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Erik van Erp, Robert Yuncken

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# A GROUPOID APPROACH TO PSEUDODIFFERENTIAL CALCULI

ERIK VAN ERP AND ROBERT YUNCKEN

**ABSTRACT.** We give an algebraic/geometric characterization of the classical pseudodifferential operators on a smooth manifold in terms of the tangent groupoid and its natural  $\mathbb{R}_+^\times$ -action. Specifically, we show that a properly supported semiregular distribution on  $M \times M$  is the Schwartz kernel of a classical pseudodifferential operator if and only if it extends to a smooth family of distributions on the range fibres of the tangent groupoid which is homogeneous for the  $\mathbb{R}_+^\times$ -action modulo smooth functions. Moreover, we show that the basic properties of pseudodifferential operators can be proven directly from this characterization. Finally, we show that with the appropriate generalization of the tangent bundle, the same definition applies without change to define pseudodifferential calculi on arbitrary filtered manifolds, and in particular the Heisenberg calculus.

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## 1. INTRODUCTION

The goal of this paper is to simplify and systematize the construction of pseudodifferential calculi.

For present purposes, a *pseudodifferential calculus* means simply an enlargement of the graded algebra of differential operators with two key properties. Firstly, the algebra should contain parametrices for some desired class of ‘generalized elliptic’ operators. Secondly, the pseudodifferential operators should be pseudolocal, that is, their

Schwartz kernels should be  $C^\infty$  off the diagonal. It is a simple exercise, for instance, to deduce hypoellipticity for the generalized elliptic operators from these two abstract properties.

Defining the classes of pseudodifferential operators can be a highly technical matter. In this article we shall propose a simple geometric/algebraic definition of pseudodifferential operators which recovers the classical pseudodifferential operators, but which also applies without change to define the Heisenberg calculus (cf. [BG88, Tay]) and its generalizations to other graded nilpotent groups (cf. [Cum89, CGGP92, Mel82, Pon08]).

The method we shall describe is based upon the tangent groupoid. The tangent groupoid was used by Alain Connes [Con94] as a geometric device for glueing a pseudodifferential operator to its principal symbol. We flip this around by showing that classical pseudodifferential operators can be completely characterized using the tangent groupoid. In this approach, the *smoothness* of the tangent groupoid accounts for the full asymptotic expansion of the symbol of a pseudodifferential operator, not just its principal term.

Furthermore, we show that one can prove the basic properties of pseudodifferential operators directly from this geometric characterization. This allows us to define more general pseudodifferential calculi without needing to prescribe symbol or kernel classes explicitly.

**1.1. Outline of the strategy.** Let  $P: C^\infty(M) \rightarrow C^\infty(M)$  be a differential operator of order  $m$ , given in local coordinates by

$$P = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha.$$

Its highest order part at each point  $x \in M$ ,

$$P_x = \sum_{\alpha=m} a_\alpha(x) \partial^\alpha$$

can be viewed as a smooth family of constant coefficient operators  $P_x$  on the tangent spaces  $T_x M$ . The family  $\{P_x\}_{x \in M}$  is the principal ‘cosymbol’ of  $P$ , i.e. the inverse Fourier transform of the principal symbol of  $P$ . The operator  $P$  and its cosymbol  $\{P_x\}_{x \in M}$  can be assembled into a single differential operator  $\mathbb{P}$  on the tangent groupoid  $\mathbb{T}M$ .

Algebraically,  $\mathbb{T}M$  is the disjoint union of the abelian groups  $T_x M \cong \mathbb{R}^n$  and a one parameter family of pair groupoids  $M \times M$ :

$$\mathbb{T}M = TM \sqcup (M \times M \times \mathbb{R}^\times).$$

The non-trivial aspect of the tangent groupoid is its smooth structure.

Let  $\mathbb{P}$  be the differential operator on  $\mathbb{T}M$  that restricts to  $P_x$  on  $T_xM$  and that restricts to  $t^m P$  on  $M \times M \times \{t\}$  with  $t \neq 0$ , acting on the first factor in  $M \times M$ . The smooth structure on  $\mathbb{T}M$  is precisely such that the coefficients of  $\mathbb{P}$  are  $C^\infty$  functions on  $\mathbb{T}M$ . Thus  $\mathbb{P}$  is a right-invariant differential operator on  $\mathbb{T}M$  that glues the operator  $P$  (at  $t = 1$ ) to its principal cosymbol (at  $t = 0$ ).

We reformulate this well-known construction in terms of distributional kernels on the groupoid  $\mathbb{T}M$ . Since each  $P_x$  is a translation invariant operator on the abelian group  $T_xM$ , there exists a distribution  $u_x$  on  $T_xM$  such that  $P_x$  acts as convolution with  $u_x$ ,

$$P_x \varphi = u_x * \varphi \quad u_x = P_x^t \delta_0 \in \mathcal{E}'(T_xM).$$

Likewise,  $P$  has a Schwartz kernel  $k$ , which is semiregular in the sense that  $k \in C^\infty(M; \mathcal{E}'(M))$ ; specifically,

$$k : x \mapsto k_x = P^t \delta_x \in \mathcal{E}'(M).$$

If we now define

$$\begin{aligned} \mathbb{P}_{(x,0)} &= u_x \in \mathcal{E}'(T_xM), & t &= 0 \\ \mathbb{P}_{(x,t)} &= t^m k_x \in \mathcal{E}'(M), & t &\neq 0 \end{aligned} \quad (1)$$

then  $(\mathbb{P}_{(x,t)})_{(x,t) \in M \times \mathbb{R}}$  is a smooth family of compactly supported distributions in the fibres of the range map  $r : \mathbb{T}M \rightarrow \mathbb{T}M^{(0)}$ , parameterized by the units  $(x, t) \in M \times \mathbb{R} = \mathbb{T}M^{(0)}$ .

The set of such smooth families of distributions on the  $r$ -fibres is denoted  $\mathcal{E}'_r(\mathbb{T}M)$ ; see Sections 2.1, 2.6 for a precise definition. In analogy with the situation for Lie groups, these distributions form a convolution algebra in which the smooth functions—more accurately, the smooth densities tangent to the  $r$ -fibres—sit as a right ideal. A detailed study of these objects appears in the recent work of Lescure-Manchon-Vassout [LMV15].

We need one more piece of structure. The tangent groupoid has a one parameter family of Lie groupoid automorphisms

$$\alpha : \mathbb{R}_+^\times \mapsto \text{Aut}(\mathbb{T}M) \quad (2)$$

defined by

$$\begin{aligned} \alpha_\lambda(x, y, t) &= (x, y, \lambda^{-1}t) & (x, y, t) &\in M \times M \times \mathbb{R}^\times, \\ \alpha_\lambda(x, \zeta) &= (x, \delta_\lambda \zeta) & \zeta &\in T_xM, \end{aligned}$$

where  $\delta_\lambda$  denotes dilation of tangent vectors,  $\delta_\lambda \zeta = \lambda \zeta$ . Note that the smooth family of distributions (1) associated to a differential operator  $P$  is homogeneous of weight  $m$  for the action  $\alpha$ . In fact, the differential operators are characterized by this property (see Section 7.3).

**Proposition 1.** *A semiregular kernel  $P \in \mathcal{E}'_r(M \times M)$  is the Schwartz kernel of a differential operator of order  $m$  if and only if  $P = \mathbb{P}|_{t=1}$  for some properly supported  $\mathbb{P} \in \mathcal{E}'_r(\mathbb{T}M)$  such that  $\alpha_{\lambda*}\mathbb{P} = \lambda^m\mathbb{P}$  for all  $\lambda \in \mathbb{R}_+^\times$ .*

The motivating observation of this paper is that pseudodifferential operators admit a similar characterization.

**Theorem 2.** *A semiregular kernel  $P \in \mathcal{E}'_r(M \times M)$  is the Schwartz kernel of a properly supported classical pseudodifferential operator of order  $m$  if and only if  $P = \mathbb{P}|_{t=1}$  for some properly supported  $\mathbb{P} \in \mathcal{E}'_r(\mathbb{T}M)$  such that  $\alpha_{\lambda*}\mathbb{P} - \lambda^m\mathbb{P}$  is a smooth density for all  $\lambda \in \mathbb{R}_+^\times$ .*

Moreover, if we replace  $\mathbb{T}M$  by the appropriate tangent groupoid  $\mathbb{T}_H M$  for a contact manifold [vE05, Pon06], or a more general filtered manifold [CP15], then the *same* definition can be used to produce versions of the Heisenberg calculus [BG88, Tay] or Melin's calculus [Mel82]. Furthermore, we will prove that the basic properties of pseudodifferential operators follow directly from our geometric definition.

To cite an instance of a need for such exotic pseudodifferential calculi, we point to Julg's proof of the Baum–Connes conjecture for  $\mathrm{Sp}(n, 1)$  [Jul02], which requires a quaternionic analogue of the Heisenberg calculus. Our goal is to streamline such constructions. We want to demonstrate that, once an appropriate tangent groupoid has been constructed, the details of the corresponding pseudodifferential calculus follow automatically.

In particular, we prove, in the general setting of filtered manifolds, that our pseudodifferential operators form a  $\mathbb{Z}$ -filtered algebra  $\Psi_H^\bullet(M)$ ; that restriction to  $t = 0$  in  $\mathcal{E}'_r(\mathbb{T}_H M)$  assigns a principal cosymbol to an operator in the calculus; that there is a short exact sequence

$$0 \rightarrow \Psi_H^{m-1}(M) \rightarrow \Psi_H^m(M) \xrightarrow{\sigma_m} \Sigma_H^m(M) \rightarrow 0; \quad (3)$$

where  $\Sigma_H^m(M)$  is the (noncommutative) algebra of principal cosymbols; that the intersection  $\bigcap_{m \in \mathbb{Z}} \Psi^m(M)$  is the ideal of properly supported smoothing operators  $\Psi^{-\infty}(M)$ ; that  $\Psi_H^\bullet(M)$  is complete with respect to asymptotic expansions; and that if an operator in  $\Psi_H^m(M)$  has invertible principal cosymbol, then it has a parametrix in  $\Psi_H^{-m}(M)$ .

As an application, we immediately obtain the hypoellipticity of 'elliptic' elements in our calculi, where ellipticity here is taken in the abstract sense of having an invertible principal cosymbol. Moreover, given that there is a tangent groupoid associated to these calculi, Connes tangent groupoid method [Con94] should yield abstract index theorems for such operators, as is the case for subelliptic operators on a contact manifold [vE10].

*Remark 3.* A different geometric approach to the Heisenberg calculus is that of Epstein and Melrose [EM] using compactifications of the cotangent bundle. This approach is currently limited to the standard calculus and the Heisenberg calculus, but it has other notable advantages, for instance an explicit noncommutative product on the symbol algebra and even a fusion of the standard and Heisenberg calculi. For comments on the relationship between the two, see Section 5.

**1.2. Structure of the paper.** In Section 2 we describe the appropriate convolution algebra of distributions on Lie groupoids. As mentioned above, this convolution algebra also appears in the preprint [LMV15]. For the sake of remaining self-contained we will describe the main points from the point of view motivated by our application, referring to [LMV15] for a fuller treatment.

In Section 3 we describe the tangent groupoid associated to a general filtered manifold. This groupoid has been recently constructed by Choi and Ponge [CP15] using systems of privileged local coordinates. We present an intrinsic algebraic construction here. See also [Con94, vE05, Pon06, JvE10] for historical antecedents.

The main new results are contained in Sections 4–6, where we define the pseudodifferential operators and prove their fundamental properties. In Section 7 we discuss the case of differential operators.

Finally, in Section 8 we prove that our construction recovers the classical pseudodifferential operators in the case of the classical tangent groupoid.

**1.3. Acknowledgements.** This article was inspired by an observation of Debord and Skandalis in their paper [DS14] which provides the first abstract characterization of the classical pseudodifferential operators in terms of the  $\mathbb{R}_+^\times$ -action on the tangent groupoid. We wish to thank them for many discussions, particularly during the time that the first author was *Professor Invité* at the Université Blaise Pascal, Clermont-Ferrand II. Sincere thanks also go to Jean-Marie Lescure and Nigel Higson for their useful insights.

## 2. CONVOLUTION OF DISTRIBUTIONS ON LIE GROUPOIDS

In this section we cover the basic theory of the convolution algebras of distributions on the fibres of a Lie groupoid. Another point of view on this material can be found in the preprint [LMV15], to which we shall frequently refer.

### 2.1. Smooth families of distributions.

**Definition 4.** Let  $\pi : M \rightarrow B$  be a submersion of  $C^\infty$  manifolds. A  $\pi$ -fibred distribution on  $M$  is a continuous map

$$u : C^\infty(M) \rightarrow C^\infty(B)$$

that is  $C^\infty(B)$ -linear.

The set of  $\pi$ -fibred distributions on  $M$  over  $B$  will be denoted by  $\mathcal{E}'_\pi(M)$ .

Continuity of  $u$  means that for every compact subset  $K_1 \subset B$  and every non-negative integer  $N_1$  there exists a compact subset  $K_2 \subset M$ , an integer  $N_2$ , and real number  $C > 0$  such that

$$\sup_{K_1} \sum_{|\alpha| \leq N_1} |\partial^\alpha \langle u, f \rangle| \leq C \sup_{K_2} \sum_{|\alpha| \leq N_2} |\partial^\alpha f|$$

for all  $f \in C^\infty(M)$ .

In particular, for every compact  $K_1 \subset B$  there exists a compact  $K_2 \subset M$  such that if  $f \in C^\infty(M)$  vanishes on  $K_2 \subset M$  then  $\langle u, f \rangle$  vanishes on  $K_1 \subset B$ . Thus  $u$  has fibrewise compact supports. More precisely,  $u$  has  $\pi$ -proper support in the following sense.

**Definition 5.** A set  $X \subset M$  is  $\pi$ -proper if the restriction  $\pi|_X : X \rightarrow B$  is a proper map.

*Remark 6.* Fibred distributions appeared in [AS11], in the guise of “distributions on  $M$  transversal to  $\pi$ ” (Definition 1.2.1 of that paper), although without the imposition of  $\pi$ -proper supports. Support conditions aside, the equivalence between the definition of [AS11] and the present one is explained in [LMV15].

The vector space  $\mathcal{E}'_\pi(M)$  inherits a locally convex topology as a closed linear subspace of the bounded linear maps between the Fréchet spaces  $C^\infty(M)$  and  $C^\infty(B)$ .

A second interpretation of  $\pi$ -fibred distributions is as smooth families of distributions. The following lemma follows directly from the  $C^\infty(B)$ -linearity of  $u \in \mathcal{E}'_\pi(M)$ .

**Lemma 7.** *If  $f \in C^\infty(M)$  vanishes on the fibre  $M_b := \pi^{-1}(b)$  then  $\langle u, f \rangle$  vanishes at  $b \in B$ .*

This lemma allows us to restrict a  $\pi$ -fibred distribution  $u$  to a compactly supported distribution  $u_b$  on each fibre  $M_b := \pi^{-1}(b)$ . Specifically, if  $g \in C^\infty(M_b)$ , and if  $f \in C^\infty(M)$  is any function on  $M$  that restricts to  $g$  on  $M_b$ , then  $u_b \in \mathcal{E}'(M_b)$  is well-defined by

$$\langle u_b, g \rangle = \langle u, f \rangle(b). \quad (4)$$

Conversely, a family of compactly supported distributions  $(u_b)_{b \in B}$  determines a  $\pi$ -fibred distribution as long as it is smooth in  $b$ , in the following sense.



**Lemma 8.** *For a product fibration  $\pi : M = B \times F \rightarrow B$ , we have*

$$\mathcal{E}'_{\pi}(M) \cong C^{\infty}(B; \mathcal{E}'(F))$$

by associating to  $u \in \mathcal{E}'_{\pi}(M)$  the map  $b \mapsto u_b$ .

For a general fibration  $M$ , a family  $(u_b)_{b \in B}$  with  $u_b \in \mathcal{E}'(M_b)$  defines an element  $u \in \mathcal{E}'_{\pi}(M)$  if and only if its restriction to any trivializing chart for the fibration is smooth in this sense, and the support  $\bigcup_{b \in B} \text{supp } u_b$  is  $\pi$ -proper.

*Proof.* Proposition 3 of [LMV15] shows that continuous  $C^{\infty}(B)$ -linear maps  $C_c^{\infty}(M) \rightarrow C^{\infty}(B)$  are in bijection with smooth families of distributions  $(u_b)_{b \in B}$  with  $u_b \in \mathcal{D}'(M_b)$ . It is straightforward to check that such a map extends continuously to  $C^{\infty}(M)$  if and only if the support is  $\pi$ -proper.  $\square$

We shall call a collection  $(u_b)_{b \in B}$  of compactly supported distributions on the fibres of  $\pi : M \rightarrow B$  a “smooth family” if it satisfies the conditions of the lemma. Thus, a fibred distribution is the same thing as a smooth family of compactly supported distributions, and we shall use the two notions interchangeably.

**2.2. Pullback of a fibred distribution.** Let  $\pi : M \rightarrow B$  be a submersion. Given a second submersion  $\rho : B' \rightarrow B$ , the pull back of the fibration  $\pi : M \rightarrow B$  to  $B'$  along  $\rho$  is the submersion  $\sigma : Z \rightarrow B'$ ,  $\sigma(m, x) = x$ , where  $Z$  is the fibred product  $Z = M \times_B B' = \{(m, x) \in M \times B' \mid \pi(m) = \rho(x)\}$ . The fibre  $Z_x = \sigma^{-1}(x)$  is naturally identified with  $M_{\rho(x)}$ .

Given  $u \in \mathcal{E}'_{\pi}(M)$  and the corresponding family  $(u_b)_{b \in B}$ , we define the pull-back  $\rho^*u$  to be the smooth family  $(u_{\rho(x)})_{x \in B'}$ , i.e.  $(\rho^*u)_x = u_{\rho(x)} \in \mathcal{E}'(Z_x) = \mathcal{E}'(M_{\rho(x)})$ .

**Lemma 9.** *If  $u \in \mathcal{E}'_{\pi}(M)$  then  $\rho^*u \in \mathcal{E}'_{\sigma}(Z)$ .*

*Proof.* This follows from [LMV15, Proposition 7] and the fact that the pull-back of a  $\pi$ -proper set by  $\rho$  is  $\sigma$ -proper.  $\square$

**2.3. Lie groupoids.** We refer to [Mac87] or [MM03] for the basic theory of Lie groupoids. If  $G$  is a Lie groupoid, we write  $G^{(0)}$  for its unit space;  $r, s : G \rightarrow G^{(0)}$  for the range and source maps;  $G^x = r^{-1}(\{x\})$  and  $G_x = s^{-1}(\{x\})$  for the range and source fibres of  $x \in G^{(0)}$ ;  $G^{(2)} = G_s \times_r G \subset G \times G$  for the space of composable pairs; and  $\mathfrak{A}G = (\ker ds)|_{G^{(0)}}$  for the Lie algebroid, with Lie bracket  $[\cdot, \cdot]$  on the smooth sections  $\Gamma(\mathfrak{A}G)$  and anchor  $\rho : \mathfrak{A}G \rightarrow TG^{(0)}$ .

Essentially, we will only need two examples, and combinations thereof.

**Example 10.** The *pair groupoid* of a smooth manifold  $M$  is  $G = M \times M \rightrightarrows G^{(0)} = M$  with range and source maps  $r(x, y) = x$ ,  $s(x, y) = y$  and composition  $(x, y)(y, z) = (x, z)$ . Its Lie algebroid is  $TM$  with the usual bracket of vector fields and the identity map as anchor. The pair groupoid is the carrier for Schwartz kernels of operators on  $C^\infty(M)$ ; see Section 2.10.

**Example 11.** A groupoid  $G \rightrightarrows G^{(0)} = M$  with  $r = s$  is a *smooth bundle of Lie groups*. The fibres  $G_x = G^x$  are Lie groups, and the Lie algebroid is a smooth bundle of Lie algebras with the pointwise Lie bracket and the zero map as anchor. In particular, a vector bundle is a groupoid when viewed as a smooth bundle of abelian groups.

**2.4. Convolution of fibred distributions on groupoids.** The source and range maps  $r, s : G \rightarrow G^{(0)}$  of a Lie groupoid  $G$  are submersions. Thus, we obtain spaces  $\mathcal{E}'_r(G)$  and  $\mathcal{E}'_s(G)$  of  $r$ - and  $s$ -fibred distributions on  $G$ . We denote smooth families in  $\mathcal{E}'_r(G)$  as  $(u^x)_{x \in G^{(0)}}$  and smooth families in  $\mathcal{E}'_s(G)$  as  $(u_x)_{x \in G^{(0)}}$ .

Before we define convolution on Lie groupoids, first recall the case where  $G$  is a Lie group. Let  $m : G \times G \rightarrow G$  denote the multiplication  $m(x, y) = xy$ , and  $m^* : C^\infty(G) \rightarrow C^\infty(G \times G)$  the coproduct  $(m^*f)(x, y) = f(xy)$ . Then if  $u, v \in \mathcal{E}'(G)$  are compactly supported distributions on  $G$ , the convolution product  $u * v \in \mathcal{E}'(G)$  is given by

$$u * v = (u \otimes v) \circ m^*$$

with  $u \otimes v \in \mathcal{E}'(G \times G)$ .

If  $G$  is a smooth groupoid, the groupoid multiplication  $m : G^{(2)} \rightarrow G$  induces the coproduct  $m^* : C^\infty(G) \rightarrow C^\infty(G^{(2)})$ . To generalize convolution of distributions to groupoids, we replace  $u \otimes v$  with an appropriate fibred distribution on  $G^{(2)} \rightarrow G^{(0)}$ .

**Definition 12.** The convolution product  $u * v$  of two  $r$ -fibred distributions  $u, v \in \mathcal{E}'_r(G)$  is the  $r$ -fibred distribution

$$u * v = u \circ s^*v \circ m^*$$

where  $s^*v$  denotes the pullback

$$s^*v : C^\infty(G^{(2)}) \rightarrow C^\infty(G)$$

of  $v$  along the source map  $s : G \rightarrow G^{(0)}$ .

Recall that  $s^*v$  is given by  $(s^*v)\gamma = v^{s(\gamma)}$ . Thus, if  $\varphi \in C^\infty(G)$  then  $\psi = (s^*v \circ m^*)(\varphi)$  is the function

$$\psi(\gamma) = \langle v^{s(\gamma)}, \varphi^\gamma \rangle, \quad \text{where } \varphi^\gamma(\beta) := \varphi(\gamma\beta).$$

We obtain a convenient formula for the convolution product:

$$\langle u * v, \varphi \rangle(x) = \langle u^x(\gamma), \langle v^{s(\gamma)}(\beta), \varphi(\gamma\beta) \rangle \rangle, \quad \varphi \in C^\infty(G), x \in G^{(0)}.$$

or more succinctly,

$$\langle u * v, \varphi \rangle = \langle u(\gamma), \langle v(\beta), \varphi(\gamma\beta) \rangle \rangle.$$

Associativity of the convolution product is verified easily. We therefore have an algebra structure on  $\mathcal{E}'_s(G)$ .

Likewise, convolution of  $s$ -fibred distributions  $u, v \in \mathcal{E}'_s(G)$  is defined by

$$u * v = v \circ r^* u \circ m^* : C^\infty(G) \rightarrow C^\infty(G^{(0)})$$

or more explicitly

$$\langle u * v, \varphi \rangle(x) = \langle v_x(\gamma), \langle u_{r(\gamma)}(\beta), \varphi(\beta\gamma) \rangle \rangle, \quad \varphi \in C^\infty(G), x \in G^{(0)}.$$

In shorthand,

$$\langle u * v, \varphi \rangle = \langle v(\gamma), \langle u(\beta), \varphi(\beta\gamma) \rangle \rangle.$$

## 2.5. Smooth densities as an ideal.

**Definition 13.** A subset  $X \subseteq G$  is called *proper* if it is both  $r$ - and  $s$ -proper, i.e. if the restrictions  $r|_X, s|_X : X \rightarrow G^{(0)}$  are both proper maps.

