



HAL
open science

Second-order in time schemes for gradient flows in Wasserstein and geodesic metric spaces

Guillaume Legendre, Gabriel Turinici

► **To cite this version:**

Guillaume Legendre, Gabriel Turinici. Second-order in time schemes for gradient flows in Wasserstein and geodesic metric spaces. 2016. hal-01317769v2

HAL Id: hal-01317769

<https://hal.science/hal-01317769v2>

Preprint submitted on 13 Nov 2016 (v2), last revised 24 Feb 2017 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Second-order in time schemes for gradient flows in Wasserstein and geodesic metric spaces

Guillaume Legendre^a, Gabriel Turinici^{a,b}

^a *Université Paris-Dauphine, PSL Research University, CNRS, UMR 7534, CEREMADE, 75016 Paris, France*

^b *Institut Universitaire de France*

November 10, 2016

Abstract

The time discretization of gradient flows in metric spaces uses variants of the celebrated implicit Euler-type scheme of Jordan, Kinderlehrer and Otto [9]. We propose in this Note a different approach which allows to construct two second-order in time numerical schemes. In a metric space framework, we see that the schemes are well defined and prove convergence for one of them under some regularity assumptions. For the particular case of a Fokker–Planck gradient flow in the Wasserstein space, we obtain (theoretically and numerically) the second-order convergence.

To cite this article: G. Legendre, G. Turinici, C. R. Acad. Sci. Paris, Ser. I 340 (2016).

Résumé

Schémas d’ordre deux en temps pour des flots de gradient dans des espaces métriques géodésiques et de Wasserstein. La discrétisation temporelle des flots de gradient dans des espaces métriques utilise des variantes du schéma d’Euler implicite issu du travail séminal de Jordan, Kinderlehrer et Otto [9]. Nous proposons dans cette Note une approche différente permettant de construire deux schémas numériques d’ordre deux en temps. Dans un cadre d’espace métrique, nous montrons que les schémas sont bien définis et prouvons la convergence de l’un d’entre eux sous des hypothèses de régularité. Pour le cas particulier d’un flot de gradient Fokker–Planck dans l’espace de Wasserstein, nous obtenons (théoriquement et numériquement) la convergence à l’ordre deux.

Pour citer cet article : G. Legendre, G. Turinici, C. R. Acad. Sci. Paris, Ser. I 340 (2016).

Email addresses: Guillaume.Legendre@dauphine.fr (Guillaume Legendre), Gabriel.Turinici@dauphine.fr (Gabriel Turinici).

1. Background on gradient flows and relationship with the implicit Euler scheme

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a smooth convex function and \bar{x} be a point in \mathbb{R}^d ; a gradient flow starting from \bar{x} is a curve $(x_t)_{t \geq 0}$ solution to the Cauchy problem $x'_t = -\nabla f(x_t)$ for $t > 0$, $x_0 = \bar{x}$. A topic that has received considerable attention lately is the extension of this definition to a Polish metric space (\mathcal{X}, d) and a functional $F : (\mathcal{X}, d) \rightarrow \mathbb{R} \cup \{+\infty\}$ (see [3,14,2,13] for instance). An important advance was in particular the work of Jordan, Kinderlehrer and Otto [9], who introduced the following numerical scheme: given a time step $\tau > 0$ and \bar{x} in \mathcal{X} , define recursively the sequence $(x_n^\tau)_{n \in \mathbb{N}}$ such that $x_0^\tau = \bar{x}$ and, for any natural integer n , x_{n+1}^τ is selected as a minimizer of the functional

$$x \mapsto P_F^{JKO}(x; x_n^\tau, \tau) := \frac{1}{2\tau} d^2(x_n^\tau, x) + F(x). \quad (1)$$

When the metric space is a Hilbert space and the functional F is smooth enough, the above scheme amounts to the implicit Euler scheme, i.e., $\frac{x_{n+1}^\tau - x_n^\tau}{\tau} = -\nabla F(x_{n+1}^\tau)$. As a consequence, the Jordan–Kinderlehrer–Otto method (named JKO from now on) can be seen as a variational generalization of the implicit Euler scheme in a metric space. Coupled with a suitable interpolation, the sequence constructed by this scheme defines, as the time step τ tends to zero, a (metric) gradient flow, the convergence being of order one in τ (see [3, Theorem 4.0.4] in a particular setting for a sharp estimate). As a natural numerical counterpart of the theoretical procedure, the JKO scheme was consequently thus used to compute the gradient flow in many applications (see [10,5,4] for instance).

However, while the JKO scheme has theoretical advantages, its first-order convergence may be judged insufficient in practice. When the gradient flow is regular enough, one may seek to replace it with a second-order alternative, or even combine these two schemes in order to construct a time step selection mechanism (see [12, Chapter 17, Section 17.2] for instance). The purpose of this Note is to propose two second-order schemes, which are both easy to implement (one of them directly using a step of the JKO scheme), and provide several preliminary theoretical and numerical results pertaining to them.

2. Second-order numerical schemes on metric spaces

Let (\mathcal{X}, d) be a Polish metric space. We recall (see [2, page 37]) that a curve $\gamma : [0, 1] \rightarrow \mathcal{X}$ is called a (constant speed) geodesic provided that $d(\gamma(t), \gamma(s)) = |t - s| \cdot d(\gamma(0), \gamma(1))$, and that the space (\mathcal{X}, d) is called geodesic if, for any couple of points $(x, y) \in \mathcal{X}^2$, there exists a geodesic γ connecting them, that is, such that $\gamma(0) = x$ and $\gamma(1) = y$. We will denote by $Geod_{x,y}$ the set of all geodesics connecting x and y . Moreover, the set $\{\gamma(\frac{1}{2}) \mid \gamma \in Geod_{x,y}\}$ will be called the set of midpoints of the couple (x, y) and will be denoted $\frac{x+y}{2}$ (see [7, Chapter 2] for a definition in general metric spaces and [1] in the particular case of Wasserstein spaces).

