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# EXACT GROUPOIDS

CLAIRE ANANTHARAMAN-DELAROCHE

ABSTRACT. Our purpose is to introduce and study in the setting of locally compact groupoids the analogues of the well known equivalent definitions of exactness for discrete groups. The best results are obtained for a class of étale groupoids that we call weakly inner amenable, since for locally compact groups this notion is weaker than the notion of inner amenability. We give examples of such groupoids which include transformation groupoids associated to actions of discrete groups by homeomorphisms on locally compact spaces. We have no example of étale groupoids which are not weakly inner amenable. For weakly inner amenable étale groupoids we prove the equivalence of six natural notions of exactness: (1) strong amenability at infinity; (2) amenability at infinity; (3) nuclearity of the uniform (Roe) algebra of the groupoid; (4) exactness of this  $C^*$ -algebra; (5) exactness of the groupoid in the sense of Kirchberg-Wassermann; (6) exactness of its reduced  $C^*$ -algebra. We end by several illustrations of our results and open questions.

## INTRODUCTION

Motivated by the study of the continuity of fibrewise crossed product  $C^*$ -bundles, Kirchberg and Wassermann introduced in [33] the notion of exact locally compact group and they proved, among many other results, that a discrete group  $G$  is exact if and only if the corresponding reduced  $C^*$ -algebra  $C_r^*(G)$  is exact. Soon after, it was proved that the exactness of  $C_r^*(G)$  is equivalent to the nuclearity of the uniform Roe  $C^*$ -algebra  $C_u^*(G)$  and also to the fact that the group  $G$  admits an amenable action on a compact space (see [20, 21, 53, 3]). As a consequence, every discrete exact group is uniformly embeddable into a Hilbert space and therefore the Novikov higher signature conjecture holds for countable exact discrete groups ([71, 24, 22, 66]).

The notion of amenability, the Baum-Connes and the Novikov conjectures have been generalized to locally compact groupoids (see [2] as regards to amenability and [67] for the conjectures). Concerning the extension of the notion of exactness to groupoids the situation is not so clear. We presented some results on this subject in a talk given at the MSRI in 2000 [5], but we have never written down all the

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details. Our goal in this paper is to provide these details. In the meantime we have improved some of our results but also discovered several technical difficulties.

The analogue of a discrete group is an étale groupoid (see Section 1.3). So, we are mainly concerned with such groupoids and we add a separability assumption when necessary. In this introduction, to make easier the discussion,  $\mathcal{G}$  will be a second countable étale groupoid.

A discrete group  $G$  is said to be *amenable at infinity* if it has an amenable action on a compact space  $Y$ . This is equivalent to the fact that the natural action of  $G$  on its Stone-Čech compactification  $\beta G$  is amenable. An action of  $\mathcal{G}$  involves a fibre space  $(Y, p)$  over the space  $X = \mathcal{G}^{(0)}$  of units of  $\mathcal{G}$ . The analogue of compactness is the property for  $p$  to be proper, in which case we say that  $(Y, p)$  is *fibrewise compact*, following the terminology of [26, Definition 3.1]. We say that  $\mathcal{G}$  is *amenable at infinity* if it has an amenable action on a fibrewise compact fibre space. A fibre space  $(Y, p)$  has a greatest fibrewise compactification called the Stone-Čech fibrewise compactification of  $(Y, p)$ . This compactification carries a natural  $\mathcal{G}$ -action (*i.e.*, is a  $\mathcal{G}$ -space) if  $(Y, p)$  is a  $\mathcal{G}$ -space (see [4]). We view  $\mathcal{G}$  as  $\mathcal{G}$ -space fibred on  $X$  by the range map  $r$  and we denote by  $(\beta_r \mathcal{G}, r_\beta)$  its Stone-Čech fibrewise compactification<sup>a</sup>. We do not know whether the amenability at infinity of  $\mathcal{G}$  implies that its natural action on  $(\beta_r \mathcal{G}, r_\beta)$  is amenable. This holds if and only if  $\mathcal{G}$  has an amenable action on a fibrewise compact fibre space  $(Y, p)$  such that  $p$  admits a continuous section. If this is the case we say that  $\mathcal{G}$  is *strongly amenable at infinity*. For groups, this makes no difference. Although  $\beta_r \mathcal{G}$  is a rather ugly space, the nice feature of strong amenability at infinity is that it has an intrinsic characterization (Theorem 3.16).

There is a natural notion of exact groupoid that we call *KW-exactness*<sup>b</sup> (see Definition 6.6). We show that  $\mathcal{G}$  is KW-exact whenever it is amenable at infinity and that, if  $\mathcal{G}$  is KW-exact, then its reduced  $C^*$ -algebra  $C_r^*(\mathcal{G})$  is exact (Proposition 6.7). The proof is an easy adaptation of the known proof for groups (see [3, Theorem 7.2] for instance).

Recall that the exactness of the reduced  $C^*$ -algebra of a discrete group  $G$  implies the amenability of the action of  $G$  on its Stone-Čech compactification. Its classical proof uses the following remark, essentially contained in [20]: if  $\Phi : C_r^*(G) \rightarrow \mathcal{B}(\ell^2(G))$  is a completely positive map such that  $\Phi(\delta_s) = 0$  except for  $s$  in a finite subset  $F$  of  $G$ , then the kernel  $k : (s, t) \in G \times G \rightarrow \mathbb{C}$  defined by  $k(s, t) = \langle \delta_{s^{-1}}, \Phi(\delta_{s^{-1}t})\delta_{t^{-1}} \rangle$  is positive definite and has its support in the tube

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<sup>a</sup>We point out that in our definition of a fibre space  $(Y, p)$  we do not require  $p$  to be open since in general  $r_\beta : \beta_r \mathcal{G} \rightarrow \mathcal{G}^{(0)}$  is not open.

<sup>b</sup>KW stands for exactness in the sense of Kirchberg and Wassermann who introduced this notion for groups.

$\{(s, t) \in G \times G : s^{-1}t \in F\}$ . The second ingredient of the proof is the characterization of the amenability at infinity of  $G$  in term of nets of such positive definite kernels on the product space  $G \times G$  [24, 3].

This characterization has an analogue in the setting of étale groupoids (Theorem 3.16) in term of positive definite kernels on the subspace  $\mathcal{G} *_r \mathcal{G}$  of pairs  $(\gamma, \gamma_1) \in \mathcal{G} \times \mathcal{G}$  with the same range. On the other hand, in the groupoid case, the definition of a kernel  $k$  on  $\mathcal{G} *_r \mathcal{G}$ , starting from a completely positive map  $\Phi$  from  $C_r^*(\mathcal{G})$  into the  $C^*$ -algebra (analogous to  $\mathcal{B}(\ell^2(G))$ ) containing the regular representation  $\Lambda$  of  $C_r^*(\mathcal{G})$  defined in Section 5.1, raises difficulties. Indeed, for  $\gamma, \gamma_1$  with the same range, we need an analogue  $f_{\gamma^{-1}\gamma_1} \in C_r^*(\mathcal{G})$  of  $\delta_{s^{-1}t}$ . We observe that if  $G$  is a discrete group and if  $g$  denotes the characteristic function of the diagonal of  $G \times G$  then  $\delta_{s^{-1}t}$  is the function  $y \mapsto g(s^{-1}t, y)$  defined on  $G$ . This function  $g$  is (of course) continuous and positive definite on the product group  $G \times G$ . Moreover this function is properly supported in the sense that for every compact (*i.e.*, finite) subset  $F$  of  $G$ , the intersections of the support of  $g$  with  $F \times G$  and with  $G \times F$  are compact. Following an idea of Jean Renault developed in an unpublished note, we introduce, for a locally compact groupoid, a property similar to the existence of  $g$ , that we call *weak inner amenability* (see Definition 4.2). The definition is justified by the fact that inner amenable locally compact groups are weakly inner amenable. This includes every discrete group. On the other hand a connected locally compact group is weakly inner amenable if and only if it is amenable [3, Theorem 7.3]. Given a discrete group  $G$  acting by homeomorphisms on a locally compact space  $X$ , the transformation groupoid  $X \rtimes G$  is weakly inner amenable. We do not know whether every étale groupoid is weakly inner amenable. Weak inner amenability insures the existence of functions on the groupoid product  $\mathcal{G} \times \mathcal{G}$  that behaves like the characteristic function of the diagonal for  $G \times G$ .

In order to get rid of  $\beta_r \mathcal{G}$ , we introduce the groupoid analogue  $C_u^*(\mathcal{G})$  of the uniform Roe algebra generated by the operators of finite propagation in case of a discrete group. This  $C^*$ -algebra  $C_u^*(\mathcal{G})$ , which is canonically isomorphic to the reduced  $C^*$ -algebra  $C_r^*(\beta_r \mathcal{G} \rtimes \mathcal{G})$  of the semi-direct product groupoid (Theorem 5.3), has the advantage over  $C_r^*(\beta_r \mathcal{G} \rtimes \mathcal{G})$  to be more elementarily defined.

Our main result states that if  $\mathcal{G}$  is equivalent to a weakly inner amenable étale groupoid, then the following three conditions are equivalent:  $\mathcal{G}$  is amenable at infinity;  $\mathcal{G}$  is KW-exact;  $C_r^*(\mathcal{G})$  is exact. If  $\mathcal{G}$  is weakly inner amenable (or even equivalent in a stronger sense to a weakly inner amenable groupoid, then these three conditions are also equivalent to the three following ones:  $\mathcal{G}$  is strongly amenable at infinity;  $C_u^*(\mathcal{G})$  is nuclear;  $C_u^*(\mathcal{G})$  is exact (see Theorem 7.6 and Corollary 7.8). This is applied to show (Corollary 7.7) that if  $\mathcal{G}$  is such that there exists a locally proper continuous homomorphism from itself into an exact discrete group, then the above six conditions hold. This includes examples of groupoids associated with semigroups (see Remark 8.7).

Non-exact discrete groups are quite exotic. The first examples were exhibited by Gromov in [19], often named Gromov monster groups. New examples have been constructed by Osajda [52]. But still, there are very few examples, whereas in the setting of étale groupoids, it is easy to find examples of KW-exact and non KW-exact groupoids. Let us consider first the case of an étale groupoid  $\mathcal{G}$  which is a groupoid group bundle  $(\mathcal{G}(x))_{x \in X}$  (Definition 8.9). Recall that  $\mathcal{G}$  is amenable if and only if each group  $\mathcal{G}(x)$  is amenable [2, 60], and in this case,  $C_r^*(\mathcal{G})$  is a continuous field of  $C^*$ -algebras, with fibres  $C_r^*(\mathcal{G}(x))$ ,  $x \in X$  [56, 39]. The discussion relative to exactness is more subtle. If  $C_r^*(\mathcal{G})$  is exact, that each quotient  $C^*$ -algebra  $C_r^*(\mathcal{G}(x))$  is exact and therefore the groups  $\mathcal{G}(x)$  are exact. If  $\mathcal{G}$  is KW-exact, then  $C_r^*(\mathcal{G})$  is a still a continuous field of  $C^*$ -algebras (see Corollary 8.13). However, the exactness of the groups  $\mathcal{G}(x)$  is not sufficient to ensure that  $C_r^*(\mathcal{G})$  is exact and that it is a continuous field of  $C^*$ -algebras with fibres  $C_r^*(\mathcal{G}(x))$ ,  $x \in X$ , as shown by examples arising from a construction due to Higson, Lafforgue and Skandalis [23] (see Proposition 8.14). A second class of interesting examples are groupoids associated with semigroups, as constructed and studied by many authors (see Section 8.3).

An aspect that has not been considered in this paper is the behaviour of crossed products under the action of an exact groupoid, for which we refer to [35].

This paper is organized as follows. Sections 1 to 4 concern exclusively locally compact groupoids. In the first section we provide the definitions and the notation relative to these groupoids and their actions, that will be used in the rest of the paper. In Section 2, we recall some facts about amenable groupoids and amenable actions of groupoids. The notions of amenability at infinity and of weak inner amenability are introduced and studied in Sections 3 and 4 respectively. Section 5 is devoted to the description of the different  $C^*$ -algebras associated with a groupoid: full and reduced  $C^*$ -algebras, the uniform Roe  $C^*$ -algebra, and crossed products relative to actions on  $C^*$ -algebras. In Sections 6 and 7, we study the relations between the different natural notions of exactness that we have defined and Section 8 is devoted to examples.

## 1. PRELIMINARIES

**1.1. Groupoids.** We assume that the reader is familiar with the basic definitions about groupoids. For details we refer to [57], [55]. Let us recall some notation and terminology. A *groupoid* consists of a set  $\mathcal{G}$ , a subset  $\mathcal{G}^{(0)}$  called the set of *units*, two maps  $r, s : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$  called respectively the *range* and *source* maps, a *composition law*  $(\gamma_1, \gamma_2) \in \mathcal{G}^{(2)} \mapsto \gamma_1 \gamma_2 \in \mathcal{G}$ , where

$$\mathcal{G}^{(2)} = \{(\gamma_1, \gamma_2) \in \mathcal{G} \times \mathcal{G} : s(\gamma_1) = r(\gamma_2)\},$$

and an *inverse* map  $\gamma \mapsto \gamma^{-1}$ . These operations satisfy obvious rules, such as the facts that the composition law (*i.e.*, product) is associative, that the elements of  $\mathcal{G}^{(0)}$  act as units (*i.e.*,  $r(\gamma)\gamma = \gamma = \gamma s(\gamma)$ ), that  $\gamma\gamma^{-1} = r(\gamma)$ ,  $\gamma^{-1}\gamma = s(\gamma)$ , and so on (see [57, Definition 1.1]). For  $x \in \mathcal{G}^{(0)}$  we set  $\mathcal{G}^x = r^{-1}(x)$  and  $\mathcal{G}_x = s^{-1}(x)$ . Usually,  $X$  will denote the set of units of  $\mathcal{G}$ .

A subgroupoid  $\mathcal{H}$  of a groupoid  $\mathcal{G}$  is a subset of  $\mathcal{G}$  which is stable under product and inverse. For instance, let  $Y$  be a subset of  $X = \mathcal{G}^{(0)}$ . We set  $\mathcal{G}(Y) = r^{-1}(Y) \cap s^{-1}(Y)$ . Then  $\mathcal{G}(Y)$  is a subgroupoid of  $\mathcal{G}$  called the *reduction of  $\mathcal{G}$  by  $Y$* . When  $Y$  is reduced to a single element  $x$ , then  $\mathcal{G}(x) = r^{-1}(x) \cap s^{-1}(x)$  is a group called the *isotropy group of  $\mathcal{G}$  at  $x$* .

A *locally compact groupoid* is a groupoid  $\mathcal{G}$  equipped with a locally compact topology such that the structure maps are continuous, where  $\mathcal{G}^{(2)}$  has the topology induced by  $\mathcal{G} \times \mathcal{G}$  and  $\mathcal{G}^{(0)}$  has the topology induced by  $\mathcal{G}$ . We assume that the range (and therefore the source) map is open, which is a necessary condition for the existence of a Haar system.

A *locally compact subgroupoid*  $\mathcal{H}$  of  $\mathcal{G}$  is a locally closed subgroupoid of  $\mathcal{G}$  such that the restriction of the range map is open from  $\mathcal{H}$  onto  $\mathcal{H}^{(0)}$ . For instance, if  $E$  is an invariant locally compact subset of  $\mathcal{G}^{(0)}$  (that is,  $r(\gamma) \in E$  if and only if  $s(\gamma) \in E$ ), then the reduction  $\mathcal{G}(E)$  is a subgroupoid. Also, the isotropy groups  $\mathcal{G}(x)$  are subgroupoids, for any  $x \in \mathcal{G}^{(0)}$ .

For us, unless explicitly mentioned to be false, *we shall always assume the locally compact spaces to be Hausdorff*. Given a locally compact space  $Y$ , we denote by  $\mathcal{C}_b(Y)$  the algebra of continuous bounded complex valued functions on  $Y$ , by  $\mathcal{C}_0(Y)$  its subalgebra of functions vanishing at infinity, and by  $\mathcal{C}_c(Y)$  the subalgebra of continuous functions with compact support. The support of  $f \in \mathcal{C}_b(Y)$  will be denoted by  $\text{Supp}(f)$ .

**Definition 1.1.** Let  $\mathcal{G}$  be a locally compact groupoid. A *Haar system* on  $\mathcal{G}$  is a family  $\lambda = (\lambda^x)_{x \in X}$  of measures on  $\mathcal{G}$ , indexed by the set  $X = \mathcal{G}^{(0)}$  of units satisfying the following conditions:

- *Support:*  $\lambda^x$  has exactly  $\mathcal{G}^x$  as support, for every  $x \in X$ ;
- *Continuity:* for every  $f \in \mathcal{C}_c(\mathcal{G})$ , the function  $x \mapsto \lambda(f)(x) = \int_{\mathcal{G}^x} f \, d\lambda^x$  is continuous;
- *Invariance:* for  $\gamma \in \mathcal{G}$  and  $f \in \mathcal{C}_c(\mathcal{G})$ , we have

$$\int_{\mathcal{G}^{s(\gamma)}} f(\gamma\gamma_1) \, d\lambda^{s(\gamma)}(\gamma_1) = \int_{\mathcal{G}^{r(\gamma)}} f(\gamma_1) \, d\lambda^{r(\gamma)}(\gamma_1).$$

In this paper, we shall almost always limit ourselves to the case of étale groupoids in which case there is a canonical Haar system (see Subsection 1.3).

**1.2. Actions of groupoids on spaces.** Let  $X$  be a locally compact space. A *fibre space* over  $X$  is a pair  $(Y, p)$  where  $Y$  is a locally compact space and  $p$  is a continuous surjective map  $p$  from  $Y$  onto  $X$ . For  $x \in X$  we denote by  $Y^x$  the *fibre*  $p^{-1}(x)$ .

If  $(Y_i, p_i)$ ,  $i = 1, 2$ , are two fibre spaces over  $X$ , we denote by  $Y_1 \underset{p_1}{*} \underset{p_2}{*} Y_2$  (or  $Y_1 * Y_2$  when there is no ambiguity) the *fibred product*  $\{(y_1, y_2) \in Y_1 \times Y_2 : p_1(y_1) = p_2(y_2)\}$  equipped with the topology induced by the product topology. For subsets  $A_1$  and  $A_2$  of  $Y_1$  and  $Y_2$  respectively, we use similarly the notation  $A_1 \underset{p_1}{*} \underset{p_2}{*} A_2$ .

**Definition 1.2.** Let  $\mathcal{G}$  be a locally compact groupoid. A *left  $\mathcal{G}$ -space* is a fibre space  $(Y, p)$  over  $X = \mathcal{G}^{(0)}$ , equipped with a continuous map  $(\gamma, y) \mapsto \gamma y$  from  $\mathcal{G} \underset{s}{*} \underset{p}{*} Y$  into  $Y$ , satisfying the following conditions:

- $p(\gamma y) = r(\gamma)$  for  $(\gamma, y) \in \mathcal{G} \underset{s}{*} \underset{p}{*} Y$ , and  $p(y)y = y$  for  $y \in Y$ ;
- if  $(\gamma_1, y) \in \mathcal{G} \underset{s}{*} \underset{p}{*} Y$  and  $(\gamma_2, \gamma_1) \in \mathcal{G}^{(2)}$ , then  $(\gamma_2 \gamma_1)y = \gamma_2(\gamma_1 y)$ .

The map  $p$  will be called the *momentum* of the  $\mathcal{G}$ -space. We shall also say that the map  $(\gamma, y) \mapsto \gamma y$  is a *continuous  $\mathcal{G}$ -action on  $Y$* .

A *continuous  $\mathcal{G}$ -equivariant morphism* from a left  $\mathcal{G}$ -space  $(Y_1, p_1)$  to a left  $\mathcal{G}$ -space  $(Y_2, p_2)$  is a continuous map  $f : Y_1 \rightarrow Y_2$  such that  $p_2 \circ f = p_1$  and  $f(\gamma y) = \gamma f(y)$  for every  $(\gamma, y) \in \mathcal{G} \underset{s}{*} \underset{p}{*} Y$ .

Right  $\mathcal{G}$ -spaces are defined similarly. Without further precisions, a  $\mathcal{G}$ -space will be a left  $\mathcal{G}$ -space. Let us observe that  $(\mathcal{G}, r)$  is a  $\mathcal{G}$ -space in an obvious way, as well as  $X$ . In this latter case, the action of  $\gamma \in s^{-1}(x)$  onto  $x \in X$  will be denoted by  $\gamma \cdot x$ , in order to distinguish it from  $\gamma x = \gamma$ . By definition, we have  $\gamma \cdot x = r(\gamma)$ . We also note that if  $(Y, p)$  is a  $\mathcal{G}$ -space, then  $p$  is equivariant:  $p(\gamma y) = \gamma \cdot p(y)$ .

*Remark 1.3.* We warn the reader that our definition of  $\mathcal{G}$ -space is different from the usual definition found in the literature where it is required that  $p$  is an open map. We do not assume this property because the momentum of the action of a locally compact groupoid on its Stone-Ćech fibrewise compactification is not always open (see Example 3.2). Moreover, it is a fact that, even if  $p$  is not open, the semi-direct product groupoid has an open range (see Proposition 1.4).

Given a  $\mathcal{G}$ -space  $(Y, p)$  we define a new locally compact groupoid  $Y \rtimes \mathcal{G}$ , called the *semi-direct product groupoid*. As a topological space, it is  $Y \underset{p}{*} \underset{r}{*} \mathcal{G}$ . The range of  $(y, \gamma)$  is  $(y, r(\gamma)) = (y, p(y))$  and its source is  $(\gamma^{-1}y, s(\gamma))$ . The product is given by

$$(y, \gamma)(\gamma^{-1}y, \gamma_1) = (y, \gamma\gamma_1)$$

and the inverse is given by

$$(y, \gamma)^{-1} = (\gamma^{-1}y, \gamma^{-1}).$$

Observe that  $(y, p(y)) \mapsto y$  is a homeomorphism from  $(Y \rtimes \mathcal{G})^{(0)}$  onto  $Y$ . Therefore we shall identify these two spaces. Sometimes we shall denote by the boldface letter  $\mathbf{r}$  the range map of  $Y \rtimes \mathcal{G}$  to distinguish it from the range map  $r$  of  $\mathcal{G}$  (and similarly for  $s$ ).

We may equivalently consider the groupoid  $\mathcal{G} \rtimes Y$  which is  $\mathcal{G}_{s *_p} Y$  as a topological space. Here the range of  $(\gamma, y)$  is  $\gamma y$  and its source is  $y$ . The product is given by

$$(\gamma, \gamma_1 y)(\gamma_1, y) = (\gamma \gamma_1, y).$$

Note that  $(y, \gamma) \mapsto (\gamma, \gamma^{-1}y)$  is an isomorphism of groupoids from  $Y \rtimes \mathcal{G}$  onto  $\mathcal{G} \rtimes Y$ .

An important and well known particular case is when  $\mathcal{G}$  is a locally compact group  $G$  acting continuously on a locally compact space  $Y$ . Then  $Y \rtimes G$  is called the *transformation groupoid* associated with the action of  $G$  on  $Y$ .

**Proposition 1.4.** *Let  $\mathcal{G}$  be a locally compact groupoid and  $(Y, p)$  a  $\mathcal{G}$ -space.*

- (i) *The range map  $\mathbf{r} : Y \rtimes \mathcal{G} \rightarrow Y$  is open.*
- (ii) *If  $\mathcal{G}$  has a Haar system  $(\lambda^x)_{x \in X}$ , then  $\delta_y \times \lambda^{p(y)}$  is a Haar system for  $Y \rtimes \mathcal{G}$ .*

*Proof.* (i) Let  $\Omega$  be an open subset of  $Y \rtimes \mathcal{G}$ . Let  $y_0 \in \mathbf{r}(\Omega)$  and let  $\gamma_0$  be such that  $(y_0, \gamma_0) \in \Omega$ . There exist open neighborhoods  $U$  of  $y_0$  and  $V$  of  $\gamma_0$  such that  $U_{p *_r} V \subset \Omega$ . Then we have  $p^{-1}(r(V)) \cap U \subset \mathbf{r}(\Omega)$  and  $p^{-1}(r(V)) \cap U$  is an open neighborhood of  $y_0$  since  $r$  is open.

We leave the easy proof of (ii) to the reader. □

**1.3. Étale groupoids.** An *étale groupoid* is a locally compact groupoid  $\mathcal{G}$  such that the range (and therefore the source) map is a local homeomorphism from  $\mathcal{G}$  into  $\mathcal{G}^{(0)}$ . In this situation, the fibres  $\mathcal{G}^x = r^{-1}(x)$  with their induced topology are discrete and  $\mathcal{G}^{(0)}$  is open in  $\mathcal{G}$ . The family of counting measures  $\lambda^x$  on  $\mathcal{G}^x$  form a Haar system (see [57, Proposition 2.8]), which will be implicit in the sequel. Étale groupoids are sometimes called *r-discrete*.

