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THE EVENTUAL STABILITY OF DEPTH, ASSOCIATED PRIMES AND COHOMOLOGY OF A GRADED MODULE

MARC CHARDIN, JEAN-PIERRE JOUANOLOU, AND AHAD RAHIMI

INTRODUCTION

The asymptotic stability of several homological invariants of the graded pieces of a graded module has attracted quite a lot of attention over the last decades. An early important result was the proof by Brodmann of the eventual stabilization of the associated primes of the powers of an ideal in a Noetherian ring ([1]).

We provide in this text several stability results together with estimates of the degree from which it stabilizes. One of our initial goals was to obtain a simple proof of the tameness result of Brodmann in [2] for graded components of cohomology over rings of dimension at most two. This is achieved in the last section, and gives a slight generalization of what is known, as our result (Theorem 7.4) applies to Noetherian rings of dimension at most two that are either local or the epimorphic image of a Gorenstein ring. Recall that Cutkosky and Herzog provided examples in [3] showing that tameness does not hold over rings of dimension three (even over nice local such rings).

Besides this result, we establish, for a graded module \( M \) over a polynomial ring \( S \) (in finitely many variables, with its standard grading) over a commutative ring \( R \), stability results for the depth and cohomological dimension of graded pieces with respect to a finitely generated \( R \)-ideal \( I \). It follows from our results that the cohomological dimension of \( M_\mu \) with respect to \( I \) is constant for \( \mu > \text{reg}(M) \), and the depth with respect to \( I \) is at least equal to its eventual value for \( \mu > \text{reg}(M) \) and stabilizes when it reaches this value for some \( M_\mu \) with \( \mu > \text{reg}(M) \). See 3.1 and 4.9 for more precise results.

Recall that \( \text{reg}(M) \in \mathbb{Z} \) when \( M \neq 0 \) is finitely generated and \( R \) is Noetherian.

When \( R \) is Noetherian, \( \mathfrak{p} \in \text{Spec}(R) \) is associated to \( M_\mu \) for some \( \mu \) if and only if \( \mathfrak{p} = \mathfrak{P} \cap R \) for \( \mathfrak{P} \) associated to \( M \) in \( S \) and the set associated primes of \( M_\mu \) is non decreasing for \( \mu > \text{reg}(M) \). It implies that this set eventually stabilizes when \( M \) is finitely generated.

Before we establish these regularity results in sections 4 and 5, we prove several facts about depth and cohomological dimension with respect to a finitely generated ideal and about Castelnuovo-Mumford regularity of a graded module. Our definition of depth agrees with the one introduced by Northcott. These results are stated in a quite general setting and self contained proofs are given. Our arguments are often at least as simple as the ones proposed under stronger hypotheses in classical
references. We are in particular careful about separating statements where a finiteness hypothesis is needed (notably in terms of finite generation, finite presentation, or Noetherianity) from others that do not require it. We show that several basic results on regularity hold without any finiteness hypothesis, and that many results on the asymptotic behaviour hold for modules of finite regularity.

In Section 6, we give pretty general duality statements that encapsulate the Herzog-Rahimi spectral sequence we use in the last section to derive tameness from our previous stability results.

1. Local cohomology and depth

Let $A$ be a commutative ring (with unit) and $M$ a $A$-module. If $a = (a_1, \ldots, a_r)$ is a $r$-tuple of elements of $A$, $K^*(a; M)$ is the Koszul complex and $H^i(a; M)$ its $i$-th cohomology module. Also, $C^*_a(M)$ is the Čech complex. This complex is isomorphic to $\lim_{\rightarrow n} K^*(a^n_1, \ldots, a^n_r; M)$. If $a$ and $b$ generates two ideals with same radical, then $H^i(C^*_a(M)) \simeq H^i(C^*_b(M))$ for all $i$. Moreover this isomorphism is graded (of degree 0) if $A$, $M$ and the ideals generated by $a$ and $b$ are graded. This for instance follows from [7, 1.2.3 and 1.4.1]. It can also be proved in an elementary way as follows: first notice that it is sufficient to prove that if $y \in \sqrt{(x_1, \ldots, x_t)}$ then $H^i(C^*_{(x_1, \ldots, x_t)}(M)) \simeq H^i(C^*_{(x_1, \ldots, x_t, y)}(M))$, second show that $C^*_{(x_1, \ldots, x_t)}(M)$ is acyclic if $y \in \sqrt{(x_1, \ldots, x_t)}$, and conclude using that $C^*_{(x_1, \ldots, x_t)}(M) \rightarrow C^*_{(x_1, \ldots, x_t, y)}(M)$.

We will denote by $H^i(M)$ the $i$-th homology module of $H^i(C^*_a(M))$, if $a$ generates the ideal $I$.

The $i$-th right derived functor of the left exact functor $H^i_{\mathcal{Y}}$ coincides with the functor $T^i(\mathcal{Y}) := \lim_{\to\mathcal{Y}} \text{Ext}^i_A(A/I^n, \mathcal{Y})$. It coincides with $H^i_{\mathcal{Y}}$ if and only if $H^i_{\mathcal{Y}}(N) = 0$ whenever $i > 0$ and $N$ is injective, and this holds if $A$ is Noetherian or $I$ is generated by a regular sequence.

If $X := \text{Spec}(A)$ and $Y := V(I) \subset X$, one has an isomorphism

$$H^i_{\mathcal{Y}}(M) \simeq H^i_{\mathcal{Y}}(X, \widetilde{M}).$$

Indeed Serre affineness theorem and Cartan-Leray theorem (see e.g. [9] or [13], and [6, 5.9.1]) provide isomorphisms

$$H^i_{\mathcal{Y}}(X, \widetilde{M}) \simeq H^i(C^*_{(a_1, \ldots, a_r)}(M)) \simeq H^i(M \otimes^L_A C^*_{(a_1, \ldots, a_r)}(A)),$$

as $C^*_{(a_1, \ldots, a_r)}(A)$ is a complex of flat modules. These isomorphisms show that the functor $M \mapsto H^i_{\mathcal{Y}}(M)$ commutes with direct sums and filtered inductive limits and provide a spectral sequence

$$E_{2}^{p,q} = \text{Tor}^A_{-p}(M, H^q_{\mathcal{Y}}(A)) \Rightarrow H^{p+q}_{\mathcal{Y}}(M).$$

Also notice that the isomorphism $H^i_{\mathcal{Y}}(M) \simeq \lim_{\to\mathcal{Y}} H^i(a^n_1, \ldots, a^n_r; M)$ shows that any element of $H^i_{\mathcal{Y}}(M)$ is annihilated by a power of the ideal $I$. 

Lemma 1.2. If $I$ is generated by $a = \langle a_1, \ldots, a_r \rangle$, then
def$depth_I(M) = \max\{p \in \mathbb{N} \cup \{\infty\} \mid H^i(M) = 0, \forall i < p\}$.

Proof. Let $d := \max\{p \mid H^i(a; M) = 0, \forall i < p\}$. Recall that for positive integers $l_i$, $H^i(a_1^{l_1}, \ldots, a_r^{l_r}; M) = 0$ if and only if $H^i(a; M) = 0$. It follows that $H^i_I(M) = \lim_{\to} H^i(a_1^{l_1}, \ldots, a_r^{l_r}; M) = 0$ if $H^i(a; M) = 0$. Notice that $d = \infty$ if and only if $d > r$, in which case $H^i(a; M) = 0$ for all $i$. Hence $depth_I(M) \geq d$.

We now assume $d < \infty$. As $I$ annihilates $H^i(a; M)$ for any $i$, the totalisation of the complex $\mathcal{C}^*K^*(a; M)$ has cohomology isomorphic to the one of $K^*(a; M)$. It provides a spectral sequence

$$'E^{p,q}_1 = H^p(q^p(a; M)) \Rightarrow H^{p+q}(a; M).$$

As $H^p(q^p(a; M)) = 0$ for $q < d$, this in turn provides a natural into map $H^d(a; M) \to H^d_I(M)$ which shows that $depth_I(M) \leq d$. \qed

Corollary 1.3. If $I$ is a finitely generated $A$-ideal, then for any $A$-module $M$,

def$depth_I(M) = \min_{p \in \mathcal{V}(I)} \{depth_{I_p}(M_p)\}$.

To show that this notion agrees with the one introduced by Northcott, we first prove a lemma.

Lemma 1.4. Let $N$ be a $A$-module and $a \in I$ a non zero divisor on $N$. Then

def$depth_I(N/aN) = depth_I(N) - 1$.

Proof. Consider the exact sequence

$$0 \longrightarrow N \xrightarrow{\times a} N \longrightarrow N/aN \longrightarrow 0$$

and the induced long exact sequence on cohomology with support in $I$,

$$\cdots \longrightarrow H^{i-1}_I(N) \longrightarrow H^{i-1}_I(N/aN) \longrightarrow H^i_I(N) \xrightarrow{\times a} H^i_I(N) \longrightarrow \cdots$$

and let $r := depth_I(N)$. The above sequence shows that $depth_I(N/aN) \geq r - 1$. Furthermore, if $r < \infty$, $H^{i-1}_I(N/aN) = 0$ if and only if the multiplication by
a is injective on $H_I^r(N)$. But this does not hold since any element of $H_I^r(N)$ is annihilated by a power of $a$ and $H_I^r(N) \neq 0$ by definition. □

We will also use a version of the Dedekind-Mertens Lemma, that we now recall in its general form, together with immediate corollaries that are useful in this text.

