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# CLASSIFICATION OF RANDOM TIMES AND APPLICATIONS

ANNA AKSAMIT, TAHIR CHOULLI AND MONIQUE JEANBLANC

ABSTRACT. The paper gathers together ideas related to thin random time, i.e., random time whose graph is contained in a thin set. The concept naturally completes the studies of random times and progressive enlargement of filtrations. We develop classification and  $(*)$ -decomposition of random times, which is analogous to the decomposition of a stopping time into totally inaccessible and accessible parts, and we show applications to the hypothesis  $(\mathcal{H}')$ , honest times and informational drift via entropy.

## 1. INTRODUCTION

The paper develops ideas related to thin random times. The concept naturally fits and completes the studies of random times and progressive enlargement of filtrations. A random time defined on a filtered probability space  $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$  is simply a random variable with values in  $[0, \infty]$ . In the literature of enlargement of filtration, it is common to assume that a given random time  $\tau$  avoids all  $\mathbb{F}$ -stopping times, i.e.,  $\mathbb{P}(\tau = T < \infty) = 0$  for any  $\mathbb{F}$ -stopping time  $T$ . Here, we take a closer look at this condition and make it a starting point to define two classes of random times. We come up with strict random times which basically satisfy avoidance condition and thin random times which satisfy *counter* property, i.e., they are fully built of  $\mathbb{F}$ -stopping times. The notion of thin random time was mentioned, but not developed, for the first time in Dellacherie and Meyer [10] under the name *variable aléatoire arlequine* referring to the costume of the Harlequin which is made of patches of different colors.

We begin, in Section 2, with defining and studying  $(*)$ -decomposition of a random time into strict and thin parts. The  $(*)$ -decomposition is congruent with the decomposition of a stopping time into totally inaccessible and accessible parts. Our study strongly relies on the notion of dual optional projection, we make use of other processes linked to general theory of stochastic processes, in particular, to enlargement of filtration theory. The main result, stated in Theorem 2.4, says that any random time can be  $(*)$ -decomposed and gives strict and thin random time characterisations in terms of its dual optional projection.

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*Key words and phrases.* thin random times, avoidance of stopping time, dual optional projection, progressive enlargement of filtration, hypothesis  $(\mathcal{H}')$ , honest times, entropy.

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In Section 3 we relate to progressive enlargement of filtration and the hypothesis  $(\mathcal{H}')$ . For a random time  $\tau$ ,  $\mathbb{F}^\tau$  is the filtration  $\mathbb{F}$  progressively enlarged with  $\tau$ , i.e.,

$$(1.1) \quad \mathcal{F}_t^\tau := \bigcap_{s>t} (\mathcal{F}_s \vee \sigma(\tau \wedge s)) \quad \text{for any } t.$$

The hypothesis  $(\mathcal{H}')$  holds for  $(\mathbb{F}, \mathbb{F}^\tau)$  if any  $\mathbb{F}$ -martingale is an  $\mathbb{F}^\tau$ -semimartingale. The main result here is Theorem 3.1 where we establish the hypothesis  $(\mathcal{H}')$  for thin random times. It extends previous results by Jeulin [17] which deal with countably valued random times. Instead of countably many real values we chose countably many  $\mathbb{F}$ -stopping times which are already captured in reference filtration  $\mathbb{F}$ . We may see Theorem 3.1 as an alternative direction of development of the result by Jacod (see [17, Theorem 3,2] and [21]) on the hypothesis  $(\mathcal{H}')$  in initial enlargement with atomic  $\sigma$ -field to the direction based on density hypothesis in progressive setting (initial times from [15]).

Honest times are important and a well studied class of random times; roughly speaking they are last passage times and we refer to them in Section 4. Adopting the notion of jumping filtration from Jacod and Skorokhod [14] we show in Theorem 4.8 that such a filtration can only support honest times which are thin. That includes compound Poisson process filtration. In [14] the link between jumping filtration and finite variation martingales is established; problems related to purely discontinuous martingale filtrations are treated in Hannig [12]. In Section 4 we also exploit two examples of thin honest times: last passage time at a barrier  $a$  of a Compound Poisson process and an example based on Brownian motion local time approximation. Some auxiliary results on honest times are collected in Appendix B.

The additional information carried by enlarged filtration and its measurement was studied by several authors. Already in Meyer [21] and Yor [25], the question on stability of martingale spaces with respect to initial enlargement with atomic  $\sigma$ -field was asked. From more recent studies, generalizing and applying previous results in different contexts, we refer the reader to [3, 4, 5, 6]. Here we define the entropy of a thin random time by (5.2) and we prove in Theorem 5.2 that its finiteness is enough for stability of some martingale spaces in progressive setting. Theorem 5.2 reveals that the notion of the entropy of thin random time is a correct one and is an answer to the question asked in [21] about additional knowledge associated with a partition and disclosed in progressive manner:

*Un problème voisin, mais plus intéressant peut-être, consiste à mesurer le bouleversement produit, sur un système probabiliste, non pas en forçant des connaissances à l'instant 0, mais en les forçant progressivement dans le système.*<sup>1</sup>

In Section 6 we collect in Theorems 6.2 and 6.3 the results concerning the hypothesis  $(\mathcal{H}')$  and entropy for more general progressive enlargement of filtration. In contrary to previous results, adding several members of a partition at the same time is allowed in this case.

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<sup>1</sup>For the reader's convenience we provide an English translation:

*A similar problem, but perhaps of more interest, consists in measuring the resulting perturbation, in a probabilistic system, not by requiring knowledge at the instant 0, but by adding them progressively to the system.*

For any càdlàg process  $X$  we will denote by  $X_-$  the left-continuous version of  $X$ , by  $\Delta X$  the jump of  $X$  and by  $X_\infty$  the limit  $\lim_{t \rightarrow \infty} X_t$  if it exists. The process  $X$  is said to be increasing if for almost all  $\omega$  it satisfies  $X_t(\omega) \geq X_s(\omega)$  for all  $t \geq s$ . The random variable is said to be positive if it has values in  $[0, \infty)$ .

## 2. THE $(*)$ -DECOMPOSITION

Let  $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$  be a filtered probability space, where  $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$  denotes a filtration satisfying the usual conditions, and such that  $\mathcal{F}_\infty \subset \mathcal{G}$ . Consider a random time  $\tau$ , i.e., a random variable with values in  $[0, \infty]$ . Note that a random time  $\tau$  is not necessarily  $\mathcal{F}_\infty$ -measurable. For a random time  $\tau$  we denote by  $\llbracket \tau \rrbracket$  its graph.

The following definition contains the leading idea of the paper. It discriminates two classes of random times using a criterion based on  $\mathbb{F}$ -stopping times.

**Definition 2.1.** *A random time  $\tau$  is called*

(a) *a strict random time if  $\llbracket \tau \rrbracket \cap \llbracket T \rrbracket = \emptyset$  for any  $\mathbb{F}$ -stopping time  $T$ , i.e., if it avoids all  $\mathbb{F}$ -stopping times.*

(b) *a thin random time if its graph  $\llbracket \tau \rrbracket$  is contained in a thin set, i.e., if there exists a sequence of  $\mathbb{F}$ -stopping times  $(T_n)_{n=1}^\infty$  with disjoint graphs such that  $\llbracket \tau \rrbracket \subset \bigcup_n \llbracket T_n \rrbracket$ . We say that such a sequence  $(T_n)_n$  exhausts the thin random time  $\tau$  or that  $(T_n)_n$  is an exhausting sequence of the thin random time  $\tau$ .*

Note that a thin random time  $\tau$  is built of  $\mathbb{F}$ -stopping times, i.e.,  $\tau = \infty \mathbb{1}_{C_0} + \sum_n T_n \mathbb{1}_{C_n}$  where  $(T_n)_n$  is an exhausting sequence for  $\tau$ , and

$$(2.1) \quad C_0 := \{\tau = \infty\} \quad \text{and} \quad C_n := \{\tau = T_n < \infty\} \quad \text{for } n \geq 1.$$

We denote by  $z^n$  the càdlàg  $\mathbb{F}$ -martingale with terminal value  $\mathbb{P}(C_n | \mathcal{F}_\infty)$ , namely

$$(2.2) \quad z_t^n := \mathbb{P}(C_n | \mathcal{F}_t).$$

Let us also remark that an exhausting sequence  $(T_n)_n$  of a thin random time is not unique.

The straightforward observation that the two classes of random times have trivial intersection is stated in the following lemma.

**Lemma 2.2.** *A random time  $\tau$  belongs to the class of strict random times and to the class of thin random times if and only if  $\tau = \infty$ .*

The main concept of this section, the  $(*)$ -decomposition, is presented in the next definition. It is followed by the result stating the existence of such a decomposition for any random time and some equivalent characterisations of two classes of random times.

**Definition 2.3.** *Consider a random time  $\tau$ . A pair of random times  $(\tau_1, \tau_2)$  is called a  $(*)$ -decomposition of  $\tau$  if  $\tau_1$  is a strict random time,  $\tau_2$  is a thin random time, and*

$$\tau = \tau_1 \wedge \tau_2 \quad \tau_1 \vee \tau_2 = \infty.$$

Before presenting Theorem 2.4 let us recall, following [17], some useful processes associated with  $\tau$ . For the process  $A := \mathbb{1}_{\llbracket \tau, \infty \llbracket}$ , we denote by  $A^p$  its  $\mathbb{F}$ -dual predictable projection and by  $A^o$  its  $\mathbb{F}$ -dual optional projection (see Appendix A). By the abuse of language,  $A^o$  is also called the dual optional projection of the random time  $\tau$ . We also define two  $\mathbb{F}$ -supermartingales  $Z$  and  $\tilde{Z}$  as optional projections of process  $1 - A_-$  and  $1 - A$  respectively, i.e.,

$$Z_t := {}^o \left[ \mathbb{1}_{\llbracket 0, \tau \llbracket} \right]_t = \mathbb{P}(\tau > t | \mathcal{F}_t) \quad \text{and} \quad \tilde{Z}_t := {}^o \left[ \mathbb{1}_{\llbracket 0, \tau \llbracket} \right]_t = \mathbb{P}(\tau \geq t | \mathcal{F}_t).$$

Since the dual optional projection  $A^o$  will play a crucial role in the paper, we recall two equalities where it appears:

$$(2.3) \quad A^o = m - Z \quad \text{and} \quad \Delta A^o = \tilde{Z} - Z,$$

where  $m$  is a BMO  $\mathbb{F}$ -martingale. Furthermore,  $\tilde{Z} = Z_- + \Delta m$ .

**Theorem 2.4.** (a) Any random time  $\tau$  has a  $(*)$ -decomposition  $(\tau_1, \tau_2)$  which is unique on the set  $\{\tau < \infty\}$ .

(b) A random time is a strict random time if and only if its dual optional projection is a continuous process.

(c) A random time is a thin random time if and only if its dual optional projection is a pure jump process.

*Proof.* (a) It is enough to take  $\tau_1$  and  $\tau_2$  of the following form  $\tau_1 = \tau_{\{\Delta A^o = 0\}}$  and  $\tau_2 = \tau_{\{\Delta A^o > 0\}}$ , where  $\tau_C$  is the restriction of the random time  $\tau$  to the set  $C$ , defined as  $\tau_C = \tau \mathbb{1}_C + \infty \mathbb{1}_{C^c}$ . Properties of dual optional projection ensure that  $\tau_1$  and  $\tau_2$  satisfy the required conditions. More precisely, the time  $\tau_1$  is a strict random time as, for any  $\mathbb{F}$ -stopping time  $T$ ,

$$\begin{aligned} \mathbb{P}(\tau_1 = T < \infty) &= \mathbb{E} \left[ \mathbb{1}_{\{\tau = T\} \cap \{\Delta A^o = 0\}} \mathbb{1}_{\{T < \infty\}} \right] \\ &= \mathbb{E} \left[ \int_0^\infty \mathbb{1}_{\{u = T\} \cap \{\Delta A_u^o = 0\}} dA_u^o \right] = 0. \end{aligned}$$

and the time  $\tau_2$  is a thin random time as

$$\llbracket \tau_2 \rrbracket = \llbracket \tau \rrbracket \cap \{\Delta A^o > 0\} = \llbracket \tau \rrbracket \cap \bigcup_n \llbracket T_n \rrbracket \subset \bigcup_n \llbracket T_n \rrbracket,$$

where the sequence  $(T_n)_n$  exhausts the jumps of the càdlàg process  $A^o$ , i.e.,  $\{\Delta A^o > 0\} = \bigcup_n \llbracket T_n \rrbracket$ .

(b) Let  $T$  be an  $\mathbb{F}$ -stopping time. Since  $\mathbb{E}(\Delta A_T^o \mathbb{1}_{\{T < \infty\}}) = \mathbb{P}(\tau = T < \infty)$  and  $A^o$  is an increasing process we deduce that

$$\mathbb{P}(\tau = T < \infty) = 0 \quad \text{if and only if} \quad \Delta A_T^o \mathbb{1}_{\{T < \infty\}} = 0 \quad \mathbb{P}\text{-a.s.}$$

Since  $\{\Delta A^o > 0\}$  is an optional set, optional section theorem [13, Theorem 4.7] implies that  $\{\Delta A^o > 0\}$  is exhausted by disjoint graphs of  $\mathbb{F}$ -stopping times. Thus, we conclude

that  $\tau$  is a strict random time if and only if  $A^\circ$  is continuous.

(c) For  $(T_n)_n$  a sequence of  $\mathbb{F}$ -stopping times with disjoint graphs, we have

$$\sum_n \mathbb{P}(\tau = T_n < \infty) = \sum_n \mathbb{E}(\Delta A_{T_n}^\circ \mathbb{1}_{\{T_n < \infty\}}).$$

Since  $\mathbb{E}(A_\infty^\circ) = \mathbb{P}(\tau < \infty)$ , by definition of the dual optional projection and using the fact that  $A^\circ$  is an increasing process, we conclude that the sequence  $(T_n)_n$  satisfies the condition  $\sum_n \mathbb{P}(\tau = T_n < \infty) = \mathbb{P}(\tau < \infty)$  if and only if it satisfies the condition  $\mathbb{E}(A_\infty^\circ) = \sum_n \mathbb{E}(\Delta A_{T_n}^\circ \mathbb{1}_{\{T_n < \infty\}})$ . In other words,  $\tau$  is a thin random time if and only if  $A^\circ$  is a pure jump process.  $\square$

For  $i \in \{1, 2\}$ , corresponding to the two  $(*)$ -parts of a random time, i.e.,  $\tau_1$  and  $\tau_2$ , we define  $A^i := \mathbb{1}_{\llbracket \tau_i, \infty \rrbracket}$ . Then  $A^{i,p}$  and  $A^{i,o}$  are respectively the  $\mathbb{F}$ -dual predictable projection and the  $\mathbb{F}$ -dual optional projection of  $A^i$ . Let us denote by  $Z^i$  and  $\tilde{Z}^i$  the supermartingales associated with  $\tau_i$ . Then, the following relations hold.