Examples of étale groupoids are plentiful. Let us mention groupoids associated with discrete group actions, local homeomorphisms, pseudo-groups of partial homeomorphisms, topological Markov shifts, graphs, inverse semigroups, metric spaces with bounded geometry,... (see [57], [14], [34], [8], [59], [55], [30] for a non exhaustive list). For a brief account on the notion of étale groupoid see also [11, Section 5.6].

A *bisection* is a subset  $S$  of  $\mathcal{G}$  such that the restriction of  $r$  and  $s$  to  $S$  is injective. Given an open bisection  $S$ , we shall denote by  $r_S^{-1}$  the inverse map, defined on the open subset  $r(S)$  of  $\mathcal{G}^{(0)}$ , of the restriction of  $r$  to  $S$ . Note that  $r_S^{-1}$  is continuous.



An étale groupoid  $\mathcal{G}$  has a cover by open bisections. These open bisections form an inverse semigroup<sup>c</sup> with composition law

$$ST = \left\{ \gamma_1 \gamma_2 : (\gamma_1, \gamma_2) \in (S \times T) \cap \mathcal{G}^{(2)} \right\},$$

the inverse  $S^{-1}$  of  $S$  being the image of  $S$  under the inverse map (see [55, Proposition 2.2.4]). A compact subset  $K$  of  $\mathcal{G}$  is covered by a finite number of open bisections. Thus, using partitions of unity, we see that every element of  $\mathcal{C}_c(\mathcal{G})$  is a finite sum of continuous functions whose compact support is contained in some open bisection.

Semi-direct products relative to actions of étale groupoids are themselves étales.

**Proposition 1.5.** *Let  $\mathcal{G}$  be an étale groupoid and  $(Y, p)$  a  $\mathcal{G}$ -space. Then the groupoid  $Y \rtimes \mathcal{G}$  is étale.*

*Proof.* We shall show that every  $(y_0, \gamma_0) \in Y \rtimes \mathcal{G}$  has an open neighborhood  $\Omega$  such that the restriction of  $\mathbf{r}$  to  $\Omega$  is a homeomorphism onto its image, which is open in  $Y$ . We first choose an open bisection  $S$  of  $\mathcal{G}$  which contains  $\gamma_0$  and we set  $W = p^{-1}(r(S))$ . Note that  $W$  is an open subset of  $Y$ . We set  $\Omega = W_p *_r S$ . It is an open subset of  $Y \rtimes \mathcal{G}$ . Its range  $\mathbf{r}(\Omega)$  is  $W_p *_r r(S) = \{(y, p(y)) : y \in W\}$ . It is open in  $(Y \rtimes \mathcal{G})^{(0)}$ . Obviously,  $\mathbf{r}$  is a continuous bijection from  $\Omega$  onto its image. Its inverse map is  $(y, p(y)) \mapsto (y, r_S^{-1}(p(y)))$  which is continuous on  $\mathbf{r}(\Omega)$ .  $\square$

Let  $\mathcal{G}$  and  $(Y, p)$  be as in the previous proposition. An open bisection  $S$  of  $\mathcal{G}$  defines a homeomorphism  $\alpha_S$  from  $p^{-1}(s(S))$  onto  $p^{-1}(r(S))$ , sending  $y$  onto  $\gamma y$ , where  $\gamma$  is the unique element of  $S$  such that  $s(\gamma) = p(y)$ . For simplicity of notation, we usually write  $Sy$  instead of  $\alpha_S(y)$ . When  $(Y, p) = (\mathcal{G}^{(0)}, \text{Id})$ , we rather use the notation  $x \mapsto S \cdot x$ . For every subset  $W$  of  $p^{-1}(s(S))$ , we shall use the equality

$$p(SW) = S \cdot p(W). \quad (1)$$

## 2. AMENABLE GROUPOIDS AND AMENABLE ACTIONS

The reference for this section is [2]. Here  $\mathcal{G}$  will be a locally compact groupoid and we set  $X = \mathcal{G}^{(0)}$ . For simplicity, the reader may assume that  $\mathcal{G}$  is étale since we are mainly interested in this class of groupoids. The notion of amenable locally compact groupoid has many equivalent definitions. We shall recall three of them. Before let us recall a notation: given a locally compact groupoid  $\mathcal{G}$ ,  $\gamma \in \mathcal{G}$  and  $\mu$  a measure on  $\mathcal{G}^{s(\gamma)}$ , then  $\gamma\mu$  is the measure on  $\mathcal{G}^{r(\gamma)}$  defined by  $\int f d\gamma\mu = \int f(\gamma\gamma_1) d\mu$ .

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<sup>c</sup>The definition is recalled in 8.3.

**Definition 2.1.** ([2, Definitions 2.2.2, 2.2.8]) We say that  $\mathcal{G}$  is *amenable* if there exists a net  $(m_i)$ , where  $m_i = (m_i^x)_{x \in X}$  is a family of probability measures  $m_i^x$  on  $\mathcal{G}^x$ , such that

- (i) each  $m_i$  is continuous in the sense that for all  $f \in \mathcal{C}_c(\mathcal{G})$ , the function  $x \mapsto \int f dm_i^x$  is continuous;
- (ii)  $\lim_i \left\| \int \gamma m_i^{s(\gamma)} - m_i^{r(\gamma)} \right\|_1 = 0$  uniformly on the compact subsets of  $\mathcal{G}$ .

We say that  $(m_i)_i$  is an *approximate invariant continuous mean* on  $\mathcal{G}$ .

We say that a function  $h$  on a groupoid  $\mathcal{G}$  is *positive definite* if for every  $x \in \mathcal{G}^{(0)}$ ,  $n \in \mathbb{N}$ , and  $\gamma_1, \dots, \gamma_n \in \mathcal{G}^x$ , the  $n \times n$  matrix  $[h(\gamma_i^{-1}\gamma_j)]$  is non-negative, that is,

$$\sum_{i,j=1}^n \overline{\alpha_i} \alpha_j h(\gamma_i^{-1}\gamma_j) \geq 0$$

for  $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ .

A second useful characterization of amenability is as follows.

**Proposition 2.2.** ([2, Proposition 2.2.13]) *Let  $\mathcal{G}$  be a locally compact groupoid with Haar system. Then  $\mathcal{G}$  is amenable if and only if there exists a net  $(h_i)$  of continuous positive definite functions with compact support in  $\mathcal{G}$  such that  $\lim_n h_i = 1$  uniformly on the compact subsets of  $\mathcal{G}$ .*

Finally, we shall also need the following characterization of amenability.

**Proposition 2.3.** ([2, Proposition 2.2.13]) *Let  $(\mathcal{G}, \lambda)$  be a locally compact groupoid with Haar system. Then  $\mathcal{G}$  is amenable if and only if there exists a net  $(g_i)$  of non-negative functions in  $\mathcal{C}_c(\mathcal{G})$  such that*

- (a)  $\int g_i d\lambda^x \leq 1$  for every  $x \in \mathcal{G}^{(0)}$ ;
- (b)  $\lim_i \int g_i d\lambda^x = 1$  uniformly on the compact subsets of  $\mathcal{G}^{(0)}$ ;
- (c)  $\lim_i \int |g_i(\gamma^{-1}\gamma_1) - g_i(\gamma_1)| d\lambda^{r(\gamma)}(\gamma_1) = 0$  uniformly on the compact subsets of  $\mathcal{G}$ .

*Remark 2.4.* We shall very often consider second countable locally compact groupoids. In this case (or more generally in the case of  $\sigma$ -compact groupoids), in the above statements, we may replace nets by sequences.

**Definition 2.5.** We say that a continuous  $\mathcal{G}$ -action on a fibre space  $(Y, p)$  is *amenable* if the groupoid  $Y \rtimes \mathcal{G}$  is amenable.

Recall that  $Y$  is the set of units of the semi-direct product  $Y \rtimes \mathcal{G}$  and that  $r^{-1}(y)$  is canonically identified to  $\mathcal{G}^{p(y)}$ . If  $\mathcal{G}$  has a Haar system  $\lambda$ , then  $Y \rtimes \mathcal{G}$  has a Haar system which is given by the family  $(\lambda^{p(y)})_{y \in Y}$ . A positive definite function on  $Y \rtimes \mathcal{G}$  is a function  $h$  such that, for every  $y \in Y$ ,  $n \in \mathbb{N}$ , and  $\gamma_1, \dots, \gamma_n \in \mathcal{G}^{p(y)}$ ,

the  $n \times n$  matrix  $[h(\gamma_i^{-1}y, \gamma_i^{-1}\gamma_j)]$  is non negative. Then characterizations given in Propositions 2.2 and 2.3 are easily spelled out for the groupoid  $Y \rtimes \mathcal{G}$ .

Finally, let us recall that if  $\mathcal{G}$  is an amenable locally compact groupoid, then for every  $\mathcal{G}$ -space  $(Y, p)$ , the groupoid  $Y \rtimes \mathcal{G}$  is also amenable [2, Corollary 2.2.10].

### 3. AMENABLE ACTIONS ON FIBREWISE COMPACT SPACES

#### 3.1. Definitions and first results.

**Definition 3.1.** A fibre space  $(Y, p)$  over a locally compact space  $X$  is said to be *fibrewise compact* if  $p$  is a proper map.

Let  $\mathcal{G}$  be a locally compact groupoid. A *fibrewise compact  $\mathcal{G}$ -space* is a  $\mathcal{G}$ -space which is fibrewise compact.

In particular, fibrewise compactness implies that all the fibres  $Y^x$  are compact, but the converse is not true.

To an étale groupoid  $\mathcal{G}$  are associated two important fibrewise compact  $\mathcal{G}$ -spaces, namely its *Alexandroff fibrewise compactification*  $(\mathcal{G}_r^+, r^+)$  and its *Stone-Čech fibrewise compactification*  $(\beta_r \mathcal{G}, r_\beta)$ . For details about these notions we refer to [4]. We recall that  $\mathcal{G}_r^+$  is the Gelfand spectrum of the abelian  $C^*$ -algebra of continuous bounded functions on  $\mathcal{G}$  of the form  $f \circ r + g$  with  $f \in \mathcal{C}_0(\mathcal{G}^{(0)})$  and  $g \in \mathcal{C}_0(\mathcal{G})$ . On the other hand,  $\beta_r \mathcal{G}$  is the Gelfand spectrum of the abelian  $C^*$ -algebra of continuous bounded functions  $f$  on  $\mathcal{G}$  such that for every  $\varepsilon > 0$  there exists a compact subset  $K$  of  $\mathcal{G}^{(0)}$  satisfying  $|f(\gamma)| \leq \varepsilon$  if  $\gamma \notin r^{-1}(K)$ . Obviously, when  $\mathcal{G}$  is a discrete group  $G$ , then  $\mathcal{G}_r^+$  is the Alexandroff compactification  $G^+$  of  $G$  and  $\beta_r \mathcal{G}$  is its Stone-Čech compactification  $\beta G$ .

**Example 3.2.** The following example will show that the maps  $r^+$  and  $r_\beta$  are not always open. We consider the topological subspace  $\mathcal{G} = [0, 1] \times \{0\} \cup ]1/2, 1] \times \{1\}$  of  $[0, 1] \times \{0, 1\}$ , that we view as a bundle of groups over  $X = [0, 1] \times \{0\} \equiv [0, 1]$ , the fibres over  $[0, 1/2]$  being the trivial group, and those over  $]1/2, 1]$  being the group with two elements. Then we have  $\mathcal{G}_r^+ = [0, 1] \times \{0\} \cup ]1/2, 1] \times \{1\}$  and  $r^+$  is the projection onto  $[0, 1]$ , obviously not open. Similarly,  $\beta_r \mathcal{G}$  has the same fibres as  $\mathcal{G}$ , except over  $\{1/2\}$  where  $r_\beta^{-1}(1/2) = \{1/2\} \times \left( \{0\} \sqcup (\beta(]1/2, 1]) \setminus ]1/2, 1]) \right)$  and  $r_\beta$  is not open.

**Proposition 3.3.** *An étale groupoid  $\mathcal{G}$  is amenable if and only if the  $\mathcal{G}$ -space  $\mathcal{G}_r^+$  is amenable.*

*Proof.* The structure of  $\mathcal{G}_r^+$  is described in [4]. One of its main features is that it contains a  $\mathcal{G}$ -invariant closed subset  $F$  such that the restriction of  $p = r^+$  to  $F$  is a  $\mathcal{G}$ -equivariant homeomorphism onto an invariant closed subset  $F'$  of  $X$ . Moreover  $U = X \setminus F'$  is the greatest open subset of  $X$  such that the restriction

of  $r$  to  $\mathcal{G}(U)$  is proper<sup>d</sup> (see [4, Propositions 1.10 and 2.2]). Assume that the  $\mathcal{G}$ -space  $\mathcal{G}_r^+$  is amenable. Let  $(h_i)$  be a net of continuous positive definite functions on  $\mathcal{G}_r^+ \rtimes \mathcal{G}$ , with compact support, that satisfies the conditions of Proposition 2.2. Let us observe that  $F_{p*_r} \mathcal{G} = \{(y, \gamma) \in F \times \mathcal{G} : p(y) = r(\gamma)\}$  is canonically identified to the groupoid  $\mathcal{G}(F')$  thanks to the map  $(y, \gamma) \mapsto \gamma$ . We denote by  $k_i$  the restriction of  $h_i$  to  $F_{p*_r} \mathcal{G}$ , and we view it as a function on  $\mathcal{G}(F')$ . It is a net of positive type functions on  $\mathcal{G}(F')$  that satisfies the conditions of Proposition 2.2. So the groupoid  $\mathcal{G}(F')$  is amenable. On the other hand  $\mathcal{G}(U)$  is amenable since it is proper. It follows that  $\mathcal{G}$  is amenable.  $\square$

**Definition 3.4.** We say that a locally compact groupoid  $\mathcal{G}$  is *amenable at infinity* if there is an amenable  $\mathcal{G}$ -space  $Y$  such  $p : Y \rightarrow X = \mathcal{G}^{(0)}$  is proper. Whenever  $(Y, p)$  can be chosen such that, in addition, there exists a continuous section  $\sigma : X \rightarrow Y$  of  $p$ , then we say that  $\mathcal{G}$  is *strongly amenable at infinity*.

The interest of this latter notion will become clear later (see Proposition 3.7 and Theorem 3.16).

**Proposition 3.5.** *Let  $\mathcal{G}$  be an amenable (resp. strongly amenable) at infinity locally compact groupoid. Let  $\mathcal{H}$  be a locally compact subgroupoid of  $\mathcal{G}$ . Then  $\mathcal{H}$  is amenable (resp. strongly amenable) at infinity.*

*Proof.* Let  $(Y, p)$  be a fibrewise compact amenable  $\mathcal{G}$ -space and set  $Y_E = p^{-1}(E)$  where  $E = \mathcal{H}^{(0)}$ . Note that the restriction of  $p$  to  $Y_E$  is proper and that it has a continuous section whenever  $p$  has a continuous section. Moreover, it is easily seen that  $Y_E \rtimes \mathcal{H}$  is a subgroupoid of  $Y \rtimes \mathcal{G}$ . Therefore, by [2, Proposition 5.1], the groupoid  $Y_E \rtimes \mathcal{H}$  is amenable.  $\square$

**Proposition 3.6.** *Let  $\mathcal{G}$  be a locally compact groupoid.*

- (i) *Assume that  $\mathcal{G}$  is amenable (resp. strongly amenable) at infinity. Then for every  $\mathcal{G}$ -space  $(Z, q)$ , the semi-direct product groupoid  $Z \rtimes \mathcal{G}$  is amenable (resp. strongly amenable) at infinity.*
- (ii) *Let  $Z$  be a  $\mathcal{G}$ -space whose momentum map  $q$  is proper. Assume that  $Z \rtimes \mathcal{G}$  is amenable at infinity. Then  $\mathcal{G}$  is amenable at infinity.*
- (iii) *Let  $Z$  be a  $\mathcal{G}$ -space whose momentum map  $q$  is proper and has a continuous section. Assume that  $Z \rtimes \mathcal{G}$  is strongly amenable at infinity. Then  $\mathcal{G}$  is strongly amenable at infinity.*

*Proof.* (i) Let  $(Y, p)$  be a fibrewise compact amenable  $\mathcal{G}$ -space. We denote by  $l : Y * Z \rightarrow Z$  the projection. We observe that  $l$  is proper, and that whenever  $p$

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<sup>d</sup>When  $\mathcal{G}$  is an infinite discrete group  $G$ , then  $F = G^+ \setminus G$  is reduced to the point at infinity,  $F' = \{e\}$ ,  $U$  is empty and  $\mathcal{G}(F') = G$ .

has a continuous section  $\sigma$ , then  $z \mapsto (\sigma(q(z)), z)$  is a continuous section for  $l$ . We let  $Z \rtimes \mathcal{G}$  acts on  $Y * Z$  by

$$(z, \gamma)(y, \gamma^{-1}z) = (\gamma y, z).$$

To prove (i), it suffices to show that this action is amenable. The map

$$((y, z), (z, \gamma)) \mapsto ((y, z), \gamma)$$

is an isomorphism of groupoids from  $(Y * Z) \rtimes (Z \rtimes \mathcal{G})$  onto  $(Y * Z) \rtimes \mathcal{G}$ , where  $\mathcal{G}$  acts diagonally on  $Y * Z$ . Similarly, the groupoids  $(Y * Z) \rtimes \mathcal{G}$  and  $(Y * Z) \rtimes (Y \rtimes \mathcal{G})$  are isomorphic and they are amenable since  $Y \rtimes \mathcal{G}$  is amenable.

(ii) Let  $(Y, p)$  be a fibrewise compact amenable  $Z \rtimes \mathcal{G}$ -space. We define a continuous  $\mathcal{G}$ -action on  $Y$ , whose momentum map is  $q \circ p$  by

$$\gamma y = (\gamma p(y), \gamma)y.$$

The map  $q \circ p$  is proper. Moreover, the map  $(y, \gamma) \mapsto (y, (p(y), \gamma))$  is an isomorphism of groupoids from  $Y \rtimes \mathcal{G}$  onto  $Y \rtimes (Z \rtimes \mathcal{G})$  and therefore  $Y \rtimes \mathcal{G}$  is amenable.

(iii) is proved in the same way.  $\square$

Of course, (ii) and (iii) do not extend to the case where  $q$  is not proper: for every group  $G$  acting by left translations onto itself, the transformation groupoid  $G \rtimes G$  is amenable (even proper), although there exist discrete groups that are not amenable at infinity, for instance the Gromov monster groups [19] or the groups introduced in [52].

**Proposition 3.7.** *An étale groupoid  $\mathcal{G}$  is strongly amenable at infinity if and only if the Stone-Čech fibrewise compactification  $(\beta_r \mathcal{G}, r_\beta)$  is an amenable  $\mathcal{G}$ -space.*

*Proof.* In one direction, we note that the inclusions  $\mathcal{G}^{(0)} \subset \mathcal{G} \subset \beta_r \mathcal{G}$  provide a continuous section for  $r_\beta$  and therefore the amenability of the  $\mathcal{G}$ -space  $\beta_r \mathcal{G}$  implies the strong amenability at infinity of  $\mathcal{G}$ . Conversely, assume that  $(Y, p, \sigma)$  satisfies the conditions of the definition 3.4. We define a continuous  $\mathcal{G}$ -equivariant morphism  $\varphi : (\mathcal{G}, r) \rightarrow (Y, p)$  by

$$\varphi(\gamma) = \gamma \sigma \circ s(\gamma).$$

Then, by [4, Proposition 2.8],  $\varphi$  extends in a unique way to a continuous  $\mathcal{G}$ -equivariant morphism  $\Phi$  from  $(\beta_r \mathcal{G}, r_\beta)$  into  $(Y, p)$ . Now, it follows from [2, Proposition 2.2.9 (i)] that  $\mathcal{G} \curvearrowright \beta_r \mathcal{G}$  is amenable, since  $\mathcal{G} \curvearrowright Y$  is amenable.  $\square$

However,  $\beta_r \mathcal{G}$  has the serious drawback that it is not second countable in most of the cases but only  $\sigma$ -compact when  $\mathcal{G}$  is second countable. On the other hand, it has the advantage that it is well determined by  $\mathcal{G}$  alone. The following technical result shows how to build a *second countable* amenable fibrewise compact  $\mathcal{G}$ -space out of any amenable fibrewise compact  $\mathcal{G}$ -space.

**Lemma 3.8.** *Let  $\mathcal{G}$  be a second countable étale groupoid that acts amenably on a fibrewise compact fibre space  $(Z, q)$ . Then there is a second countable fibrewise compact  $\mathcal{G}$ -space  $(Y, p)$  and a  $\mathcal{G}$ -equivariant proper surjective continuous morphism  $q_Y : Z \rightarrow Y$  such that  $\mathcal{G}$  also acts amenably on  $(Y, p)$  and  $q = p \circ q_Y$ . Moreover, if there is a continuous section  $\sigma$  for  $q$ , then there is a continuous section for  $p$ , namely  $q_Y \circ \sigma$ .*

*Proof.* The space  $Z$  is  $\sigma$ -compact and therefore the semi-direct product groupoid  $Z \rtimes \mathcal{G}$  is also  $\sigma$ -compact.

We shall use the characterization of an amenable action on a  $\sigma$ -compact fibre space recalled in Proposition 2.3: there exists a sequence  $(g_n)$  of non negative functions in  $\mathcal{C}_c(Z \rtimes \mathcal{G})$  such that

- (a)  $\int g_n(z, \gamma) d\lambda^{q(z)}(\gamma) \leq 1$  for every  $z \in Z$ ;
- (b)  $\lim_n \int g_n(z, \gamma) d\lambda^{q(z)}(\gamma) = 1$  uniformly on the compact subsets of  $Z$ ;
- (c)  $\lim_n \int |g_n(\gamma^{-1}z, \gamma^{-1}\gamma_1) - g_n(z, \gamma_1)| d\lambda^{q(z)}(\gamma_1) = 0$  uniformy on the compact subsets of  $Z \rtimes \mathcal{G}$ .

*First observation.* Let  $g : Z \rtimes \mathcal{G} \rightarrow \mathbb{C}$ . Then  $g$  is continuous if and only if for every open bisection  $S$  of  $\mathcal{G}$ , the function  $z \mapsto g(z, r_S^{-1}(q(z)))$  is continuous on  $q^{-1}(r(S))$ . This is immediate since  $z \mapsto (z, r_S^{-1}(q(z)))$  is a homeomorphism from the open subset  $q^{-1}(r(S))$  of  $Z$  onto the open subset  $q^{-1}(r(S))_{q *_r S}$  of  $Z \rtimes \mathcal{G}$ .

*Construction of  $Y$ .* Let  $\mathcal{F}$  be a countable family of open bisections of  $\mathcal{G}$  which covers  $\mathcal{G}$ , is stable under product and inverse and contains a basis  $\mathcal{B}$  of open sets relative to the topology of  $X = \mathcal{G}^{(0)}$ . For  $S, S' \in \mathcal{F}$  we denote by  $g_{n,S,S'}$  the function  $z \mapsto g_n(Sz, r_{S'}^{-1}(q(Sz)))$ . The domain of definition  $\Omega_{S,S'}$  of this function is the set of  $z \in Z$  such that  $q(z) \in s(S)$  and  $S \cdot q(z) \in r(S')$ , that is,  $q^{-1}(S^{-1} \cdot (r(S) \cap r(S')))$ . We observe that if  $S$  is an open subset of  $X$  then  $\Omega_{S,S} = q^{-1}(S)$ .

We endow  $Z$  with the weakest topology  $\mathcal{T}$  that contains the  $\Omega_{S,S'}$  as open subsets and makes continuous the functions  $g_{n,S,S'}$  on  $\Omega_{S,S'}$ . It is second countable but not necessarily Hausdorff. Now, on  $Z$  we define the following equivalence relation:

$$z_1 \sim z_2 \quad \text{if} \quad q(z_1) = q(z_2) \quad \text{and} \quad g_{n,S,S'}(z_1) = g_{n,S,S'}(z_2)$$

for all  $n$  and all  $S, S'$  with  $z_1, z_2 \in \Omega_{S,S'}$ . We denote by  $Y$  the quotient space endowed with the quotient topology and by  $q_Y : Z \rightarrow Y$  the quotient map. Let us observe that  $\mathcal{T}$  is generated by subsets that are saturated with respect to this equivalence relation and therefore  $q_Y$  is an open map. We denote by  $\tilde{g}_{n,S,S'}$  the continuous function on  $q_Y(\Omega_{S,S'})$ , deduced from  $g_{n,S,S'}$  by passing to the quotient. We also introduce the map  $p$  from  $Y$  onto  $X$  such that  $p(q_Y(z)) = q(z)$ . We remark that  $q$  is still continuous when  $Z$  is equipped with the topology  $\mathcal{T}$ , and therefore  $p$  is continuous.