**Theorem 1.5.** (Generalized Dedekind-Mertens Lemma, [8 3.2.1]). Let $A$ be a ring, $M$ be a $A$-module and $T$ a set of variables. For $P \in A[T]$ and $Q \in M[T]$, let $c(P)$ be the $A$-ideal generated by the coefficients of $P$, $c(Q)$ be the submodule of $M$ generated by the coefficients of $Q$ and $\ell(Q)$ be the number of non zero coefficients of $Q$.

Then one has the equality

$$c(P)\ell(Q)^{-1}c(PQ) = c(P)\ell(Q)c(Q).$$

In particular, the kernel of the multiplication by $P$ in $M[T]$ is supported in $V(c(P))$.

**Corollary 1.6.** Let $A$ be a ring, $I = (a_0, \ldots, a_p)$ be a $A$-ideal and $M$ be a $A$-module. Set $\xi := a_0 + a_1T + \cdots + a_pT^p \in A[T]$. Then

$$\ker(M[T] \xrightarrow{\times \xi} M[T]) \subset H_I^0(M[T]) = H_I^0(M)[T].$$

Let $S = R[X_1, \ldots, X_n]$ be a polynomial ring over a commutative ring $R$ and set $S_+ := (X_1, \ldots, X_n)$.

**Corollary 1.7.** Let $M$ be a graded $S$-module. Set $\ell := T_1X_1 + \cdots + T_nX_n$ with $\deg T_i = 0$. Then the kernel of the map,

$$M[T_1, \ldots, T_n] \xrightarrow{\times \xi} M[T_1, \ldots, T_n][1]$$

is a graded $S[T_1, \ldots, T_n]$-submodule of $H_{S_+}^0(M)[T_1, \ldots, T_n]$.

**Corollary 1.8.** Consider indeterminates $(U_{i,j})_{1 \leq i, j \leq n}$, $\xi_i := \sum_{1 \leq j \leq n} U_{i,j}X_j$, $\Delta := \det(U_{i,j})_{1 \leq i, j \leq n}$ and $R' := R[(U_{i,j})_{1 \leq i, j \leq n}]\Delta$. Let $S' := R'[X_1, \ldots, X_n]$ and set $M' := M \otimes_R R'$ for any $S$-module $M$. Then $(\xi_1, \ldots, \xi_n)$ is $M'$-regular off $V(S_+)$.

The following proposition shows that the above definition of depth agrees with the one introduced by Northcott in [10].

**Proposition 1.9.** Let $r \geq 1$ be an integer and $I$ be a finitely generated $A$-ideal. The following are equivalent,

1. $\text{depth}_I(M) \geq r$,
2. There exists a faithfully flat extension $B$ of $A$ and a regular sequence $f_1, \ldots, f_r$ on $B \otimes_A M$ contained in $IB$,
3. There exists a polynomial extension $B$ of $A$ and a regular sequence $f_1, \ldots, f_r$ on $B \otimes_A M$ contained in $IB$,
4. There exists a regular sequence $f_1, \ldots, f_r$ on $M[T_1, \ldots, T_n]$, where the $T_i$'s are variables, contained in $IA[T_1, \ldots, T_n]$. 


Remark 1.10. Let \( M \) be a \( S \)-module. Consider the following properties,

(i) \( M_{\mu} = 0 \) for \( \mu \gg 0 \),
(ii) \( M = H^0_{S_+}(M) \),
(iii) \( H^i_{S_+}(M) = 0 \) for \( i > 0 \).

Then (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (i) if \( M \) is finitely generated or \( \text{reg}(M) < \infty \), and (iii) \( \Rightarrow \) (ii) if \( M_{\mu} = 0 \) for \( \mu \ll 0 \).

Proof. (i) \( \Rightarrow \) (ii) is clear since any homogeneous element in \( M \) is killed by a power of any element of \( S_+ \) if (i) holds. If (ii) holds, then \( M_x = 0 \) for any \( x \in S_+ \), hence the Čech complex on generators of \( S_1 \) (as an \( R \)-module) is concentrated in homological degree 0, which shows (iii).

If (ii) holds, any element in \( M \) is killed by a power of \( S_+ \), hence if \( M \) is finitely generated by \( (m_t)_{t \in T} \), any generator \( m_t \) is killed by \( S_+^{N_t} \), for some \( N_t \in \mathbb{N} \). It then follows that any element in \( M \) of degree bigger than \( \max_{t \in T} \{ \deg(m_t) + N_t \} \) is 0. If \( \text{reg}(M) < \infty \), (ii) \( \Rightarrow \) (i) follows trivially from the definition of \( \text{reg}(M) \).

If (iii) holds set \( N := M/H^0_{S_+}(M) \). The exact sequence \( 0 \rightarrow H^0_{S_+}(M) \rightarrow M \rightarrow N \rightarrow 0 \) gives rise to a long exact sequence in local cohomology showing that \( H^i_{S_+}(N) = 0 \) for all \( i \). As \( \text{depth}_{S_+}(N) = +\infty \), Lemma 1.2 shows that \( N = S_+ N \). This implies that \( N = 0 \) as \( N_{\mu} = 0 \) for \( \mu \ll 0 \).

The following two propositions extend classical results on regularity.

Proposition 2.2. Let \( \ell \geq 1 \) and \( m \) be integers and \( M \) be a graded \( S \)-module.

If \( H^i_{S_+}(M)_{m-i} = 0 \) for \( i \geq \ell \), then \( H^i_{S_+}(M)_{\mu-i} = 0 \) for \( \mu \geq m \) and \( i \geq \ell \).

Assume that \( H^i_{S_+}(M)_{m-i} = 0 \) for all \( i \) and let \( \mu \geq m \). Then \( H^i_{S_+}(M)_{\mu-i} = 0 \) for \( i > 0 \) and \( H^0_{S_+}(M)_{\mu} = M_{\mu}/S_1 M_{\mu-1} \).

Proof. We may assume that \( S = R[X_1, \ldots, X_n] \) is a polynomial ring over \( R \). We then prove the assertion by induction on \( n \).
When $n = 0$, $M = H_{S+}^0(M)$ and $H_{S+}^i(M) = 0$ for $i \neq 0$, and the claim follows in both cases.

Next assume that $n \geq 1$ and the assertion is true for $n - 1$ over any commutative ring. Let

$$\xi := X_1T^{n-1} + \cdots + X_{n-1}T + X_n \in R[T,X_1,\ldots,X_n].$$

The Dedekind-Mertens Lemma implies that

$$\ker(\xi : M[T](-1) \rightarrow M[T]) \subset H_{S+}^0(M[T]),$$

by Corollary 1.6. But $H_{S+}^i(N[T]) = H_{S+}^i(N)[T]$ for any $S$-module $N$ and any $i$. Hence, replacing $R$ by $R[T]$, $M$ by $M[T]$ and $X_n$ by $\xi$, we may assume that

$$K := \ker(X_n : M(-1) \rightarrow M) \subset H_{S+}^0(M).$$

We then have $H_{S+}^i(K) = 0$ for $i \neq 0$ and the exact sequence

$$0 \longrightarrow K \longrightarrow M(-1) \overset{X_n}{\longrightarrow} M \longrightarrow Q \longrightarrow 0$$

induces for all $i$ an exact sequence

$$H_{S+}^i(M)(-1) \overset{X_n}{\longrightarrow} H_{S+}^i(M) \longrightarrow H_{S+}^i(Q) \longrightarrow H_{S+}^{i+1}(M)(-1).$$

For $i + j = m$, the equalities

$$H_{S+}^i(M)j = 0 \text{ and } H_{S+}^{i+1}(M)(-1)j = H_{S+}^{i+1}(M)j-1 = 0$$

imply $H_{S+}^i(Q)j = 0$. Hence $H_{S+}^i(Q)j = 0$ for $i \geq \ell$ and $i + j = m$. As $Q$ is annihilated by $X_n$, setting $n := (X_1,\ldots,X_{n-1})$ one has $H_{S+}^i(Q) = H_n^i(Q)$ for all $i$. Applying the recursion hypothesis to the $R[X_1,\ldots,X_{n-1}]$-module $Q$, it follows that

$$H_{S+}^i(Q)j = H_n^i(Q)j = 0, \quad \forall i \geq \ell, \forall j \geq m - i.$$}

Hence $X_n : H_{S+}^i(M)j \rightarrow H_{S+}^i(M)j$ is onto for $i \geq \ell$ and $i + j \geq m$. As $H_{S+}^i(M)_{m-i} = 0$ for $i \geq \ell$, this proves our claim. \qed

**Remark 2.3.** Notice that the exact sequence $0 \rightarrow S_+M \rightarrow M \rightarrow M/S_+M \rightarrow 0$ induces an exact sequence

$$0 \rightarrow H_{S+}^0(S_+M) \rightarrow H_{S+}^0(M) \rightarrow M/S_+M \rightarrow H_{S+}^1(S_+M) \rightarrow H_{S+}^1(M) \rightarrow 0$$

**Proposition 2.4.** If $S = R[X_1,\ldots,X_n]$ is a polynomial ring, then for any graded $S$-module $M$

(i) $\text{reg}(M) = \sup\{\mu \mid \exists i, \text{ Tor}^S_i(M,R)_{\mu+i} \neq 0\}$,

(ii) $\text{reg}(M) = \sup\{\mu \mid \exists i, \text{ H}_i(X_1,\ldots,X_n;M)_{\mu+i} \neq 0\}$,

(iii) $\text{reg}(M) = \sup\{\mu \mid \exists j, \text{ H}_j^1(X_1,\ldots,X_n;M)_{\mu-j} \neq 0\}$,

(iv) $\text{reg}(M) = \sup\{\mu \mid \exists j, \text{ Ext}^1_S(R,M)_{\mu-j} \neq 0\}$.