Let  $\Omega_r$  denote the bundle of densities tangent to the range fibres of  $G$ . The space of properly supported sections of  $\Omega_r$  will be denoted  $C_p^\infty(G; \Omega_r)$ . There is a canonical inclusion

$$C_p^\infty(G; \Omega_r) \hookrightarrow \mathcal{E}'_r(G)$$

given by integration along the range fibres:

$$\langle f, \varphi \rangle(x) = \int_{\gamma \in G^x} \varphi(\gamma) f(\gamma), \quad f \in C_p^\infty(G; \Omega_r), \varphi \in C^\infty(G).$$

Likewise, there is a canonical inclusion  $C_p^\infty(G; \Omega_s) \hookrightarrow \mathcal{E}'_s(G)$  where  $\Omega_s$  is the bundle of density tangent to the  $s$ -fibres.

**Proposition 14** (Lescure-Manchon-Vassout [LMV15]).  $C_p^\infty(G, \Omega_r)$  is a right ideal in  $\mathcal{E}'_r(G)$  and  $C_p^\infty(G, \Omega_s)$  is a left ideal in  $\mathcal{E}'_s(G)$ .

We give a quick proof because it yields a formula which will be useful later.

*Proof.* Let  $\lambda = (\lambda^x)_{x \in G^{(0)}} \in C^\infty(G; \Omega_r)$  be a left-invariant Haar system. For any  $f \in C_p^\infty(G)$  and  $u \in \mathcal{E}'_r(G)$  an application of Fubini's Theorem for distributions yields

$$\langle u * (f\lambda), \varphi \rangle(x) = \int_{\gamma \in G^x} \langle u^x(\beta), f(\beta^{-1}\gamma) \rangle \varphi(\gamma) d\lambda^x(\gamma).$$

That is,  $u * (f\lambda) = (u * f)\lambda$  where the function  $u * f$  is defined independently of  $\lambda$  as

$$(u * f)(\gamma) = \langle u(\beta), f(\beta^{-1}\gamma) \rangle. \quad (5)$$

If we write this as

$$u * f = (r^*(u \circ \iota^*) \circ m^*)(f),$$

where  $\iota : G \rightarrow G$  is the inverse map, it follows that  $u * f$  is a smooth function. It is easy to check that  $u * f$  has proper support.

The case of  $s$ -fibred distributions is analogous.  $\square$

*Remark 15.* The map  $\pi(u) : f \mapsto u * f$  will be called the left convolution representation of  $u \in \mathcal{E}'(G)$  on  $C_p^\infty(G)$ . Note that the value of  $(u * f)(\gamma)$  depends only on the values of  $f$  on the source fibre  $G_{s(\gamma)}$ . Thus,  $\pi$  restricts to a representation  $\pi_x : \mathcal{E}'(G) \rightarrow \mathbf{L}(C_c^\infty(G_x))$  for each  $x \in G^{(0)}$ , called the *regular representation at  $x$* .

*Remark 16.* If  $f \in C_p^\infty(G; \Omega_r)$  and  $u \in \mathcal{E}'_r(G)$  then  $f * u$  is generally not smooth. Likewise,  $u * f$  is generally not smooth for  $u \in \mathcal{E}'_s(G)$  and  $f \in C_p^\infty(G; \Omega_s)$ .

As a simple example, consider the pair groupoid  $G = \mathbb{R} \times \mathbb{R}$ . Let  $u$  be the  $r$ -fibred distribution with  $u^x = \delta_0$  for every  $x \in \mathbb{R}$ . If  $f \in C_p^\infty(G; \Omega_r)$  then  $(u * f)(x, y) = f(0, y)$  is again a smooth density. But  $f * u = v$  with  $v^x = F(x)\delta_0$ , where  $F(x) = \int_{y \in M} f(x, y)$ . Therefore  $f * u$  is not smooth unless  $F = 0$ .

Such examples can be avoided if we restrict attention to distributions which can be disintegrated as smooth families of distributions for both the  $r$ - and  $s$ -fibrations. This is the purpose of the next section.

**2.6. Proper distributions on a Lie groupoid.** A *transverse measure* on a Lie groupoid  $G$  is a positive density  $\mu$  on the object space  $G^{(0)}$ . By abuse of notation, we also denote by  $\mu$  the integral

$$\mu : C_c^\infty(G^{(0)}) \rightarrow \mathbb{R}; \quad \varphi \mapsto \int_{G^{(0)}} \varphi(x) d\mu(x).$$

If we fix a choice of transverse measure,  $r$ -fibred and  $s$ -fibred distributions give rise to global distributions on  $G$ . Specifically, we have the inclusions

$$\begin{aligned} \mu_r : \mathcal{E}'_r(G) &\rightarrow \mathcal{D}'(G); & u &\mapsto \mu \circ u = \int_{G^{(0)}} \langle u, \cdot \rangle(x) d\mu(x), \\ \mu_s : \mathcal{E}'_s(G) &\rightarrow \mathcal{D}'(G); & u &\mapsto \mu \circ u = \int_{G^{(0)}} \langle u, \cdot \rangle(x) d\mu(x). \end{aligned}$$

Observe that the image of the map  $\mu_r$  in  $\mathcal{D}'(G)$  is independent of the choice of transverse measure  $\mu$ , and likewise for  $\mu_s$ .

**Definition 17.** The set of *proper distributions* on  $G$ , denoted  $\mathcal{D}'_p(G)$ , is the intersection

$$\mathcal{D}'_p(G) = \mu_r(\mathcal{E}'_r(G)) \cap \mu_s(\mathcal{E}'_s(G)) \subset \mathcal{D}'(G).$$

An  $r$ -fibred distribution  $u \in \mathcal{E}'_r(G)$  or an  $s$ -fibred distribution  $v \in \mathcal{E}'_s(G)$  is called *proper* if  $\mu_r(u)$  or  $\mu_s(v)$  is proper, respectively. We denote the set of proper  $r$ -fibred distributions as  $\mathcal{E}'_{r,s}(G)$ , and the set of proper  $s$ -fibred distributions as  $\mathcal{E}'_{s,r}(G)$ .

In other words, a properly supported distribution  $u \in \mathcal{D}'(G)$  is proper if it can be disintegrated along the  $r$ -fibres as well as along the  $s$ -fibres of  $G$  (in the language of [AS11, LMV15] it is “transversal to both  $r$  and  $s$ ”). The definition of properness is independent of the choice of transverse measure  $\mu$ , although the specific disintegrations into  $r$ - and  $s$ -fibred distributions will depend on  $\mu$ .

Note also that proper distributions always have proper support in the sense of Definition 13, although proper support is a strictly weaker condition.

**2.7. Transposition.** The inverse map  $\iota : G \rightarrow G$ ,  $\iota(\gamma) = \gamma^{-1}$  induces maps on distribution spaces:

$$\begin{aligned} \iota_* : \mathcal{E}'_s(G) &\rightarrow \mathcal{E}'_r(G), & u &\mapsto u^t = u \circ \iota^*, \\ \iota_* : \mathcal{E}'_r(G) &\rightarrow \mathcal{E}'_s(G), & v &\mapsto v^t = v \circ \iota^*, \\ \iota_* : \mathcal{D}'(G) &\rightarrow \mathcal{D}'(G), & w &\mapsto w^t = w \circ \iota^*. \end{aligned} \quad (6)$$

Each of these maps will be called *transposition*.

Observe that  $\mathcal{E}'_r(G)$  has a canonical algebra structure, but it is not closed under transposition. The same holds for  $\mathcal{E}'_s(G)$ . Conversely,  $\mathcal{D}'(G)$  is closed under transposition, but has no canonical algebra structure. However, the bijections

$$\mathcal{E}'_{r,s}(G) \xrightarrow{\mu_r} \mathcal{D}'_p(G) \xrightarrow{\mu_s} \mathcal{E}'_{s,r}(G) \quad (7)$$

are compatible with the algebra structure and transposition, as we now show.

**Proposition 18.** Fix a transverse measure  $\mu$  for  $G$ .

- (a) The map  $\mu_s^{-1} \circ \mu_r : \mathcal{E}'_{r,s}(G) \rightarrow \mathcal{E}'_{s,r}(G)$  is an algebra isomorphism.
- (b) If  $u \in \mathcal{E}'_{r,s}(G)$  then  $u^t \in \mathcal{E}'_{s,r}(G)$  and  $\mu_r(u)^t = \mu_s(u^t)$  in  $\mathcal{D}'_p(G)$ .

*Proof.* Let  $u, v \in \mathcal{E}'_{r,s}(G)$  and write  $\tilde{u} = \mu_s^{-1} \circ \mu_r(u)$ ,  $\tilde{v} = \mu_s^{-1} \circ \mu_r(v)$ . The relation  $\mu_r(u) = \mu_s(\tilde{u})$  amounts to

$$\int_{x \in G^{(0)}} \langle u^x(\gamma), \varphi(\gamma) \rangle d\mu(x) = \int_{y \in G^{(0)}} \langle \tilde{u}_y(\gamma), \varphi(\gamma) \rangle d\mu(y)$$

for all  $\varphi \in C_c^\infty(G)$ . We obtain

$$\begin{aligned} \langle \mu_r(u * v), \varphi \rangle &= \int_{x \in G^{(0)}} \langle u^x(\gamma), \langle v^{s(\gamma)}(\beta), \varphi(\gamma\beta) \rangle \rangle d\mu(x) \\ &= \int_{y \in G^{(0)}} \langle \tilde{u}_y(\gamma), \langle v^y(\beta), \varphi(\gamma\beta) \rangle \rangle d\mu(y) \\ &= \int_{y \in G^{(0)}} \langle v^y(\beta), \langle \tilde{u}_y(\gamma), \varphi(\gamma\beta) \rangle \rangle d\mu(y) \\ &= \int_{z \in G^{(0)}} \langle \tilde{v}_z(\beta), \langle \tilde{u}_{r(\beta)}(\gamma), \varphi(\gamma\beta) \rangle \rangle d\mu(z) = \langle \mu_s(\tilde{u} * \tilde{s}), \varphi \rangle. \end{aligned}$$

where the third equality is Fubini's Theorem for distributions. This proves (a). A similar calculation shows that  $\mu_s(u^t) = \mu_r(\tilde{u}^t) = \mu_r(u)^t$ , which proves (b).  $\square$

Thus, by identifying the three sets of distributions via a choice of transverse measure we obtain an algebra with transposition. The identifications (7) restrict to

$$C_p^\infty(G; \Omega_r) \xrightarrow{\mu_r} C_p^\infty(G; \Omega) \xrightarrow{\mu_s} C_p^\infty(G; \Omega_s).$$

The following is immediate.

**Proposition 19.** *The algebras  $C_p^\infty(G; \Omega_r)$  and  $C_p^\infty(G; \Omega_s)$  are two-sided ideals in  $\mathcal{E}_{r,s}(G)$  and  $\mathcal{E}_{s,r}(G)$ , respectively.*

**2.8. Unit.** Restriction to the unit space

$$I : C^\infty(G) \rightarrow C^\infty(G^{(0)}), \quad \langle I, \varphi \rangle(x) := \varphi(x)$$

is both an  $r$ -fibred and an  $s$ -fibred distribution. In either case it corresponds to the smooth family of distributions  $I = (\delta_x)_{x \in G^{(0)}}$  where  $\delta_x$  is the Dirac distribution  $\langle \delta_x, \varphi \rangle := \varphi(x)$  on  $G^x$  or  $G_x$  respectively.

It is easy to check that  $I$  is the unit element of the convolution algebras  $\mathcal{E}'_r(G)$  and  $\mathcal{E}'_s(G)$ . If  $\mu$  is a transverse measure, then  $\mu_r(I) = \mu_s(I) = I_\mu$  with

$$\langle I_\mu, \varphi \rangle = \int_{G^{(0)}} \varphi(x) d\mu(x).$$

Therefore  $I$  is proper.

**2.9. Lie groupoid automorphisms.** A *Lie groupoid automorphism* is a diffeomorphism  $\alpha : G \rightarrow G$  which restricts to a diffeomorphism of  $G^{(0)}$  and commutes with the range, source and multiplication maps.

**Lemma 20.** *Let  $\alpha : G \rightarrow G$  be a Lie groupoid automorphism. There is an induced map  $\alpha_*$  on  $\mathcal{E}'_r(G)$  given by*

$$\alpha_* u = (\alpha^{-1})^* \circ u \circ \alpha^*,$$

where recall  $\alpha$  acts both on  $G$  and on  $G^{(0)}$ . Specifically,

$$\langle \alpha_* u, \varphi \rangle(x) = \langle u, \alpha^* \varphi \rangle(\alpha^{-1}x), \quad u \in \mathcal{E}'_r(G), \varphi \in C^\infty(G), x \in G^{(0)}.$$

It is an automorphism of the convolution algebra:

$$\alpha_*(u * v) = (\alpha_* u) * (\alpha_* v), \quad u, v \in \mathcal{E}'_r(G).$$

The analogous statement holds for  $\mathcal{E}'_s(G)$ . Moreover, if  $\mu$  is a transverse measure on  $G^{(0)}$  then

$$\begin{aligned} \alpha_*(\mu \circ u) &= (\alpha_* \mu) \circ (\alpha_* u), & (u \in \mathcal{E}'_r(G)), \\ \alpha_*(\mu \circ v) &= (\alpha_* \mu) \circ (\alpha_* v), & (v \in \mathcal{E}'_s(G)). \end{aligned}$$

Therefore,  $\alpha_*$  is an algebra automorphism of  $\mathcal{E}_{r,s}(G)$  and  $\mathcal{E}'_{s,r}(G)$ .

The proof is a direct calculation.

**2.10. Properly fibred Schwartz kernels.** Let  $M$  be a smooth manifold. Recall ([Trè67, Proposition 50.5 and Theorem 44.1]) that continuous linear operators  $T : C^\infty(M) \rightarrow C^\infty(M)$  can be characterized by the fact that they have Schwartz kernels that are *semiregular in the first variable*, i.e. there is  $u \in C^\infty(M, \mathcal{E}'(M))$  with

$$Tf(x) = \int_{y \in M} u(x, y) f(y), \quad \text{for all } f \in C^\infty(M). \quad (8)$$

From Lemma 8,  $C^\infty(M, \mathcal{E}'(M))$  is isomorphic to the space  $\mathcal{E}'_r(M \times M)$  of  $r$ -fibred distributions on the pair groupoid. We will write

$$\text{Op} : \mathcal{E}'_r(M \times M) \rightarrow \mathbf{L}(C^\infty(M))$$

for the isomorphism which sends  $u \in \mathcal{E}'_r(M \times M)$  to the operator  $T$  of Equation (8). Note that composition of operators corresponds to convolution of  $r$ -fibred distributions,

$$\text{Op}(u) \text{Op}(v) = \text{Op}(u * v), \quad u, v \in \mathcal{E}'_r(M \times M),$$

so that  $\text{Op}$  is an algebra isomorphism.

*Remark 21.* In fact,  $\text{Op}$  is the left-regular representation of  $\mathcal{E}'_r(G)$  for  $G = M \times M$  on the smooth functions on any source fibre  $G_x = M$  (see Remark 15).

A distribution on the pair groupoid  $M \times M$  is proper if and only if it is properly supported and semiregular in both variables. The following proposition, which characterizes operators with proper Schwartz kernels, follows from the discussion in [Trè67, Ch. 51], and we will omit the proof. We must fix a transverse measure, i.e. a positive density  $\mu$  on  $M$ . This allows us to identify smooth

functions and smooth densities on  $M$ , via  $f \mapsto f\mu$ , and yields the bilinear form

$$\langle f, g \rangle = \int_M f(x)g(x) d\mu(x), \quad f, g \in C_c^\infty(M).$$

**Proposition 22.** *The Schwartz kernel of an operator  $T : C_c^\infty(M) \rightarrow \mathcal{D}'(M)$  is a proper distribution on  $M \times M$  if and only if*

- *$T$  is a continuous linear operator  $C^\infty(M) \rightarrow C^\infty(M)$*
- *$T$  has a transpose  $T^t : C^\infty(M) \rightarrow C^\infty(M)$  such that  $\langle f, Tg \rangle = \langle T^t f, g \rangle$  for any  $f, g \in C_c^\infty(M)$ .*
- *The Schwartz kernel of  $T$  is properly supported.*

*Remark 23.* A direct calculation shows that if  $T = \text{Op}(u)$  with  $u \in \mathcal{E}'_{r,s}(M \times M)$  then  $T^t = \text{Op}(\tilde{u}^t)$  where  $\tilde{u} = \mu_s^{-1} \circ \mu_r(u)$  and  $\tilde{u}^t$  denotes the distributional transpose of Equation (6).

The properties listed in Proposition 22 are equivalent to the fact that (an extension of)  $T$  maps each of  $C^\infty(M)$ ,  $C_c^\infty(M)$ ,  $\mathcal{D}'(M)$  and  $\mathcal{E}'(M)$  to itself. In particular, a pseudodifferential operator (in any ‘reasonable’ calculus on a manifold without boundary) has these properties. We therefore treat various calculi of pseudodifferential operators as subalgebras of  $\mathcal{E}'_{r,s}(M \times M)$ . The 2-sided ideal  $C_p^\infty(M \times M, \Omega_r)$  corresponds, of course, to the properly supported *smoothing operators*  $\Psi^{-\infty}(M)$ .

### 3. THE OSCULATING AND $H$ -TANGENT GROUPOIDS

As mentioned in the introduction, pseudodifferential operators are pseudolocal, that is, their Schwartz kernels have singular support on the diagonal. A pseudodifferential calculus can be defined by prescribing certain asymptotic expansions of the singularity of Schwartz kernels near the diagonal in  $M \times M$ . These asymptotics have a very simple characterization if we extend Schwartz kernels to the algebra  $\mathcal{E}'_{r,s}(\mathbb{T}M)$  associated to the tangent groupoid  $\mathbb{T}M$ . Because we wish to demonstrate that our approach works without change for the Heisenberg calculus and its generalizations, we will work from the beginning in the context of *filtered manifolds*.

In a recent preprint, Choi and Ponge [CP15] have constructed the tangent groupoid for filtered manifolds (which they call *Carnot manifolds*) using privileged local coordinate systems. The approach that we will take here is more algebraic, and is a development of ideas from [vE05].

**Notation.** We begin with some remarks on notation. Let  $M$  be a manifold. We will be considering bundles and fibrations over  $M \times \mathbb{R}$ . These are to be regarded as families of fibrations over  $M$



indexed by  $t \in \mathbb{R}$ . If  $\pi : E \rightarrow M \times \mathbb{R}$  is a submersion, we shall denote by  $E|_t = \pi^{-1}(M \times \{t\})$  its restriction to  $t \in \mathbb{R}$ . Similarly, we write  $E|_I = \pi^{-1}(M \times I)$  when  $I \subset \mathbb{R}$ . If  $\mathbb{X} : M \times \mathbb{R} \rightarrow E$  is a section of  $E$ , we will denote by  $\mathbb{X}_t$  its restriction to  $M \times \{t\}$ . If  $u \in \mathcal{E}'_\pi(E)$  is a fibred distribution on  $E \rightarrow M \times \mathbb{R}$ , we will write  $u_t$  for its restriction to a fibred distribution on  $E|_t \rightarrow M$ .

**3.1. The classical tangent groupoid.** Our results on pseudodifferential operators are new even in the case of the classical pseudodifferential calculus. For the sake of the reader who is interested only in classical pseudodifferential operators, we start with a brief review of the classical tangent groupoid which should be sufficient for reading the rest of the paper.

Let  $M$  be a smooth manifold without boundary. Algebraically, the tangent groupoid  $\mathbb{T}M$  is the disjoint union

$$\mathbb{T}M = TM \times \{0\} \sqcup M \times M \times \mathbb{R}^\times.$$

The  $t = 0$  fiber  $TM \times \{0\}$  is glued to  $M \times M \times \mathbb{R}^\times$  by blowing up the diagonal in  $M \times M$ . To make this precise, it is easier to first construct the Lie algebroid  $\mathfrak{t}M$  of  $\mathbb{T}M$ .

As a vector bundle over  $M \times \mathbb{R}$ , the Lie algebroid  $\mathfrak{t}M$  of the tangent groupoid  $\mathbb{T}M$  is

$$\mathfrak{t}M = TM \times \mathbb{R} \rightarrow M \times \mathbb{R} \tag{9}$$

The bracket of two smooth sections  $\mathbb{X}, \mathbb{Y} \in \Gamma(\mathfrak{t}M)$  is rescaled by  $t$ ,

$$[\mathbb{X}, \mathbb{Y}]_t = t [\mathbb{X}_t, \mathbb{Y}_t],$$

where the right hand bracket is the ordinary bracket of vector fields  $\mathbb{X}_t$  and  $\mathbb{Y}_t$  in  $\Gamma(TM)$ . The anchor on  $\mathfrak{t}M$  is

$$\rho(\mathbb{X})_t = t \mathbb{X}_t$$

Note that multiplication by  $t$  is *injective* on *smooth* sections  $\Gamma(\mathfrak{t}M)$ , and that the image of this map is

$$\mathfrak{X} = \{\mathbb{X} \in \Gamma(TM \times \mathbb{R}) \mid \mathbb{X}|_{t=0} = 0\}$$

We may define  $\mathfrak{t}M$  efficiently but indirectly as the vector bundle on  $M \times \mathbb{R}$  underlying the  $C^\infty(M \times \mathbb{R})$  module  $\mathfrak{X}$ . Note that the isomorphism

$$t : \Gamma(\mathfrak{t}M) \cong \mathfrak{X} \quad \mathbb{X} \mapsto t\mathbb{X}$$

intertwines the Lie algebroid bracket and anchor of  $\mathfrak{t}M$  with the standard (constant in  $t$ ) bracket and anchor of the product Lie algebroid  $TM \times \mathbb{R}$ . Thus, the  $C^\infty(M \times \mathbb{R})$ -module  $\mathfrak{X}$  implicitly defines the Lie algebroid  $\mathfrak{t}M$ . This indirect definition of  $\mathfrak{t}M$  generalizes quite naturally to filtered manifolds (see section 3.5).

General results on integrability of Lie algebroids imply the existence of a minimal Lie groupoid with Lie algebroid  $\mathfrak{t}M$  (see e.g. [CF03, Nis00, Deb01a, Deb01b]). This Lie groupoid is the tangent groupoid  $\mathbb{T}M$ .

Explicitly, the smooth structure of  $\mathbb{T}M$  is as follows. Let  $\nabla$  be a connection on  $TM$  and  $\exp_x^\nabla : T_x M \rightarrow M$  the associated exponential maps. An exponential map from the Lie algebroid  $\mathfrak{t}M$  to the Lie groupoid  $\mathbb{T}M$  is

$$\begin{aligned} \mathbb{E}xp^\nabla : \mathfrak{t}M = TM \times \mathbb{R} &\rightarrow \mathbb{T}M; \\ (x, v, t) &\mapsto (\exp_x^\nabla(tv), x, t), & \text{if } t \neq 0 \\ (x, v, 0) &\mapsto (x, v, 0), & \text{if } t = 0. \end{aligned}$$

Note that  $\mathbb{E}xp^\nabla$  is injective in a neighbourhood of  $TM \times \{0\}$ , and we use this to define the smooth structure of  $\mathbb{T}M$  in such a neighborhood. One should check that this smooth structure is independent of the choice of connection  $\nabla$ ; we will leave this verification for the general treatment in Sections 3.5 and 3.6. Away from  $t = 0$  the subset  $M \times M \times \mathbb{R}^\times$  is an open subset of  $\mathbb{T}M$  with the evident manifold structure.

