon_2} \|T\|_{L^8(\mathbf{Q}_\tau)}^8 \\ &\quad + \frac{35\epsilon_2}{8b^4} \tau |\Omega| \|(\partial_t + \Delta) g^4\|_{L^\infty(\mathbf{Q}_\tau)}^{8/7} \\ &\quad + \frac{1}{b^4\epsilon_3} \|T(\tau)\|_{L^5(\Omega)}^5 + \frac{4}{b^4}\epsilon_3 |\Omega| \|g\|_{L^\infty(\mathbf{Q}_\tau)}^5 \\ &\quad + \frac{5\tau}{b^5} |\partial\Omega| \|g\partial_n g^4\|_{L^\infty((0,\tau)\times\bar{\Omega})} \\ &\quad + \|T_0\|_{L^5(\Omega)}^5. \end{aligned}$$

Choosing $\epsilon_1 = \frac{5}{b^4}$, $\epsilon_2 = \frac{5}{24\pi\theta b^4}$ and $\epsilon_3 = \frac{2}{b^4}$, then

$$\begin{aligned} 5\pi\theta \|T\|_{L^8(\mathbf{Q}_\tau)}^8 &\leq \frac{5\theta}{8\pi} \|G\|_{L^2(\mathbf{Q}_\tau)}^2 + \frac{10}{b^8} \pi\theta\tau |\Omega| \|g\|_{L^\infty(\mathbf{Q}_\tau)}^8 \\ &\quad + \frac{35}{192\pi\theta b^8} \tau |\Omega| \|(\partial_t + \Delta) g^4\|_{L^\infty(\mathbf{Q}_\tau)}^{8/7} \\ &\quad + \frac{8}{5b^8} |\Omega| \|g\|_{L^\infty(\mathbf{Q}_\tau)}^5 + \frac{5\tau}{b^5} |\partial\Omega| \|g\partial_n g^4\|_{L^\infty((0,\tau)\times\bar{\Omega})} + \|T_0\|_{L^5(\Omega)}^5. \end{aligned}$$

Thus

$$\begin{aligned}
\|T\|_{L^8(\mathbf{Q}_\tau)}^8 &\leq \frac{1}{8\pi^2} \|G\|_{L^2(\mathbf{Q}_\tau)}^2 + \frac{2}{b^8} \tau |\Omega| \|g\|_{L^\infty(\mathbf{Q}_\tau)}^8 \\
&\quad + \frac{7}{192\pi^2\theta^2 b^8} \tau |\Omega| \|(\partial_t + \Delta)g^4\|_{L^\infty(\mathbf{Q}_\tau)}^{8/7} \\
&\quad + \frac{8}{25\pi\theta b^8} |\Omega| \|g\|_{L^\infty(\mathbf{Q}_\tau)}^5 + \frac{\tau}{b^5\pi\theta} |\partial\Omega| \|g\partial_n g^4\|_{L^\infty((0,\tau)\times\bar{\Omega})} \\
&\quad + \frac{1}{5\pi\theta} \|T_0\|_{L^5(\Omega)}^5.
\end{aligned}$$

Since $G \in E_3^\tau$, g and T_0 satisfy (10), it follows that

$$\|T\|_{L^8(\mathbf{Q}_\tau)}^8 \leq M_\tau^8. \quad (50)$$

Hence, we have $T \in E_1$. Then the map \mathcal{H}_3 is well-posed.

Finally, for any type of boundary conditions, we conclude that there exists $M_\tau > 0$ such that $T \in E_1$. Then the map \mathcal{H}_3 is well-posed. \square

Remark 3.5. *In immediate consequence of Theorem 3.4 is that we can reduce the regularity of g and always in 2-dimensional case, it suffices to take $g \in H^{1/4}(0, \tau; L^2(\partial\Omega)) \cap L^\infty(\Sigma_\tau) \cap L^2(0, \tau; H^{1/2}(\Omega))$ for Robin and Neumann boundary conditions case and take $g \in H^{3/4}(0, \tau; L^2(\partial\Omega)) \cap L^\infty(\Sigma_\tau) \cap L^2(0, \tau; H^{3/2}(\Omega))$ for Dirichlet boundary conditions. For more informations on the regularity of the trace operator we refer the reader to [20].*

Proposition 3.6. *Under the assumptions of Theorem 3.4, \mathcal{H}_3 is a continuous map from E_3^τ to E_1^τ .*

Proof. Let $G_1, G_2 \in E_3^\tau$, $T_1 = \mathcal{H}_3(G_1)$ and $T_2 = \mathcal{H}_3(G_2)$. Let us set $w = T_1 - T_2$, then w is solution of the following equation

$$\begin{aligned}
\partial_t w(t, \mathbf{x}) - \Delta w(t, \mathbf{x}) &= -2\pi\theta(T_1^4 - T_2^4)(t, \mathbf{x}) + \theta(G_1 - G_2)(t, \mathbf{x}) && \text{in } \mathbf{Q}_\tau, \\
w(0, \mathbf{x}) &= 0 && \text{in } \Omega.
\end{aligned}$$

supplemented to homogenous boundary conditions (homogeneous Dirichlet, Neumann, Robin). So we have, [16]

$$w(t) = -2\pi\theta \int_0^t \mathbb{T}(t-s)(T_1^4 - T_2^4)(s)ds + \theta \int_0^t \mathbb{T}(t-s)(G_1 - G_2)(s)ds$$

where $\mathbb{T}(t)$ is a semigroup of contraction in $L^2(\Omega)$ generated by the operator \mathbf{A} defined by

$$D(\mathbf{A}) = \{T \in H^2(\Omega), \text{ and } a\partial_n T + bT = 0, \text{ on } \partial\Omega\},$$

for all $a, b \in \mathbb{R}_+$.