In particular, $M$ is generated in degrees at most $\text{reg}(M)$ (when $\text{reg}(M) = -\infty$, it means that $M$ is generated in degrees at most $\mu$, for any $\mu \in \mathbb{Z}$).
Proof. We first show this equality if \( \text{reg}(M) < \infty \). Let \( K_\bullet(M) := K_\bullet(X_1, \ldots, X_n; M) \) and \( K_\bullet(M) := K_\bullet(X_1, \ldots, X_n; M) \). As \( K_\bullet(M) = K_\bullet(S) \otimes S M \), \( K_\bullet(M) = \text{Hom}_S(K_\bullet(S), M) \) and \( K_\bullet(S) \) is a free \( S \)-resolution of \( R \), it follows that \( H_i(K_\bullet(M)) \simeq \text{Tor}_i^S(M, R) \) and \( H^j(K_\bullet(M)) \simeq \text{Ext}_j^R(R, M) \). Furthermore, the complexes \( K_\bullet(M) \) and \( K_\bullet(M) \) are isomorphic, up to a shift in homological degree and internal degree: \( K_\bullet(M) \simeq K_{n-\bullet}(M)(n) \), proving that

\[
\text{Ext}_j^R(R, M)_{\mu-j} \simeq H^j(K_\bullet(M))_{\mu-j} \simeq H_{n-j}(K_\bullet(M))_{\mu-j+n} \simeq \text{Tor}_n^S(M, R)_{\mu-j+n}
\]

and the equivalence of the four items.

To prove (ii), the double complex \( C_{(X_1, \ldots, X_n)K_\bullet(M)} \) gives rise to two spectral sequences whose first terms are respectively

\[
'E_1^{p,q} = K_{-p}(X_1, \ldots, X_n; H^q_{S+}(M)), \quad 'E_2^{p,q} = \text{Tor}_p^S(H^q_{S+}(M), R)
\]

and

\[
''E_1^{p,q} = C_{(X_1, \ldots, X_n) \text{Tor}_p^R}(M, R), \quad ''E_2^{p,q} = H^p_{S+}(\text{Tor}_q^R(M, R)).
\]

Recall that \( K_\bullet(M)_{X_i} \) is acyclic for any \( i \), hence \( ''E_1^{p,q} = 0 \) for \( p \neq 0 \) which implies that \( ''E_2^{p,q} = 0 \) for \( p \neq 0 \) and \( ''E_2^{q,\mu} \simeq \text{Tor}_q^S(M, R) \).

On the other hand, \( (''E_1^{p,q})_\mu = 0 \) for \( \mu > \text{reg}(M) - q - p \) as \( H^q_{S+}(M) \) lives in degrees at most \( \text{reg}(M) - q \). It follows first that \( \text{Tor}_n^S(M, R)_{\mu+n} \) gives in degrees at most \( \text{reg}(M) - j \) showing that \( \text{Tor}_n^S(M, R)_{\mu+n} = 0 \) for \( \mu > \text{reg}(M) \).

To conclude, choose \( j \) such that \( H^q_{S+}(M)_{\text{reg}(M)-j} \neq 0 \). Set \( \mu := \text{reg}(M) - j + n \) and notice that \( (''E_1^{p,q})_\mu = 0 \) when \( p+q = -n+j+1 \). As \( 'E_1^{p,q} = 0 \) for \( p < -n \) it follows that \( 0 \neq H^q_{S+}(M)_{\text{reg}(M)-j} = (''E_1^{p,q})_\mu = (''E_2^{p,q})_\mu \) which shows that \( ''E_2^{q,\mu} \simeq \text{Tor}_n^S(M, R)_{\mu} \).

To finish the proof, we must show that \( \text{reg}(M) < \infty \) if there exists \( \mu_0 \) such that \( \text{Tor}_n^S(M, R)_{\mu} = 0 \) for all \( \mu \geq \mu_0 \).

We first show that in this case, there exists a graded free \( S \)-resolution \( F_\bullet \) of \( M \) with \( F_i = \bigoplus_{j \in I} S(-d_{ij}) \), and \( d_{ij} < \mu_0 \) for all \( i \) and \( j \). Notice that if \( M \) is graded and \( (M/S+)_{\mu_0} = 0 \), then \( M \) is generated in degree at most \( \nu \), showing the existence of a graded epimorphism \( \phi : F_0 \to M \) with \( F_0 \) as claimed. The exact sequence,

\[
0 \to \ker(\phi) \to F_0 \to M \to 0
\]

gives rise to another

\[
\text{Tor}_n^S(M, R) \to \ker(\phi)/S_+ \ker(\phi) \to F_0/S_+ F_0
\]

and proves the existence of \( F_1 \) as claimed such that \( F_1 \to \ker(\phi) \) is a graded epi-

morphism. As \( \text{Tor}_n^S(M, R) \simeq \text{Tor}_{n-1}^S(\ker(\phi), R) \), for \( j \geq 2 \), the conclusion follows

by induction on \( i \).

Finally, it suffices to remark that if \( F_\bullet \) is a graded resolution as above, then

\[
H^q_{S+}(M)_{\mu} \simeq H_{n-q}(H^q_{S+}(F_\bullet)_{\mu}) \text{ vanishes in degree bigger than } -n + \max_i \{d_{ij}\} \leq -n + \mu_0, \text{ as } H^q_{S+}(F_\bullet)_{\mu} = 0 \text{ for all } i \text{ for such a } \mu. \text{ Hence } \text{reg}(M) \leq \mu_0. \quad \square
\]
Theorem 3.3. Let $\mu \geq 0$. The conclusion follows.

Furthermore, $\text{reg}(N) = \text{reg}(N \otimes_R R_\mathfrak{p})$ for some $\mathfrak{p} \in \text{Spec}(R)$ if $\text{reg}(N) < \infty$.

Proof. For $\mathfrak{p} \in \text{Spec}(R)$,

$$H^i_{S_+}(N \otimes_R R_\mathfrak{p}) \simeq (H^i_{S_+}(N)_\mathfrak{p}) \otimes_R R_\mathfrak{p}$$

from which the claim directly follows. \quad \square

3. Depth of the Graded Components of a Graded Module

As in the previous section, $S$ is a finitely generated standard graded algebra over a commutative ring $R$.

Proposition 3.1. Let $I$ be a finitely generated $R$-ideal, $M$ be a graded $S$-module and $d$ be an integer. Assume that $\text{depth}_I(M_\mu) \geq d$ for $\mu \gg 0$. Then

(i) $\text{depth}_I(M_\mu) \geq d$ for any $\mu$ such that $H^0_{S_+}(M)_\mu = 0$ for $q < d$,

(ii) if $\text{depth}_I(M_\mu) = d$ for some $\mu$ such that $H^0_{S_+}(M)_\mu = 0$ for $q \leq d$, then $\text{depth}_I(M_\nu) \leq d$ for any $\nu \geq \mu$.

Proof. Let $a = (a_1, \ldots, a_r)$ be generators of $I$. Recall that $H^p(a; M)_\mu = H^p(a; M_\mu)$, as $I$ is an $R$-ideal. Hence, if $\text{depth}_I(M_\mu) \geq d$ for $\mu \gg 0$, the $S$-modules $H^p(a; M)$ are supported in $V(S_+)$ for $p < d$, by Lemma 1.2 and (i) $\Rightarrow$ (ii) in Lemma 2.1. Comparing the two spectral sequences obtained from the double complex $C^*_S K^*(a; M)$, computing the hypercohomology $H^i$ of $K^*$ we obtain on one hand that $H^i \simeq H^i(a; M)$ for $i < d$ and $H^d \simeq H^d_{S_+}(H^d(a; M))$. On the other hand, one has a spectral sequence

$$\epsilon E^{p,q}_1 = H^q_{S_+} K^p \Rightarrow H^{p+q}$$

which shows that $(H^i)_\mu = 0$ for $i < d$ if $H^0_{S_+}(M)_\mu = 0$ for $q < d$, proving (i).

For (ii), the condition $H^0_{S_+}(M)_\mu = 0$ for $q \leq d$ implies that $H^0_{S_+}(H^d(a; M))_\mu = H^0_{S_+}(H^d(a; M)) = 0$. Hence if $H^d(a; M_\mu) \neq 0$, then $0 \neq (S_+)^{d-p} H^d(a; M)_\mu \subseteq H^d(a; M)_\mu$, which shows (ii) by Lemma 1.2 \quad \square

Corollary 3.2. Let $I$ be an $R$-ideal and $M$ be a graded $S$-module.

Then $\text{depth}_I(M_{\geq \mu})$ is independent of $\mu$ for $\mu > \text{reg}(M) - \text{depth}_{S_+}(M)$.

Theorem 3.3. Let $I$ be an $R$-ideal and $M$ be a graded $S$-module with $\text{reg}(M) < \infty$. Set $r := \text{reg}(M) - \text{depth}_{S_+}(M)$, $d := \min_{\nu \geq r} \{\text{depth}_I(M_\nu)\} = \text{depth}_I(M_{>r})$ and $\mu_0 := \inf\{\nu > r \mid \text{depth}_I(M_\nu) = d\}$.

Then $\text{depth}_I(M_\mu) = d$ for all $\mu \geq \mu_0$.