**Lemma 2.5.** *Let  $\tau$  be a random time and  $(\tau_1, \tau_2)$  its  $(*)$ -decomposition. Then, the supermartingales  $Z$  and  $\tilde{Z}$  can be decomposed in terms of the supermartingales  $Z^1, Z^2$  and  $\tilde{Z}^1, \tilde{Z}^2$  as:*

$$Z = Z^1 + Z^2 - 1 \quad \text{and} \quad \tilde{Z} = \tilde{Z}^1 + \tilde{Z}^2 - 1.$$

*Proof.* The result follows from the property that  $\tau_1 \vee \tau_2 = \infty$ .  $\square$

The next result gives the supermartingales  $Z$  and  $\tilde{Z}$  of a thin random time and their decompositions into  $\mathbb{F}$ -martingale  $m$  and finite variation process  $A^\circ$  in terms of the exhausting sequence of  $\tau$  and of the associated  $\mathbb{F}$ -martingales defined in (2.2).

**Lemma 2.6.** *Let  $\tau$  be a thin random time with exhausting sequence  $(T_n)_{n \geq 1}$  and  $(z^n)_{n \geq 1}$  be the family of  $\mathbb{F}$ -martingales associated with  $\tau$  through (2.2). Then*

- (a)  $z^n > 0$  and  $z_-^n > 0$  a.s. on  $C_n$  for each  $n$ ,
- (b)  $1 - Z_\tau > 0$  a.s. on  $\{\tau < \infty\}$ ,
- (c)  $\tilde{Z}_t = \sum_n \mathbb{1}_{\{t \leq T_n\}} z_t^n$ ,  $Z_t = \sum_n \mathbb{1}_{\{t < T_n\}} z_t^n$ ,  $A_t^\circ = \sum_n \mathbb{1}_{\{t \geq T_n\}} z_{T_n}^n$  and  $m_t = \sum_n z_{t \wedge T_n}^n$ .

*Proof.* (a) Define the  $\mathbb{F}$ -stopping time

$$(2.4) \quad R^n := \inf\{t \geq 0 : z_t^n = 0\}.$$

As  $z^n$  is a positive càdlàg martingale, by [23, Proposition (3.4) p.70], it vanishes on  $\llbracket R^n, \infty \rrbracket$ . Since  $z^n$  is bounded  $z_\infty^n$  exists and:

$$\{R^n < \infty\} = \{\inf_t z_t^n = 0\} = \{z_\infty^n = 0\}.$$

Moreover, the equality  $0 = \mathbb{E}(z_\infty^n \mathbb{1}_{\{z_\infty^n = 0\}}) = \mathbb{E}(\mathbb{1}_{C_n} \mathbb{1}_{\{z_\infty^n = 0\}})$  implies that  $C_n \cap \{z_\infty^n = 0\} = \emptyset$ , so as well  $C_n \cap \{\inf_t z_t^n = 0\} = \emptyset$ . We obtain that  $z^n > 0$  and  $z_-^n > 0$  a.s. on  $C_n$ .

(b) We have  $Z_\tau \mathbb{1}_{\{\tau < \infty\}} = \sum_n \mathbb{1}_{C_n} Z_{T_n}$  and, on  $\{T_n < \infty\}$ , we have

$$1 - Z_{T_n} = \mathbb{P}(\tau \leq T_n | \mathcal{F}_{T_n}) \geq \mathbb{P}(\tau = T_n | \mathcal{F}_{T_n}) = z_{T_n}^n.$$

From part (a), this implies that  $1 - Z_\tau > 0$  a.s. on  $\{\tau < \infty\}$ .

We omit the proof of (c) as it is straightforward.  $\square$

The following result describes how, after a thin random time, the conditional expectations with respect to elements of  $\mathbb{F}^\tau$  can be expressed in terms of the conditional expectations with respect to elements of  $\mathbb{F}$ .

**Lemma 2.7.** *Let  $\tau$  be a thin random time with exhausting sequence  $(T_n)_{n \geq 1}$  and  $(z^n)_{n \geq 1}$  be the family of  $\mathbb{F}$ -martingales associated with  $\tau$  through (2.2). Then, for any  $\mathcal{G}$ -measurable integrable random variable  $X$  and  $s \leq t$  we have*

$$\mathbb{E}[X|\mathcal{F}_t^\tau] \mathbb{1}_{\{s \geq T_n\} \cap C_n} = \mathbb{1}_{\{s \geq T_n\} \cap C_n} \frac{\mathbb{E}[X \mathbb{1}_{C_n} | \mathcal{F}_t]}{z_t^n}.$$

*Proof.* Note that

$$\mathcal{F}_t^\tau = \bigcap_{u > t} \mathcal{F}_u \vee \sigma(C_n \cap \{T_n \leq s\}, s \leq u, n \in \mathbb{N}).$$

Thus, by Monotone Class Theorem, for each  $G \in \mathcal{F}_t^\tau$  there exists  $F \in \mathcal{F}_t$  such that

$$(2.5) \quad G \cap \{T_n \leq s\} \cap C_n = F \cap \{T_n \leq s\} \cap C_n.$$

Then, we have to show that

$$\mathbb{E}[X \mathbb{1}_{\{s \geq T_n\} \cap C_n} z_t^n | \mathcal{F}_t^\tau] = \mathbb{1}_{\{s \geq T_n\} \cap C_n} \mathbb{E}[X \mathbb{1}_{\{s \geq T_n\} \cap C_n} | \mathcal{F}_t].$$

For any  $G \in \mathcal{F}_t^\tau$ , we choose  $F \in \mathcal{F}_t$  satisfying (2.5), and we obtain

$$\begin{aligned} \mathbb{E}[X \mathbb{1}_{\{s \geq T_n\} \cap C_n \cap G} z_t^n] &= \mathbb{E}[X \mathbb{1}_{\{s \geq T_n\} \cap C_n \cap F} \mathbb{E}[\mathbb{1}_{C_n} | \mathcal{F}_t]] \\ &= \mathbb{E}[\mathbb{1}_{\{s \geq T_n\} \cap F} \mathbb{E}[\mathbb{1}_{C_n} | \mathcal{F}_t] \mathbb{E}[X \mathbb{1}_{C_n} | \mathcal{F}_t]] \\ &= \mathbb{E}[\mathbb{1}_{\{s \geq T_n\} \cap C_n \cap F} \mathbb{E}[X \mathbb{1}_{C_n} | \mathcal{F}_t]] \\ &= \mathbb{E}[\mathbb{1}_{\{s \geq T_n\} \cap C_n \cap G} \mathbb{E}[X \mathbb{1}_{C_n} | \mathcal{F}_t]] \end{aligned}$$

which ends the proof.  $\square$

We end this subsection with a remark on the (\*)-decomposition of a random time  $\tau$  as an  $\mathbb{F}^\tau$ -stopping time, where  $\mathbb{F}^\tau$  is the filtration  $\mathbb{F}$  progressively enlarged with  $\tau$  as in (1.1).

**Remark 2.8.** *We can also decompose the random time  $\tau_2$  into accessible and totally inaccessible parts. Then, we consider a decomposition of  $\tau$  onto three parts as:*

$$\tau_1 = \tau_{\{\Delta A_\tau^c = 0\}}, \quad \tau_2^i = \tau_{\{\Delta A_\tau^c > 0, \Delta A_\tau^c = 0\}} \quad \text{and} \quad \tau_2^a = \tau_{\{\Delta A_\tau^c > 0, \Delta A_\tau^c > 0\}}.$$

Then  $\tau_1 \wedge \tau_2^i$  is an  $\mathbb{F}^\tau$ -totally inaccessible part and  $\tau_2^a$  is an  $\mathbb{F}^\tau$ -accessible part of the  $\mathbb{F}^\tau$ -stopping time  $\tau$ . These types of results were already shown in [17, p.65] and [9]. We note that  $\tau$  is an  $\mathbb{F}^\tau$ -predictable stopping time if and only if  $\tau$  is an  $\mathbb{F}$ -predictable stopping time. Moreover, a filtration  $\mathbb{F}^\tau$  such that  $\tau_2^a$  the accessible thin part of  $\tau$  is not an  $\mathbb{F}$ -stopping time is not quasi-left continuous. The last observation provides a systemic way to construct examples of non quasi-left continuous filtrations.

### 3. THE HYPOTHESIS $(\mathcal{H}')$

We exploit here the hypothesis  $(\mathcal{H}')$  in progressive enlargement in the connection to the  $(*)$ -decomposition of a random time. Let us first recall that hypothesis  $(\mathcal{H}')$  holds for  $(\mathbb{F}, \mathbb{F}^\tau)$  if any  $\mathbb{F}$ -martingale is an  $\mathbb{F}^\tau$ -semimartingale. First, in subsection 3.1, we examine the case of thin random times. Then, in subsection 3.2, we work with general random times.

**3.1. Thin random time.** Before formulating the result of this subsection we must recall a vital result by Jacod (see [17, Theorem 3,2] and [21]) on the hypothesis  $(\mathcal{H}')$  in initial enlargement with atomic  $\sigma$ -field.

Let  $\mathbb{F}^{\mathcal{C}}$  denote the initial enlargement of the filtration  $\mathbb{F}$  with the atomic  $\sigma$ -field  $\mathcal{C} := \sigma(C_n, n \geq 0)$  with  $C_n$  defined in (2.1), i.e.,

$$(3.1) \quad \mathcal{F}_t^{\mathcal{C}} := \bigcap_{s>t} \mathcal{F}_s \vee \sigma(C_n, n \geq 0).$$

In this case of enlargement, Jacod's result says that the hypothesis  $(\mathcal{H}')$  holds for  $(\mathbb{F}, \mathbb{F}^{\mathcal{C}})$  and the decomposition of any  $\mathbb{F}$ -martingale  $X$  as an  $\mathbb{F}^{\mathcal{C}}$ -semimartingale is

$$(3.2) \quad X_t = \widehat{X}_t + \sum_n \mathbb{1}_{C_n} \int_0^t \frac{1}{z_{s-}^n} d\langle X, z^n \rangle_s,$$

where  $\widehat{X}$  is an  $\mathbb{F}^{\mathcal{C}}$ -local martingale and  $z^n$  are given in (2.2).

**Theorem 3.1.** *Let  $\tau$  be a thin random time. Then  $\mathbb{F} \subset \mathbb{F}^\tau \subset \mathbb{F}^{\mathcal{C}}$  and the hypothesis  $(\mathcal{H}')$  is satisfied for  $(\mathbb{F}, \mathbb{F}^\tau)$ . Moreover, for each  $\mathbb{F}^\tau$ -predictable and bounded process  $G$  and each  $\mathbb{F}$ -local martingale  $Y$  the integral  $X := G \cdot Y$  is an  $\mathbb{F}^\tau$ -semimartingale with canonical decomposition*

$$(3.3) \quad X_t = \widehat{X}_t + \int_0^{t \wedge \tau} \frac{1}{Z_{s-}} d\langle X, m \rangle_s + \sum_n \mathbb{1}_{C_n} \int_0^t \mathbb{1}_{\{s>T_n\}} \frac{1}{z_{s-}^n} d\langle X, z^n \rangle_s$$

where  $\widehat{X}$  is an  $\mathbb{F}^\tau$ -local martingale.

*Proof.* The first part follows from Jacod's result (3.2) and Stricker's Theorem [22, Theorem 4, Chapter II] since  $\mathbb{F}^\tau \subset \mathbb{F}^{\mathcal{C}}$ . Let  $H$  be an  $\mathbb{F}^\tau$ -predictable bounded process. Then, [17, Lemma (4,4)] implies that

$$H_t = \mathbb{1}_{\{t \leq \tau\}} J_t + \mathbb{1}_{\{\tau < t\}} K_t(\tau) \quad t \geq 0$$

where  $J$  is an  $\mathbb{F}$ -predictable bounded process and  $K : \mathbb{R}_+ \times \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}$  is  $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable and bounded. Note that, since  $\{t \leq \tau\} \subset \{Z_{t-} > 0\}$ ,  $J$  can be chosen to satisfy  $J_t = J_t \mathbb{1}_{\{Z_{t-} > 0\}}$ . Since  $\tau$  is a thin random time, we can rewrite the process  $H$  as

$$H_t = J_t \mathbb{1}_{\{t \leq \tau\}} + \sum_n \mathbb{1}_{\{T_n < t\}} K_t(T_n) \mathbb{1}_{C_n}$$

with  $C_n = \{\tau = T_n\}$ . Note that each process  $K_t^n := \mathbb{1}_{\{T_n < t\}} K_t(T_n)$  is  $\mathbb{F}$ -predictable and bounded and, since  $C_n \subset \{z_{t-}^n > 0\}$ ,  $K^n$  can be chosen to satisfy  $K_t^n = K_t^n \mathbb{1}_{\{z_{t-}^n > 0\}}$ .

Let  $X$  be an  $H^1$   $\mathbb{F}$ -martingale. Then stochastic integrals  $J \cdot X$  and  $K^n \cdot X$  are well defined and each of them is  $H^1$   $\mathbb{F}$ -martingale. For each  $n$  and for each bounded  $\mathbb{F}$ -martingale  $N$ , by integration by parts, we have that

$$(3.4) \quad \mathbb{E}[\mathbb{1}_{C_n} N_\infty] = \mathbb{E}[[z^n, N]_\infty] = \mathbb{E}[\langle z^n, N \rangle_\infty].$$

Since  $N \rightarrow \mathbb{E}(\mathbb{1}_{C_n} N_\infty)$  is a linear form, by the duality  $(H^1, BMO)$  implies that (3.4) holds for any  $H^1$   $\mathbb{F}$ -martingale  $N$ . Similarly, by [2, Proposition 1.32], for any  $H^1$   $\mathbb{F}$ -martingale  $N$ , the process  $\langle N, m \rangle$  exists and we have

$$\mathbb{E}(N_\tau) = \mathbb{E}([N, m]_\infty) = \mathbb{E}(\langle N, m \rangle_\infty)$$

where  $m$  is given in (2.3). Therefore

$$\begin{aligned} \mathbb{E} \left[ \int_0^\infty H_s dX_s \right] &= \mathbb{E} \left[ \int_0^\tau J_s dX_s \right] + \sum_n \mathbb{E} \left[ \mathbb{1}_{C_n} \int_0^\infty K_s^n dX_s \right] \\ &= \mathbb{E} \left[ \int_0^\infty J_s d\langle m, X \rangle_s \right] + \sum_n \mathbb{E} \left[ \int_0^\infty K_s^n d\langle z^n, X \rangle_s \right] \end{aligned}$$

Then, since for any predictable finite variation process  $V$ ,  $\mathbb{E}[\int_0^\infty h_s dV_s] = \mathbb{E}[\int_0^\infty {}^p h_s dV_s]$ , we deduce

$$\begin{aligned} \mathbb{E} \left[ \int_0^\infty H_s dX_s \right] &= \mathbb{E} \left[ \int_0^\infty \frac{Z_{s-}}{Z_{s-}} \mathbb{1}_{\{Z_{s-} > 0\}} J_s d\langle m, X \rangle_s \right] \\ &\quad + \sum_n \mathbb{E} \left[ \int_0^\infty \frac{z_{s-}^n}{z_{s-}^n} \mathbb{1}_{\{z_{s-}^n > 0\}} K_s^n d\langle z^n, X \rangle_s \right] \\ &= \mathbb{E} \left[ \int_0^\tau \frac{1}{Z_{s-}} J_s d\langle m, X \rangle_s \right] + \sum_n \mathbb{E} \left[ \mathbb{1}_{C_n} \int_0^\infty \frac{1}{z_{s-}^n} K_s^n d\langle z^n, X \rangle_s \right]. \end{aligned}$$

For any  $H^1$   $\mathbb{F}$ -martingale  $Y$  and  $\mathbb{F}^\tau$ -predictable process  $G \equiv 1$  the assertion of the theorem follows as for any  $s \leq t$  and  $F \in \mathcal{F}_s^\tau$  the process  $H = \mathbb{1}_{(s,t]} \mathbb{1}_F$  is clearly  $\mathbb{F}^\tau$ -predictable. To end the proof we recall that any local martingale is locally in  $H^1$ .  $\square$

**Remark 3.2.** [17, Lemma (4,11)] is a special case of Theorem 3.1 where the random time with countably many values is considered. It corresponds to the situation of thin random time whose graph is included in countable union of constant sections, i.e.,  $[[\tau]] \subset \bigcup_n [[t_n]]$  with  $[[t_n]] = \{(\omega, t_n) : \omega \in \Omega\}$ . Our proof is similar to the proof of [17, Lemma (4,11)].