Now, using the regularizing effects of the heat equation, see [16, proposition 3.5.7, p.44] with $p = 8$ and $q = 2$, we deduce the following inequality

$$\begin{aligned} \|w(t)\|_{L^8(\Omega)} &\leq 2\pi\theta \int_0^t \frac{1}{(4\pi(t-s))^{\frac{1}{2}-\frac{1}{8}}} \|T_1^4(s) - T_2^4(s)\|_{L^2(\Omega)} ds \\ &\quad + \theta \int_0^t \frac{1}{(4\pi(t-s))^{\frac{1}{2}-\frac{1}{8}}} \|G_1(s) - G_2(s)\|_{L^2(\Omega)} ds. \end{aligned} \quad (51)$$

In view of the Hölder's inequality, the Cauchy-Schwarz inequality and (25), then (51) becomes

$$\begin{aligned} \|w(t)\|_{L^8(\Omega)} &\leq 2\pi\theta \int_0^t \frac{1}{(4\pi(t-s))^{\frac{1}{2}-\frac{1}{8}}} \|w(s)\|_{L^8(\Omega)} \\ &\quad \times \|T_1(s) + T_2(s)\|_{L^8(\Omega)} \|(T_1^2 + T_2^2)(s)\|_{L^4(\Omega)} ds \\ &\quad + \theta \left(\int_0^t \frac{ds}{(4\pi(t-s))^{\frac{3}{4}}} \right)^{\frac{1}{2}} \|G_1 - G_2\|_{L^2(\mathbf{Q}_\tau)}. \end{aligned} \quad (52)$$

We have

$$\left(\int_0^t \frac{ds}{(4\pi(t-s))^{\frac{3}{4}}} \right)^{\frac{1}{2}} = \left(\frac{4\sqrt[4]{t}}{(2\pi)^{\frac{3}{4}}} \right)^{\frac{1}{2}} = \frac{2\sqrt[8]{t}}{(2\pi)^{\frac{3}{8}}}. \quad (53)$$

Thanks to the generalized Hölder's inequality, we get

$$\begin{aligned} \int_0^t \frac{1}{(4\pi(t-s))^{\frac{1}{2}-\frac{1}{8}}} \|w(s)\|_{L^8(\Omega)} \|(T_1 + T_2)(s)\|_{L^8(\Omega)} \|(T_1^2 + T_2^2)(s)\|_{L^4(\Omega)} ds \\ \leq \frac{2\sqrt[8]{t}}{(2\pi)^{\frac{3}{8}}} \left(\int_0^t \|w(s)\|_{L^8(\Omega)}^8 ds \right)^{\frac{1}{8}} \|T_1 + T_2\|_{L^8(\mathbf{Q}_\tau)} \|T_1^2 + T_2^2\|_{L^4(\mathbf{Q}_\tau)}. \end{aligned} \quad (54)$$

We substitute (53) and (54) into (52), we obtain

$$\begin{aligned} \|w(t)\|_{L^8(\Omega)} &\leq 2\pi\theta \frac{2\sqrt[8]{t}}{(2\pi)^{\frac{3}{8}}} \left(\int_0^t \|w(s)\|_{L^8(\Omega)}^8 ds \right)^{\frac{1}{8}} \|T_1 + T_2\|_{L^8(\mathbf{Q}_\tau)} \|T_1^2 + T_2^2\|_{L^4(\mathbf{Q}_\tau)} \\ &\quad + \theta \frac{2\sqrt[8]{t}}{(2\pi)^{\frac{3}{8}}} \|G_1 - G_2\|_{L^2(\mathbf{Q}_\tau)}. \end{aligned}$$

The estimations (26) and (27) give

$$\begin{aligned}
\|w(t)\|_{L^8(\Omega)} &\leq 2\pi\theta \frac{2\sqrt[8]{t}}{(2\pi)^{\frac{3}{8}}} 4M^3 \left(\int_0^t \|w(s)\|_{L^8(\Omega)}^8 ds \right)^{\frac{1}{8}} \\
&\quad + \theta \frac{2\sqrt[8]{t}}{(2\pi)^{\frac{3}{8}}} \|G_1 - G_2\|_{L^2(\mathbf{Q}_\tau)} \\
&\leq 2\pi\theta \frac{2\sqrt[8]{\tau}}{(2\pi)^{\frac{3}{8}}} 4M^3 \left(\int_0^t \|w(s)\|_{L^8(\Omega)}^8 ds \right)^{\frac{1}{8}} \\
&\quad + \theta \frac{2\sqrt[8]{\tau}}{(2\pi)^{\frac{3}{8}}} \|G_1 - G_2\|_{L^2(\mathbf{Q}_\tau)}.
\end{aligned}$$