Proof. We may assume that $d < \infty$. By definition of $d$, $\text{depth}_I(M_\mu) \geq d$ for $\mu \geq \mu_0$, as $\mu_0 > r$. On the other hand, $\text{depth}_I(M_\mu) \leq d$ for $\mu \geq \nu$ by Proposition 3.1 (ii). The conclusion follows. \quad \square
4. Cohomological dimension

Let $A$ be a commutative ring (with unit), $I$ a finitely generated ideal and $M$ a $A$-module. First remark that it follows from (1) that if $M$ is annihilated by an ideal $J$, for instance if $J = \text{ann}_A(M)$, considering $M$ as an $A/J$-module one has

$$cd^A_I(M) = cd^{A/J}_{I+J/J}(M).$$

Furthermore,

**Proposition 4.1.** Let $M$ be a $A$-module.

(a) \[ cd_I(M) \leq \max_{E \subseteq M, E \text{ f.g.}} \{ cd_I(E) \}. \]

(b) \[ cd_I(M) \leq cd_I(A/\text{ann}_A M) \leq cd_I(A). \]

(c) If $M$ is finitely generated, then

$$cd_I(M) = cd_I(A/\text{ann}_A M).$$

**Proof.** (a) $M$ is the filtered inductive limit of its submodules of finite type (for the inclusion), and local cohomology commutes with filtered inductive limits.

(b) The spectral sequence (2) shows that $cd_I(N) \leq cd_I(A)$ for any module $N$, in particular $cd_I(A/\text{ann}_A(M)) \leq cd_I(A)$. Together with (3) applied with $J := \text{ann}_A(M)$, we get $cd^A_I(M) = cd^{A/J}_{I+J/J}(M) \leq cd^{A/J}_{I+J/J}(A/J) = cd_I(A/J)$.

(c) According to (b), it suffices to show that $cd_I(A/\text{ann}_A M) \leq cd_I(M)$. Replacing $A$ by $A/\text{ann}_A(M)$, we may assume that $M$ is faithful.

We will show that $cd_I(M) \leq r$ implies $cd_I(A) \leq r$. This is clear for $r \geq \text{ara}_A(I)$, and we now perform a descending recursion on $r$. Assume this is true for $r + 1$. If $cd_I(M) \leq r$, by recursion hypothesis we now that $cd_I(A) \leq r + 1$, hence spectral sequence (2) implies that

$$0 = H^{r+1}_I(M) = M \otimes_A H^{r+1}_I(A).$$

As $M$ is faithful and of finite type, [13, 4.3] shows that $H^{r+1}_I(A) = 0$, which implies that $cd_I(A) \leq r$. \[ \square \]

The following corollary has been proved by M. Dibaei and A. Vahidi in the Noetherian case in [12, 2.2].

**Corollary 4.2.** Let $M$ be a $A$-module and $I, J$ two finitely generated ideals. Then

$$cd_{I+J}(M) \leq cd_I(A/\text{ann}_A M) + cd_J(M),$$

and $cd_{I+J}(M) \leq cd_I(M) + cd_J(M)$ if $M$ is finitely generated.

**Proof.** We may assume that $M$ is faithful. If $a$ is a finite set of generators of $I$ and $b$ a finite set of generators of $J$, the double complex with components $C^a_\bullet(A) \otimes_A C^b_\bullet(M)$ gives rise to a spectral sequence with second term $H^1_I(H^1_J(M))$. 


that abuts to $H^i_{I+J}(M)$. Hence,

$$cd_{I+J}(M) \leq \max\{i + j \mid H^i_J(H^j_I(M)) \neq 0\} \leq cd_I(A) + cd_J(M),$$

by Proposition 4.1 (b).

Furthermore, by Proposition 4.1 (c), $cd_I(A) = cd_I(M)$ whenever $M$ is faithful and finitely generated. □

Corollary 4.3. Let $M$ be a finitely generated $A$-module and $\mathfrak{a}$ an ideal of $A$ such that $a^tM = 0$ for some $t$. Then

$$cd_I(M) = cd_I(M/\mathfrak{a}M).$$

Proof. Let $d := cd_I(M) = cd_I(A/\text{ann}_A(M))$ (by Proposition 4.1 (c)). As $M/\mathfrak{a}M$ is annihilated by $\text{ann}_A(M)$, it follows from Proposition 4.1 (b) that $cd_I(M/\mathfrak{a}M) \leq d$. Furthermore, Proposition 4.1 (b) shows that the functor $N \mapsto H^d_I(N)$ restricted to the category of $A$-modules annihilated by $\text{ann}_A(M)$ is right exact. It implies that $H^d_I(M/\mathfrak{a}M) = H^d_I(M) \otimes_A A/\mathfrak{a} = H^d_I(M)/\mathfrak{a}H^d_I(M)$.

If $cd_I(M/\mathfrak{a}M) < d$ it implies

$$H^d_I(M) = aH^d_I(M) = a^2H^d_I(M) = \cdots = a^tH^d_I(M) = 0,$$

which contradicts the definition of $d$. □

Proposition 4.4. Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence of $A$-modules. Then

$$cd_I(M) \leq \max\{cd_I(M'), cd_I(M'')\} \leq cd_I(A/\text{ann}_A(M)).$$

Furthermore all inequalities are equalities if $M$ is finitely generated.

Proof. First the exact sequences

$$H^i_I(M') \rightarrow H^i_I(M) \rightarrow H^i_I(M''), \quad i \in \mathbb{Z},$$

show the inequality on the left. As both $A$-modules $M'$ and $M''$ are annihilated by $\text{ann}_A(M)$, the inequality on the right follows from Proposition 4.1 (b).

Finally, the extreme terms are equal according to Proposition 4.1 (c) if $M$ is finitely generated. □

Remark 4.5. If $A$ is a domain, distinct from its field of fractions $K$ and $I$ a proper finitely generated ideal, then $A$ is a submodule of $K$ such that

$$-\infty = cd_I(K) < 0 \leq cd_I(A).$$

Corollary 4.6. If $M$ is a Noetherian $A$-module, then

$$cd_I(M) = \max_{p \in \text{Supp}_A(M)} cd_I(A/p) = \max_{p \in \text{Ass}_A(M)} cd_I(A/p) = \max_{p \in \text{Min}_A(M)} cd_I(A/p).$$

Proof. Let $\text{Min}_A(M)$ be the minimal primes in the support of $M$. Every $p \in \text{Supp}_A(M)$ contains some $q \in \text{Min}_A(M)$, and the canonical epimorphism $A/q \rightarrow A/p$
gives an inequality $\text{cd}_I(A/p) \leq \text{cd}_I(A/q)$ by Proposition 4.4. It follows that

$$\max_{p \in \text{Supp}_A(M)} \text{cd}_I(A/p) = \max_{p \in \text{Min}_A(M)} \text{cd}_I(A/p).$$

On the other hand $\text{Min}_A(M) \subset \text{Ass}_A(M)$ and for $p \in \text{Ass}_A(M)$ the existence of a monomorphism $A/p \to M$ implies by Proposition 4.4 that $\text{cd}_I(A/p) \leq \text{cd}_I(M)$. Hence

$$\max_{p \in \text{Min}_A(M)} \text{cd}_I(A/p) \leq \max_{p \in \text{Ass}_A(M)} \text{cd}_I(A/p).$$

and it remains to show that $\text{cd}_I(M) \leq \max_{p \in \text{Supp}_A(M)} \text{cd}_I(A/p)$.

Observe that, if $M \neq 0$, it admits a finite filtration by cyclic modules $A/p_i$ ($1 \leq i \leq t$) with $p_i \in \text{Supp}_A(M)$ for all $i$. Again by Proposition 4.4, we obtain

$$\text{cd}_I(M) = \max_{1 \leq i \leq t} \{\text{cd}_I(A/p_i)\} \leq \max_{p \in \text{Supp}_A(M)} \{\text{cd}_I(A/p)\}.$$ 

This concludes the proof. \(\square\)

The following result generalizes the main theorem of [5], which applies in the case of two finitely generated modules over a Noetherian ring.

**Proposition 4.7.** Let $M$ and $N$ be $A$-modules. Assume $M$ is finitely presented and $\text{Supp}_A(N) \subset \text{Supp}_A(M)$. Then

$$\text{cd}_I(N) \leq \text{cd}_I(M).$$

**Proof.** Let $E$ be a finitely generated submodule of $N$. The inclusion $\text{Supp}_A(N) \subset \text{Supp}_A(M)$ implies that

$$\sqrt{\text{ann}_A(M)} \subset \sqrt{\text{ann}_A(E)}.$$ 

As $M$ is finitely generated, the Fitting ideal $\text{Fitt}^{t}_A(M)$ has the same radical as $\text{ann}_A(M)$ and contains a power of $\text{ann}_A(M)$. Furthermore, as $M$ is finitely presented, this ideal is finitely generated. It then follows from (4) that there exist $t$ such that $\text{ann}_A(M)^t \subset \text{ann}_A(E)$. By Corollary 4.3 and Proposition 4.1 (b) and (c), one has

$$\text{cd}_I(E) \leq \text{cd}_I(E/\text{ann}_A(M)E) \leq \text{cd}_I(A/\text{ann}_A(M)) = \text{cd}_I(M).$$

The conclusion follows by Proposition 4.1 (a) applied to the $A$-module $N$. \(\square\)

Now, let $S$ be a finitely generated standard graded algebra over a commutative ring $R$.

