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EXTREME VALUE LAWS FOR SEQUENCES OF INTERMITTENT MAPS

ANA CRISTINA MOREIRA FREITAS, JORGE MILHAZES FREITAS, AND SANDRO VAIENTI

Abstract. We study non-stationary stochastic processes arising from sequential dynamical systems built on maps with a neutral fixed points and prove the existence of Extreme Value Laws for such processes. We use an approach developed in [FFV16], where we generalised the theory of extreme values for non-stationary stochastic processes, mostly by weakening the uniform mixing condition that was previously used in this setting. The present work is an extension of our previous results for concatenations of uniformly expanding maps obtained in [FFV16].

1. Introduction

The erratic behaviour of chaotic dynamical systems motivated the use of probabilistic tools to study the statistical behaviour of such systems. The time evolution of chaotic systems gives rise to time series resulting from evaluating an observable function along the orbits of the system. The statistical properties of these stochastic processes, in particular, the existence of limiting distributions has become a very important topic in Ergodic Theory.

The mixing features of the systems determine the dependence structure of the processes, leading, usually, to some sort of asymptotic independence that, often, allows to recover the behaviour of purely random, independent and identically distributed sequences of random variables.

The ergodic properties of the systems are tied to the existence of invariant measures, which endow the stochastic processes arising from such systems with stationarity. In some sense, the invariant measures, which usually have some physical significance, determine the system itself. However, sometimes the exact formula for the invariant measure is not accessible and one has to rely on reference measures with respect to which these processes are not stationary anymore.

Relaxing stationarity gives rise to non-autonomous dynamical systems for which the study of limit theorems is just at the beginning. Here, we will focus on the particular problem of studying the existence of limiting Extreme Value Laws (EVL), which, as shown in [FFT10, FFT11], is related to the occurrence of rare events and the study of Hitting and Return Time Statistics.

The study of the extremal properties of non-stationary stochastic processes was introduced by Hüsler in [Hüs83, Hüs86] and the theory was built up on this initial effort, which generalised

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Leadbetter's conditions and approach to deal with general stationary stochastic processes. This fact precluded its application in a dynamical setting. In [FFV16], the authors developed a more general theory, based on necessary adjustments to Leadbetter's conditions and a much more refined way of dealing with clustering, originally developed in [FFT12, FFT15], which, ultimately, allowed the application to non-autonomous dynamical systems.

We will use the theory established in [FFV16] to study *sequential dynamical systems* originated by the composition of intermittent maps. Sequential dynamical systems were introduced by Berend and Bergelson [BB84], as a non-stationary system in which a concatenation of maps is applied to a given point in the underlying space, and the probability is taken as a conformal measure, which allows the use of the transfer operator (Perron-Fröbenius) as a useful tool to quantify the loss of memory of any prescribed initial observable. The theory of sequential systems was later developed in the fundamental paper by Conze and Raugi [CR07], where a few limit theorems, in particular the Central Limit Theorem, were proved for concatenations of one-dimensional dynamical systems, each possessing a transfer operator with a quasi-compact structure on a suitable Banach space. For the same systems and others, even in higher dimensions, the Almost Sure Invariance Principle was subsequently shown in [HNTV16].

Both papers [CR07, HNTV16] dealt however with uniformly expanding maps, for which the transfer operators admits a spectral gap and the correlations decays exponentially. In a different direction, a class of sequential systems given by composition of non-uniformly expanding maps of Pomeau-Manneville type was studied in [AHN$^+$15], by perturbing the slope at the indifferent fixed point 0. Polynomial decay of correlations was proved for particular classes of centred observables, which could also be interpreted as the decay of the iterates of the transfer operator on functions of zero (Lebesgue) average, and this fact is better known as *loss of memory.* In the successor paper [NTV16], a (non-stationary) central limit theorem was shown for sums of centred observables and with respect to the Lebesgue measure.

We continue here the statistical analysis of these indifferent transformations by proving the existence of limiting extreme value distributions under suitable normalisation for the threshold of the exceedances.

2. Conditions for the existence of extreme value laws for non-stationary processes

In this section, we revise the general theory developed in [FFV16] in order to prove the existence of EVL for non-stationary processes, which is particularly suitable for application to processes arising from non-autonomous systems. However, since in our application there is no clustering of exceedances, we simplify the exposition by adapting the general conditions and setting to this particular case of absence of clustering.

Let $X_0, X_1, \dots$ be a stochastic process, where each r.v. $X_i : \mathcal{Y} \to \mathbb{R}$ is defined on the measure space $(\mathcal{Y}, \mathcal{B}, \mathbb{P})$. We assume that $\mathcal{Y}$ is a sequence space with a natural product structure so that each possible realisation of the stochastic process corresponds to a unique element of $\mathcal{Y}$ and there exists a measurable map $\mathcal{T} : \mathcal{Y} \to \mathcal{Y}$, the time evolution map, which can be seen as the passage of one unit of time, so that $X_{i-1} \circ \mathcal{T} = X_i, \quad$ for all $i \in \mathbb{N}$. The σ-algebra $\mathcal{B}$ can also be seen as a product σ-algebra adapted to the X_i's. For the purpose of this paper, $X_0, X_1, \dots$ is possibly non-stationary. Stationarity would mean that $\mathbb{P}$ is $\mathcal{T}$-invariant. Note that $X_i = X_0 \circ \mathcal{T}_i$, for all $i \in \mathbb{N}_0$, where $\mathcal{T}_i$ denotes the i-fold composition of $\mathcal{T}$, with the

convention that $\mathcal{T}_0$ denotes the identity map on $\mathcal{Y}$. In the application below to sequential dynamical systems, we will have that $\mathcal{T}_i = T_i \circ \ldots \circ T_1$ will be the concatenation of i possibly different transformations $T_1, \ldots, T_i$.

Each random variable X_i has a marginal distribution function (d.f.) denoted by F_i, *i.e.*, $F_i(x) = \mathbb{P}(X_i \leq x)$. Note that the F_i, with $i \in \mathbb{N}_0$, may all be distinct from each other. For a d.f. F we let $\bar{F} = 1 - F$. We define $u_{F_i} = \sup\{x : F_i(x) < 1\}$ and let $F_i(u_{F_i}-) := \lim_{h\to 0, h>0} F_i(u_{F_i} - h) = 1$ for all i.