The space  $Y$  is Hausdorff, second countable. Let  $q_Y(z_1) \neq q_Y(z_2)$  in  $Y$ . If  $q(z_1) \neq q(z_2)$ , then  $q_Y(z_1)$  and  $q_Y(z_2)$  belongs to disjoint open subsets of the form  $q_Y(\Omega_{S,S})$  since  $\mathcal{F}$  contains a basis of the topology of  $X$ . Now, if  $q(z_1) = q(z_2)$ , there exists  $S, S' \in \mathcal{F}$  and  $n$  such that  $z_1, z_2 \in \Omega_{S,S'}$  and  $g_{n,S,S'}(z_1) \neq g_{n,S,S'}(z_2)$ . It follows that  $\tilde{g}_{n,S,S'}(q_Y(z_1)) \neq \tilde{g}_{n,S,S'}(q_Y(z_2))$  and therefore  $q_Y(z_1)$  and  $q_Y(z_2)$  have disjoint neighborhoods. Therefore,  $Y$  is Hausdorff. Since  $Z$  is second countable and  $q_Y$  is open, we see that  $Y$  is second countable.

The map  $p$  is proper and  $Y$  is locally compact. Let  $K$  be a compact subset of  $X$ . Then  $p^{-1}(K) = q_Y(q^{-1}(K))$ . Since  $q$  is proper,  $q^{-1}(K)$  is compact for the initial topology and therefore  $q_Y(q^{-1}(K))$  is compact. It follows that  $Y$  is locally compact, because  $p$  is a continuous proper map onto the locally compact space  $X$ .

We also see that  $q_Y$  is proper when  $Z$  is endowed with its initial topology, since we have

$$q_Y^{-1}(K) \subset q_Y^{-1}(p^{-1}(p(K))) = q^{-1}(p(K)),$$

for every compact subset  $K$  of  $Y$ .

The groupoid  $\mathcal{G}$  acts continuously on  $(Y, p)$ . First, if  $z_1 \sim z_2$ , we check that for  $\gamma$  such that  $s(\gamma) = q(z_1)$  then  $\gamma z_1 \sim \gamma z_2$ . Of course, we have  $q(\gamma z_1) = r(\gamma) = q(\gamma z_2)$ . If  $S, S' \in \mathcal{F}$  are such that  $r(\gamma) \in S^{-1} \cdot (r(S) \cap r(S'))$ , we have to verify that  $g_{n,S,S'}(\gamma z_1) = g_{n,S,S'}(\gamma z_2)$ , that is,

$$g_n(S\gamma z_1, r_{S'}^{-1}(q(S\gamma z_1))) = g_n(S\gamma z_2, r_{S'}^{-1}(q(S\gamma z_2))).$$

Let  $T \in \mathcal{F}$  such that  $\gamma \in T$ . Then  $g_{n,S,S'}(\gamma z_1) = g_{n,ST,S'}(z_1)$  and similarly for  $z_2$ . Since  $ST \in \mathcal{F}$  we get the wanted equality.

Now, we set  $\gamma q_Y(z) = q_Y(\gamma z)$  if  $s(\gamma) = q(z) = p(q_Y(z))$ . We have to show that the map  $(z, \gamma) \mapsto q_Y(\gamma z)$  is continuous from  $Z_{q^*s}\mathcal{G}$  into  $Y$ , when  $Z$  is equipped with the topology  $\mathcal{T}$ . We check the continuity in  $(z_0, \gamma_0)$ . We consider a neighborhood of  $\gamma_0 z_0$  of the form

$$\{z \in \Omega_{S,S'} : |g_{n,S,S'}(z) - g_{n,S,S'}(\gamma_0 z_0)| \leq \varepsilon\}.$$

Such a neighborhood is denoted by  $V(n, S, S', \varepsilon; \gamma_0 z_0)$ . We note that the finite intersections of such  $V(n, S, S', \varepsilon; \gamma_0 z_0)$  with  $r(\gamma_0) \in S^{-1} \cdot (r(S) \cap r(S'))$  form a basis of neighborhoods of  $\gamma_0 z_0$ . Let  $T$  be an open bisection which contains  $\gamma_0$ . Then if  $(z, \gamma) \in V(n, ST, S', \varepsilon; z_0)_{q^*s} T$  we have

$$|g_{n,S,S'}(\gamma z) - g_{n,S,S'}(\gamma_0 z_0)| = |g_{n,ST,S'}(z) - g_{n,ST,S'}(z_0)| \leq \varepsilon.$$

This proves the continuity of the action.

The action of  $\mathcal{G}$  on  $Y$  is amenable. For  $(q_Y(z), \gamma) \in Y \rtimes \mathcal{G}$  we set  $\tilde{g}_n(q_Y(z), \gamma) = g_n(z, \gamma)$ . This is well defined. Indeed, let  $z_1 \sim z_2$ . Let  $S \in \mathcal{B}$  such that  $q(z_1) \in S$  and let  $S'$  be an open bisection containing  $\gamma$ . Then

$$g_n(z_1, \gamma) = g_{n,S,S'}(z_1) = g_{n,S,S'}(z_2) = g_n(z_2, \gamma).$$

It is now straightforward to see that  $(\tilde{g}_n)$  is a sequence in  $\mathcal{C}(Y \rtimes \mathcal{G})$  which satisfies the appropriate version of the above conditions (a), (b) and (c).  $\square$

**3.2. Invariance under equivalence.** We are now going to show that amenability at infinity is invariant under equivalence of groupoids. We first recall this notion of *equivalence of groupoids*. Let  $\mathcal{G}$  be a locally compact groupoid and  $(Z, q)$  a left  $\mathcal{G}$ -space. We say that the action is *free* if  $\gamma z = z$  implies that  $\gamma \in \mathcal{G}^{(0)}$ . We say that it is *proper* if the map from  $\mathcal{G}_s *_q Z$  to  $Z \times Z$  given by  $(\gamma, z) \mapsto (\gamma z, z)$  is proper. We say that  $(Z, q)$  is a *principal  $\mathcal{G}$ -space* if the action is free and proper. Similarly, one defines principal right  $\mathcal{G}$ -spaces.

**Definition 3.9.** Let  $\mathcal{G}$  and  $\mathcal{H}$  be two locally compact groupoids. We say that a locally compact space  $Z$  is a  *$\mathcal{G}$ - $\mathcal{H}$ -equivalence* if

- (a)  $Z$  is a left principal  $\mathcal{G}$ -space whose momentum  $q_{\mathcal{G}} : Z \rightarrow \mathcal{G}^{(0)}$  is open;
- (b)  $Z$  is a right principal  $\mathcal{H}$ -space whose momentum  $q_{\mathcal{H}} : Z \rightarrow \mathcal{H}^{(0)}$  is open;
- (c) the  $\mathcal{G}$  and  $\mathcal{H}$  actions commute;
- (d) the map  $q_{\mathcal{G}}$  induces a homeomorphism from  $Z/\mathcal{H}$  onto  $\mathcal{G}^{(0)}$ ;
- (e) the map  $q_{\mathcal{H}}$  induces a homeomorphism from  $\mathcal{G} \setminus Z$  onto  $\mathcal{H}^{(0)}$ .

We say that  $\mathcal{G}$  and  $\mathcal{H}$  are *equivalent* if there exists a  $\mathcal{G}$ - $\mathcal{H}$ -equivalence.

**Proposition 3.10.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two equivalent locally compact groupoids. Assume that  $\mathcal{H}$  is amenable at infinity. Then  $\mathcal{G}$  is also amenable at infinity.*

*Proof.* We keep the notation of Definition 3.9. Let  $Y$  be an amenable left  $\mathcal{H}$ -space such that the momentum map  $p : Y \rightarrow \mathcal{H}^{(0)}$  is proper. We set

$$W = \{(z, y) \in Z \times Y : q_{\mathcal{H}}(z) = p(y)\}.$$

Then  $\mathcal{H}$  acts to the right on  $W$  by  $(z, y)\gamma = (z\gamma, \gamma^{-1}y)$  and this action is proper. It follows that the orbit space, denoted  $Z *_H Y$ , is locally compact. We denote by  $l : W \rightarrow Z *_H Y$  the quotient map.

Now, we define a continuous left action of the groupoid  $\mathcal{G}$  on  $Z *_H Y$  as follows. The momentum map  $q : Z *_H Y \rightarrow \mathcal{G}^{(0)}$  is  $q : l(z, y) \mapsto q_{\mathcal{G}}(z)$  and, for  $\gamma \in \mathcal{G}$  such that  $s(\gamma) = q_{\mathcal{G}}(z)$ , we set

$$\gamma l(z, y) = l(\gamma z, y).$$

Obviously, these maps are well defined. Let us check that  $q$  is proper. Let  $K$  be a compact subset of  $\mathcal{G}^{(0)}$ . Then we have

$$\{(z, y) \in W : q_{\mathcal{G}}(z) \in K\} \subset q_{\mathcal{G}}^{-1}(K) \times p^{-1}(q_{\mathcal{H}}(q_{\mathcal{G}}^{-1}(K))).$$

It follows that  $t : (z, y) \in W \mapsto q_{\mathcal{G}}(z)$  is proper since  $q_{\mathcal{G}}$  and  $p$  are proper and that  $q$  is also proper since  $q^{-1}(K) = l(t^{-1}(K))$ .

To end the proof, it remains to show that the groupoids  $\mathcal{G}' = (Z *_H Y) \rtimes \mathcal{G}$  and  $\mathcal{H}' = Y \rtimes \mathcal{H}$  are equivalent, because amenability is invariant under equivalence



[2, Theorem 2.2.17]. The equivalence is defined via the locally compact space  $W$ , with momentum maps  $q_{\mathcal{G}'} = l$  and  $q_{\mathcal{H}'} : (z, y) \mapsto y$ . The left and right actions of  $\mathcal{G}'$  and  $\mathcal{H}'$  are respectively defined by

$$(\gamma l(z, y), \gamma)(z, y) = (\gamma z, y), \quad \text{for } (\gamma l(z, y), \gamma) \in \mathcal{G}', (z, y) \in W,$$

and

$$(z, y)(y, \gamma) = (z\gamma, \gamma^{-1}y), \quad \text{for } (z, y) \in W, (y, \gamma) \in \mathcal{H}'.$$

Let us show that  $q_{\mathcal{H}'}$  is open. Let  $\Omega_Z$  and  $\Omega_Y$  be open subsets of  $Z$  and  $Y$  respectively. Then

$$q_{\mathcal{H}'}((\Omega_Z \times \Omega_Y) \cap W) = p^{-1}(q_{\mathcal{H}}(\Omega_Z)) \cap \Omega_Y$$

is open since  $q_{\mathcal{H}}$  is open. To show that  $q_{\mathcal{G}'} = l$  is open, we observe that, given an open subset  $\Omega$  of  $W$  we have

$$l^{-1}(l(\Omega)) = \{\omega\gamma : \omega \in \Omega, \gamma \in r^{-1}(q(\omega))\} = \mathbf{s}(\mathbf{r}^{-1}(\Omega))$$

where  $\mathbf{s}, \mathbf{r}$  are here the source and range maps respectively of the groupoid  $W \rtimes \mathcal{H}$ .

The other conditions insuring that  $W$  is a  $\mathcal{G}'$ - $\mathcal{H}'$ -equivalence are easily established.  $\square$

*Remark 3.11.* Assume that in the previous proposition the groupoid  $\mathcal{H}$  is strongly amenable at infinity and that the  $\mathcal{G}$ - $\mathcal{H}$ -equivalence is such that there exists a continuous map  $\sigma_{\mathcal{G}} : \mathcal{G}^{(0)} \rightarrow Z$  such that  $q_{\mathcal{G}} \circ \sigma_{\mathcal{G}}(x) = x$  for all  $x \in \mathcal{G}^{(0)}$ . Then  $\mathcal{G}$  is also strongly amenable at infinity. Indeed, keeping the notation of the proof of Proposition 3.10, it suffices to show that  $q : Z *_H Y \rightarrow \mathcal{G}^0$  has a continuous section. Let  $\sigma_Y$  be continuous section for  $p : Y \rightarrow \mathcal{H}^{(0)}$ . Then we set

$$\sigma(x) = l\left(\sigma_{\mathcal{G}}(x), \sigma_Y(q_{\mathcal{H}}(\sigma_{\mathcal{G}}(x)))\right).$$

Obviously,  $\sigma$  is a continuous section for  $q$ .

**3.3. A characterization of strong amenability at infinity.** Let  $\mathcal{G}$  be an étale groupoid. It is strongly amenable at infinity if and only if the semi-direct product groupoid  $\beta_r \mathcal{G} \rtimes \mathcal{G}$  is amenable. This can be defined in terms of positive definite functions as recalled in Proposition 2.3. We are going to provide a characterization where the obscure space  $\beta_r \mathcal{G}$  does not occur. To that end we introduce the dense open subspace

$$\mathcal{G} *_r \mathcal{G} = \{(\gamma, \gamma_1) \in \mathcal{G} \times \mathcal{G} : r(\gamma) = r(\gamma_1)\}$$

of  $\beta_r \mathcal{G} \rtimes \mathcal{G}$ . For simplicity, we shall use the notation  $q$  instead of  $r_{\beta} : \beta_r \mathcal{G} \rightarrow X = \mathcal{G}^{(0)}$ . We recall that the restriction of  $q$  to  $\mathcal{G}$  is the range map  $r$ .

For  $f \in \mathcal{C}_c(\beta_r \mathcal{G} \rtimes \mathcal{G})$ , we denote by  $\rho(f)$  its restriction to  $\mathcal{G} *_r \mathcal{G}$ . Let us observe that the support of  $f$  is contained in  $q^{-1}(r(K)) *_r K$ <sup>e</sup> for some compact subset  $K$  of  $\mathcal{G}$ .

<sup>e</sup>If  $A \subset \beta_r \mathcal{G}$  and  $B \subset \mathcal{G}$ , we write  $A *_r B$  instead of  $A_q *_r B$ .

**Lemma 3.12.** *Let  $\mathcal{G}$  be an étale groupoid. Let  $h$  be a bounded continuous function on  $\mathcal{G} *_r \mathcal{G}$  with support in  $r^{-1}(r(K)) *_r K$  for some compact subset  $K$  of  $\mathcal{G}$ . Then  $h$  extends continuously to an element  $H$  of  $\mathcal{C}_c(\beta_r \mathcal{G} \rtimes \mathcal{G})$ .*

*Proof.* Let  $K \subset \cup_{i=1}^n S_i$  be a covering of  $K$  by a finite number of open bisections, and let  $\psi_i, i = 1, \dots, n$ , be continuous non-negative functions on  $\mathcal{G}$ , the support of  $\psi_i$  being compact and contained in  $S_i$ , such that  $\sum_{i=1, \dots, n} \psi_i(\gamma) = 1$  if  $\gamma \in K$ . We shall denote by  $S_{i,r}(\gamma)$  the unique element  $\gamma'$  of  $S_i$  such that  $r(\gamma') = r(\gamma)$  whenever  $r(\gamma) \in r(S_i)$ . Then  $S_{i,r}$  is a continuous map from  $r^{-1}(r(S_i))$  onto  $S_i$ .

For  $i = 1, \dots, n$ , we define a continuous function  $k_i$  on  $\mathcal{G}$  as follows:

$$\begin{aligned} k_i(\gamma) &= h(\gamma, S_{i,r}(\gamma)) \sqrt{\psi_i(S_{i,r}(\gamma))} \quad \text{if } r(\gamma) \in r(S_i) \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Note that  $k_i$  is a continuous bounded function on  $\mathcal{G}$  supported in  $r^{-1}(r(\text{Supp}(\psi_i)))$ . Thus  $k_i$  extends to an element of  $\mathcal{C}_0(\beta_r \mathcal{G})$ , still denoted by  $k_i$ , whose support is compact, since it is contained into  $q^{-1}(r(\text{Supp}(\psi_i)))$ .

For  $(z, \gamma) \in \beta_r \mathcal{G} \rtimes \mathcal{G}$ , let us set

$$H(z, \gamma) = \sum_{i=1}^n k_i(z) \sqrt{\psi_i(\gamma)}.$$

Obviously,  $H$  is a continuous function on  $\beta_r \mathcal{G} \rtimes \mathcal{G}$  with compact support, and for  $(\gamma, \gamma_1) \in \mathcal{G} *_r \mathcal{G}$  we have

$$\begin{aligned} H(\gamma, \gamma_1) &= \sum_{i=1}^n k_i(\gamma) \sqrt{\psi_i(\gamma_1)} \\ &= \sum_{i=1}^n h(\gamma, S_{i,r}(\gamma)) \sqrt{\psi_i(S_{i,r}(\gamma))} \sqrt{\psi_i(\gamma_1)}. \end{aligned}$$

Since  $\gamma_1 = S_{i,r}(\gamma)$  for every  $i$  such that  $\gamma_1 \in S_i$  it follows that

$$H(\gamma, \gamma_1) = \sum_{i=1}^n h(\gamma, \gamma_1) \psi_i(\gamma_1) = h(\gamma, \gamma_1).$$

□

**Definition 3.13.** Let  $\mathcal{G}$  be a locally compact groupoid. We say that a function  $k : \mathcal{G} *_r \mathcal{G} \rightarrow \mathbb{C}$  is a *positive definite kernel* if for every  $x \in X$ ,  $n \in \mathbb{N}$  and  $\gamma_1, \dots, \gamma_n \in \mathcal{G}^x$ , the matrix  $[k(\gamma_i, \gamma_j)]$  is non-negative, that is

$$\sum_{i,j=1}^n \overline{\alpha_i} \alpha_j k(\gamma_i, \gamma_j) \geq 0$$

for  $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ .

**Definition 3.14.** Let  $\mathcal{G}$  be a locally compact groupoid. A *tube* is a subset of  $\mathcal{G} *_r \mathcal{G}$  whose image by the map  $(\gamma, \gamma_1) \mapsto \gamma^{-1}\gamma_1$  is relatively compact in  $\mathcal{G}$ .

We denote by  $\mathcal{C}_t(\mathcal{G} *_r \mathcal{G})$  the space of continuous bounded functions on  $\mathcal{G} *_r \mathcal{G}$  with support in a tube.

We define a linear bijection from  $\mathcal{C}_b(\mathcal{G} *_r \mathcal{G})$  onto itself by setting

$$\theta(f)(\gamma, \gamma_1) = f(\gamma^{-1}, \gamma^{-1}\gamma_1).$$

Note that  $\theta = \theta^{-1}$ .

**Theorem 3.15.** *Let  $\mathcal{G}$  be an étale groupoid.*

- (i) *The map  $\Theta : f \mapsto \theta(\rho(f))$  is a linear bijection from  $\mathcal{C}_c(\beta_r \mathcal{G} \rtimes \mathcal{G})$  onto  $\mathcal{C}_t(\mathcal{G} *_r \mathcal{G})$ .*
- (ii) *The map  $\Theta$  induces a bijection between the space of continuous positive definite functions with compact support on the groupoid  $\beta_r \mathcal{G} \rtimes \mathcal{G}$  and the space of continuous positive definite kernels contained in  $\mathcal{C}_t(\mathcal{G} *_r \mathcal{G})$ .*

*Proof.* Let us prove (i). We observe that  $h \in \mathcal{C}_t(\mathcal{G} *_r \mathcal{G})$  if and only if there exists a compact subset  $K$  of  $\mathcal{G}$  such that the support of  $\theta(h)$  is contained in  $r^{-1}(r(K)) *_r K$ . Let  $f \in \mathcal{C}_c(\beta_r \mathcal{G} \rtimes \mathcal{G})$ . Then its support is contained in some  $q^{-1}(r(K)) *_r K$  where  $K$  is a compact subset  $\mathcal{G}$  and therefore we have  $\rho(f) \in \theta(\mathcal{C}_t(\mathcal{G} *_r \mathcal{G}))$ . Moreover,  $\rho$  is injective since  $\mathcal{G} *_r \mathcal{G}$  is dense into  $\beta_r \mathcal{G} \rtimes \mathcal{G}$ . Thus  $\Theta$  is injective. Its surjectivity follows from the previous lemma.

(ii) Assume that  $f \in \mathcal{C}_c(\beta_r \mathcal{G} \rtimes \mathcal{G})$  is positive definite, that is, for every  $z \in \beta_r \mathcal{G}$ ,  $n \in \mathbb{N}$  and  $\gamma_1, \dots, \gamma_n \in \mathcal{G}^{q(z)}$ , the matrix  $[f(\gamma_i^{-1}z, \gamma_i^{-1}\gamma_j)]$  is non-negative. Observe that  $\Theta(f)(\gamma_i, \gamma_j) = f(\gamma_i^{-1}, \gamma_i^{-1}\gamma_j)$  and so, by taking  $z = r(\gamma_i) \in \mathcal{G}^{(0)} \subset \mathcal{G} \subset \beta_r \mathcal{G}$ , we get the non-negativity of  $[\Theta(f)(\gamma_i, \gamma_j)]$ .

Conversely, assume that the kernel  $\Theta(f)$  is positive definite. Let  $z \in \beta_r \mathcal{G}$  and  $\gamma_1, \dots, \gamma_n \in \mathcal{G}^{q(z)}$  as above. Let  $(z_\alpha)$  be a net in  $\mathcal{G}$  such that  $\lim z_\alpha = z$ . For each  $i$  we choose an open bisection  $S_i$  in  $\mathcal{G}$  with  $\gamma_i \in S_i$  and for  $\alpha$  large enough we set  $\gamma_{i,\alpha} = r_{S_i}^{-1}(r(z_\alpha))$ . Given  $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ , we have

$$\sum \bar{\lambda}_i \lambda_j f(\gamma_{i,\alpha}^{-1}z_\alpha, \gamma_{i,\alpha}^{-1}\gamma_{j,\alpha}) = \sum \bar{\lambda}_i \lambda_j \Theta(f)(\tilde{\gamma}_{i,\alpha}, \tilde{\gamma}_{j,\alpha}) \geq 0,$$

where we have set  $\tilde{\gamma}_{i,\alpha} = z_\alpha^{-1}\gamma_{i,\alpha}$ . The fact that  $\sum \bar{\lambda}_i \lambda_j f(\gamma_i^{-1}z, \gamma_i^{-1}\gamma_j) \geq 0$  is obtained by passing to the limit.  $\square$

**Theorem 3.16.** *Let  $\mathcal{G}$  be an étale groupoid. The following conditions are equivalent:*

- (i)  *$\mathcal{G}$  is strongly amenable at infinity;*

- (ii) *there exists a net  $(k_i)_{i \in I}$  of bounded positive definite continuous kernels on  $\mathcal{G} *_r \mathcal{G}$  supported in tubes such that  $\lim_i k_i = 1$  uniformly on tubes.*

*Proof.* By Proposition 2.2, the groupoid  $\beta_r \mathcal{G} \rtimes \mathcal{G}$  is amenable if and only if there exists a net  $(h_i)_{i \in I}$  of continuous positive definite functions in  $\mathcal{C}_c(\beta_r \mathcal{G} \rtimes \mathcal{G})$  such that  $\lim_i h_i = 1$  uniformly on every compact subset of  $\beta_r \mathcal{G} \rtimes \mathcal{G}$ .

Therefore, to prove the theorem we just have to check that  $\lim_i h_i = 1$  uniformly on every compact subset of  $\beta_r \mathcal{G} \rtimes \mathcal{G}$  if and only if  $\lim_i k_i = 1$  uniformly on tubes where we set  $k_i = \Theta(h_i)$ . First, given a compact subset  $K$  of  $\mathcal{G}$ , if  $i$  is such that  $|h_i(z, \gamma) - 1| \leq \varepsilon$  on the compact set  $q^{-1}(r(K)) *_r K$ , then we have  $|k_i(\gamma_1, \gamma_2) - 1| \leq \varepsilon$  whenever  $\gamma_1^{-1} \gamma_2 \in K$ . Conversely, let  $K$  be a compact subset of  $\mathcal{G}$  and let  $\Omega$  be an open relatively compact subset of  $\mathcal{G}$  containing  $K$ . Let  $i$  be such that  $|k_i(\gamma_1, \gamma_2) - 1| \leq \varepsilon$  whenever  $\gamma_1^{-1} \gamma_2 \in \Omega$ . Then we have  $|h_i(\gamma, \gamma') - 1| \leq \varepsilon$  for  $(\gamma, \gamma') \in \mathcal{G} *_r \Omega$ . Since  $\mathcal{G} *_r \Omega$  is dense in  $\beta_r \mathcal{G} *_r K$ , we get  $|h_i(z, \gamma') - 1| \leq \varepsilon$  for  $(z, \gamma') \in \beta_r \mathcal{G} *_r K$ . To conclude, we observe that every compact subset of  $\beta_r \mathcal{G} \rtimes \mathcal{G}$  is contained in such a subset  $\beta_r \mathcal{G} *_r K$ .  $\square$

*Remark 3.17.* This theorem can be used to show that strongly amenable at infinity étale groupoids are uniformly embeddable in continuous fields of Hilbert spaces.