**Definition 4.8.** For a graded $S$-module $M$,

$$a_{\geq i}^S(M) := \sup\{\mu \mid H_\mu^S(M)_\mu \neq 0\},$$

so that $\text{reg}(M) = \max_{i} \{a_{\geq i}^S(M) + i\}$.

**Proposition 4.9.** Let $M$ be a finitely generated graded $S$-module and $I$ be a finitely generated $R$-ideal. Then

(a) $\text{cd}_I(M_\mu)$ is a non decreasing function of $\mu$ for $\mu > a_{\geq \mu}^S(M)$,
(b) \( \text{cd}_I(M_\mu) \) is constant for \( \mu \geq \text{reg}(M) + n - \text{depth}_{S_+}(M) \) if \( n > 0 \).

Proof. We may, and will, assume that \( \text{cd}_{S_+}(M) > 0 \), as the proposition is immediate when \( \text{cd}_{S_+}(M) = 0 \) by Lemma 2.1. We also remark that it suffices to prove the claim after the faithfully flat base change \( R \to R' \), hence we may further assume (making an invertible linear change of coordinates) by Remark 1.8 that the sequence \( (X_1, \ldots, X_n) \) is \( M \)-regular off \( V(S_+) \).

In particular the kernel of the map \( M \to M(1) \) induced by multiplication by \( X_n \) is contained in \( H^0_{S_+}(M) \). It follows an injection \( M_\nu \to M_{\nu+1} \) for \( \nu > a_0^{S_+}(M) \) which proves (a) by Proposition 4.4 as \( M_{\nu+1} \) is finitely generated over \( R \).

To prove (b) we consider the two converging spectral sequences arising from the double complex \( C^*_\bullet K_\bullet(X_1, \ldots, X_n; M) \). They have as respective second terms

\[
\text{Tor}_i^S(M_\nu(R), R) \quad \text{and} \quad H^0_i(M_\nu(R), R).
\]

Let \( d := \max_{\nu > a_0^{S_+}(M)} \{ \text{cd}_I(M_\mu) \} \). We may assume \( d \geq 0 \). It follows from the comparison of the spectral sequences that

\[
\text{Tor}_{i+1}^S(H^0_i(M_\nu(R), R), R) = H^0_i(M_{\nu+1})/S_iH^0_i(M_\nu) = 0
\]

if \( H^0_{i+1}(M_{\nu-i}) = 0 \) for \( 1 \leq i \leq n - 1 \) and \( H^0_{i+1}(\text{Tor}_i^S(M, R), R) = 0 \) for all \( i \).

But \( \text{Tor}_i(M, R) = 0 \) for \( \nu \geq \text{reg}(M) + i \) and \( \text{Tor}_i(M, R) = 0 \) for \( i > n - \text{depth}_{S_+}(M) \) by Lemma 1.2.

It follows that \( H^0_i(M_\nu) = 0 \) implies \( H^0_i(M_{\nu+1}) = 0 \) if

\[
\nu \geq \max\{a_0^{S_+}(M) + n, \text{reg}(M) + n - \text{depth}_{S_+}(M)\} = \text{reg}(M) + n - \text{depth}_{S_+}(M).
\]

This implies (b), in view of (a). \( \square \)

5. Associated Primes of the Graded Components of a Graded Module

Let \( S \) be a standard graded Noetherian algebra over a commutative ring \( R \).

Theorem 5.1. Let \( M \) be a graded \( S \)-module. then

\[
\bigcup_{\mu \in \mathbb{Z}} \text{Ass}_R(M_\mu) = \{ \mathfrak{p} \cap R, \mathfrak{p} \in \text{Ass}(M) \}.
\]

Proof. For \( \mu \in \mathbb{Z} \), let \( \mathfrak{p} \in \text{Ass}_R(M_\mu) \). There exists \( x \in M_\mu \) with \( \mathfrak{p} = \text{ann}_R(x) \).

Hence \( \mathfrak{p}R_\mathfrak{p} = \text{ann}_{R_\mathfrak{p}}(x) \). Let \( \Omega \) be a \( S \otimes_R R_\mathfrak{p} \)-ideal, maximal among those of the form \( \text{ann}_{S \otimes_R R_\mathfrak{p}}(y), y \in M \otimes_R R_\mathfrak{p} \), that contains \( \text{ann}_{R_\mathfrak{p}}(x) \). The ideal \( \Omega \) is associated to \( M \otimes_R R_\mathfrak{p} \), hence \( \mathfrak{p} := \Omega \cap S \) is associated to \( M \) and \( \mathfrak{p} \cap R = \mathfrak{p} \). One inclusion follows.

Conversely, let \( \mathfrak{p} \) be an ideal associated to \( M \). We need to show that \( \mathfrak{p} := \mathfrak{p} \cap R \) is associated to \( M_\mu \) for some \( \mu \). This will be the case if \( \mathfrak{p}R_\mathfrak{p} \) is associated \( (M_\mu)_\mathfrak{p} \), so that we may assume that \( R \) is local with maximal ideal \( \mathfrak{p} \). Let \( m \neq 0 \) in \( M \) such that \( \mathfrak{p}m = 0 \). If \( m_\nu \) is the degree \( \nu \) component of \( m \), one has \( \mathfrak{p}m_\nu = 0 \). Hence choosing \( \mu \) such that \( m_\mu \neq 0 \), one has \( \mathfrak{p} \subseteq \text{ann}_R(m_\mu) \), hence \( \mathfrak{p} = \text{ann}_R(m_\mu) \), as \( \mathfrak{p} \) is maximal. \( \square \)
Theorem 5.2. Let $M$ be a graded $S$-module. If $H^0_M(M) = 0$, then $\text{Ass}_R(M) \subseteq \text{Ass}_R(M_\mu)$ for all $\mu \geq \nu$.

Proof. Let $x_1, \ldots, x_n$ be generators of $S_1$ as an $R$-module and $T_1, \ldots, T_n$ be variables. Set $P := \sum_{\mu, \mu, \nu} x^\alpha T^\beta$. Then for any ideal $P$ of the same length as $P$, the polynomial $P$ is of bidegree $(\mu - \nu, \mu - \nu)$ for the bigraduation defined by setting $\deg(x_i) := (1, 0)$ and $\deg(T_i) := (0, 1)$, and $c(P) = (S_+)^{\mu - \nu}$. Theorem 5.3 shows that the kernel $K$ of the map

$$M[T] \xrightarrow{\times P} M[T]$$

is a submodule of $H^0_M(M)[T]$. Hence $K$ vanishes in bidegree $(\nu, \theta)$ for any $\theta$. In particular, it provides an injective map

$$M_\nu = M[T]_{\nu, 0} \xrightarrow{\times P} M[T]_{\mu, \mu - \nu}.$$

As $M[T]_{\mu, \mu - \nu}$ is a finite direct sum of copies of $M_\mu$, it follows that any associated prime of $M_\nu$ is associated to $M_\mu$. 

Theorem 5.3. Let $M$ be a finitely generated graded $S$-module and $A$ be the finite set $\cup_{\mu \in \mathbb{Z}} \text{Ass}_R(M_\mu)$ (Theorem 5.4). Set

$$j(M) := \max_{\mu \in A} \{a^0_{\mu}(H^0_R(M_\mu \otimes_R R_p))\} = a^0_{S_+}(M).$$

Then for any ideal $p \in \text{Spec}(R)$, $\ell_{R_p}(H^0_R(M_\mu \otimes_R R_p))$ is a nondecreasing function of $\mu$, for $\mu > j(M)$.

Proof. First notice that $H^0_R(M_\mu \otimes_R R_p) = 0$ if $p \notin A$. Let $p \in A$. We will prove that $\ell_{R_p}(H^0_R(M_\mu \otimes_R R_p))$ is a nondecreasing function of $\mu$, for $\mu > a^0_{S_+}(H^0_R(M_\mu \otimes_R R_p))$. The proof of Theorem 5.2 applied to the $S \otimes_R R_p$-module $H^0_R(M_\mu \otimes_R R_p)$, with $P := \sum x_i T_i$ provides an injective morphism of $R_p[T]$-modules

$$H^0_R(M_\mu \otimes_R R_p)_\mu[T] \xrightarrow{\times P} H^0_R(M_\mu \otimes_R R_p)_{\mu + 1}[T].$$

Let $R_p(T) := S^{-1}R_p[T]$ with $S$ the multiplicative system of polynomials whose coefficient ideal is the unit ideal. The above injection induces an injective morphism of $R_p(T)$-modules of finite length

$$H^0_R(M_\mu \otimes_R R_p)_{\mu} \otimes_{R_p} R_p(T) \xrightarrow{\times P} H^0_R(M_\mu \otimes_R R_p)_{\mu + 1} \otimes_{R_p} R_p(T).$$

For any $R_p$-module $N$ of finite length, the $R_p(T)$-module $N \otimes_{R_p} R_p(T)$ is a module of the same length as the $R_p$-module $N$. The conclusion follows. 

Corollary 5.4. Let $M$ be a finitely generated graded $S$-module. Then for any $p \in \text{Spec}(R)$ of height 0, $\ell_{R_p}(M_\mu \otimes_R R_p)$ is a nondecreasing function of $\mu$, for $\mu > a^0_{S_+}(M)$. In particular $M_\mu \otimes_R R_p = 0$ for all $\mu > \text{reg}(M)$ or $M_\mu \otimes_R R_p \neq 0$ for all $\mu > \text{reg}(M)$.