2.1. Definition of the schemes

Given a time step τ , we define the *Variational Implicit Midpoint (VIM) scheme*, starting from \bar{x} in \mathcal{X} , for the (gradient flow of the) functional F by setting $x_0^\tau = \bar{x}$ and, for any natural integer n , recursively choosing

$$x_{n+1}^\tau \in \operatorname{argmin}_{y \in \mathcal{X}} P_F^{VIM}(y; x_n^\tau, \tau), \quad (2)$$

where

$$\forall (x, y) \in \mathcal{X}^2, P_F^{VIM}(y; x, \tau) = \inf \left\{ \frac{d^2(x, y)}{2\tau} + 2F(u) \mid u \in \frac{x+y}{2} \right\}. \quad (3)$$

In the particular case of a Hilbert space and a smooth functional F , the point x_{n+1}^τ is effectively a solution of the equation $\frac{x-x_n^\tau}{\tau} + \nabla F(\frac{x_n^\tau+x}{2}) = 0$, which characterizes the implicit midpoint rule method, a one-stage implicit Runge–Kutta method of second order (see [8, Chapter II, Section 7]).

We now prove that the VIM scheme is well-defined by establishing the existence of a solution to problem (2).

Theorem 2.1 *Let (\mathcal{X}, d) be a Polish metric space such that,*

$$\text{for any } x \text{ in } \mathcal{X}, \text{ the set } \bigcup_{y \in \mathcal{X}} \frac{x+y}{2} \text{ is closed.} \quad (4)$$

Assume that the functional F is lower semicontinuous, bounded from below, and such that, for all $r > 0$, and $c \in \mathbb{R}$, the set $\{y \in \mathcal{X} \mid F(y) \leq c, d(\tilde{x}, y) \leq r\}$ is compact for some \tilde{x} in \mathcal{X} . Then, there exists x in \mathcal{X} such that $P_F^{VIM}(x; \tilde{x}, \tau) = \inf_{y \in \mathcal{X}} P_F^{VIM}(y; \tilde{x}, \tau)$.

Proof Let $(x_n, u_n)_{n \in \mathbb{N}}$ be a minimizing sequence for $\inf_{y \in \mathcal{X}} P_F^{VIM}(y; \tilde{x}, \tau)$. In particular, the sequence $(\frac{1}{2} d^2(\tilde{x}, x_n))_{n \in \mathbb{N}}$ is bounded and so is $(d(\tilde{x}, u_n))_{n \in \mathbb{N}}$. On the other hand, the sequence $(F(u_n))_{n \in \mathbb{N}}$ is also bounded, so that, by hypothesis, the sequence $(u_n)_{n \in \mathbb{N}}$ lives in a compact set and thus converges, up to a subsequence, to some u . In addition, since $\cup_{y \in \mathcal{X}} \frac{\tilde{x}+y}{2}$ is closed, there exists x in \mathcal{X} such that $u \in \frac{\tilde{x}+x}{2}$. Recalling that $u_n \in \frac{\tilde{x}+x_n}{2}$, we obtain that $d(\tilde{x}, x) = 2 d(\tilde{x}, u) = 2 \lim_{n \rightarrow \infty} d(\tilde{x}, u_n) = \lim_{n \rightarrow \infty} d(\tilde{x}, x_n)$. The lower semicontinuity property of F then allows to conclude. ■

Remark 1 *The assumptions on the functional F in the above theorem are classical (see [2, Section 4.2.2] for instance), but can be weakened. In particular, we may not suppose F to be bounded from below.*

One may observe that, for any natural integer n and any point x in \mathcal{X} and any midpoint u in $\frac{x_n^\tau+x}{2}$, it holds

$$\frac{1}{2\tau} d^2(x_n^\tau, x) + 2F(u) = \frac{2}{\tau} d^2(x_n^\tau, u) + 2F(u) = 2P_F^{JKO}\left(u; x_n^\tau, \frac{\tau}{2}\right). \quad (5)$$

As a consequence, another interpretation of the VIM scheme is to view the midpoint u as computed by a JKO scheme with a halved time step $\frac{\tau}{2}$ and to *extrapolate* it as follows.

Recalling that a geodesic space (\mathcal{X}, d) is called non-branching (see [2, Definition 3.15] or [3, Theorem 11.2.10]) if, for any $t \in]0, 1[$, a constant speed geodesic γ is uniquely determined by its initial point $\gamma(0)$ and by the point $\gamma(t)$. In particular, if $u \in \frac{x+y_1}{2} \cap \frac{x+y_2}{2}$ then $y_1 = y_2$. By definition, the 2-extrapolate of the couple (x, u) (denoted $2u - x$ hereafter) is the unique point y such that $u \in \frac{x+y}{2}$. On the contrary, when there exists no such point, the 2-extrapolate is defined as the point y with largest distance to x such that u is on a geodesic from x to y . When the space is non-branching and complete, using the properties of the distance between two points on a geodesic and the fact that any Cauchy sequence converges, we obtain that, for fixed points x and u , the set $\{d(x, y) \mid \exists \gamma \in \text{Geod}_{x,y}, u \in \gamma\}$ is closed. The 2-extrapolate is thus well-defined and unique. It moreover satisfies $d(x, u) \leq d(x, 2u - x) \leq 2d(x, u)$.

These considerations lead to the definition of the *Extrapolated Variational Implicit Euler (EVIE) scheme*, in which x_{n+1}^τ is now the 2-extrapolate $2u - x_n^\tau$ of the point u computed by the JKO scheme with time step $\frac{\tau}{2}$. Implementation-wise, this last scheme has the obvious advantage of using the JKO scheme. From a theoretical point of view, the existence and uniqueness of its solution in a non-branching space is assured under the same hypotheses that those required for the JKO scheme. While adaptations would certainly allow to define it in branching spaces, uniqueness may be lost, even when the JKO scheme possesses a unique solution at each step.