We end this section with a result linking processes in  $\mathbb{F}^\tau$  and  $\mathbb{F}^C$ . It can be used as an alternative approach to show the decomposition in Theorem 3.1 using (3.2). It is related to the ideas in [8].

**Proposition 3.3.** *Let  $\tau$  be a thin random time and  $X$  be a process such that  $X = \mathbb{1}_{] \tau, \infty[} \cdot X$ . Then*

(a) *The process  $X$  is an  $\mathbb{F}^C$ -(super-, sub-) martingale if and only if the process  $X$  is an  $\mathbb{F}^\tau$ -(super-, sub-) martingale.*

(b) *Let  $\vartheta$  be an  $\mathbb{F}^C$ -stopping time. Then  $\vartheta \vee \tau$  is an  $\mathbb{F}^\tau$ -stopping time.*

(c) The process  $X$  is an  $\mathbb{F}^C$ -local martingale if and only if the process  $X$  is an  $\mathbb{F}^\tau$ -local martingale.

*Proof.* (a) Note that the filtrations  $\mathbb{F}^\tau$  and  $\mathbb{F}^C$  are equal after  $\tau$ , i.e., for each  $t$  and for each set  $G \in \mathcal{F}_t^C$  there exists a set  $F \in \mathcal{F}_t^\tau$  such that

$$(3.5) \quad \{\tau \leq t\} \cap G = \{\tau \leq t\} \cap F.$$

To show (3.5), by Monotone Class Theorem, it is enough to consider  $G = C_n$  and to take  $F = C_n \cap \{\tau \leq t\}$  which belongs to  $\mathcal{F}_t^\tau$  as  $C_n \in \mathcal{F}_\tau^\tau$  by [13, Corollary 3.5]. That implies that the process  $\mathbb{1}_{\llbracket \tau, \infty \llbracket} \cdot X$  is  $\mathbb{F}^\tau$ -adapted if and only if it is  $\mathbb{F}^C$ -adapted. The equivalence of (super-, sub-) martingale property comes from (3.5).

(b) For each  $t$  we have  $\{\vartheta \vee \tau \leq t\} = \{\vartheta \leq t\} \cap \{\tau \leq t\} \in \mathcal{F}_t^\tau$  by (3.5).

(c) We combine the two previous points.  $\square$

**3.2. General random time.** In this section we work with the  $(*)$ -decomposition  $(\tau_1, \tau_2)$  of the random time  $\tau$ . We define three enlarged filtrations  $\mathbb{F}^{\tau_1} := (\mathcal{F}_t^{\tau_1})_{t \geq 0}$ ,  $\mathbb{F}^{\tau_2} := (\mathcal{F}_t^{\tau_2})_{t \geq 0}$  and  $\mathbb{F}^{\tau_1, \tau_2} := (\mathcal{F}_t^{\tau_1, \tau_2})_{t \geq 0}$  as

$$\begin{aligned} \mathcal{F}_t^{\tau_i} &:= \bigcap_{s > t} \mathcal{F}_s \vee \sigma(\tau_i \wedge s) \quad \text{for } i = 1, 2 \\ \mathcal{F}_t^{\tau_1, \tau_2} &:= \bigcap_{s > t} \mathcal{F}_s \vee \sigma(\tau_1 \wedge s) \vee \sigma(\tau_2 \wedge s). \end{aligned}$$

Obviously,  $\mathbb{F} \subset \mathbb{F}^{\tau_i} \subset \mathbb{F}^{\tau_1, \tau_2}$  for  $i = 1, 2$ .

**Theorem 3.4.** *Let  $\tau$  be a random time and  $(\tau_1, \tau_2)$  its  $(*)$ -decomposition. Then, the hypothesis  $(\mathcal{H}')$  is satisfied for  $(\mathbb{F}, \mathbb{F}^\tau)$  if and only if the hypothesis  $(\mathcal{H}')$  is satisfied for  $(\mathbb{F}, \mathbb{F}^{\tau_1})$ .*

*Proof.* In the first step, we show the following inclusions of filtrations for  $i = 1, 2$ :

$$\mathbb{F} \subset \mathbb{F}^{\tau_i} \subset \mathbb{F}^{\tau_1, \tau_2} = \mathbb{F}^\tau.$$

Let  $A^\circ$  be an  $\mathbb{F}$ -dual optional projection of  $\tau$ . Note that

$$\mathbb{1}_{\llbracket \tau_1, \infty \llbracket} = \mathbb{1}_{\llbracket \tau, \infty \llbracket} \mathbb{1}_{\{\Delta A^\circ = 0\}} \quad \text{and} \quad \mathbb{1}_{\llbracket \tau_2, \infty \llbracket} = \mathbb{1}_{\llbracket \tau, \infty \llbracket} \mathbb{1}_{\{\Delta A^\circ > 0\}},$$

thus, since  $\Delta A^\circ \in \mathcal{F}_\tau^\tau$ , the processes  $\mathbb{1}_{\llbracket \tau_1, \infty \llbracket}$  and  $\mathbb{1}_{\llbracket \tau_2, \infty \llbracket}$  are  $\mathbb{F}^\tau$ -adapted which implies that  $\mathbb{F}^{\tau_1, \tau_2} \subset \mathbb{F}^\tau$ . On the other hand we have

$$\mathbb{1}_{\llbracket \tau_1, \infty \llbracket} + \mathbb{1}_{\llbracket \tau_2, \infty \llbracket} = \mathbb{1}_{\llbracket \tau, \infty \llbracket}$$

which implies that  $\mathbb{F}^{\tau_1, \tau_2} \supset \mathbb{F}^\tau$ .

In the second step, note that if an  $\mathbb{F}$ -martingale is an  $\mathbb{F}^\tau$ -semimartingale, by Stricker's Theorem [22, Theorem 4, Ch II, p. 53], it is as well an  $\mathbb{F}^{\tau_1}$ -semimartingale. Thus the necessary condition follows. Since  $\tau_2$  is a thin random time, the previous step and Theorem 3.1 imply that the hypothesis  $(\mathcal{H}')$  is satisfied for  $(\mathbb{F}^{\tau_1}, \mathbb{F}^\tau)$ . Thus the sufficient condition follows.  $\square$

In the next proposition we see that  $\tau_1$  and  $\tau_2$  are in some sense orthogonal (in terms of semimartingale decomposition and associated supermartingales, which is due to  $\tau_1 \vee \tau_2 = \infty$ ).

**Proposition 3.5.** *The  $\mathbb{F}$ -supermartingale  $Z^2$  of a thin random time  $\tau_2$  coincides with the  $\mathbb{F}^{\tau_1}$ -supermartingale  $Z^{2, \mathbb{F}^{\tau_1}}$  of  $\tau_2$ , i.e.,  $\mathbb{P}(\tau_2 > t | \mathcal{F}_t) = \mathbb{P}(\tau_2 > t | \mathcal{F}_t^{\tau_1})$ .*

*Proof.* Let  $T$  be an  $\mathbb{F}$ -stopping time. For each  $A \in \mathcal{F}_t^{\tau_1}$ , there exists  $B \in \mathcal{F}_t$  such that  $A \cap \{\tau = T\} = B \cap \{\tau = T\}$ , so  $\mathbb{P}(\tau = T | \mathcal{F}_t) = \mathbb{P}(\tau = T | \mathcal{F}_t^{\tau_1})$  which ends the proof.  $\square$

#### 4. THIN HONEST TIMES

Here we restrict our attention to a special class of random times, namely to honest times. We recall its definition below (see [17, p. 73]) and some alternative characterizations in Appendix B.

**Definition 4.1.** *A random time  $\tau$  is an  $\mathbb{F}$ -honest time if for every  $t > 0$  there exists an  $\mathcal{F}_t$ -measurable random variable  $\tau_t$  such that  $\tau = \tau_t$  on  $\{\tau < t\}$ . Then, it is always possible to choose  $\tau_t$  such that  $\tau_t \leq t$ .*

We refer to the Appendix B for further helpful results on honest times.

**4.1. Fundamental properties.** Let us start with some characterisation and properties of (thin) honest times.

**Proposition 4.2.** (a) *Let  $\tau$  be an honest time and denote by  $(\tau_1, \tau_2)$  its  $(*)$ -decomposition. Then, the times  $\tau_1$  and  $\tau_2$  are honest times.*  
 (b) *A random time  $\tau$  is a strict honest time if and only if  $Z_\tau = 1$  a.s. on  $\{\tau < \infty\}$ .*  
 (c) *Let  $\tau$  be an honest time with  $(*)$ -decomposition  $(\tau_1, \tau_2)$ . Then,  $Z_\tau = 1$  on  $\{\tau = \tau_1 < \infty\}$  and  $Z_\tau < 1$  on  $\{\tau = \tau_2 < \infty\}$ .*

*Proof.* (a) On the set  $\{\tau < \infty\}$ ,  $\tau$  is equal to  $\gamma$ , the end of the optional set  $\Gamma$  (Theorem B.2). Then, as  $\{\tau_1 < \infty\} \subset \{\tau < \infty\}$ , on the set  $\{\tau_1 < \infty\}$ , one has  $\tau_1 = \gamma$ , so  $\tau_1$  is an honest time. Same argument for  $\tau_2$ .

(b) Assume that  $\tau$  is a strict honest time. Then, the honest time property presented in Theorem B.2 (c) implies that  $\tilde{Z}_\tau = 1$  and the strict time property implies, by Theorem 2.4 (b), the continuity of  $A^\circ$ . Therefore, the equality  $\tilde{Z} = Z + \Delta A^\circ$  leads to equality  $Z_\tau = 1$  a.s. on  $\{\tau < \infty\}$ .

Assume now that  $Z_\tau = 1$  on the set  $\{\tau < \infty\}$ . Then, on  $\{\tau < \infty\}$  we have  $1 = Z_\tau \leq \tilde{Z}_\tau \leq 1$ , so  $\tilde{Z}_\tau = 1$  and  $\tau$  is an honest time. Furthermore, as  $\Delta A_\tau^\circ = \tilde{Z}_\tau - Z_\tau = 0$ , for each  $\mathbb{F}$ -stopping time  $T$  we have

$$\begin{aligned} \mathbb{P}(\tau = T < \infty) &= \mathbb{E}(\mathbb{1}_{\{\tau=T\}} \mathbb{1}_{\{\Delta A_\tau^\circ=0\}} \mathbb{1}_{\{T<\infty\}}) \\ &= \mathbb{E}\left(\int_0^\infty \mathbb{1}_{\{u=T\}} \mathbb{1}_{\{\Delta A_u^\circ=0\}} dA_u^\circ\right) = 0. \end{aligned}$$

So  $\tau$  is a strict random time.

(c) From the honest time property of  $\tau$  and Lemma 2.5, on the set  $\{\tau < \infty\}$

$$1 = \tilde{Z}_\tau = \tilde{Z}_\tau^1 + \tilde{Z}_\tau^2 - 1.$$

On the set  $\{\tau = \tau_1 < \infty\}$ , we have

$$1 = \tilde{Z}_{\tau_1}^1 + \tilde{Z}_{\tau_1}^2 - 1 = \tilde{Z}_{\tau_1}^2,$$

where the second equality comes from (c) in Theorem B.2. Now let us compute  $Z_{\tau_1}^2$

$$Z_{\tau_1}^2 = \tilde{Z}_{\tau_1}^2 - \Delta A_{\tau_1}^{2,o} = \tilde{Z}_{\tau_1}^2 = 1,$$

where we have used the strict random time property of  $\tau_1$ , i.e.,  $\{\Delta A^{2,o} > 0\} = \bigcup_n \llbracket T_n \rrbracket$  (with  $(T_n)$  being an exhausting sequence of  $\tau_2$ ) and  $\mathbb{P}(\tau_1 = T_n < \infty) = 0$ . Finally, on  $\{\tau = \tau_1 < \infty\}$

$$Z_\tau = Z_{\tau_1}^1 + Z_{\tau_1}^2 - 1 = 1.$$

On the set  $\{\tau = \tau_2 < \infty\}$ ,

$$Z_\tau = Z_{\tau_2}^1 + Z_{\tau_2}^2 - 1 \leq Z_{\tau_2}^2 < 1,$$

where the last inequality is due to Lemma 2.6 (b).  $\square$

**Remark 4.3.** *We would like to remark that the condition that  $Z_\tau < 1$  for an honest time  $\tau$  – which, by Proposition 4.2 (c), is equivalent to the condition that  $\tau$  is a thin honest time – is an essential assumption in [1].*

**Lemma 4.4.** *Let  $\tau$  be a thin honest time and  $\tau_t$  be associated with  $\tau$  as in Definition 4.1. Then, for each  $n$ :*

- (a) on  $\{T_n = \tau_t\} = \{T_n = \tau_t \leq t\}$  we have  $z_t^n = 1 - Z_t$ ,  $A_t^o = z_{T_n}^n$  and  $1 - m_t = z_t^n - z_{T_n}^n$ ;  
 (b) on  $\{T_n < t\}$  we have  $z_t^n = \mathbb{1}_{\{\tau_t = T_n\}}(1 - \tilde{Z}_t)$  and  $z_{t-}^n = \mathbb{1}_{\{\tau_t = T_n\}}(1 - Z_{t-})$ ; in particular

$$1 - \tilde{Z}_t = \sum_n \mathbb{1}_{\{\tau_t = T_n < t\}}(1 - \tilde{Z}_t) \quad \text{and} \quad 1 - Z_{t-} = \sum_n \mathbb{1}_{\{\tau_t = T_n < t\}}(1 - Z_{t-}).$$

*Proof.* (a) Using properties of  $\tau_t$  we deduce that

$$\begin{aligned} \mathbb{1}_{\{T_n = \tau_t\}} z_t^n &= \mathbb{P}(T_n = \tau_t \leq t, \tau = T_n < \infty | \mathcal{F}_t) \\ &= \mathbb{P}(\tau \leq t, T_n = \tau_t = \tau | \mathcal{F}_t) \\ &= \mathbb{P}(\tau \leq t, T_n = \tau_t | \mathcal{F}_t) \\ &= \mathbb{1}_{\{T_n = \tau_t\}}(1 - Z_t) \end{aligned}$$

where the first equality is due to  $\tau_t \leq t$ , the third one follows by  $\tau_t = \tau$  on  $\{\tau \leq t\}$  and the last one is true since  $T_n \wedge t$  and  $\tau_t$  are two  $\mathcal{F}_t$ -measurable random variable and

$$\begin{aligned} \{T_n = \tau_t\} &= \{T_n = \tau_t < t\} \cup \{T_n = \tau_t = t\} \\ &= \{\{T_n \wedge t = \tau_t\} \cap \{\tau_t < t\}\} \cup \{\{T_n = t\} \cap \{\tau_t = t\}\}. \end{aligned}$$

The dual optional projection of a thin random time satisfies

$$\mathbb{1}_{\{T_n = \tau_t\}} A_t^o = \sum_k \mathbb{1}_{\{T_n = \tau_t, T_k \leq t\}} z_{T_k}^k = \mathbb{1}_{\{T_n = \tau_t\}} z_{T_n}^n,$$

where the second equality is due to the fact that for  $n \neq k$  we have

$$\mathbb{1}_{\{T_n = \tau_t, T_k \leq t\}} z_{T_k}^k = \mathbb{1}_{\{T_n = \tau_t\}} \mathbb{E}(\mathbb{1}_{\{\tau = T_k \leq t\}} | T_k)$$

$$= \mathbb{1}_{\{T_n = \tau = T_k\}} \mathbb{E}(\mathbb{1}_{\{\tau = T_k \leq t\}} | T_k) = 0$$

since  $T_n$  and  $T_k$  have disjoint graphs and  $\tau$  is an honest time. Combining the two previous points, we conclude that  $1 - m_t = 1 - Z_t - A_t^o = z_t^n - z_{T_n}^n$  on the set  $\{T_n = \tau_t\}$ .