Proof. Notice that $M \otimes_R R_p = H^0_R(M \otimes_R R_p)$ as $p$ is of height 0. Also recall that $M$ (hence $M \otimes_R R_p$) is generated in degrees at most $\text{reg}(M)$. 

□
Lemma 5.5. Let $M$ be a graded $S$-module generated in degree at most $B$. Then,

(i) $M_\mu \neq 0 \Rightarrow M_{\mu + 1} \neq 0$, if $\mu > a^B_{<+}(M),$
(ii) $M_\mu \neq 0 \Leftrightarrow M_{\mu + 1} \neq 0$ if $\mu > \max\{B, a^B_{<+}(M)\},$
(iii) $M_\mu \neq 0 \Leftrightarrow M_{\mu + 1} \neq 0$ if $\mu > \text{reg}(M).

Proof. (i) follows from Theorem 5.2 as $R$ is Noetherian. Now (ii) and (iii) follows from (i), as $M_\mu = 0 \Rightarrow M_{\mu + 1} = 0$ for $\mu \geq B$ and $M$ is generated in degrees at most $\text{reg}(M)$. □

Remark 5.6. The proof of Theorem 5.2 shows that the above lemma holds without assuming that $R$ is Noetherian.

6. Duality results

6.1. Preliminaries on $\text{RHom}$. Let $(X, \mathcal{O}_X)$ be a ringed space and $Y$ be closed in $X$. For any complex $K^\bullet \in D(X)$, let $C(K^\bullet)$ be the Godement resolution of $K^\bullet$, and set

$- \text{R}\Gamma(X, K^\bullet) := C(K^\bullet)$ and $\text{R}\Gamma_Y(X, K^\bullet) := \Gamma_Y(X, C(K^\bullet))$ in $D(X),$

$- \text{R}\Gamma(X, K^\bullet) := \Gamma(X, C(K^\bullet))$ and $\text{R}\Gamma_Y(X, K^\bullet) := \Gamma_Y(X, C(K^\bullet))$ in $D(\Gamma(X, \mathcal{O}_X)).$

Notice that a flasque resolution of $K^\bullet$ in $D^+(X)$ (e.g. an injective resolution) can be used in place of $C(K^\bullet)$ if $K^\bullet \in D^+(X).

We set $H^i(X, K^\bullet) := H^i(\text{R}\Gamma(X, K^\bullet)), \quad H^i_Y(X, K^\bullet) := H^i(\text{R}\Gamma_Y(X, K^\bullet))$ and $\text{H}^i(X, K^\bullet) := H^i(\text{R}\Gamma(X, K^\bullet))$, which coincides with the usual notations for $\mathcal{O}_X$-modules when considered as complexes concentrated in degree 0.

If there exists $d$ such that, for any $\mathcal{O}_X$-module $\mathcal{E}, H^i(X, \mathcal{E}) = 0$ (resp. $H^i_Y(X, \mathcal{E}) = 0$) for $i > d$ then any flasque resolution of $K^\bullet$ can be used in place of $C(K^\bullet)$ to compute $H^i(X, K^\bullet)$ and $\text{H}^i(X, K^\bullet)$ (resp. $H^i_Y(X, K^\bullet)$).

Given $K^\bullet$ in $D(X)$ and $L^\bullet$ in $D^+(X)$ one checks that the class in $D(X)$ (resp. in $D(\Gamma(X, \mathcal{O}_X))$) of the complex $\text{Hom}^\bullet_{\mathcal{O}_X}(K^\bullet, I^\bullet)$ (resp. $\text{Hom}^\bullet_{\mathcal{O}_X}(K^\bullet, I^\bullet)$) is independent of the choice of an injective resolution $I^\bullet$ of $L^\bullet$ in $D^+(X)$, and one set

$\text{RHom}^\bullet_{\mathcal{O}_X}(K^\bullet, L^\bullet) := \text{Hom}^\bullet_{\mathcal{O}_X}(K^\bullet, I^\bullet)$ and $\text{RHom}^\bullet_{\mathcal{O}_X}(K^\bullet, L^\bullet) := \text{Hom}^\bullet_{\mathcal{O}_X}(K^\bullet, I^\bullet).

When the components of the complex $K^\bullet$ are locally free $\mathcal{O}_X$-modules, there is a quasi-isomorphism

$\text{RHom}^\bullet_{\mathcal{O}_X}(K^\bullet, L^\bullet) \simeq \text{Hom}^\bullet_{\mathcal{O}_X}(K^\bullet, L^\bullet).

For a pair $(\mathcal{E}, \mathcal{F})$ of $\mathcal{O}_X$-modules, one defines

$\text{Hom}_Y(\mathcal{E}, \mathcal{F}) := \Gamma_Y(X, \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})) = \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \Gamma_Y \mathcal{F})$

and

$\text{Hom}_Y(\mathcal{E}, \mathcal{F}) := \Gamma_Y \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) = \text{Hom}_{\mathcal{O}_X}(\mathcal{E}, \Gamma_Y \mathcal{F}),$

and then extends these definitions to pairs of complexes of $\mathcal{O}_X$-modules as usual.
Assume $L^\bullet$ is bounded below. Given a bounded below injective resolution $I^\bullet$ of $L^\bullet$, the components of the complex $\mathcal{H}om_{\mathcal{O}_X}(K^\bullet, I^\bullet)$ are flasques hence

$$R\Gamma_Y(R\mathcal{H}om_{\mathcal{O}_X}(K^\bullet, L^\bullet)) = R\Gamma_Y(\mathcal{H}om_{\mathcal{O}_X}(K^\bullet, I^\bullet)) = \Gamma_Y(\mathcal{H}om_{\mathcal{O}_X}(K^\bullet, I^\bullet)) = \mathcal{H}om_Y(K^\bullet, I^\bullet),$$

in the following cases:

(a) $K^\bullet$ is bounded below,

(b) there exists $d$ such that, for any $\mathcal{O}_X$-module $E$, $H^i_Y(X, E) = 0$ for $i > d$.

In cases (a) and (b) $\mathcal{H}om_Y(K^\bullet, I^\bullet)$ is independent of the choice of $I^\bullet$ (up to an isomorphism in $D(X)$), and one sets

$$R\Gamma_Y(K^\bullet, L^\bullet) := \mathcal{H}om_Y(K^\bullet, I^\bullet).$$

As the components of $\Gamma_Y I^\bullet$ are injective $\mathcal{O}_X$-modules, $\mathcal{H}om_Y(K^\bullet, I^\bullet) \simeq \mathcal{H}om_{\mathcal{O}_X}(K^\bullet, \Gamma_Y I^\bullet) \simeq R\mathcal{H}om_{\mathcal{O}_X}(K^\bullet, R\Gamma_Y L^\bullet)$. In other words,

$$R\Gamma_Y(K^\bullet, L^\bullet) \simeq R\mathcal{H}om_{\mathcal{O}_X}(K^\bullet, R\Gamma_Y L^\bullet)$$

if (a) or (b) holds.

### 6.2. Some spectral sequences.

Let $A$ be a commutative ring (with unit) and $I$ be a finitely generated $A$-ideal. Set $(X, \mathcal{O}_X) := (\text{Spec}(A), \hat{A})$ and $Y := V(I) \subset X$.

Assume that there exists $n$ such that

$$H^i_Y(A) = H^i_Y(X, \mathcal{O}_X) = 0, \text{ for } i \neq n.$$ 

Then $R\Gamma_Y(\mathcal{O}_X) \simeq H^0_Y(\mathcal{O}_X)[-n]$ in $D^+(X, \mathcal{O}_X)$. Given a complex $K^\bullet \in D^-(X, \mathcal{O}_X)$, it follows that

$$R\Gamma_Y(R\mathcal{H}om_{\mathcal{O}_X}(K^\bullet, \mathcal{O}_X)) \simeq R\mathcal{H}om_{\mathcal{O}_X}(K^\bullet, R\Gamma_Y(\mathcal{O}_X)) \simeq R\mathcal{H}om_{\mathcal{O}_X}(K^\bullet, H^0_Y(\mathcal{O}_X))[-n]$$

in $D(X, \mathcal{O}_X)$. Such an isomorphism holds for $K^\bullet \in D(X, \mathcal{O}_X)$ when $X$ has finite homological dimension, hence, for instance, if $A$ is Noetherian of finite dimension.

Assuming further that the components of $K^\bullet$ are locally free $\mathcal{O}_X$-modules of finite type, under one of the two hypotheses above, one has

$$R\Gamma_Y(\mathcal{H}om_{\mathcal{O}_X}(K^\bullet, \mathcal{O}_X)) \simeq \mathcal{H}om_{\mathcal{O}_X}(K^\bullet, H^0_Y(\mathcal{O}_X))[-n] \text{ in } D(X, \mathcal{O}_X).$$

This provides a spectral sequence

$$E_2^{p,q} = H^p_Y(H^q_Y(\mathcal{H}om_{\mathcal{O}_X}(K^\bullet, \mathcal{O}_X))) \Rightarrow H^{p+q-n}(\mathcal{H}om_{\mathcal{O}_X}(K^\bullet, H^0_Y(\mathcal{O}_X))),$$

in the two cases above.