2.2. Convergence of the VIM scheme

In this section, we prove the convergence of the VIM scheme in a metric space setting by showing that a natural interpolation of its discrete solution converges to a gradient flow written in the integral form

of an Energy Dissipation Inequality (EDI) (see [2, Section 4.2]). More precisely, we recall that, given a functional $F : (\mathcal{X}, d) \rightarrow \mathbb{R} \cup \{+\infty\}$ with domain $D(F) = \{x \in \mathcal{X}, F(x) < +\infty\}$, and a point $\bar{x} \in D(F)$, the curve $(x_t)_{t \geq 0}$ in \mathcal{X} is a gradient flow in the EDI sense starting at \bar{x} provided that it is a locally absolutely continuous curve, $x_0 = \bar{x}$ and

$$\forall s \geq 0, F(x_s) + \frac{1}{2} \int_0^s |x'_r| \, dr + \frac{1}{2} \int_0^s |\nabla F|^2(x_r) \, dr \leq F(\bar{x}), \quad (6)$$

$$\text{a.e. } t > 0, \forall s \geq t, F(x_s) + \frac{1}{2} \int_t^s |x'_r| \, dr + \frac{1}{2} \int_t^s |\nabla F|^2(x_r) \, dr \leq F(x_t), \quad (7)$$

where, for any point x in $D(F)$, the slope of F at x is

$$|\nabla F|(x) = \limsup_{z \rightarrow x} \frac{(F(x) - F(z))^+}{d(x, z)} = \max \left\{ \limsup_{z \rightarrow x} \frac{F(x) - F(z)}{d(x, z)}, 0 \right\},$$

and the metric derivative of x at r is

$$|x'_r| = \lim_{h \rightarrow 0} \frac{d(x_{r+h}, x_r)}{|h|}. \quad (8)$$

The time step τ being fixed with $0 < \tau < \bar{\tau}$, we introduce a variational interpolation *à la* De Giorgi of the discrete solution of the VIM scheme by defining the curve $(x_t^\tau)_{t \geq 0}$ as follows:

- $x_0^\tau = \bar{x}$,
 - for $n \in \mathbb{N}$, $x_{(n+1)\tau}^\tau \in \operatorname{argmin}_{y \in \mathcal{X}} P_F^{VIM}(y; x_{n\tau}^\tau, \tau)$,
 - for $n \in \mathbb{N}$ and $t \in]n\tau, (n+1)\tau[$, $x_t^\tau \in \operatorname{argmin}_{y \in \mathcal{X}} P_F^{VIM}(y; x_{n\tau}^\tau, t - n\tau)$.
- For such a map, we define the discrete speed $\operatorname{Dsp}^\tau : [0, +\infty) \rightarrow [0, +\infty)$ by

$$\operatorname{Dsp}_t^\tau = \frac{d(x_{n\tau}^\tau, x_{(n+1)\tau}^\tau)}{\tau} \text{ for } t \text{ in } (n\tau, (n+1)\tau),$$

and the discrete slope $\operatorname{Dsl}^\tau : [0, +\infty) \rightarrow [0, +\infty)$ by

$$\operatorname{Dsl}_t^\tau = \frac{d(x_{n\tau}^\tau, x_t^\tau)}{t - n\tau} \text{ for } t \text{ in } (n\tau, (n+1)\tau).$$

In order to avoid unnecessary technicalities in the forthcoming analysis, we will suppose from now on that midpoints are unique, that is

$$\forall (x, y) \in \mathcal{X}^2, \text{ the set } \frac{x+y}{2} \text{ is a singleton.} \quad (9)$$

We finally introduce a notion of slope of F at the midpoint of (x, y) by setting

$$|\nabla^M F|(x, y) = \limsup_{z \rightarrow y} \frac{(F(\frac{x+y}{2}) - F(\frac{x+z}{2}))^+}{d(\frac{x+y}{2}, \frac{x+z}{2})}.$$

Theorem 2.2 *Let $T > 0$ be fixed and (\mathcal{X}, d) be a Polish metric space satisfying hypotheses (4) and (9). We moreover assume that*

(i) *F is lower semicontinuous, bounded from below, and such that:*

$$\forall r > 0, \forall c \in \mathbb{R}, \forall x \in \mathcal{X} \text{ the set } \{y \in \mathcal{X} \mid F(y) \leq c, d(x, y) \leq r\} \text{ is compact,} \quad (10)$$

(ii) *F has the following continuity property*

$$\text{if } x_n \rightarrow x, \text{ and } \sup\{|\nabla F|(x_n), E(x_n)\} < \infty \text{ then } F(x_n) \rightarrow F(x); \quad (11)$$

(iii) $|\nabla^M F| : D(F) \times D(F) \rightarrow [0, \infty)$ has the following semicontinuity property: for any x in $D(F)$ and any two sequences $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ in $D(F)$ converging to x , it holds

$$|\nabla^M F|(x, x) \leq \liminf_{n \rightarrow \infty} |\nabla^M F|(x_n, y_n),$$

(iv) if any two of the elements $x, y, \frac{x+y}{2}$ belong to $D(F)$, then the third also does and:

$$\left| \frac{F(x) + F(y) - 2F\left(\frac{x+y}{2}\right)}{d^2(x, y)} \right| \leq H, \quad (12)$$

where H is a constant independent of x and y .

Then, for some $\bar{\tau} > 0$, the set of curves $\{(x_t^\tau)_{t \in [0, T]}; 0 \leq \tau \leq \bar{\tau}\}$ is relatively compact (with respect to the local uniform convergence) and any limit curve is a gradient flow in the EDI formulation (6)–(7).

Remark 2 Assumptions *i* and *ii* are classical and used in [2, Assumption 4.13] while *iii* is a generalization of the second assumption in [2, Assumption 4.13]. Item *iv* is however a regularity property specific to our setting; it could be weakened to $F(x) + F(y) - 2F\left(\frac{x+y}{2}\right) = o(d(x, y))$.

Proof We will follow closely and adapt where necessary the method of proof given in [2, Subsection 4.2.2]. The proof is divided into several steps, starting with the derivation of some key properties of the variational interpolation $(x_t^\tau)_{t \geq 0}$ introduced above.

(I) First, we shall show that, the positive real number τ and the natural integer n being fixed, the function $(0, 1] \ni \theta \mapsto \frac{1}{2\theta\tau} d^2(x_{n\tau}^\tau, x_{(n+\theta)\tau}^\tau) + 2F\left(\frac{x_{n\tau}^\tau + x_{(n+\theta)\tau}^\tau}{2}\right)$ is locally Lipschitz and that its derivative is given by

$$\frac{d}{d\theta} \left(\frac{1}{2\theta\tau} d^2(x_{n\tau}^\tau, x_{(n+\theta)\tau}^\tau) + 2F\left(\frac{x_{n\tau}^\tau + x_{(n+\theta)\tau}^\tau}{2}\right) \right) = -\frac{1}{2\theta^2\tau} d^2(x_{n\tau}^\tau, x_{(n+\theta)\tau}^\tau). \quad (13)$$