Our main goal is to determine the limiting law of

$$\mathbf{P}_n = \mathbb{P}(X_0 \leq u_{n,0}, X_1 \leq u_{n,1}, \ldots, X_{n-1} \leq u_{n,n-1})$$

as $n \to \infty$, where $\{u_{n,i}, i \leq n-1, n \geq 1\}$ is considered a real-valued boundary. We assume throughout the paper that

$$\bar{F}_{\max} := \max\{\bar{F}_i(u_{n,i}), i \leq n-1\} \to 0 \text{ as } n \to \infty, \tag{2.1}$$

which is equivalent to $u_{n,i} \to u_{F_i}$ as $n \to \infty$, uniformly in i. Let us denote $F_n^* := \sum_{i=0}^{n-1} \bar{F}_i(u_{n,i})$, and assume that there is $\tau > 0$ such that

$$F_n^* := \sum_{i=0}^{n-1} \bar{F}_i(u_{n,i}) \to \tau, \qquad \text{as } n \to \infty. \tag{2.2}$$

In what follows, for every $A \in \mathcal{B}$, we denote the complement of A as $A^c := \mathcal{Y} \setminus A$. Let $\mathbb{A} := (A_0, A_1, \ldots)$ be a sequence of events such that $A_i \in \mathcal{T}_i^{-1}\mathcal{B}$. For some $s, \ell \in \mathbb{N}_0$, we define

$$\mathscr{W}_{s,\ell}(\mathbb{A}) = \bigcap_{i=s}^{s+\ell-1} A_i^c. \tag{2.3}$$

We will write $\mathscr{W}_{s,\ell}^c(\mathbb{A}) := (\mathscr{W}_{s,\ell}(\mathbb{A}))^c$. We consider $\mathbb{A}_n^{(0)} := (A_{n,0}^{(0)}, A_{n,1}^{(0)}, \ldots)$, where the event $A_{n,i}^{(0)}$ is defined as $A_{n,i}^{(0)}(u_{n,i}) := \{X_i > u_{n,i}\}$.

Now, we recall a mixing condition, introduced in [FFV16], which was specially designed for the application to the dynamical setting.

Condition ($Д_0(u_{n,i})$)**.** We say that $Д_0(u_{n,i})$ holds for the sequence $X_0, X_1, \ldots$ if for every $\ell, t, n \in \mathbb{N}$,

$$\left|\mathbb{P}\left(A_{n,i}^{(q)} \cap \mathscr{W}_{i+t,\ell}\left(\mathbb{A}_n^{(0)}\right)\right) - \mathbb{P}\left(A_{n,i}^{(q)}\right)\mathbb{P}\left(\mathscr{W}_{i+t,\ell}\left(\mathbb{A}_n^{(0)}\right)\right)\right| \leq \gamma_i(n,t), \tag{2.4}$$

where $\gamma_i(n,t)$ is decreasing in t for each n and each i and there exists a sequence $(t_n^*)_{n\in\mathbb{N}}$ such that $t_n^*\bar{F}_{\max} \to 0$ and $\sum_{i=0}^{n-1} \gamma_i(n, t_n^*) \to 0$ when $n \to \infty$.

In order to prove the existence of a distributional limit for $\mathbf{P}_n$, in [FFV16], we used as usual a blocking argument that splits the data into k_n blocks separated by time gaps of size larger than t_n^*, which are created by simply disregarding the observations in the time frame occupied by the gaps. The precise construction of the blocks is given in [FFV16, Section 2.2] but we briefly describe here some of the key properties of this construction.

In the stationary context, one takes blocks of equal size, which in particular means that the expected number of exceedances within each block is $n\mathbb{P}(X_0 > u_n)/k_n \sim \tau/k_n$. Here the

blocks may have different sizes, which we denote by $\ell_{n,1},\dots,\ell_{n,k_n}$ but, as in [Hüs83, Hüs86], these are chosen so that the expected number of exceedances is again $\sim \tau/k_n$. Also, for $i=1,\dots,k_n$, let $\mathcal{L}_{n,i}=\sum_{j=1}^{i}\ell_{n,j}$ and $\mathcal{L}_{n,0}=0$. See beginning of Section 2.2 of [FFV16] for the precise definition of these quantities.

We recall now a condition that imposes some restrictions on the speed of recurrence within each block, which, in the present context, precludes the existence of clustering.

Consider the sequence $(t_n^*)_{n\in\mathbb{N}}$, given by condition $Д_0(u_{n,i})$ and let $(k_n)_{n\in\mathbb{N}}$ be another sequence of integers such that

$$k_n\to\infty \quad\text{and}\quad k_n t_n^*\bar{F}_{\max}\to 0, \quad \text{as } n\to\infty. \tag{2.5}$$

Condition ($Д_0'(u_{n,i})$)**.** We say that $Д_0'(u_{n,i})$ holds for the sequence $X_0,X_1,X_2,\dots$ if there exists a sequence $(k_n)_{n\in\mathbb{N}}$ satisfying (2.5) and such that

$$\lim_{n\to\infty}\sum_{i=1}^{k_n}\sum_{j=0}^{\ell_i-1}\sum_{r>j}^{\ell_i-1}\mathbb{P}(A^{(0)}_{\mathcal{L}_{i-1}+j}\cap A^{(0)}_{\mathcal{L}_{i-1}+r})=0. \tag{2.6}$$

Condition $Д_0'(u_{n,i})$ precludes the occurrence of clustering of exceedances.

The following is a corollary of [FFV16, Theorem 2.4], in the particular case of absence of clustering and which we will use below to obtain the existence of EVL.

Theorem 2.1. *Let $X_0,X_1,\dots$ be a stationary stochastic process and suppose* (2.1) *and* (2.2) *hold for some $\tau>0$. Assume that conditions $Д_0(u_{n,i})$ e $Д_0'(u_{n,i})$ are satisfied. Then*

$$\lim_{n\to\infty}\mathbf{P}_n=e^{-\tau}.$$

3. Sequential systems on intermittent maps: statement of the main result

We consider maps with indifferent fixed points in the formulation proposed in [LSV99]. Namely, for $\alpha\in(0,1)$,

$$T_\alpha(x)=\begin{cases} x(1+2^\alpha x^\alpha) & \text{for } x\in[0,1/2)\\ 2x-1 & \text{for } x\in[1/2,1]\end{cases} \tag{3.1}$$

and we concatenate them. For each $i\in\mathbb{N}$, let $T_i=T_{\alpha_i}$, with $\alpha_i\in(0,\alpha^*)$, where $\alpha^*=1/7$.

This countable sequence of maps $\{T_i\}_{i\in\mathbb{N}}$ allows us to define a *sequential dynamical system.* A *sequential orbit* of $x\in X$ will be defined by the concatenation

$$\mathcal{T}_n(x):=T_n\circ\cdots\circ T_1(x),\ n\geq 1. \tag{3.2}$$

We denote by P_j the Perron-Fröbenius (transfer) operator associated to T_j defined by the duality relation

$$\int_X P_j f\ g\ dm\ =\ \int_X f\ g\circ T_j\ dm, \quad \text{for all } f\in L^1_m,\ g\in L^\infty_m.$$

Note that here the transfer operator P_j is defined with respect to the reference Lebesgue measure m.