As the $\mathcal{O}_X$-modules taking place in this spectral sequence are quasi-coherent, and the components of $K^\bullet$ are finitely presented (recall that $\mathcal{H}om_{\hat{A}}(M, N) = \mathcal{H}om_{\mathcal{O}_X}(\hat{M}, \hat{N})$ if $M$ is finitely presented) the above results shows the following Proposition.
Proposition 6.1. Let $A$ be a commutative ring and $I$ a finitely generated $A$-ideal. Assume that there exists $n$ such that

$$H_i^1(A) = H_i^1(X, O_X) = 0 \text{ for } i \neq n.$$ 

Then, for any bounded below complex of projective $A$-modules of finite type (resp. for any complex of projective $A$-modules of finite type if $A$ is Noetherian of finite dimension) $K^\bullet$, there exists a spectral sequence

$$E_2^{pq} = H^q_p(H^q\text{Hom}^\bullet_A(K^\bullet, A)) \Rightarrow H^{p+q-n}(\text{Hom}^\bullet_A(K^\bullet, H_i^1(A))).$$

Corollary 6.2. Assume $(A, m)$ is local Gorenstein of dimension $n$. Then for any complex $K^\bullet$ of free $A$-modules of finite type, there is a spectral sequence

$$E_2^{pq} = H^q_m(H^q\text{Hom}^\bullet_A(K^\bullet, A)) \Rightarrow H^{p+q-n}(\text{Hom}^\bullet_A(K^\bullet, H_m^n(A))).$$

Proof. Under the hypotheses of the Corollary, $H_m^n(A) = 0$ for $i \neq n$ and $H_m^n(A)$ is injective. □

Example 6.3. In the context of the Corollary, taking for $K^\bullet$ the dual of a resolution of finitely generated module $M$ by free modules of finite type it gives the local duality

$$H^n_0(M) \cong \text{Hom}_A(\text{Ext}_A^{n-p}(M, A), H_m^n(A)).$$

Corollary 6.4. Assume $I$ is generated by a weakly regular sequence of length $n$, then for any bounded below complex $K^\bullet$ of projective $A$-modules of finite type, there is a spectral sequence

$$E_2^{pq} = H^q_p(H^q\text{Hom}^\bullet_A(K^\bullet, A)) \Rightarrow H^{p+q-n}(\text{Hom}^\bullet_A(K^\bullet, H_i^1(A))).$$

Example 6.5. Let $R$ be a commutative ring, $X_i$ for $1 \leq i \leq n$ be indeterminates, set $A := R[X_1, \ldots, X_n]$, with its standard grading and $p := (X_1, \ldots, X_n)$. As $H_p^n(A) \cong (X_1 \cdots X_n)^{-1}R[X_1^{-1}, \ldots, X_n^{-1}]$, it follows that for any graded free $A$-module of finite type $F$ and every integer $\nu$, the pairing

$$\text{Hom}_A(F, H_p^n(A))_{-\nu-n} \otimes_R F_{\nu} \longrightarrow H_p^n(A)_{-n} \cong R$$

$$(u : F \to H_p^n(A)(-\nu-n)) \otimes_R x \longmapsto u(x)$$

defines a perfect duality between $R$-modules of finite type, and this duality is functorial in the free graded $A$-module $F$. It gives, for each integer $\nu$, an isomorphism of complexes of $R$-modules

$$\text{Hom}^\bullet_A(K^\bullet, H_p^n(A))_{-\nu-n} \cong \text{Hom}^\bullet_R(K^\bullet, R).$$

Together with Corollary 6.4 it gives for any $\nu$ a spectral sequence of $R$-modules

$$E_2^{pq} = H^q_p(H^q\text{Hom}^\bullet_A(K^\bullet, A))_{\nu} \Rightarrow H^{p+q-n}(\text{Hom}^\bullet_R(K^\bullet_{-\nu-n}, R)).$$

Replacing $K^\bullet$ by its dual $F^\bullet$, one deduces of a spectral sequence

$$E_2^{pq} = H^q_p(H^q(F^\bullet))_{\nu} \Rightarrow H^{p+q-n}(\text{Hom}^\bullet_R(\text{Hom}^\bullet_A(F^\bullet, A)_{-\nu-n}, R)).$$
For instance, if $M$ is a graded $A$-module admitting a resolution by free modules of finite rank, taking for $K^\bullet$ such a resolution, that one may assume to be graded, it follows a spectral sequence:

$$E_2^{p,q} = H^p_\nu(\text{Ext}_A^q(M, A)) \Rightarrow \text{Ext}_{n\nu}^{p+q-n}(M_{-\nu-n}, R).$$

6.3. The Herzog-Rahimi spectral sequences. We keep notations as in the preceding subsection. For any graded complex $K^\bullet$ whose components are of finite type, we have established an isomorphism

$$R\Gamma_p(\text{Hom}_A^\bullet(K^\bullet, A)) \cong \text{Hom}_A^\bullet(K^\bullet, H_p^n(A))[-n] \text{ in } D(A),$$

whenever $K^\bullet$ is bounded above or $A$ is Noetherian of finite dimension.

In each of these cases, it follows, for any integer $\nu$, isomorphisms

$$R\Gamma_p(\text{Hom}_A^\bullet(K^\bullet, A))_\nu \cong \text{Hom}_R^\bullet((K^\bullet, A)_{-\nu-n}, R)[-n] \text{ in } D(R).$$

Hence if $F^\bullet$ is a graded complex of finite free $A$-modules, and either $A$ is Noetherian of finite dimension or $F^\bullet$ is bounded below, one has isomorphisms

$$R\Gamma_p(F^\bullet)_\nu \cong \text{Hom}_R^\bullet((\text{Hom}_A(F^\bullet, A)_{-\nu-n}, R)[-n]$$

$$\cong R\text{Hom}_R^\bullet((\text{Hom}_A(F^\bullet, A)_{-\nu-n}, R), R)[-n] \text{ in } D(R).$$

Now, assume further that $(R, m)$ is local Gorenstein of dimension $d$. The above isomorphisms then give

$$R\text{Hom}_R^\bullet(R\Gamma_p(F^\bullet)_\nu, H^d_m(R)) \cong R\text{Hom}_R^\bullet(R\Gamma_p(F^\bullet)_\nu, R\Gamma_m(R))[d]$$

$$\cong R\Gamma_m(R\text{Hom}_R^\bullet(R\Gamma_p(F^\bullet)_\nu), R)[d]$$

$$\cong R\Gamma_m(R\text{Hom}_R^\bullet(R\text{Hom}_R^\bullet((\text{Hom}_A(F^\bullet, A)_{-\nu-n}, R), R), R), R)))[n+d]$$

$$\cong R\Gamma_m(\text{Hom}_R^\bullet(F^\bullet, A)_{-\nu-n})[n+d].$$

As $R$ is Gorenstein, $H^d_m(R)$ is the injective envelope of the residue field of $R$, and we obtain a spectral sequence

$$E_2^{p,q} = H^p_m(H^q(\text{Hom}_A^\bullet(F^\bullet, A)_{-\nu-n})) \Rightarrow \text{Hom}_R(H^{n+d-p-q}(R\Gamma_p(F^\bullet)_\nu), H^d_m(R)).$$

If $M$ is graded $A$-module with free resolution $F^\bullet$, this spectral sequence takes the form

$$E_2^{p,q} = H^p_m(\text{Ext}_A^q(M, A)_{-\nu-n}) \Rightarrow \text{Hom}_R(H^{n+d-p-q}(M)_\nu, H^d_m(R)).$$

Which is the Herzog-Rahimi spectral sequence, as $\omega_A \cong A(-n)$ in this situation.

7. Tameness of local cohomology over Noetherian rings

In this section $S$ is a finitely generated standard graded algebra over an epimorphic image $R$ of a Gorenstein ring.

**Theorem 7.1.** Let $(R, m)$ be a local Noetherian Gorenstein ring of dimension $d$, $S$ be a finitely generated standard graded Cohen-Macaulay algebra over $R$ and $M$ be a finitely generated graded $S$-module. Set $-^\nu := \text{Hom}_R(-, H^d_m(R))$. Then, there
is a spectral sequence

\[ E_2^{ij} = H_m(\text{Ext}^j_S(M, \omega_S)) \Rightarrow (H^{\dim S-(i+j)}_S(M))^{\gamma}. \]

**Proof.** See [R, section 4] or the previous section. \(\square\)

For a \(R\)-module or a \(S\)-module \(M\), we set

\[ H_0^0(M) := \bigcup_{i \leq R, \dim(R/I) \leq i} H_i^0(M) \subseteq M. \]

We will need the following facts about the functor \(H_0^0(\_\_\_)\).

**Lemma 7.2.** Let \(M\) be a graded \(S\)-module. Then

(i) \(H_0^0(M)^{\gamma} = H_0^0(M_\gamma).\)

(ii) If \(p\) a prime ideal of \(R\) with \(\dim(R/p) = i\), then

\[ H_0^0(M) \otimes_R R_p = H_0^0(M \otimes_R R_p). \]

**Proof.** Claim (i) follows from the fact that, for any \(R\)-ideal \(I\), \(H_0^0(M)^{\gamma} = H_0^0(M_\gamma).\) For (ii), recall that inductive limits commute with tensor products and notice that if \(\dim(R/I) \leq i\), then \(H_0^0(M \otimes_R R_p) = 0\) if \(I \not\subseteq p\), and \(H_0^0(M \otimes_R R_p) = H_0^0(M \otimes_R R_p)\) else. \(\square\)

**Theorem 7.3.** Let \(S\) be a polynomial ring in \(n\) variables over an equidimensional Gorenstein ring \(R\) of dimension \(d\). Let \(M\) be a finitely generated graded \(S\)-module.