Indeed, let $0 < \theta_0 < \theta < \theta_1 \leq 1$. By the definition of the variational interpolation, one has

$$\frac{1}{2\theta_0\tau} d^2(x_{n\tau}^\tau, x_{(n+\theta_0)\tau}^\tau) + 2F\left(\frac{x_{n\tau}^\tau + x_{(n+\theta_0)\tau}^\tau}{2}\right) \leq \frac{1}{2\theta_0\tau} d^2(x_{n\tau}^\tau, x_{(n+\theta_1)\tau}^\tau) + 2F\left(\frac{x_{n\tau}^\tau + x_{(n+\theta_1)\tau}^\tau}{2}\right),$$

so that

$$\begin{aligned} \frac{1}{2\theta_0\tau} d^2(x_{n\tau}^\tau, x_{(n+\theta_0)\tau}^\tau) + 2F\left(\frac{x_{n\tau}^\tau + x_{(n+\theta_0)\tau}^\tau}{2}\right) - \frac{1}{2\theta_0\tau} d^2(x_{n\tau}^\tau, x_{(n+\theta_1)\tau}^\tau) - 2F\left(\frac{x_{n\tau}^\tau + x_{(n+\theta_1)\tau}^\tau}{2}\right) \\ \leq \frac{(\theta_1 - \theta_0)\tau}{2\theta_0\theta_1\tau^2} d^2(x_{n\tau}^\tau, x_{(n+\theta_1)\tau}^\tau). \end{aligned}$$

Arguing symmetrically, one also has

$$\begin{aligned} \frac{1}{2\theta_0\tau} d^2(x_{n\tau}^\tau, x_{(n+\theta_0)\tau}^\tau) + 2F\left(\frac{x_{n\tau}^\tau + x_{(n+\theta_0)\tau}^\tau}{2}\right) - \frac{1}{2\theta_0\tau} d^2(x_{n\tau}^\tau, x_{(n+\theta_1)\tau}^\tau) - 2F\left(\frac{x_{n\tau}^\tau + x_{(n+\theta_1)\tau}^\tau}{2}\right) \\ \geq \frac{(\theta_1 - \theta_0)\tau}{2\theta_0\theta_1\tau^2} d^2(x_{n\tau}^\tau, x_{(n+\theta_0)\tau}^\tau), \end{aligned}$$

and the function is thus locally Lipschitz. Moreover, dividing by $\theta_1 - \theta_0$ the last two inequalities and passing to the limits $\theta_0 \uparrow \theta$ and $\theta_1 \downarrow \theta$, we obtain the proposed expression for the derivative.

(II) Next, we establish that, with the same notations as above, $\theta \mapsto d(x_{n\tau}^\tau, x_{(n+\theta)\tau}^\tau)$ is non decreasing, $\theta \mapsto F\left(\frac{x_{n\tau}^\tau + x_{(n+\theta)\tau}^\tau}{2}\right)$ is non increasing and that it holds

$$\forall \theta \in (0, 1], \quad |\nabla^M F|(x_{n\tau}^\tau, x_{(n+\theta)\tau}^\tau) \leq \frac{d(x_{n\tau}^\tau, x_{(n+\theta)\tau}^\tau)}{\theta\tau}. \quad (14)$$

Let $0 < \theta_0 < \theta_1 \leq 1$. By the respective minimality properties of $x_{(n+\theta_0)\tau}^\tau$ and $x_{(n+\theta_1)\tau}^\tau$, we have

$$\begin{aligned} \frac{1}{2\theta_0\tau} d^2(x_{n\tau}^\tau, x_{(n+\theta_0)\tau}^\tau) + 2F\left(\frac{x_{n\tau}^\tau + x_{(n+\theta_0)\tau}^\tau}{2}\right) &\leq \frac{1}{2\theta_0\tau} d^2(x_{n\tau}^\tau, x_{(n+\theta_1)\tau}^\tau) + 2F\left(\frac{x_{n\tau}^\tau + x_{(n+\theta_1)\tau}^\tau}{2}\right), \\ \frac{1}{2\theta_1\tau} d^2(x_{n\tau}^\tau, x_{(n+\theta_1)\tau}^\tau) + 2F\left(\frac{x_{n\tau}^\tau + x_{(n+\theta_1)\tau}^\tau}{2}\right) &\leq \frac{1}{2\theta_1\tau} d^2(x_{n\tau}^\tau, x_{(n+\theta_0)\tau}^\tau) + 2F\left(\frac{x_{n\tau}^\tau + x_{(n+\theta_0)\tau}^\tau}{2}\right). \end{aligned}$$

Adding the last two inequalities, we get

$$\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right) \left(d^2(x_{n\tau}^\tau, x_{(n+\theta_0)\tau}^\tau) - d^2(x_{n\tau}^\tau, x_{(n+\theta_1)\tau}^\tau)\right) \leq 0,$$

so that $d(x_{n\tau}^\tau, x_{(n+\theta_0)\tau}^\tau) \leq d(x_{n\tau}^\tau, x_{(n+\theta_1)\tau}^\tau)$. From this, we now have

$$\frac{1}{2\theta_1\tau} d^2(x_{n\tau}^\tau, x_{(n+\theta_0)\tau}^\tau) + 2F\left(\frac{x_{n\tau}^\tau + x_{(n+\theta_1)\tau}^\tau}{2}\right) \leq \frac{1}{2\theta_1\tau} d^2(x_{n\tau}^\tau, x_{(n+\theta_1)\tau}^\tau) + 2F\left(\frac{x_{n\tau}^\tau + x_{(n+\theta_1)\tau}^\tau}{2}\right),$$

which implies, using a previous inequality, that $F\left(\frac{x_{n\tau}^\tau + x_{(n+\theta_1)\tau}^\tau}{2}\right) \leq F\left(\frac{x_{n\tau}^\tau + x_{(n+\theta_0)\tau}^\tau}{2}\right)$.