### 3.4. Examples induced by locally proper homomorphisms.

**Definition 3.18.** Let  $\rho : \mathcal{G} \rightarrow \mathcal{H}$  be a continuous homomorphism (also called cocycle) between locally compact groupoids. We denote by  $\psi$  the map  $\gamma \mapsto (r(\gamma), \rho(\gamma), s(\gamma))$  from  $\mathcal{G}$  into  $\mathcal{G}^{(0)} \times \mathcal{H} \times \mathcal{G}^{(0)}$ . We say that  $\rho$  *locally proper* is  $\psi$  is proper or, equivalently, if for every compact subset  $K$  of  $\mathcal{G}$ , the restriction of  $\rho$  to the reduction  $\mathcal{G}(K)$  of  $\mathcal{G}$  by  $K$  is proper. Following [30, 61], we say that  $\rho$  is *faithful* if  $\psi$  is injective.

Note that a continuous homomorphism is faithful and locally proper if and only if the map  $\psi$  is faithful and closed.

**Examples 3.19.** (a) If  $\mathcal{G}$  is a closed subgroupoid of  $\mathcal{H}$ , then the inclusion map is faithful and locally proper.

(b) Let  $E$  be a locally compact subset of  $\mathcal{H}^{(0)}$  such that the reduction  $\mathcal{H}(E)$  is a locally compact subgroupoid of  $\mathcal{H}$  (*i.e.*, the range map of  $\mathcal{H}(E)$  is open). Then the inclusion map from  $\mathcal{H}(E)$  into  $\mathcal{H}$  is faithful and locally proper. Indeed, it suffices to observe that if  $K$  is a compact subset of  $E$ , then  $\mathcal{H}(E)(K) = \mathcal{H}(K)$ .

(c) If  $Y$  is a  $\mathcal{H}$ -space, the obvious homomorphism from  $Y \rtimes \mathcal{H} \rightarrow \mathcal{H}$  is faithful and locally proper.

**Proposition 3.20.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be locally compact groupoids. Assume that there exists a continuous, faithful and locally proper homomorphism  $\rho : \mathcal{G} \rightarrow \mathcal{H}$ .*

- (i) The groupoid  $\mathcal{G}$  is equivalent to a semi-direct product groupoid  $Y \rtimes \mathcal{H}$  via an equivalence  $Z$  such that there exists a continuous section  $\sigma : \mathcal{G}^{(0)} \rightarrow Z$  of  $q_{\mathcal{G}} : Z \rightarrow \mathcal{G}^{(0)}$ .
- (ii) If  $\mathcal{H}$  is amenable (resp. strongly amenable) at infinity then  $\mathcal{G}$  is amenable (resp. strongly amenable) at infinity.

*Proof.* We follow the arguments of the proof of [30, Theorem 1.8] (see also [61]). We set  $X = \mathcal{G}^{(0)}$  and we underline the elements of  $\mathcal{H}$  in order to distinguish them from the elements of  $\mathcal{G}$ . We set  $Z = \{(x, \underline{\gamma}) \in X \times \mathcal{H} : \rho(x) = r(\underline{\gamma})\}$ . The groupoid  $\mathcal{G}$  acts to the left on  $Z$  by

$$\gamma(s(\underline{\gamma}), \underline{\gamma}) = (r(\gamma), \rho(\gamma)\underline{\gamma}).$$

This action is free and proper exactly when  $\rho$  is faithful and locally proper. We set  $Y = \mathcal{G} \setminus Z$ . It is a locally compact space, since the  $\mathcal{G}$ -action is proper.

The groupoid  $\mathcal{H}$  acts to the right on  $Z$  by  $(x, \underline{\gamma})\underline{\gamma}' = (x, \underline{\gamma}\underline{\gamma}')$ . Obviously, this action is free and proper. It follows that  $Z$  is a  $\mathcal{G}$ - $(Y \rtimes \mathcal{H})$  equivalence. Observe that the momentum map  $q_{\mathcal{G}} : Z \rightarrow \mathcal{G}^{(0)}$  admits the continuous section  $x \mapsto (x, \rho(x))$ . Then, the conclusion follows from Proposition 3.10, Remark 3.11 and Proposition 3.6 (i).  $\square$

For étale groupoids, we have the following stronger result.

**Proposition 3.21.** *Let  $\mathcal{G}$  be an étale groupoid. Assume that there exists a locally proper continuous homomorphism  $\rho$  from  $\mathcal{G}$  into a strongly amenable at infinity étale groupoid  $\mathcal{H}$ . Then  $\mathcal{G}$  is strongly amenable at infinity.*

*Proof.* Let  $(k_i)_{i \in I}$  be a net of kernels on  $\mathcal{H} *_r \mathcal{H}$  satisfying the condition (ii) of the theorem 3.16 and let  $(\varphi_j)_{j \in J}$  be a net of continuous functions  $\varphi_j : \mathcal{G}^{(0)} \rightarrow [0, 1]$  with compact support and such that for every compact subset  $K$  of  $\mathcal{G}^{(0)}$  there exists  $j_0$  for which  $\varphi_j(x) = 1$  if  $x \in K$  and  $j \geq j_0$ . For  $(\gamma_1, \gamma_2) \in \mathcal{G} *_r \mathcal{G}$ , we set

$$h_{i,j}(\gamma_1, \gamma_2) = \varphi_j(s(\gamma_1))\varphi_j(s(\gamma_2))k_i(\rho(\gamma_1), \rho(\gamma_2)).$$

Then  $(h_{i,j})_{(i,j) \in I \times J}$  is a net of kernels satisfying the condition (ii) of the theorem 3.16.  $\square$

**Corollary 3.22.** *Let  $\mathcal{G}$  be an étale groupoid such that there exists a locally proper continuous homomorphism  $\rho$  from  $\mathcal{G}$  into a countable exact discrete group  $G$ . Then  $\mathcal{G}$  is strongly amenable at infinity.*

*Proof.* This follows from the fact that one of the equivalent definitions of exactness for a discrete group is (strong) amenability at infinity.  $\square$

## 4. WEAKLY INNER AMENABLE GROUPOIDS

We shall use in Section 7 another kind of amenability. Since for locally compact groups it is weaker than inner amenability, we shall call it weak inner amenability<sup>f</sup>.

Let us first recall some facts about inner amenability. Let  $G$  be a locally compact group,  $\lambda$  its left regular representation and  $\rho$  its right regular representation. These representations act in the Hilbert space  $L^2(G)$  of square integrable functions. For  $\xi \in L^2(G)$  and  $s, y \in G$ , we have

$$(\lambda_s \xi)(y) = \xi(s^{-1}y), \quad (\rho_s \xi)(y) = \Delta(s)^{1/2} \xi(ys)$$

where  $\Delta$  is the modular function of  $G$ .

Following [54, page 84], we say that  $G$  is *inner amenable* if there exists an inner invariant mean on  $L^\infty(G)$ , that is, a state  $m$  such that  $m(sfs^{-1}) = m(f)$  for every  $f \in L^\infty(G)$  and  $s \in G$ , where  $(sfs^{-1})(y) = f(s^{-1}ys)$ . This is equivalent to the existence of a net  $(\xi_i)$  in  $\mathcal{C}_c(G)$  such that  $\|\xi_i\|_2 = 1$  and  $\langle \xi_i, \lambda_s \rho_s \xi_i \rangle$  goes to one uniformly on compact subsets of  $G$  (see [45]).

Amenable groups are inner amenable. Every discrete group  $G$  is inner amenable in this sense, since the Dirac measure  $\delta_e$  is an inner invariant mean<sup>g</sup>. On the other hand, inner amenable connected groups are amenable [45].

Let  $G$  be a locally compact group and let  $(\xi_i)$  be a net of elements of  $\mathcal{C}_c(G)$  satisfying the above condition. Let us define  $f_i$  on  $G \times G$  by

$$f_i(s, t) = \langle \xi_i, \lambda_s \rho_t \xi_i \rangle.$$

Then  $f_i$  is a positive definite function on the product group  $G \times G$ . For each compact subset  $K$  of  $G$ , the intersections with  $K \times G$  and with  $G \times K$  of the support of  $f_i$  are compact. Moreover,  $\lim_i f_i = 1$  uniformly on the diagonal.

To extend these properties to locally compact groupoids, we introduce the following definitions.

**Definition 4.1.** Let  $\mathcal{G}$  be a locally compact groupoid. Following [64, Definition 2.1], we say that a closed subset  $A$  of  $\mathcal{G} \times \mathcal{G}$  is *proper* if for every compact subset  $K$  of  $\mathcal{G}$ , the sets  $(K \times \mathcal{G}) \cap A$  and  $(\mathcal{G} \times K) \cap A$  are compact. We say that a function  $f : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{C}$  is *properly supported* if its support is proper.

Given a groupoid  $\mathcal{G}$ , let us observe that the product  $\mathcal{G} \times \mathcal{G}$  has an obvious structure of groupoid, with  $X \times X$  as set of units, where  $X = \mathcal{G}^{(0)}$ . Observe that a map  $f : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{C}$  is positive definite if and only if, given an integer  $n$ ,  $(x, y) \in X \times X$  and  $\gamma_1, \dots, \gamma_n \in \mathcal{G}^x$ ,  $\eta_1, \dots, \eta_n \in \mathcal{G}^y$ , the matrix  $[f(\gamma_i^{-1} \gamma_j, \eta_i^{-1} \eta_j)]_{i,j}$  is non-negative.

<sup>f</sup>This notion was considered in [3] for transformation groupoids, under the name of Property (W).

<sup>g</sup>Effros [16] excludes this trivial inner invariant mean in his definition of inner amenability.

**Definition 4.2.** We say that a locally compact groupoid  $\mathcal{G}$  is *weakly inner amenable* if for every compact subset  $K$  of  $\mathcal{G}$  and for every  $\varepsilon > 0$  there exists a continuous bounded positive definite function  $f$  on the product groupoid  $\mathcal{G} \times \mathcal{G}$ , properly supported, such that  $|f(\gamma, \gamma) - 1| < \varepsilon$  for all  $\gamma \in K$ .

*Remark 4.3.* (1) Every amenable locally compact groupoid  $\mathcal{G}$  with Haar system is weakly inner amenable since the groupoid  $\mathcal{G} \times \mathcal{G}$  is amenable and therefore Proposition 2.2 applies to this groupoid.

(2) Inner amenable locally compact groups are weakly inner amenable. A locally compact group which is either almost connected or type  $I$  is weakly inner amenable if and only if it is amenable (see [3, Remark 5.10]). We do not know whether there exist weakly inner amenable groups that are not inner amenable. Such examples should be sought in the class of totally disconnected locally compact (non discrete) groups.

**Proposition 4.4.** *Let  $\rho : \mathcal{G} \rightarrow \mathcal{H}$  be a locally proper continuous homomorphism between locally compact groupoids. Assume that  $\mathcal{H}$  is weakly inner amenable. Then the groupoid  $\mathcal{G}$  is also weakly inner amenable.*

*Proof.* Let  $K$  be a compact subset of  $\mathcal{G}$  and let  $\varepsilon > 0$  be given. We choose a continuous function  $\varphi : \mathcal{G}^{(0)} \rightarrow [0, 1]$  with compact support  $K'$ , such that  $\varphi(x) = 1$  if  $x \in r(K) \cup s(K)$ . Let  $f : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$  be a continuous bounded positive definite function on the product groupoid  $\mathcal{H} \times \mathcal{H}$ , properly supported, such that  $|f(\gamma, \gamma) - 1| < \varepsilon$  for all  $\gamma \in \rho(K)$ . We define a positive definite function  $F$  on  $\mathcal{G} \times \mathcal{G}$  by

$$F(\gamma_1, \gamma_2) = \varphi \circ r(\gamma_1) \varphi \circ s(\gamma_1) f(\rho(\gamma_1), \rho(\gamma_2)) \varphi \circ r(\gamma_2) \varphi \circ s(\gamma_2).$$

We have obviously  $|F(\gamma, \gamma) - 1| < \varepsilon$  if  $\gamma \in K$ . Let us check that  $F$  is properly supported. We denote by  $S_F$  and  $S_f$  the supports of  $F$  and  $f$  respectively. We fix a compact subset  $K_1$  of  $\mathcal{G}$ . Let  $C$  be a compact subset of  $\mathcal{H}$  such that

$$(\rho(K_1) \times \mathcal{H}) \cap S_f \subset C \times C.$$

Then  $(K_1 \times \mathcal{G}) \cap S_F$  is contained into  $(\rho^{-1}(C) \cap \mathcal{G}(K')) \times (\rho^{-1}(C) \cap \mathcal{G}(K'))$  and therefore is compact. The case of  $(\mathcal{G} \times K_1) \cap S_F$  is similar.  $\square$

Applying this result to the examples given in 3.19 we obtain the following corollaries.

**Corollary 4.5.** *Let  $\mathcal{H}$  be a weakly inner amenable locally compact groupoid.*

- (i) *Every closed subgroupoid of  $\mathcal{H}$  is weakly inner amenable.*
- (ii) *Let  $E$  be a locally compact subset of  $\mathcal{H}^{(0)}$  such that the reduced groupoid  $\mathcal{H}(E)$  is a subgroupoid. Then  $\mathcal{H}(E)$  is weakly inner amenable.*

**Corollary 4.6.** *Let  $\mathcal{H}$  be a weakly inner amenable locally compact groupoid and  $(Y, p)$  a  $\mathcal{G}$ -space. Then the semi-direct product groupoid  $Y \rtimes \mathcal{H}$  is weakly inner amenable.*

*Remark 4.7.* In particular, we see that if  $\mathcal{G}$  is weakly inner amenable, all its isotropy subgroups must be amenable whenever they are connected. We do not know whether every étale groupoid is weakly inner amenable.

## 5. GROUPOIDS AND $C^*$ -ALGEBRAS

**5.1. Full, reduced and uniform  $C^*$ -algebras.** Let  $(\mathcal{G}, \lambda)$  be a locally compact groupoid with a Haar system  $\lambda$ . We set  $X = \mathcal{G}^{(0)}$ . The space  $\mathcal{C}_c(\mathcal{G})$  is an involutive algebra with respect to the following operations for  $f, g \in \mathcal{C}_c(\mathcal{G})$ :

$$(f * g)(\gamma) = \int f(\gamma_1)g(\gamma_1^{-1}\gamma)d\lambda^{r(\gamma)}(\gamma_1)$$

$$f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

We define a norm on  $\mathcal{C}_c(\mathcal{G})$  by

$$\|f\|_1 = \max \left\{ \sup_{x \in X} \int |f(\gamma)| d\lambda^x(\gamma), \sup_{x \in X} \int |f(\gamma^{-1})| d\lambda^x(\gamma) \right\}.$$

The full  $C^*$ -algebra  $C^*(\mathcal{G})$  of the groupoid  $(\mathcal{G}, \lambda)$  is the enveloping  $C^*$ -algebra of the Banach  $*$ -algebra obtained by completion of  $\mathcal{C}_c(\mathcal{G})$  with respect to the norm  $\|\cdot\|_1$ .

For the notion of Hilbert  $C^*$ -module  $\mathcal{H}$  over a  $C^*$ -algebra  $A$  (or Hilbert  $A$ -module) that we use in the sequel, we refer to [38]. We shall denote by  $\mathcal{B}_A(\mathcal{H})$  the  $C^*$ -algebra of  $A$ -linear adjointable maps from  $\mathcal{H}$  into itself.

Let  $\mathcal{E}$  be the Hilbert  $C^*$ -module<sup>h</sup>  $L^2_{\mathcal{C}_0(X)}(\mathcal{G}, \lambda)$  over  $\mathcal{C}_0(X)$  obtained by completion of  $\mathcal{C}_c(\mathcal{G})$  with respect to the  $\mathcal{C}_0(X)$ -valued inner product

$$\langle \xi, \eta \rangle(x) = \int_{\mathcal{G}^x} \overline{\xi(\gamma)}\eta(\gamma) d\lambda^x(\gamma).$$

The  $\mathcal{C}_0(X)$ -module structure is given by

$$(\xi f)(\gamma) = \xi(\gamma)f \circ r(\gamma).$$

Let us observe that  $L^2_{\mathcal{C}_0(X)}(\mathcal{G}, \lambda)$  is the space of continuous sections vanishing at infinity of a continuous field of Hilbert spaces with fibre  $L^2(\mathcal{G}^x, \lambda^x)$  at  $x \in X$ .

We let  $\mathcal{C}_c(\mathcal{G})$  act on  $\mathcal{E}$  by the formula

$$(\Lambda(f)\xi)(\gamma) = \int f(\gamma^{-1}\gamma_1)\xi(\gamma_1) d\lambda^{r(\gamma)}(\gamma_1).$$

---

<sup>h</sup>When  $\mathcal{G}$  is étale, we shall use the notation  $\ell^2_{\mathcal{C}_0(X)}(\mathcal{G})$  rather than  $L^2_{\mathcal{C}_0(X)}(\mathcal{G}, \lambda)$



Then,  $\Lambda$  extends to a representation of  $C^*(\mathcal{G})$  in the Hilbert  $\mathcal{C}_0(X)$ -module  $\mathcal{E}$ , called the *regular representation of  $(\mathcal{G}, \lambda)$* . Its range is denoted by  $C_r^*(\mathcal{G})$  and called the *reduced  $C^*$ -algebra of the groupoid  $\mathcal{G}$* . Note that  $\Lambda(C^*(\mathcal{G}))$  acts fibrewise on the corresponding continuous field of Hilbert spaces with fibres  $L^2(\mathcal{G}^x, \lambda^x)$  by the formula

$$(\Lambda_x(f)\xi)(\gamma) = \int_{\mathcal{G}^x} f(\gamma^{-1}\gamma_1)\xi(\gamma_1) d\lambda^x(\gamma_1)$$

for  $f \in \mathcal{C}_c(\mathcal{G})$  and  $\xi \in L^2(\mathcal{G}^x, \lambda^x)$ . Moreover, we have  $\|\Lambda(f)\| = \sup_{x \in X} \|\Lambda_x(f)\|$ .

We now introduce a third  $C^*$ -algebra associated with a groupoid, which extends the notion of uniform Roe algebra associated with a countable discrete group [64]. Recall that  $\mathcal{C}_t(\mathcal{G} *_r \mathcal{G})$  denotes the space of continuous bounded functions on  $\mathcal{G} *_r \mathcal{G}$  with support in a tube. We define on  $\mathcal{C}_t(\mathcal{G} *_r \mathcal{G})$  the following operations:

$$(f * g)(\gamma, \gamma') = \int f(\gamma, \gamma_1)g(\gamma_1, \gamma') d\lambda^{r(\gamma)}(\gamma_1)$$

$$f^*(\gamma, \gamma') = \overline{f(\gamma', \gamma)},$$

which make  $\mathcal{C}_t(\mathcal{G} *_r \mathcal{G})$  a  $*$ -algebra.

For  $f \in \mathcal{C}_t(\mathcal{G} *_r \mathcal{G})$  and  $\xi \in \mathcal{E}$ , we set

$$(T(f)\xi)(\gamma) = \int f(\gamma, \gamma_1)\xi(\gamma_1) d\lambda^{r(\gamma)}(\gamma_1).$$

Then  $T$  is a  $*$ -homomorphism from  $\mathcal{C}_t(\mathcal{G} *_r \mathcal{G})$  into  $\mathcal{B}_{\mathcal{C}_0(X)}(\mathcal{E})$ .

**Definition 5.1.** The *uniform  $C^*$ -algebra of  $\mathcal{G}$*  is the  $C^*$ -subalgebra  $C_u^*(\mathcal{G})$  of  $\mathcal{B}_{\mathcal{C}_0(X)}(\mathcal{E})$  generated by the operators  $T(f)$  associated with the bounded continuous kernels supported in tubes.

Our goal is now to prove that the  $C^*$ -algebras  $C_u^*(\mathcal{G})$  and  $C_r^*(\beta_r \mathcal{G} \rtimes \mathcal{G})$  are canonically isomorphic when  $\mathcal{G}$  is an étale groupoid. We keep the notation of Theorem 3.15. Recall that we have set  $q = r_\beta : \beta_r \mathcal{G} \rightarrow X$ . We shall have to apply the definition of the reduced  $C^*$ -algebra to the groupoid  $\beta_r \mathcal{G} \rtimes \mathcal{G}$ . It is represented by  $\Lambda'$  in the Hilbert  $\mathcal{C}_0(\beta_r \mathcal{G})$ -module  $\mathcal{E}'$  which is the completion of  $\mathcal{C}_c(\beta_r \mathcal{G} \rtimes \mathcal{G})$  with respect to the inner product

$$\langle \xi, \eta \rangle(z) = \int \overline{\xi(z, \gamma)}\eta(z, \gamma) d\lambda^{q(z)}(\gamma).$$

**Lemma 5.2.** *Let  $\mathcal{G}$  be an étale groupoid.*

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<sup>i</sup>Very often, the Hilbert  $\mathcal{C}_0(X)$ -module  $L^2_{\mathcal{C}_0(X)}(\mathcal{G}, \lambda^{-1})$  is considered in order to define the reduced  $C^*$ -algebra (see for instance [30, 31]). We pass to this setting to ours (which we think more convenient for our purpose) by considering the isomorphism  $U : L^2_{\mathcal{C}_0(X)}(\mathcal{G}, \lambda^{-1}) \rightarrow L^2_{\mathcal{C}_0(X)}(\mathcal{G}, \lambda)$  such that  $(U\xi)(\gamma) = \xi(\gamma^{-1})$ .

- (i) The  $*$ -algebra  $\mathcal{C}_c(\mathcal{G})$  is canonically embedded into the  $*$ -algebras  $\mathcal{C}_c(\beta_r\mathcal{G} \rtimes \mathcal{G})$  and  $\mathcal{C}_t(\mathcal{G} *_r \mathcal{G})$  (via  $*$ -homomorphisms).
- (ii) These embeddings extend into embeddings of  $C_r^*(\mathcal{G})$  into  $C_r^*(\beta_r\mathcal{G} \rtimes \mathcal{G})$  and  $C_u^*(\mathcal{G})$ .
- (iii) The map  $\Theta : f \mapsto \theta(\rho(f))$  is an isomorphism of  $*$ -algebras from  $\mathcal{C}_c(\beta_r\mathcal{G} \rtimes \mathcal{G})$  onto  $\mathcal{C}_t(\mathcal{G} *_r \mathcal{G})$  which preserves the above mentioned embeddings of  $\mathcal{C}_c(\mathcal{G})$ .

*Proof.* (i) The embedding from  $\mathcal{C}_c(\mathcal{G})$  into  $\mathcal{C}_c(\beta_r\mathcal{G} \rtimes \mathcal{G})$  is given by  $f \mapsto f \circ \pi$  where  $\pi : \beta_r\mathcal{G} \rtimes \mathcal{G} \rightarrow \mathcal{G}$  is the second projection, which is proper. We embed  $\mathcal{C}_c(\mathcal{G})$  into  $\mathcal{C}_t(\mathcal{G} *_r \mathcal{G})$  by sending  $f \in \mathcal{C}_c(\mathcal{G})$  onto  $\tilde{f}$  such that  $\tilde{f}(\gamma, \gamma_1) = f(\gamma^{-1}\gamma_1)$ .

(ii) For  $f \in \mathcal{C}_c(\mathcal{G})$ , we have  $\|\Lambda'(f \circ \pi)\| = \sup_{z \in \beta_r\mathcal{G}} \|\Lambda'_z(f \circ \pi)\|$  with

$$(\Lambda'_z(f \circ \pi)\xi)(z, \gamma) = \int f(\gamma^{-1}\gamma_1)\xi(z, \gamma_1) d\lambda^{q(z)}(\gamma_1).$$

Observe that  $\xi \in \ell^2((\beta_r\mathcal{G} \rtimes \mathcal{G})^z)$  can be identified to the element  $\gamma \mapsto \xi(z, \gamma)$  of  $\ell^2(\mathcal{G}^{q(z)})$ . It follows that  $\|\Lambda'_z(f \circ \pi)\| = \|\Lambda_{q(z)}(f)\|$  and so  $\|\Lambda'(f \circ \pi)\| = \|\Lambda(f)\|$ . The second assertion of (ii) is immediate.