Similarly to (3.2), we define the composition of operators as

$$\Pi_n:=P_n\circ\cdots\circ P_1,\ n\geq 1. \tag{3.3}$$

It is easy to check that duality persists under concatenation, namely

$$\int_X g(\mathcal{T}_n)\, f\, dm = \int_X g(T_n \circ \cdots \circ T_1)\, f\, dm \;=\; \int_X g(\, P_n \circ \cdots \circ P_1 f)\, dm \;=\; \int_X g\,(\Pi_n f)\, dm. \quad (3.4)$$

We would like to point out that there exists another possible and interesting way to *perturb* the map (3.1). Let us suppose that we define it on the unit circle S^1 for a given slope α; then we can construct close maps by adding *noise* $\epsilon : T_\epsilon(x) = T_\alpha(x) + \epsilon \bmod 1$, $x \in S^1$ and ϵ taking values in some interval Δ around zero. To our knowledge there are no results available in this setting to control the rate of mixing (e.g. *loss of memory*, see (4.1) below), which is a main ingredient in establishing extreme value distribution. Instead the additive noise has been studied whenever the maps T_ϵ are chosen in an i.i.d. way with a smooth density function for the distribution of ϵ on Δ. In this situation one has a random dynamical system with a stationary measure which is absolutely continuous with respect to Lebesgue. The question is therefore to investigate the stochastic stability whenever the *size* of the noise $|\Delta|$ goes to zero. After a former result by Araújo and Tahzibi [AT05], who established weak convergence of the stationary measure to the invariant measure of T_α, Shen and Van Strien [SvS13] finally obtained the convergence of the densities of the two measures in the L^1 norm.

We now return to our perturbation *by changing the slope* and we note that is has also be considered for other interesting purposes. The first result, by Freitas and Todd [FT09] is about *statistical stability*, which establishes the continuity in L^1 of the densities of the absolutely continuous invariant measures when the parameter α changes. A strong achievement in this direction has been obtained, independently, by Baladi and Todd [BT16], Korepanov [Kor15] and, more recently, Bahsoun and Saussol [BS15], with the proof of the differentiability of the function $\alpha \to \int \psi d\mu_\alpha$, where μ_α is the absolutely continuous invariant measure for T_α and ψ is a function in some L^q; we defer to those papers for the precise definition and for the differences among them. We just stress that as a consequence, it is possible to obtain linear response and, in particular, [BT16] gives a formula for the value of the derivative.

Let us now focus on the situation of our interest, namely the sequential or random composition of these kind of maps. Whenever a finite number of them are chosen in an i.i.d. way and with a position dependent probability distribution $\mathbb{P}$, the stochastic stability was proven by Duan [Dua13]. Still in this framework and by considering the *annealed situation* where the statistics is insured by the direct product of $\mathbb{P}$ with the stationary measure, Bahsoun, Bose and Duan [BBD14] proved polynomial decay of correlations, and successively Bahsoun and Bose [BB16] got a central limit theorem. The latter was successively generalized in the *quenched* case (with respect to the stationary measure and for almost all the realizations), by Nicol, Torok and Vaienti [NTV16]; this paper contains also a proof of the central limit theorem for sequential systems and its results will be used again in the next section. Still in this context we also quote the paper by Leppänen and Stenlund [LS15] where a few results on the continuity of the densities and their pushforward with respect to the parameter α are proved.

We now turn to the context of extreme value analysis. Similarly to [FFT10] (in the context of stationary deterministic systems), we consider that the time series $X_0, X_1, \ldots$ arises from these sequential systems simply by evaluating a given observable $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$ along the sequential orbits,

$$X_n = \varphi \circ \mathcal{T}_n, \quad \text{for each } n \in \mathbb{N}. \quad (3.5)$$

Note that, on the contrary to the setup in [FFT10], the stochastic process $X_0, X_1, \dots$ defined in this way is not necessarily stationary, because m is not an invariant measure for any of the T_i.

We assume that the r.v. $\varphi : X \to \mathbb{R} \cup \{\pm\infty\}$ achieves a global maximum at $\zeta \in [0,1]$ (we allow $\varphi(\zeta) = +\infty$) being of following form:

$$\varphi(x) = g\big(\mathrm{dist}(x,\zeta)\big), \tag{3.6}$$

where ζ is a chosen point in the phase space $[0,1]$ and the function $g : [0,+\infty) \to \mathbb{R} \cup \{+\infty\}$ is such that 0 is a global maximum ($g(0)$ may be $+\infty$); g is a strictly decreasing continuous bijection $g : V \to W$ in a neighbourhood V of 0; and has one of the following three types of behaviour:

Type g_1: there exists some strictly positive function $h : W \to \mathbb{R}$ such that for all $y \in \mathbb{R}$

$$\lim_{s \to g_1(0)} \frac{g_1^{-1}(s + yh(s))}{g_1^{-1}(s)} = \mathrm{e}^{-y}; \tag{3.7}$$

Type g_2: $g_2(0) = +\infty$ and there exists $\beta > 0$ such that for all $y > 0$

$$\lim_{s \to +\infty} \frac{g_2^{-1}(sy)}{g_2^{-1}(s)} = y^{-\beta}; \tag{3.8}$$

Type g_3: $g_3(0) = D < +\infty$ and there exists $\gamma > 0$ such that for all $y > 0$

$$\lim_{s \to 0} \frac{g_3^{-1}(D - sy)}{g_3^{-1}(D - s)} = y^{\gamma}. \tag{3.9}$$

It may be shown that no non-degenerate limit applies if $\int_0^{g_1(0)} g_1^{-1}(s)ds$ is not finite. Hence, an appropriate choice of h in the Type 1 case is given by $h(s) = \int_s^{g_1(0)} g_1^{-1}(t)dt / g_1^{-1}(s)$ for $s < g_1(0)$.

Examples of each one of the three types are as follows: $g_1(x) = -\log x$ (in this case (3.7) is easily verified with $h \equiv 1$), $g_2(x) = x^{-1/\alpha}$ for some $\alpha > 0$ (condition (3.8) is verified with $\beta = \alpha$) and $g_3(x) = D - x^{1/\alpha}$ for some $D \in \mathbb{R}$ and $\alpha > 0$ (condition (3.9) is verified with $\gamma = \alpha$).