Finally, by the definition of $x_{(n+\theta)\tau}^\tau$, one has

$$\forall y \in \mathcal{X}, \quad \frac{1}{2\theta\tau} d^2(x_{n\tau}^\tau, x_{(n+\theta)\tau}^\tau) + 2F\left(\frac{x_{n\tau}^\tau + x_{(n+\theta)\tau}^\tau}{2}\right) \leq \frac{1}{2\theta\tau} d^2(x_{n\tau}^\tau, y) + 2F\left(\frac{x_{n\tau}^\tau + y}{2}\right),$$

so that, by the very definition of the set of midpoints between two points,

$$\frac{2}{\theta\tau} d^2\left(x_{n\tau}^\tau, \frac{x_{n\tau}^\tau + x_{(n+\theta)\tau}^\tau}{2}\right) + F\left(\frac{x_{n\tau}^\tau + x_{(n+\theta)\tau}^\tau}{2}\right) \leq \frac{2}{\theta\tau} d^2\left(x_{n\tau}^\tau, \frac{x_{n\tau}^\tau + y}{2}\right) + 2F\left(\frac{x_{n\tau}^\tau + y}{2}\right).$$

Hence, we get

$$\begin{aligned} \frac{F\left(\frac{x_{n\tau}^\tau + x_{(n+\theta)\tau}^\tau}{2}\right) - F\left(\frac{x_{n\tau}^\tau + y}{2}\right)}{d\left(\frac{x_{n\tau}^\tau + x_{(n+\theta)\tau}^\tau}{2}, \frac{x_{n\tau}^\tau + y}{2}\right)} &\leq \\ \frac{1}{\theta\tau} \frac{\left(d\left(x_{n\tau}^\tau, \frac{x_{n\tau}^\tau + x_{(n+\theta)\tau}^\tau}{2}\right) - d\left(x_{n\tau}^\tau, \frac{x_{n\tau}^\tau + y}{2}\right)\right) \left(d\left(x_{n\tau}^\tau, \frac{x_{n\tau}^\tau + x_{(n+\theta)\tau}^\tau}{2}\right) + d\left(x_{n\tau}^\tau, \frac{x_{n\tau}^\tau + y}{2}\right)\right)}{d\left(\frac{x_{n\tau}^\tau + x_{(n+\theta)\tau}^\tau}{2}, \frac{x_{n\tau}^\tau + y}{2}\right)} & \end{aligned}$$

Using the reverse triangular inequality and taking the upper limit as y tends to $x_{(n+\theta)\tau}^\tau$, we obtain that

$$\begin{aligned} |\nabla^M F|(x_{n\tau}^\tau, x_{(n+\theta)\tau}^\tau) &\leq \frac{1}{\theta\tau} \limsup_{y \rightarrow x_{(n+\theta)\tau}^\tau} \left(d\left(x_{n\tau}^\tau, \frac{x_{n\tau}^\tau + x_{(n+\theta)\tau}^\tau}{2}\right) + d\left(x_{n\tau}^\tau, \frac{x_{n\tau}^\tau + y}{2}\right)\right) \\ &= \frac{2}{\theta\tau} d\left(x_{n\tau}^\tau, \frac{x_{n\tau}^\tau + x_{(n+\theta)\tau}^\tau}{2}\right) = \frac{1}{\theta\tau} d(x_{n\tau}^\tau, x_{(n+\theta)\tau}^\tau). \end{aligned}$$

(III) We now prove that for any integers n and m such that $0 \leq n < m$:

$$\frac{1}{2} \int_{n\tau}^{m\tau} (\text{Dsp}_r^\tau)^2 dr + 2 \sum_{k=n}^{m-1} F\left(\frac{x_{k\tau}^\tau + x_{(k+1)\tau}^\tau}{2}\right) - 2 \sum_{k=n}^{m-1} F(x_{k\tau}^\tau) = -\frac{1}{2} \int_{n\tau}^{m\tau} (\text{Dsl}_r^\tau)^2 dr. \quad (15)$$

From (13), we infer that

$$\forall k \in \mathbb{N}, \quad \frac{1}{2\tau} d(x_{k\tau}^\tau, x_{(k+1)\tau}^\tau) + 2F\left(\frac{x_{k\tau}^\tau + x_{(k+1)\tau}^\tau}{2}\right) - 2F(x_{k\tau}^\tau) = -\int_0^1 \frac{d^2(x_{k\tau}^\tau, x_{(k+\theta)\tau}^\tau)}{2\theta^2\tau} d\theta,$$

and, by a change of variable and the respective definitions of the discrete speed and slopes,

$$\forall k \in \mathbb{N}, \forall t \in (k\tau, (k+1)\tau), \frac{\tau}{2}(\text{Dsp}_t^\tau)^2 + 2F\left(\frac{x_{k\tau}^\tau + x_{(k+1)\tau}^\tau}{2}\right) - 2F(x_{k\tau}^\tau) = -\frac{1}{2} \int_{k\tau}^{(k+1)\tau} (\text{Dsl}_r^\tau)^2 dr. \quad (16)$$

The results follows from summing from $k = n$ to $m - 1$.

(IV) Combining relation (15) with hypothesis (12), we obtain

$$\left| \frac{1}{2} \int_{n\tau}^{m\tau} (\text{Dsp}_r^\tau)^2 dr + F(x_{m\tau}^\tau) - F(x_{n\tau}^\tau) + \frac{1}{2} \int_{n\tau}^{m\tau} (\text{Dsl}_r^\tau)^2 dr \right| \leq H\tau \int_{n\tau}^{m\tau} (\text{Dsp}_r^\tau)^2 dr. \quad (17)$$

Therefore, for $t \leq T$ and $T = n\tau$, it holds

$$d^2(x_t^\tau, x_0^\tau) \leq \left(\int_0^T \text{Dsp}_r^\tau dr \right)^2 \leq T \int_0^T (\text{Dsp}_r^\tau)^2 dr \leq \frac{2T}{1 - 2H\tau} (F(x_0^\tau) - \inf F), \quad (18)$$

which shows that, for any T , the set $\{x_t^\tau\}_{0 \leq t \leq T}$ is bounded (uniformly with respect to τ). Using (10), we conclude that it is relatively compact. Moreover, further exploitation of inequality (17) shows that for $t = n\tau < m\tau = s$, one has

$$d^2(x_t^\tau, x_s^\tau) \leq \left(\int_t^s \text{Dsp}_r^\tau dr \right)^2 \leq \frac{2(s-t)}{1 - 2H\tau} (F(x_0^\tau) - \inf F), \quad (19)$$

which gives equicontinuity and, by the Arzelà–Ascoli theorem, the relative compactness of the set of curves $\{(x_t^\tau)_{0 \leq t \leq T}; 0 \leq \tau \leq \bar{\tau}\}$ with respect to the local uniform convergence.