Taking into account Theorem 3.15, straightforward computations prove (iii).  $\square$

**Theorem 5.3.** *Let  $\mathcal{G}$  be an étale groupoid. The map  $\Lambda'(f) \mapsto T(\Theta(f))$  defined on  $\Lambda'(\mathcal{C}_c(\beta_r\mathcal{G} \rtimes \mathcal{G}))$  extends to an isomorphism from  $C_r^*(\beta_r\mathcal{G} \rtimes \mathcal{G})$  onto  $C_u^*(\mathcal{G})$ , which is the identity on  $C_r^*(\mathcal{G})$ .*

*Proof.* Let  $\Phi$  be the faithful non-degenerate homomorphism from  $\mathcal{C}_0(\beta_r\mathcal{G})$  into  $\mathcal{B}_{\mathcal{C}_0(X)}(\mathcal{E})$  defined by

$$\Phi(f)\xi(\gamma) = f(\gamma^{-1})\xi(\gamma).$$

The relative (or interior) tensor product

$$\mathcal{H} = \mathcal{E}' \otimes_{\mathcal{C}_0(\beta_r\mathcal{G})} \mathcal{E},$$

is a Hilbert  $\mathcal{C}_0(X)$ -module (see [38]) whose  $\mathcal{C}_0(X)$ -inner product is defined, for  $\xi, \xi' \in \mathcal{C}_c(\beta_r\mathcal{G} \rtimes \mathcal{G})$  and  $\eta, \eta' \in \mathcal{C}_c(\mathcal{G})$ , by

$$\begin{aligned} \langle \xi \otimes \eta, \xi' \otimes \eta' \rangle(x) &= \int_{\mathcal{G}^x} \overline{\eta(\gamma)} \langle \xi, \xi' \rangle(\gamma^{-1}) \eta'(\gamma) d\lambda^x(\gamma) \\ &= \int_{\mathcal{G}^x} \overline{\eta(\gamma)} \eta'(\gamma) \left( \int \overline{\xi(\gamma^{-1}, \gamma_1)} \xi'(\gamma^{-1}, \gamma_1) d\lambda^{s(\gamma)}(\gamma_1) \right) d\lambda^x(\gamma). \end{aligned}$$

We shall first check that  $\mathcal{H}$  is isomorphic to the Hilbert  $\mathcal{C}_0(X)$ -module

$$\tilde{\mathcal{H}} = L_{\mathcal{C}_0(X)}^2(\mathcal{G} *_r \mathcal{G}, \lambda \otimes \lambda)$$

which is defined as the completion of  $\mathcal{C}_c(\mathcal{G} *_r \mathcal{G})$  with respect to the  $\mathcal{C}_0(X)$ -valued inner product

$$\langle \xi, \xi' \rangle(x) = \int \overline{\xi(\gamma, \gamma_1)} \xi'(\gamma, \gamma_1) d\lambda^x(\gamma) d\lambda^x(\gamma_1).$$

The  $\mathcal{C}_0(X)$ -module structure is defined by

$$(\xi f)(\gamma, \gamma_1) = \xi(\gamma, \gamma_1) f \circ r(\gamma).$$

For  $\xi \in \mathcal{C}_c(\beta_r \mathcal{G} \rtimes \mathcal{G})$  and  $\eta \in \mathcal{C}_c(\mathcal{G})$  we set

$$(W(\xi \otimes \eta))(\gamma_1, \gamma_2) = \xi(\gamma_2^{-1}, \gamma_2^{-1} \gamma_1) \eta(\gamma_2).$$

We have  $W(\xi \otimes \eta) \in \mathcal{C}_c(\mathcal{G} *_r \mathcal{G})$ . A straightforward computation shows that  $W$  extends to an isomorphism of Hilbert  $\mathcal{C}_0(X)$ -module from  $\mathcal{H}$  onto  $\tilde{\mathcal{H}}$ .

We also observe that the map from  $\mathcal{C}_c(\mathcal{G}) \times \mathcal{C}_c(\mathcal{G})$  into  $\mathcal{C}_c(\mathcal{G} *_r \mathcal{G})$  sending  $(\xi, \eta)$  to  $(\gamma, \gamma_1) \in \mathcal{G} *_r \mathcal{G} \rightarrow \xi(\gamma) \eta(\gamma_1)$  defines an isomorphism of Hilbert  $\mathcal{C}_0(X)$ -module from  $\mathcal{E} \otimes_{\mathcal{C}_0(X)} \mathcal{E}$  onto  $\tilde{\mathcal{H}}$ . We identify these two Hilbert  $\mathcal{C}_0(X)$ -modules.

For  $f \in \mathcal{C}_c(\beta_r \mathcal{G} \rtimes \mathcal{G})$ , we claim that

$$W \circ (\Lambda'(f) \otimes \text{Id}_{\mathcal{E}}) = (T(\Theta(f)) \otimes \text{Id}_{\mathcal{E}}) \circ W.$$

This will imply that  $\Lambda'(f) \mapsto T(\Theta(f))$  is isometric and thus extends to an isomorphism of the completions. This isomorphism will be the identity on  $C_r^*(\mathcal{G})$  by Lemma 5.2 (iii). Let us prove our claim. Given  $\xi \in \mathcal{C}_c(\beta_r \mathcal{G} \rtimes \mathcal{G})$  and  $\eta \in \mathcal{C}_c(\mathcal{G})$ , we have

$$\begin{aligned} W \circ (\Lambda'(f) \otimes \text{Id})(\xi \otimes \eta)(\gamma_1, \gamma_2) &= (\Lambda'(f)\xi)(\gamma_2^{-1}, \gamma_2^{-1} \gamma_1) \eta(\gamma_2) \\ &= \left( \int f((\gamma_1^{-1}, \gamma_1^{-1} \gamma_2)(\gamma_2^{-1}, \gamma)) \xi(\gamma_2^{-1}, \gamma) d\lambda^{s(\gamma_2)}(\gamma) \right) \eta(\gamma_2) \\ &= \left( \int f(\gamma_1^{-1}, \gamma_1^{-1} \gamma_2 \gamma) \xi(\gamma_2^{-1}, \gamma) d\lambda^{s(\gamma_2)}(\gamma) \right) \eta(\gamma_2) \\ &= \left( \int f(\gamma_1^{-1}, \gamma_1^{-1} \gamma) \xi(\gamma_2^{-1}, \gamma_2^{-1} \gamma) d\lambda^{r(\gamma_2)}(\gamma) \right) \eta(\gamma_2). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \left( (T(\Theta(f)) \otimes \text{Id}_{\mathcal{E}}) \circ W \right) (\xi \otimes \eta)(\gamma_1, \gamma_2) &= \int \Theta(f)(\gamma_1, \gamma) (W(\xi \otimes \eta))(\gamma, \gamma_2) d\lambda^{r(\gamma_2)}(\gamma) \\ &= \int f(\gamma_1^{-1}, \gamma_1^{-1} \gamma) \xi(\gamma_2^{-1}, \gamma_2^{-1} \gamma) \eta(\gamma_2) d\lambda^{r(\gamma_2)}(\gamma), \end{aligned}$$

and so our claim is proved.  $\square$

## 5.2. Groupoid actions on $C^*$ -algebras and crossed products.

**Definition 5.4.** Let  $X$  be a locally compact space. A  $\mathcal{C}_0(X)$ -algebra is a  $C^*$ -algebra  $A$  equipped with a  $*$ -homomorphism  $\rho$  from  $\mathcal{C}_0(X)$  into the centre of the multiplier algebra of  $A$  which is non-degenerate in the sense that there exists an approximate unit  $(u_\lambda)$  of  $\mathcal{C}_0(X)$  such that  $\lim_\lambda \rho(u_\lambda)a = a$  for every  $a \in A$ .

Given  $f \in \mathcal{C}_0(X)$  and  $a \in A$ , for simplicity we shall write  $fa$  instead of  $\rho(f)a$ .

We begin by recalling some facts and definitions that are mostly borrowed from [41, 31]. Let  $U$  be an open subset of  $X$  and  $F = X \setminus U$ . We view  $\mathcal{C}_0(U)$  as an ideal of  $\mathcal{C}_0(X)$  and we denote by  $\mathcal{C}_0(U)A$  the closed linear span of  $\{fa : f \in \mathcal{C}_0(U), a \in A\}$ . It is a closed ideal of  $A$  and in fact, we have  $\mathcal{C}_0(U)A = \{fa : f \in \mathcal{C}_0(U), a \in A\}$  (see [9, Corollaire 3.9]). We set  $A_F = A/\mathcal{C}_0(U)A$  and whenever  $F = \{x\}$  we write  $\mathcal{C}_x(X)$  instead of  $\mathcal{C}_0(X \setminus \{x\})$  and  $A_x$  instead of  $A_{\{x\}}$ . We denote by  $e_x : A \rightarrow A_x$  the quotient map and for  $a \in A$  we set  $a(x) = e_x(a)$ . Recall that the map  $a \mapsto (a(x))_{x \in X}$  from  $A$  into  $\prod_{x \in X} A_x$  is injective and that  $x \mapsto \|a(x)\|$  is upper semi-continuous (see [63]).

Let  $A$  and  $B$  be two  $\mathcal{C}_0(X)$ -algebras. A *morphism*  $\alpha : A \rightarrow B$  of  $\mathcal{C}_0(X)$ -algebras is a morphism of  $C^*$ -algebras which is  $\mathcal{C}_0(X)$ -linear, that is,  $\alpha(fa) = f\alpha(a)$  for  $f \in \mathcal{C}_0(X)$  and  $a \in A$ . For  $x \in X$ , in this case  $\alpha$  factors through a morphism  $\alpha_x : A_x \rightarrow B_x$  such that  $\alpha_x(a(x)) = \alpha(a)(x)$ .

Let  $X, Y$  be two locally compact spaces and  $p : Y \rightarrow X$  a continuous map. Then  $\mathcal{C}_0(Y)$  has an obvious structure of  $\mathcal{C}_0(X)$ -algebra: for  $f \in \mathcal{C}_0(X)$  and  $g \in \mathcal{C}_0(Y)$  we set  $(\rho(f)g)(y) = f \circ p(y)g(y)$ .

Let  $A$  be a  $\mathcal{C}_0(X)$ -algebra. Then  $A \otimes \mathcal{C}_0(Y)$  is a  $\mathcal{C}_0(X \times Y)$ -algebra. We set  $F = \{(p(y), y) : y \in Y\}$ . It is a closed subset of  $X \times Y$ . We put  $p^*A = (A \otimes \mathcal{C}_0(Y))_F$ . With its natural structure of  $\mathcal{C}_0(Y)$ -algebra,  $p^*A$  is called the *pull-back of  $A$  via  $p$* . Let us observe that  $(p^*A)_y = A_{p(y)}$ .

Let  $\alpha : A \rightarrow B$  be a morphism of  $\mathcal{C}_0(X)$ -algebras. Then

$$\alpha \otimes \text{Id} : A \otimes \mathcal{C}_0(Y) \rightarrow B \otimes \mathcal{C}_0(Y)$$

passes to the quotient and defines a morphism of  $\mathcal{C}_0(Y)$ -algebras  $p^*\alpha : p^*A \rightarrow p^*B$ .

**Definition 5.5.** ([41]) Let  $(\mathcal{G}, \lambda)$  be a locally compact groupoid with a Haar system and  $X = \mathcal{G}^{(0)}$ . An *action* of  $\mathcal{G}$  on a  $C^*$ -algebra  $A$  is given by a structure of  $\mathcal{C}_0(X)$ -algebra on  $A$  and an isomorphism  $\alpha : s^*A \rightarrow r^*A$  of  $\mathcal{C}_0(\mathcal{G})$ -algebras such that for every  $(\gamma_1, \gamma_2) \in \mathcal{G}^{(2)}$  we have  $\alpha_{\gamma_1\gamma_2} = \alpha_{\gamma_1}\alpha_{\gamma_2}$ , where  $\alpha_\gamma : A_{s(\gamma)} \rightarrow A_{r(\gamma)}$  is the isomorphism deduced from  $\alpha$  by factorization.

When  $A$  is equipped with such an action, we say that  $A$  is a  $\mathcal{G}$ - $C^*$ -algebra.

Let  $A$  be a  $\mathcal{G}$ - $C^*$ -algebra. We set  $\mathcal{C}_c(r^*(A)) = \mathcal{C}_c(\mathcal{G})r^*(A)$ . It is the space of the continuous sections with compact support of the upper semi-continuous field

of  $C^*$ -algebras defined by the  $\mathcal{C}_0(\mathcal{G})$ -algebra  $r^*A$ . Then,  $\mathcal{C}_c(\mathcal{G})r^*(A)$  is a  $*$ -algebra with respect to the following operations:

$$(f * g)(\gamma) = \int f(\gamma_1)\alpha_{\gamma_1}(g(\gamma_1^{-1}\gamma)) d\lambda^{r(\gamma)}(\gamma_1)$$

and

$$f^*(\gamma) = \alpha_\gamma(f(\gamma^{-1})^*)$$

(see [49, Proposition 4.4]). We define a norm on  $\mathcal{C}_c(r^*(A))$  by

$$\|f\|_1 = \max \left\{ \sup_{x \in X} \int \|f(\gamma)\| d\lambda^x(\gamma), \sup_{x \in X} \int \|f(\gamma^{-1})\| d\lambda^x(\gamma) \right\}.$$

The *full crossed product*  $A \rtimes \mathcal{G}$  is the enveloping  $C^*$ -algebra of the Banach  $*$ -algebra obtained by completion of  $\mathcal{C}_c(r^*(A))$  with respect to  $\|\cdot\|_1$ .

For  $x \in X$ , we consider the Hilbert  $A_x$ -module  $L^2(\mathcal{G}^x, \lambda^x) \otimes A_x$ . It is the completion of the space  $\mathcal{C}_c(\mathcal{G}^x, A_x)$  of continuous compactly supported functions on  $\mathcal{G}^x$  with values in  $A_x$ , with respect to the  $A_x$ -valued inner product

$$\langle \xi, \eta \rangle = \int \xi(\gamma)^* \eta(\gamma) d\lambda^x(\gamma).$$

For  $f \in \mathcal{C}_c(r^*(A))$  and  $\xi \in \mathcal{C}_c(\mathcal{G}^x, A_x)$ , we set

$$\Lambda_x(f)\xi(\gamma) = \int \alpha_\gamma(f(\gamma^{-1}\gamma_1))\xi(\gamma_1) d\lambda^{r(\gamma)}(\gamma_1). \quad (2)$$

Then  $\Lambda_x(f)$  extends to an element of  $\mathcal{B}_{A_x}((L^2(\mathcal{G}^x, \lambda^x) \otimes A_x))$  still denoted by  $\Lambda_x(f)$ . Moreover,  $\Lambda_x$  is a representation of  $A \rtimes \mathcal{G}$ . The *reduced crossed product*  $A \rtimes_r \mathcal{G}$  (also denoted by  $C_r^*(\mathcal{G}, A)$  in the sequel) is the quotient of  $A \rtimes \mathcal{G}$  with respect to the family of representations  $(\Lambda_x)_{x \in X}$  (see [31, Section 3.6]<sup>j</sup>).

As explained in [31] this family of representations come from a representation  $\Lambda$  of  $A \rtimes \mathcal{G}$  in the Hilbert  $A$ -module  $L_A^2(\mathcal{G}, \lambda)$  which is defined by completion of the right  $A$ -module  $\mathcal{C}_c(r^*(A))$  with respect to the  $A$ -valued inner product

$$\langle \xi, \eta \rangle(x) = \int_{\mathcal{G}^x} \xi(\gamma)^* \eta(\gamma) d\lambda^x(\gamma),$$

the structure of right  $A$ -module being given by  $(\xi a)(\gamma) = \xi(\gamma)a \circ r(\gamma)$ . We let  $\mathcal{C}_c(r^*(A))$  act on  $L_A^2(\mathcal{G}, \lambda)$  by

$$\Lambda(f)\xi = f \otimes \xi,$$

where  $(f \otimes \xi)(\gamma) = \int \alpha_\gamma(f(\gamma^{-1}\gamma_1))\xi(\gamma_1) d\lambda^{r(\gamma)}(\gamma_1)$ .

For  $x \in X$ , the map sending  $\xi \otimes b \in L_A^2(\mathcal{G}, \lambda) \otimes_{e_x} A_x$  onto

$$\gamma \in \mathcal{G}^x \mapsto \xi(\gamma)b$$

---

<sup>j</sup>As for the reduced  $C^*$ -algebra, we have made a different choice from that in [31] for the construction of  $A \rtimes_r \mathcal{G}$ . However, the constructions are easily seen to be isomorphic.

induces an isomorphism of Hilbert  $A_x$ -modules from  $L_A^2(\mathcal{G}, \lambda) \otimes_{e_x} A_x$  onto  $L^2(\mathcal{G}^x, \lambda^x) \otimes A_x$ . Under this identification,  $\Lambda(f) \otimes_{e_x} \text{Id}$  becomes  $\Lambda_x(f)$ . It follows that  $\Lambda$  extends to a  $*$ -homomorphism from  $A \rtimes \mathcal{G}$  into  $\mathcal{B}_A(L_A^2(\mathcal{G}, \lambda))$  with range isomorphic to  $A \rtimes_r \mathcal{G}$ .

**Lemma 5.6.** *Let  $(\mathcal{G}, \lambda)$  be a locally compact groupoid with Haar system. Let  $(Z, q_Z)$  and  $(Y, q_Y)$  be two  $\mathcal{G}$ -spaces and let  $p : Z \rightarrow Y$  be a continuous  $\mathcal{G}$ -equivariant morphism. Let  $A$  be a  $(Y \rtimes \mathcal{G})$ - $C^*$ -algebra.*

- (i)  $p^*A$  is in a natural way a  $(Z \rtimes \mathcal{G})$ - $C^*$ -algebra.
- (ii) If  $p$  is proper, then  $C_r^*(Y \rtimes \mathcal{G}, A)$  embeds canonically into  $C_r^*(Z \rtimes \mathcal{G}, p^*A)$ .

*Proof.* Let us first observe that  $Y$  and  $Z$  are  $(Y \rtimes \mathcal{G})$ -spaces in an obvious way, and that, by replacing  $Y \rtimes \mathcal{G}$  by  $\mathcal{G}$ , we may assume that  $Y = \mathcal{G}^{(0)}$  and therefore  $p = q_Z$ .

(i) We have to show that  $p^*A$  has a natural structure of  $(Z \rtimes \mathcal{G})$ -space. Let us denote for the moment by  $\mathcal{H}$  the groupoid  $Z \rtimes \mathcal{G}$  and by  $\underline{s}, \underline{r}$  its source and range maps respectively, in order to distinguish them from the source and range maps  $s : \mathcal{G} \rightarrow X$  and  $r : \mathcal{G} \rightarrow X$  respectively. Let  $P : \mathcal{H} \rightarrow \mathcal{G}$  be the groupoid homomorphism defined by  $P(z, \gamma) = \gamma$ . We observe that  $p \circ \underline{s} = s \circ P$  and that that  $p \circ \underline{r} = r \circ P$ . It follows that

$$\underline{s}^*(p^*A) = (p \circ \underline{s})^*A = (s \circ P)^*A = P^*(s^*A)$$

and similarly for  $r$  instead of  $s$ . Now, the structure of  $\mathcal{H}$ -algebra on  $p^*A$  is defined by the isomorphism  $\beta = P^*\alpha : P^*(s^*A) \rightarrow P^*(r^*A)$ . Note that  $(P^*(s^*A))_{(z, \gamma)} = A_{s(\gamma)}$ , that  $(P^*(r^*A))_{(z, \gamma)} = A_{r(\gamma)}$ , and that  $\beta_{(z, \gamma)} = \alpha_\gamma$ . For these facts we refer to [41].

(ii) Note that since  $p$  is proper, the map  $P$  is still proper. We define a  $*$ -homomorphism  $\Phi$  from the  $*$ -algebra  $\mathcal{C}_c(r^*(A))$  into the  $*$ -algebra

$$\mathcal{C}_c(\underline{r}^*p^*A) = \mathcal{C}_c(P^*r^*A)$$

by  $f \mapsto f \circ P$ . We have to show that this map is isometric. We have

$$\|f\|_{\mathcal{C}_r^*(\mathcal{G}, A)} = \sup_{x \in \mathcal{G}^{(0)}} \|\Lambda_x(f)\|$$

where  $\Lambda_x(f)$  acts on the completion  $L^2(\mathcal{G}^x, \lambda^x) \otimes A_x$  of  $\mathcal{C}_c(\mathcal{G}^x, A_x)$ . Now, we observe that if  $z \in Z$  is such that  $p(z) = x$ , then  $(Z \rtimes \mathcal{G})^z$  is canonically identified with  $\mathcal{G}^x$  and that  $(p^*A)_z = A_x$ . It follows that  $L^2(\mathcal{G}^x, \lambda^x) \otimes A_x$  is canonically identified with  $L^2((Z \rtimes \mathcal{G})^z, \lambda^{p(z)}) \otimes (p^*A)_z$  and that  $\Lambda_z(\Phi(f)) = \Lambda_x(f)$ . This concludes the proof since  $\|\Phi(f)\|_{\mathcal{C}_r^*(Z \rtimes \mathcal{G}, p^*A)} = \sup_{z \in Z} \|\Lambda_z(\Phi(f))\|$ .

□

## 6. AMENABILITY AT INFINITY VERSUS EXACTNESS

**6.1. Nuclearity.** One of our goals in this section is to show that if  $\mathcal{G}$  is a second countable amenable at infinity étale groupoid, then  $C_r^*(\beta_r\mathcal{G} \rtimes \mathcal{G})$  (or  $C_u^*(\mathcal{G})$ ) is nuclear. It is proved in [2, Corollary 6.2.14] that if  $Y$  is a second countable  $\mathcal{G}$ -space with  $\mathcal{G}$  étale and second countable, then the groupoid  $Y \rtimes \mathcal{G}$  is amenable (*i.e.*, the  $\mathcal{G}$ -action on  $Y$  is amenable) if and only if the  $C^*$ -algebra  $C_r^*(Y \rtimes \mathcal{G})$  is nuclear. The proof given there relies heavily on the separability assumption. Unfortunately, here  $\beta_r\mathcal{G}$  is not second countable and we have to extend the result of [2, Corollary 6.2.14] to this case.

Let us first recall the definition of nuclearity.

**Definition 6.1.** Let  $\Phi : A \rightarrow B$  be a completely positive contraction between two  $C^*$ -algebras. We say that  $\Phi$  is *factorable* if there exists an integer  $n$  and completely positive contractions  $\psi : A \rightarrow M_n(\mathbb{C})$ ,  $\varphi : M_n(\mathbb{C}) \rightarrow B$  such that  $\Phi = \varphi \circ \psi$ .

We say that  $\Phi$  is *nuclear* if there exists a net of factorable completely positive contractions  $\Phi_i : A \rightarrow B$  such that

$$\forall a \in A, \quad \lim_i \|\Phi(a) - \Phi_i(a)\| = 0.$$

We say that a  $C^*$ -algebra  $A$  is *nuclear* if  $\text{Id}_A : A \rightarrow A$  is nuclear.

An equivalent definition of nuclearity for a  $C^*$ -algebra  $A$  is the fact that, for every  $C^*$ -algebra  $B$  there is only one  $C^*$ -norm on the algebraic tensor product  $A \odot B$  (see [11, Theorem 3.8.7]).

**Proposition 6.2.** *Let  $\mathcal{G}$  be a second countable locally compact groupoid with Haar system and let  $(Z, q)$  be a fibrewise compact amenable  $\mathcal{G}$ -space. Then  $C_r^*(Z \rtimes \mathcal{G})$  is nuclear in either case:*

- (i)  $Z$  is second countable;
- (ii)  $\mathcal{G}$  is étale.

*Proof.* (i) follows from [2, Corollary 6.2.14], whose proof requires the separability assumptions. To prove (ii), the only difficulty is that we have no separability assumption on  $Z$ . We denote by  $\mathcal{F}$  the set of finite subsets of  $\mathcal{C}_0(Z)$ , ordered by inclusion. We shall construct a family  $(A_F)_{F \in \mathcal{F}}$  of nuclear  $C^*$ -subalgebras of  $C_r^*(Z \rtimes \mathcal{G})$  such that  $A_{F_1} \subset A_{F_2}$  when  $F_1 \subset F_2$  and  $C_r^*(Z \rtimes \mathcal{G}) = \overline{\cup_{F \in \mathcal{F}} A_F}$ . Then  $C_r^*(Z \rtimes \mathcal{G})$  will be nuclear by [37].

*Construction of  $A_F$ .* We choose a second countable fibrewise compact amenable  $\mathcal{G}$ -space  $(Y, p)$  and  $q_Y : Z \rightarrow Y$  as in Lemma 3.8. Recall that  $p$  and  $q_Y$  are proper. We view  $Y$  and  $Z$  as  $(Y \rtimes \mathcal{G})$ -spaces in an obvious way. We denote by  $B_F$  the smallest  $C^*$ -subalgebra of  $\mathcal{C}_0(Z)$  which contains  $q_Y^* \mathcal{C}_0(Y) = \{f \circ q_Y : f \in \mathcal{C}_0(Y)\}$

and  $F$ , and is stable under convolution by the elements of  $\mathcal{C}_c(Y \rtimes \mathcal{G})$ . It is a separable abelian  $C^*$ -algebra and therefore its spectrum  $Y_F$  is second countable. Let  $q_F : Z \rightarrow Y_F$  (resp.  $p_F : Y_F \rightarrow Y$ ) be the continuous surjective map corresponding to the inclusion  $B_F \subset \mathcal{C}_0(Z)$  (resp.  $\mathcal{C}_0(Y) \subset B_F$ ). Note that  $p_F \circ q_F = q_Y$  is a proper map and therefore  $p_F$  and  $q_F$  are proper. Since  $B_F$  is stable under convolution by the elements of  $\mathcal{C}_c(Y \rtimes \mathcal{G})$ , using [4, Proposition 2.9] we see that  $(Y_F, p_F)$  has a unique structure of  $(Y \rtimes \mathcal{G})$ -space (or equivalently of  $\mathcal{G}$ -space) which makes  $q_F$  equivariant. By [2, Proposition 2.2.9], the  $\mathcal{G}$ -space  $Y_F$  is amenable and so, by [2, Corollary 6.2.14], the  $C^*$ -algebra  $C_r^*(Y_F \rtimes \mathcal{G})$  is nuclear.