Finally, as mentioned in the introduction, the map  $\alpha : \mathbb{R}_+^\times \mapsto \text{Aut}(\mathbb{T}M)$  defined by

$$\begin{aligned} \alpha_\lambda(y, x, t) &= (y, x, \lambda^{-1}t) & (y, x, t) \in M \times M \times \mathbb{R}^\times, \\ \alpha_\lambda(x, v, 0) &= (x, \delta_\lambda v, 0) & \zeta \in T_x M, \end{aligned}$$

as in Equation (2) defines a one-parameter family of groupoid automorphisms on  $\mathbb{T}M$ , which we refer to as the *zoom* action. To see that  $\alpha_\lambda$  is a diffeomorphism, we express it in exponential coordinates as

$$\tilde{\alpha}_\lambda : TM \times \mathbb{R} \rightarrow TM \times \mathbb{R}; \quad (x, v, t) \mapsto (x, \lambda v, \lambda^{-1}t).$$

Then  $\alpha_\lambda \circ \mathbb{E}xp^\nabla = \mathbb{E}xp^\nabla \circ \tilde{\alpha}_\lambda$  for all  $\lambda \in \mathbb{R}_+^\times$ . Since  $\tilde{\alpha}$  is clearly a diffeomorphism, so is  $\alpha$ .

The rest of this section describes the construction of the tangent groupoid of a filtered manifold from this point of view. It may be skipped if the reader is only interested in the classical pseudodifferential calculus.

**3.2. Graded vector spaces and vector bundles.** The following terminology and notation will be used throughout the rest of the paper.

Let  $V$  be a finite-dimensional graded vector space. For us, this means that  $V$  decomposes as a direct sum  $\bigoplus_{i=1}^N V_i$ . We refer to  $N$  as the *depth* of the grading. We write  $\deg(\zeta) = i$  when  $\zeta \in V_i$ . A

*graded basis* for  $V$  is a basis  $\{e_1, \dots, e_n\}$  with each element contained in some  $V_i$ . The corresponding linear coordinates  $\xi = (\xi_1, \dots, \xi_n)$  will be called a *graded coordinate system*. We will write  $d_j = \deg(e_j)$ . The *homogeneous dimension* of  $V$  is  $d_H = \sum_{j=1}^n d_j = \sum_{i=1}^N i \dim V_i$ .

If  $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$  is a multi-index with respect to these coordinates, we write  $|\mathbf{a}| = \sum_j a_j$  for its *usual order* and  $|\mathbf{a}|_H = \sum_j a_j d_j$  for its *homogeneous order*.

The *canonical dilations* on a graded vector space  $V$  are the linear endomorphisms  $\delta_\lambda$  ( $\lambda \in \mathbb{R}$ ) where  $\delta_\lambda$  acts on  $V_i$  by multiplication by  $\lambda^i$ .

A *homogeneous norm* on  $V$  is a continuous function  $\|\cdot\|_H : V \rightarrow [0, \infty)$  which is nonzero on  $V \setminus \{0\}$  and satisfies  $\|\delta_\lambda(\xi)\|_H = \lambda \|\xi\|_H$  for all  $\lambda > 0$ ,  $\xi \in V$ . For instance, one could define  $\|\xi\|_H = \sum_j |\xi_j|^{1/d_j}$ . Any two homogeneous norms  $\|\cdot\|_H$  and  $\|\cdot\|'_H$  are equivalent, in the sense that there exists a constant  $C > 0$  such that  $C^{-1} \|\xi\|_H \leq \|\xi\|'_H \leq C \|\xi\|_H$  for all  $\xi \in V$ .

Likewise, if  $M$  is a smooth manifold, a *grading* on a vector bundle  $\pi : E \rightarrow M$  is a decomposition into subbundles  $E = \bigoplus_{i=1}^N E_i$ . The canonical dilations  $\delta_\lambda \in \text{End}(E)$  are defined fibrewise just as above. A *homogeneous norm* is a continuous function  $\|\cdot\|_H : E \rightarrow [0, \infty)$  which is nonzero off the zero section and satisfies  $\|\delta_\lambda(\xi)\|_H = \lambda \|\xi\|_H$  for all  $\lambda > 0$ ,  $\xi \in V$ . Any two homogeneous norms are equivalent on  $E|_K$  for any compact  $K \subseteq M$ .

### 3.3. The osculating groupoid of a filtered manifold.

**Definition 24.** ([Mel82]) A *filtered manifold* is a smooth manifold  $M$  equipped with a filtration of the tangent bundle  $TM$  by vector bundles  $M \times \{0\} = H^0 \leq H^1 \leq \dots \leq H^N = TM$  such that  $\Gamma(H^\bullet)$  is a Lie algebra filtration, i.e.

$$[\Gamma(H^i), \Gamma(H^j)] \subseteq \Gamma(H^{i+j})$$

Here we are using the convention that  $H^i = TM$  for  $i > N$ .

Sections  $X \in \Gamma(H^i)$  will be referred to as *vector fields of  $H$ -order  $\leq i$* , written  $\text{ord}_H(X) \leq i$ .

- Example 25.**
- (1) Any manifold  $M$  can be equipped with the trivial filtration of depth one:  $H^1 = TM$ . This is the relevant structure for describing the classical pseudodifferential operators on  $M$  in terms of Connes' tangent groupoid.
  - (2) Contact manifolds and, more generally, Heisenberg manifolds are filtered manifolds of depth 2. See [FS74, BG88].
  - (3) A foliation of  $M$  gives rise to a filtration of depth two,  $H^1 = F \leq H^2 = TM$ , where  $F$  is the tangent space to the leaves.

(We note, however, that the pseudodifferential calculus produced by our machine will be the Heisenberg calculus, not the calculus of families of operators tangent to the leaves.)

- (4) Let  $G$  be a complex semisimple Lie group and  $\{\alpha_1, \dots, \alpha_r\}$  be a set of simple roots for its Lie algebra  $\mathfrak{g}$ . If  $\beta$  is a root (or  $\beta = 0$ ), let us write  $|\beta| = \sum_i b_i$  where  $\beta = \sum_i b_i \alpha_i$  is the decomposition of  $\beta$  in terms of simple roots. Let us write  $\mathfrak{g}^j = \bigoplus_{|\beta| \leq j} \mathfrak{g}_\beta$ . Then  $\mathfrak{b} := \mathfrak{g}^0$  is a Borel subalgebra, and  $(\mathfrak{g}^j / \mathfrak{g}^0)_{j \geq 0}$  is a filtration of  $\mathfrak{g} / \mathfrak{b}$ . This induces a  $G$ -invariant filtration of the tangent bundle  $T(G/B)$  which makes the flag variety into a filtered manifold.

This example can be generalized to parabolic geometries; see [ČS09].

If  $M$  is a filtered manifold, we write

$$\mathfrak{t}_H M = \bigoplus_{i=1}^n H^i / H^{i-1}$$

for the associated graded vector bundle. The *homogeneous dimension*  $d_H$  of  $M$  is defined to be the homogeneous dimension of  $\mathfrak{t}_H M$ .

We write  $\sigma_i : H^i \rightarrow H^i / H^{i-1}$  for the grading maps<sup>1</sup>. A calculation shows that for any  $X \in \Gamma(H^i)$ ,  $Y \in \Gamma(H^j)$ , and  $f, g \in \Gamma(M)$ ,

$$[fX, gY] \equiv fg[X, Y] \pmod{\Gamma(H^{i+j-1})}, \quad (10)$$

so the Lie bracket is a pointwise operation on  $\mathfrak{t}_H M$ . In other words,  $\mathfrak{t}_H M$  is a smooth bundle of graded nilpotent Lie algebras over  $M$ . Equipped with the zero anchor,  $\mathfrak{t}_H M$  is thus a Lie algebroid.

**Definition 26.** The *osculating groupoid*  $T_H M$  is the smooth bundle of connected, simply connected nilpotent Lie groups obtained by exponentiating the Lie algebroid  $\mathfrak{t}_H M$ .

Explicitly,  $T_H M = \mathfrak{t}_H M$  as a smooth fibre bundle, and each fibre is equipped with the group law given by the Baker-Campbell-Hausdorff formula. Thus  $T_H M$  is a Lie groupoid, with range and source maps both equal to the bundle projection.

Being a graded vector bundle,  $\mathfrak{t}_H M$  is equipped with a canonical family of dilations  $(\delta_\lambda)_{\lambda \in \mathbb{R}}$ , as described in Section 3.2. The dilations are Lie algebroid endomorphisms (automorphisms for  $\lambda \neq 0$ ). They therefore exponentiate to a smooth family of groupoid endomorphisms of  $T_H M$  (i.e. fibrewise group endomorphisms), which we again call *dilations* and denote by the same notation  $(\delta_\lambda)_{\lambda \in \mathbb{R}}$ .

<sup>1</sup> The letter  $\sigma$  is chosen to agree with the notation for the principal (co)symbol maps that we will encounter in later sections. The two maps coincide for differential operators; see Section 7.2 for details.

**3.4. The  $H$ -tangent groupoid.** Algebraically, the  $H$ -tangent groupoid is the disjoint union of the osculating groupoid  $T_H M$  with a family of pair groupoids

$$\mathbb{T}_H M = T_H M \times \{0\} \sqcup M \times M \times \mathbb{R}^\times \quad (11)$$

This is to be understood as a family of Lie groupoids over  $\mathbb{R}$ , in that the allowable compositions are

$$\begin{aligned} (z, y, t)(y, x, t) &= (z, x, t), & x, y, z \in M, t \in \mathbb{R}^\times, \\ (x, \zeta, 0)(x, \eta, 0) &= (x, \zeta\eta, 0), & x \in M, \zeta, \eta \in T_H M_x. \end{aligned}$$

To glue the  $t = 0$  fiber  $T_H M \times \{0\}$  to  $M \times M \times \mathbb{R}^\times$  we blow up the diagonal in  $M \times M$  by means of the dilations  $\delta_t^{-1}$ , which is made precise as follows.

**Definition 27.** A *degrading* of  $\mathfrak{t}_H M$  is an isomorphism of vector bundles  $\psi : \mathfrak{t}_H M \rightarrow TM$  such that for each  $i$ , the restriction of  $\psi$  to  $H^i/H^{i-1}$  is right inverse to the grading map  $\sigma_i : H^i \rightarrow H^i/H^{i-1}$ .

Fixing a choice of degrading allows us to transport the dilations  $\delta_\lambda$  of  $\mathfrak{t}_H M$  to endomorphisms of  $TM$ ,  $\delta_\lambda^\psi := \psi \circ \delta_\lambda \circ \psi^{-1}$ . Let  $\nabla$  be a connection on  $TM$  that is compatible with the degrading, in the sense that  $\nabla$  commutes with  $\delta_\lambda^\psi$ :

$$\nabla_X(\delta_\lambda^\psi Y) = \delta_\lambda^\psi \nabla_X(Y) \quad X, Y \in \Gamma(TM).$$

For such a choice of connection, use the exponential maps

$$\exp_x^\nabla : T_x M \rightarrow M$$

to define

$$\begin{aligned} \mathbb{E}\exp^{\psi, \nabla} : \mathfrak{t}_H M \times \mathbb{R} &\longrightarrow T_H M \times \{0\} \sqcup M \times M \times \mathbb{R}^\times; & (12) \\ (x, v, t) &\longmapsto \begin{cases} (\exp_x^\nabla(\psi(\delta_t v)), x, t) & \text{if } t \neq 0, \\ (x, v, 0) & \text{if } t = 0, \end{cases} \end{aligned}$$

The smooth structure of  $\mathfrak{t}_H M \times \mathbb{R}$  induces a smooth structure in an open neighborhood of the  $t = 0$  fiber of  $\mathbb{T}_H M$ . Away from  $t = 0$  the open subset  $M \times M \times \mathbb{R}^\times$  of  $\mathbb{T}_H M$  has the evident smooth manifold structure.

**Theorem 28.** *With the above charts  $\mathbb{T}_H M$  is a smooth groupoid.*

We call  $\mathbb{T}_H M$  the  *$H$ -tangent groupoid*. It coincides with the tangent groupoid of [CP15]. Their construction involves Taylor expansions that are not easy to handle. We find it easiest to first construct the Lie algebroid  $\mathfrak{t}_H M$  of  $\mathbb{T}_H M$  and appeal to general results about integration of Lie algebroids.

**Example 29.** If  $M$  is trivially filtered, i.e.  $H^1 = TM$ , then  $\mathbb{T}_H M$  is Connes' tangent groupoid ([Con94]). If  $M$  is a contact manifold or a Heisenberg manifold then  $\mathbb{T}_H M$  is the Heisenberg tangent groupoid of [vE05] or [Pon06].

**3.5. The  $H$ -tangent Lie algebroid.** As in section 3.1, we shall construct the Lie algebroid  $\mathfrak{t}_H M$  indirectly, starting with its module of sections. Let  $\mathfrak{X}_H$  be the  $C^\infty(M \times \mathbb{R})$  module of sections of  $TM \times \mathbb{R}$  whose degree of vanishing at  $t = 0$  is controlled by the filtration as follows

$$\mathfrak{X}_H = \{\mathbb{X} \in \Gamma(TM \times \mathbb{R}) \mid \partial_t^k \mathbb{X}|_{t=0} \in \Gamma(H^k) \text{ for all } k \geq 0\} \quad (13)$$

where  $\partial_t = \frac{\partial}{\partial t}$ . Note that the  $k = 0$  equation implies  $\mathbb{X}|_{t=0} = 0$ . The bundle  $TM \times \mathbb{R} \rightarrow M \times \mathbb{R}$  is a Lie algebroid with constant bracket and anchor

$$\begin{aligned} [\mathbb{X}, \mathbb{Y}]_t &= [\mathbb{X}_t, \mathbb{Y}_t] & \mathbb{X}, \mathbb{Y} \in \Gamma(TM \times \mathbb{R}) \\ \rho(\mathbb{X})_t &= \mathbb{X}_t \end{aligned}$$

The bracket and anchor of  $\Gamma(TM \times \mathbb{R})$  restrict to  $\mathfrak{X}_H$ ,

$$[\cdot, \cdot] : \mathfrak{X}_H \times \mathfrak{X}_H \rightarrow \mathfrak{X}_H \quad (14)$$

$$\rho : \mathfrak{X}_H \rightarrow \Gamma(TM \times \mathbb{R}) \subset \Gamma(T(M \times \mathbb{R})). \quad (15)$$

We will show that the Lie algebra  $\mathfrak{X}_H$  is the module of sections of a Lie algebroid  $\mathfrak{t}_H M \rightarrow M \times \mathbb{R}$ . We first need to show that  $\mathfrak{X}_H$  is indeed the module of sections of a vector bundle on  $M \times \mathbb{R}$ .

**Lemma 30.** *Fix a degrading  $\psi$ . The associated map*

$$\Delta^\psi : \Gamma(\mathfrak{t}_H M \times \mathbb{R}) \rightarrow \mathfrak{X}_H \quad (\Delta^\psi \mathbb{X})(x, t) = \psi(\delta_t(\mathbb{X}(x, t)))$$

*is an isomorphism of  $C^\infty(M \times \mathbb{R})$ -modules.*

*Proof.* Let  $H_i := \psi(H^i/H^{i+1})$ . Then  $TM = \bigoplus_{i=1}^N H_i$ . For  $\mathbb{X} \in \Gamma(\mathfrak{t}_H M \times \mathbb{R})$  write  $\psi(\mathbb{X}) = \sum_{i=1}^N \mathbb{X}_i$  with  $\mathbb{X}_i \in \Gamma(H_i \times \mathbb{R})$ . We compute

$$\partial_t^k (\Delta^\psi \mathbb{X})|_{t=0} = \sum_{i=1}^N \partial_t^k (t^i \mathbb{X}_i)|_{t=0} = k! \mathbb{X}_k|_{t=0} \in \Gamma(H_k) \subseteq \Gamma(H^k),$$

so  $\Delta^\psi \mathbb{X} \in \mathfrak{X}_H$ .

Clearly,  $\Delta^\psi$  is  $C^\infty(M \times \mathbb{R})$ -linear. Since  $\psi \circ \delta_t$  is invertible for  $t \neq 0$ ,  $\Delta^\psi$  is injective. We need to show it is surjective.

Let  $\mathbb{Y} \in \mathfrak{X}_H$  and write  $\mathbb{Y} = \sum_{i=1}^N \mathbb{Y}_i$  with  $\mathbb{Y}_i \in \Gamma(H_i \times \mathbb{R})$ . Since  $\partial_t^k \mathbb{Y}|_{t=0} \in \Gamma(H^k)$ , we have  $\partial_t^k \mathbb{Y}_i|_{t=0} = 0$  for all  $k < i$ , and therefore  $\mathbb{Y}_i(x, t) = t^i \mathbb{X}_i(x, t)$  for some  $\mathbb{X}_i \in \Gamma(H_i \times \mathbb{R})$ . Thus  $\mathbb{Y} = \Delta^\psi \mathbb{X}$  for  $\mathbb{X} = \sum_{i=1}^N \psi^{-1}(\mathbb{X}_i)$ .  $\square$

We claim that  $\mathfrak{X}_H$  is the module of sections of a Lie algebroid  $\mathbb{t}_H M$ .

Let us introduce the restriction maps

$$\text{ev}_t : \mathfrak{X}_H \subset \Gamma(TM \times \mathbb{R}) \rightarrow \Gamma(TM); \quad \mathbb{X} \mapsto \mathbb{X}_t.$$

For  $t \neq 0$ ,  $\text{ev}_t$  induces a *canonical* identification of Lie algebroids  $\mathbb{t}_H M|_t = TM$ . At  $t = 0$ ,  $\text{ev}_0$  is zero on  $\mathfrak{X}_H$ . On the other hand, Lemma 30 gives a vector space isomorphism  $\mathbb{t}_H M|_0 \cong \mathfrak{t}_H M$  that turns out to be canonical as well, i.e. it does *not* depend on the choice of degrading  $\psi$ .

**Lemma 31.** *The map  $\text{ev}_0^H : \mathfrak{X}_H \rightarrow \Gamma(\mathfrak{t}_H M)$  defined by*

$$\text{ev}_0^H = \text{ev}_0 \circ (\Delta^\psi)^{-1}$$

*is independent of  $\psi$ . Moreover,  $\text{ev}_0^H$  is a Lie algebra homomorphism,*

*where  $\mathfrak{X}_H$  has the bracket restricted from  $\Gamma(TM \times \mathbb{R})$  as in equation (14).*

*Proof.* We could define  $\text{ev}_0^H$  independently of  $\psi$  as the map

$$\text{ev}_0^H = \bigoplus_{i=1}^N \text{ev}_0^{(i)} : \mathfrak{X}_H \rightarrow \bigoplus_{i=1}^N \Gamma(H^i / H^{i-1}) = \Gamma(\mathfrak{t}_H M), \quad (16)$$

with components

$$\text{ev}_0^{(i)} : \mathfrak{X}_H \rightarrow \Gamma(H^i / H^{i-1}); \quad \mathbb{X} \mapsto \sigma_i \left( \frac{1}{i!} \partial_t^i \mathbb{X}|_{t=0} \right).$$

To see that  $\text{ev}_0^H \circ \Delta^\psi = \text{ev}_0$  is a direct calculation, similar to the calculation in the proof of Lemma 30.

To verify that  $\text{ev}_0^H$  is a Lie algebra homomorphism, we compute:

$$\begin{aligned} \text{ev}_0^H[\mathbb{X}, \mathbb{Y}] &= \bigoplus_{i=1}^N \sigma_i \left( \frac{1}{i!} \partial_t^i [\mathbb{X}, \mathbb{Y}]|_{t=0} \right) = \bigoplus_{i=1}^N \sigma_i \left( \frac{1}{i!} \sum_{j=0}^i \binom{i}{j} [\partial_t^j \mathbb{X}, \partial_t^{i-j} \mathbb{Y}] \right) \\ &= \bigoplus_{i=1}^N \sum_{j=0}^i \left[ \sigma_j \left( \frac{1}{j!} \partial_t^j \mathbb{X} \right), \sigma_{i-j} \left( \frac{1}{(i-j)!} \partial_t^{i-j} \mathbb{Y} \right) \right] = [\text{ev}_0^H \mathbb{X}, \text{ev}_0^H \mathbb{Y}]. \end{aligned}$$

□

The family of restrictions  $\text{ev}_0^H$  and  $\text{ev}_t$  ( $t \neq 0$ ) therefore gives a *canonical* identification of Lie algebroids

$$\mathfrak{X}_H \cong \Gamma(\mathbb{t}_H M); \quad \mathbb{t}_H M = \mathfrak{t}_H M \times \{0\} \sqcup TM \times \mathbb{R}^\times \quad (17)$$

which defines the algebraic structure of  $\mathbb{t}_H M$ . The smooth structure of  $\mathbb{t}_H M$  is implicit in the identification  $\mathfrak{X}_H \cong \Gamma(\mathbb{t}_H M)$  and the isomorphism  $\Delta^\psi$ . Explicitly, there is a *non-canonical* isomorphism of vector bundles that depends on  $\psi$ , and we use this to determine the

smooth structure of the disjoint union in equation 17. To be explicit, we declare that the following map be a diffeomorphism:

$$\begin{aligned} \Phi^\psi : \mathfrak{t}_H M \times \mathbb{R} &\longrightarrow \mathfrak{t}_H M = \mathfrak{t}_H M \times \{0\} \sqcup TM \times \mathbb{R}^\times; \\ (x, v, t) &\longmapsto \begin{cases} (x, \psi(\delta_t v), t) & \text{if } t \neq 0, \\ (x, v, 0) & \text{if } t = 0, \end{cases} \end{aligned} \quad (18)$$

where  $x \in M$ ,  $v \in \mathfrak{t}_H M_x$ ,  $t \in \mathbb{R}$ .

**Lemma 32.** *The above smooth structure on  $\mathfrak{t}_H M$  is independent of the degrading  $\psi$ . That is, if  $\varphi$  is another degrading then the map*

$$(\Phi^\varphi)^{-1} \circ \Phi^\psi : \mathfrak{t}_H M \times \mathbb{R} \rightarrow \mathfrak{t}_H M \times \mathbb{R}$$

*is a smooth bundle automorphism.*

*Proof.* We calculate

$$(\Phi^\varphi)^{-1} \circ \Phi^\psi : (x, v, t) \mapsto \begin{cases} (x, \delta_t^{-1} \circ \varphi^{-1} \circ \psi \circ \delta_t(v), t), & t \neq 0 \\ (x, v, t), & t = 0. \end{cases}$$

Fix a degree  $i$ . By definition,  $\sigma_i \circ \varphi = \sigma_i \circ \psi = \text{id}$  on  $H^i/H^{i-1}$ , so we can write

$$\varphi^{-1} \circ \psi|_{H^i/H^{i-1}} = \text{id} + \epsilon_{i-1} + \cdots + \epsilon_0,$$

where  $\epsilon_j : H^i/H^{i-1} \rightarrow H^j/H^{j-1}$ . Then for  $t \neq 0$  we have

$$\delta_t^{-1} \circ \varphi^{-1} \circ \psi \circ \delta_t|_{H^i/H^{i-1}} = \text{id} + \sum_{j < i} t^{i-j} \epsilon_j.$$

This extends smoothly to  $\text{id}$  at  $t = 0$ , so  $(\Phi^\varphi)^{-1} \circ \Phi^\psi$  is smooth.  $\square$

**3.6. Integration of Lie algebroids.** In this section and the next we confirm that the groupoid  $\mathbb{T}_H M$  of Equation 11 admits a unique smooth structure such that  $\mathfrak{t}_H M$  is its Lie algebroid. Specifically, the map of Equation 12 is the composition of the map of equation 18 with an exponential map  $\mathfrak{t}_H M \rightarrow \mathbb{T}_H M$  from the Lie algebroid to the Lie groupoid.

