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Hussein Cheikh Ali

ON A NONLINEAR CRITICAL ELLIPTIC EQUATION WITH A PERTURBED HARDY POTENTIAL

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ABSTRACT. In this work, we use a minimisation argument to show the existence of a nontrivial solution to

$$\begin{cases} -\Delta u - \frac{a(x)}{|x|^2}u = u^{2^\star-1} & \text{in } \Omega \\ u \geq 0 & \text{a.e. in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a smooth domain.

1. Introduction

Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$. We let $a \in C^{0,\theta}(\bar{\Omega})$ for some $\theta \in]0,1[$ be a Hölder continuous function. In this note, we investigate the existence of weak solutions $u \in H_0^1(\Omega)$, $u \not\equiv 0$, to the problem

$$(H) \begin{cases} -\Delta u - \frac{a(x)}{|x|^2}u = u^{2^\star-1} & \text{in } \Omega \\ u \geq 0 & \text{a.e. in } \Omega \\ u = 0 & \text{in } \partial\Omega \end{cases}$$

where $2^\star := \frac{2n}{n-2}$ and $H_0^1(\Omega)$ is the completion of $C_c^\infty(\Omega)$ for the norm $u \mapsto \|\nabla u\|_2$. The exponent $2^\star$ is critical from the viewpoint of Sobolev embeddings: $H_0^1(\Omega)$ is continuously embedded in $L^q(\Omega)$ for all $1 \leq q \leq 2^\star$, and the embedding is compact if $1 \leq q < 2^\star$. Such type of equations has been investigated by many authors since the pioneer work of Brezis-Nirenberg [1] (which is the case $a(x) := \lambda|x|^2$ for some $\lambda \in \mathbb{R}$). For the case of nontrivial Hardy potential, without claiming any exhaustivity, we refer to Jannelli [4], Ruiz-Willem [5] (both for the case $0 \in \Omega$), the recent survey by Ghoussoub-Robert [3] and the references therein.

Concerning terminology, we say that $u \in H_0^1(\Omega)$ is a weak solution to (H) if

$$\int_\Omega \nabla u \nabla v dx - \int_\Omega \frac{a(x)}{|x|^2} uv dx = \int_\Omega u^{2^\star-1} v dx \text{ for all } v \in H_0^1(\Omega).$$

This definition makes sense because of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2^\star}(\Omega)$ and the Hardy inequality

$$\frac{(n-2)^2}{4} \int_\Omega \frac{v^2}{|x|^2} dx \leq \int_\Omega |\nabla v|^2 dx$$

for all $v \in H_0^1(\Omega)$. Since $0 \in \partial\Omega$, this classical inequality is improved and one has

$$\gamma_H(\Omega) \int_\Omega \frac{v^2}{|x|^2} dx \leq \int_\Omega |\nabla v|^2 dx \tag{1}$$

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for all $v \in H_0^1(\Omega)$, where

$$\frac{(n-2)^2}{4} < \gamma_H(\Omega) \leq \frac{n^2}{4}$$

is the best possible constant such that (1) holds. Note that when Ω is convex, or more generally when $\Omega \subset \mathbb{R}^n_+ := \{x \in \mathbb{R}^n / \, x_1 > 0\}$, then $\gamma_H(\Omega)$ achieves the maximal value $\frac{n^2}{4}$. We refer to Ghoussoub-Robert [2] for discussions and references about the value of $\gamma_H(\Omega)$.

In the sequel, we assume that the operator $-\Delta - \frac{a(x)}{|x|^2}$ is coercive, that is there exists $c > 0$ such that

$$\int_\Omega \left(|\nabla v|^2 - \frac{a(x)v^2}{|x|^2} \right) dx \geq c \int_\Omega |\nabla v|^2 dx \text{ for all } v \in H_0^1(\Omega).$$

We investigate weak solutions to (H) as minima of the functional defined for $u \in H_0^1(\Omega) \setminus \{0\}$ by

$$J_a^\Omega(u) := \frac{\int_\Omega |\nabla u|^2 dx - \int_\Omega \frac{a(x)u^2}{|x|^2} dx}{\left(\int_\Omega |u|^{2^\star} dx \right)^{\frac{2}{2^\star}}}.$$

We define

$$\mu_a(\Omega) := \inf \left\{ J_a^\Omega(u) | u \in H_0^1(\Omega) \setminus \{0\} \right\}.$$

In the following $D^{1,2}(\mathbb{R}^n_+)$ is the completion of $C_0^\infty(\mathbb{R}^n_+)$ for the norm $u \mapsto \|\nabla u\|_2$. We prove the following result.

Theorem 1.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$. Let $a \in C^{0,\theta}(\overline{\Omega})$ for some $\theta \in (0,1)$. We assume that $-\Delta - \frac{a(x)}{|x|^2}$ is coercive and that $\gamma := a(0) < \frac{n^2}{4}$. We assume that*

$$\mu_a(\Omega) < \frac{1}{K(n,\gamma)}$$

where

$$\frac{1}{K(n,\gamma)} := \inf_{u \in D^{1,2}(\mathbb{R}^n_+) \setminus \{0\}} \frac{\int_{\mathbb{R}^n_+} (|\nabla v|^2 - \frac{\gamma v^2}{|x|^2}) dx}{(\int_{\mathbb{R}^n_+} |v|^{2^\star} dx)^{\frac{2}{2^\star}}}.$$

Then there exists $u \in H_0^1(\Omega) \setminus \{0\}$ that is a weak solution of (H). Moreover, u can be achieved as a minimizer for $\mu_a(\Omega)$.