Since  $q_F$  is proper,  $A_F = C_r^*(Y_F \rtimes \mathcal{G})$  is canonically embedded in  $C_r^*(Z \rtimes \mathcal{G})$  by Lemma 5.6. Moreover, if  $F_1 \subset F_2$  we have  $A_{F_1} \subset A_{F_2}$ .

*Proof of  $C_r^*(Z \rtimes \mathcal{G}) = \overline{\cup_{F \in \mathcal{F}} A_F}$ .* It suffices to show that every  $f \in \mathcal{C}_c(Z \rtimes \mathcal{G})$  belongs to  $\cup_{F \in \mathcal{F}} A_F$ . There exists a compact subset  $K$  of  $\mathcal{G}$  such that the support of  $f$  is contained in  $q^{-1}(r(K)) * K$ . Using a finite covering of  $K$  by open bisections and a corresponding partition of units, it suffices to consider the case where  $K$  is contained in an open bisection  $S$ . For  $z \in Z$ , we set  $\tilde{f}(z) = f(z, r_S^{-1}(q(z)))$  if  $z \in q^{-1}(r(S))$  and  $\tilde{f}(z) = 0$  otherwise. Then  $\tilde{f} \in \mathcal{C}_c(Z)$  and therefore belongs to some  $\mathcal{C}_c(Y_F)$ . Let  $\varphi \in \mathcal{C}_c(\mathcal{G})$ , with support contained in  $S$  and equals to 1 on  $K$ . Then for  $(z, \gamma) \in Z \rtimes \mathcal{G}$ , we have  $f(z, \gamma) = \tilde{f}(z)\varphi(\gamma) \in \mathcal{C}_c(Y_F \rtimes \mathcal{G})$ . It follows that  $f \in A_F$ .  $\square$

**6.2. Exactness.** Let us recall the definition of this notion, which is weaker than nuclearity.

**Definition 6.3.** We say that a  $C^*$ -algebra  $A$  is *exact* if for every short exact sequence

$$0 \rightarrow J \rightarrow B \rightarrow B/J \rightarrow 0$$

of  $C^*$ -algebras, the following sequence

$$0 \rightarrow A \otimes J \rightarrow A \otimes B \rightarrow A \otimes (B/J) \rightarrow 0$$

is exact, where  $\otimes$  denotes the minimal (or spatial) tensor product.

A deep result of Kirchberg states that  $A$  is exact if and only if there exists an Hilbert space  $\mathcal{H}$  and a nuclear embedding of  $A$  into  $\mathcal{B}(\mathcal{H})$  (or equivalently a nuclear embedding in some  $C^*$ -algebra). In particular, every  $C^*$ -subalgebra of a nuclear  $C^*$ -algebra is exact. We refer to [69] for more details relative to these results.

Recall that if  $(Y, p)$  is a fibrewise compact  $\mathcal{G}$ -space, then  $C_r^*(\mathcal{G})$  embeds canonically into  $C_r^*(Y \rtimes \mathcal{G})$ . As an immediate consequence of Proposition 6.2 we get:

**Corollary 6.4.** *Let  $\mathcal{G}$  be a second countable étale groupoid and consider the following conditions:*

- (1)  $\mathcal{G}$  is strongly amenable to infinity.



- (2)  $\mathcal{G}$  is amenable to infinity.
- (3)  $C_u^*(\mathcal{G})$  is nuclear.
- (4)  $C_u^*(\mathcal{G})$  is exact.
- (5)  $C_r^*(\mathcal{G})$  is exact.

Then (1)  $\Rightarrow$  (3)  $\Rightarrow$  (4)  $\Rightarrow$  (5) and (1)  $\Rightarrow$  (2)  $\Rightarrow$  (5) .

**Definition 6.5.** Let  $\mathcal{G}$  be a locally compact groupoid and  $A, B$  two  $\mathcal{G}$ - $C^*$ -algebras. A  $\mathcal{G}$ -equivariant morphism  $\rho : A \rightarrow B$  is a morphism of  $\mathcal{C}_0(X)$ -algebras such that  $\beta \circ (s^* \rho) = (r^* \rho) \circ \alpha$ , where  $\alpha$  and  $\beta$  denote the  $\mathcal{G}$ -actions on  $A$  and  $B$  respectively.

**Definition 6.6.** We say that a locally compact groupoid  $\mathcal{G}$  with Haar system is  $C^*$ -exact if  $C_r^*(\mathcal{G})$  is exact. We say that it is exact in the sense of Kirchberg and Wassermann (or  $KW$ -exact) if for every  $\mathcal{G}$ -equivariant exact sequence

$$0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$$

of  $\mathcal{G}$ - $C^*$ -algebras, the corresponding sequence

$$0 \rightarrow C_r^*(\mathcal{G}, I) \rightarrow C_r^*(\mathcal{G}, A) \rightarrow C_r^*(\mathcal{G}, B)$$

of reduced crossed products is exact.

**Proposition 6.7.** Let  $\mathcal{G}$  be a second countable locally compact groupoid with Haar system. Consider the following conditions:

- (1)  $\mathcal{G}$  acts amenably on a second countable fibrewise compact fibre space (for instance  $\mathcal{G}$  is étale and amenable at infinity).
- (2)  $\mathcal{G}$  is  $KW$ -exact.
- (3)  $C_r^*(\mathcal{G})$  is exact.

Then (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3).

If moreover  $\mathcal{G}$  is étale, then (2)  $\Rightarrow C_u^*(\mathcal{G})$  exact  $\Rightarrow$  (3).

Before proceeding to the proof, we need some preliminaries.

**Lemma 6.8.** Let  $A$  be a  $\mathcal{G}$ - $C^*$ -algebra with  $\mathcal{G}$ -action  $\alpha$ .

- (i) Given  $a \in A$  and  $\xi \in \mathcal{C}_c(r^*(A))$ , we set  $(m_a \xi)(\gamma) = \alpha_\gamma(a(s(\gamma)))\xi(\gamma)$ . Then  $m_a$  extends to an operator  $M_a \in \mathcal{B}_A(L_A^2(\mathcal{G}, \lambda))$ .
- (ii)  $M_a$  is a two-sided multiplier of  $C_r^*(\mathcal{G}, A)$ . More precisely, for  $f \in \mathcal{C}_c(r^*(A))$  and  $a \in A$  we have

$$\Lambda(f)M_a = \Lambda(f \cdot a), \quad \text{where } (f \cdot a)(\gamma) = f(\gamma)\alpha_\gamma(a(s(\gamma))),$$

and

$$M_a \Lambda(f) = \Lambda(a \cdot f), \quad \text{where } (a \cdot f)(\gamma) = a(r(\gamma))f(\gamma).$$

- (iii) Let  $(u_k)$  be an approximate unit of  $A$ . Then, for  $f \in C_r^*(\mathcal{G}, A)$ , we have

$$\lim_k \|u_k \cdot f - f\|_{C_r^*(\mathcal{G}, A)} = 0.$$

*Proof.* (i) is obvious.

(ii) Given  $f, \xi \in \mathcal{C}_c(r^*(A))$ ,  $a \in A$ , we have

$$\begin{aligned} (\Lambda(f)M_a\xi)(\gamma) &= (f \otimes M_a\xi)(\gamma) \\ &= \int \alpha_\gamma(f(\gamma^{-1}\gamma_1))\alpha_{\gamma_1}(s(\gamma_1))\xi(\gamma_1) d\lambda^{r(\gamma)}(\gamma_1) \\ &= \int \alpha_\gamma\left(f(\gamma^{-1}\gamma_1)\alpha_{\gamma^{-1}\gamma_1}(s(\gamma^{-1}\gamma_1))\right)\xi(\gamma_1) d\lambda^{r(\gamma)}(\gamma_1) \\ &= \Lambda(f \cdot a)\xi(\gamma). \end{aligned}$$

Similarly, one shows that  $M_a\Lambda(f) = \Lambda(a \cdot f)$ . It follows that  $M_a$  is a two-sided multiplier of  $C_r^*(\mathcal{G}, A)$ .

(iii) It suffices to consider the case where  $f \in \mathcal{C}_c(r^*(A))$ . We have to show that  $\lim_k \|u_k f - f\|_{C_r^*(\mathcal{G}, A)} = 0$  (where we write  $u_k f$  instead of  $u_k \cdot f$ . By [31, Section 3.6], we have

$$\|u_k f - f\|_{C_r^*(\mathcal{G}, A)} \leq \|u_k f - f\|_1.$$

We are going to show that  $\lim_k \|u_k f - f\|_1 = 0$ . Let  $K$  be a compact subset of  $\mathcal{G}$  which contains the support of  $f$  and let  $c > 0$  be such that

$$\max(\sup_{x \in X} \lambda^x(K), \sup_{x \in X} \lambda_x(K)) \leq c.$$

We set  $c' = \sup_{\gamma \in \mathcal{G}} \|f(\gamma)\|$ .

Recall that

$$(u_k f - f)(\gamma) = u_k(r(\gamma))f(\gamma) - f(\gamma) \in A_{r(\gamma)}.$$

We fix  $\gamma \in K$  and choose  $a \in A$  with  $a(r(\gamma)) = f(\gamma)$ . Given  $\varepsilon' > 0$ , there exists  $k_\gamma$  such that  $\|u_{k_\gamma} a - a\| < \varepsilon'$ . In particular, we have  $\|u_{k_\gamma}(r(\gamma))f(\gamma) - f(\gamma)\| < \varepsilon'$ . Using the upper semi-continuity of the norm, we see that there is a neighborhood  $V_\gamma$  of  $\gamma$  such that

$$\|u_{k_\gamma}(r(\gamma'))f(\gamma') - f(\gamma')\| \leq \varepsilon'$$

for  $\gamma' \in V_\gamma$ .

The compact space  $K$  is covered by finitely many such  $V_\gamma$ 's, that are denoted  $V_{\gamma_i}$ ,  $i = 1, \dots, n$ . Let  $k$  be such that  $\|u_k u_{k_{\gamma_i}} - u_{k_{\gamma_i}}\| \leq \varepsilon'$  for  $i = 1, \dots, n$ . Take  $\gamma \in K$  and choose  $V_{\gamma_i}$  such that  $\gamma \in V_{\gamma_i}$ . We have

$$\begin{aligned} \|u_k(r(\gamma))f(\gamma) - f(\gamma)\| &\leq \left\| u_k(r(\gamma))\left(f(\gamma) - u_{k_{\gamma_i}}(r(\gamma))f(\gamma)\right) \right\| + \\ &\quad \left\| \left(u_k(r(\gamma))u_{k_{\gamma_i}}(r(\gamma)) - u_{k_{\gamma_i}}(r(\gamma))\right)f(\gamma) \right\| + \left\| u_{k_{\gamma_i}}(r(\gamma))f(\gamma) - f(\gamma) \right\| \\ &\leq 2\varepsilon' + \varepsilon'c'. \end{aligned}$$

It follows that  $\sup_{\gamma \in \mathcal{G}} \|(u_k f - f)(\gamma)\| \leq 2\varepsilon' + \varepsilon' c'$  and therefore  $\|(u_k f - f)\|_1 \leq (2\varepsilon' + \varepsilon' c')c$ .  $\square$

*Proof of Proposition 6.7.* (1)  $\Rightarrow$  (2) Let  $Y$  be an amenable second countable  $\mathcal{G}$ -space with  $p : Y \rightarrow X = \mathcal{G}^{(0)}$  proper and let

$$0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$$

be an equivariant exact sequence of  $\mathcal{G}$ - $C^*$ -algebras. Then

$$0 \rightarrow p^* I \rightarrow p^* A \rightarrow p^* B \rightarrow 0$$

is an equivariant exact sequence of  $(Y \rtimes \mathcal{G})$ - $C^*$ -algebras. Taking the reduced crossed products we obtain the commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & C_r^*(\mathcal{G}, I) & \rightarrow & C_r^*(\mathcal{G}, A) & \rightarrow & C_r^*(\mathcal{G}, B) \rightarrow 0 \\ & & i_I \downarrow & & i_A \downarrow & & i_B \downarrow \\ 0 & \rightarrow & C_r^*(Y \rtimes \mathcal{G}, p^* I) & \rightarrow & C_r^*(Y \rtimes \mathcal{G}, p^* A) & \rightarrow & C_r^*(Y \rtimes \mathcal{G}, p^* B) \rightarrow 0 \end{array}$$

The vertical arrows were introduced in Lemma 5.6 and shown to be injective, due to the fact that  $p$  is proper. Since  $Y \rtimes \mathcal{G}$  is second countable and amenable the reduced crossed products of the second line are also full crossed products [2, Corollary 6.2.14], and therefore the second line is exact (see [2, Lemma 6.3.2]).

Let us show that the first line is exact in the middle. Assume that  $a \in C_r^*(\mathcal{G}, A)$  is sent onto  $0 \in C_r^*(\mathcal{G}, B)$ . Then  $i_A(a)$  belongs to  $C_r^*(Y \rtimes \mathcal{G}, p^* I)$  due to the exactness of the second line. This forces  $a$  to belong to  $C_r^*(\mathcal{G}, I)$ . Indeed, let  $(u_k)$  be a bounded approximate unit of  $I$ . Then, by Lemma 6.8, we have

$$i_A(a) = \lim_k i_I(u_k) i_A(a) = \lim_k i_A(u_k a).$$

Next we observe that  $u_k a \in C_r^*(\mathcal{G}, I)$ . It suffices to consider the case where  $a \in \mathcal{C}_c(r^* A)$ . But then, for  $\gamma \in \mathcal{G}$ , we have  $(u_k a)(\gamma) = u_k(r(\gamma))a(\gamma) \in I_{r(\gamma)}$  and so, by [13, Lemma 2.1 (iii)], we see that  $u_k a \in \mathcal{C}_c(r^* I)$ . Therefore, we have

$$i_A(a) = \lim_k i_I(u_k a) \in i_I(C_r^*(\mathcal{G}, I))$$

and we conclude that  $a \in C_r^*(\mathcal{G}, I)$ .

(2)  $\Rightarrow$  (3). Let

$$0 \rightarrow I \rightarrow A \rightarrow B \rightarrow 0$$

be an exact sequence of  $C^*$ -algebras. Then

$$0 \rightarrow I \otimes \mathcal{C}_0(X) \rightarrow A \otimes \mathcal{C}_0(X) \rightarrow B \otimes \mathcal{C}_0(X) \rightarrow 0$$

is a  $\mathcal{G}$ -equivariant exact sequence of  $\mathcal{G}$ - $C^*$ -algebras. Assuming that (2) holds we see that

$$0 \rightarrow C_r^*(\mathcal{G}, I \otimes \mathcal{C}_0(X)) \rightarrow C_r^*(\mathcal{G}, A \otimes \mathcal{C}_0(X)) \rightarrow C_r^*(\mathcal{G}, B \otimes \mathcal{C}_0(X)) \rightarrow 0$$

is an exact sequence. Then we have just to observe that it coincides with the sequence

$$0 \rightarrow I \otimes C_r^*(\mathcal{G}) \rightarrow A \otimes C_r^*(\mathcal{G}) \rightarrow B \otimes C_r^*(\mathcal{G}) \rightarrow 0.$$

Assume now that  $\mathcal{G}$  is étale and that (2) holds. Then  $C_u^*(\mathcal{G}) = C_r^*(\beta_r \mathcal{G} \rtimes \mathcal{G})$ . Using [4, Proposition 2.9] as in the proof of Proposition 6.2, we see that there is an increasing net  $(A_F)$  of crossed products  $A_F = C_r^*(Y_F \rtimes \mathcal{G})$  with each  $Y_F$  separable, such that  $C_r^*(\beta_r \mathcal{G} \rtimes \mathcal{G}) = \overline{\bigcup_{F \in \mathcal{F}} A_F}$ . Since  $\mathcal{G}$  is KW-exact, then each  $A_F$  is exact and therefore  $C_r^*(\beta_r \mathcal{G} \rtimes \mathcal{G})$  is exact.

Finally, if  $C_u^*(\mathcal{G})$  is exact, then  $C_r^*(\mathcal{G})$ , which is contained in  $C_u^*(\mathcal{G})$ , is exact.  $\square$

*Remark 6.9.* KW-exactness of groupoids has been studied by Lalonde in [35, 36]. In [36], it is proved that equivalence of groupoids preserves KW-exactness. In [35], it is proved that the reduced crossed product  $C_r^*(\mathcal{G}, A)$  is exact whenever  $\mathcal{G}$  is a KW-exact second countable locally compact groupoid acting on a separable exact  $C^*$ -algebra  $A$ .

*Remark 6.10.* We end this section by stating immediate consequences of amenability at infinity and KW-exactness.

Let  $\mathcal{G}$  be a locally compact groupoid with a Haar system. We say that a subset  $E$  of  $X = \mathcal{G}^{(0)}$  is *invariant* if  $s(\gamma) \in E$  if and only if  $r(\gamma) \in E$ . Let  $F$  be a closed invariant subset of  $X$  and set  $U = X \setminus F$ . It is well known that the inclusion  $\iota : \mathcal{C}_c(\mathcal{G}(U)) \rightarrow \mathcal{C}_c(\mathcal{G})$  extends to an injective homomorphism from  $C^*(\mathcal{G}(U))$  into  $C^*(\mathcal{G})$  and from  $C_r^*(\mathcal{G}(U))$  into  $C_r^*(\mathcal{G})$ . Similarly, the restriction map  $\pi : \mathcal{C}_c(\mathcal{G}) \rightarrow \mathcal{C}_c(\mathcal{G}(F))$  extends to a surjective homomorphism from  $C^*(\mathcal{G})$  onto  $C^*(\mathcal{G}(F))$  and from  $C_r^*(\mathcal{G})$  onto  $C_r^*(\mathcal{G}(F))$ . Moreover the sequence

$$0 \rightarrow C^*(\mathcal{G}(U)) \rightarrow C^*(\mathcal{G}) \rightarrow C^*(\mathcal{G}(F)) \rightarrow 0$$

is exact. For these facts, we refer to [57, page 102], [25, Section 2.4], or to [56, Proposition 2.4.2] for a detailed proof. On the other hand, the corresponding sequence with respect to the reduced  $C^*$ -algebras is not always exact, as shown by Skandalis in the Appendix of [58]. However, it is exact whenever  $\mathcal{G}$  is KW-exact.

Assume now that  $\mathcal{G}$  is amenable at infinity. Then all its isotropy groups  $\mathcal{G}(x)$ ,  $x \in X = \mathcal{G}^{(0)}$ , are amenable at infinity by Proposition 3.5 applied to  $E = \{x\}$ . It follows that  $\mathcal{G}(x)$  is KW-exact and that  $C_r^*(\mathcal{G}(x))$  is exact (see [3, Theorem 7.2]).

7. A SUFFICIENT CONDITION FOR EXACTNESS TO IMPLY AMENABILITY AT INFINITY

The main result of this section is Theorem 7.6. Its proof requires some preliminaries. First, we shall need the following slight extension of the classical Kirchberg's characterization of exactness.

**Lemma 7.1.** *Let  $A, B$  be two separable  $C^*$ -algebras, where  $B$  is nuclear. Let  $\mathcal{E}$  be a countably generated Hilbert  $C^*$ -module over  $B$ . Let  $\iota : A \rightarrow \mathcal{B}_B(\mathcal{E})$  be an embedding of  $C^*$ -algebras. Then  $A$  is exact if and only if  $\iota$  is nuclear.*

*Proof.* If  $\iota$  is nuclear, it is well known that  $A$  is exact (see [69, Proposition 7.2]). Conversely, assume that  $A$  is exact. We identify  $A$  with its image by  $\iota$ . We denote by  $\tilde{A}$  the  $C^*$ -algebra obtained by adjunction of a unit 1 to  $A$ . We assume that  $\tilde{A}$  is embedded into  $\mathcal{B}(H)$  for some separable Hilbert space  $H$  and that 1 is the unit of  $\mathcal{B}(H)$ . We set  $H_\infty = \ell^2(\mathbb{N}) \otimes H$  and denote by  $i$  the embedding of  $\tilde{A}$  into  $\mathcal{B}(H_\infty)$  sending  $a$  to  $1_{\ell^2(\mathbb{N})} \otimes a$ . Observe that  $i(\tilde{A}) \cap \mathcal{K}(H_\infty) = \{0\}$ , where  $\mathcal{K}(H_\infty)$  is the  $C^*$ -algebra of compact operators on  $H_\infty$ . We now identify  $\tilde{A}$  with its image in  $\mathcal{B}(H_\infty)$ .

Using Kasparov's stabilization theorem [38, Corollary 6.3],  $\mathcal{E}$  is a direct factor of the Hilbert  $B$ -module  $H_\infty \otimes B$ . So we may assume that  $\mathcal{E} = H_\infty \otimes B$ . We identify  $\mathcal{E}$  to  $B \oplus \mathcal{E}'$  with  $\mathcal{E}' = H_\infty \otimes B$  and we define a unital representation  $\pi$  of  $\tilde{A}$  into  $\mathcal{B}_B(\mathcal{E})$  by

$$(\lambda, a)(b, \xi) = (\lambda b, \lambda \xi + a \xi),$$

where  $\lambda, \mu \in \mathbb{C}$ ,  $a \in A$ ,  $b \in B$  and  $\xi \in \mathcal{E}'$ .

Recall that  $\mathcal{B}_B(\mathcal{E})$  is the multiplier algebra  $M(\mathcal{K}(H_\infty) \otimes B)$  of  $\mathcal{K}(H_\infty) \otimes B$  (see [38, Theorem 2.4]). We denote by  $\tau$  the embedding of  $\tilde{A}$  into  $\mathcal{B}_B(\mathcal{E})$  obtained by composition of the canonical embeddings of  $\tilde{A}$  into  $\mathcal{B}(H_\infty)$  and of  $\mathcal{B}(H_\infty)$  into  $\mathcal{B}_B(\mathcal{E})$ .

Let  $\varepsilon > 0$  and  $a_1, \dots, a_n \in A$  be given. Since  $B$  is nuclear, the Kasparov-Voiculescu theorem [27, Theorem 6] implies the existence of a unitary operator  $U \in \mathcal{B}_B(\mathcal{E}, \mathcal{E} \oplus \mathcal{E})$  such that

$$\text{for } 1 \leq k \leq n, \quad \|U\tau(a_k)U^* - (\tau \oplus \pi)(a_k)\| \leq \varepsilon.$$

Since  $\tilde{A}$  is exact and since  $\tau$  factors through  $\mathcal{B}(H_\infty)$ , there exists a factorable completely positive contraction  $\Phi : \tilde{A} \rightarrow \mathcal{B}_B(\mathcal{E})$  such that

$$\text{for } 1 \leq k \leq n, \quad \|\Phi(a_k) - \tau(a_k)\| \leq \varepsilon.$$

It follows that

$$\text{for } 1 \leq k \leq n, \quad \|U\Phi(a_k)U^* - \tau(a_k) \oplus \pi(a_k)\| \leq 2\varepsilon.$$

Denoting by  $P$  the projection of  $\mathcal{E} \oplus \mathcal{E}$  onto its second component, we get

$$\text{for } 1 \leq k \leq n, \quad \|PU\Phi(a_k)U^*P - \pi(a_k)\| \leq 2\varepsilon.$$

For  $a \in A \subset \tilde{A}$  and  $\xi \in H_\infty \otimes B = \mathcal{E}'$ , we have  $\pi(a)(0, \xi) = (0, a\xi)$ . Let  $Q$  be the projection of  $\mathcal{E}$  onto its summand  $\mathcal{E}'$ . Then we get

$$\text{for } 1 \leq k \leq n, \quad \|QPU\Phi(a_k)U^*PQ - a_k\| \leq 2\varepsilon.$$

This shows that the embedding of  $A$  into  $\mathcal{B}_B(H_\infty \otimes B)$  is nuclear.  $\square$

The major part of the rest of this section 7 is adapted from an unpublished note of Jean Renault that we thank for allowing us to use his ideas.

**Definition 7.2.** Let  $(\mathcal{G}, \lambda)$  be a locally compact groupoid with a Haar system and let  $B$  be a  $C^*$ -algebra. A map  $\Phi : C_r^*(\mathcal{G}) \rightarrow B$  is said to have a *compact support* if there exists a compact subset  $K$  of  $\mathcal{G}$  such that  $\Phi(f) = 0$  for every  $f \in \mathcal{C}_c(\mathcal{G})$  with  $(\text{Supp } f) \cap K = \emptyset$ .