According to Lie's Third Theorem, every Lie algebra integrates to a unique simply connected Lie group. Similarly, under mild hypotheses, every Lie algebroid determines a unique  $s$ -simply connected Lie groupoid, i.e. a Lie groupoid whose source fibers are simply connected [CF03] [Nis00].

The two components of the Lie algebroid

$$\mathfrak{t}_H M = \mathfrak{t}_H M \times \{0\} \sqcup TM \times \mathbb{R}^\times$$

each determine an  $s$ -simply connected Lie groupoid. At  $t = 0$ , the Lie groupoid  $T_H M$  is  $s$ -simply connected, since each of the nilpotent groups  $T_H M_x$  is, by construction, simply connected. The  $s$ -simply connected groupoid that integrates  $TM$  is not the pair groupoid



$M \times M$ , but the *fundamental groupoid*  $\Pi_M$ . Elements in  $\Pi_M$  are homotopy classes of paths in  $M$  with fixed end points. Source and range of an arrow  $\gamma = [c]$  represented by a path  $c: [0, 1] \rightarrow M$  are the end points of the path:  $s(\gamma) = c(0)$ ,  $r(\gamma) = c(1)$ . Composition of paths gives the groupoid multiplication. The covering map

$$\Pi_M \rightarrow M \times M \quad \gamma \mapsto (r(\gamma), s(\gamma))$$

is a groupoid homomorphism, and a local diffeomorphism. In a neighborhood of the unit space the groupoids  $\Pi_M$  and  $M \times M$  are naturally isomorphic and diffeomorphic.

By Theorem 2 and 3 in [Nis00], the disjoint union

$$\tilde{\mathcal{G}} = T_H M \times \{0\} \sqcup \Pi_M \times \mathbb{R}^\times$$

has a unique smooth manifold structure that makes it a Lie groupoid with Lie algebroid  $\mathfrak{t}_H M$ . Following [Nis00], the smooth structure of  $\tilde{\mathcal{G}}$  is induced by an exponential map from the Lie algebroid  $\mathfrak{t}_H M$  to  $\tilde{\mathcal{G}}$  as follows.

Let  $\nabla$  be a connection on  $TM$  that is compatible with a degrading  $\psi: \mathfrak{t}_H M \rightarrow TM$  in the sense that  $\nabla$  commutes with the dilations  $\delta_t^\psi$ . The connection  $\nabla + \frac{d}{dt}$  on  $TM \times \mathbb{R}$  induces a connection on  $\mathfrak{t}_H M$  via the degrading and the non-canonical isomorphism  $\Delta^\psi$ .

A connection on a Lie algebroid induces an exponential map to an associated Lie groupoid. See [Nis00] for details of the general construction. We describe here the exponential map

$$\exp^{\tilde{\mathcal{G}}} : \mathfrak{t}_H M \rightarrow \tilde{\mathcal{G}}.$$

When restricted to  $t = 0$  the exponential map  $\mathfrak{t}_H M \rightarrow T_H M$  does not depend on  $\nabla$ , and is simply the identification  $\mathfrak{t}_H M = T_H M$ .

Because  $\nabla$  commutes with the dilation automorphisms  $\delta_t^\psi$ , the restriction of the connection on  $\mathfrak{t}_H M$  to  $TM \times \{t\}$  is again  $\nabla$ , independent of  $t \neq 0$ . Therefore for  $t \in \mathbb{R}^\times$  the exponential map

$$\exp_t^{\tilde{\mathcal{G}}} : TM \rightarrow \Pi_M$$

is fixed, and is the lift of the exponential map

$$\text{Exp}^\nabla : TM \rightarrow M \times M \quad (x, v) \mapsto (\exp_x^\nabla(v), x) \quad (19)$$

We obtain a commutative diagram

$$\begin{array}{ccc} \mathfrak{t}_H M & \xrightarrow{\exp^{\tilde{\mathcal{G}}}} & \tilde{\mathcal{G}} \\ \uparrow & & \downarrow \\ \mathfrak{t}_H M \times \mathbb{R} & \xrightarrow{\text{Exp}^{\psi, \nabla}} & T_H M \end{array}$$

with the left vertical arrow as in Equation 18, and the bottom arrow as in Equation 12.

The condition for integrability<sup>2</sup> of the Lie algebroid  $\mathfrak{t}_H M$  is that the exponential map  $\exp^{\tilde{\mathcal{G}}}: \mathfrak{t}_H M \rightarrow \tilde{\mathcal{G}}$  be *injective* in a neighborhood of the zero section of  $\mathfrak{t}_H M$ .

**Definition 33.** An open neighbourhood of the zero section  $U \subseteq TM$  will be called a *domain of injectivity* for  $\nabla$  if  $\text{Exp}^\nabla|_U$  is injective.

If  $U$  is a domain of injectivity, then  $\mathbb{E}x\text{p}^{\psi, \nabla}$  is injective on the open set

$$\{(x, v, t) \in \mathfrak{t}_H M \times \mathbb{R} \mid (x, \psi(\delta_t v)) \in U\}, \quad (20)$$

with image

$$\mathbb{U} := (T_H M \times \{0\}) \sqcup (\text{Exp}^\nabla(U) \times \mathbb{R}^\times) \subset \mathbb{T}_H M. \quad (21)$$

Such a set  $\mathbb{U}$  will be called a *global exponential coordinate patch* for  $\mathbb{T}_H M$ . It is an open neighbourhood of both the unit space  $\mathbb{T}_H M^{(0)} \cong M \times \mathbb{R}$  and the osculating groupoid  $\mathbb{T}_H M|_{t=0} = T_H M \times \{0\}$ .

By the results in [Nis00],  $\tilde{\mathcal{G}}$  is a smooth groupoid with the smooth structure in a neighborhood of  $t = 0$  induced by  $\exp^{\tilde{\mathcal{G}}}$ . Theorem 28 is an immediate corollary: just observe that  $\mathbb{U} \subset \mathbb{T}_H M$  contains the entire  $t = 0$  fiber  $T_H M$ , and that in this neighborhood the groupoid operations of  $\mathbb{T}_H M$  agree with those of  $\tilde{\mathcal{G}}$ .

**3.7. Exponential along the range fibres.** In the preceding, we have followed the usual conventions and defined our groupoid exponentials along the source fibres. However, Schwartz kernels are most conveniently described as smooth families of distributions along the range fibres of the pair groupoid—see Section 2.10. For this reason, we will now abruptly reverse conventions and redefine the groupoid exponential as

$$\begin{aligned} \mathbb{E}x\text{p}^{\psi, \nabla} : \mathfrak{t}_H M \times \mathbb{R} &\longrightarrow \mathbb{T}_H M; \\ (x, v, t) &\longmapsto \begin{cases} (x, \exp_x^\nabla(-\psi(\delta_t v)), t) & \text{if } t \neq 0, \\ (x, v, 0) & \text{if } t = 0. \end{cases} \end{aligned} \quad (22)$$

This is obtained from the source exponential by precomposing with negation on  $\mathfrak{t}_H M \times \mathbb{R}$  and postcomposing with the inverse on  $\mathbb{T}_H M$ . It is again smooth, and trivial on  $\mathbb{T}_H M|_0 = T_H M$ .

---

<sup>2</sup>This necessary hypothesis is not explicitly mentioned in [Nis00], but the correction was published in [BN03].

Given this new convention, an exponential coordinate patch will still mean a neighbourhood of the form given in Equation (21), although we should now demand that  $-U$  be a domain of injectivity for  $\nabla$ .

**3.8. The zoom action on  $\mathbb{T}_H M$ .** The following action, in the case of the classical tangent groupoid, was of central importance in [DS14].

**Definition 34.** The *zoom action*  $\alpha : \mathbb{R}_+^\times \rightarrow \text{Aut}(\mathbb{T}_H M)$  is the one-parameter family of groupoid automorphisms defined as follows:

$$\begin{aligned} \alpha_\lambda(x, y, t) &= (x, y, \lambda^{-1}t) && \text{if } (x, y) \in M \times M, t \neq 0, \\ \alpha_\lambda(x, \zeta, 0) &= (x, \delta_\lambda(\zeta), 0) && \text{if } (x, \zeta) \in T_H M, t = 0. \end{aligned} \quad (23)$$

It is clear that this action is smooth away from  $t = 0$ . We claim that it is also smooth at  $t = 0$ . If  $\nabla$  is a filtered connection and  $\psi$  a degrading, we define an  $\mathbb{R}_+^\times$ -action  $\tilde{\alpha}$  on  $\mathfrak{t}_H M \times \mathbb{R}$  by

$$\tilde{\alpha}_\lambda(x, \zeta, t) = (x, \delta_\lambda(\zeta), \lambda^{-1}t). \quad (24)$$

Then  $\alpha_\lambda \circ \mathbb{E}\text{xp}^{\psi, \nabla} = \mathbb{E}\text{xp}^{\psi, \nabla} \circ \tilde{\alpha}_\lambda$  everywhere on  $\mathbb{T}_H M$ . Thus, in exponential coordinates the action  $\alpha$  pulls back to  $\tilde{\alpha}$ , which is clearly smooth.

The zoom action induces one-parameter families of automorphisms  $(\alpha_{\lambda*})_{\lambda \in \mathbb{R}_+^\times}$  on the convolution algebra  $\mathcal{E}'_{r,s}(\mathbb{T}_H M)$ ; see Section 2.9. Specifically, for  $\mathfrak{u} \in \mathcal{E}'_{r,s}(\mathbb{T}_H M)$ ,  $f \in C^\infty(\mathbb{T}_H M)$ ,

$$\langle \alpha_{\lambda*} \mathfrak{u}, f \rangle(x, t) = \langle \mathfrak{u}, \alpha_\lambda^* f \rangle(x, \lambda t) \quad \text{for } (x, t) \in M \times \mathbb{R}.$$

The ideal of properly supported smooth densities is invariant under the zoom action.

**3.9. Local exponential coordinates.** To conclude this section, we describe a convenient class of local coordinate charts on  $\mathbb{T}_H M$  which will be used in later sections.

Let  $\mathbb{R}^n \supset M_0 \hookrightarrow M$  be a coordinate chart for  $M$  such that its image has compact closure, and such that it is contained in a larger coordinate chart  $\mathbb{R}^n \supset M_1 \hookrightarrow M$  upon which all of the bundles  $H^i$  are trivializable. We identify  $M_0$  with its image in  $M$ . We thus have a trivialization of graded vector bundles

$$M_0 \times V \xrightarrow{\cong} \mathfrak{t}_H M|_{M_0}$$

where  $V$  is a graded vector space.

Let us fix a filtered connection  $\nabla$ , a degrading  $\psi$ , and a domain of injectivity  $U \subseteq TM$  as in Section 3.4. Using the exponential  $\mathbb{E}\text{xp}^{\psi, \nabla}$  of Equation (22) we obtain a smooth map

$$M_0 \times V \times \mathbb{R} \xrightarrow{\cong} \mathfrak{t}_H M|_{M_0} \times \mathbb{R} \xrightarrow{\mathbb{E}\text{xp}^{\psi, \nabla}} \mathbb{T}_H M, \quad (25)$$

which we refer to as a *local exponential coordinate chart*. It has the following properties:

- It is a diffeomorphism from some neighbourhood of  $(M_0 \times V \times \{0\}) \cup (M_0 \times \{0\} \times \mathbb{R})$  onto  $\mathbb{U}_0 := \mathbb{U} \cap r^{-1}(M_0 \times \mathbb{R})$  where  $\mathbb{U}$  is the exponential coordinate patch of (21).
- It respects the  $r$ -fibration in the sense that the following diagram commutes:

$$\begin{array}{ccc} M_0 \times V \times \mathbb{R} & \longrightarrow & \mathbb{T}_H M \\ \downarrow & & \downarrow r \\ M_0 \times \mathbb{R} & \hookrightarrow & M \times \mathbb{R}. \end{array}$$

We shall reuse  $r$  to denote the left-hand fibration  $r : M_0 \times V \times \mathbb{R} \rightarrow M_0 \times \mathbb{R}$ .

- The set  $\mathbb{U}_0 \subseteq \mathbb{T}_H M$  is invariant under the zoom action, and the map (25) intertwines the action  $\tilde{\alpha}_\lambda : (x, \xi, t) \mapsto (x, \delta_\lambda(\xi), \lambda^{-1}t)$  on  $M_0 \times V \times \mathbb{R}$  with the zoom action  $\alpha_\lambda$  on  $\mathbb{T}_H M$ .

We choose coordinates  $x = (x_i)$  for  $M_0$ , graded coordinates  $\xi = (\xi_i)$  for  $V$  and let  $t$  be the canonical coordinate on  $\mathbb{R}$ . Such a system  $(x, \xi, t)$  will be referred to as *local exponential coordinates*.

#### 4. PSEUDODIFFERENTIAL OPERATORS

In this section we define a class of pseudodifferential operators associated to a filtered manifold  $M$ . We also state the main properties of these pseudodifferential operators. The proofs will be deferred to the next section.

##### 4.1. Definition of pseudodifferential operators.

**Definition 35.** A properly supported  $r$ -fibred distribution  $\mathbb{P} \in \mathcal{E}'_r(\mathbb{T}_H M)$  is *essentially homogeneous of weight  $m \in \mathbb{R}$*  if  $\alpha_{\lambda*} \mathbb{P} - \lambda^m \mathbb{P} \in C_p^\infty(\mathbb{T}_H M; \Omega_r)$  for all  $\lambda \in \mathbb{R}_+^\times$ . The space of such distributions will be denoted  $\Psi_H^m(M)$ .

We recall that *proper support* on a groupoid means that both the range and the source maps are proper on the support of  $\mathbb{P}$ .

**Definition 36.** Given  $\mathbb{P} \in \Psi_H^m(M)$ , the function  $F : \mathbb{R}_+^\times \rightarrow C_p^\infty(\mathbb{T}_H M; \Omega_r)$  defined by

$$F : \lambda \mapsto F_\lambda := \lambda^{-m} \alpha_{\lambda*} \mathbb{P} - \mathbb{P} \quad (26)$$

will be called the *cocycle associated to  $\mathbb{P} \in \Psi_H^m(M)$* .

These cocycles satisfy the cocycle condition

$$F_{\lambda_1 \lambda_2} = \lambda_1^{-m} \alpha_{\lambda_1*} F_{\lambda_2} + F_{\lambda_1}. \quad (27)$$

As previously mentioned we will denote by  $\mathbb{P}_t$  the restriction of  $\mathbb{P}$  to the fibre  $\mathbb{T}_H M|_t$ , so that

$$\begin{aligned}\mathbb{P}_t &\in \mathcal{E}'_r(M \times M) \quad \text{for } t \neq 0, \\ \mathbb{P}_0 &\in \mathcal{E}'_r(T_H M).\end{aligned}$$

Recall from Section 2.10 that for  $P \in \mathcal{E}'_r(M \times M)$ ,  $\text{Op}(P)$  is the continuous linear map

$$\text{Op}(P) : C^\infty(M) \rightarrow C^\infty(M); \quad (\text{Op}(P)f)(x) = \int_{y \in M} P(x, y)f(y).$$

We will generally blur the distinction between the distribution  $P$  and the associated operator  $\text{Op}(P)$ .

**Definition 37.** A Schwartz kernel  $P \in \mathcal{E}'_r(M \times M)$  will be called an *H-pseudodifferential kernel* of order  $m$  (or, by abuse of terminology, an *H-pseudodifferential operator* of order  $m$ ) if  $P = \mathbb{P}_1$  for some  $\mathbb{P} \in \Psi_H^m(M)$ . In this case, we say that  $\mathbb{P}$  *extends*  $P$ . The set of H-pseudodifferential kernels of order  $m$  will be denoted  $\Psi_H^m(M)$ .

**Example 38.** If  $P$  is (the Schwartz kernel of) a differential operator of (classical) order  $m$ , then the discussion in the introduction shows that it extends to a smooth family of  $r$ -fibred distributions on  $\mathbb{T}M$ ,

$$\mathbb{P}_t = \begin{cases} t^m P, & t \neq 0, \\ \sigma_m(P), & t = 0, \end{cases}$$

where  $\sigma(P) = (P_x)_{x \in M}$  is the principal symbol. The resulting  $r$ -fibred distribution  $\mathbb{P}$  is homogeneous on the nose, in the sense that  $\alpha_{\lambda*} \mathbb{P} - \lambda^m \mathbb{P} = 0$  for all  $\lambda \in \mathbb{R}_+^\times$ . Thus differential operators are pseudodifferential operators, according to our definition.

We will discuss this example in more detail in Section 7.

**Example 39.** Let  $M = \mathbb{R}^n$  with the trivial filtration:  $H^1 = TM$ . With the standard connection, the exponential map is a global diffeomorphism

$$\begin{aligned}\text{Exp} : TM \times \mathbb{R} &\rightarrow \mathbb{T}M; \\ (x, \zeta, t) &\mapsto (x, x - t\zeta, t), & \text{if } t \neq 0, \\ (x, \zeta, 0) &\mapsto (x, \zeta, 0), & \text{if } t = 0.\end{aligned}$$

Consider the fibred distribution  $\mathbb{P} \in \mathcal{E}'_r(M)$  whose pull-back  $\tilde{\mathbb{P}} := \text{Exp}^* \mathbb{P}$  via exponential coordinates is

$$\tilde{\mathbb{P}}(x, \zeta, t) = \log |\zeta| d\zeta.$$

In standard coordinates,

$$\begin{aligned}\mathbb{P}(x, y, t) &= \log(|t|^{-1}|x - y|) dy, & \text{if } t \neq 0, \\ \mathbb{P}(x, \zeta, t) &= \log |\zeta| d\zeta, & \text{if } t \neq 0.\end{aligned}$$

From Equation (24) we see that

$$\tilde{\alpha}_{\lambda*}\tilde{\mathbb{P}}(x, \zeta, t) = \log |\lambda^{-1}\zeta| d(\lambda^{-1}\zeta) = \lambda^{-n}(\log |\zeta| - \log |\lambda|) d\zeta,$$

so

$$(\tilde{\alpha}_{\lambda*}\tilde{\mathbb{P}} - \lambda^{-n}\tilde{\mathbb{P}})(x, \zeta, t) = -\lambda^{-n} \log |\lambda| d\zeta$$

which is a smooth density for all  $\lambda \in \mathbb{R}_+^\times$ . Thus  $\mathbb{P}$  is essentially homogeneous<sup>3</sup> of weight  $-n$ . It is not properly supported, but one can easily resolve this by multiplying  $\mathbb{P}$  by an exponential cut-off function (see the next section). Thus, after a smooth cut-off,  $\mathbb{P}_1$  is a pseudodifferential operator according to our definition.

*Remark 40.* In other kernel-based approaches to pseudodifferential calculi (e.g. [CGGP92]), logarithmic singularities need to be included specially. One of the advantages of the present approach is that they enter naturally.

Note that the definition of  $\Psi_H^m(M)$  forces the singular support of an  $H$ -pseudodifferential kernel to lie on the diagonal:

**Proposition 41.** *Every  $\mathbb{P} \in \Psi_H^m(M)$  is smooth off the unit space, in the sense that the restriction of  $\mathbb{P}$  to  $\mathbb{T}_H M \setminus \mathbb{T}_H M^{(0)}$  is equal to a smooth density. Hence every  $P \in \Psi_H^m(M)$  is smooth off the diagonal.*

*Proof.* The singular support of  $\mathbb{P} \in \Psi_H^m(M)$  is invariant under the zoom action. If  $\gamma \in \mathbb{T}_H M \setminus \mathbb{T}_H M^{(0)}$  then its orbit under the zoom action is not proper. Since  $\mathbb{P}$  has proper support by definition, the result follows.  $\square$

A more profound consequence of essential homogeneity is that  $H$ -pseudodifferential kernels are proper distributions, i.e. that they are both  $r$ - and  $s$ -fibered, despite the fact that the definition only demands  $r$ -fibered.

**Proposition 42.** *We have  $\Psi_H^m(M) \subseteq \mathcal{E}'_{r,s}(\mathbb{T}_H M)$ . Therefore elements of  $\Psi_H^m(M)$  are Schwartz kernels which are semiregular in both variables (see Proposition 22).*

---

<sup>3</sup> The log distribution is typically viewed as order 0. The discrepancy here occurs because we are working with the density  $\log |\zeta| d\zeta$  rather than the function  $\log |\zeta|$ . The Lebesgue measure  $d\zeta$  is homogeneous of order  $-n$ .

*Remark 43.* Of course, we could have avoided this technicality by demanding properness in our definitions of  $H$ -pseudodifferential operators (Definitions 35 and 37), but this would come at the expense of making it far more difficult to determine whether a given Schwarz kernel  $P$  on  $M$  is  $H$ -pseudodifferential. The minimalist definitions we have chosen here will pay off in later analysis.

The proof of Proposition 42 requires some additional tools (we will in fact prove the conormality of the wavefront set, as one should expect for pseudodifferential operators). To avoid interrupting the flow of exposition we will defer this until Section 5.7.

Granted this, we now have an algebra structure on the  $H$ -pseudodifferential operators.

**Theorem 44.** *For any  $m, m' \in \mathbb{R}$  we have  $\Psi_H^m(M) * \Psi_H^{m'}(M) \subseteq \Psi_H^{m+m'}(M)$ . Thus  $\Psi_H^\bullet(M)$  is a graded algebra.*

*Proof.* The first statement follows from the facts that  $\alpha_{\lambda^*}$  is an algebra automorphism of  $\mathcal{E}'_{r,s}(\mathbb{T}_H M)$  and that  $C_p^\infty(\mathbb{T}_H M; \Omega_r)$  is a two-sided ideal. The second statement is immediate upon restriction to  $t = 1$ .  $\square$

**4.2. Exponential cut-off functions.** If we consider smoothing kernels as negligible, the pertinent data of an element  $\mathbb{P} \in \Psi_H^m(M)$  is carried in a neighbourhood of the unit space  $\mathbb{T}_H M^{(0)}$  and the osculating groupoid  $\mathbb{T}_H M|_{t=0} = T_H M$ . The exponential coordinate charts of Section 3.6 are such neighbourhoods, and it will be useful to cut off  $\mathbb{P}$  by a bump function supported in such a neighbourhood.

To be specific, let us fix a degrading  $\psi : \mathbb{T}_H M \rightarrow TM$ , a compatible connection  $\nabla$  on  $TM$  and let  $U \subset TM$  be a domain of injectivity for  $\nabla$  as in Definition 33. Let  $\varphi_1 \in C^\infty(M \times M)$  be a smooth bump function which equals 1 on a neighbourhood of the diagonal and zero outside the image of the domain of injectivity  $U$ . Define the function  $\varphi$  on  $\mathbb{T}_H M$  by

$$\begin{cases} \varphi(x, y, t) = \varphi_1(x, y), & \text{if } t \neq 0, \\ \varphi(x, \xi, 0) = 1 & \text{if } t = 0. \end{cases} \quad (28)$$

It is constant equal to 1 on a neighbourhood of  $\mathbb{T}_H M^{(0)} \cup \mathbb{T}_H M|_0$ , and equal to 0 outside of the exponential coordinate patch  $\mathbb{U} = (T_H M \times \{0\}) \sqcup (\text{Exp}^\nabla(U) \times \mathbb{R}^\times)$ . Moreover, it is invariant under the zoom action. Such a  $\varphi$  will be called an *exponential cut-off function*.

**4.3. Principal cosymbols.** The following proposition will allow us to unambiguously define the principal part of an  $H$ -pseudodifferential operator.