(V) We finally pass to the limit. Let $(\tau_n)_{n \in \mathbb{N}}$ be a decreasing sequence tending to zero such that $(x_t^{\tau_n})$ converges to a limit curve x_t locally uniformly as n tends to infinity. Using inequality (19), this curve is absolutely continuous and satisfies

$$\forall 0 \leq t < s, \int_t^s |x_r'| dr \leq \liminf_{n \rightarrow +\infty} \int_t^s (\text{Dsp}_r^{\tau_n})^2 dr. \quad (20)$$

In addition, set t and, $\forall n \in \mathbb{N}$, let $N(n) \in \mathbb{N}$ be such that $N(n)\tau_n < t \leq (N(n) + 1)\tau_n$, so that $N(n)\tau_n$ tends to t as n tends to infinity. Using the lower semicontinuity of $|\nabla^M F|$ (assumption iii) and (14), we have on the one hand

$$|\nabla^M F|(x_t, x_t) \leq \liminf_{n \rightarrow +\infty} |\nabla^M F|(x_{N(n)\tau_n}^{\tau_n}, x_t^{\tau_n}) \leq \liminf_{n \rightarrow +\infty} \frac{d(x_{N(n)\tau_n}^{\tau_n}, x_t^{\tau_n})}{t - N(n)\tau_n} = \liminf_{n \rightarrow +\infty} \text{Dsl}_t^{\tau_n}.$$

On the other hand, it follows from (12) that

$$\begin{aligned} \forall t, |\nabla^M F|(x_t, x_t) &= \limsup_{z \rightarrow x_t} \frac{(F(x_t) - F(\frac{x_t+z}{2}))^+}{d(x_t, \frac{x_t+z}{2})} = \limsup_{z \rightarrow x_t} \frac{2(F(x_t) - F(\frac{x_t+z}{2}))^+}{d(x_t, z)} \\ &= \limsup_{z \rightarrow x_t} \frac{(2F(x_t) - (F(x_t) + F(z)))^+}{d(x_t, z)} = \limsup_{z \rightarrow x_t} \frac{(F(x_t) - F(z))^+}{d(x_t, z)} = |\nabla F|(x_t). \end{aligned}$$

By Fatou's lemma and (17), we thus obtain that, for $\tau_n \leq 1/(2H)$,

$$\forall t < s, \int_t^s |\nabla F|^2(x_r) dr \leq \liminf_{n \rightarrow +\infty} \int_t^s (\text{Dsl}_r^{\tau_n})^2 dr \leq 2(F(x_0^\tau) - \inf F). \quad (21)$$

Passing to the limit in (17) using (20) and (21) for $t = 0$ and s arbitrary we obtain (6). Note that the same technique does not yet work for $t > 0$, since F is not necessarily continuous. From (21), it follows that, for almost every $t > 0$ (taken as fixed from now on), there exists a subsequence τ_{n_k} with $\sup_k |\nabla F|(x_t^{\tau_{n_k}}) < \infty$. By assumption (11), we then obtain (7). \blacksquare

3. Second-order schemes for the Fokker–Planck equation as a gradient flow in the Wasserstein space

As a complement to the above results, we consider in this final section a particular gradient flow in the one-dimensional case, for which the results of Section 2.2 do not apply but which provides evidence that the VIM and EVIE schemes have good numerical properties. We refer to [11,4] for approaches in higher dimensions compatible with these schemes.

The space \mathcal{X} is now set to be the Wasserstein space of order 2, that is the set of probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with finite second-order moment, $\mathcal{P}_2(\mathbb{R})$, endowed with the 2-Wasserstein distance \mathcal{W}_2 (see [13, Chapter 5] or [9,14]). Note that, from now on, the variable x will denote the position over the real line.

Consider a probability measure ν in $\mathcal{P}_2(\mathbb{R})$. When ν is absolutely continuous with respect to the Lebesgue measure, with probability density ρ , the functional F is defined by

$$F(\nu) = E(\rho) + S(\rho) \text{ with } E(\rho) = \int_{\mathbb{R}} V(x)\rho(x) dx \text{ and } S(\rho) = \frac{\sigma^2}{2} \int_{\mathbb{R}} \rho(x) \log(\rho(x)) dx. \quad (22)$$

Here, the term $E(\rho)$ corresponds to a potential energy, the function V being the potential in question, and the term $S(\rho)$ corresponds to an internal energy, the scalar σ being a positive real number. Otherwise, we set $F(\nu) = +\infty$. Correspondingly, if the measures ν_1 and ν_2 are absolutely continuous with respect to the Lebesgue measure, with respective densities ρ_1 and ρ_2 , we denote by $\mathcal{W}_2(\rho_1, \rho_2)$ the Wasserstein distance between ν_1 and ν_2 and by $Geod_{\rho_1, \rho_2}$ the set of geodesics connecting ν_1 to ν_2 .

For any smooth potential V , it is well-known (see [9]) that the gradient flow in $\mathcal{P}_2(\mathbb{R})$ of the functional F is a curve $t \mapsto \nu(t)$ in $\mathcal{P}_2(\mathbb{R})$, such that, at almost any time t , the measure $\nu(t)$ is absolutely continuous with respect to the Lebesgue measure, with probability density $\rho(t, \cdot)$, and that ρ is solution of the Fokker–Planck partial differential equation of the form

$$\frac{\partial \rho}{\partial t}(t, x) = \frac{\partial}{\partial x}[V'(x)\rho(t, x)] + \frac{\sigma^2}{2} \frac{\partial^2 \rho}{\partial x^2}(t, x), \quad (23)$$

which is itself related to the following stochastic differential equation

$$dX(t) = -V'(X(t)) dt + \sigma dW(t), \quad (24)$$

the stochastic process W being a standard one-dimensional Wiener process.