(b) Again using properties of random variable  $\tau_t$  we derive

$$\begin{aligned} \mathbb{1}_{\{T_n < t\}} z_t^n &= \mathbb{P}(\tau = T_n = \tau_t < t | \mathcal{F}_t) = \mathbb{1}_{\{T_n = \tau_t < t\}} (1 - \tilde{Z}_t), \\ \mathbb{1}_{\{T_n < t\}} z_{t-}^n &= \mathbb{P}(\tau = T_n = \tau_t < t | \mathcal{F}_{t-}) = \mathbb{1}_{\{T_n = \tau_t < t\}} (1 - Z_{t-}). \end{aligned}$$

Then, Lemma 2.6 (c) completes the proof.  $\square$

For progressive enlargement with an honest time, the hypothesis  $(\mathcal{H}')$  is satisfied (not only for  $\mathbb{F}$ -local martingales stopped at  $\tau$ ), and the following decomposition is given in [17, Theorem (5,10)]. Let  $M$  be an  $\mathbb{F}$ -local martingale. Then, there exists an  $\mathbb{F}^\tau$ -local martingale  $\widehat{M}$  such that:

$$(4.1) \quad M_t = \widehat{M}_t + \int_0^{t \wedge \tau} \frac{1}{Z_{s-}} d\langle M, m \rangle_s - \int_0^t \mathbb{1}_{\{s > \tau\}} \frac{1}{1 - Z_{s-}} d\langle M, m \rangle_s.$$

**Remark 4.5.** For a thin honest time  $\tau$ , the two decomposition formulas, first given in Theorem 3.1 and second given in (4.1), coincide. It is enough to show that

$$\int_0^t \mathbb{1}_{\{s > \tau\}} \frac{1}{1 - Z_{s-}} d\langle X, 1 - m \rangle_s = \sum_n \mathbb{1}_{C_n} \int_0^t \mathbb{1}_{\{s > T_n\}} \frac{1}{z_{s-}^n} d\langle X, z^n \rangle_s.$$

This is a simple consequence of the set inclusion  $\{\tau < s\} \cap \{\tau = T_n\} \subset \{T_n = \tau_s \leq s\}$  and Lemma 4.4 (a):

$$\begin{aligned} \int_0^t \mathbb{1}_{\{s > \tau\}} \frac{1}{1 - Z_{s-}} d\langle X, 1 - m \rangle_s &= \sum_n \int_0^t \mathbb{1}_{\{s > \tau\} \cap \{\tau = T_n\}} \frac{1}{1 - Z_{s-}} d\langle X, 1 - m \rangle_s \\ &= \sum_n \int_0^t \mathbb{1}_{\{s > \tau\} \cap \{\tau = T_n\}} \frac{1}{z_{s-}^n} d\langle X, z^n \rangle_s \\ &= \sum_n \mathbb{1}_{C_n} \int_0^t \mathbb{1}_{\{s > T_n\}} \frac{1}{z_{s-}^n} d\langle X, z^n \rangle_s. \end{aligned}$$

**4.2. Jumping filtration.** In this section we develop the relationship between jumping filtration and thin honest times. Let us first recall the definition of a jumping filtration and the main result obtained in Jacod and Skorokhod [14].

**Definition 4.6.** A filtration  $\mathbb{F}$  is called a jumping filtration if there exists a localizing sequence  $(\theta_n)_n$ , i.e., a sequence of stopping times increasing a.s. to  $\infty$ , with  $\theta_0 = 0$  and such that for all  $n$  and  $t > 0$  the  $\sigma$ -fields  $\mathcal{F}_t$  and  $\mathcal{F}_{\theta_n}$  coincide up to null sets on  $\{\theta_n \leq t < \theta_{n+1}\}$ . The sequence  $(\theta_n)_n$  is then called a jumping sequence.

There exists an important alternative characterization of jumping filtration in terms of martingale's variation ([14, Theorem 1]).

**Theorem 4.7.** The two following conditions are equivalent:

- (a) a filtration  $\mathbb{F}$  is a jumping filtration;
- (b) all martingales in the filtration  $\mathbb{F}$  are a.s. of locally finite variation.

In the remaining part of this subsection we investigate relationship between jumping filtration and honest times. We show that there does not exist strict honest time in a jumping filtration and that there exists a strict honest time in a filtration which admits a non-constant continuous martingale (in particular such a filtration is not a jumping filtration).

**Theorem 4.8.** *The following assertions hold.*

- (a) *If  $\mathbb{F}$  is a jumping filtration, then all  $\mathbb{F}$ -honest times are thin.*  
 (b) *If all  $\mathbb{F}$ -honest times are thin, then all non-constant  $\mathbb{F}$ -local martingales are purely discontinuous.*

*Proof.* (a) Let  $\tau$  be an honest time. Then, take the same process  $\alpha$  as in the proof of Proposition B.1, i.e.,  $\alpha$  is an increasing, càdlàg, adapted process such that  $\alpha_t = \tau$  on  $\{\tau \leq t\}$  and  $\tau = \sup\{t : \alpha_t = t\}$ . Let us define the partition  $(C_n)_{n=0}^\infty$  such that

$$C_n = \{\theta_{n-1} \leq \tau < \theta_n\}$$

for  $n \geq 1$  and  $C_0 = \{\tau = \infty\}$  with  $(\theta_n)_n$  being a jumping sequence for the jumping filtration  $\mathbb{F}$ . On each  $C_n$  with  $n \geq 1$  we have

$$\tau = T_n := \inf\{t \geq \theta_{n-1} : t = \alpha_{\theta_{n-}}\}.$$

From the jumping filtration property, we know that  $\alpha_{\theta_{n-}}$  is  $\mathcal{F}_{\theta_{n-}}$ -measurable so each  $T_n$  is a stopping time and  $[\tau] \subset \bigcup_{n=1}^\infty [T_n]$  which shows that the honest time  $\tau$  is a thin random time.

(b) The proof by contradiction is based on [23, Exercice (1.26) p.235]. Assume that  $M$  is a non-constant continuous  $\mathbb{F}$ -local martingale with  $M_0 = 0$ . Define the  $\mathbb{F}$ -stopping time  $S_1 = \inf\{t > 0 : \langle M \rangle_t = 1\}$ . Then, define the  $\mathbb{F}$ -honest time

$$\tau := \sup\{t \leq S_1 : M_t = 0\}.$$

Since  $M$  is continuous,  $\tau$  is not equal to infinity with strictly positive probability. We now show that  $\tau$  is an  $\mathbb{F}$ -strict honest time. Let us denote  $\mathcal{Z}(\omega) := \{t : M_t(\omega) = 0\}$ . The set  $\mathcal{Z}(\omega)$  is closed and  $\mathcal{Z}^c(\omega)$  is the union of countably many open intervals. We call  $G(\omega)$  the set of left ends of these open intervals. In what follows we show that for any  $\mathbb{F}$ -stopping time  $T$  we have  $\mathbb{P}(T \in G) = 0$ . Define the  $\mathbb{F}$ -stopping time

$$D_T := \inf\{t > T : M_t = 0\}$$

and note that

$$\{T \in G\} = \{M_T = 0\} \cap \{T < D_T\} \in \mathcal{F}_T.$$

Assume  $\mathbb{P}(T \in G) = p > 0$ . Then the process

$$Y_t = \mathbb{1}_{\{T \in G\}} |M_{T+t}| \mathbb{1}_{\{0 \leq t \leq D_T - T\}}$$

is an  $(\mathcal{F}_{T+t})_{t \geq 0}$ -martingale. Indeed for  $s \leq t$  we have

$$\begin{aligned} \mathbb{E}(Y_t | \mathcal{F}_{T+s}) &= \mathbb{1}_{\{T \in G\}} \text{sgn}(M_{T+t}) \mathbb{E}(M_{T+t} \mathbb{1}_{\{t \leq D_T - T\}} | \mathcal{F}_{T+s}) \\ &= \mathbb{1}_{\{T \in G\}} \text{sgn}(M_{T+t}) \\ &\quad \left[ M_{T+s} \mathbb{1}_{\{s \leq D_T - T\}} - \mathbb{E}(\mathbb{1}_{\{s \leq D_T - T\}} \mathbb{1}_{\{t > D_T - T\}} \mathbb{E}(M_{T+t} | \mathcal{F}_{D_T}) | \mathcal{F}_{T+s}) \right] \end{aligned}$$

$$\begin{aligned} &= Y_s - \mathbb{1}_{\{T \in G\}} \operatorname{sgn}(M_{T+t}) \mathbb{E}(\mathbb{1}_{\{s \leq D_T - T\}} \mathbb{1}_{\{t > D_T - T\}} M_{D_T} | \mathcal{F}_{T+s}) \\ &= Y_s \end{aligned}$$

where we have used the martingale property of  $M$  and  $M_{D_T} = 0$ . Moreover  $Y_0 = 0$  and there exists  $\varepsilon > 0$  such that

$$\mathbb{P}(M_T = 0, D_T - T > \varepsilon) \geq \frac{p}{2} > 0.$$

Since  $Y_\varepsilon = \mathbb{1}_{\{M_T=0\}} \mathbb{1}_{\{D_T-T \geq \varepsilon\}} |M_{T+\varepsilon}| \geq 0$  and  $\mathbb{P}(Y_\varepsilon > 0) > 0$ , we have  $\mathbb{E}(Y_\varepsilon) > 0 = Y_0$ . So,  $\mathbb{P}(T \in G) = 0$ . Finally, as  $\tau \in G$  a.s. we conclude that  $\tau$  is a strict honest time.  $\square$

Finally we give two examples of strict honest times originating from purely discontinuous semimartingales of infinite variation. In the first Example 4.9, we study the case of Azéma's martingale (see [22, IV.8 p.232-237]). In the second Example 4.10, we recall the example 2.1 from [18] on *Maximum of downwards drifting spectrally negative Lévy processes with paths of infinite variation*.

**Example 4.9.** *Let  $B$  be a Brownian motion and  $\mathbb{F}$  its natural filtration. Define the process*

$$g_t := \sup\{s \leq t : B_s = 0\}.$$

The process

$$\mu_t := \operatorname{sgn}(B_t) \sqrt{t - g_t}$$

is a martingale with respect to the filtration  $\mathbb{G} := (\mathcal{F}_{g_t})_{t \geq 0}$  and is called the Azéma martingale. Then, the random time

$$\tau := \sup\{t \leq 1 : \mu_t = 0\}$$

is clearly a  $\mathbb{G}$ -honest time. Note that  $\tau = \tau^B := \sup\{t \leq 1 : B_t = 0\}$  and  $\tau^B$  is an  $\mathbb{F}$ -strict honest time (see in [20, Table 1 $\alpha$  1), p.32] that  $\tau^B$  has continuous  $\mathbb{F}$ -dual optional projection). Thus, since  $\mathbb{G} \subset \mathbb{F}$ ,  $\tau$  is a  $\mathbb{G}$ -strict honest time.

**Example 4.10.** *Let  $X$  be a Lévy process with characteristic triplet  $(\alpha, \sigma^2 = 0, \nu)$  satisfying  $\nu((0, \infty)) = 0$ ,  $\alpha + \int_{-\infty}^{-1} x \nu(dx) < 0$  and  $\int_{-1}^0 |x| \nu(dx) = \infty$ . Then,  $\rho = \sup\{t : X_{t-} = X_{t-}^*\}$  with  $X_t^* = \sup_{s \leq t} X_s$  is a strict honest time as shown in [18, Section 2.1].*

### 4.3. Examples of thin honest times.

4.3.1. *Compound Poisson process: last passage time at a barrier  $a$ .* Let us consider the filtration  $\mathbb{F}$  generated by a Compound Poisson Process (CCP)  $X$ , defined as

$$X_t = \sum_{n=1}^{N_t} Y_n,$$

where  $N$  is a Poisson process with parameter  $\eta$  and sequence of jump times  $(\theta_n)_{n=1}^\infty$ , and  $(Y_n)_{n=1}^\infty$  are i.i.d. strictly positive integrable random variables, independent from  $N$ , with cumulative distribution function  $F$ . We will study a thin honest time in the filtration of  $X$  which is not a stopping time. In a progressive enlargement framework, in order to study

the  $\mathbb{F}^\tau$ -semimartingale decomposition of  $\mathbb{F}$ -martingales before  $\tau$ , one needs to compute the martingale  $m = Z + A^\circ$ . Therefore, we shall present the computations of  $A^\circ$  (hence  $m$ ).

Define a random time  $\tau$  as

$$(4.2) \quad \tau := \sup\{t : \mu t - X_t \leq a\}$$

with  $a > 0$  and a constant  $\mu$ . From now on we assume that  $\mu > \eta \mathbb{E}(Y_1)$ . Under this condition, the random time  $\tau$  is finite a.s.. Since  $\tau$  is a last passage time, it is an honest time in the filtration  $\mathbb{F}$ . Furthermore, since the process  $\mu t - X_t$  has only negative jumps, one has  $\mu\tau - X_\tau = a$ . The random time  $\tau$  is thin as we shall see below.