**Proposition 45.** *Let  $\mathbb{P}, \mathbb{Q} \in \Psi_H^m(M)$ . If  $\mathbb{P}_1 \equiv \mathbb{Q}_1$  modulo  $C_p^\infty(M \times M; \Omega_r)$  then  $\mathbb{P}_0 \equiv \mathbb{Q}_0$  modulo  $C_p^\infty(T_H M; \Omega_r)$ .*

*Proof.* After replacing  $\mathbb{P}$  by  $\mathbb{P} - \mathbb{Q}$  it suffices to show that if  $\mathbb{P}_1$  is a smooth density on  $M \times M$  then  $\mathbb{P}_0$  is a smooth density on  $T_H M$ .

Here is the strategy. From essential homogeneity, we know that  $\mathbb{P}_t \in C_p^\infty(M \times M; \Omega_r)$  for all  $t \neq 0$ . We also know that  $\mathbb{P}_t \rightarrow \mathbb{P}_0$  as  $t \rightarrow 0$  as a distribution. So, if we can prove that some sequence  $\mathbb{P}_{t_k}$  with  $t_k \rightarrow 0$  admits a smooth limit, then by uniqueness of limits  $\mathbb{P}_0$  must be smooth. We will work with  $t_k = 2^{-k}$ , since the differences

$$\mathbb{P}_{2^{-k+1}} - 2^{-m} \mathbb{P}_{2^{-k}}$$

(and similar differences for the derivatives of  $\mathbb{P}_{2^{-k}}$ ) can all be controlled by a single globally smooth density, namely  $F_{\frac{1}{2}}$ . For  $m$  sufficiently large, this will yield convergence. We now give the details.

We begin by trivializing  $T_H M$ . Using an exponential cut-off function, we assume that  $\mathbb{P}$  is supported in an exponential coordinate patch, and we write  $\tilde{\mathbb{P}} \in \mathcal{E}'_r(M_0 \times V \times \mathbb{R}; \Omega_r)$  for the pullback of  $\mathbb{P}$  via local exponential coordinates (see Section 3.9). We trivialize the vertical density bundle by letting  $\omega = (\omega_{x,t})_{(x,t) \in M_0 \times \mathbb{R}}$  denote the constant family of Lebesgue measures on  $V$ . We can then write  $\tilde{\mathbb{P}}|_{\mathbb{R}^\times} = p\omega$ , where  $p \in C^\infty(M_0 \times V \times \mathbb{R}^\times)$ . As always, we will write  $p_t$  for the restriction of  $p$  to  $M_0 \times V \times \{t\}$ .

Likewise, we write the cocycle associated to  $\tilde{\mathbb{P}}$  as  $F_\lambda = f_\lambda \omega$  where  $f_\lambda \in C^\infty(M_0 \times V \times \mathbb{R})$ . Recall that the zoom action in exponential coordinates is given by Equation (24):

$$\tilde{\alpha}_\lambda(x, \zeta, t) = (x, \delta_\lambda(\zeta), \lambda^{-1}t).$$

The action on the fibrewise Lebesgue measures is therefore  $\tilde{\alpha}_{\lambda*} \omega = \lambda^{-d_H} \omega$ . It follows that

$$f_\lambda = \lambda^{-m-d_H} \tilde{\alpha}_{\lambda^{-1}}^* p - p \tag{29}$$

on  $M \times V \times \mathbb{R}^\times$ .

Differentiating Equation (29) gives

$$\partial_x^{\mathbf{a}} \partial_\zeta^{\mathbf{b}} f_\lambda(x, \zeta, t) = \lambda^{-m-d_H-|\mathbf{b}|_H} (\partial_x^{\mathbf{a}} \partial_\zeta^{\mathbf{b}} p)(x, \delta_{\lambda^{-1}}(\zeta), \lambda t) - (\partial_x^{\mathbf{a}} \partial_\zeta^{\mathbf{b}} p)(x, \zeta, t)$$

for any multi-indices  $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$ . Putting  $\lambda = \frac{1}{2}$ , we have the following estimate in uniform norms:

$$\|\partial_x^{\mathbf{a}} \partial_\zeta^{\mathbf{b}} f_{\frac{1}{2}}\|_{\overline{M_0 \times V \times [0,1]}} \geq \left| 2^{m+d_H+|\mathbf{b}|_H} \|\partial_x^{\mathbf{a}} \partial_\zeta^{\mathbf{b}} p_{\frac{1}{2}t}\|_{\overline{M_0 \times V}} - \|\partial_x^{\mathbf{a}} \partial_\zeta^{\mathbf{b}} p_t\|_{\overline{M_0 \times V}} \right|$$



for all  $t \in [0, 1]$ . We write  $m' = m + d_H + |\mathbf{b}|_H$ . Then for every  $k \in \mathbb{N}$ , we get

$$\begin{aligned}
& \|\partial_x^{\mathbf{a}} \partial_{\xi}^{\mathbf{b}} p_{2^{-k}}\|_{\overline{M_0} \times V} \\
&= \left( \|\partial_x^{\mathbf{a}} \partial_{\xi}^{\mathbf{b}} p_{2^{-k}}\|_{\overline{M_0} \times V} - 2^{-m'} \|\partial_x^{\mathbf{a}} \partial_{\xi}^{\mathbf{b}} p_{2^{-(k-1)}}\|_{\overline{M_0} \times V} \right) \\
&\quad + \left( 2^{-m'} \|\partial_x^{\mathbf{a}} \partial_{\xi}^{\mathbf{b}} p_{2^{-(k-1)}}\|_{\overline{M_0} \times V} - 2^{-2m'} \|\partial_x^{\mathbf{a}} \partial_{\xi}^{\mathbf{b}} p_{2^{-(k-2)}}\|_{\overline{M_0} \times V} \right) \\
&\quad \cdots + \left( 2^{-(k-1)m'} \|\partial_x^{\mathbf{a}} \partial_{\xi}^{\mathbf{b}} p_{2^{-1}}\|_{\overline{M_0} \times V} - 2^{-km'} \|\partial_x^{\mathbf{a}} \partial_{\xi}^{\mathbf{b}} p_1\|_{\overline{M_0} \times V} \right) \\
&\quad + 2^{-km'} \|\partial_x^{\mathbf{a}} \partial_{\xi}^{\mathbf{b}} p_1\|_{\overline{M_0} \times V} \\
&\leq \left( \sum_{j=1}^k 2^{-jm'} \right) \|\partial_x^{\mathbf{a}} \partial_{\xi}^{\mathbf{b}} f_{\frac{1}{2}}\|_{\overline{M_0} \times V \times [0,1]} + 2^{-km'} \|\partial_x^{\mathbf{a}} \partial_{\xi}^{\mathbf{b}} p_1\|_{\overline{M_0} \times V}.
\end{aligned}$$

If  $m' > 0$ , this is bounded as  $k \rightarrow \infty$ . Thus, for all multi-indices  $\mathbf{a}, \mathbf{b}$  with  $|\mathbf{b}|_H > -m - d_H$ , the sequence of functions  $(p_{2^{-k}})_{k \in \mathbb{N}}$  is bounded with respect to the seminorm  $\|\partial_x^{\mathbf{a}} \partial_{\xi}^{\mathbf{b}} \bullet\|_{\overline{M_0} \times V}$ . Since  $\text{supp}(p) \cap (\overline{M_0} \times V \times [0, 1])$  is compact, the derivatives  $\partial_x^{\mathbf{a}} \partial_{\xi}^{\mathbf{b}} p$  with  $|\mathbf{b}|_H \leq -m - d$  are bounded in terms of those with  $|\mathbf{b}|_H > -m - d_H$ . Thus,  $(p_{2^{-k}})_{k \in \mathbb{N}}$  is a bounded sequence in  $C^\infty(\overline{M_0} \times V)$ . By the Arzela-Ascoli Theorem, it admits a limit point  $p_0$ . But  $\tilde{\mathbb{P}}_{2^{-k}} \rightarrow \tilde{\mathbb{P}}_0$  as distributions, so  $\tilde{\mathbb{P}}_0 = p_0 \omega$ . This completes the proof.  $\square$

**Definition 46.** For  $m \in \mathbb{R}$ , we denote

$$\Sigma_H^m(M) := \{A \in \mathcal{E}'_r(T_H M) / C_p^\infty(T_H M; \Omega_r) \mid \delta_{\lambda^*} A - \lambda^m A = 0\}.$$

Elements of  $\Sigma_H^m(M)$  will be called *cosymbols of order  $m$*  on  $M$ .

We will not bother to use a separate notation for an essentially homogeneous  $r$ -fibred distribution  $A \in \mathcal{E}'_r(T_H M)$  and its class in  $\Sigma_H^m(M)$ .

*Remark 47.* We note that the order  $m$  in Definition 46 is the order as a pseudodifferential operator. This is not the same as the usual analyst's notion of the order of  $A \in \mathcal{E}'_r(T_H M)$  as an essentially homogeneous distribution on the graded vector bundle  $T_H M$ , which would be  $-m - d_H$ .

To explain this further, recall that in the analyst's definition one identifies distributions with "generalized" functions via some fibre-wise Lebesgue measure  $\omega$ , which is of order  $-d_H$  in the sense of Definition 46. The order of homogeneity is then naturally defined with respect to the pull-back by  $\delta_\lambda$  rather than the push-forward, so reversing the sign.

We will only need the pseudodifferential order in this work.

**Definition 48.** We define the *cosymbol map*

$$\sigma_m : \Psi_H^m(M) \rightarrow \Sigma_H^m(M)$$

as follows. For  $P \in \Psi_H^m(M)$ , choose any  $\mathbb{P} \in \Psi_H^m(M)$  with  $\mathbb{P}_1 = P$  and set  $\sigma_m(P) := \mathbb{P}_0$ . It is well-defined modulo  $C_p^\infty(T_H M; \Omega_r)$  by Proposition 45.

**Proposition 49.** *If  $P \in \Psi_H^m(M)$  and  $Q \in \Psi_H^{m'}(M)$  then  $\sigma_{m+m'}(P * Q) = \sigma_m(P) * \sigma_{m'}(Q)$ .*

*Proof.* If  $\mathbb{P} \in \Psi_H^m$  and  $\mathbb{Q} \in \Psi_H^{m'}$  extend  $P$  and  $Q$ , respectively, then  $\mathbb{P} * \mathbb{Q} \in \Psi_H^{m+m'}$  extends  $P * Q$ . The result follows.  $\square$

**Lemma 50.** *The cosymbol map  $\sigma_m$  is surjective for every  $m \in \mathbb{R}$ .*

*Proof.* Let  $A \in \mathcal{E}'_r(T_H M)$  with  $\delta_{\lambda*} A - \lambda^m A \in C_p^\infty(T_H M; \Omega_r)$  for all  $\lambda \in \mathbb{R}_+^\times$ . Define the constant family  $\tilde{\mathbb{A}} \in \mathcal{E}'_r(\mathfrak{t}_H M \times \mathbb{R})$  by  $\tilde{\mathbb{A}}_t = A$  for all  $t \in \mathbb{R}$ . Then  $\tilde{\mathbb{A}}$  is essentially homogeneous for the action  $(\tilde{\alpha}_\lambda)$  of Equation (24).

Now fix a global exponential coordinate patch and let  $\tilde{\varphi} \in C^\infty(\mathfrak{t}_H M \times \mathbb{R})$  be (the pullback of) an exponential cut-off function as in Equation (28). Then  $\tilde{\varphi}\tilde{\mathbb{A}}$  is again essentially homogeneous of weight  $m$ , so its push-forward  $\mathbb{A}$  via exponential coordinates  $\mathbb{A}$  is in  $\Psi_H^m(M)$  and has  $\mathbb{A}_0 = A$ .  $\square$

**Proposition 51.** *Let  $\mathbb{P} \in \Psi_H^m(M)$  and  $k \in \mathbb{Z}$ . The following are equivalent:*

- (1)  $\mathbb{P} = t^k \mathbb{Q}$  for some  $\mathbb{Q} \in \mathcal{E}'_r(\mathfrak{t}_H M)$ ,
- (2) For any (or for every) pullback  $\tilde{\mathbb{P}} \in \mathcal{E}'_r(\mathfrak{t}_H M \times \mathbb{R})$  of  $\mathbb{P}$  via global exponential coordinates, we have  $(\partial_t^j \tilde{\mathbb{P}})|_{t=0} = 0$  for  $j = 0, \dots, k - 1$ .

In this case,  $\mathbb{Q} \in \Psi_H^{m-k}(M)$ .

*Remark 52.* Before defining the pullback  $\tilde{\mathbb{P}}$ , one should as usual multiply  $\mathbb{P}$  by an exponential cut-off function. We shall refrain from repeating this in the sequel.

*Proof.* After pulling back via exponential coordinates we have  $\tilde{\mathbb{P}} \in \mathcal{E}'_r(\mathfrak{t}_H M \times \mathbb{R}) \cong C^\infty(\mathbb{R}, \mathcal{E}'_r(\mathfrak{t}_H M))$ . The equivalence of the vanishing conditions (1) and (2) is immediate.

Now let  $F$  be the cocycle associated to  $\mathbb{P}$  and  $\tilde{F}$  its pull-back via exponential coordinates. For each  $\lambda \in \mathbb{R}_+^\times$  we have  $(\partial_t^j \tilde{F}_\lambda)|_{t=0} = 0$  for  $j = 0, \dots, k - 1$ . It follows that  $F_\lambda = t^k G_\lambda$  for some  $G_\lambda \in$

$C_p^\infty(\mathbb{T}_H M; \Omega_r)$ . Thus, for each  $t \neq 0$ ,

$$\begin{aligned} G_\lambda|_t &= t^{-k}(\lambda^{-m}\alpha_{\lambda^*}\mathbb{P} - \mathbb{P})_t = t^{-k}(\lambda^{-m}\mathbb{P}_{\lambda t} - \mathbb{P}_t) \\ &= \lambda^{-m+k}\mathbb{Q}_{\lambda t} - \mathbb{Q}_t = (\lambda^{-m+k}\alpha_{\lambda^*}\mathbb{Q} - \mathbb{Q})_t, \end{aligned}$$

and this extends by continuity to  $t = 0$ . Thus  $\mathbb{Q} \in \Psi_H^{m-k}(M)$ .  $\square$

**Corollary 53.** *We have  $\Psi_H^{m-1}(M) = \ker \sigma_m \subseteq \Psi_H^m(M)$  for every  $m \in \mathbb{Z}$ .*

*Proof.* The map  $\Psi_H^{m-1}(M) \hookrightarrow \Psi_H^m(M)$  given by  $\mathbb{P} \mapsto t\mathbb{P}$  yields the inclusion  $\Psi_H^{m-1}(M) \hookrightarrow \ker \sigma_m$ . It is surjective by Proposition 51.  $\square$

This proves the short exact sequence (3) of the introduction.

## 5. SYMBOLS

Although Fourier transforms—"symbols"—play no role in our definition of pseudodifferential operators, they are (unsurprisingly) central to proving the main analytic properties. The major simplification afforded by the passage to the tangent groupoid is that the usual analytic estimates on symbol class functions which are necessary in standard approaches can be completely replaced by the condition of essential homogeneity.

We begin by collecting some preliminaries on Fourier transforms on graded vector bundles.

**5.1. Fourier transform.** Let  $E \xrightarrow{\pi} M$  be a smooth vector bundle over a manifold  $M$  and  $E' \xrightarrow{\pi'} M$  its dual bundle. The fibrewise Fourier transform  $\mathcal{F}_\pi : \mathcal{E}'_\pi(E) \rightarrow C^\infty(E')$ ;  $u \mapsto \hat{u}$  is defined by

$$\hat{u}(\eta) = (u_x, e^{-i(\eta, \cdot)}), \quad \text{where } \eta \in E'_x, x \in M. \quad (30)$$

**5.2. Schwartz functions on a vector bundle.** We recall the definition of the space of Schwartz functions on a smooth vector bundle (cf. [CR08, DS14]). Consider first a trivial bundle  $U \times \mathbb{R}^k \xrightarrow{\pi} U$  over an open set  $U \subseteq \mathbb{R}^n$ . We define  $\mathcal{S}_\pi(U \times \mathbb{R}^k) = C^\infty(U, \mathcal{S}(\mathbb{R}^k))$ , with the usual topology for  $C^\infty$ -functions valued in a Fréchet space. A defining family of seminorms on  $\mathcal{S}_\pi(U \times \mathbb{R}^k)$  is given by

$$\|f\|_{K, \mathbf{a}, \mathbf{b}, \mathbf{c}} = \sup_{x \in K, \xi \in \mathbb{R}^k} |\xi^{\mathbf{a}} \partial_\xi^{\mathbf{b}} \partial_x^{\mathbf{c}} f(x, \xi)|, \quad (31)$$

where  $K \subset U$  is compact and  $\mathbf{a}, \mathbf{b} \in \mathbb{N}^k$  and  $\mathbf{c} \in \mathbb{N}^n$  are multi-indices. This topology is invariant under smooth bundle isomorphisms: if  $\phi : U \times \mathbb{R}^k \rightarrow U' \times \mathbb{R}^k$  is a smooth linear bundle isomorphism with  $U' \subseteq \mathbb{R}^n$  then  $\phi^* : \mathcal{S}(U' \times \mathbb{R}^k) \rightarrow \mathcal{S}(U \times \mathbb{R}^k)$  is a continuous linear bijection.

Thus, if  $\pi : E \rightarrow M$  is a smooth vector bundle, we define  $\mathcal{S}_\pi(E)$  to be the space of all functions  $f : E \rightarrow \mathbb{C}$  such that for every trivializing chart  $\phi : U \times \mathbb{R}^k \rightarrow E$  we have  $\phi^* f \in \mathcal{S}_\pi(U \times \mathbb{R}^k)$ . It inherits a Fréchet topology from the spaces  $\mathcal{S}(U \times \mathbb{R}^k)$ .

We will also need Schwartz densities on a vector bundle. This is possible since the bundle of vertical densities  $\Omega_\pi$  is canonically trivial on each fibre. The above definitions generalize directly to define the space of Schwartz class vertical densities  $\mathcal{S}_\pi(E; \Omega_\pi)$ . The fibrewise Fourier transform induces an isomorphism of Fréchet spaces  $\mathcal{S}_\pi(E; \Omega_\pi) \rightarrow \mathcal{S}_{\pi'}(E')$ .

The space  $\mathcal{S}'_\pi(E)$  of  $\pi$ -fibred tempered distributions is the set of continuous  $C^\infty(M)$ -linear maps from  $\mathcal{S}_\pi(E)$  to  $C^\infty(M)$ . It is a subspace of  $\mathcal{D}'_\pi(E)$ . The fibrewise Fourier transform extends to an isomorphism  $\mathcal{S}'_\pi(E) \rightarrow \mathcal{S}'_{\pi'}(E'; \Omega_{\pi'}^{-1})$ .

**5.3. Fourier transform on graded vector bundles.** Now suppose  $E$  is a graded vector bundle. The dual bundle  $E'$  inherits a canonical grading  $E' = \bigoplus_i E'_i$  where

$$E'_i = \{\eta \in E' \mid (\eta, \xi) = 0 \text{ for all } \xi \in E_j, j \neq i \text{ with } \pi(\xi) = \pi'(\eta)\}.$$

The associated family of dilations on  $E'$  will be denoted by  $(\delta'_\lambda)_{\lambda \in \mathbb{R}}$ . They satisfy

$$(\delta'_\lambda \eta, \xi) = (\eta, \delta_\lambda \xi) \quad \text{for all } x \in M, \xi \in E_x, \eta \in E'_x \text{ and } \lambda \in \mathbb{R}.$$

Moreover, for any  $u \in \mathcal{E}'_\pi(E)$  we have

$$\widehat{\delta_{\lambda*} u} = \delta'_\lambda{}^* \hat{u}. \quad (32)$$

**5.4. Full symbols.** We can now use the fibrewise Fourier transform to define the full symbol of an  $H$ -pseudodifferential operator  $P \in \Psi_H^m(M)$ . More important is the Fourier transform of the family  $\mathbb{P} \in \Psi_H^m(M)$ , since this is where we see the essential homogeneity. In either case, some choices are necessary in the definition, as we now make precise.

Let  $\mathbb{P} \in \Psi_H^m(M)$ . As usual, we use an exponential cut-off function to ensure that  $\mathbb{P}$  is supported in an associated global exponential coordinate patch. Let  $\tilde{\mathbb{P}} \in \mathcal{E}'_r(\mathfrak{t}_H M \times \mathbb{R})$  be the pullback of  $\mathbb{P}$  via an exponential map  $\mathbb{E}xp^{\psi, \nabla}$  and let  $\hat{\mathbb{P}} \in C^\infty(\mathfrak{t}_H^* M \times \mathbb{R})$  be the fibrewise Fourier transform of  $\tilde{\mathbb{P}}$ . For brevity, we will refer to  $\hat{\mathbb{P}}$  as the Fourier transform of  $\mathbb{P}$  with respect to the given exponential.

**Definition 54.** If  $P = \mathbb{P}_1 \in \Psi_H^m(M)$ , we refer to  $\hat{P} = \hat{\mathbb{P}}_1 \in C^\infty(\mathfrak{t}_H^* M)$  as the *full symbol* of  $P$ , conscious of the fact that this depends on the choice of exponential coordinates and cut-off.

**Example 55.** Consider the case of  $M = \mathbb{R}^n$  with trivial filtration. We have  $\mathfrak{t}_H M = TM$ , and using the standard connection  $\nabla$ , the exponential map is

$$\begin{aligned} \mathbb{E}xp^\nabla : \mathfrak{t}_H M \times \mathbb{R} &\rightarrow \mathbb{T}_H M \\ (x, \zeta, t) &\mapsto (x, x - t\zeta, t), & t \neq 0 \\ (x, \zeta, 0) &\mapsto (x, \zeta, 0), & t = 0. \end{aligned}$$

If we write  $\text{Exp} = \mathbb{E}xp^\nabla|_{t=1}$  then  $\text{Exp}^{-1} : (x, y, 1) \mapsto (x, x - y, 1)$ . Therefore, the full cosymbol  $\hat{P}$  of  $P \in \Psi_H^m(M)$  in the sense of Definition 54 is defined by exactly the same formula which converts a classical pseudodifferential kernel to its symbol—see Equation (18.1.7) of [Hör85].

From Equation (24), the zoom action on  $\tilde{\mathbb{P}}$  is given by

$$(\tilde{\alpha}_{\lambda*} \tilde{\mathbb{P}})|_t = \delta_{\lambda*}(\tilde{\mathbb{P}}_{\lambda t}) \quad \text{for all } t \in \mathbb{R}, \lambda \in \mathbb{R}^\times.$$

Taking fibrewise Fourier transforms we get

$$\begin{aligned} \widehat{(\tilde{\alpha}_{\lambda*} \tilde{\mathbb{P}})}(x, \eta, t) &= (\delta_{\lambda*} \tilde{\mathbb{P}}_{(x, \lambda t)}, e^{-i(\eta, \cdot)}) \\ &= (\tilde{\mathbb{P}}_{(x, \lambda t)}, e^{-i(\delta'_\lambda(\eta), \cdot)}) = \hat{\mathbb{P}}(x, \delta'_\lambda \eta, \lambda t), \end{aligned}$$

for all  $(x, \eta, t) \in \mathfrak{t}_H^* M \times \mathbb{R}$ . That is,

$$\widehat{\tilde{\alpha}_{\lambda*} \tilde{\mathbb{P}}} = \beta_\lambda^* \hat{\mathbb{P}}, \quad (33)$$

where  $\beta_\lambda : \mathfrak{t}_H^* M \times \mathbb{R} \rightarrow \mathfrak{t}_H^* M \times \mathbb{R}$  is defined by

$$\beta_\lambda(\eta, x, t) = (x, \delta'_\lambda(\eta), \lambda t). \quad (34)$$

Therefore, under the Fourier transform, the essential homogeneity condition of Definition 35 gives the following.