In what follows, we consider quadratic potentials of the form $V(x) = \theta \frac{(x-\mu)^2}{2}$, where θ and μ are given constants. We denote by $\mathcal{N}(a, b^2)$ the normal (Gaussian) distribution with mean a and variance b^2 and also, by abusing the notation, the associated density (with respect to the Lebesgue measure)

$$\rho(x) = \frac{1}{\sqrt{2\pi b^2}} e^{-\frac{(x-a)^2}{2b^2}}.$$

Proposition 3.1 *If $\rho(0, \cdot) = \mathcal{N}(\mu_0, \sigma_0^2)$, any intermediary state of the semi-discrete VIM and EVIE schemes has a Gaussian distribution and the error of both schemes has order $O(\tau^2)$.*

Proof We will prove the assertion for the VIM scheme, the proof for the EVIE one being similar. The potential V being quadratic, the solution of (24) is a Ornstein–Uhlenbeck process, given by $X(t) = X(0)e^{-\theta t} + \mu(1 - e^{-\theta t}) + \frac{\sigma}{\sqrt{2\theta}}e^{-\theta t}W(e^{2\theta t} - 1)$. In particular, the exact evolution from $\rho_n = \mathcal{N}(\mu_n, \sigma_n^2)$ after τ time units is

$$\rho_{n+1}^{exact} = \mathcal{N}\left(\mu_n e^{-\theta\tau} + \mu(1 - e^{-\theta\tau}), e^{-2\theta\tau}\sigma_n^2 + \frac{\sigma^2}{2\theta}(1 - e^{-2\theta\tau})\right).$$

On the other hand, ρ_n being absolutely continuous with respect to the Lebesgue measure, the set $Geod_{\rho_n, \rho}$ is a singleton for any ρ in $\mathcal{P}_2(\mathbb{R})$ (and so is the set $\frac{\rho_n + \rho}{2}$). Hence, the unique geodesic γ_ρ in $Geod_{\rho_n, \rho}$ is determined by a map $T : \mathbb{R} \rightarrow \mathbb{R}$ (which is the gradient of a convex function, see [6,14]), $\gamma_\rho(t)$ being the

push-forward of ρ_n by $x \mapsto (1-t)x + tT(x)$. The minimization in (2) can thus be expressed in terms of the map T , for instance $\frac{1}{2} d^2(\rho_n, \rho) = \int_{\mathbb{R}} \frac{(T(x)-x)^2}{2} \rho_n(x) dx$. After some tedious (but rather straightforward) computations, one obtains that the unique element ρ_{n+1}^{VIM} minimizing $\frac{1}{2\tau} d^2(\rho_n, \rho) + 2F\left(\frac{\rho_n + \rho}{2}\right)$ also has a normal distribution, that is $\rho_{n+1}^{VIM} = \mathcal{N}(\mu_{n+1}, \sigma_{n+1}^2)$, with $\mu_{n+1} = \left(1 + \frac{\theta\tau}{2}\right)^{-1} \left(\mu_n \left(1 - \frac{\theta\tau}{2}\right) + \frac{\mu\theta\tau}{2}\right)$ and $\sigma_{n+1} = \left(1 + \frac{\theta\tau}{2}\right)^{-1} \sqrt{\sigma_n^2 + \sigma^2\tau \left(1 + \frac{\theta\tau}{2}\right) - \frac{\sigma_n\theta\tau}{2}}$. Consequently, ρ_{n+1}^{VIM} is an approximation of ρ_{n+1}^{exact} exact to the second order in τ and, by the usual Gronwall inequalities, the error of the VIM scheme is globally of order $O(\tau^2)$. \blacksquare

We performed numerical simulations to support these results. Roughly speaking, we employed a continuous piecewise affine (P1) finite element discretization for the cumulative distribution functions of elements in $\mathcal{P}_2(\mathbb{R})$, which amounts to replace the space $\mathcal{P}_2(\mathbb{R})$ by a discrete one, denoted $\mathcal{P}_2(\mathbb{R})_m$ and defined as follows. Let $M \in \mathbb{N}^*$ and $m = (m_1, \dots, m_M)$ be a (ordered) mass distribution such that $m_k > 0$ for any integer k in $\{1, \dots, M\}$ and $\sum_{k=1}^M m_k = 1$, fixed once and for all. An element in $\mathcal{P}_2(\mathbb{R})_m$ with density ρ_X is then represented by a vector $X = (x_0, \dots, x_M) \in \mathbb{R}^{M+1}$, with $x_0 < x_1 < \dots < x_{M-1} < x_M$, by setting ρ_X constant and equal to $\frac{m_k}{x_{k+1} - x_k}$ on the segment $[x_k, x_{k+1})$.

Approximating the integral defining the discrete potential energy $E(\rho_X)$ using the composite Simpson quadrature rule (see [12, Chapter 4, Subsection 4.1.3]), one obtains

$$E_m(X) = \sum_{k=0}^M \frac{m_k}{6} \left[V(x_k) + 4V\left(\frac{x_k + x_{k+1}}{2}\right) + V(x_{k+1}) \right]. \quad (25)$$

On the other hand, the discrete internal energy $S(\rho_X)$ can be computed exactly and has the simple form

$$S_m(X) = \sum_{k=0}^M \frac{m_k}{x_{k+1} - x_k} (x_{k+1} - x_k) \log \left[\frac{m_k}{x_{k+1} - x_k} \right] = \sum_{k=0}^M m_k \log(m_k) - \sum_{k=0}^M m_k \log(x_{k+1} - x_k). \quad (26)$$

To compute the Wasserstein distance \mathcal{W}_2 between the discrete densities ρ_{X_1} and ρ_{X_2} , we note that the map T is the unique piecewise linear function on any interval $[x_k^1, x_{k+1}^1]$, which associates x_k^1 to x_k^2 and x_{k+1}^1 to x_{k+1}^2 respectively. Straightforward computations then show that the squared distance $\mathcal{W}_2^2(\rho_{X_1}, \rho_{X_2}) = \int_{\mathbb{R}} (T(x) - x) \rho_{X_1}(x) dx$ reduces to

$$\mathcal{W}_2^2(\rho_{X_1}, \rho_{X_2}) = \frac{1}{3} \sum_{k=0}^M m_k \left[(x_k^2 - x_k^1)^2 + (x_{k+1}^2 - x_{k+1}^1) (x_k^2 - x_k^1) + (x_{k+1}^2 - x_{k+1}^1)^2 \right]. \quad (27)$$

For the numerical results presented in Figure 1, we have set the final time equal to 1, $\sigma = 1$, $\theta = \frac{1}{2}$, $\mu = 5$, $M = 32$, a uniform mass step $m_k = \frac{1}{M}$, and the projection on the discrete space of a centered reduced Gaussian distribution as the initial datum. Second-order in time convergence is confirmed for both schemes.