We now choose *time-dependent levels* $u_{n,i}$ given by $m(X_i > u_{n,i}) = \tau/n$, where $\tau \geq 0$. Let $\delta_{n,i} = g^{-1}(u_{n,i})$ so that

$$m(X_i > u_{n,i}) = \int \mathbf{1}_{(\zeta - \delta_{n,i}, \zeta + \delta_{n,i})} \Pi_i(1) dm = \frac{\tau}{n}. \tag{3.10}$$

Observe that $\delta_{n,0} = \frac{\tau}{2n}$ and, by Lemma 4.4, which appears below, for n sufficiently large, we have that for some constants $0 < c < C'$,

$$\frac{\tau}{2C'n} \leq \delta_{n,i} \leq \frac{\tau}{2cn}. \tag{3.11}$$

Note that this choice for the levels $u_{n,i}$ guarantees that condition (2.2) is trivially satisfied.

We are now in condition of stating and proving our main result.

Theorem 3.1. *Consider the family of maps given by* (3.1) *and the sequential dynamics given by* $\mathcal{T}_n = T_n \circ \ldots \circ T_1$*, where* $T_i = T_{\alpha_i}$*, with* $\alpha_i \in (0, \alpha^*)$ *and* $\alpha^* = 1/7$*. Let* $X_1, X_2, \ldots$ *be defined by* (3.5)*, where the observable function* φ*, given by* (3.6)*, achieves a global maximum at a chosen* $\zeta \in (0,1]$*. For m-a.e.* $\zeta \in (0,1]$*, we may define the levels* $(u_{n,i})_{n,i\in\mathbb{N}}$ *such that* (3.10) *holds for some* $\tau \geq 0$*, conditions* $Д_0(U_{n,i})$ *and* $Д_0'(U_{n,i})$ *hold and consequently:*

$$\lim_{n\to\infty} m(X_0 \leq u_{n,0}, X_1 \leq u_{n,1}, \ldots, X_{n-1} \leq u_{n,n-1}) = \mathrm{e}^{-\tau}.$$

Remark 3.2. We emphasise that this restriction on α ($\alpha < 1/7$) is rather technical and is due to the use of the blocking argument and of decay of correlations, which is proved only on sufficiently regular Banach spaces of functions. We remark that the same techniques gave rise to similar restrictions on α even in the stationary setting, where the orbits are obtained by iterations of the same Liverani-Saussol-Vaienti map (see [HNT12, Section 3.4]). It is interesting to observe that the threshold value $\alpha < 1/7$ is the same appearing in [NTV16] in order to establish the central limit theorem for smooth observable.

Remark 3.3. Let us consider a sequential sequence with the same map T_α, namely we simply iterate it. We could then compare our previous theorem with the results we got in our recent work [FFT16] and for the same type of observable 3.6. We proved there that for any target point $\zeta \neq 0$ we have the so-called *dichotomy*, namely the distribution of the maxima converges to $e^{-\tau}$ when ζ is not periodic, and to $e^{-\theta}\tau$ when ζ is periodic with period p and the *extremal index* θ is given by $\theta = 1 - 1/|DT_\alpha^p(\zeta)|$. For the special case in which ζ is the indifferent fixed point, we prove that there exists an extremal index equal to zero, which corresponds to a degenerate limit law, when the usual normalising time-level sequence (u_n) is used [1]. Moreover, we have shown that by changing the definition of (u_n) in a suitable way, we recover a non-degenerate standard exponential limit. We defer to [FFT16] for the precise definition of such u_n.

4. Proof of the theorem

By Theorem 2.1, to prove Theorem 3.1 we only need to check conditions $Д_0(u_{n,i})$ and $Д_0'(u_{n,i})$.

4.1. **Verification of $Д_0(u_{n,i})$.** The intermittent map introduced above exhibits polynomial decay of correlations, which can be obtained by considering decay of the L^1 norm of the concatenation of the Perron-Frobenius operators: this fact is also known as *loss of memory*. We will be interested in the kind of correlations given in [FFV16, Proposition 4.3], which reads

$$\begin{aligned} DC(\phi,\psi,i,t) &:= \int \phi\circ\mathcal{T}_i\psi\circ\mathcal{T}_{i+t}dm - \int \phi\circ\mathcal{T}_i dm \int \psi\circ\mathcal{T}_{i+t}dm \\ &= \int \left(\psi - \int \psi\Pi_{i+t}(1)dm\right) P_{i+t}\ldots P_{i+1}\left(\Pi_i(1)\left(\phi - \int \phi\Pi_i(1)dm\right)\right). \end{aligned}$$

Let $\tilde{\phi} = \phi - \int \phi\Pi_i(1)dm$. Observe that $\int \Pi_i(1)\tilde{\phi}dm = 0$. This means that the observable function $\Pi_i(1)\tilde{\phi} \in \mathcal{V}_0$, where $\mathcal{V}_0$ is the set of functions with 0 integral that was defined in [CR07, Lemma 2.12].

Now, contrary to what we did in the case of uniformly expanding maps, we will consider decay

[1] In the stationary case this sequence is defined by requiring that for a given $\tau > 0$ we have $n\mu_\alpha(\varphi > u_n) \to \tau$, when $n \to \infty$ and where μ_α is the invariant measure of T_α.

of the L^1 norm of the concatenation of the PF operators, namely we will consider, having set $\tilde{\psi} = \psi - \int \psi \Pi_i(1) dm$:

$$|DC(\phi, \psi, i, t)| = \left| \int \tilde{\psi}\, P_{i+t} \dots P_{i+1} \left(\Pi_i(1) \tilde{\phi} \right) dm \right| \tag{4.1}$$

$$\leq \|P_{i+t} \dots P_{i+1}(\Pi_i(1)\tilde{\phi})\|_1 \; ||\psi||_\infty. \tag{4.2}$$

To deal with such correlations we apply the following result proved in [AHN$^+$15]:

Theorem 4.1 ([AHN$^+$15])**.** *Suppose ψ, ϕ are in the cone $\mathcal{C}_a$ (see below), for some a and with equal expectation $\int \phi dm = \int \psi dm$. Then for any $0 < \alpha^* < 1$ and for any sequence $T_1, \cdots, T_n$, $n \geq 1$, of maps of Pomeau-Manneville type with $0 < \alpha_k \leq \alpha^* < 1$, $k \in [1, n]$, we have*

$$\int |\Pi_n(\phi) - \Pi_n(\psi)| dm \leq C_{\alpha^*} (\|\phi\|_1 + \|\psi\|_1) n^{-\frac{1}{\alpha}+1} (\log n)^{\frac{1}{\alpha}}, \tag{4.3}$$

where the constant C_{α^*} depends only on the map T_{α^*}.