**Lemma 7.3** (Renault). *Let  $(\mathcal{G}, \lambda)$  be a locally compact groupoid with a Haar system, and let  $\Phi : C_r^*(\mathcal{G}) \rightarrow M_n(\mathbb{C})$  be a completely positive map. Then for every  $\varepsilon > 0$  and every finite subset  $F$  of  $C_r^*(\mathcal{G})$  there exists a completely positive map  $\Psi : C_r^*(\mathcal{G}) \rightarrow M_n(\mathbb{C})$  with compact support such that  $\|\Psi(a) - \Phi(a)\| \leq \varepsilon$  for  $a \in F$ .*

*Proof.* Using the Stinespring dilation theorem, we get a representation  $\rho$  of  $C_r^*(\mathcal{G})$  into some Hilbert space  $H_\rho$  and vectors  $e_1, \dots, e_n \in H_\rho$  such that, for  $a \in C_r^*(\mathcal{G})$ , we have

$$\Phi(a) = [\langle e_i, \rho(a)e_j \rangle] \in M_n(\mathbb{C}).$$

Let  $\mathcal{E}$  be the Hilbert module  $L_{\mathcal{C}_0(X)}^2(\mathcal{G}, \lambda)$  over  $\mathcal{C}_0(X)$ . Let  $\mu$  be a probability measure on  $X = \mathcal{G}^{(0)}$  with support  $X$  and define on  $\mathcal{G}$  the measure  $\mu \circ \lambda$  by

$$\int_{\mathcal{G}} f \, d\mu \circ \lambda = \int_X \left( \int_{\mathcal{G}^x} f(\gamma) \, d\lambda^x(\gamma) \right) d\mu(x).$$

We consider the faithful representation  $f \mapsto \tilde{\Lambda}(f) = \Lambda(f) \otimes \text{Id}$  in the Hilbert space  $\mathcal{E} \otimes_{\mathcal{C}_0(X)} L^2(X, \mu)$ . Observe that this Hilbert space is canonically isomorphic to the Hilbert space  $L^2(\mathcal{G}, \mu \circ \lambda)$  and that, for  $f \in \mathcal{C}_c(\mathcal{G})$  and  $\xi \in L^2(\mathcal{G}, \mu \circ \lambda)$  we have

$$(\tilde{\Lambda}(f)\xi)(\gamma) = \int f(\gamma^{-1}\gamma_1)\xi(\gamma_1) \, d\lambda^{r(\gamma)}(\gamma_1).$$

Since the representation  $\rho$  is weakly contained in  $\tilde{\Lambda}$ , given  $\varepsilon' > 0$ , there exists a multiple  $\tilde{\Lambda}_K = \tilde{\Lambda} \otimes \text{Id}_K$ , where  $K$  is some separable Hilbert space, and vectors  $\xi_1, \dots, \xi_n$  in  $L^2(\mathcal{G}, \mu \circ \lambda) \otimes K = L^2(\mathcal{G}, \mu \circ \lambda, K)$  such that

$$\left| \langle e_i, \rho(a)e_j \rangle - \langle \xi_i, \tilde{\Lambda}_K(a)\xi_j \rangle \right| < \varepsilon'$$

for  $a \in F$  and  $i, j \in \{1, \dots, n\}$ . Moreover, we may choose the  $\xi_i$ 's to have a compact support. For  $f \in \mathcal{C}_c(\mathcal{G})$ , we have

$$\langle \xi_i, \tilde{\Lambda}_K(f)\xi_j \rangle = \int \langle \xi_i(\gamma), \xi_j(\gamma_1) \rangle f(\gamma^{-1}\gamma_1) d\lambda^x(\gamma) d\lambda^x(\gamma_1) d\mu(x).$$

It follows that the completely positive map

$$a \mapsto \Psi(a) = \left[ \langle \xi_i, \tilde{\Lambda}_K(a)\xi_j \rangle \right]$$

has a compact support and satisfies  $\|\Psi(a) - \Phi(a)\| \leq \varepsilon$  for  $a \in F$  if  $\varepsilon'$  is small enough.  $\square$

**Corollary 7.4.** *Let  $B$  be a  $C^*$ -algebra and  $\Phi : C_r^*(\mathcal{G}) \rightarrow B$  be a nuclear completely positive map. Then for every  $\varepsilon > 0$  and every  $a_1, \dots, a_k \in C_r^*(\mathcal{G})$  there exists a factorable completely positive map  $\Psi : C_r^*(\mathcal{G}) \rightarrow B$ , with compact support, such that  $\|\Psi(a_i) - \Phi(a_i)\| \leq \varepsilon$  for  $i = 1, \dots, k$ .*

The notion of positive definite function extends to the case of  $f : \mathcal{G} \rightarrow B$  in an obvious way: it is *positive definite* if for every  $x \in X$ ,  $n \in \mathbb{N}$  and  $\gamma_1, \dots, \gamma_n \in \mathcal{G}^x$ , then  $[f(\gamma_i^{-1}\gamma_j)]$  is a positive element of  $M_n(B)$ . In the next lemma, we shall use the following observation:  *$f$  is positive definite if and only if for every finite set  $I$  and every groupoid homomorphism  $\theta : I \times I \rightarrow \mathcal{G}$ , where  $I \times I$  is the trivial groupoid on  $I$ , the element  $[f \circ \theta(i, j)]$  of the  $C^*$ -algebra  $M_I(B)$  of matrices over  $I \times I$  with coefficients in  $B$  is positive.* Indeed, assume that this property holds. Let  $\gamma_1, \dots, \gamma_n \in \mathcal{G}^x$  be given. We set  $I = \{1, \dots, n\}$  and  $\theta(i, j) = \gamma_i^{-1}\gamma_j$ . Then  $\theta : I \times I \rightarrow \mathcal{G}$  is a groupoid homomorphism and the matrix  $[f(\gamma_i^{-1}\gamma_j)] = [f \circ \theta(i, j)]$  is positive. The converse assertion is also easy.

Given  $f : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{C}$ , we set  $f_\gamma(\gamma') = f(\gamma, \gamma')$ .

**Lemma 7.5** (Renault). *Let  $(\mathcal{G}, \lambda)$  be a locally compact groupoid with a Haar system.*

- (a) *Let  $f \in \mathcal{C}_c(\mathcal{G})$  be a continuous positive definite function. Then,  $f$  viewed as an element of  $C_r^*(\mathcal{G})$  is a positive element.*
- (b) *Let  $f : \mathcal{G} \times \mathcal{G} \rightarrow \mathbb{C}$  be a properly supported positive definite function. Then  $\gamma \mapsto f_\gamma$  is a continuous positive definite function from  $\mathcal{G}$  into  $C_r^*(\mathcal{G})$ .*

*Proof.* (a) Let  $\mathcal{E} = L_{\mathcal{C}_0(X)}^2(\mathcal{G}, \lambda)$ . We have to show that  $\Lambda(f) \in \mathcal{B}_{\mathcal{C}_0(X)}(\mathcal{E})$  is positive, which amounts to prove that for every  $x \in \mathcal{G}^{(0)}$  and  $\xi \in L^2(\mathcal{G}^x, \lambda^x)$ , we have  $\langle \xi, \Lambda_x(f)\xi \rangle \geq 0$ . It suffices to consider the case where  $\xi \in \mathcal{C}_c(\mathcal{G}^x)$ . Let  $K$  be the support of  $\xi$ . Since

$$\langle \xi, \Lambda_x(f)\xi \rangle = \int \overline{\xi(\gamma)} f(\gamma^{-1}\gamma_1) \xi(\gamma_1) d\lambda^x(\gamma) d\lambda^x(\gamma_1),$$

by approximating the restriction of  $\lambda^x$  to  $K$  by positive measures with finite support, we get the wanted positivity result.

(b) Since  $f$  is properly supported, the map  $\gamma \mapsto f_\gamma$  is continuous from  $\mathcal{G}$  into  $\mathcal{C}_c(\mathcal{G})$  endowed with the inductive limit topology, and therefore from  $\mathcal{G}$  into  $C_r^*(\mathcal{G})$  endowed with its norm topology. To show that  $\gamma \mapsto F(\gamma) = f_\gamma$  is a continuous positive definite function, let  $I$  be a finite set and  $\theta : I \times I \rightarrow \mathcal{G}$  be a groupoid homomorphism. We have to check that  $[F \circ \theta(i, j)] \in M_I(C_r^*(\mathcal{G}))$  is positive. But  $M_I(C_r^*(\mathcal{G})) = C_r^*((I \times I) \times \mathcal{G})$ . Therefore, by the first part of the lemma applied to the product groupoid  $(I \times I) \times \mathcal{G}$ , it suffices to show that

$$((i, j), \gamma) \mapsto f_{\theta(i, j)}(\gamma) = f(\theta(i, j), \gamma)$$

belongs to  $\mathcal{C}_c((I \times I) \times \mathcal{G})$  and is positive definite. But this is clear, since this function is obtained by composing  $f$  with the homomorphism  $\theta \times \text{Id} : (I \times I) \times \mathcal{G} \rightarrow \mathcal{G} \times \mathcal{G}$ .  $\square$

**Theorem 7.6.** *Let  $\mathcal{G}$  be a second countable weakly inner amenable étale groupoid. Then the following conditions are equivalent:*

- (1)  $\mathcal{G}$  is strongly amenable at infinity.
- (2)  $\mathcal{G}$  is amenable at infinity.
- (3)  $C_u^*(\mathcal{G})$  is nuclear.
- (4)  $C_u^*(\mathcal{G})$  is exact.
- (5)  $\mathcal{G}$  is KW-exact.
- (6)  $C_r^*(\mathcal{G})$  is exact.

*Proof.* By Corollary 6.4 and Proposition 6.7 it suffices to show that (6) implies (1). Therefore, let us assume that  $C_r^*(\mathcal{G})$  is exact. The proof is adapted from ideas of Jean Renault.

We fix a compact subset  $K$  of  $\mathcal{G}$  and  $\varepsilon > 0$ . We want to find a continuous bounded positive definite kernel  $k \in \mathcal{C}_t(\mathcal{G} *_r \mathcal{G})$  such that  $|k(\gamma, \gamma_1) - 1| \leq \varepsilon$  whenever  $\gamma^{-1}\gamma_1 \in K$ .

We set  $\mathcal{E} = \ell_{\mathcal{C}_0(X)}^2(\mathcal{G})$  with  $X = \mathcal{G}^{(0)}$ . Recall that  $\lambda^x$  is the counting measure on  $\mathcal{G}^x$ . We first choose a bounded, continuous positive definite function  $f$  on  $\mathcal{G} \times \mathcal{G}$ , properly supported, such that  $|f(\gamma, \gamma) - 1| \leq \varepsilon/2$  for  $\gamma \in K$ . Let  $\Phi : C_r^*(\mathcal{G}) \rightarrow \mathcal{B}_{\mathcal{C}_0(X)}(\mathcal{E})$  be a compactly supported completely positive map such that<sup>k</sup>

$$\|\Phi(f_\gamma) - f_\gamma\| \leq \varepsilon/2$$

for  $\gamma \in K$ . We also choose a continuous function  $\xi : \mathcal{G}^{(0)} \rightarrow [0, 1]$  with compact support such that  $\xi(x) = 1$  if  $x \in s(K) \cup r(K)$ .

Let  $(\gamma, \gamma_1) \in \mathcal{G} *_r \mathcal{G}$ . We choose an open bisection  $S$  such that  $\gamma \in S$  and a continuous function  $\varphi : X \rightarrow [0, 1]$ , with compact support in  $r(S)$  such that

<sup>k</sup>We write  $f_\gamma$  instead of  $\Lambda(f_\gamma)$  for simplicity of notation.



$\varphi(x) = 1$  on a neighborhood of  $r(\gamma)$ . We denote  $\xi_\varphi$  the continuous function on  $\mathcal{G}$  with compact support (and thus  $\xi_\varphi \in \mathcal{E}$ ) such that

$$\xi_\varphi(\gamma') = 0 \text{ if } \gamma' \notin S, \quad \xi_\varphi(\gamma') = \varphi \circ r(\gamma') \xi \circ s(\gamma') \text{ if } \gamma' \in S.$$

Note that  $\|\xi_\varphi\|_{\mathcal{E}} \leq 1$ . We define  $\xi_{\varphi_1}$  similarly with respect to  $\gamma_1$ .

We set, for  $(\gamma, \gamma_1) \in \mathcal{G} *_r \mathcal{G}$ ,

$$\begin{aligned} k(\gamma, \gamma_1) &= \langle \xi_\varphi, \Phi(f_{\gamma^{-1}\gamma_1}) \xi_{\varphi_1} \rangle(r(\gamma)) \\ &= \xi \circ s(\gamma) (\Phi(f_{\gamma^{-1}\gamma_1}) \xi_{\varphi_1})(\gamma). \end{aligned}$$

We observe that  $k(\gamma, \gamma_1)$  does not depend on the choices of  $S, \varphi, S_1, \varphi_1$ .

We see that  $k$  is continuous since  $\gamma \mapsto \Phi(f_\gamma)$  is continuous. Moreover, since  $\Phi$  is completely positive and compactly supported, we see that  $k$  is bounded, positive definite, and supported in a tube.

Let  $(\gamma, \gamma_1) \in \mathcal{G} *_r \mathcal{G}$  such that  $\gamma^{-1}\gamma_1 \in K$ . Then we have

$$|k(\gamma, \gamma_1) - 1| \leq \varepsilon/2 + |\langle \xi_\varphi, (\Lambda(f_{\gamma^{-1}\gamma_1}) \xi_{\varphi_1}) \rangle(r(\gamma)) - 1|,$$

and

$$\langle \xi_\varphi, (\Lambda(f_{\gamma^{-1}\gamma_1}) \xi_{\varphi_1}) \rangle(r(\gamma)) = \xi \circ s(\gamma) \xi \circ s(\gamma_1) f(\gamma^{-1}\gamma_1, \gamma^{-1}\gamma_1).$$

We fix  $(\gamma, \gamma_1)$  such that  $\gamma^{-1}\gamma_1 \in K$ . Observe that  $s(\gamma) \in r(K)$  and  $s(\gamma_1) \in s(K)$  and therefore  $\xi \circ s(\gamma) = 1 = \xi \circ s(\gamma_1)$ . It follows that

$$|k(\gamma, \gamma_1) - 1| \leq \varepsilon/2 + |f(\gamma^{-1}\gamma_1, \gamma^{-1}\gamma_1) - 1| \leq \varepsilon.$$

□

**Corollary 7.7.** *Let  $\mathcal{G}$  be a second countable étale groupoid such that there exists a locally proper continuous homomorphism  $\rho$  from  $\mathcal{G}$  into a countable discrete group  $G$ . Then the six conditions of Theorem 7.6 are equivalent. Moreover, they hold when  $G$  is exact.*

*Proof.* We observe that  $G$  is weakly inner amenable. Then, by Proposition 4.4 the groupoid  $\mathcal{G}$  is weakly inner amenable. □

**Corollary 7.8.** *Let  $\mathcal{G}$  be a second countable locally compact groupoid with Haar system.*

- (i) *Assume that  $\mathcal{G}$  is equivalent to a second countable weakly inner amenable étale groupoid  $\mathcal{H}$ . Then the conditions (2), (5) and (6) of the previous theorem are equivalent.*
- (ii) *Assume that  $\mathcal{G}$  is equivalent to a second countable weakly inner amenable étale groupoid  $\mathcal{H}$  via a  $\mathcal{G}$ - $\mathcal{H}$ -equivalence  $Z$  such that  $q_{\mathcal{G}} : Z \rightarrow \mathcal{G}^{(0)}$  admits a continuous section. Then all the conditions of the previous theorem are equivalent.*

Before beginning the proof, let us recall some facts. Let  $A, B$  be two  $C^*$ -algebras. They are said to be *Morita equivalent* if there exists an  $A$ - $B$  imprimitivity bimodule. For details about this notion due to Rieffel, we refer to [62]. In fact, since we mainly consider separable  $C^*$ -algebras, we shall use the Brown-Green-Rieffel theorem saying that two separable  $C^*$ -algebras  $A$  and  $B$  are Morita-equivalent if and only if  $A \otimes \mathcal{K}$  and  $B \otimes \mathcal{K}$  are isomorphic, where  $\mathcal{K}$  is the  $C^*$ -algebra of compact operators on a separable Hilbert space [10]. In particular, in this case, it is obvious that exactness is preserved under Morita equivalence.

*Proof.* (i) Assume that  $\mathcal{G}$  is equivalent to a second countable étale groupoid  $\mathcal{H}$  which is weakly inner amenable and assume that  $C_r^*(\mathcal{G})$  is exact. Then the  $C^*$ -algebras  $C_r^*(\mathcal{G})$  and  $C_r^*(\mathcal{H})$  are Morita equivalent (see [48], [68] or [65]). It follows from Theorem 7.6 that  $\mathcal{H}$  is strongly amenable at infinity and therefore  $\mathcal{G}$  is amenable at infinity by Proposition 3.10.

To prove (ii), we use the remark 3.11 instead. □

*Remark 7.9.* With similar techniques as those used to prove Theorem 7.6 we can prove the following result.

**Proposition 7.10.** *Let  $\mathcal{G}$  be a second countable locally compact groupoid with Haar system. The following conditions are equivalent:*

- (i)  $\mathcal{G}$  is amenable;
- (ii)  $C_r^*(\mathcal{G})$  is nuclear and  $\mathcal{G}$  is weakly inner amenable.

The case of a transformation groupoid was dealt with in [3, Theorem 5.8].

## 8. EXAMPLES

**8.1. Groupoids equivalent to transformation groupoids.** Let  $(\mathcal{G}, \lambda)$  be second countable locally compact groupoid which is equivalent to a transformation groupoid  $X \rtimes G$  where  $G$  is discrete. This groupoid  $X \rtimes G$  is weakly inner amenable. Then Corollary 7.8 applies and the conditions (2), (5) and (6) of the theorem 7.6 are equivalent. In particular they hold if  $G$  is exact since  $X \rtimes G$  is then amenable at infinity by Corollary 3.22. Moreover, if  $X$  is compact, then  $\mathcal{G}$  is amenable at infinity if and only if  $G$  is exact by Proposition 3.6.

**Example 8.1. Transitive groupoids.** Assume that  $\mathcal{G}$  is transitive, that is, for every  $x, y \in X$  there exists  $\gamma \in \mathcal{G}$  such that  $r(\gamma) = x$  and  $s(\gamma) = y$ . Then all the isotropy groups of  $\mathcal{G}$  are isomorphic. We know that when  $\mathcal{G}$  is amenable at infinity, these isotropy groups are KW-exact (see Remark 6.10). If they are discrete, let us observe that the converse is true: if any of its isotropy group is exact, then  $\mathcal{G}$  is amenable at infinity. Indeed, fixing  $x \in X$ , the space  $\mathcal{G}_x$  is a  $\mathcal{G}$ - $\mathcal{G}(x)$ -equivalence, where  $q_{\mathcal{G}}$  is the restriction of  $r$  to  $\mathcal{G}_x$ , and where the left action of  $\mathcal{G}$  and the right

action of  $\mathcal{G}(x)$  are obviously defined (see [48]). So  $\mathcal{G}$  is equivalent to the group  $\mathcal{G}(x)$  and therefore is amenable at infinity whenever  $\mathcal{G}(x)$  is exact. Note that, by the corollary 7.8, this latter property is equivalent to the KW-exactness of  $\mathcal{G}$  and also to the exactness of  $C_r^*(\mathcal{G})$ . The equivalence of these two last properties has been also established in [36].

**8.2. Groupoids from partial group actions.** A *partial action* of a discrete group  $G$  on a locally compact space  $X$  is a pair  $\beta = (\{X_t\}_{t \in G}, \{\beta_t\}_{t \in G})$  such that

- $X_t$  is open in  $X$  and  $\beta_t : X_{t^{-1}} \rightarrow X_t$  is a homeomorphism for every  $t \in G$ ;
- $X_e = X$  and  $\beta_e = \text{Id}_X$ , where  $e$  is the unit of  $G$ ;
- $\beta_{st}$  is an extension of  $\beta_s \circ \beta_t$  for every  $s, t \in G$ .

The following groupoid  $G \ltimes_{\beta} X$  is associated to  $\beta$ . It is defined as the topological subspace

$$G \ltimes_{\beta} X = \{(x, t, y) : t \in G, y \in X_{t^{-1}}, x = \beta_t(y)\}$$

of  $X \times G \times X$ . We have  $r((x, t, y)) = x$  (where  $x$  is identified with  $(x, e, x)$ ),  $s((x, t, y)) = y$ , the composition law is given by  $(x, s, y)(y, t, z) = (x, st, z)$  and the inverse is given by  $(x, t, y)^{-1} = (y, t^{-1}, x)$ . As such,  $G \ltimes_{\beta} X$  is an étale groupoid (see [1]). The  $C^*$ -algebra  $C_r^*(G \ltimes_{\beta} X)$  is isomorphic to the reduced crossed product  $\mathcal{C}_0(X) \rtimes_r G$  with respect to the partial action of  $G$  (see [44, Proposition 2.2]) and so  $C_r^*(G \ltimes_{\beta} X)$  is exact whenever the group  $G$  is exact, by [7, Corollary 5.3]. It would be nice to have a direct proof of this fact (not using Fell bundles as in [7]). Although we have not checked all the details, it seems that  $G \ltimes_{\beta} X$  is KW-exact when  $G$  is exact. Indeed, actions of  $G \ltimes_{\beta} X$  on  $C^*$ -algebras give rise to partial actions of  $G$  and we use the fact that exact sequences which are equivariant under partial actions of an exact group yield exact sequences of the corresponding reduced crossed products (see [18, Theorem 22.9]).

Note that the cocycle  $(x, t, y) \mapsto t$  is faithful. It is locally proper if and only if for every  $t \in G$  the graph  $\{(\beta_t(y), y) : y \in X_{t^{-1}}\}$  is closed in  $X \times X$ . In this case, the groupoid  $G \ltimes_{\beta} X$  is equivalent to a transformation groupoid  $Y \rtimes G$  which is explicitly constructed and the groupoid  $G \ltimes_{\beta} X$  is strongly amenable at infinity if and only if  $Y \rtimes G$  is strongly amenable at infinity (see Proposition 3.20 (i)). This holds when  $G$  is exact. In general, we do not know whether  $G \ltimes_{\beta} X$  is amenable at infinity when  $G$  is exact.

In the next section, we show that many semigroups provide groupoids defined by partial actions.

### 8.3. Groupoids from semigroups.

**8.1.2.1. Inverse semigroups.** A semigroup  $S$  is an inverse semigroup if for each  $u \in S$  there exists a unique  $u^* \in S$  (called the inverse of  $u$ ) such that  $u = uu^*u$  and  $u^* = u^*uu^*$ . The set  $E_S$  of idempotents of  $S$  plays a crucial role. It is an

abelian sub-semigroup of  $S$ . There is a natural partial order on  $S$  defined by  $u \leq v$  if there exists an idempotent  $e \in S$  such that  $u = ve$ . One defines an equivalence relation  $\sigma$  on  $S$  by saying that  $u \sim_\sigma v$  whenever there exists an idempotent  $e \in S$  such that  $ue = ve$ . The quotient  $S/\sigma$  is a group, denoted by  $G_S$ , and called the *maximal group homomorphic image of  $S$*  (or the minimum group congruence). For these facts and more on the theory of inverse semigroups we refer to [40].

By an abuse of notation,  $\sigma$  will also denote the quotient map from  $S$  onto  $S/\sigma$ . If  $S$  has a zero, we denote by  $S^\times$  the set  $S \setminus \{0\}$ . When  $S$  does not have a zero, we set  $S^\times = S$ .

To each inverse semigroup  $S$  is associated in an explicit way a groupoid  $\mathcal{G}_S$ . We recall its construction, which is described in detail in [55]. We denote by  $X$  the space of non-zero maps  $\chi$  from  $E_S$  into  $\{0, 1\}$  such that  $\chi(e_f) = \chi(e)\chi(f)$  and  $\chi(0) = 0$  whenever  $S$  has a zero. Equipped with the topology induced from the product space  $\{0, 1\}^E$ , the space  $X$ , called the *spectrum* of  $S$ , is locally compact and totally disconnected. Note that when  $S$  is a monoid (*i.e.*, has a unit element 1) then  $\chi$  is non-zero if and only if  $\chi(1) = 1$ , and therefore  $X$  is compact.

The semigroup  $S$  acts on  $X$  as follows. The domain (open and compact) of  $t \in S$  is  $D_{t^*t} = \{\chi \in X : \chi(t^*t) = 1\}$  and we set  $\theta_t(\chi)(e) = \chi(t^*et)$ . We define on  $\Xi = \{(t, \chi) \in S \times X : \chi \in D_{t^*t}\}$  the equivalence relation  $(t, \chi) \sim (t_1, \chi_1)$  if  $\chi = \chi_1$  and there exists  $e \in E_S$  with  $\chi(e) = 1$  and  $te = t_1e$ . Then  $\mathcal{G}_S$  is the quotient of  $\Xi$  with respect to this equivalence relation, equipped with the quotient topology. The range of the class  $[t, \chi]$  of  $(t, \chi)$  is  $\theta_t(\chi)$  and its source is  $\chi$ . The composition law is given by  $[u, \chi][v, \chi'] = [uv, \chi']$  if  $\theta_v(\chi') = \chi$  (see [55] or [17] for details). In general,  $\mathcal{G}_S$  is not Hausdorff. However, in many common examples it is the case.