**Lemma 56.** *Let  $\mathbb{P} \in \Psi_H^m(M)$ . Its Fourier transform (with respect to any exponential) satisfies*

$$\beta_\lambda^* \hat{\mathbb{P}} - \lambda^m \hat{\mathbb{P}} \in \mathcal{S}_{r'}(\mathfrak{t}_H^* M \times \mathbb{R}) \quad \text{for all } \lambda \in \mathbb{R}_+^\times.$$

Importantly, the maps  $\beta_\lambda$  are themselves dilations associated to a graded vector bundle, but over  $M$  not over  $M \times \mathbb{R}$  (see Figure 1). Specifically, we change our point of view to consider  $\mathfrak{t}_H^* M \times \mathbb{R}$  as a vector bundle over  $M$  via the bundle projection

$$\rho : \mathfrak{t}_H^* M \times \mathbb{R} \rightarrow M; \quad (x, \eta, t) \mapsto x. \quad (35)$$

Then  $(\beta_\lambda)$  is the family of dilations associated to the grading of  $\mathfrak{t}_H^* M \times \mathbb{R}$  where  $\mathfrak{t}_H^* M$  has its original grading and the complementary bundle  $M \times \mathbb{R}$  is given grading degree 1.

FIGURE 1. The zoom action in exponential coordinates on  $t_H M \times \mathbb{R}$ , and its Fourier transform on  $t_H^* M \times \mathbb{R}$ .

*Remark 57.* Classical symbols of order  $m$  are usually defined in terms of the symbol inequalities

$$|\partial_x^{\mathbf{a}} \partial_\eta^{\mathbf{b}} a(x, \eta)| \leq C(1 + \|\eta\|)^{m - |\mathbf{b}|}, \quad (x, \eta) \in T^*M. \quad (36)$$

Note that these inequalities would be satisfied automatically if  $a \in C^\infty(T^*M)$  were a homogeneous function of order  $m$  at infinity, perhaps with some Schwartz-class perturbation. A full symbol  $a := \hat{P} \in C^\infty(T^*M)$  is generally not homogeneous in this way, but its extension  $\hat{\mathbb{P}}$  to  $C^\infty(T^*M \times \mathbb{R})$  will be homogeneous at infinity with respect to the *augmented dilations*  $(\beta_\lambda)$ ; see Proposition 66 below for a precise statement. And  $(\beta_\lambda)$ -homogeneity implies all the symbol inequalities (36) upon restricting to  $t = 1$ .

This simple observation explains the economy which is gained by passing to the tangent groupoid. For instance, in the case of the Heisenberg calculus, a great deal of the technical analysis of the book [BG88] is absorbed by this remark.

However, we should point out that the homogeneity condition of Lemma 56 for  $\hat{\mathbb{P}}$  is not *a priori* the same as saying that  $\hat{\mathbb{P}}$  is equal to a Schwartz class perturbation of a genuinely homogeneous function. For this, we need an embellishment of a well-known lemma (see e.g. [Tay, Proposition 2.2]) which lets us extract a homogeneous part from  $\hat{\mathbb{P}}$ . We tackle this in the next section.

### 5.5. Essential homogeneity of Fourier transforms.

**Lemma 58.** *Let  $\pi : E \rightarrow M$  be a graded vector bundle, and let  $\delta_\lambda$  and  $\delta'_\lambda$  denote the associated dilations on  $E$  and  $E'$ . The following conditions on a  $\pi$ -fibred tempered distribution  $u \in \mathcal{S}'_\pi(E)$  are equivalent:*

- (1)  $u \in \mathcal{E}'_\pi(E) + \mathcal{S}_\pi(E; \Omega_\pi)$  and satisfies

$$\delta_{\lambda*} u - \lambda^m u \in \mathcal{S}_\pi(E; \Omega_\pi) \quad \text{for all } \lambda \in \mathbb{R}_+^\times,$$

- (2)  $\hat{u} \in C^\infty(E')$  and satisfies

$$\delta_\lambda^* \hat{u} - \lambda^m \hat{u} \in \mathcal{S}_{\pi'}(E') \quad \text{for all } \lambda \in \mathbb{R}_+^\times,$$

- (3)  $\hat{u} \in C^\infty(E')$  and there exists a function  $f \in C^\infty(E' \setminus (M \times \{0\}))$  which is homogeneous on the nose, i.e.

$$\delta_\lambda^* f - \lambda^m f = 0 \quad \text{for all } \lambda \in \mathbb{R}_+^\times,$$

and such that  $\hat{u} - f$  is Schwartz class at infinity, i.e. given any smooth bump function  $\phi \in C^\infty(E)$  which is 0 on some neighbourhood of  $M \times \{0\}$  and 1 outside some  $\pi'$ -proper set, we have  $\hat{u} - \phi f \in \mathcal{S}_{\pi'}(E')$ .

When these conditions hold,  $\text{sing-supp}(u) \subseteq M \times \{0\}$ .

**Definition 59.** The function  $f \in C^\infty(E' \setminus M \times \{0\})$  in Condition (3) of Lemma 58 will be called the *homogeneous part* of  $\hat{u}$ .

*Remark 60.* Note that there is no analogue of the equivalence (2)  $\Leftrightarrow$  (3) for the singularity of  $u$  at 0: the logarithmic kernel  $\hat{\mathbb{P}}$  of Example 39 does not differ from any genuinely homogeneous distribution on  $TM \times \mathbb{R}$  by a smooth density. Thus, one can extract a homogeneous part from  $\hat{\mathbb{P}}$  at infinity, but not from  $\mathbb{P}$  near the diagonal.

*Proof of Lemma 58.* (1)  $\Rightarrow$  (2): Immediate from Equation (32).

(2)  $\Rightarrow$  (3): For  $\lambda > 1$ , let  $g_\lambda = \lambda^{-m} \delta_\lambda^{*} \hat{u} - \hat{u}$ . For any  $n \in \mathbb{N}$  we have

$$\lambda^{-mn} \delta_{\lambda^n}^{*} \hat{u}(x, \eta) = \hat{u}(x, \eta) + \sum_{k=0}^{n-1} \lambda^{-mk} \delta_{\lambda^k}^{*} g_\lambda(x, \eta), \quad (37)$$

where  $(x, \eta) \in E'$ . Since  $g_\lambda$  is Schwartz class, the sum on the right hand side of (37) converges as  $n \rightarrow \infty$  for any  $(x, \eta) \in E' \setminus (M \times \{0\})$ , and the limit  $\sum_{k=0}^{\infty} \lambda^{-mk} \delta_{\lambda^k}^{*} g_\lambda$  is a function in  $C^\infty(E' \setminus (M \times \{0\}))$  all of whose derivatives decay superpolynomially at infinity. In particular, if we define

$$f = \lim_{n \rightarrow \infty} 2^{-mn} \delta_{2^n}^{*} \hat{u} \in C^\infty(E \setminus (M \times \{0\})), \quad (38)$$

then  $f - \hat{u}$  is Schwartz class at infinity.

Note that we also have  $f = \lim_{n \rightarrow \infty} 2^{-qmn} \delta_{2^{qn}}^{*} \hat{u}$  for any positive rational number  $q$ , since this sequence also converges and has a common subsequence with (38). It follows that  $\lambda^{-m} \delta_\lambda^{*} f = f$  for any  $\lambda = 2^q$  with  $q \in \mathbb{Q}$ . By continuity, this holds for all  $\lambda \in \mathbb{R}$ .

(3)  $\Rightarrow$  (1): The essential homogeneity condition on  $u$  is straightforward. The non-trivial part is to show that  $u \in \mathcal{E}'_\pi(E) + \mathcal{S}_\pi(E; \Omega_\pi)$ .

We work in a trivializing bundle coordinates  $(x, \eta) \in M_0 \times \mathbb{R}^k \hookrightarrow E'$ , where  $M_0 \subset \mathbb{R}^n$  is open with relatively compact image in  $M$ . Fix multi-indices  $\mathbf{a}, \mathbf{b} \in \mathbb{N}^k$ ,  $\mathbf{c} \in \mathbb{N}^n$ . Then  $\partial_\eta^{\mathbf{a}} \eta^{\mathbf{b}} \partial_x^{\mathbf{c}} f$  is homogeneous of degree  $m - |\mathbf{a}|_H + |\mathbf{b}|_H$ , and  $\partial_\eta^{\mathbf{a}} \eta^{\mathbf{b}} \partial_x^{\mathbf{c}} \hat{u} - \partial_\eta^{\mathbf{a}} \eta^{\mathbf{b}} \partial_x^{\mathbf{c}} f$  is Schwartz class at infinity. In particular, if  $|\mathbf{a}|_H \geq m + |\mathbf{b}|_H + d_H + 1$  we obtain a bound

$$|\partial_\eta^{\mathbf{a}} \eta^{\mathbf{b}} \partial_x^{\mathbf{c}} \hat{u}(x, \eta)| \leq C(1 + \|\eta\|_H)^{-d_H - 1}, \quad \text{for all } (x, \eta) \in E',$$

where  $\|\cdot\|_H$  is any homogeneous norm on  $E$  and  $d_H$  is the homogeneous dimension. This decay is sufficient to be integrable in

$\eta$  (see [NS79]). By the Lebesgue Dominated Convergence Theorem,  $x \mapsto \partial_{\eta}^{\mathbf{a}} \eta^{\mathbf{b}} \partial_x^{\mathbf{c}} \hat{u}(x, \cdot)$  is continuous function from  $M_0$  to  $L^1(\mathbb{R}^k)$ . Taking the inverse Fourier transform, for any fixed  $\mathbf{b}, \mathbf{c}$  we have that  $\zeta^{\mathbf{a}} \partial_{\zeta}^{\mathbf{b}} \partial_x^{\mathbf{c}} u$  is a bounded continuous vertical density on  $M_0 \times \mathbb{R}^k$  for all  $|\mathbf{a}|_H$  sufficiently large. This implies that  $u$  is smooth outside of  $\zeta = 0$ , and also Schwartz class at infinity. It follows that  $u \in \mathcal{E}'_{\pi}(E) + \mathcal{S}_{\pi}(E; \Omega_{\pi})$ .

This final argument also shows that the singular support of  $u$  lies in the zero section.  $\square$

*Remark 61.* If we wished, we could use Lemma 58 to define principal symbols of  $H$ -pseudodifferential operators, as follows. If  $P \in \Psi_H^m(M)$ , then its cosymbol  $\sigma_m(P)$  is represented by an essentially homogeneous proper  $r$ -fibred distribution on  $T_H M \cong \mathfrak{t}_H M$ , unique modulo smooth vertical densities. We may therefore define the *principal symbol*  $\hat{\sigma}_m(P) \in C^{\infty}(\mathfrak{t}_H^* M \setminus (M \times \{0\}))$  of  $P$  to be the homogeneous part of the fibrewise Fourier transform of its cosymbol. We may further identify this principal symbol with a function on the cosphere bundle  $S^* M$  if we identify the cosphere bundle with the unit sphere bundle for some homogeneous norm on  $\mathfrak{t}_H^* M$ .

Note, though, that the composition of  $H$ -pseudodifferential operators does not correspond to the pointwise product of principal symbols unless the osculating groups  $T_H M_x$  are all abelian. One would need a noncommutative product to obtain such a correspondence in general. In the case of the Heisenberg calculus, the appropriate non-commutative product is described explicitly by Epstein and Melrose in their unpublished book [EM].

We will not pursue this further here.

Our application of Lemma 58 will instead be to the full family  $\hat{\mathbb{P}}$ . Unfortunately, one technical point prevents a direct application of Lemma 58 to  $\hat{\mathbb{P}}$ : the essential homogeneity of  $\hat{\mathbb{P}}$  in Lemma 56 is modulo  $\mathcal{S}_{\rho'}(\mathfrak{t}_H^* M \times \mathbb{R})$ , but we need it modulo  $\mathcal{S}_{\rho}(\mathfrak{t}_H^* M \times \mathbb{R})$  where  $\rho : \mathfrak{t}_H^* M \times \mathbb{R} \rightarrow M$  is the augmented bundle fibration of (35).

The resolution of this issue is to modify  $\mathbb{P}$  near  $t = \pm\infty$  so that the associated cocycle  $F_{\lambda} = \lambda^{-m} \alpha_{\lambda*} \mathbb{P} - \mathbb{P}$  is actually zero for  $t$  outside of some compact interval.

**Definition 62.** An  $r$ -fibred distribution  $\mathbb{P} \in \Psi_H^m(M)$  is called *homogeneous on the nose outside*  $[-1, 1]$  if for all  $\lambda > 1$ ,

$$\text{supp}(\alpha_{\lambda*} \mathbb{P} - \lambda^m \mathbb{P}) \subset \mathbb{T}_H M|_{[-1, 1]}.$$

**Proposition 63.** Any  $H$ -pseudodifferential operator  $P$  admits an extension to  $\mathbb{P} \in \Psi_H^m(M)$  which is homogeneous on the nose outside  $[-1, 1]$ .



This is not a profound point, but it is surprisingly technical. The key difficulty will come down to showing that the cocycle

$$F : \mathbb{R}_+^\times \rightarrow C_p^\infty(\mathbb{T}_H M; \Omega_r)$$

is smooth as a function of  $\lambda \in \mathbb{R}_+^\times$ . The appropriate topology on  $C_p^\infty(\mathbb{T}_H M; \Omega_r)$  is the  $LF$ -topology given by

$$C_p^\infty(\mathbb{T}_H M; \Omega_r) = \varinjlim C^\infty(X_i; \Omega_r),$$

where  $(X_i)_{i \in \mathbb{N}}$  is a countable exhaustion of  $\mathbb{T}_H M$  by proper subsets which are the closure of their interiors. Note that it is obvious that  $F$  is smooth with respect to the topology on  $\mathcal{E}'_r(\mathbb{T}_H M)$ .

Again, we could avoid this technicality by forcing the smoothness of  $\lambda \mapsto F_\lambda$  into the definition of  $\Psi_H^m(M)$ . The reader willing to do so, may then skip over the following lemma and go directly to the proof of Proposition 63.

**Lemma 64.** *For any  $\mathbb{P} \in \Psi_H^m(M)$ , the associated cocycle  $F : \lambda \mapsto \lambda^{-m} \alpha_{\lambda*} \mathbb{P} - \mathbb{P}$  is a smooth function of  $\mathbb{R}_+^\times$  into  $C_p^\infty(\mathbb{T}_H M; \Omega_r)$ .*

*Proof.* Note that, for  $\lambda$  in any fixed closed interval  $[a, b] \subset \mathbb{R}_+^\times$ , the functions  $F_\lambda$  are all supported in a common proper set  $X := \bigcup_{\lambda \in [a, b]} \alpha_\lambda(\text{supp } \mathbb{P})$ , so it will suffice to prove that  $F$  is smooth as a function into  $C^\infty(\mathbb{T}_H M; \Omega_r)$ . To this end, we consider the restriction of  $F_\lambda$  to the compact set  $X_0 = X \cap r^{-1}(\overline{M_0})$  where  $M_0$  is any relatively compact open subset of  $M \times \mathbb{R}$ .

The key step is to prove the continuity of  $F$ , for which we use a Baire category argument. Fix  $R > 0$  and  $k \in \mathbb{N}$ . We put

$$I_{k,R} = \{\lambda \in \mathbb{R}_+^\times \mid \|F_\lambda|_{X_0}\|_{C^k} \leq R\} \subseteq \mathbb{R}_+^\times.$$

The set of functions

$$\{F_\lambda|_{X_0} \mid \lambda \in I_{k,R}\}$$

is relatively compact in the  $C^{k-1}$ -topology by the Arzela-Ascoli Theorem. Suppose now that  $\mu \in \mathbb{R}_+^\times$  is in the closure of  $I_{k,R}$ . Let  $\lambda_i \in I_{k,R}$  with  $\lambda_i \rightarrow \mu$ . Then  $F_{\lambda_i} \rightarrow F_\mu$  in the  $\mathcal{E}'_r$ -topology but also a subsequence of  $F_{\lambda_i}|_{X_0}$  converge in the  $C^{k-1}$ -topology. Therefore  $F_\mu|_{X_0} \in C^{k-1}(X_0; \Omega_r)$ . This proves that  $\overline{I_{k,R}} \subseteq I_{k-1,R}$ . Moreover,  $\lambda \mapsto F_\lambda|_{X_0}$  is continuous from  $\overline{I_{k,R}}$  to  $C^{k-1}(X_0; \Omega_r)$ .

Since  $\mathbb{R}_+^\times = \bigcup_{R \in \mathbb{N}} I_{k,R}$ , the Baire Category Theorem shows there is some  $R \in \mathbb{N}$  such that  $\overline{I_{k,R}}$  contains an open interval  $(a, b)$ . Now let  $t \in \mathbb{R}_+^\times$  be arbitrary and consider the function

$$\mathbb{R}_+^\times \rightarrow C^{k-1}(X_0; \Omega_r) \tag{39}$$

$$\lambda \mapsto F_{\lambda t}|_{X_0} = \lambda^{-m} \alpha_{\lambda*} F_t|_{X_0} + F_\lambda|_{X_0}$$

where the latter equality is the cocycle relation (27). Note that  $\lambda \mapsto \lambda^{-m} \alpha_{\lambda*} F_t$  is a smooth function of  $\mathbb{R}_+^\times$  into  $C^\infty(\mathbb{T}_H M; \Omega_r)$ , so by the previous paragraph (39) is continuous for  $\lambda \in (a, b)$ . In other words  $\lambda \mapsto F_\lambda|_{X_0}$  is continuous from  $(ta, tb)$  to  $C^{k-1}(X_0; \Omega_r)$ , and hence continuous on all of  $\mathbb{R}_+^\times$ . This is true for any  $k$  and any  $X_0$ , so  $F : \mathbb{R}_+^\times \rightarrow C^\infty(\mathbb{T}_H M; \Omega_r)$  is continuous.

To pass from continuity to smoothness will use a convolution trick. Pick a compactly supported bump function  $\phi \in C_c^\infty(\mathbb{R}_+^\times)$  with  $\int_{\mathbb{R}_+^\times} \phi(t) \frac{dt}{t} = 1$ , and define  $\mathbb{Q} \in \mathcal{E}'(\mathbb{T}_H M; \Omega_r)$  as an average of translates of  $\mathbb{P}$ :

$$\mathbb{Q} := \int_{\mathbb{R}_+^\times} \phi(t) t^{-m} (\alpha_{t*} \mathbb{P}) \frac{dt}{t} \in \mathcal{E}'_s(\mathbb{T}_H M).$$

Then

$$\mathbb{Q} - \mathbb{P} = \int_{\mathbb{R}_+^\times} \phi(t) F_t \frac{dt}{t} \in C^\infty(\mathbb{T}_H M; \Omega_s),$$

since the integrand is continuous and compactly supported. Therefore  $\lambda \mapsto \lambda^{-m} \alpha_{\lambda*} (\mathbb{Q} - \mathbb{P})$  is a smooth function  $\mathbb{R}_+^\times \rightarrow C^\infty(\mathbb{T}_H M; \Omega_s)$ . Also,

$$\begin{aligned} (\lambda^{-m} \alpha_{\lambda*} \mathbb{Q}) - \mathbb{P} &= \int_{\mathbb{R}_+^\times} \phi(t) ((\lambda t)^{-m} \alpha_{\lambda t*} \mathbb{P} - \mathbb{P}) \frac{dt}{t} \\ &= \int_{\mathbb{R}_+^\times} \phi(\lambda^{-1} t) F_t \frac{dt}{t}, \end{aligned}$$

and by differentiation under the integral we see that this is a smooth function of  $\lambda \in \mathbb{R}_+^\times$  into  $C^\infty(\mathbb{T}_H M; \Omega_s)$ . Writing  $F_\lambda = \lambda^{-m} \alpha_{\lambda*} (\mathbb{P} - \mathbb{Q}) + (\lambda^{-m} \alpha_{\lambda*} \mathbb{Q} - \mathbb{P})$ , we obtain that  $F$  is smooth.  $\square$

*Proof of Proposition 63.* Let  $P \in \Psi_H^m(M)$ . Choose any  $\mathbb{Q} \in \Psi_H^m(M)$  with  $\mathbb{Q}_1 = P$ . Define  $r$ -fibred distributions  $\mathbb{P}^\pm$  on  $\mathbb{T}_H M|_{\mathbb{R}_\pm^\times} = M \times M \times \mathbb{R}_\pm^\times$  by  $\mathbb{P}_t^\pm = |t|^m \mathbb{Q}_{\pm 1}$ . These are homogeneous of weight  $m$  on the nose:  $\alpha_{\lambda*} \mathbb{P}^\pm = \lambda^m \mathbb{P}^\pm$  for all  $\lambda > 0$ . Moreover, for all  $t > 0$  we have

$$\mathbb{P}_{\pm t}^\pm - \mathbb{Q}_{\pm t} = (t^m \alpha_{t^{-1}*} \mathbb{Q} - \mathbb{Q})_{\pm t},$$

which, by the previous lemma, is a smooth function of  $t$  valued in  $C_p^\infty(M \times M; \Omega_r)$ . That is,  $\mathbb{P}^\pm - \mathbb{Q}|_{\mathbb{R}_\pm^\times} \in C_p^\infty(\mathbb{T}_H M|_{\mathbb{R}_\pm^\times}; \Omega_r)$ .

Now let  $\{\rho^-, \rho^0, \rho^+\}$  be a smooth partition of unity on  $\mathbb{R}$  with supports in  $(-\infty, -\frac{1}{2})$ ,  $(-1, 1)$ ,  $(\frac{1}{2}, \infty)$ , respectively. Define  $\mathbb{P}$  by

$$\mathbb{P}_t = \rho^-(t) \mathbb{P}_t^- + \rho^0(t) \mathbb{Q}_t + \rho^+(t) \mathbb{P}_t^+ \quad (t \in \mathbb{R}).$$

Then  $\mathbb{P} - \mathbb{Q} \in C_p^\infty(\mathbb{T}_H M; \Omega_r)$ , so  $\mathbb{P} \in \Psi_H^m(M)$ . It is homogeneous on the nose outside  $[-1, 1]$ , and  $\mathbb{P}_1 = P$ .  $\square$

*Remark 65.* Note also that our exponential cut-off functions (Section 4.2) are also homogeneous on the nose. So after a cut-off we may assume  $P \in \Psi_H^m(M)$  extends to  $\mathbb{P} \in \Psi_H^m(M)$  which is simultaneously homogeneous on the nose outside  $[-1, 1]$  and supported in a global exponential coordinate patch.

After this rather lengthy digression, we can finally apply Lemma 58 to the an element  $\mathbb{P} \in \Psi_H^m(M)$ .

**Proposition 66.** *Let  $\mathbb{P} \in \Psi_H^m(M)$  be homogeneous on the nose outside  $[-1, 1]$ . Its Fourier transform  $\hat{\mathbb{P}} \in C^\infty(\mathfrak{t}_H^*M \times \mathbb{R})$  (with respect to any exponential coordinates) satisfies*

$$\beta_\lambda^* \hat{\mathbb{P}} - \lambda^m \hat{\mathbb{P}} \in \mathcal{S}_\rho(\mathfrak{t}_H^*M \times \mathbb{R}) \quad \text{for all } \lambda \in \mathbb{R}_+^\times.$$

Therefore  $\hat{\mathbb{P}}$  is equal, modulo a function in  $\mathcal{S}_\rho(\mathfrak{t}_H^*M \times \mathbb{R})$ , to a smooth function on  $\mathfrak{t}_H^*M \times \mathbb{R}$  which is genuinely homogeneous of order  $m$  at infinity with respect to the dilations  $\beta_\lambda$ .