The cone $\mathcal{C}_a$ contains functions given by (here $X(x) = x$ denotes the identity function):

$$\mathcal{C}_a = \{ f \in C^0((0,1]) \cap L^1(m) \mid \; f \geq 0, \; f \text{ decreasing}, \; X^{\alpha+1} f \text{ increasing}, \; f(x) \leq a x^{-\alpha} \int f dm \}$$

Having fixed $0 < \alpha < 1$, it was proven in [AHN$^+$15] that, provided a is large enough, the cone $\mathcal{C}_a$ is preserved by all operators P_k.

We are now ready to verify Д$_0(u_{n,i})$. Note that $A^{(0)}_{n,i} = \{X_i > u_{n,i}\} =: U_{n,i}$ is an interval.

We will apply the bound (4.1). We begin to observe than in our case ϕ is not in the cone $\mathcal{C}_a$; we therefore approximate it with a function χ which is C^1 and with compact support, equal to 1 on $U_{n,i}$ and rapidly decreasing to zero on a set Λ of diameter Δ in the complement of $U_{n,i}$. [2] We have that $||\chi||_\infty = 1$, $||\chi'||_\infty = O(\Delta^{-1})$ and finally $||\phi - \chi||_1 = O(\Delta)$. In this way we have:

$$\Pi_i(1)\tilde{\phi} = \Pi_i(1)\chi - \Pi_i(1) \int \chi \Pi_i(1) dm + \Pi_i(1)[\phi - \chi] - \Pi_i(1) \int [\phi - \chi] \Pi_i(1) dm.$$

[2]This can be achieved for instance in this way. Let $U_n = (a_n, b_n)$ and $U_n^\Delta = (a_n - \Delta, b_n + \Delta)$. Define

$$\chi(x) = \begin{cases} 1 & \text{for } x \in (a_n, b_n) \\ \mathrm{e}^{-\frac{1}{1-\left(\frac{x-b_n}{\Delta}\right)^2}} & \text{for } x \in [b_n, b_n + \Delta) \\ \mathrm{e}^{-\frac{1}{1-\left(\frac{x-a_n}{\Delta}\right)^2}} & \text{for } x \in (a_n - \Delta, a_n] \\ 0 & \text{for } x \in \mathbb{R} \setminus U_n^\Delta \end{cases}.$$

Note that $\Delta U_n := \{x : \; \chi(x) - \mathbf{1}_{U_n}(x) > 0\} = U_n^\Delta \setminus [a_n, b_n]$ and $m(\Delta U_n) = 2\Delta$. We have $\chi \in \mathcal{C}^\infty$, $\chi''(b_n + \frac{\Delta}{3^{1/4}}) = 0 = \chi''(a_n - \frac{\Delta}{3^{1/4}})$ and

$$\max\{\chi'(x)\} = \chi'(b_n + \frac{\Delta}{3^{1/4}}) = \chi'(a_n - \frac{\Delta}{3^{1/4}}) = \frac{2\mathrm{e}^{-\frac{1}{1-1/\sqrt{3}}}}{3^{1/4}(1 - 1/\sqrt{3})^2} \frac{1}{\Delta} = O(1/\Delta).$$

.

To this quantity we have to apply the power $\Pi_t := P_{i+t} \dots P_{i+1}$ and then take the L^1 norm: for the last two terms in the preceding identity this contribution will be of order 2Δ. Now, generalizing an argument in [LSV99], it can be shown, as in [NTV16], that there are constants $\lambda < 0, \nu > 0, \delta > 0$ such that, having set $\chi' := \chi - \int \chi \Pi_i(1) dm$, the functions

$$F := \chi' \Pi_i(1) + \lambda X \Pi_i(1) + \nu \Pi_i(1) + \delta; \; G := \lambda X \Pi_i(1) + \nu \Pi_i(1) + \delta$$

are *pushed into the cone* $\mathcal{C}_a$, in such a way that

$$\Pi_t(\Pi_i(1)\chi') = \Pi_t(F) - \Pi_t(G),$$

and, by the above theorem on loss of memory,

$$||\Pi_t(\Pi_i(1)\chi'||_1 = ||\Pi_t(F) - \Pi_t(G)||_1 \leq C_{\alpha^*}(\|F\|_1 + \|G\|_1) t^{-\frac{1}{\alpha^*}+1} (\log t)^{\frac{1}{\alpha^*}}.$$

It is important to notice that the constants λ, ν, δ

- are independent of i;
- are affine functions of the C^1 norm of χ, with multiplicative constants depending only on α^*.

In conclusion, this means that we can write

$$||\Pi_t(\Pi_i(1)\chi'||_1 \leq C_{\alpha^*}[A_{\alpha^*}||\chi||_\infty + B_{\alpha^*}||\chi'||_\infty + D_{\alpha^*}] t^{-\frac{1}{\alpha^*}+1} (\log t)^{\frac{1}{\alpha^*}},$$

where the factors $A_{\alpha^*}, B_{\alpha^*}, D_{\alpha^*}$ depend only on α^*. Therefore, and taking into account the bounds on χ, there will be new constants C_1, C_2, C_3 depending only on α^* such that

$$||\Pi_t(\Pi_i(1)\tilde{\phi}||_1 \leq 2\Delta \; + C_1 t^{-\frac{1}{\alpha^*}+1} (\log t)^{\frac{1}{\alpha^*}} \; + C_2 \Delta^{-1} t^{-\frac{1}{\alpha^*}+1} (\log t)^{\frac{1}{\alpha^*}} \; + C_3 t^{-\frac{1}{\alpha^*}+1} (\log t)^{\frac{1}{\alpha^*}}.$$

Returning to (4.1), it follows that there exists C^* (depending only on α^*) such that

$$DC(\phi, \psi, i, t) \leq \left(2\Delta + C^* \Delta^{-1} t^{-\frac{1}{\alpha^*}+1} (\log t)^{\frac{1}{\alpha^*}}\right) \|\psi\|_\infty. \tag{4.4}$$

In order to verify condition Д$_0(u_n, i)$, we let $\Delta = n^{1+\eta}$, for some $\eta > 0$, $t_n = n^\kappa$, for some $0 < \kappa < 1$ and for each n, i, ℓ set $\phi_i = \mathbf{1}_{(\zeta - \delta_{n,i}, \zeta + \delta_{n,i})}$ and $\psi_i = \mathbf{1}_{(\zeta - \delta_{n,i+t_n}, \zeta + \delta_{n,i+t_n})} \cdot \dots \cdot \mathbf{1}_{(\zeta - \delta_{n,i+t_n+\ell}, \zeta + \delta_{n,i+t_n+\ell})} \circ (T_{i+t_n+\ell} \circ \dots \circ T_{i+t_n+1})$. Then we can write:

$$DC(\phi_i, \psi_i, i, t_n) \leq 2n^{-(1+\eta)} + C^* n^{1+\eta} n^{(-\frac{1}{\alpha^*}+1)\kappa} (\kappa \log n)^{\frac{1}{\alpha^*}} =: \gamma_i(n, t_n).$$

Then, for some $C^{**} > 0$, we have

$$\sum_{i=0}^{n-1} \gamma_i(n, t_n) \leq 2n^{-\eta} + C^{**} n^{2+2\eta} n^{(-\frac{1}{\alpha^*}+1)\kappa} \to 0, \qquad \text{as } n \to \infty,$$

as long as α is sufficiently small so that $(-\frac{1}{\alpha^*} + 1)\kappa + 2 + 2\eta < 0$, which ultimately settles condition Д$_0(u_{n,i})$.