Then there exists \(A, B, C, D, E\) defined below such that:

(1a) For \(\gamma > A\), \(\dim H_0^0(S \otimes_{S \otimes R} (\text{Ext}^j_S(M, \omega_S))) \leq \text{end}(H_0^0(S \otimes_{S \otimes R} (\text{Ext}^j_S(M, \omega_S))))\) and \(r_j^i := \text{reg}(H_0^0(S \otimes_{S \otimes R} (\text{Ext}^j_S(M, \omega_S))))\). Then one has:

\[ A := a_{0,0}^{n_0}, B := r_0^{n_0}, C := \max\{a_1^{n_1+1}, a_0^{n_0-1}\}, D := \max\{r_1^{n_1+1}, r_0^{n_0-1}\} \text{ and } E := \max\{a_0^{n_0}, r_0^{n_1+1} - 1, r_2^{n_0+1} - 1, a_2^{n_2+2}\}. \]

**Proof.** Recall that if \(N\) is a finitely generated \(R\)-module, \(\dim N < r\) if and only if \(N_p = 0\) for all \(p \in \text{Spec}(R)\) with \(\dim(R/p) = r\). Furthermore, it follows from Lemma 7.2 and from Lemma 2.5 and its proof that for any \(p \in \text{Spec}(R)\) with \(\dim(R/p) = r\) and for any \(\ell\), one has

\[ a_{S \otimes_R R_p}^d(H_0^0(\text{Ext}^i_{S \otimes_R R_p}(M \otimes_R R_p, \omega_{S \otimes_R R_p}))) \leq a_{S \otimes_R R_p}^d(H_0^0(\text{Ext}^i_{S \otimes_R R_p}(M \otimes_R R_p, \omega_{S \otimes_R R_p}))). \]

Therefore, \(\text{reg}(H_0^0(\text{Ext}^i_{S \otimes_R R_p}(M \otimes_R R_p, \omega_{S \otimes_R R_p}))) \leq r_i^{n_0-1}\), and, as a consequence, it suffices to prove (1a) and (1b) when \(\dim R = 0\), (2a) and (2b) when \(\dim R = 1\), and (3) when \(\dim R = 2\).
Notice that $H^0_N(N) = N$ for any $S$-module $N$.

If $\dim R = 0$, by Theorem 5.1, $(H^i_{p_0}(M)_{-\gamma})^\vee \simeq \text{Ext}^{n-i}_S(M, \omega_S)_{\gamma}$, and the result follows from Lemma 5.2(i) and (iii).

If $\dim R = 1$, Theorem 5.1 provides exact sequences

$$0 \to H^1_m(\text{Ext}^{n-i}_S(M, \omega_S)_{\gamma}) \to (H^2_{p_0}(M)_{-\gamma})^\vee \to H^1_m(\text{Ext}^{n+1}_S(M, \omega_S)_{\gamma}) \to 0.$$

The result follows applying Lemma 5.2(i) and (iii) to $H^0_m(\text{Ext}^{n-i}_S(M, \omega_S))$ and Corollary 5.4 to $\text{Ext}^{n-i}_S(M, \omega_S)_{\gamma}$. Indeed, $H^1_m(\text{Ext}^{n-i}_S(M, \omega_S)_{\gamma})$ is zero if and only if $\dim \text{Ext}^{n-i}_S(M, \omega_S)_{\gamma} < 1$.

We now assume that $\dim R = 2$. In this case Theorem 5.1 provides a spectral sequence which converges to $(H^i_{p_0}(M)_{-\gamma})^\vee$.

$$\cdots \Rightarrow H^0_m(\text{Ext}^{n+1}_S(M, \omega_S)_{\gamma}) \Rightarrow H^1_m(\text{Ext}^{n+1}_S(M, \omega_S)_{\gamma}) \Rightarrow \cdots$$

It provides a filtration $F^0_\gamma \subseteq F^1_\gamma \subseteq F^2_\gamma = (H^2_{p_0}(M)_{-\gamma})^\vee$, by graded $S$-modules, such that $F^2_\gamma / F^1_\gamma \simeq \ker(\psi^{n-i+1}_\gamma)$, $F^1_\gamma / F^0_\gamma \simeq H^1_m(\text{Ext}^{n+1}_S(M, \omega_S)_{\gamma})$ and $F^0_\gamma \simeq \text{coker}(\psi^{n-i+1}_\gamma)$.

We will show that, the three modules satisfy:

(i) $F^0_\gamma \neq 0 \Rightarrow F^0_{\gamma+1} \neq 0$, if $\gamma > a^0_{n-i}$,

(ii) $F^1_\gamma / F^0_\gamma \neq 0 \Rightarrow F^1_{\gamma+1} / F^1_{\gamma+1} \neq 0$ if $\gamma > \max\{r^0_{n-i+1} - 1, r^n_{n-i+1} - 2\}$,

(iii) $F^2_\gamma / F^1_\gamma \neq 0 \Rightarrow F^2_{\gamma+1} / F^2_{\gamma+1} \neq 0$ if $\gamma > a^2_{n-i+2}$.

For (i) notice that $\text{coker}(\psi^{n-i+1}_\gamma) = 0$ if and only if $H^1_m(\text{Ext}^{n-i}_S(M, \omega_S)_{\gamma}) = 0$, hence if and only if $\dim(\text{Ext}^{n-i}_S(M, \omega_S)_{\gamma}) < 2$. Hence (i) follows from Corollary 5.4.

For (ii), let $N := \text{Ext}^{n-i+1}_S(M, \omega_S) / H^1_m(\text{Ext}^{n-i+1}_S(M, \omega_S))$. The exact sequence

$$0 \to H^1_m(\text{Ext}^{n-i+1}_S(M, \omega_S)) \to \text{Ext}^{n-i+1}_S(M, \omega_S) \to N \to 0$$

shows that $H^0_m(N) = 0$ (hence depth($N) \geq 1$),

$$F^1_\gamma / F^0_\gamma = H^1_m(\text{Ext}^{n-i+1}_S(M, \omega_S)) \simeq H^1_m(N)$$

and

$$a^1_{S_+}(N) \leq \max\{a^1_{S_+}(\text{Ext}^{n-i+1}_S(M, \omega_S)), a^2_{S_+}(H^0_m(\text{Ext}^{n-i+1}_S(M, \omega_S))))\} \leq \max\{r^0_{n-i+1} - 1, r^n_{n-i+1} - 2\}.$$ 

Hence, Proposition 3.1(ii) implies (ii).

For (iii), let $\alpha \in H^0_m(\text{Ext}^{n-i+2}_S(M, \omega_S)) = H^0_m(\text{Ext}^{n-i+2}_S(M, \omega_S))_{\gamma}$.
Let \( x_1, \ldots, x_t \) be generators of \( S_1 \) as an \( R \)-module and set \( \ell := \sum_i x_i T_i \in S[T] \).

For \( \gamma > a_S^0 (H^0_m(\text{Ext}_S^{n-i+2}(M, \omega_S))) \), \( \ell \alpha \) is not zero in \( H^0_m(\text{Ext}_S^{n-i+2}(M, \omega_S)_{\gamma+1})[T] \) by Theorem 1.5 (see also the proof of Theorem 5.3). The commutative diagram

\[
\begin{array}{ccc}
H^0_m(\text{Ext}_S^{n-i+2}(M, \omega_S)_\gamma) & \xrightarrow{\times \ell} & H^0_m(\text{Ext}_S^{n-i+2}(M, \omega_S)_{\gamma+1})[T] \\
\psi_{n-i+2}^\gamma & \downarrow & \\
H^2_m(\text{Ext}_S^{n-i+1}(M, \omega_S)_\gamma) & \xrightarrow{\times \ell} & H^2_m(\text{Ext}_S^{n-i+1}(M, \omega_S)_{\gamma+1})[T]
\end{array}
\]

then shows that \( \psi_{n-i+2}^\gamma (\alpha) \neq 0 \) if \( \psi_{n-i+2}^{\gamma+1} \) is injective. Hence, \( \psi_{n-i+2}^\gamma \) is injective if \( \psi_{n-i+2}^{\gamma+1} \) is. Claim (iii) follows. \( \square \)

**Theorem 7.4.** Let \( S \) be a Noetherian standard graded algebra over a commutative ring \( R \). Assume \( R \) has dimension at most two and either \( R \) is an epimorphic image of a Gorenstein ring or \( R \) is local. Let \( M \) be a finitely generated graded \( S \)-module.

Then there exists \( \gamma_0 \) such that, for any \( i \),

\[
\{ H^i_{S_k}(M)_\gamma = 0 \text{ for } \gamma < \gamma_0 \} \text{ or } \{ H^i_{S_k}(M)_\gamma \neq 0 \text{ for } \gamma < \gamma_0 \}.
\]

**Proof.** First, if \( R \) is local, then we can complete \( R \) to reduce to the case where \( R \) is a quotient of a regular ring (by Cohen structure theorem), hence an epimorphic image of a Gorenstein ring.

As a Gorenstein ring is a finite product of equidimensional Gorenstein rings, and each such ring is itself a quotient of a Gorenstein ring of any bigger dimension, \( R \) is also a quotient of an equidimensionnal Gorenstein ring \( R' \). We further remark that \( R \) is the epimorphic image of \( R'/K \), where \( K \) is generated by a regular sequence of length \( \dim R' - 2 \) in \( R' \).

Thus we may, and will, assume that \( R \) is an equidimensional Gorenstein ring of dimension at most two. Now \( S \) is an epimorphic image of a polynomial ring in a finite number of variables over \( R \), so that we may, and will, also assume that \( S \) is a polynomial ring over \( R \).

The result then follows from Theorem 5.3. \( \square \)

**References**


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