**Lemma 4.11.** *The honest time  $\tau$  is a thin random time with exhausting sequence  $(T_n)_{n \geq 1}$  given by*

$$(4.3) \quad \begin{aligned} T_1 &:= \inf\{t > 0 : \mu t - X_t = a\} \quad \text{and} \\ T_n &:= \inf\{t > T_{n-1} : \mu t - X_t = a\} \quad \text{for } n > 1. \end{aligned}$$

Each  $T_n$  is a predictable stopping time. Furthermore  $Z_\tau < 1$ .

*Proof.* Random time  $\tau$  is honest since on  $\{\tau < t\}$  it is equal to  $\tau_t := \{s \leq t : \mu s - X_s \leq a\}$ . The sets  $(C_n)_{n=0}^\infty$  with  $C_n = \{\tau = T_n\}$  form a partition of  $\Omega$ . Then,  $\tau = \sum_{n=0}^\infty T_n \mathbb{1}_{C_n}$ . Thus  $\tau$  is a thin random time. Note that  $\tau$  is not an  $\mathbb{F}$ -stopping time as  $C_n \notin \mathcal{F}_{T_n}$  for any  $n$ . To show that each  $T_n$  is predictable, let us define the stopping times  $J_d$  and  $J_u$  as

$$\begin{aligned} J_d &= \inf\{t > 0 : \mu t - X_{t-} = a, \mu t - X_t < a\} \\ J_u &= \inf\{t > 0 : \mu t - X_{t-} > a, \mu t - X_t = a\}. \end{aligned}$$

First observe that  $\llbracket J_d \rrbracket \subset \bigcup_n \llbracket \theta_n \rrbracket$  and  $\llbracket J_u \rrbracket \subset \bigcup_n \llbracket \theta_n \rrbracket$ . For each  $n$ , we have  $\mathbb{P}(J_d = \theta_n) = 0$  as

$$\begin{aligned} \mathbb{P}(J_d = \theta_n) &= \mathbb{E} \left[ \mathbb{P} [J_d = \theta_n | \mathcal{F}_{\theta_{n-1}}] \right] \\ &= \mathbb{E} \left[ \mathbb{1}_{\{J_d > \theta_{n-1}\}} \mathbb{P} [\mu\theta_n - X_{\theta_{n-1}} = a | \mathcal{F}_{\theta_{n-1}}] \right] \\ &\leq \mathbb{E} \left[ \mathbb{P} \left[ \theta_n - \theta_{n-1} = \frac{a - \mu\theta_{n-1} + X_{\theta_{n-1}}}{\mu} | \mathcal{F}_{\theta_{n-1}} \right] \right] = 0, \end{aligned}$$

so we conclude that  $J_d = \infty$  a.s. For each  $n$ , we have  $\mathbb{P}(J_u = \theta_n) = 0$  as

$$\begin{aligned} \mathbb{P}(J_u = \theta_n) &= \mathbb{E} \left[ \mathbb{P} [J_u = \theta_n | \mathcal{F}_{\theta_{n-1}} \vee \sigma(Y_n)] \right] \\ &= \mathbb{E} \left[ \mathbb{1}_{\{J_u > \theta_{n-1}\}} \mathbb{P} [\mu\theta_n - X_{\theta_n} = a | \mathcal{F}_{\theta_{n-1}} \vee \sigma(Y_n)] \right] \\ &\leq \mathbb{E} \left[ \mathbb{P} \left[ \theta_n - \theta_{n-1} = \frac{a - \mu\theta_{n-1} + X_{\theta_{n-1}} + Y_n}{\mu} | \mathcal{F}_{\theta_{n-1}} \vee \sigma(Y_n) \right] \right] \\ &= 0, \end{aligned}$$

so we conclude that  $J_u = \infty$  a.s.. Now, for each  $n \geq 1$ , we simply define an announcing sequence  $(T_{n,m})_{m \geq 1}$  for  $T_n$  as

$$T_{n,m} = \inf\{t > T_{n-1} : \mu t - X_t \geq a - \frac{1}{m}\}$$

with  $T_0 = 0$ . We see that  $J_d = \infty$  and  $J_u = \infty$  a.s. ensure that each sequence  $(T_{n,m})_m$  is indeed an announcing sequence of  $T_n$ .  $\square$

Let us remark that in fact the random time  $\tau$  (defined in (4.2)) can be seen as the end of the optional set  $\Gamma = \bigcup_n \llbracket T_n \rrbracket$  as

$$\tau(\omega) = \sup\{t : (\omega, t) \in \Gamma\}.$$

**Proposition 4.12.** *The supermartingales  $Z$  and  $\tilde{Z}$  associated with the honest time  $\tau$  are given by*

$$(4.4) \quad Z_t = \Psi(\mu t - X_t - a) \mathbb{1}_{\{\mu t - X_t \geq a\}} + \mathbb{1}_{\{\mu t - X_t < a\}},$$

$$(4.5) \quad \tilde{Z}_t = \Psi(\mu t - X_t - a) \mathbb{1}_{\{\mu t - X_t > a\}} + \mathbb{1}_{\{\mu t - X_t \leq a\}},$$

where  $\Psi(x)$  is the ruin probability associated with the process  $\mu t - X_t$ , i.e., for every  $x \geq 0$

$$(4.6) \quad \Psi(x) := \mathbb{P}(t^x < \infty) \quad \text{with} \quad t^x := \inf\{t : x + \mu t - X_t < 0\}.$$

The function  $\Psi$  satisfies the following properties:

(a) for  $x < 0$  we have  $\Psi(x) = 1$ ;

(b) the function  $\Psi$  is continuous and decreasing on  $(0, \infty)$ ;

(c) for  $x = 0$ , we have  $\Psi(0) = \frac{\eta \mathbb{E}(Y_1)}{\mu} < 1$ .

In particular,  $Z_\tau = \frac{1}{1+\kappa}$  where  $\kappa = \frac{\mu}{\eta \mathbb{E}(Y_1)} - 1$ .

The supermartingale  $Z$  admits the decomposition  $Z = m - A^\circ$  where

$$m_t = (1 - \Psi(0)) \sum_n \mathbb{1}_{\{t \geq T_n\}} + \Psi(\mu t - X_t - a) \mathbb{1}_{\{\mu t - X_t \geq a\}} + \mathbb{1}_{\{\mu t - X_t < a\}}$$

$$A_t^\circ = (1 - \Psi(0)) \sum_n \mathbb{1}_{\{t \geq T_n\}}.$$

The  $\mathbb{F}$ -dual optional projection and the  $\mathbb{F}$ -dual predictable projection of  $\mathbb{1}_{\llbracket \tau, \infty \rrbracket}$  are equal, i.e.  $A^\circ = A^p$ .

*Proof.* The form of  $Z$  follows from the stationary and independent increments property of  $\mu t - X_t$

$$\begin{aligned} \mathbb{P}(\tau > t | \mathcal{F}_t) &= \mathbb{P}(\inf_{s \geq t} (\mu s - X_s) < a | \mathcal{F}_t) \\ &= \mathbb{P}(\inf_{s \geq t} (\mu(s-t) - (X_s - X_t)) < a - \mu t + X_t | \mathcal{F}_t) \\ &= \Psi(\mu t - X_t - a) \mathbb{1}_{\{\mu t - X_t \geq a\}} + \mathbb{1}_{\{\mu t - X_t < a\}}. \end{aligned}$$

Let us now compute the dual optional projection  $A^\circ$  of the process  $\mathbb{1}_{\llbracket \tau, \infty \rrbracket}$ . For any bounded optional process  $X$  we have

$$\mathbb{E}[X_\tau] = \sum_n \mathbb{E}[\mathbb{1}_{\{\tau = T_n\}} X_{T_n}] = \sum_n \mathbb{E}[X_{T_n} \mathbb{E}[\mathbb{1}_{\{\tau = T_n\}} | \mathcal{F}_{T_n}]]$$

which implies that

$$A^\circ = \sum_n \mathbb{P}(\tau = T_n | \mathcal{F}_{T_n}) \mathbb{1}_{\llbracket T_n, \infty \rrbracket}.$$

To compute  $\mathbb{P}(\tau = T_n | \mathcal{F}_{T_n})$  let us define, for any  $x$  and  $T$ , the stopping time  $S_x^T = \inf\{t > T : x + \mu t - X_t < 0\}$  and notice that

$$\mathbb{P}(\tau = T_n | \mathcal{F}_{T_n}) = \mathbb{E}(S_{-a}^{T_n} = \infty | \mathcal{F}_{T_n}) = \mathbb{E}(S_0^0 = \infty) = 1 - \Psi(0).$$

Then,  $A^o = (1 - \Psi(0)) \sum_n \mathbb{1}_{\llbracket T_n, \infty \rrbracket}$ . This is also the dual predictable projection as, by previous lemma,  $T_n$  are predictable stopping times. The martingale  $m = Z + A^o$  equals then

$$\begin{aligned} m_t &= \mathbb{E} \left[ (1 - \Psi(0)) \sum_n \mathbb{1}_{\{T_n < \infty\}} | \mathcal{F}_t \right] \\ &= (1 - \Psi(0)) \sum_n \mathbb{1}_{\{t \geq T_n\}} + \Psi(\mu t - X_t - a) \mathbb{1}_{\{\mu t - X_t \geq a\}} + \mathbb{1}_{\{\mu t - X_t < a\}}. \end{aligned}$$

Finally, from the general relation  $\tilde{Z} = Z + \Delta A^o$ , we conclude the form of  $\tilde{Z}$ .  $\square$

**4.3.2. Brownian motion: local time approximation.** We give an example related to an approximation result for the local time. Let  $B$  be a Brownian motion. For  $\varepsilon > 0$ , define a double sequence of stopping times by

$$\begin{aligned} U_0^\varepsilon &= 0, \quad V_0^\varepsilon = 0 \\ U_n^\varepsilon &= \inf\{t \geq V_{n-1}^\varepsilon : B_t = \varepsilon\}, \quad V_n^\varepsilon = \inf\{t \geq U_n^\varepsilon : B_t = 0\}. \end{aligned}$$

We consider the random time

$$(4.7) \quad \tau^\varepsilon := \sup\{V_n^\varepsilon : V_n^\varepsilon \leq T_1\}$$

with  $T_1 = \inf\{t : B_t = 1\}$ . From the definition we easily see that it is an honest thin random time.

Let us introduce the processes  $X^\varepsilon$ ,  $Y^\varepsilon$  and  $J^\varepsilon$

$$\begin{aligned} X_t^\varepsilon &:= \sup\{s \leq t \wedge T_1 : B_s = \varepsilon\} \\ Y_t^\varepsilon &:= \sup\{s \leq t \wedge T_1 : B_s = 0\} \\ J_t^\varepsilon &:= \mathbb{1}_{\{X_t^\varepsilon > Y_t^\varepsilon\}} \end{aligned}$$

and the function  $\zeta$

$$\zeta(x) := \mathbb{P}^x(T_0 < T_1) = 1 - x, \quad \text{for } x \in [0, 1].$$

The supermartingale  $Z^\varepsilon$  associated with  $\tau^\varepsilon$  is equal to

$$\begin{aligned} Z_t^\varepsilon &= J_t^\varepsilon \zeta(B_{t \wedge T_1}) + (1 - J_t^\varepsilon) \zeta(\varepsilon) \\ &= J_t^\varepsilon (1 - B_{t \wedge T_1}) + (1 - J_t^\varepsilon) (1 - \varepsilon) \\ &= 1 - J_t^\varepsilon B_{t \wedge T_1} - (1 - J_t^\varepsilon) \varepsilon. \end{aligned}$$

Let us define the process  $D_t^\varepsilon = \max\{n : V_n^\varepsilon \leq t\}$  which is equal to the number of down-crossings of Brownian motion from level  $\varepsilon$  to level 0 before time  $t$ . By integration by parts we obtain

$$B_t J_t^\varepsilon + \varepsilon(1 - J_t^\varepsilon) = \int_0^t J_s^\varepsilon dB_s + \int_0^t B_s dJ_s^\varepsilon + \varepsilon(1 - J_t^\varepsilon)$$

$$= \int_0^t J_s^\varepsilon dB_s + \varepsilon D_t^\varepsilon + \varepsilon.$$

The dual optional projection of  $\tau^\varepsilon$  equals

$$A_t^{o,\varepsilon} = (\mathbb{1}_{\{t \geq \tau^\varepsilon\}})^p = \varepsilon D_{t \wedge T_1}^\varepsilon + \varepsilon$$

and we easily see that it is a pure jump process with the property

$$\{\Delta A^{o,\varepsilon} > 0\} = \llbracket 0, T_1 \rrbracket \cap \bigcup_{n=0}^{\infty} \llbracket V_n^\varepsilon \rrbracket.$$

We can interpret the sequence  $\tau^\varepsilon$  with  $\varepsilon$  going to zero as an approximation of the strict honest time  $\tau$  given by

$$(4.8) \quad \tau := \sup\{t < T_1 : B_t = 0\},$$

as  $\tau^\varepsilon \rightarrow \tau$   $\mathbb{P}$  a.s. (by time reversal at  $\tau$ ). The supermartingale  $Z$  associated with  $\tau$  equals

$$Z_t = 1 - \int_0^{t \wedge T_1} \mathbb{1}_{\{B_s > 0\}} dB_s - \frac{1}{2} L_{t \wedge T_1}^0$$

and, by [23, Chapter VI Theorem (1.10)] and the fact that  $\mathbb{E}(\sqrt{T_1}) < \infty$ , we have the following convergence for dual optional projections

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[ \sup_t |\varepsilon D_{t \wedge T_1}^\varepsilon - \frac{1}{2} L_{t \wedge T_1}^0| \right] = 0.$$

In order to study the relationship between progressive enlargements with  $\tau^\varepsilon$ , let us recall the definition of weak convergence of  $\sigma$ -fields (see Definition 1 in [19] and references therein).