Note that in order to optimise the choice of the α^* (which we want as large as possible), we need to choose η close to 0 and κ close to 1, which means that $\alpha^* < \frac{1}{3}$. However, in order to prove Д$'_0(u_{n,i})$ we still need further restrictions on α.

4.2. **Verification of $Д_0'(u_{n,i})$.** We will begin with a lemma that adjusts to the sequential setting the argument used in [HNT12, Lemma 3.10]. Essentially, it says that the Lebesgue measure of the points that after n iterations by the sequential intermittent maps return to an ε neighbourhood of themselves scales like a power of ε that depends on α^*.

Let $\mathcal{E}_n(\varepsilon) := \{x \in [0,1] : \ |\mathcal{T}_n(x) - x| \leq \varepsilon\}$.

Lemma 4.2. *There exists some $C > 0$ such that for all $n \in \mathbb{N}$, we have*

$$m(\mathcal{E}_n(\varepsilon)) \leq C\varepsilon^{1/(1+\alpha^*)}.$$

Proof. Let $J_1, J_2, \ldots, J_k$ be the domains of injectivity of $\mathcal{T}_n$, ordered from the left to the right, *i.e.*, $J_i = [a_i, b_i)$ and $0 = a_1 < b_1 = a_2 < \ldots < b_{k-1} = a_k < b_k = 1$. Note that $\mathcal{T}_n$ is full branched map, in particular, each branch $\mathcal{T}_n|_{J_i}$ is a convex map where for each $i \neq 1$ we have $D\mathcal{T}_n(x) > \gamma > 1$ but when $i = 1$, we have $D\mathcal{T}_n(0) = 1$.

We consider now an ε-neighbourhood of the diagonal and the intersection of its boundary with the full branches of $\mathcal{T}_n$, *i.e.*, we define for each $i = 1, \ldots, k$, the points $x_i^{\pm} \in J_i$ such that $\mathcal{T}_n(x_i^{\pm}) = x_i^{\pm} \pm \varepsilon$, whenever this intersection is well defined. Note that, whenever both points $x_i^{\pm}$ exist then $\mathcal{E}_n(\varepsilon) \cap J_i \subset [x_i^-, x_i^+]$.

Let $x \geq x_i^-$ in J_i. By convexity of $\mathcal{T}_n|_{J_i}$, we have

$$D\mathcal{T}_n(x) \geq D\mathcal{T}_n(x_i^-) \geq \frac{x_i^- - \varepsilon - \mathcal{T}_n(a_i)}{x_i^- - a_i},$$

hence

$$D\mathcal{T}_n(x) - 1 \geq \frac{x_i^- - \varepsilon - \mathcal{T}_n(a_i)}{x_i^- - a_i} - 1 = \frac{a_i - \varepsilon - \mathcal{T}_n(a_i)}{x_i^- - a_i} \geq \frac{a_i - \varepsilon - \mathcal{T}_n(a_i)}{m(J_i)}.$$

It follows that

$$2\varepsilon = \int_{x_i^-}^{x_i^+} D\mathcal{T}_n(x) - 1dx \geq m([x_i^-, x_i^+])\frac{a_i - \varepsilon - \mathcal{T}_n(a_i)}{m(J_i)},$$

which implies

$$\mathcal{E}_n(\varepsilon) \cap J_i \leq \frac{2\varepsilon}{a_i - \varepsilon - \mathcal{T}_n(a_i)} m(J_i).$$

This estimate is useful whenever $a_i - \varepsilon - \mathcal{T}_n(a_i)$ is not small. Hence, we define

$$V^\eta = \cup\{a_i : \ |a_i - \mathcal{T}_n(a_i)| < \varepsilon + \eta\} \quad \text{and} \quad Z^\eta = \cup_{a_i \in V^\eta} J_i.$$

Then

$$m(\mathcal{E}_n(\varepsilon)) = m(\mathcal{E}_n(\varepsilon) \cap Z^\eta) + m(\mathcal{E}_n(\varepsilon) \cap (Z^\eta)^c) \leq m(Z^\eta) + \frac{2\varepsilon}{\eta} m((Z^\eta)^c).$$

Now we estimate these sets in two different ways depending on whether n is small or large. Assume that $\varepsilon < \eta$ and n is sufficiently large so that $\max_i |J_i| \leq \varepsilon$, where $|J_i| = b_i - a_i$. Recall that $\mathcal{T}_n(a_i) = 0$ for all i. Since $a_i \in V^\eta$ means that $a_i < \eta + \varepsilon$ then

$$m(\mathcal{E}_n(\varepsilon)) \leq 2\eta + \frac{2\varepsilon}{\eta}.$$

Optimising over $\eta \in (0,1)$ we have that $\eta = O(\sqrt{\varepsilon})$ is the best choice and gives

$$m(\mathcal{E}_n(\varepsilon)) \leq C\sqrt{\varepsilon} \leq C\varepsilon^{1/(1+\alpha^*)},$$

since as mentioned above we have $\alpha^* < 1/2$, which implies that $1/(1+\alpha^*) > 2/3 > 1/2$.

When n is small then the worst case scenario happens on J_1. In this case x_1^- is not defined and $\mathcal{E}_n(\varepsilon) \cap J_1 = [0, x_1^+]$. In this case, we have:

$$\varepsilon = \mathcal{T}_n(x_1^+) - x_1^+ \geq T_{\alpha^*}(x_1^+) - x_1^+ = 2^{\alpha^*}(x_1^+)^{1+\alpha^*},$$

which implies that $x_1^+ = \left(\frac{\varepsilon}{2\alpha^*}\right)^{\frac{1}{1+\alpha^*}}$ and ultimately, for $\alpha \in (0,1)$, taking $\eta = \sqrt{\varepsilon}$, we have $m(\mathcal{E}_n(\varepsilon)) \leq \varepsilon^{\frac{1}{1+\alpha^*}}$. □

We now follow the argument originally used by Collet in [Col01] and further developed in [HNT12]. Let $0 < \beta < 1$, $0 < \kappa < \beta$ and $0 < \xi < 1$ such that $\kappa(1+\xi) < \beta$. We define the set of points that recur too fast:

$$E_j = \left\{ x \in [0,1] : \ |\mathcal{T}_i(x) - x| \leq \frac{2}{j} \text{ for some } i \leq j^{\kappa(1+\xi)} \right\}.$$

By Lemma 4.2, we have that

$$m(E_j) \leq \sum_{i=1}^{j^{\kappa(1+\xi)}} m(\mathcal{E}_i(2/j)) \leq \frac{C}{j^\varsigma},$$

where $\varsigma = \frac{1}{1+\alpha^*} - \kappa(1+\xi)$ and for some $C > 0$.