Since $(c+d)^8 \leq 128(c^8 + d^8)$ for all $(c, d) \in \mathbf{R}_+^2$, it follows that

$$\|w(t)\|_{L^8(\Omega)}^8 \leq \frac{\theta^8}{\pi^3} 2^{25} M^{24} \tau \int_0^t \|w(s)\|_{L^8(\Omega)}^8 ds + \frac{\theta^8}{\pi^3} 2^9 \tau \|G_1 - G_2\|_{L^2(\mathbf{Q}_\tau)}^2.$$

Applying the Gronwall's inequality, we deduce

$$\|\mathcal{H}_3(G_1) - \mathcal{H}_3(G_2)\|_{L^8(\mathbf{Q}_\tau)}^8 \leq \frac{\theta^8}{\pi^3} 2^9 \tau^2 e^{\frac{\theta^8}{\pi^3} 2^{25} M^{24} \tau^2} \|G_1 - G_2\|_{L^2(\mathbf{Q}_\tau)}^2. \quad (55)$$

□

3.3 Existence and uniqueness of the solution

Now, we may give a very direct proof of Theorem 2.1 using Schauder's theorem.

Proof of Theorem 2.1. $\mathcal{H} = \mathcal{H}_3 \circ \mathcal{H}_2 \circ \mathcal{H}_1$ is a well-posed continuous map because it is composed by a three well-posed continuous maps.

Moreover, $\mathcal{H}(E_1^\tau)$ is a relatively compact. Indeed, $\mathcal{H}(E_1^\tau) \subset W_2^{2,1}(\mathbf{Q}_\tau)$, see (39). The embedding $W_2^{2,1}(\mathbf{Q}_\tau)$ in $L^p(\mathbf{Q}_\tau)$ is compact, for all $p \in [2, \infty[$ in two dimensional case, for more details see [32, Lemma 3.3].

Consequently, $W_2^{2,1}(\mathbf{Q}_\tau)$ is compactly embedded in $L^8(\mathbf{Q}_\tau)$.

All conditions of Schauder fixed Theorem are checked. Then, \mathcal{H} admits a fixed point T such that $\mathcal{H}(T) = T$.

Now, we prove the uniqueness of the solution for the coupled system (1)-(5).

Let us consider $(T_1, T_2) \in E_1^{\tau^2}$, $(I_1, I_2) \in E_2^{\tau^2}$ and $(G_1, G_2) \in E_3^{\tau^2}$ such that

$$\begin{cases} I_1 = \mathcal{H}_1(T_1), & I_2 = \mathcal{H}_1(T_2) \\ G_1 = \mathcal{H}_2(I_1), & G_2 = \mathcal{H}_2(I_2) \\ T_1 = \mathcal{H}_3(G_1), & T_2 = \mathcal{H}_3(G_2) \\ T_1 = \mathcal{H}(T_1), & T_2 = \mathcal{H}(T_2). \end{cases}$$

$\mathcal{H}_1, \mathcal{H}_2$ and \mathcal{H}_3 are a continuous maps , then from (28), (29) and (55) it follows that

$$\begin{aligned} \|I_1 - I_2\|_{L^2(0,\tau;L^2(\mathbf{x}))} &\leq 4\sqrt{2\pi}M_\tau^3\|T_1 - T_2\|_{L^8(\mathbf{Q}_\tau)}, \\ \|G_1 - G_2\|_{L^2(\mathbf{Q}_\tau)} &\leq \sqrt{2\pi}\|I_1 - I_2\|_{L^2(0,\tau;L^2(\mathbf{x}))}, \\ \|T_1 - T_2\|_{L^8(\mathbf{Q}_\tau)}^8 &\leq \frac{\theta^8}{\pi^3}2^9\tau^2e^{\frac{\theta^8}{\pi^3}2^{25}M_\tau^{24}\tau^2}\|G_1 - G_2\|_{L^2(\mathbf{Q}_\tau)}^8. \end{aligned}$$

Hence, under the assumptions (7),(10), there exists $M_\tau > 0$ such that \mathcal{H} is a contraction map in E_1^τ . Then, we deduce the uniqueness of the solution.

Finally, \mathcal{H} admits a unique fixed point T such that $\mathcal{H}(T) = T$. Then, the system (1)-(5) has a unique solution $(T, I) \in E_1^\tau \times E_2^\tau$. Therefore, by Theorem 3.1 and Proposition 3.3, it follows that $I \in L^2(0, \tau; \mathbf{W})$ and $T \in W_2^{2,1}(\mathbf{Q}_\tau)$. \square

Remark 3.7. *We find it important to remark here that the existence and uniqueness of the solution for radiative conductive heat transfer system is established for all $\tau > 0$. Then, the existence and uniqueness result is global.*

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