The internal energy $S(\rho_X)$ constitutes a barrier enforcing the constraint $x_k < x_{k+1}$. As such the (continuous in time) dynamics of the probability density can also be viewed as a dynamics of the vector X which follows a, cumbersome but regular, system of ordinary differential equations. The proposed schemes provide a consistent, second-order in time, discretization of this system, and we have obtained the following result.

Corollary 3.1 *The scheme (2) discretized using (25), (26), and (27) converges at second order in time to the exact solution of the semi-discrete gradient flow of the functional F in the Wasserstein space of order 2.*

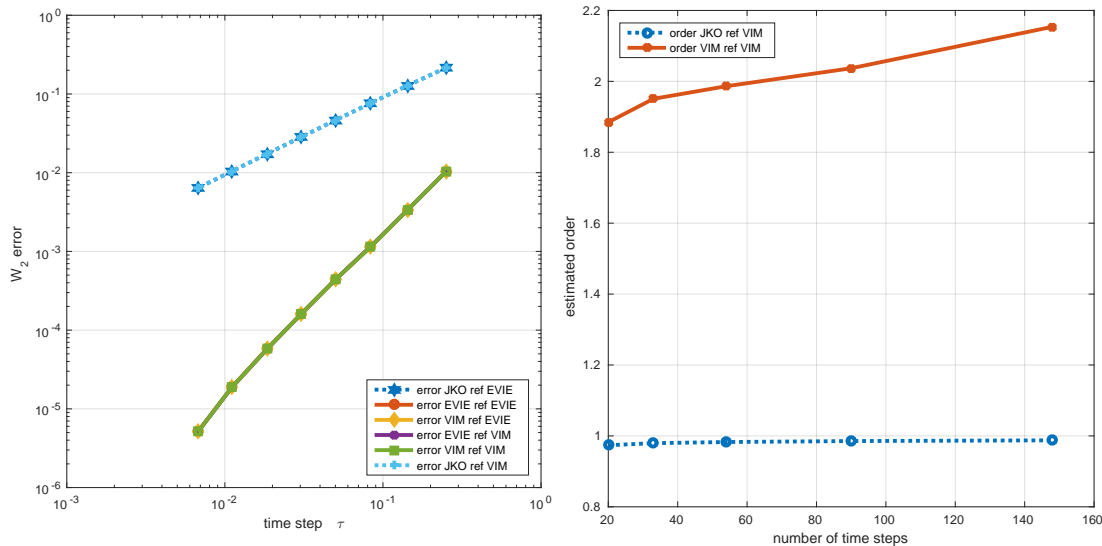


Figure 1. Results of numerical simulation for the case presented in Section 3. **Left:** error of the JKO scheme (dotted line) and of the VIM and EVIE schemes (solid lines) schemes as a function of the length of the time step τ . The error is the W_2 distance between the numerical solution computed by the schemes on grids containing respectively 4, 7, 12, 20, 33, 54, 90 and 148 time steps and two numerical reference solutions obtained using both the VIM and EVIE schemes on a finer grid (244 time steps). We observe that any of the second-order schemes on a grid with four steps is as accurate as the standard JKO scheme on a grid with 90 steps. Similarly, the VIM or EVIE scheme on a grid with seven steps is as good as the JKO scheme on a grid with 148 steps. Note that the points of the VIM and EVIE schemes are indistinguishable to machine precision. **Right:** numerical estimation of the order of convergence of the JKO scheme (dotted line) and of the VIM and EVIE schemes (solid lines).

Acknowledgements

G. Turinici acknowledges support from the Agence Nationale de la Recherche (ANR), projects EMAQS ANR-2011-BS01-017-01 and CINE-PARA 2015-19.

References

- [1] Martial Agueh and Guillaume Carlier. Barycenters in the Wasserstein space. *SIAM J. Math. Anal.*, 43(2):904–924, 2011.
- [2] Luigi Ambrosio and Nicola Gigli. A user’s guide to optimal transport. In Benedetto Piccoli and Michel Rascle, editors, *Modelling and Optimisation of Flows on Networks: Cetraro, Italy 2009*, Lecture notes in mathematics, pages 1–155. Springer, 2008.
- [3] Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré. *Gradient flows in metric spaces and in the space of probability measures*. Lectures in mathematics. ETH Zürich. Birkhäuser Basel, second edition, 2008.
- [4] Jean-David Benamou, Guillaume Carlier, Quentin Mérigot, and Édouard Oudet. Discretization of functionals involving the Monge–Ampère operator. *Numer. Math.*, 134(3):611–636, 2016.
- [5] Adrien Blanchet, Vincent Calvez, and José A. Carrillo. Convergence of the mass-transport steepest descent scheme for the subcritical Patlak–Keller–Segel model. *SIAM J. Numer. Anal.*, 46(2):691–721, 2008.
- [6] Yann Brenier. Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.*, 44(4):375–417, 1991.

- [7] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in metric geometry*, volume 33 of *Graduate studies in mathematics*. American Mathematical Society, 2001.
- [8] Ernst Hairer, Syvert P. Nørsett, and Gerhard Wanner. *Solving ordinary differential equations I. Nonstiff problems*, volume 8 of *Springer series in computational mathematics*. Springer, 1993.
- [9] Richard Jordan, David Kinderlehrer, and Felix Otto. The variational formulation of the Fokker–Planck equation. *SIAM J. Math. Anal.*, 29(1):1–17, 1998.
- [10] David Kinderlehrer and Noel J. Walkington. Approximation of parabolic equations using the Wasserstein metric. *ESAIM Math. Model. Numer. Anal.*, 33(4):837–852, 1999.
- [11] Jean-Marie Mirebeau. Discretization of the 3D Monge-Ampere operator, between wide stencils and power diagrams. *ESAIM Math. Model. Numer. Anal.*, 49(5):1511–1523, 2015.
- [12] William H. Press, Saul A. Teukolsky, William T. Vetterling, and Brian P. Flannery. *Numerical recipes: the art of scientific computing*. Cambridge University Press, third edition, 2007.
- [13] Filippo Santambrogio. *Optimal transport for applied mathematicians. Calculus of variations, PDEs, and modeling*, volume 87 of *Progress in nonlinear differential equations and their applications*. Birkhäuser, 2015.
- [14] Cédric Villani. *Optimal transport. Old and new*, volume 338 of *Grundlehren der mathematischen Wissenschaften*. Springer, 2009.