**Definition 4.13.** *A sequence of  $\sigma$ -fields  $\mathcal{G}_n$  converges weakly to a  $\sigma$ -field  $\mathcal{G}$  if and only if for all  $G \in \mathcal{G}$ ,  $\mathbb{E}(\mathbb{1}_G | \mathcal{G}_n)$  converges in probability to  $\mathbb{1}_G$ . We write  $\mathcal{G}_n \xrightarrow{w} \mathcal{G}$ .*

**Lemma 4.14.** *Let  $\mathbb{F}^n$  be a progressive enlargement of the filtration  $\mathbb{F}$  with random time  $\tau^{1/n}$  defined in (4.7) and  $\mathbb{F}^\infty$  be the progressive enlargement of the filtration  $\mathbb{F}$  with random time  $\tau$  defined in (4.8). Then, for each  $t$ , the sequence of  $\sigma$ -fields  $(\mathcal{F}_t^n)_n$  converges weakly to  $\mathcal{F}_t^\infty$ .*

*Proof.* We have to check that for each  $F \in \mathcal{F}_t^\infty$ ,  $\mathbb{P}(F | \mathcal{F}_t^n)$  converges in probability to  $\mathbb{1}_F$ . We limit our attention to the sets  $F$  belonging to the generator of  $\mathcal{F}_t^\infty$ . If  $F \in \mathcal{F}_t$ , the condition is obviously satisfied. If  $F = \{\tau \leq s\}$  for  $s \leq t$ , using Proposition B.3 and the honesty of  $\tau^{1/n}$ , we have

$$\begin{aligned} \mathbb{E}(\mathbb{1}_{\{\tau \leq s\}} | \mathcal{F}_t^n) &= \mathbb{1}_{\{\tau^{1/n} \leq s\}} \mathbb{E}(\mathbb{1}_{\{\tau \leq s\}} | \mathcal{F}_t^n) \\ &= \mathbb{1}_{\{\tau^{1/n} \leq s\}} \frac{\mathbb{E}(\mathbb{1}_{\{\tau \leq s\}} | \mathcal{F}_t)}{\mathbb{P}(\tau^{1/n} \leq s | \mathcal{F}_t)} \xrightarrow[n \rightarrow \infty]{a.s.} \mathbb{1}_{\{\tau \leq s\}} \quad a.s. \end{aligned}$$

where the convergence comes from  $\tau^{1/n} \rightarrow \tau$  a.s. □

## 5. ENTROPY OF A THIN RANDOM TIME

The additional information carried by enlarged filtration and its measurement was studied by several authors. Already in Meyer [21] and Yor [25], the question on stability of martingale spaces with respect to initial enlargement with atomic  $\sigma$ -field was asked. Here we complete previous studies by giving a simple connection between progressive enlargement with thin random time and conditional entropy of a partition associated to this time.

In the case of initial enlargement with a partition  $\mathcal{C} := (C_n)_n$ , the additional knowledge is measured by entropy, namely

$$H(\mathcal{C}) := - \sum_n \mathbb{P}(C_n) \log \mathbb{P}(C_n).$$

In the case of progressive enlargement with a thin random time  $\tau$ , we suggest the measurement of additional knowledge by the entropy of a thin random time defined through:

$$(5.1) \quad H(\tau) := - \sum_n \mathbb{E} [\mathbb{1}_{C_n} \log z_{T_n}^n],$$

where  $C_n$  and  $z_n$  are defined in (2.1) and (2.2). Let us remark that the condition  $H(\tau) < \infty$  is weaker than the condition  $H(\mathcal{C}) < \infty$ .

To state the main result of this section in Theorem 5.2, which consists of a generalisation of [25, Theorem 2] we need to define more general object, namely

$$(5.2) \quad H_\gamma(\tau) := \sum_n \mathbb{E} \left[ \mathbb{1}_{C_n} \left[ \log \frac{1}{z_{T_n}^n} \right]^\gamma \right] \quad \gamma > 0.$$

**Remark 5.1.** (a) If  $\tau$  is an  $\mathbb{F}$ -stopping time then  $H_\gamma(\tau) = 0$ .

(b) If for any  $n$  the set  $C_n$  is in  $\mathcal{F}_{T_n}$ , then we do not gain any additional information since  $\mathbb{1}_{C_n} \log z_{T_n}^n = \mathbb{1}_{C_n} \log \mathbb{1}_{C_n} = 0$ .

(c) As noted in Section 2, the exhausting sequence  $(T_n)_n$  of a thin random time is not unique. However  $H_\gamma(\tau)$  is invariant under different decompositions of  $\tau$  since for  $\mathbb{F}$ -stopping times  $T, T_1$  and  $T_2$  such that  $\llbracket T_1 \rrbracket \cap \llbracket T_2 \rrbracket = \emptyset$  and  $\{\tau = T\} = \{\tau = T_1\} \cup \{\tau = T_2\}$  we have

$$\begin{aligned} & \mathbb{1}_{\{\tau=T\}} \log^\gamma \mathbb{P}(\tau = T | \mathcal{F}_T) \\ &= \mathbb{1}_{\{\tau=T_1\}} \log^\gamma \mathbb{P}(\tau = T | \mathcal{F}_T) + \mathbb{1}_{\{\tau=T_2\}} \log^\gamma \mathbb{P}(\tau = T | \mathcal{F}_T) \\ &= \mathbb{1}_{\{\tau=T_1\}} \log^\gamma \mathbb{P}(\tau = T_1 | \mathcal{F}_{T_1}) + \mathbb{1}_{\{\tau=T_2\}} \log^\gamma \mathbb{P}(\tau = T_2 | \mathcal{F}_{T_2}). \end{aligned}$$

The entropy of thin random time reveals to be an adequate notion to treat the stability of martingale spaces with respect to progressive enlargement of filtration (with thin random time). In this section we work under standing assumption that

(C) all  $\mathbb{F}$ -martingales are continuous

Then we introduce some related notations. For any  $p \in [1, \infty)$ , we denote  $H^p$  and  $S^p$  the Banach spaces consisting respectively of continuous local martingales and continuous semimartingales equipped with the following norms:

(a) a continuous  $\mathbb{F}$ -local martingale  $X$  belongs to  $H^p$  if

$$\|X\|_{H^p} := \|\langle X \rangle_\infty^{1/2}\|_{L^p} < \infty;$$

(b) a continuous  $\mathbb{F}$ -semimartingale  $X$ , with canonical decomposition  $X = M + V$ , belongs to  $S^p$  if

$$\|X\|_{S^p} := \|\langle M \rangle_\infty^{1/2}\|_{L^p} + \left\| \int_0^\infty |dV_t| \right\|_{L^p} < \infty.$$

We are ready to state the main result of this section. It is a generalisation of [25, Theorem 2] and the proof here is based on the original proof.

**Theorem 5.2.** *Let  $\tau$  be a thin random time with an exhausting sequence  $(T_n)_n$  and family of  $\mathbb{F}$ -martingales  $(z_n)_n$  satisfying  $z_\infty^n = \mathbb{1}_{C_n}$ . Assume **(C)** and let  $r \in [1, \infty)$ ,  $p, \gamma > 0$  satisfy  $\frac{1}{r} = \frac{1}{p} + \frac{1}{\gamma}$ . Then the following conditions are equivalent:*

- (a) *for each  $\mathbb{F}$ -local martingale  $Y$  and each  $\mathbb{F}^\tau$ -predictable process  $G$ , the  $\mathbb{F}^\tau$ -semimartingale  $X := G \cdot Y$  satisfies:*

$$\|X\|_{S^r(\mathbb{F}^\tau)} \leq C_{p,r} \|\langle X \rangle_\infty^{1/2}\|_{L^p};$$

- (b)  $H_{\gamma/2}(\tau) < \infty$ .

*In particular, if the conditions (a) and (b) are satisfied, then  $H^p(\mathbb{F})$  is continuously embedded in  $S^r(\mathbb{F}^\tau)$ .*

*Proof.* By Theorem 3.1, under assumption **(C)**, each  $X$  of the form  $X = G \cdot Y$  where  $Y$  is an  $\mathbb{F}$ -local martingale and  $G$  is an  $\mathbb{F}^\tau$ -predictable process, has the decomposition

$$(5.3) \quad X = \widehat{X} + \langle X, Y^b \rangle + \langle X, Y^a \rangle$$

where  $Y^b$  and  $Y^a$  are  $\mathbb{F}^\tau$ -local martingales given by

$$Y^b := \int_0^{\tau \wedge t} \frac{1}{Z_{s-}} d\widehat{m}_s \quad \text{and} \quad Y_t^a := \sum_n \mathbb{1}_{C_n} \int_{T_n}^t \frac{1}{z_s^n} d\widehat{z}_s^n$$

with  $\widehat{X}$ ,  $\widehat{m}$  and  $\widehat{z}^n$  the  $\mathbb{F}^\tau$ -local martingale parts from Doob-Meyer decomposition of corresponding  $\mathbb{F}^\tau$ -semimartingales  $X$ ,  $m$  and  $z^n$ . Since  $r < p$  and continuity of  $X$  it always holds that  $\|\langle \widehat{X} \rangle^{1/2}\|_{L^r} = \|\langle X \rangle^{1/2}\|_{L^r} = \|\langle X \rangle^{1/2}\|_{L^p}$ . Thus, showing (a)  $\iff$  (b) is equivalent to showing that

$$(5.4) \quad \left\| \int_0^\infty |d\langle X, Y^b \rangle_t + \langle X, Y^a \rangle_t| \right\|_{L^r} \leq C_{p,r} \|\langle X \rangle_\infty^{1/2}\|_{L^p}$$

holds for any  $X$ . By [25, Lemma 2] and the fact that stochastic intervals  $\llbracket 0, \tau \rrbracket$  and  $\llbracket T_n, \infty \rrbracket \cap C_n$  for  $n \in \mathbb{N}$  are pairwise disjoint, inequality (5.4) holds for any adequate  $X$  if and only if

$$\mathbb{E} [\langle Y^b \rangle_\infty^{\gamma/2}] < \infty \quad \text{and} \quad \mathbb{E} [\langle Y^a \rangle_\infty^{\gamma/2}] < \infty.$$

Firstly we show that

$$(5.5) \quad \|\langle Y^b \rangle_\infty^{1/2}\|_{L^\gamma} < \infty \quad \forall \gamma > 0.$$

By [25, Remark 5.1 2) p.123], since, for  $\gamma > 2$ ,  $x^{\gamma/2}$  is moderate Orlicz function, we have

$$\left\| \left[ \int_0^\tau \frac{1}{Z_s^2} d\langle m \rangle_s \right]^{1/2} \right\|_{L^\gamma} \leq C \left[ 1 + \left\| \left[ \log \frac{1}{U} \right]^{1/2} \right\|_{L^\gamma} \right]$$

where  $U$  is random variable with uniform distribution on  $[0, 1]$ . Note that  $\int_0^1 (-\log x)^{\gamma/2} < \infty$  and thus, by using the fact that  $L^\mu \subset L^\gamma$  for  $0 < \mu < \gamma < \infty$ , the (5.5) holds.

Showing that  $\mathbb{E} \left[ \langle Y^a \rangle_\infty^{\gamma/2} \right] < \infty$  if and only if  $H_{\gamma/2}(\tau) < \infty$  for  $\gamma = 2$  is a simpler special case which will be useful afterwards and we start with it. By properties of dual predictable projection, we have that

$$\mathbb{E}(\langle Y^a \rangle_\infty) = \sum_n \mathbb{E} \left[ \int_{T_n}^\infty \mathbb{1}_{C_n} \frac{1}{(z_t^n)^2} d\langle z^n \rangle_t \right] = \sum_n \mathbb{E} \left[ \int_{T_n}^\infty \frac{1}{z_t^n} d\langle z^n \rangle_t \right].$$

Consider the function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  defined as  $f(x) = x - x \log x$  for  $x > 0$  and  $f(0) = 0$ . Then, Itô's formula for  $z^n$  implies

$$\mathbb{1}_{C_n} = z_t^n - z_t^n \log z_t^n - \int_t^\infty \log z_s^n dz_s^n - \frac{1}{2} \int_t^\infty \frac{1}{z_s^n} d\langle z^n \rangle_s.$$

We deduce, by taking conditional expectation with respect to  $\mathcal{F}_t$ , that

$$(5.6) \quad \mathbb{E} \left[ \int_t^\infty \frac{1}{z_s^n} d\langle z^n \rangle_s \middle| \mathcal{F}_t \right] = 2z_t^n \log \frac{1}{z_t^n}.$$

Finally

$$\begin{aligned} \mathbb{E}(\langle Y^a \rangle_\infty) &= \sum_n \mathbb{E} \left[ \int_{T_n}^\infty \frac{1}{z_t^n} d\langle z^n \rangle_t \right] \\ &= -2 \sum_n \mathbb{E} [z_{T_n}^n \log z_{T_n}^n] = 2H(\tau) < \infty. \end{aligned}$$

In order to complete the proof it remains to show that  $H_\gamma(\tau) < \infty$  if and only if  $\mathbb{E} [\langle Y^a \rangle^\gamma] < \infty$ , first notice that Lemma C.1 implies that it is equivalent to proving  $H_\gamma(\tau) < \infty$  if and only if  $\mathbb{E} \left[ \left[ \log \frac{1}{I} \right]^\gamma \right] < \infty$ . To this end, note that, by Lemma C.2, we have

$$\begin{aligned} \mathbb{E} \left[ \left[ \log \frac{1}{I} \right]^\gamma \right] &= \sum_n \mathbb{E} \left[ \mathbb{E} \left[ \mathbb{1}_{C_n} \left[ \log \frac{1}{I_n} \right]^\gamma \middle| \mathcal{F}_{T_n} \right] \right] \\ &= \sum_n \mathbb{E} \left[ \mathbb{1}_{C_n} \frac{1 - z_{T_n}^n}{z_{T_n}^n} \int_0^{z_{T_n}^n} \left[ \log \frac{1}{\beta} \right]^\gamma \frac{1}{(1 - \beta)^2} d\beta \right]. \end{aligned}$$

Denoting by  $z = \sum_n \mathbb{1}_{C_n} z_{T_n}^n$ , taking any  $\varepsilon \in (0, 1)$  and defining  $f(x) = \int_0^x \left[ \log \frac{1}{\beta} \right]^\gamma d\beta$  for  $x \in (0, 1)$ , we further obtain

$$\begin{aligned} \mathbb{E} \left[ \left[ \log \frac{1}{I} \right]^\gamma \right] &= \mathbb{E} \left[ \frac{1 - z}{z} \int_0^z \left[ \log \frac{1}{\beta} \right]^\gamma \frac{1}{(1 - \beta)^2} d\beta \right] \\ &\leq \mathbb{E} \left[ \mathbb{1}_{\{z > \varepsilon\}} \left[ \frac{f(\varepsilon)}{\varepsilon(1 - \varepsilon)^2} + \frac{1}{1 - \varepsilon} \left[ \log \frac{1}{\varepsilon} \right]^\gamma \right] \right] + \mathbb{E} \left[ \mathbb{1}_{\{z \leq \varepsilon\}} \frac{f(z)}{z(1 - \varepsilon)^2} \right] \\ &\leq C_1 + C_2 \mathbb{E} \left[ \mathbb{1}_{\{z \leq \varepsilon\}} \left[ \log \frac{1}{z} \right]^\gamma \right] \\ &\leq C_1 + C_2 H_\gamma(\tau). \end{aligned}$$

Thus we conclude that  $H_\gamma(\tau) < \infty$  if and only if  $\mathbb{E} \left[ \left[ \log \frac{1}{I} \right]^\gamma \right] < \infty$  and the proof is complete.  $\square$

## 6. GENERAL PROGRESSIVE ENLARGEMENT WITH A PARTITION

In this section we consider more general progressive enlargement of filtration with given a partition  $(C_m)_m$  of  $\Omega$ , which, in contrary to previous results, allows for adding several members of a partition at the same time.