The core of Collet's argument is based on the use of Hardy-Littlewood maximal functions to obtain, from an estimate on the measure of the sets E_j, an estimate for the conditional measure on balls of radius $1/j$, centred on m-a.e point ζ, of the intersection of these sets E_j with the corresponding balls.

Lemma 4.3. *Assume that $(E_n)_{n\in\mathbb{N}}$ is a sequence of measurable sets such that*

$$m(E_j) \leq \frac{C}{j^\varsigma},$$

for some $C, \varsigma > 0$. Then for $0 < \beta < \varsigma$ and $\gamma > 1/(\varsigma - \beta)$, we have that for m-a.e. $\zeta \in [0,1]$, there exists $N(\zeta)$ such that for all $j \geq N(\zeta)$

$$m(\{|x - \zeta| \leq j^{-\gamma}\} \cap E_{j^\gamma}) \leq \frac{2}{j^{\gamma+\gamma\beta}}.$$

Proof. Define the Hardy-Littlewood maximal function:

$$L_n(x) = \sup_{\ell>0} \frac{1}{2\ell} \int_{x-\ell}^{x+\ell} \mathbf{1}_{E_n}(z)dz.$$

By the Theorem of Hardy-Littlewood we have

$$m(L_n > \lambda) \leq \frac{C}{\lambda} \|\mathbf{1}_{E_n}\|_{L^1} = \frac{C}{\lambda} m(E_n).$$

Taking $\lambda = n^{-\beta}$ with $0 < \beta < \varsigma$, we have

$$m(L_n > n^{-\beta}) \leq \frac{c}{n^{-\beta}} m(E_n) \leq \frac{C}{n^{\varsigma-\beta}}.$$

Hence, taking $n = j^\gamma$, we have $m(L_{j^\gamma} > j^{-\beta\gamma}) \leq \frac{C}{j^{\gamma(\varsigma-\beta)}}$ and assuming that $\gamma(\varsigma - \beta) > 1$ it follows that

$$\sum_j m(L_{j^\gamma} > j^{-\beta\gamma}) \leq \sum_j \frac{C}{j^{\gamma(\varsigma-\beta)}} < \infty.$$

Hence, by the Borel-Cantelli lemma we have that for m-a.e. ζ there exists $N(\zeta)$ such that for all $j \geq N(\zeta)$ we have $\zeta \in \{L_{j^\gamma} \leq j^{-\beta\gamma}\}$.

Choosing $\ell = j^{-\gamma}$, by definition of the function L, we have for m-a.e. ζ

$$\int_{x-\ell}^{x+\ell} \mathbf{1}_{E_n}(z)dz = m((\zeta - j^{-\gamma}, \zeta + j^{-\gamma}) \cap E_{j^\gamma}) \leq 2j^{-\gamma(1+\beta)}.$$

□

Lemma 4.4. *There exist constants $c, C, C', C'' > 0$ such that for all $i \in \mathbb{N}$ and $x \in [0,1]$ we have*

$$c \leq \Pi_i(1)(x) \leq Cx^{-\alpha}.$$

In particular, for $x \in U_n$ and n sufficiently large, we can write

$$c \leq \Pi_i(1)(x) \leq C',$$

where $C' = C''\zeta^{-\alpha}$.

Proof. It is enough to prove the first inequalities. The upper bound follows because the constant function 1 is in the cone $\mathcal{C}_a$ and therefore for any $P_i : (P_i 1)(x) \leq ax^\alpha \int P_i 1 dm \leq ax^\alpha$; in this case $C = a$. The lower bound is the content of Lemma 2.4 in [LSV99] with $c = \min\left\{a, \left[\frac{\alpha(1+\alpha)}{a^\alpha}\right]^{\frac{1}{1-\alpha}}\right\}$. □

Lemma 4.5. *There exists a constant $C > 0$ such that for m-a.e. $\zeta \in (0,1]$, for all $\ell \in \mathbb{N}$ and all n sufficiently large, we have*

$$n\sum_{i=1}^{n^\kappa} m\left(\{x : \ |\mathcal{T}_\ell(x) - \zeta| \leq \delta_{n,\ell} \text{ and } |\mathcal{T}_{i+\ell}(x) - \zeta| \leq \delta_{n,i+\ell}\}\right) \leq C\frac{n^\kappa}{n^\beta} \xrightarrow{n\to\infty} 0.$$

Proof. Let $j = \left(\frac{cn}{\tau}\right)^{1/\gamma}$ so that $j^{-\gamma} = \tau/(cn)$. Also observe that $n^\kappa = (\tau j^\gamma/c)^\kappa \leq j^{\gamma\kappa(1+\xi)}$, if n is large enough. Hence, for such sufficiently large n, we have:

$$\begin{aligned}
V_n :=& \{x : \ |x - \zeta| \leq \frac{\tau}{cn} \text{ and } |\mathcal{T}_i(x) - \zeta| \leq \frac{\tau}{cn} \text{ for some } i \leq n^\kappa\} \\
\subset& \{x : \ |x - \zeta| \leq j^{-\gamma} \text{ and } |\mathcal{T}_i(x) - \zeta| \leq j^{-\gamma} \text{ for some } i \leq n^\kappa\} \\
\subset& \{x : \ |x - \zeta| \leq j^{-\gamma} \text{ and } |\mathcal{T}_i(x) - x| \leq 2j^{-\gamma} \text{ for some } i \leq n^\kappa\} \\
\subset& \{x : \ |x - \zeta| \leq j^{-\gamma} \text{ and } |\mathcal{T}_i(x) - x| \leq 2j^{-\gamma} \text{ for some } i \leq j^{\gamma\kappa(1+\xi)}\} \\
=& \{x : \ |x - \zeta| \leq j^{-\gamma}\} \cap E_{j^\gamma}.
\end{aligned}$$