## 5.6. Symbol estimates.

**Corollary 67.** *Let  $\mathbb{P} \in \Psi_H^m(M)$  be homogeneous on the nose outside  $[-1, 1]$ . Fix a local exponential coordinate system  $(x, \xi, t) \in M_0 \times V \times \mathbb{R} \hookrightarrow \mathfrak{t}_H^*M \times \mathbb{R}$  with  $\overline{M_0}$  compact, as in Section 3.9 and denote the dual coordinates by  $(x, \eta, t) \in M_0 \times V' \times \mathbb{R} \hookrightarrow \mathfrak{t}_H^*M \times \mathbb{R}$ . The fibrewise Fourier transform of  $\mathbb{P}$  in these coordinates satisfies the following estimates: for any multi-indices  $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{N}^n$ ,  $k \in \mathbb{N}$ , there is  $C = C_{M_0, \mathbf{a}, \mathbf{b}, \mathbf{c}, d} > 0$  such that for all  $(x, \eta, t) \in \overline{M_0} \times V' \times \mathbb{R}$ ,*

$$|\eta^{\mathbf{a}} \partial_\eta^{\mathbf{b}} \partial_x^{\mathbf{c}} \partial_t^k \hat{\mathbb{P}}(x, \eta, t)| \leq C(1 + \|\eta\|_H + |t|)^{m + |\mathbf{a}|_H - |\mathbf{b}|_H - k}, \quad (40)$$

where  $\|\cdot\|_H$  is any homogeneous norm on  $V'$ .

*Proof.* From Proposition 66 the function  $\eta^{\mathbf{a}} \partial_\eta^{\mathbf{b}} \partial_x^{\mathbf{c}} \partial_t^k \hat{\mathbb{P}}$  is homogeneous at infinity of order  $m + |\mathbf{a}|_H - |\mathbf{b}|_H - k$ , modulo a function in  $\mathcal{S}_\rho(\mathfrak{t}_H^*M \times \mathbb{R})$ .  $\square$

We are now very close to the classical theory of pseudodifferential operators of type  $\Psi_{\rho, \delta}^m$ . Rather than making a beeline for the established theory, let us prove the fundamental properties of pseudodifferential operators directly from the analysis above. To the expert, the following results will be no surprise, in light of Corollary 67.

**5.7. Conormality.** The first consequence of the estimates of Corollary 67 is that our  $H$ -psuedodifferential kernels are proper distributions, i.e. both  $r$ - and  $s$ -fibered. Recall that this was already stated in Proposition 42. We will prove this by showing they have wavefront sets which are conormal to the diagonal.

In order to set this up, let us return briefly to fibred distributions on a smooth fibration  $\pi : M \rightarrow B$  as in Section 2.1. Recall that one can use a positive smooth density  $\mu$  on the base  $B$  to convert  $\pi$ -fibred distributions into ordinary distributions via the map

$$\mu_\pi : \mathcal{E}'_\pi(M) \rightarrow \mathcal{D}'(M); \quad \langle \mu_\pi u, f \rangle = \int_{x \in B} \langle u, f \rangle(x) \mu(x) \quad (41)$$

where  $u \in \mathcal{E}'_\pi(M)$ ,  $f \in C_c^\infty(M)$ . In [LMV15] it is shown that one can use wavefront conditions to determine whether a global distribution  $v \in \mathcal{D}'(M)$  is in the image of  $\mu_\pi$ . For our purposes, the following sufficient condition will be appropriate.

**Lemma 68.** *Suppose  $B \subset M$  and that the submersion  $\pi : M \rightarrow B$  is a smooth retract. Suppose  $v \in \mathcal{D}'(M)$  has  $\pi$ -proper support, singular support contained in  $B$  and wave front set contained in the conormal to  $B$ :*

$$\text{WF}(v) \subseteq \{(b, \eta) \in T^*M|_B \mid (\eta, \xi) = 0 \text{ for all } \xi \in T_b B\}. \quad (42)$$

Then  $v = \mu_\pi u$  for some  $u \in \mathcal{E}'_\pi(M)$ .

*Proof.* Since  $TB$  is complementary to the tangent space to the fibres at every point, the conormal bundle of  $B$  intersects trivially with the conormal bundle of any fibre. Now apply [LMV15, Proposition 7].  $\square$

We now prove that the conormality condition (42) holds for essentially homogeneous  $r$ -fibred distributions on  $\mathbb{T}_H M$ . Let us fix any transverse measure  $\mu$  on the base of  $\mathbb{T}_H M$ .

**Proposition 69.** *Let  $\mathbb{P} \in \Psi_H^m(M)$ . The wave front set of the distribution  $\mu_r(\mathbb{P}) \in \mathcal{D}'(\mathbb{T}_H M)$  is conormal to the object space  $\mathbb{T}_H M^{(0)} \cong M \times \mathbb{R}$ , i.e.*

$$\begin{aligned} \text{WF}(\mu_r(\mathbb{P})) \subseteq \{ \eta \in T_{(x,t)}^*(\mathbb{T}_H M) \mid (x,t) \in \mathbb{T}_H M^{(0)} \\ \text{and } (\eta, \xi) = 0 \text{ for all } \xi \in T_{(x,t)}(\mathbb{T}_H M^{(0)}) \}. \end{aligned}$$

*Proof of Proposition 69.* We work in local exponential coordinates as in Section 3.9. We assume, without loss of generality, that  $\mathbb{P}$  is homogeneous on the nose outside  $[-1, 1]$ . We know that  $\mu_r(\mathbb{P})$  is smooth off the zero section (Proposition 41). Fix  $(x_0, 0, t_0) \in M_0 \times V \times \mathbb{R}$ . Let  $\chi \in C^\infty(M \times \mathbb{R})$  be a smooth bump function which is 1 on some neighbourhood of  $(x, t)$  and supported in  $M_0 \times I$  for some compact interval  $I \subset \mathbb{R}$ . Note that multiplication by  $\chi$  commutes with the fibrewise Fourier transform in  $\xi \in V$ , so  $\widehat{\chi \mathbb{P}}$  still satisfies the decay conditions of Corollary (67) in  $M_0 \times V' \times I$ . In particular, for any  $\mathbf{c} \in \mathbb{N}^n, k \in \mathbb{N}$

$$|\partial_x^{\mathbf{c}} \partial_t^k \widehat{\chi \mathbb{P}}(x, \eta, t)| \leq C_{\mathbf{c},k} (1 + \|\eta\|_{V'})^m \quad \text{for all } (x, \eta, t) \in M_0 \times V' \times I,$$

for some constant  $C_{\mathbf{c},k}$ .

Now we apply the Fourier transform  $\mathcal{F}_{(x,t) \rightarrow (y,s)}$  in the base variables. We obtain bounds

$$|y^{\mathbf{c}} s^k \mathcal{F}_{(x,\xi,t) \rightarrow (y,\eta,s)}(\chi^{\mathbb{P}})(y, \eta, s)| \leq C_{\mathbf{c},k} \mu(M_1 \times I) (1 + \|\eta\|_{V'})^m$$

for every  $\mathbf{c} \in \mathbb{N}^n$ ,  $k \in \mathbb{N}$ . It follows that every  $(y, \eta, s) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$  with  $(y, s) \neq (0, 0)$  admits a conical neighbourhood upon which  $\mathcal{F}_{(x,\xi,t) \rightarrow (y,\eta,s)}(\chi^{\tilde{\mathbb{P}}})$  has rapid decay. This is precisely the conormality of the wave front set.  $\square$

The properness of  $H$ -pseudodifferential operators (Proposition 42) is now an immediate corollary of Proposition 69 and Lemma 68.

**5.8. Regularity.** The second consequence of the decay conditions (40) is the regularity of  $H$ -pseudodifferential kernels of large negative order.

**Lemma 70.** *Let  $M$  be a filtered manifold of dimension  $n$  and let  $\mathbb{P} \in \Psi_H^{-d_H-k-1}(M)$  for some  $k \geq 0$ . Let  $M_0 \times V \times \mathbb{R} \rightarrow \mathbb{T}_H M$  be any local exponential coordinate system with coordinates  $(x, \xi, t)$ . For any multi-indices  $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$ ,  $c \in \mathbb{N}$  with  $|\mathbf{a}|_H \leq k$ , we have*

$$\partial_{\xi}^{\mathbf{a}} \partial_x^{\mathbf{b}} \partial_t^c \tilde{\mathbb{P}} \in C(M_0 \times V \times \mathbb{R}; \Omega_s),$$

where  $\tilde{\mathbb{P}}$  is the pullback of  $\mathbb{P}$  via local exponential coordinates.

*Proof.* We assume  $\mathbb{P}$  is supported in a global exponential coordinate chart and that it is homogeneous on the nose outside  $[-1, 1]$ . Corollary 67 applies. We deduce that for any multi-indices  $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$ ,  $c \in \mathbb{N}$  with  $|\mathbf{a}|_H \leq k$ , for each  $(x, t) \in M_0 \times \mathbb{R}$ , the function  $\eta \mapsto \eta^{\mathbf{a}} \partial_x^{\mathbf{b}} \partial_t^c \hat{\mathbb{P}}(x, \eta, t)$  is bounded in absolute value by  $C(1 + \|\eta\|_H)^{-d_H-1}$ , which is integrable (see [NS79]). By the dominated convergence theorem we obtain  $\eta^{\mathbf{a}} \partial_x^{\mathbf{b}} \partial_t^c \hat{\mathbb{P}} \in C(M_0 \times \mathbb{R}; L^1(V'))$ . Taking fibrewise Fourier transforms, we obtain  $\partial_{\xi}^{\mathbf{a}} \partial_x^{\mathbf{b}} \partial_t^c \tilde{\mathbb{P}} \in C(M_0 \times V \times \mathbb{R}; \Omega_r)$ .  $\square$

**Theorem 71.** *Let  $M$  be a smooth manifold equipped with a Lie filtration of depth  $N$  and homogeneous dimension  $d_H$ . For any  $k \geq 0$  we have  $\Psi_H^{-d_H-kN-1}(M) \subseteq C^k(\mathbb{T}_H M; \Omega_r)$  and  $\Psi_H^{-d_H-kN-1}(M) \subseteq C^k(M \times M; \Omega_r)$ .*

*Proof.* Follows immediately from Lemma 70 and  $|\mathbf{a}|_H \leq N|\mathbf{a}|$ .  $\square$

**Definition 72.** We define  $\Psi_H^{-\infty}(M) = \bigcap_{m \in \mathbb{Z}} \Psi_H^m(M)$ .

**Corollary 73.** *We have  $\Psi_H^{-\infty}(M) = \Psi^{-\infty}(M) = C_p^{\infty}(M \times M)$ , the algebra of properly supported smoothing operators on  $M$ .*

6.  $H$ -ELLIPTICITY

**6.1. Asymptotic expansions.** In order to construct parametrices, we need convergence of asymptotic expansions. We will be working with cosymbols rather than symbols, which means that we want to make sense of asymptotic series of the form

$$P \sim Q_m + Q_{m-1} + Q_{m-2} + \cdots$$

where  $Q_k \in \Psi_H^k(M)$  and the sign  $\sim$  signifies that  $P - \sum_{i=0}^{j-1} Q_{m-i} \in \Psi_H^{m-j}(M)$  for each  $j \in \mathbb{N}$ . We can phrase this succinctly as follows.

**Definition 74.** We write  $\Phi_H^m(M) = \Psi_H^m(M)/\Psi^{-\infty}(M)$ . The class  $[P] \in \Phi_H^m(M)$  of an  $H$ -pseudodifferential operator  $P \in \Psi_H^m(M)$  will be called the *full cosymbol* of  $P$ .

The graded algebra of full cosymbols is  $\Phi_H^\infty(M) = \bigcup_{m \in \mathbb{Z}} \Phi_H^m(M)$ . The family of subspaces  $(\Phi_H^m(M))_{m \in \mathbb{Z}}$  form a local base at 0 for a vector space topology on  $\Phi_H^\infty(M)$ , which we call the *topology of asymptotic convergence*.

Concretely, for  $(P_n)_{n \in \mathbb{N}} \subset \Psi_H^\infty(M)$  the full cosymbols  $[P_n]$  converge asymptotically to  $[P]$  if and only if for all  $m \in \mathbb{Z}$  there is  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$ ,  $P - P_n \in \Psi_H^{-m}(M)$ . For brevity, we say  $(P_n)$  converges asymptotically to  $P$ .

*Remark 75.* Note that we have restricted our attention here to integer orders  $m \in \mathbb{Z}$ . The case of non-integer orders will be discussed in Remark 77.

A sequence  $([P_n])_{n \in \mathbb{N}} \subset \Phi_H^\infty(M)$  is *Cauchy* if for all  $m \in \mathbb{Z}$  there exists  $n_0 \in \mathbb{N}$  such that for all  $n, n' \geq n_0$  we have  $P_n - P_{n'} \in \Psi_H^{-m}(M)$ .

**Theorem 76.** *The space  $\Phi_H^\infty(M)$  of full cosymbols is complete in the topology of asymptotic convergence, in the sense that every Cauchy sequence admits a limit.*

*Proof.* By rearranging terms and neglecting a finite number of them, it suffices to prove that a sum  $\sum_{k=0}^{\infty} [P_k]$  admits a limit for any sequence  $(P_k)_{k \geq 0}$  with  $P_k \in \Psi_H^{-d_H - kN - 1}(M)$ .

For each  $k$ , choose  $Q_k \in \Psi_H^{-d_H - kN - 1}(M)$  extending  $P_k$ . We assume that  $Q_k$  is homogeneous on the nose outside  $[-1, 1]$  and supported in an exponential coordinate patch. We write  $\tilde{Q}_k$  for the pullback of  $Q_k$  to some local exponential coordinates  $(x, \xi, t) \in M_0 \times V \times \mathbb{R} \hookrightarrow \mathfrak{t}_H M \times \mathbb{R}$ . We trivialize the  $r$ -density bundle  $\Omega_r$  by using the constant family of Lebesgue measures  $\omega$  on  $V$ . By Theorem 71,  $\tilde{Q}_k = q_k \omega$  for some  $q_k \in C^k(M_0 \times V \times \mathbb{R})$ .

Let  $a_k : M_0 \times V \times \mathbb{R} \rightarrow \mathbb{C}$  be the function which is polynomial of order  $k$  in  $\xi \in V$  and whose vertical derivatives to order  $k$  agree with those of  $q_k$  on the base space, i.e.

$$\partial_{\xi}^{\mathbf{a}} a_k(x, 0, t) = \partial_{\xi}^{\mathbf{a}} q_k(x, 0, t) \quad \text{for all } (x, t) \in M_0 \times \mathbb{R} \text{ and } |\mathbf{a}| \leq k.$$

Note that  $a_k$  is  $C^\infty$  by Lemma 70. Multiplying by a smooth bump function with proper support which is 1 in a neighbourhood of  $M_0 \times \{0\} \times \mathbb{R}$  and which is homogeneous on the nose outside  $[-1, 1]$ , we can produce a properly supported function  $a'_k \in C^\infty(M_0 \times V \times \mathbb{R})$  with the same vertical derivatives on the unit space. Put  $q'_k = q_k - a'_k$ . Then  $\tilde{Q}'_k = q'_k \omega$  represents the same full cosymbol as  $\tilde{Q}_k$  but satisfies

$$\partial_{\xi}^{\mathbf{a}} q'_k(x, 0, t) = 0 \quad \text{for all } (x, t) \in U \times \mathbb{R} \text{ and } |\mathbf{a}| \leq k. \quad (43)$$

The embedding  $\Psi_H^{-d_H - kN - 1} \hookrightarrow \Psi_H^{-d_H - 1}$  from Corollary 53 leads us to define

$$\tilde{\mathbb{P}}'_k = t^{kN} \tilde{Q}'_k \in \Psi_H^{-d_H - 1}(M). \quad (44)$$

So  $\tilde{\mathbb{P}}'_k = p'_k \omega$ , where  $p'_k = t^{kN} q'_k$  satisfies the same vertical vanishing conditions (43) as  $q'_k$ . An application of the Taylor Remainder Formula shows that for all  $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$ ,  $c \in \mathbb{N}$  with  $|\mathbf{a}| \leq k - 1$  there is a constant  $C_{\mathbf{a}, \mathbf{b}, c} > 0$  such that

$$\left| \partial_{\xi}^{\mathbf{a}} \partial_x^{\mathbf{b}} \partial_t^c p'_k(x, \xi, t) \right| \leq C_{\mathbf{a}, \mathbf{b}, c} \|\xi\|^{k - |\mathbf{a}|} \\ \text{for all } (x, \xi, t) \in \overline{M_0} \times V \times [1, -1],$$

where  $\|\xi\|$  denotes the Euclidean norm of  $\xi \in V$ .

Fix a smooth bump function  $\theta \in C^\infty(V)$  which is 1 on some neighbourhood of the origin and 0 outside the unit ball  $\{\xi \in V \mid \|\xi\| \leq 1\}$ . For  $r \in \mathbb{R}_+^\times$ , we define  $\theta_r(\xi) = \theta(r^{-1}\xi)$ . Then  $\theta_r$  is supported in the ball of radius  $r$  and its derivatives satisfy

$$\|\partial_{\xi}^{\mathbf{a}} \theta_r\|_{\infty} = r^{-|\mathbf{a}|} \|\partial_{\xi}^{\mathbf{a}} \theta\|_{\infty} \quad (45)$$

for any multi-index  $\mathbf{a} \in \mathbb{N}^n$ . For each  $k \geq 1$ , put

$$r_k = \min \left( 2^{-2k} \left( \max_{|\mathbf{a}| \leq k-1} \|\partial_{\xi}^{\mathbf{a}} \theta\|_{\infty} \right)^{-1} \left( \max_{|\mathbf{a}|, |\mathbf{b}|, c \leq k-1} C_{\mathbf{a}, \mathbf{b}, c} \right)^{-1}, 1 \right).$$

We then define  $p_k(x, \zeta, t) = \theta_{r_k}(\zeta)p'_k(x, \zeta, t)$  and  $\tilde{\mathbb{P}}_k = p_k\omega$ . For all  $|\mathbf{a}|, |\mathbf{b}|, c \leq k-1$  and all  $(x, \zeta, t) \in \overline{M_0} \times V \times [-1, 1]$  we have

$$\begin{aligned} \left| \partial_{\zeta}^{\mathbf{a}} \partial_x^{\mathbf{b}} \partial_t^c p_k(x, \zeta, t) \right| &= \left| \sum_{\mathbf{a}' \leq \mathbf{a}} \binom{\mathbf{a}}{\mathbf{a}'} (\partial_{\zeta}^{\mathbf{a}-\mathbf{a}'} \theta_{r_k})(\zeta) (\partial_{\zeta}^{\mathbf{a}'} \partial_x^{\mathbf{b}} \partial_t^c p'_k)(x, \zeta, t) \right| \\ &\leq \sum_{\mathbf{a}' \leq \mathbf{a}} \binom{\mathbf{a}}{\mathbf{a}'} r_k^{-|\mathbf{a}-\mathbf{a}'|} \|\partial_{\zeta}^{\mathbf{a}-\mathbf{a}'} \theta\|_{\infty} C_{\mathbf{a}', \mathbf{b}, c} r_k^{k-|\mathbf{a}'|} \\ &\leq r_k^{k-|\mathbf{a}|} \sum_{\mathbf{a}' \leq \mathbf{a}} \binom{\mathbf{a}}{\mathbf{a}'} \|\partial_{\zeta}^{\mathbf{a}-\mathbf{a}'} \theta\|_{\infty} C_{\mathbf{a}', \mathbf{b}, c} \\ &\leq 2^{-k-1}, \end{aligned} \tag{46}$$

where  $\binom{\mathbf{a}}{\mathbf{a}'} = \prod_i \binom{a_i}{a'_i}$  is a multi-binomial coefficient and the last line uses the formula  $\sum_{\mathbf{a}' \leq \mathbf{a}} \binom{\mathbf{a}}{\mathbf{a}'} = 2^{|\mathbf{a}|}$ .

Therefore, for every  $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$  and  $c \in \mathbb{N}$ , the series  $\sum_{k \geq |\mathbf{a}|} \partial_x^{\mathbf{b}} \partial_t^c p_k$  is uniformly convergent on  $\overline{M_0} \times V \times [-1, 1]$ . In particular the series  $\sum_{k \geq 0} p_k$  converges to a limit  $p \in C^{\infty}(M_0 \times [-1, 1]; C(V))$ . Therefore  $\sum_k \tilde{\mathbb{P}}_k$  converges to  $\tilde{\mathbb{P}} = p\omega \in \mathcal{E}'_r(M_0 \times V \times [-1, 1])$ . Since every  $\tilde{\mathbb{P}}_k$  is homogeneous on the nose outside  $[-1, 1]$ , this convergence extends to  $\mathcal{E}'_r(M_0 \times V \times \mathbb{R})$ .

One can readily globalize this construction, using a locally finite partition of unity subordinate to an atlas for  $M$  of charts of type  $M_0$ , to obtain a globally convergent sum  $\mathbb{P} = \sum_k \mathbb{P}_k \in \mathcal{E}'_r(\mathbb{T}_H M)$  where  $\mathbb{P}_k \in t^{kN} \Psi_H^{-d_H - kN - 1}(M) \subset \Psi_H^{-d_H - 1}(M)$  and  $\mathbb{P}_k|_1$  has the same full cosymbol as  $P_k$ . We omit the details.

Finally, we prove that  $\mathbb{P}$  is essentially homogeneous of order  $-d_H - 1$ . Fix  $k_0 \geq 0$ , and write

$$\begin{aligned} \alpha_{\lambda*} \mathbb{P} - \lambda^{-d_H - 1} \mathbb{P} &= \sum_{k=0}^{k_0-1} \left( \alpha_{\lambda*} \mathbb{P}_k - \lambda^{-d_H - 1} \mathbb{P}_k \right) \\ &\quad + \alpha_{\lambda*} \left( \sum_{k \geq k_0} \mathbb{P}_k \right) - \lambda^{-d_H - 1} \left( \sum_{k \geq k_0} \mathbb{P}_k \right). \end{aligned} \tag{47}$$

The first sum, being finite, belongs to  $C^{\infty}(\mathbb{T}_H M; \Omega_r)$ . Meanwhile, the sum  $\sum_{k \geq k_0} \mathbb{P}_k$  is convergent in  $C^{k_0}(\mathbb{T}_H M; \Omega_r)$  by the preceding calculations. Therefore, the cocycle (47) belongs to  $C^{k_0}(\mathbb{T}_H M; \Omega_r)$  for every  $k_0 \geq 0$ . Thus,  $\mathbb{P} \in \Psi_H^m(M)$ , as desired.

Therefore, putting  $P = \mathbb{P}_1 \in \Psi_H^{-d_H - 1}(M)$ , we have that  $[P] = \sum_k [P_k]$  in the topology of asymptotic convergence.  $\square$

*Remark 77.* In the case of operators of non-integer order, the subtlety is that, given our definition of  $H$ -pseudodifferential operator, we do



not have that  $\Psi_H^m(M) \subset \Psi_H^{m'}(M)$  for  $m \leq m'$  when  $m' - m \notin \mathbb{Z}$ , since fractional powers of  $t$  are not smooth. Nevertheless, one can obtain an analogue of Theorem 76 for non-integer orders by defining  $\Phi_H^{r+\infty}(M) = \bigcup_{m \in \mathbb{Z}} \Phi_H^{r+m}(M)$  for any  $r \in \mathbb{R}$ , with the analogous topology of asymptotic convergence. The proof of Theorem 76 goes through essentially without change.

**6.2.  $H$ -elliptic operators.** Recall (Section 2.8) that the identity elements  $I \in \mathcal{E}'_r(T_H M)$  and  $\mathbb{I} \in \mathcal{E}'_r(\mathbb{T}_H M)$  are given by restriction to the unit spaces of  $T_H M$  and  $\mathbb{T}_H M$ , respectively.