Let  $\tau$  be a thin random time with exhausting sequence  $(T_n)_n$  and  $\xi$  be a discrete random variable with values in  $\mathbb{N}$ . Since for each  $m \in \mathbb{N}$  we can find unique decomposition  $(n, k) \in \mathbb{N} \times \mathbb{N}$  such that  $m = 2^{n-1}(2k - 1)$ , by rearranging and renumbering terms in  $(C_m)_m$ , we can obtain double sequence  $(C_n^k)_{n,k}$  of sets such that the members  $(C_n^k)_k$  are added at time  $T_n$ . More precisely, for double sequence  $(C_n^k)_{n,k}$  and a sequence of  $\mathbb{F}$ -stopping times  $(T_n)_n$  with disjoint graphs, we define  $\xi$  and  $\tau$  by

$$\xi := \sum_{n,k} 2^{n-1}(2k - 1) \mathbb{1}_{C_n^k} \quad \text{and} \quad \tau := \sum_n T_n \mathbb{1}_{\bigcup_k C_n^k}.$$

Note that  $\{\tau = T_n < \infty\} = \{T_n < \infty\} \cap \bigcup_k C_n^k$  and  $\sigma(\xi \mathbb{1}_{\{\tau=T_n\}}) = \sigma((C_n^k)_k)$ . Analogously to (2.2) we define

$$(6.1) \quad z_t^{n,k} := \mathbb{P}(C_n^k | \mathcal{F}_t).$$

Described above situation is captured by the progressively enlarged filtration  $\mathbb{F}^{\xi, \tau} := (\mathcal{F}_t^{\xi, \tau})_{t \geq 0}$  defined through

$$(6.2) \quad \mathcal{F}_t^{\xi, \tau} := \bigcap_{s > t} \mathcal{F}_s \vee \sigma(\xi \mathbb{1}_{\{u \geq \tau\}} : u \leq s).$$

Similarly to Lemma 2.7 we obtain:

**Lemma 6.1.** *For any  $\mathcal{G}$ -measurable integrable random variable  $X$  and  $s \leq t$  we have*

$$\mathbb{E} \left[ X | \mathcal{F}_t^{\xi, \tau} \right] \mathbb{1}_{\{s \geq T_n\} \cap C_n^k} = \mathbb{1}_{\{s \geq T_n\} \cap C_n^k} \frac{\mathbb{E} \left[ X \mathbb{1}_{C_n^k} | \mathcal{F}_t \right]}{z_t^{n,k}}.$$

Theorem 3.1 can be easily extended to the case of  $\mathbb{F}^{\xi, \tau}$  as stated below.

**Theorem 6.2.** *The hypothesis  $(\mathcal{H}')$  is satisfied for  $(\mathbb{F}, \mathbb{F}^{\xi, \tau})$ . Moreover, for each  $\mathbb{F}^{\xi, \tau}$ -predictable and bounded process  $G$  and each  $\mathbb{F}$ -local martingale  $Y$  the integral  $X := G \cdot Y$  is an  $\mathbb{F}^{\xi, \tau}$ -semimartingale with canonical decomposition*

$$(6.3) \quad X_t = \widehat{X}_t + \int_0^{t \wedge \tau} \frac{1}{Z_{s-}} d\langle X, m \rangle_s + \sum_{n,k} \mathbb{1}_{C_n^k} \int_0^t \mathbb{1}_{\{s > T_n\}} \frac{1}{z_{s-}^{n,k}} d\langle X, z^{n,k} \rangle_s$$

where  $\widehat{X}$  is an  $\mathbb{F}^{\xi, \tau}$ -local martingale.

*Proof.* The proof is almost identical to the original proof of Theorem 3.1. We just give the form of  $\mathbb{F}^{\xi, \tau}$ -predictable process  $H$  by

$$H_t = \mathbb{1}_{\{t \leq \tau\}} J_t + \mathbb{1}_{\{\tau < t\}} K_t(\xi, \tau) \quad t \geq 0$$

where  $J$  is an  $\mathbb{F}$ -predictable bounded process and  $K : \mathbb{R}_+ \times \Omega \times \mathbb{N} \times \mathbb{R}_+ \rightarrow \mathbb{R}$  is  $\mathcal{P} \otimes 2^{\mathbb{N}} \otimes \mathcal{B}(\mathbb{R}_+)$ -measurable. Note that, since  $\{t \leq \tau\} \subset \{Z_{t-} > 0\}$ ,  $J$  can be chosen to satisfy  $J_t =$

$J_t \mathbb{1}_{\{Z_{t-} > 0\}}$ . Since  $\tau$  is a thin random time, we can rewrite the process  $H$  as

$$H_t = J_t \mathbb{1}_{\{t \leq \tau\}} + \sum_{n,k} \mathbb{1}_{\{T_n < t\}} K_t(2^{n-1}(2k-1), T_n) \mathbb{1}_{C_n^k}.$$

Note that each process  $K_t^{n,k} := \mathbb{1}_{\{T_n < t\}} K_t(2^{n-1}(2k-1), T_n)$  is  $\mathbb{F}$ -predictable and bounded and, since  $C_n^k \subset \{z_{t-}^{n,k} > 0\}$ ,  $K^{n,k}$  can be chosen to satisfy  $K_t^{n,k} = K_t^{n,k} \mathbb{1}_{\{z_{t-}^{n,k} > 0\}}$ .  $\square$

Likewise, Theorem 5.2 can be generalized to the case of  $\mathbb{F}^{\xi, \tau}$ . To this end let us define

$$(6.4) \quad H_\gamma(\xi, \tau) := \sum_{n,k} \mathbb{E} \left[ \mathbb{1}_{C_n^k} \left[ \log \frac{1}{z_{T_n}^{n,k}} \right]^\gamma \right] \quad \gamma > 0.$$

Then, the following result follows by the same proof as Theorem 5.2.

**Theorem 6.3.** *Let  $(C_n^k)_{n,k}$  be an  $\mathcal{F}_\infty$ -measurable partition,  $\tau$  be a thin random time with exhaustive sequence  $(T_n)_n$  and  $\mathbb{F}^{\xi, \tau}$  be given by (6.2). Assume (C) and let  $r \in [1, \infty)$ ,  $p, \gamma > 0$  satisfy  $\frac{1}{r} = \frac{1}{p} + \frac{1}{\gamma}$ . The the following conditions are equivalent:*

- (a) *for each  $\mathbb{F}$ -local martingale  $Y$  and each  $\mathbb{F}^{\xi, \tau}$ -predictable process  $G$ , the  $\mathbb{F}^{\xi, \tau}$ -semimartingale  $X := G \cdot Y$  satisfies:*

$$\|X\|_{S^r(\mathbb{F}^{\xi, \tau})} \leq C_{p,r} \| \langle X \rangle_\infty^{1/2} \|_{L^p};$$

- (b)  $H_{\gamma/2}(\xi, \tau) < \infty$ .

*In particular, if the conditions (a) and (b) are satisfied, then  $H^p(\mathbb{F})$  is continuously embedded in  $S^r(\mathbb{F}^{\xi, \tau})$ .*

## APPENDIX A. PROJECTIONS

We collect here the definitions of the key tools we have used along the paper. Projections and dual projections onto the reference filtration  $\mathbb{F}$  play an important role in the theory of enlargement of filtrations. First we recall the definition of optional and predictable projections, see [13, Theorems 5.1 and 5.2] and [16, p.264-265].

**Definition A.1.** *Let  $X$  be a measurable bounded (or positive) process. The optional projection of  $X$  is the unique optional process  ${}^oX$  such that for every stopping time  $T$  we have*

$$\mathbb{E} [X_T \mathbb{1}_{\{T < \infty\}} | \mathcal{F}_T] = {}^oX_T \mathbb{1}_{\{T < \infty\}} \quad a.s..$$

*The predictable projection of  $X$  is the unique predictable process  ${}^pX$  such that for every predictable stopping time  $T$  we have*

$$\mathbb{E} [X_T \mathbb{1}_{\{T < \infty\}} | \mathcal{F}_{T-}] = {}^pX_T \mathbb{1}_{\{T < \infty\}} \quad a.s..$$

For definition of dual optional projection and dual predictable projection see [16, p.265], [22, Chapter 3 Section 5], [11, Chapter 6 Paragraph 73 p.148], [13, Sections 5.18, 5.19]. We point out that the convention we use here allows a jump at 0.

**Definition A.2.** (a) Let  $V$  be a càdlàg pre-locally integrable variation process (not necessary adapted). The dual optional projection of  $V$  is the unique optional process  $V^o$  such that for every optional process  $H$  we have

$$\mathbb{E} \left[ \int_{[0, \infty)} H_s dV_s \right] = \mathbb{E} \left[ \int_{[0, \infty)} H_s dV_s^o \right].$$

(b) Let  $V$  be a càdlàg locally integrable variation process (not necessary adapted). The dual predictable projection of  $V$  is the unique predictable process  $V^p$  such that for every predictable process  $H$  we have

$$\mathbb{E} \left[ \int_{[0, \infty)} H_s dV_s \right] = \mathbb{E} \left[ \int_{[0, \infty)} H_s dV_s^p \right].$$

## APPENDIX B. HONEST TIMES

In this Appendix we gather complementary results on honest times. They complete or slightly extend the existing results mainly from [17]. We have made use of them along the paper.

**Proposition B.1.** (a) A random time  $\tau$  is an  $\mathbb{F}$ -honest time if and only if for every  $t > 0$  there exists an  $\mathcal{F}_{t-}$ -measurable random variable  $\tau_t$  such that  $\tau = \tau_t$  on  $\{\tau < t\}$ .

(b) A random time  $\tau$  is an  $\mathbb{F}$ -honest time if and only if for every  $t > 0$  there exists an  $\mathcal{F}_t$ -measurable random variable  $\tau_t$  such that  $\tau = \tau_t$  on  $\{\tau \leq t\}$ .

*Proof.* Sufficiency of both conditions is straightforward.

Using the notation from Definition 4.1 we introduce the process  $\alpha^-$  as  $\alpha_t^- = \sup_{r \in \mathbb{Q}, r < t} \tau_r$ . This definition implies that  $\alpha^-$  is an increasing, left-continuous, adapted process such that  $\alpha_t^- = \tau$  on  $\{\tau < t\}$  thus the necessary condition in (a) is proven.

Let us denote by  $\alpha$  the right-continuous version of  $\alpha^-$ , i.e.,  $\alpha_t = \alpha_{t+}^-$ . Then,  $\alpha$  is an increasing, càdlàg, adapted process such that  $\alpha_t = \tau$  on  $\{\tau \leq t\}$  and  $\tau = \sup\{t : \alpha_t = t\}$  thus the necessary condition in (b) is proved.  $\square$

The next theorem gives some characterisations of an honest time. We prove equivalence of the condition (f). We also refer the reader to Azéma [7, Theorem 1.4 and Proposition 1.2, p. 298-299] to compare with predictable case.

**Theorem B.2.** Let  $\tau$  be a random time. Then, the following conditions are equivalent:

- (a)  $\tau$  is an honest time;
- (b) there exists an optional set  $\Gamma$  such that  $\tau(\omega) = \sup\{t : (\omega, t) \in \Gamma\}$  on  $\{\tau < \infty\}$ ;
- (c)  $\tilde{Z}_\tau = 1$  a.s. on  $\{\tau < \infty\}$ ;
- (d)  $\tau = \sup\{t : \tilde{Z}_t = 1\}$  a.s. on  $\{\tau < \infty\}$ ;
- (e)  $\mathcal{P}^\tau|_{\llbracket 0, \infty \llbracket}$  is generated by  $\mathcal{P}|_{\llbracket 0, \infty \llbracket}$  and  $\llbracket 0, \tau \llbracket$ , where  $\mathcal{P}^\tau$  is the predictable  $\sigma$ -field linked to  $\mathbb{F}^\tau$ ;
- (f)  $A_t^o = A_{t \wedge \tau}^o$  a.s.  $\forall t \geq 0$  on the set  $\{\tau < \infty\}$ .

*Proof.* The equivalence among conditions (a), (b), (c), (d) and (e) is stated in Theorem (5,1) from Jeulin [17].

To prove that (b) $\Rightarrow$ (f) suppose that  $\tau$  is an end of the optional set  $\Gamma$  on  $\{\tau < \infty\}$ . Without loss of generality we may assume that  $\Gamma$  is left-closed, i.e., for each  $\omega$  if  $x_n \nearrow x$  and  $x_n \in \Gamma(\omega)$  then  $x \in \Gamma(\omega)$ , since left-closure of an optional set is optional. Thus one has  $[\tau] \subset \Gamma$ . Define the support of the measure  $dA^\circ$ , i.e.,

$$S(dA^\circ) := \{(\omega, t) : t \geq 0 \quad \forall \varepsilon > 0 \quad A_t^\circ(\omega) > A_{t-\varepsilon}^\circ(\omega)\},$$

with the extension of  $A^\circ$  such that  $A_t^\circ = 0$  for  $t \leq 0$ . Note that  $S(dA^\circ)$  is left-closed and optional. Moreover  $[\tau] \subset S(dA^\circ)$  since

$$\mathbb{P}(\tau < \infty) = \mathbb{E}(A_\infty^\circ) = \mathbb{E}\left[\int_{[0, \infty)} \mathbb{1}_{S(dA^\circ)}(s) dA_s^\circ\right] = \mathbb{P}\left([\tau] \subset S(dA^\circ), \tau < \infty\right).$$

Let  $\Lambda$  be any left-closed, optional set containing  $[\tau]$ . Then

$$\begin{aligned} \mathbb{E}\left[\int_{[0, \infty)} \mathbb{1}_\Lambda(s) dA_s^\circ\right] &= \mathbb{E}\left[\int_{[0, \infty)} \mathbb{1}_\Lambda(s) dA_s\right] = \mathbb{P}(\tau < \infty) \\ &= \mathbb{E}(A_\infty^\circ) = \mathbb{E}\left[\int_{[0, \infty)} \mathbb{1}_{S(dA^\circ)}(s) dA_s^\circ\right], \end{aligned}$$

where the first and the third equality comes from Definition A.2, the second is due to  $[\tau] \subset \Lambda$ . Hence  $S(dA^\circ) \subset \Lambda$ . Therefore we conclude that  $S(dA^\circ)$  is the smallest left-closed optional set containing  $[\tau]$  and we deduce that  $[\tau] \subset S(dA^\circ) \subset \Gamma$ . Since, by assumption,  $\tau$  is an end of  $\Gamma$  on  $\{\tau < \infty\}$ , we obtain that, on the set  $\{\tau < \infty\}$ ,  $A_t^\circ = A_{t \wedge \tau}^\circ$  for  $t \geq 0$ .

To finish the proof, we show implication (f) $\Rightarrow$ (b). It is straightforward to see that  $\tau$  is the end of the support of  $dA^\circ$  which is an optional set.  $\square$

For an honest time, on the set  $\{t \geq \tau\}$ , the projection on  $\mathcal{F}_t^\tau$  can be expressed in terms of projection on  $\mathcal{F}_t$  as in the following proposition. It is known that, on the set  $\{t \geq \tau\}$ ,  $1 - Z_t > 0$ .