**Definition 8.2.** An inverse semigroup  $S$  is said to be  *$E$ -unitary* if  $E_S$  is the kernel of  $\sigma : S \rightarrow S/\sigma$  (equivalently, every element greater than an idempotent is an idempotent). When  $S$  has a zero, this means that  $S = E_S$ .

**Definition 8.3.** Let  $S$  be an inverse semigroup. A *morphism* (or *grading*) is an application  $\psi$  from  $S^\times$  into a discrete group  $G$  such that  $\psi(st) = \psi(s)\psi(t)$  if  $st \neq 0$ . If in addition  $\psi^{-1}(e) = E_S^\times$ , we say that  $\psi$  is an *idempotent pure morphism*. When such an application  $\psi$  from  $S^\times$  into a group  $G$  exists, the inverse semigroup  $S$  is called *strongly  $E^*$ -unitary*.

Note that when  $S$  is without zero,  $S$  is strongly  $E^*$ -unitary if and only if it is  $E$ -unitary.

Let  $S$  be an inverse semigroup. Its reduced  $C^*$ -algebra  $C_r^*(S)$  is defined in [55]. It is canonically isomorphic to the reduced  $C^*$ -algebra of the groupoid  $\mathcal{G}_S$  (see [55, Theorem 4.4.2], [30, Theorem 3.5]).

**Proposition 8.4.** *Let  $S$  be an inverse semigroup such that there exists an idempotent pure morphism  $\psi$  from  $S^\times$  into a discrete group  $G$ . Then  $\mathcal{G}_S$  is isomorphic*

to  $G \times_{\beta} X$  for a canonical action of  $G$  on the spectrum  $X$  of  $S$ . In particular, the reduced  $C^*$ -algebra  $C_r^*(S)$  is exact when  $G$  is exact.

*Proof.* The construction of a partial action of  $G$  on the spectrum of  $S$  such that the groupoids  $\mathcal{G}_S$  and  $G \times_{\beta} X$  are canonically isomorphic is carried out in [46] (see also [6, 44]). The second assertion of the proposition is then immediate.  $\square$

**Proposition 8.5.** *Let  $S$  be a  $E$ -unitary inverse semigroup. Then  $C_r^*(S)$  is exact if and only if the maximal group homomorphic image  $S/\sigma$  is an exact group.*

*Proof.* We have to show that  $S/\sigma$  is exact when  $C_r^*(S)$  is exact. It suffices to consider the case where  $S$  does not have a 0, since  $S = E_S$  otherwise. Let  $\chi_{\infty}$  be the character such that  $\chi_{\infty}(e) = 1$  for every  $e \in E_S$ . Then  $\chi_{\infty}$  is  $\mathcal{G}_S$ -invariant in  $X = \mathcal{G}_S^{(0)}$  and  $[t, \chi_{\infty}] \mapsto \sigma(t)$  is an isomorphism from the isotropy group  $\mathcal{G}_S(\chi_{\infty})$  onto  $S/\sigma$ . But, as recalled in the remark 6.10, the  $C^*$ -algebra  $C_r^*(\mathcal{G}_S(\chi_{\infty}))$  is a quotient of  $C_r^*(\mathcal{G}_S)$ . This latter  $C^*$ -algebra is exact since it is canonically isomorphic to  $C_r^*(S)$ . It follows that the group  $S/\sigma$  is exact.  $\square$

8.1.2.1. *Sub-semigroups of a group.* In this subsection, we consider a discrete groupe  $G$  and a sub-semigroup  $P$  which contains the unit  $e$  of  $G$ . For  $p \in P$ , let  $V_p$  be the isometry in  $\mathcal{B}(\ell^2(P))$  defined by

$$V_p \delta_q = \delta_{pq}.$$

The reduced  $C^*$ -algebra or Toeplitz algebra of  $P$  is the sub- $C^*$ -algebra  $C_r^*(P)$  of  $\mathcal{B}(\ell^2(P))$  generated by these isometries.

An inverse semigroup  $S(P)$ , called the *inverse hull* of  $P$ , is attached to  $P$ . One of its definitions is

$$S(P) = \{V_{p_1}^* V_{q_1} \cdots V_{p_n}^* V_{q_n} : n \in \mathbb{N}, p_i, q_i \in P\}.$$

It is an inverse semigroup of partial isometries in  $\mathcal{B}(\ell^2(P))$  (see [51, §3.2]). An important property of  $S(P)$  is that the map  $\psi$  from  $S(P)^{\times}$  into  $G$  such that

$$\psi(V_{p_1}^* V_{q_1} \cdots V_{p_n}^* V_{q_n}) = p_1^{-1} q_1 \cdots p_n^{-1} q_n$$

is well defined and is an idempotent pure morphism (see [51, Proposition 3.2.11]). Moreover, by [51, Lemma 3.4.1], the semigroup  $S(P)$  does not have a zero if and only if  $PP^{-1}$  is a subgroup of  $G$ .

Let us denote by  $\lambda_G$  the left regular representation of  $G$  and by  $E_P$  the orthogonal projection from  $\ell^2(G)$  onto  $\ell^2(P)$ . We say that  $(P, G)$  satisfies the *Toeplitz condition* if for every  $g \in G$  such that  $E_P \lambda_g E_P \neq 0$ , there exist  $p_1, \dots, p_n, q_1, \dots, q_n \in P$  such that  $E_P \lambda_g E_P = V_{p_1}^* V_{q_1} \cdots V_{p_n}^* V_{q_n}$ . For instance the quasi-lattice ordered groups introduced by Nica in [50] satisfy this property (see [43, §8]).

**Proposition 8.6.** *Let  $(P, G)$  be as above.*

- (i) If  $G$  is exact, then the  $C^*$ -algebra  $C_r^*(P)$  is exact.
- (ii) Assume that  $G = PP^{-1}$  and that the Toeplitz condition is satisfied. Then  $C_r^*(P)$  is exact if and only if the group  $G$  is exact.

*Proof.* (i) follows from Proposition 8.4 and from the fact that  $C_r^*(P)$  is a quotient of  $C_r^*(\mathcal{G}_{S(P)})$  (see [51, Corollary 3.2.13]).

(ii) Assume now that  $G = PP^{-1}$  and that the Toeplitz condition is satisfied. Then the inverse semigroup  $S(P)$  does not have a 0. Moreover, the map  $\tau : S(P)/\sigma \rightarrow G$  such that  $\tau \circ \sigma = \psi$  is an isomorphism. Indeed  $\psi$  is surjective, so  $\tau$  is also surjective. Assume that  $\tau(\sigma(x)) = e$ , with  $x \in S(P)$ . Since  $\psi$  is idempotent pure, we see that  $x$  is an idempotent and therefore  $\sigma(x)$  is the unit of  $S(P)/\sigma$ .

It follows from Proposition 8.5 that  $C_r^*(S(P))$  is exact if and only if  $G$  is exact. Finally, we conclude by using the fact that  $C_r^*(P) = C_r^*(S(P))$  since the Toeplitz condition is satisfied (see [51, Theorem 3.2.14] and [42, Lemma 2.28]).  $\square$

*Remark 8.7.* Assume that  $(P, G)$  satisfies the Toeplitz condition. Using Proposition 8.4 and its notation, we know that the groupoid  $\mathcal{G}_{S(P)}$  is isomorphic to  $G \times_{\beta} X$ . Moreover, the Toeplitz condition implies that the cocycle  $(g, x) \mapsto g$  (of course faithful) is locally proper (see [6, Corollary 3.9]). In this case the equivalence with a transformation groupoid  $Y \rtimes G$  is such that we can apply Corollary 7.8 (see Proposition 3.20). If  $G$  is exact, the six equivalent conditions of Theorem 7.6 hold.

#### 8.4. Fields of groupoids.

**Definition 8.8.** A *field of groupoids* is a triple  $(\mathcal{G}, T, p)$  where  $\mathcal{G}$  is a groupoid,  $T$  a set and  $p : \mathcal{G}^{(0)} \rightarrow T$  is a surjective map such that  $p \circ r = p \circ s$ .

For  $t \in T$ , note that  $(p \circ r)^{-1}(t) = \mathcal{G}_t$  is the reduction  $\mathcal{G}(p^{-1}(t))$  of  $\mathcal{G}$  by  $p^{-1}(t)$ . Its set of units is  $p^{-1}(t)$ . In the case where  $T = \mathcal{G}^{(0)}$  and  $p$  is the identity map, then  $\mathcal{G}_t$  is the isotropy group of  $\mathcal{G}(t)$  at  $t$ , and we say that  $\mathcal{G}$  is a field of groups.

**Definition 8.9.** A *continuous field of groupoids* (or *groupoid bundle*) is a triple  $(\mathcal{G}, T, p)$  as in the previous definition where  $\mathcal{G}$  is a locally compact groupoid,  $T$  is locally compact and  $p$  is continuous and open.

In case  $p$  is the identity map of  $\mathcal{G}^{(0)}$  we shall say that  $(\mathcal{G}, T, p)$  is a *continuous field of groups*<sup>1</sup> or *groupoid group bundle*.

Let  $(\mathcal{G}, T, p)$  be a continuous field of groupoids. In the sequel, we shall assume that  $\mathcal{G}$  has a Haar system. In the case of a field of groups, the existence of a Haar system is equivalent to the fact that  $r = s$  is open by [58, Lemma 1.3]. Observe

<sup>1</sup>This terminology is rather misleading since, as we shall see,  $C_r^*(\mathcal{G})$  is not in general a continuous field of  $C^*$ -algebras with fibres  $C_r^*(\mathcal{G}(x))$ ,  $x \in \mathcal{G}^{(0)}$ .

that  $\mathcal{C}_c(\mathcal{G})$  has a structure of  $\mathcal{C}_0(T)$ -module by setting  $(fa)(\gamma) = f \circ p(r(\gamma))a(\gamma)$  for  $f \in \mathcal{C}_0(T)$  and  $a \in \mathcal{C}_c(\mathcal{G})$ . The map  $a \mapsto fa$  extends continuously in order to turn  $C^*(\mathcal{G})$  and  $C_r^*(\mathcal{G})$  into  $\mathcal{C}_0(T)$ -algebras. This is obvious for the reduced  $C^*$ -algebra. In the case of the full  $C^*$ -algebra, that we shall not need, one uses the same arguments as in the proof of [57, Lemma 1.13] (see also [56, Lemme 2.4.4] for details).

We shall see that  $C_r^*(\mathcal{G})$  can be viewed as a field of  $C^*$ -algebras over  $T$  in two different ways. Before, we need to recall some definitions.

**Definition 8.10.** A *field of  $C^*$ -algebras over a locally compact space  $X$*  is a triple  $\mathcal{A} = (A, \{\pi_x : A \rightarrow A_x\}_{x \in X}, X)$  where  $A, A_x$  are  $C^*$ -algebras, and where  $\pi_x$  is a surjective  $*$ -homomorphism such that

- (i)  $\{\pi_x : x \in X\}$  is faithful, that is,  $\|a\| = \sup_{x \in X} \|\pi_x(a)\|$  for every  $a \in A$ ;
- (ii) for  $f \in \mathcal{C}_0(X)$  and  $a \in A$ , there is an element  $fa \in A$  such that  $\pi_x(fa) = f(x)\pi_x(a)$  for  $x \in X$ .

We say that the field is *upper semi-continuous* (resp. *lower semi-continuous*) if the function  $x \mapsto \|\pi_x(a)\|$  is upper semi-continuous (resp. lower semi-continuous) for every  $a \in A$ .

If for each  $a \in A$ , the function  $x \mapsto \|\pi_x(a)\|$  is in  $\mathcal{C}_0(X)$ , we shall say that  $\mathcal{A}$  is a *continuous field of  $C^*$ -algebras<sup>m</sup>*.

For  $f \in \mathcal{C}_0(X)$ , denote by  $\rho_f$  the map  $a \mapsto fa$ . Then  $\rho_f$  is in the center  $Z(M(A))$  of the multiplier algebra  $M(A)$  of  $A$  and  $f \mapsto \rho_f$  is a  $*$ -homomorphism from  $\mathcal{C}_0(X)$  into  $Z(M(A))$ . In the case of a continuous field of  $C^*$ -algebras, we have  $A = \mathcal{C}_0(X)A$ , that is,  $A$  is a  $\mathcal{C}_0(X)$ -algebra (see [32, Lemma 2.1]). Observe that the converse is not true: a  $\mathcal{C}_0(X)$ -algebra only give rise to an upper semi-continuous field of  $C^*$ -algebras (see [63, Proposition 1.2]).

Let  $(\mathcal{G}, T, p)$  be a continuous field of groupoids as above. We set  $X = \mathcal{G}^{(0)}$ . For  $t \in T$ , we set  $X_t = p^{-1}(t)$  and  $U_t = X \setminus X_t$ . Let us explain now how  $C_r^*(\mathcal{G})$  can be viewed as a field of  $C^*$ -algebras over  $T$  in two different ways. First, since it is a  $\mathcal{C}_0(T)$ -algebra, we have the field  $(C_r^*(\mathcal{G}), \{e_t : C_r^*(\mathcal{G}) \rightarrow C_r^*(\mathcal{G})_t\}_{t \in T}, T)$  where  $e_t$  is the quotient map from  $C_r^*(\mathcal{G})$  onto  $C_r^*(\mathcal{G})_t = C_r^*(\mathcal{G})/\mathcal{C}_t(T)C_r^*(\mathcal{G})$ . Second, it is the field  $(C_r^*(\mathcal{G}), \{\pi_t : C_r^*(\mathcal{G}) \rightarrow C_r^*(\mathcal{G}(X_t))\}_{t \in T}, T)$ . The first field is upper semi-continuous by [63, Proposition 1.2] and the second is lower semi-continuous by [56, Théorème 2.4.6] (see also [39, Theorem 5.5]).

**Proposition 8.11.** *Let  $(\mathcal{G}, T, p)$  be a continuous field of groupoids. We assume that  $\mathcal{G}$  has a Haar system. Let  $t_0 \in T$ . The function  $t \mapsto \|\pi_t(a)\|$  is continuous at*

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<sup>m</sup>In [32], this is called a continuous bundle of  $C^*$ -algebras.

$t_0$  for every  $a \in C_r^*(\mathcal{G})$  if and only if the following sequence

$$0 \rightarrow C_r^*(\mathcal{G}(U_{t_0})) \rightarrow C_r^*(\mathcal{G}) \xrightarrow{\pi_{t_0}} C_r^*(\mathcal{G}(X_{t_0})) \rightarrow 0$$

is exact.

*Proof.* We have  $\mathcal{C}_c(\mathcal{G}(U_{t_0})) = \mathcal{C}_{t_0}(T)\mathcal{C}_c(\mathcal{G})$  and by continuity we get  $C_r^*(\mathcal{G}(U_{t_0})) = \mathcal{C}_{t_0}(T)C_r^*(\mathcal{G})$ . It follows from [32, Lemma 2.3] that the function  $t \mapsto \|\pi_t(a)\|$  is upper semi-continuous at  $t_0$  for all  $a \in C_r^*(\mathcal{G})$  if and only if the kernel of  $\pi_{t_0}$  is  $C_r^*(\mathcal{G}(U_{t_0}))$ . Since  $t \mapsto \|\pi_t(a)\|$  is always lower semi-continuous, this proves the proposition.  $\square$

*Remark 8.12.* We immediately get from Proposition 8.11 that the function  $t \mapsto \|\pi_t(a)\|$  is continuous at  $t_0$  for every  $a \in C_r^*(\mathcal{G})$  whenever the groupoid  $\mathcal{G}(X_{t_0})$  is second countable and amenable. Indeed chasing around the following commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & C^*(\mathcal{G}(U_{t_0})) & \longrightarrow & C^*(\mathcal{G}) & \longrightarrow & C^*(\mathcal{G}(X_{t_0})) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & C_r^*(\mathcal{G}(U_{t_0})) & \longrightarrow & C_r^*(\mathcal{G}) & \xrightarrow{\pi_{t_0}} & C_r^*(\mathcal{G}(X_{t_0})) \rightarrow 0 \end{array}$$

and using the facts that the first line is exact, that the vertical arrows are surjective and that  $C^*(\mathcal{G}(X_{t_0})) = C_r^*(\mathcal{G}(X_{t_0}))$  since  $\mathcal{G}(X_{t_0})$  is amenable [2, Proposition 6.1.10], we see that the second line is also an exact sequence.

This continuity result was obtained in [56, Corollary 2.4.7].

This is no longer true if  $\mathcal{G}(X_{t_0})$  is only assumed to be exact, but we have the following result.

**Corollary 8.13.** *Let  $(\mathcal{G}, T, p)$  be a continuous field of groupoids. We assume that  $\mathcal{G}$  has a Haar system and is KW-exact. Then*

$$(C_r^*(\mathcal{G}), \{\pi_t : C_r^*(\mathcal{G}) \rightarrow C_r^*(\mathcal{G}(X_t))\}_{t \in T}, T)$$

is a continuous field of  $C^*$ -algebras on  $T$ .

*Proof.* This is an immediate consequence of Remark 6.10 and Proposition 8.11.  $\square$

Consider now the case of an étale groupoid  $\mathcal{G}$  which is a groupoid group bundle. This means that for every  $\gamma \in \mathcal{G}$  we have  $r(\gamma) = s(\gamma)$ . Assume that  $C_r^*(\mathcal{G})$  is an exact  $C^*$ -algebra. Then, for every  $x \in X = \mathcal{G}^{(0)}$ , the  $C^*$ -algebra  $C_r^*(\mathcal{G}(x))$  is exact, since it is a quotient of  $C_r^*(\mathcal{G})$ . It follows that the discrete group  $\mathcal{G}(x)$  is exact and therefore  $\mathcal{G}$  is a bundle of exact discrete groups. If moreover  $\mathcal{G}$  is KW-exact, then  $C_r^*(\mathcal{G})$  is a continuous field of  $C^*$ -algebras with fibres  $C_r^*(\mathcal{G}(x))$ .



We now give an example of étale groupoid group bundle  $\mathcal{G}$  such that  $\mathcal{G}(x)$  is an exact group for every  $x \in \mathcal{G}^{(0)}$ , whereas  $\mathcal{G}$  is not KW-exact. The construction is due to Higson, Lafforgue and Skandalis [23].

We consider a residually finite group  $\Gamma$  and an decreasing sequence  $\Gamma \supset N_0 \supset N_1 \cdots \supset N_k \supset \cdots$  of finite index normal subgroups with  $\bigcap_k N_k = \{e\}$ . Let  $\widehat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$  be the Alexandroff compactification of  $\mathbb{N}$ . We set  $N_\infty = \{e\}$  and, for  $k \in \widehat{\mathbb{N}}$ , we denote by  $q_k : \Gamma \rightarrow \Gamma_k = \Gamma/N_k$  the quotient homomorphism. Let  $\mathcal{G}$  be the quotient of  $\widehat{\mathbb{N}} \times \Gamma$  with respect to the equivalence relation

$$(k, t) \sim (l, u) \text{ if } k = l \text{ and } q_k(t) = q_k(u).$$

Equipped with the quotient topology,  $\mathcal{G}$  has a natural structure of (Hausdorff) étale locally compact groupoid group bundle: its space of units is  $\widehat{\mathbb{N}}$ , the range and source maps are given by  $r([k, t]) = s([k, t]) = q_k(t)$ , where  $[k, t] = (k, q_k(t))$  is the equivalence class of  $(k, t)$ . The fibre  $\mathcal{G}(k)$  of the bundle is the quotient group  $\Gamma_k$  if  $k \in \mathbb{N}$  and  $\Gamma$  if  $k = \infty$ . We call this groupoid an HLS-*groupoid*. A basic result of [23] is that the sequence

$$0 \longrightarrow C_r^*(\mathcal{G}(\mathbb{N})) \longrightarrow C_r^*(\mathcal{G}) \longrightarrow C_r^*(\mathcal{G}(\infty)) \longrightarrow 0$$

is not exact whenever  $\Gamma$  has Kazhdan's property (T) (it is not even exact in  $K$ -theory!). Therefore  $\mathcal{G}$  is not KW-exact. As an example we can take the exact group  $\mathrm{SL}(3, \mathbb{Z})$  and any sequence  $(N_k)$  as above.

For this HLS-groupoids, the exactness of  $C_r^*(\mathcal{G})$  is a very strong condition which suffices to imply the amenability of  $\Gamma$  as shown by Willet in [70].

**Proposition 8.14.** *Let us keep the above notation and assumptions. We assume that  $\Gamma$  is finitely generated. Then the following conditions are equivalent:*

- (1) *If  $\Gamma$  is amenable;*
- (2)  *$\mathcal{G}$  is amenable;*
- (3)  *$\mathcal{G}$  is KW-exact;*
- (4)  *$C_r^*(\mathcal{G})$  is a continuous field of  $C^*$ -algebras with fibres  $C_r^*(\mathcal{G}(x))$ ;*
- (5)  *$C_r^*(\mathcal{G})$  is nuclear;*
- (6)  *$C_r^*(\mathcal{G})$  is exact.*

*Proof.* The equivalence between (1) and (2) follows from [2, Corollary 5.3.33]. That (2)  $\Rightarrow$  (3) is obvious and (3)  $\Rightarrow$  (4) is contained in Corollary 8.13. Let us prove that (4)  $\Rightarrow$  (1). Assume by contradiction that  $\Gamma$  is not amenable. We fix a symmetric probability measure  $\mu$  on  $\Gamma$  with a finite support that generates  $\Gamma$  and we choose  $n_0$  such that the restriction of  $q_n$  to the support of  $\mu$  is injective for  $n \geq n_0$ . We define  $a \in \mathcal{C}_c(\mathcal{G}) \subset C_r^*(\mathcal{G})$  such that  $a(\gamma) = 0$  except for  $\gamma = (n, q_n(s))$  with  $n \geq n_0$  and  $s \in \mathrm{Supp}(\mu)$  where  $a(\gamma) = \mu(s)$ . Then  $\pi_n(a) = 0$  if  $n < n_0$  and  $\pi_n(a) = \lambda_{\Gamma_n}(\mu) \in C_r^*(\Gamma_n) = C_r^*(\mathcal{G}(n))$  if  $n \geq n_0$ , where  $\lambda_{\Gamma_n}$  is the quasi-regular

representation of  $\Gamma$  in  $\ell^2(\Gamma_n)$ . By Kesten's result [29, 28] on the spectral radius relative to symmetric random walks, we have  $\|\lambda_{\Gamma_n}(\mu)\|_{C_r^*(\Gamma_n)} = 1$  for  $\mathbb{N} \ni n \geq n_0$  and  $\|\lambda_{\Gamma_\infty}(\mu)\|_{C_r^*(\Gamma_\infty)} < 1$  since  $\Gamma$  is not amenable. It follows that  $C_r^*(\mathcal{G})$  is not a continuous field of  $C^*$ -algebras with fibres  $C_r^*(\mathcal{G}(n))$  on  $\widehat{\mathbb{N}}$ , a contradiction.

We know that (2)  $\Rightarrow$  (5)  $\Rightarrow$  (6). The fact that (6)  $\Rightarrow$  (1) is due to Willett [70].  $\square$

## 9. OPEN QUESTIONS

(1) Let  $\mathcal{G}$  be a locally compact groupoid. Is it true that the amenability at infinity of  $\mathcal{G}$  implies its strong amenability at infinity? Note that by Theorem 7.6 this is true for every second countable weakly inner amenable étale groupoid (see also Corollary 7.8 (ii) for a more general result).

(2) Are there étale groupoids that are not weakly inner amenable? In particular, if  $G$  is a discrete group acting partially on a locally compact space  $X$ , is it true that the corresponding partial transformation groupoid is weakly inner amenable? This is true when the domains of the partial homeomorphisms are both open and closed but what happens in general?

(3) Is it true that an étale groupoid  $\mathcal{G}$  is KW-exact whenever its reduced  $C^*$ -algebra is exact? Is it true that the KW-exactness implies the amenability at infinity?

(4) In [32], Kirchberg and Wassermann have constructed examples of continuous fields of exact  $C^*$ -algebras on a locally compact space, whose  $C^*$ -algebra of continuous sections vanishing at infinity is not exact. Find examples of étale groupoid group bundles  $\mathcal{G}$ , whose reduced  $C^*$ -algebra is not exact whereas

$$(C_r^*(\mathcal{G}), \{\pi_x : C_r^*(\mathcal{G}) \rightarrow C_r^*(\mathcal{G}(x))\}_{x \in \mathcal{G}^{(0)}}, \mathcal{G}^{(0)})$$

is a continuous field of exact  $C^*$ -algebras.

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