**Lemma 78.** *Let  $u \in \mathcal{E}'_r(T_H M)$  be essentially homogeneous of weight  $m \in \mathbb{R}$  modulo  $C_p^\infty(T_H M; \Omega_r)$ , in the sense that  $\delta_{\lambda*} u - \lambda^m u \in C_p^\infty(T_H M; \Omega_r)$  for all  $\lambda \in \mathbb{R}_+^\times$ . If  $v \in \mathcal{E}'_r(T_H M)$  is a convolution inverse to  $u$  modulo  $C_p^\infty(T_H M; \Omega_r)$  then  $v$  is essentially homogeneous of weight  $-m$ .*

*Proof.* For all  $\lambda \in \mathbb{R}_+^\times$  we have, modulo  $C_p^\infty(T_H M; \Omega_r)$ ,

$$\begin{aligned} \delta_{\lambda*} v &\equiv (\delta_{\lambda*} v) * u * v \equiv (\delta_{\lambda*} v) * (\lambda^{-m} \delta_{\lambda*} u) * v \\ &\equiv \lambda^{-m} \delta_{\lambda*} (I) * v = \lambda^{-m} v. \end{aligned}$$

□

**Definition 79.** An  $H$ -pseudodifferential operator  $P \in \Psi_H^m(M)$  will be called  $H$ -elliptic if its principal cosymbol  $\sigma_m(P)$  admits a convolution inverse in  $\mathcal{E}'_s(T_H M)/C_p^\infty(T_H M; \Omega_r)$ .

**Lemma 80.** *Let  $P \in \Psi_H^m(M)$  be  $H$ -elliptic. Then there is  $Q \in \Psi_H^{-m}(M)$  such that  $P * Q - I$  and  $Q * P - I$  are in  $\Psi_H^{-1}(M)$ .*

*Proof.* Fix  $\mathbb{P} \in \Psi_H^m$  extending  $P$ . From the previous lemma and the surjectivity of the cosymbol map (Lemma 50) we can find  $\mathbb{Q} \in \Psi_H^{-m}$  such that  $\mathbb{P}_0 * \mathbb{Q}_0 \equiv I \equiv \mathbb{Q}_0 * \mathbb{P}_0$  modulo  $C_p^\infty(T_H M; \Omega_r)$ . Put  $Q = \mathbb{Q}_1$ . Then the cosymbols  $\sigma_0(P * Q - I)$  and  $\sigma_0(Q * P - I)$  vanish, so the result follows from Lemma 53. □

**Definition 81.** Let  $P \in \Psi_H^m(M)$ . An  $H$ -pseudodifferential operator  $Q \in \Psi_H^{-m}(M)$  is called a *parametrix* for  $P$  if  $P * Q - I \in \Psi^{-\infty}(M)$  and  $Q * P - I \in \Psi^{-\infty}(M)$ .

**Theorem 82.** *Every  $H$ -elliptic pseudodifferential operator  $P \in \Psi_H^m(M)$  admits a parametrix.*

*Proof.* This is a standard argument using asymptotic expansions. Using Lemma 80, pick  $Q \in \Psi_H^{-m}(M)$  such that  $R := I - P * Q$  and  $S := I - Q * P$  are in  $\Psi_H^{-1}(M)$ . Let us write  $R^k$  for the  $k$ th convolution power of  $R$ . The series  $\sum_{k=0}^\infty R^k$  is convergent in the

topology of asymptotic convergence by Theorem 76, and we let  $A \in \Psi_H^0(M)$  be an asymptotic limit.

Put  $Q' = Q * A$ . Let  $l \in \mathbb{N}$  be arbitrary. Working modulo  $\Psi_H^{-(l+1)}(M)$  we have

$$P * Q' = P * Q * A \equiv (I - R) * \sum_{k=0}^l R^k = I - R^{l+1} \equiv I.$$

Thus  $P * Q' - I \in \Psi^{-\infty}(M)$ .

Finally, note that  $Q * R = S * Q$ . Thus, modulo  $\Psi_H^{-(l+1)}(M)$  we have

$$\begin{aligned} Q' * P &= Q * A * P \equiv \left( \sum_{k=0}^l S^k \right) * Q * P \\ &= \left( \sum_{k=0}^l S^k \right) * (I - S) = I - S^{l+1} \equiv I. \end{aligned}$$

Hence  $Q' * P - I \in \Psi^\infty(M)$ .  $\square$

**Corollary 83.** *Every  $H$ -elliptic differential operator is hypoelliptic.*  $\square$

## 7. DIFFERENTIAL OPERATORS

The goal of this section is to describe the above ideas in more detail in the case of primary interest: differential operators.

Let  $M$  be a filtered manifold. The Schwartz kernel of a differential operator is supported on the unit space of the pair groupoid  $G = M \times M$ . In analogy with the case of Lie groups, distributions supported on the unit space of a Lie groupoid  $G$  are in bijective correspondence with elements of the universal enveloping algebra  $\mathcal{U}(\mathfrak{A}G)$ . This is well-known, although to our knowledge it does not appear explicitly in the literature. We therefore begin with a brief account of the bijection, as well as the bijection with left-invariant differential operators on  $G$ . The latter is well described in Section 3 of [NWX99]. Parts of the present exposition are adapted from that paper.

**7.1. The universal enveloping algebra of a Lie algebroid.** We begin with the definitions. Let  $G \rightrightarrows G^{(0)}$  be a Lie groupoid.

*Left-invariant  $r$ -vertical differential operators.* A smooth section of  $T_r G = \ker(dr)$  is called an  $r$ -vertical vector field on  $G$ . We will use the term *monomial of  $r$ -vertical vector fields* to refer to either a finite product  $X_1 \dots X_k$  of  $r$ -vertical vector fields or a smooth function  $f \in C^\infty(G)$ , the latter being considered as a monomial of length zero. A differential operator  $D : C^\infty(G) \rightarrow C^\infty(G)$  is called  $r$ -vertical if it can

be written as a locally finite sum of monomials of  $r$ -vertical vector fields. We will write  $\text{DO}_r(G)$  for the set of  $r$ -vertical differential operators of globally finite order.

An  $r$ -vertical differential operator  $D$  restricts to a differential operator on each  $r$ -fibre  $G^x$  ( $x \in G^{(0)}$ ). We say  $D$  is *left-invariant* if  $L_{\gamma^{-1}}^* \circ D|_{G^{s(\gamma)}} \circ L_\gamma^* = D|_{G^{r(\gamma)}}$  for all  $\gamma \in G$ , where  $L_\gamma : G^{s(\gamma)} \rightarrow G^{r(\gamma)}$  is left multiplication by  $\gamma$ . The subalgebra of left-invariant  $r$ -vertical differential operators will be denoted  $\text{DO}_r(G)^G$ .

*The universal enveloping algebra.* Since we have worked above with  $r$ -fibered distributions, it will be convenient here to work with the Lie algebroid  $\mathfrak{A}G^{\text{op}} := T_r G|_{G^{(0)}}$ . This is just a technicality:  $\mathfrak{A}G^{\text{op}}$  is isomorphic to  $\mathfrak{A}G := T_s G|_{G^{(0)}}$  via  $X \mapsto -X$ . The anchor  $\rho : \mathfrak{A}G^{\text{op}} \rightarrow TG^{(0)}$  is the derivative of the source map.

The universal enveloping algebra  $\mathcal{U}(\mathfrak{A}G^{\text{op}})$  is the universal algebra admitting an algebra morphism  $i_0 : C^\infty(M) \rightarrow \mathcal{U}(\mathfrak{A}G^{\text{op}})$  and a Lie algebra morphism  $i_1 : \Gamma(\mathfrak{A}G^{\text{op}}) \rightarrow \mathcal{U}(\mathfrak{A}G^{\text{op}})$  subject to the relations

$$i_0(f)i_1(X) = i_1(fX), \quad [i_1(X), i_0(f)] = i_0((\rho X)f), \quad (48)$$

for all  $f \in C^\infty(M)$ ,  $X \in \Gamma(\mathfrak{A}G^{\text{op}})$ .

*Fibred distributions supported on  $G^{(0)}$ .* We use  $\mathcal{E}'_r(G)^{(0)}$  to denote the algebra of  $r$ -fibred distributions of finite order which are supported on the unit space of  $G$ . Here, an  $r$ -fibred distribution is of *finite order* if it extends to a continuous map  $u : C^k(G) \rightarrow C^k(G^{(0)})$  for some  $k \in \mathbb{N}$ .

**Proposition 84.** *There are canonical isomorphisms*

$$\mathcal{U}(\mathfrak{A}G^{\text{op}}) \cong \text{DO}_r(G)^G \cong \mathcal{E}'_r(G)^{(0)}.$$

*Specifically, the left-hand isomorphism is the unique extension of the map  $\Gamma(\mathfrak{A}G) \rightarrow \Gamma(T_r M)$  induced by left-translation:*

$$i_1 : X \mapsto \tilde{X}, \quad \text{where } \tilde{X}_\gamma = L_{\gamma*} X_{s(\gamma)}.$$

*The right-hand isomorphism is  $D \mapsto I \circ D$  where  $I : C^\infty(G) \rightarrow C^\infty(G^{(0)})$  is restriction to the units. Its inverse is  $u \mapsto D_u$  where*

$$D_u \varphi(\gamma) := \langle u^{s(\gamma)}, \varphi(\gamma \cdot) \rangle. \quad (49)$$

*Proof.* For the left-hand isomorphism see [NWX99], Section 3. For the right-hand isomorphism, first consider  $u = (u^x) \in \mathcal{E}'_r(G)^{(0)}$ . Classical results imply that  $u^x$  is the evaluation at the unit  $1_x$  of some differential operator  $P^x$  on  $G^x$ . The smoothness and finite order of  $(u^x)$  ensure that the coefficients of the  $P^x$  vary smoothly

along  $G^{(0)}$ , so there is  $P \in \text{DO}_r(G)$  such that  $\langle u, \varphi \rangle = (P\varphi)|_{G^{(0)}}$  for all  $\varphi \in C^\infty(G)$ . Thus

$$(D_u\varphi)(\gamma) = (P(L_\gamma^*\varphi))(1_{s(\gamma)}),$$

from which it follows that  $D_u$  is a left-invariant  $r$ -vertical differential operator. Moreover, this formula shows that the map  $u \mapsto D_u$  is inverse to the restriction  $D \mapsto I \circ D$ .  $\square$

*Remark 85.* Let  $\pi(u) : \varphi \mapsto u * \varphi$  denote the left-convolution representation of  $u \in \mathcal{E}'_r(G)$  on functions  $\varphi \in C^\infty(G)$  (see Remark 15). Recall that  $\pi(u)$  restricts to a map on  $C^\infty(G_x)$  for each source fibre  $G_x$ . The  $r$ -vertical differential operator  $D_u$  is equal to

$$D_u = \iota^* \circ \pi(u) \circ \iota^*,$$

where  $\iota : \gamma \mapsto \gamma^{-1}$  is the inverse on  $G$ .

**7.2. Principal symbols.** We now apply the above to the  $H$ -tangent groupoid of a filtered manifold  $M$ .

Consider first the pair groupoid  $M \times M$ . Recall that  $\mathfrak{A}(M \times M) \cong TM$  as a Lie algebroid, so we get  $\mathcal{U}(\mathfrak{A}(M \times M)) \cong \text{DO}(M)$ , the algebra of all differential operators of finite order on  $M$ . This isomorphism can be clarified via the first isomorphism of Proposition 84: it sends  $D \in \text{DO}(M)$  to the operator  $\tilde{D} \in \text{DO}_r(M \times M)^{M \times M}$  which acts as  $D$  on the second coordinate.

The resulting isomorphism  $\text{DO}(M) \cong \mathcal{E}'_r(M \times M)^{(0)}$  sends a differential operator to its Schwartz kernel. For if  $D \in \text{DO}(M)$  has Schwartz kernel  $u_D$  then

$$\langle I \circ \tilde{D}, f \rangle(x) = (\tilde{D}f)(x, x) = \int_{y \in M} u_D(x, y) f(x, y) = \langle u_D, f \rangle(x),$$

for any  $f \in C^\infty(M \times M)$ .

The filtration on sections of the Lie algebroid  $\Gamma(TM)$  extends to a filtration on  $\mathcal{U}(TM) = \text{DO}(M)$ . We denote the space of differential operators of  $H$ -order  $\leq m$  by  $\text{DO}_H^m(M)$ . The grading maps  $\sigma_m : \Gamma(H^m) \rightarrow \Gamma(H^m/H^{m-1}) \subset \Gamma(\mathfrak{t}_H M)$  extend canonically to the universal enveloping algebras, giving

$$\sigma_m : \text{DO}_H^m(M) \rightarrow \mathcal{U}(\mathfrak{t}_H M).$$

for each  $m \in \mathbb{N}$ . This defines the *principal part* of a differential operator of  $H$ -order  $m$ .

**7.3. Differential operators as  $H$ -pseudodifferential operators.** Given  $X \in \Gamma(H^m)$ , the section  $\mathbb{X} = t^m X$  belongs to  $\mathfrak{X}_H = \Gamma(\mathfrak{t}_H M)$ , as defined in Equation (13). Let us write  $\sigma_m : \Gamma(H^m) \rightarrow \Gamma(\mathfrak{t}_H M)$  for the

map  $\sigma_m : X \mapsto \mathbb{X}$ . These maps satisfy

$$[\sigma_m(X), \sigma_{m'}(Y)] = \sigma_{m+m'}([X, Y]), \quad X \in \Gamma(H^m), Y \in \Gamma(H^{m'}).$$

It follows that they extend to maps  $\sigma_m : \mathcal{U}^m(TM) \rightarrow \mathcal{U}(\mathfrak{t}_H M) \cong \mathcal{E}'_r(\mathbb{T}_H M)^{(0)}$  satisfying  $\sigma_m(D_1)\sigma_{m'}(D_2) = \sigma_{m+m'}(D_1 D_2)$ . Moreover, evaluation at  $t = 0$ , as defined in Lemma 31, yields  $\text{ev}_0^H(\sigma_m(D)) = \sigma_m(D)$  by the extension of this property for vector fields.

To cut through the abstraction, if  $D = X_1 \dots X_k$  is a monomial of vector fields with  $\text{ord}_H(X_i) \leq m_i$ , then  $D \in \text{DO}_H^m(M)$  with  $m = \sum_i m_i$  and we have  $\sigma_m(D) = t^m D$  and  $\sigma_m(D) = \sigma_{m_1}(X_1) \dots \sigma_{m_k}(X_k)$ .

We can now prove Proposition 1 in the general context of filtered manifolds. It is almost a direct consequence of the definitions.

**Theorem 86.** *A semiregular kernel  $P \in \mathcal{E}'_r(M \times M)$  is the Schwartz kernel of a differential operator of  $H$ -order  $m$  if and only if  $P = \mathbb{P}|_{t=1}$  for some  $\mathbb{P} \in \mathcal{E}'_r(\mathbb{T}_H M)$  which is homogeneous of weight  $m$ , i.e. such that  $\alpha_{\lambda*} \mathbb{P} = \lambda^m \mathbb{P}$  for all  $\lambda \in \mathbb{R}_+^\times$ .*

*Proof.* If  $P$  is the kernel of a differential operator  $D$  of  $H$ -order  $\leq m$ , then we can put  $\mathbb{P} = \sigma_m(D)$ .

Conversely, suppose that  $\mathbb{P} \in \mathcal{E}'_s(\mathbb{T}_H M)$  is homogeneous of weight  $m$ . The support of  $\mathbb{P}$  is invariant under the zoom action on  $\mathbb{T}_H M$ , and since the only vertically compact orbits of the zoom action are those contained in the unit space,  $\text{supp}(\mathbb{P}) \subseteq \mathbb{T}_H M^{(0)}$ . Thus  $P = \mathbb{P}_1$  is the kernel of a differential operator  $D$  on  $M$ .

It remains to show that  $D$  has  $H$ -order  $\leq m$ . By homogeneity,  $\mathbb{P}_t = t^m \mathbb{P}$  for all  $t \neq 0$ . If we restrict  $D$  to an arbitrary relatively compact open subset  $M_0 \subseteq M$  then  $D$  has finite order, that is  $D \in \text{DO}_H^n(M_0)$  for some  $n$ . Suppose  $n > m$ . Put  $\mathbb{Q} = \sigma_n(D) \in \mathcal{E}'_s(\mathbb{T}_H(M_0))$ . Then  $\mathbb{Q}_t = t^{n-m} \mathbb{P}_t$  for all  $t \neq 0$ , and by continuity this holds for  $t = 0$  as well. Therefore  $\sigma_n(D) = \mathbb{Q}_0 = 0$ , so that  $D \in \text{DO}_H^{n-1}(M_0)$ . Iterating this argument, we conclude that  $D \in \text{DO}_H^m(M_0)$ .  $\square$

## 8. CLASSICAL PSEUDODIFFERENTIAL OPERATORS

We conclude with the proof that our construction recovers the classical pseudodifferential operators in the case of Connes' tangent groupoid for a trivially filtered manifold  $M$  (Theorem 2 of the Introduction).

For the terminology for classical pseudodifferential operators, we will follow the conventions of [Hör85, §18]. As usual, we work in local exponential coordinates as in Section 3.9—in particular,  $M_0$  is an open domain in  $\mathbb{R}^n$  with compact closure in  $M$ . With the trivial filtration on  $\mathbb{R}^n$ , we have  $\mathfrak{t}_H M = TM$  and we equip  $T(M_0)$  with

the standard connection  $\nabla$ . The exponential map on the tangent groupoid is then

$$\begin{aligned} \mathbb{E}\text{xp}^\nabla : \mathfrak{t}_H M \times \mathbb{R} &\rightarrow \mathbb{T}_H M \\ (x, \zeta, t) &\mapsto (x, x - t\zeta, t), & t \neq 0 \\ (x, \zeta, 0) &\mapsto (x, \zeta, 0), & t = 0. \end{aligned}$$

As remarked already in Example 55, if  $a \in C^\infty(T^*M_0)$  is a symbol of class  $S^m(M_0 \times \mathbb{R}^n)$ , then the associated pseudodifferential operator has distributional kernel

$$K_a = \text{Exp}_* \check{a},$$

where  $\check{a}(x, \zeta)$  is the fibrewise inverse Fourier transform of  $a(x, \eta)$  and  $\text{Exp} = \mathbb{E}\text{xp}|_{t=1}$ . A *classical* pseudodifferential operator of order  $m$  means one whose symbol  $a$  admits local asymptotic expansions

$$a(x, \eta) \sim \sum_j a_j(x, \eta) \quad (50)$$

where  $a_j$  is homogeneous of order  $m - j$  in  $\eta$  outside of  $\|\eta\| \leq 1$  (cf. Definition 18.1.5 of [Hör85]).

**Theorem 87.** *Let  $M$  be a smooth manifold without boundary, equipped with the trivial filtration:  $H^1 = TM$ . The algebra  $\Psi_H^\bullet(M)$  of  $H$ -pseudodifferential operators is precisely the algebra of (Schwartz kernels of) classical pseudodifferential operators on  $M$ .*

*Proof.* Without loss of generality, we suppose that  $M = M_0 \subseteq \mathbb{R}^n$ , so that  $\mathfrak{t}_H M = TM = M \times \mathbb{R}^n$ . First suppose that  $P = K_a$  is the kernel of a classical pseudodifferential operator on  $M$  with symbol  $a \sim \sum_j a_j$  as in Equation (50). Note that  $P$  is a proper distribution by Proposition 22.

For each  $j$ , let  $\tilde{\mathbb{P}}_j \in \mathcal{E}'_r(M \times \mathbb{R}^n \times \mathbb{R})$  be the constant family of distributions equal to  $\check{a}_j$  on each  $M \times \mathbb{R}^n \times \{t\}$ . This is homogeneous of weight  $m - j$  for the zoom action  $\tilde{\alpha}$  so, after cutting down by an exponential bump function, its push-forward  $\mathbb{P}_j$  via  $\mathbb{E}\text{xp}^\nabla$  is in  $\Psi_H^{m-j}(M)$ . Let  $Q \in \Psi_H^m(M)$  be an asymptotic limit of  $\sum_j \mathbb{P}_j|_{t=1}$ . By the regularity of  $H$ -pseudodifferential operators (Theorem 71) and the analogous result for classical pseudodifferential operators, we see that for  $k \gg 0$  both  $P$  and  $Q$  differ from  $\text{Exp}_*(\sum_{j=1}^k \check{a}_j)$  by a kernel in  $C^{k-m-n}(M \times M; \Omega_r)$ . Thus  $P - Q \in \Psi^\infty(M)$  and so  $P \in \Psi_H^m(M)$ .

Conversely, suppose  $P \in \Psi_H^m(M)$ . Then  $P = \mathbb{P}_1$  for some  $\mathbb{P} \in \Psi_H^m(M)$  which is supported in a global exponential coordinate patch and homogeneous on the nose outside of  $[-1, 1]$ . By Corollary 67,

the fibrewise Fourier transform  $\hat{\mathbb{P}}$  (with respect to the exponential  $\mathbb{E}\exp^\nabla$ ) satisfies the symbol bounds

$$|\partial_\eta^a \partial_x^b \hat{\mathbb{P}}(x, \eta, t)| \leq C_{a,b} (1 + \|\eta\|)^{m-|a|} \quad \text{for all } (x, \eta, t) \in M \times \hat{\mathbb{R}}^n \times \mathbb{R},$$

so that  $a := \hat{\mathbb{P}}_1$  is in the symbol class  $S^m(M \times \hat{\mathbb{R}}^n)$ . The kernel of the operator with symbol  $a$  is  $P$ , so it remains only to prove that  $a$  admits a polyhomogeneous expansion as in Equation (50).

Note that  $\hat{\mathbb{P}}_0$  is homogeneous modulo Schwartz class with respect to the standard dilations  $\delta'_\lambda$  on  $M \times \hat{\mathbb{R}}^n \times \{0\}$ . Therefore, thanks to Lemma 58, after modification by a fibrewise Schwartz function we may assume  $\hat{\mathbb{P}}_0$  is homogeneous of weight  $m$  on the nose outside of  $\|\eta\| \leq 1$ . Define

$$a_0(x, \eta, t) = \hat{\mathbb{P}}_0(x, \eta) \quad \text{for } (x, \eta, t) \in M \times \hat{\mathbb{R}}^n \times \mathbb{R}.$$

Then  $b_1 = t^{-1}(\hat{\mathbb{P}} - a_0)$  extends to a well-defined smooth function on  $M \times \hat{\mathbb{R}}^n \times \mathbb{R}$  which is homogeneous of weight  $(m-1)$  modulo Schwartz class for the dilations  $\beta_\lambda$  of Equation (34). Repeating the above arguments, we have  $b_1|_{t=1} \in S^{m-1}(M \times \hat{\mathbb{R}}^n)$  and we put  $a_1(x, \eta, t) = b_1(x, \eta, 0)$ . This we may again assume is homogeneous of weight  $m-1$  on the nose outside of  $\|\eta\| \leq 1$ . Continuing in this fashion, we obtain the asymptotic symbol expansion  $a \sim \sum_{j=0}^m a_j|_{t=1}$ .  $\square$

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DEPARTMENT OF MATHEMATICS, DARTMOUTH COLLEGE, HANOVER, NH 03755,  
USA

*E-mail address:* erikvanerp@dartmouth.edu

UNIVERSITÉ CLERMONT AUVERGNE, UNIVERSITÉ BLAISE PASCAL, BP 10448, F-  
63000 CLERMONT-FERRAND, FRANCE

*E-mail address:* yuncken@math.univ-bpclermont.fr