**Proposition B.3.** *Let  $\tau$  be an honest time. Then, for any  $\mathcal{G}$ -measurable integrable random variable  $X$  and  $s \leq t$  we have*

$$\mathbb{E}[X | \mathcal{F}_t^\tau] \mathbb{1}_{\{\tau \leq s\}} = \mathbb{1}_{\{\tau \leq s\}} \frac{\mathbb{E}[X \mathbb{1}_{\{\tau \leq s\}} | \mathcal{F}_t]}{\mathbb{P}[\tau \leq s | \mathcal{F}_t]}.$$

*Proof.* Note that for each  $G \in \mathcal{F}_t^\tau$  there exists  $F \in \mathcal{F}_t$  such that  $G \cap \{\tau \leq s\} = F \cap \{\tau \leq s\}$  as, by Monotone Class Theorem, it is enough to check it for  $G \in \mathcal{F}_t$  for which it is obviously satisfied and for  $G = \{\tau \in B\}$  where  $B$  is a Borel set for which, by honest time property, we have

$$\{\tau \in B\} \cap \{\tau \leq s\} = \{\tau_s \in B\} \cap \{\tau \leq s\}$$

with  $\{\tau_s \in B\} \in \mathcal{F}_s \subset \mathcal{F}_t$ . Then, we have to show that

$$\mathbb{E}[X \mathbb{1}_{\{\tau \leq s\}} \mathbb{P}(\tau \leq s | \mathcal{F}_t) | \mathcal{F}_t^\tau] = \mathbb{1}_{\{\tau \leq s\}} \mathbb{E}(X \mathbb{1}_{\{\tau \leq s\}} | \mathcal{F}_t).$$

For any  $G \in \mathcal{F}_t^\tau$ , we choose  $F \in \mathcal{F}_t$  such that  $G \cap \{\tau \leq s\} = F \cap \{\tau \leq s\}$ , and we obtain

$$\begin{aligned} \mathbb{E}[X \mathbb{1}_{\{\tau \leq s\}} \cap G \mathbb{P}(\tau \leq s | \mathcal{F}_t)] &= \mathbb{E}[X \mathbb{1}_{\{\tau \leq s\}} \cap F \mathbb{P}(\tau \leq s | \mathcal{F}_t)] \\ &= \mathbb{E}[\mathbb{1}_{\{\tau \leq s\}} \cap F \mathbb{E}(X \mathbb{1}_{\{\tau \leq s\}} | \mathcal{F}_t)] \end{aligned}$$

$$= \mathbb{E} \left[ \mathbb{1}_{\{\tau \leq s\} \cap G} \mathbb{E}(X \mathbb{1}_{\{\tau \leq s\}} | \mathcal{F}_t) \right].$$

which ends the proof.  $\square$

### APPENDIX C. ENTROPY OF THIN RANDOM TIME

Here we present generalisations of two auxiliary lemmas, Lemma 3 and Lemma 4, from [25]. They serve to prove Theorem 5.2.

**Lemma C.1.** *For all  $\gamma > 0$  there exist  $c_\gamma$  and  $C_\gamma$  such that*

$$(C.1) \quad c_\gamma \mathbb{E} [\langle Y^a \rangle_\infty^\gamma] \leq \mathbb{E} \left[ \left[ \log \frac{1}{I} \right]^\gamma \right] \leq C_\gamma \mathbb{E} [\langle Y^a \rangle_\infty^\gamma]$$

where  $I$  is defined as

$$(C.2) \quad I := \sum_n \mathbb{1}_{C_n} I_n \quad \text{where} \quad I_n := \inf_{t \geq T_n} z_t^n$$

*Proof. Step 1.* We first prove the first inequality in (C.1). We have that

$$\begin{aligned} \mathbb{1}_{\{t \geq T_n\} \cap C_n} \mathbb{E} \left[ \langle Y^a \rangle_\infty - \langle Y^a \rangle_t | \mathcal{F}_t^\tau \right] &= \mathbb{1}_{\{t \geq T_n\} \cap C_n} \mathbb{E} \left[ \int_t^\infty \frac{1}{(z_s^n)^2} d\langle z^n \rangle_s | \mathcal{F}_t^\tau \right] \\ &= \mathbb{1}_{\{t \geq T_n\} \cap C_n} \frac{1}{z_t^n} \mathbb{E} \left[ \mathbb{1}_{C_n} \int_t^\infty \frac{1}{(z_s^n)^2} d\langle z^n \rangle_s | \mathcal{F}_t \right] \\ &= \mathbb{1}_{\{t \geq T_n\} \cap C_n} \frac{1}{z_t^n} \mathbb{E} \left[ \int_t^\infty \frac{1}{z_s^n} d\langle z^n \rangle_s | \mathcal{F}_t \right] \\ &= 2 \mathbb{1}_{\{t \geq T_n\} \cap C_n} \log \frac{1}{z_t^n} \\ &\leq 2 \mathbb{1}_{C_n} \log \frac{1}{I_n} \end{aligned}$$

where the second equality follows from Lemma 2.7, the third one from dual predictable projection properties and the fourth one from (5.6). Therefore, for each  $\mu \in (0, 1]$  we deduce that

$$(C.3) \quad \begin{aligned} \mathbb{1}_{\{t \geq T_n\} \cap C_n} \mathbb{E} \left[ \left[ \langle Y^a \rangle_\infty - \langle Y^a \rangle_t \right]^\mu | \mathcal{F}_t^\tau \right] &\leq \mathbb{1}_{\{t \geq T_n\} \cap C_n} \left[ \mathbb{E} \left[ \langle Y^a \rangle_\infty - \langle Y^a \rangle_t | \mathcal{F}_t^\tau \right] \right]^\mu \\ &\leq 2^\mu \mathbb{1}_{C_n} \left[ \log \frac{1}{I_n} \right]^\mu. \end{aligned}$$

Consequently, the required inequality for  $\gamma \in (0, 1]$  follows by

$$\begin{aligned} \mathbb{E}(\langle Y^a \rangle_\infty^\gamma) &= \sum_n \mathbb{E} \left[ \mathbb{1}_{C_n} \mathbb{E} \left[ \left[ \langle Y^a \rangle_\infty - \langle Y^a \rangle_{T_n} \right]^\gamma | \mathcal{F}_{T_n}^\tau \right] \right] \\ &\leq \sum_n \mathbb{E} \left[ \sup_t \mathbb{1}_{\{t \geq T_n\} \cap C_n} \mathbb{E} \left[ \left[ \langle Y^a \rangle_\infty - \langle Y^a \rangle_t \right]^\gamma | \mathcal{F}_t^\tau \right] \right] \\ &\leq 2^\gamma \sum_n \mathbb{E} \left[ \mathbb{1}_{C_n} \left[ \log \frac{1}{I_n} \right]^\gamma \right] \\ &= 2^\gamma \mathbb{E} \left[ \left[ \log \frac{1}{I} \right]^\gamma \right] \end{aligned}$$

and for  $\gamma > 1$  follows by

$$\begin{aligned} \mathbb{E}(\langle Y^a \rangle_\infty^\gamma) &\leq \gamma^\gamma \sum_n \mathbb{E} \left[ \sup_t \mathbb{1}_{\{t \geq T_n\}} \cap C_n \left[ \mathbb{E} \left[ \langle Y^a \rangle_\infty - \langle Y^a \rangle_t \middle| \mathcal{F}_t^\tau \right] \right]^\gamma \right] \\ &\leq (2\gamma)^\gamma \mathbb{E} \left[ \mathbb{1}_{C_n} \left[ \log \frac{1}{I_n} \right]^\gamma \right] \\ &= (2\gamma)^\gamma \mathbb{E} \left[ \left[ \log \frac{1}{I} \right]^\gamma \right], \end{aligned}$$

where the first inequality is due to Burkholder-Gundy inequality for terminal value of increasing process and supremum of the associated potential ([11] p.188) and the second inequality is due to (C.3) for  $\mu = 1$ .

*Step 2.* We now prove the second inequality in (C.1). Let  $\mu \in (0, 1]$  and  $p > 1$  and consider  $\mathbb{F}^\tau$ -martingale  $M_t := \mathbb{E}(\langle Y^a \rangle_\infty^\mu | \mathcal{F}_t^\tau)$ . Firstly we will show that

$$(C.4) \quad \left[ \log \frac{1}{I} \right]^\mu \leq C_\mu \left[ 1 + \sup_t M_t \right].$$

By Itô's lemma and decomposition (6.3) applied to  $z^n$  we obtain that on  $\{t \geq T_n\} \cap C_n$

$$\log \frac{1}{z_t^n} = \int_t^\infty \frac{1}{z_s^n} d\hat{z}_s^n + \frac{1}{2} \int_t^\infty \frac{1}{(z_s^n)^2} d\langle z^n \rangle_s.$$

Next, by taking conditional expectation w.r.t  $\mathcal{F}_t^\tau$  and using inequality  $|x + y|^\mu \leq |x|^\mu + |y|^\mu$  for  $\mu \in (0, 1]$ , we deduce that on  $\{t \geq T_n\} \cap C_n$

$$\begin{aligned} \left[ \log \frac{1}{z_t^n} \right]^\mu &\leq \mathbb{E} \left[ \left| \int_t^\infty \frac{1}{z_s^n} d\hat{z}_s^n \right|^\mu \middle| \mathcal{F}_t^\tau \right] + \frac{1}{2^\mu} \mathbb{E} \left[ \left[ \int_t^\infty \frac{1}{(z_s^n)^2} d\langle z^n \rangle_s \right]^\mu \middle| \mathcal{F}_t^\tau \right] \\ &\leq C \left[ \mathbb{E} \left[ \left[ \int_t^\infty \frac{1}{(z_s^n)^2} d\langle z^n \rangle_s \right]^{\mu/2} \middle| \mathcal{F}_t^\tau \right] + \mathbb{E} \left[ \left[ \int_t^\infty \frac{1}{(z_s^n)^2} d\langle z^n \rangle_s \right]^\mu \middle| \mathcal{F}_t^\tau \right] \right] \\ &\leq C \left[ \mathbb{E} [\langle Y^a \rangle_\infty^{\mu/2} | \mathcal{F}_t^\tau] + \mathbb{E} [\langle Y^a \rangle_\infty^\mu | \mathcal{F}_t^\tau] \right] \end{aligned}$$

where in the second inequality we have used Burkholder-Davis-Gundy inequality for continuous local martingales (see [24, IV-42, p.93]) applied to  $\widehat{\mathbb{F}}$ -local martingale  $N \mathbb{1}_{\widehat{F}}$  for any  $\widehat{F} \in \widehat{\mathcal{F}}_0$  where

$$N_T := \int_t^{T+t} \frac{1}{z_s^n} d\hat{z}_s^n \quad \text{and} \quad \widehat{\mathbb{F}} := (\mathcal{F}_T)_{T \geq 0} \quad \text{with} \quad \widehat{\mathcal{F}}_T := \mathcal{F}_{T+t}^\tau.$$

Finally, using inequality  $x + x^2 \leq 2x^2 + 1/4$  to  $x = \langle Y^a \rangle_\infty^{\mu/2}$  we conclude that on  $\{t \geq T_n\} \cap C_n$

$$\left[ \log \frac{1}{z_t^n} \right]^\mu \leq C \left[ 1 + \mathbb{E} [\langle Y^a \rangle_\infty^\mu | \mathcal{F}_t^\tau] \right]$$

and by taking supremum over  $t$  and summing over  $n$  the inequality (C.4) follows. In order to prove general case, relying on (C.4) and the inequality  $(1 + x)^p \leq 2^{p-1}(1 + x^p)$ , we obtain

$$\left[ \log \frac{1}{I} \right]^{\mu p} \leq C_\mu^p 2^{p-1} \left[ 1 + \sup_t M_t^p \right].$$

Then, by taking expectations and applying Doob's maximal inequality to  $M$  we obtain

$$\begin{aligned} \mathbb{E} \left[ \left[ \log \frac{1}{I} \right]^{\mu p} \right] &\leq C_\mu^p 2^{p-1} \left[ 1 + \mathbb{E}(\sup_t M_t^p) \right] \\ &\leq C_\mu^p 2^{p-1} \left[ 1 + \frac{p^p}{(p-1)^p} \mathbb{E}(\langle Y^a \rangle_\infty^{\mu p}) \right]. \end{aligned}$$

That completes the proof since any  $\gamma = \mu p > 0$  can be obtained with  $\mu \in (0, 1]$  and  $p > 1$ .  $\square$

**Lemma C.2.** *Let, for each  $n$ ,  $\mathbb{Q}^n$  be an absolutely continuous measure given by  $\frac{d\mathbb{Q}^n}{d\mathbb{P}} = \frac{\mathbb{1}_{C_n}}{z_0^n}$ . Then, for each  $\mathcal{F}_{T_n}$ -measurable random variable  $\beta$  with values in random interval  $(0, z_{T_n}^n)$ , we have*

$$\mathbb{Q}^n(I_n < \beta | \mathcal{F}_{T_n}) = \frac{\beta}{1 - \beta} \frac{1 - z_{T_n}^n}{z_{T_n}^n}$$

where  $I_n$  is defined in (C.2). Or, equivalently

$$\mathbb{Q}^n(I_n \in d\beta | \mathcal{F}_{T_n}) = \frac{1 - z_{T_n}^n}{z_{T_n}^n} \frac{1}{(1 - \beta)^2} \mathbb{1}_{\{0 < \beta < z_{T_n}^n\}}.$$

*Proof.* For any  $\mathcal{F}_{T_n}$ -measurable random variable  $\beta$  with values in random interval  $(0, z_{T_n}^n)$  we define an  $\mathbb{F}$ -stopping time  $T_n^\beta$  by  $T_n^\beta := \inf\{t \geq T_n : z_t^n < \beta\}$ . Then we compute

$$\begin{aligned} \mathbb{P}(\{I_n < \beta\} \cap C_n | \mathcal{F}_{T_n}) &= \mathbb{P}(\{T_n^\beta < \infty\} \cap C_n | \mathcal{F}_{T_n}) \\ &= \mathbb{E}_{\mathbb{P}}(\mathbb{P}(\{T_n^\beta < \infty\} \cap C_n | \mathcal{F}_{T_n^\beta}) | \mathcal{F}_{T_n}) \\ &= \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{T_n^\beta < \infty\}} z_{T_n^\beta}^n | \mathcal{F}_{T_n}) \\ &= \beta \mathbb{P}(\{T_n^\beta < \infty\} | \mathcal{F}_{T_n}) \\ &= \beta \mathbb{P}(\{I_n < \beta\} \cap C_n | \mathcal{F}_{T_n}) + \beta \mathbb{P}(\{I_n < \beta\} \cap C_n^c | \mathcal{F}_{T_n}). \end{aligned}$$

Since  $\{I_n < \beta\} \cap C_n^c = C_n^c$  we deduce that

$$(1 - \beta) \mathbb{P}(\{I_n < \beta\} \cap C_n | \mathcal{F}_{T_n}) = \beta(1 - z_{T_n}^n)$$

and therefore, by Bayes' rule, we have

$$\mathbb{Q}^n(I_n < \beta | \mathcal{F}_{T_n}) = \frac{\mathbb{P}(\{I_n < \beta\} \cap C_n | \mathcal{F}_{T_n})}{z_{T_n}^n} = \frac{\beta}{1 - \beta} \frac{1 - z_{T_n}^n}{z_{T_n}^n}$$

which completes the proof.  $\square$

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