Hence, by Lemma 4.3 we have $m(V_n) \leq 2\tau^{1+\beta}/n^{1+\beta}$. It follows that taking $C = 2\tau^{1+\beta}$,

$$n\sum_{i=1}^{n^\kappa} m\left(\left\{x : \ |x - \zeta| \leq \frac{\tau}{cn} \text{ and } |\mathcal{T}_i(x) - \zeta| \leq \frac{\tau}{cn}\right\}\right) \leq n\sum_{i=1}^{n^\kappa} m(V_n) \leq n^{1+\kappa}\frac{2\tau^{1+\beta}}{n^{1+\beta}} \leq C\frac{n^\kappa}{n^\beta}. \tag{4.5}$$

Finally, we observe that the quantity we want to estimate can be written as

$$n\sum_{i=1}^{n^\kappa}\int \mathbf{1}_{B_{\delta_{n,\ell}}(\zeta)}\circ\mathcal{T}_\ell\,\mathbf{1}_{B_{\delta_{n,i+\ell}}(\zeta)}\circ\mathcal{T}_{i+\ell}dm = n\sum_{i=1}^{n^\kappa}\int \mathbf{1}_{B_{\delta_{n,\ell}}(\zeta)}\,\mathbf{1}_{B_{\delta_{n,i+\ell}}(\zeta)}\circ T_{i+\ell}\circ\ldots\circ T_{\ell+1}\,\Pi_\ell(1)dm.$$

Recalling that by (3.11) we have $\delta_{n,i}\le\frac{\tau}{cn}$, for all $i\in\mathbb{N}_0$, then, by Lemma 4.4 and (4.5), it follows that there exist $C',C''>0$ such that

$$n\sum_{i=1}^{n^\kappa}\int \mathbf{1}_{B_{\delta_{n,\ell}}(\zeta)}\circ\mathcal{T}_\ell\,\mathbf{1}_{B_{\delta_{n,i+\ell}}(\zeta)}\circ\mathcal{T}_{i+\ell}dm \le C'n\sum_{i=1}^{n^\kappa}m(V_n)\le C''\frac{n^\kappa}{n^\beta}.$$

□

Recall that we are taking: $k_n=n^{1-\beta}$ and $t_n=n^\kappa$.

From Lemma 4.4, we have that $c\mu(U_n)\le m(X_j>u_n)\le C\mu(U_n)$. Hence, if we let $L_n=\max\{\ell_i:\ i=1,\ldots,k_n\}$, we obtain that there exists a constant $\tilde C>0$ such that $L_n\le\tilde Cn^\beta$.

In order to prove Д$'_0$, we need to control the sum on the left

$$\begin{aligned}\sum_{i=1}^{k_n}\sum_{j=0}^{\ell_i-1}\sum_{r>j}^{\ell_i-1}\mathbb{P}(A^{(0)}_{n,\mathcal{L}_{i-1}+j}\cap A^{(0)}_{n,\mathcal{L}_{i-1}+r}) &\le \sum_{i=1}^{k_n}\sum_{j=0}^{L_n-1}\sum_{r>j}^{L_n-1}\mathbb{P}(A^{(0)}_{n,\mathcal{L}_{i-1}+j}\cap A^{(0)}_{n,\mathcal{L}_{i-1}+r})\\ &\le \tilde Cn\max_{\ell=1,\ldots,n}\sum_{i=1}^{\tilde Cn^\beta}\int \mathbf{1}_{U_n}\circ\mathcal{T}_\ell\,\mathbf{1}_{U_n}\circ\mathcal{T}_{i+\ell}dm.\end{aligned}$$

From Lemma 4.5 we have that

$$\lim_{n\to\infty} n\max_{\ell=1,\ldots,n}\sum_{i=1}^{n^\kappa}\int \mathbf{1}_{B_{\delta_{n,\ell}}(\zeta)}\circ\mathcal{T}_\ell\,\mathbf{1}_{B_{\delta_{n,i+\ell}}(\zeta)}\circ\mathcal{T}_{i+\ell}dm=0.$$

Hence we are left to handle $n\max_{\ell=1,\ldots,n}\sum_{i=n^\kappa}^{\tilde Cn^\beta}\int \mathbf{1}_{B_{\delta_{n,\ell}}(\zeta)}\circ\mathcal{T}_\ell\,\mathbf{1}_{B_{\delta_{n,i+\ell}}(\zeta)}\circ\mathcal{T}_{i+\ell}\,dm$ for which we use decay of correlations. Using (4.4), we have:

$$\begin{aligned}n\max_{\ell=1,\ldots,n}\sum_{i=n^\kappa}^{\tilde Cn^\beta}\int \mathbf{1}_{B_{\delta_{n,\ell}}(\zeta)}\circ\mathcal{T}_\ell\,\mathbf{1}_{B_{\delta_{n,i+\ell}}(\zeta)}\circ\mathcal{T}_{i+\ell}\,dm&\\ \le C(n^{1+\beta}n^{1+\eta}n^{\kappa(1-1/\alpha^*)}\log(n)^{1/\alpha^*}+n^{-(1+\eta)+\beta+1}+n^{-2}).&\end{aligned}$$

If we take $\eta=2\beta$ then if α^* is sufficiently small it is easy to check that the terms on right vanish as $n\to\infty$.

Now, we focus on a possible upper bound for α^*. From the first term on the rhs of the previous equation we have that

$$2+4\beta+\kappa-\kappa/\alpha^*<0 \iff \alpha^*<\frac{\kappa}{2+4\beta+\kappa}. \tag{4.6}$$

Moreover, in order to be able to apply Lemma 4.3 we need that $\varsigma>\beta$ which means that

$$\frac{1}{1+\alpha^*}-\kappa(1+\xi)>\beta \iff \alpha^*<\beta+\kappa(1+\xi)-1. \tag{4.7}$$

Recall that $\kappa(1+\xi) < \beta$ but we are free to choose any $\beta \in (0,1)$. Analysing both the expressions one obtains that the maximum range for α^* occurs for β and κ as close as possible to 1, which means that $\alpha^* \leq 1/7$.

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Ana Cristina Moreira Freitas, Centro de Matemática & Faculdade de Economia da Universidade do Porto, Rua Dr. Roberto Frias, 4200-464 Porto, Portugal

E-mail address: amoreira@fep.up.pt

URL: http://www.fep.up.pt/docentes/amoreira/

Jorge Milhazes Freitas, Centro de Matemática & Faculdade de Ciências da Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal

E-mail address: jmfreita@fc.up.pt

URL: http://www.fc.up.pt/pessoas/jmfreita/

Sandro Vaienti, Aix Marseille Université, CNRS, CPT, UMR 7332, 13288 Marseille, France and Université de Toulon, CNRS, CPT, UMR 7332, 83957 La Garde, France.

E-mail address: vaienti@cpt.univ-mrs.fr

URL: http://www.cpt.univ-mrs.fr/~vaienti/