Remark 1. *For example, if $a(x) = \gamma$ and $\gamma < \frac{(n-2)^2}{4}$ we have that $-\Delta - \frac{a(x)}{|x|^2}$ is coercive in $\mathbb{R}^n$. More generally, it is necessary and sufficient that $\gamma < \gamma_H(\Omega)$ or $\gamma_H(\Omega)$ is the Hardy-Sobolev constant.*

This note is devoted to the proof of Theorem 1.1.

2. Proof of Theorem 1.1

Let $(u_k)_{k \in \mathbb{N}} \in H_0^1(\Omega) \setminus \{0\}$ be a minimizing sequence for $\mu_a(\Omega)$, that is $J_a^\Omega(u_k) = \mu_a(\Omega) + o(1)$ as $k \to +\infty$. Without loss of generality, we can assume that

$$\|u_k\|_{2^*}^2 = 1 \text{ for all } k \in \mathbb{N},$$

and then

$$\int_\Omega |\nabla u_k|^2 - \frac{a(x)u_k^2}{|x|^2} dx = \mu_a(\Omega) + o(1) \text{ as } k \to +\infty. \tag{2}$$

Step 1: We claim that $(u_k)_{k \in \mathbb{N}}$ is bounded in $H_0^1(\Omega)$.

Proof. Since the operator is coercive, there exists $C > 0$ such that

$$\int_\Omega \left(|\nabla u_k|^2 - \frac{a(x)u_k^2}{|x|^2} \right) dx \geq C \int_\Omega |\nabla u_k|^2 dx \tag{3}$$

for all $k \in \mathbb{N}$. We combine (2) and (3) to get

$$C \int_\Omega |\nabla u_k|^2 dx \leq \mu_a(\Omega) + o(1)$$

as $k \to +\infty$. Therefore $(u_k)_{k\in\mathbb{N}}$ is bounded in $H_0^1(\Omega)$. This proves the claim. □

As a consequence, up to the extraction of a subsequence, there exists $u \in H_0^1(\Omega)$ such that

$$\begin{array}{lll} u_k \rightharpoonup u & \text{converges weakly in} & H_0^1(\Omega) \\ u_k \to u & \text{converges strongly in} & L^2(\Omega) \\ u_k \to u & \text{converges a.e in} & \Omega. \end{array}$$

We define $v_k = u_k - u$, we use equation (2) and we get

$$\int_\Omega |\nabla(v_k + u)|^2 dx - \int_\Omega \frac{a(x)(v_k+u)^2}{|x|^2} dx = \mu_a(\Omega) + o(1).$$

Therefore

$$\begin{aligned} &\int_\Omega |\nabla v_k|^2 dx + \int_\Omega |\nabla u|^2 dx + 2\int_\Omega (\nabla v_k, \nabla u) dx \\ &- \int_\Omega \frac{a(x)v_k^2}{|x|^2} dx - \int_\Omega \frac{a(x)u^2}{|x|^2} dx - 2\int_\Omega \frac{a(x)v_k u}{|x^2|} dx = \mu_a(\Omega) + o(1). \end{aligned}$$

Since $v_k \rightharpoonup 0$ converges weakly in $H_0^1(\Omega)$ as $k \to +\infty$, we get that

$$\int_\Omega |\nabla v_k|^2 dx + \int_\Omega |\nabla u|^2 dx - \int_\Omega \frac{a(x)v_k^2}{|x|^2} dx - \int_\Omega \frac{a(x)u^2}{|x|^2} dx = \mu_a(\Omega) + o(1) \tag{4}$$

as $k \to +\infty$.

Step 2: We claim that

$$\lim_{k\to+\infty} \left[\int_\Omega \frac{a(x)v_k^2}{|x|^2} dx - \int_\Omega \frac{a(0)v_k^2}{|x|^2} dx \right] = 0 \tag{5}$$

where $a(0) = \gamma$.

Proof. Since $a \in C^{0,\theta}(\bar{\Omega})$ then there exists $c > 0$ such that

$$|a(x) - a(0)| \leq c|x|^\theta \text{ for all } x \in \Omega.$$

This gives

$$\begin{aligned} \left| \int_\Omega \frac{a(x)v_k^2}{|x|^2} dx - \int_\Omega \frac{a(0)v_k^2}{|x|^2} dx \right| &= \left| \int_\Omega \frac{(a(x)-a(0))v_k^2}{|x|^2} \right| \\ &\leq \int_\Omega \frac{|a(x)-a(0)|v_k^2}{|x|^2} dx \\ &\leq c\int_\Omega \frac{|x|^\theta v_k^2}{|x|^2} dx. \end{aligned}$$

We let $\delta > 0$ to be fixed later. We have that

$$\begin{aligned}\left|\int_\Omega \frac{a(x)v_k^2}{|x|^2}dx - \int_\Omega \frac{a(0)v_k^2}{|x|^2}dx\right| &\leq c\int_{B(0,\delta)} \frac{|x|^\theta v_k^2}{|x|^2}dx + c\int_{\Omega\setminus B(0,\delta)} \frac{|x|^\theta v_k^2}{|x|^2}dx \\ &= I_1(k) + I_2(k)\end{aligned}$$

as $k \to +\infty$. Concerning $I_1(k)$, we use the Hardy inequality (1) to get

$$I_1(k) = c\int_{B(0,\delta)} \frac{|x|^\theta v_k^2}{|x|^2}dx \leq \delta^\theta \int_{B(0,\delta)} \frac{v_k^2}{|x|^2}dx \leq \frac{c}{\gamma_H(\Omega)}\delta^\theta \int_\Omega |\nabla v_k|^2 dx$$

for all $k \in \mathbb{N}$. Therefore, since $(v_k)_k$ is bounded in $H_0^1(\Omega)$, we get that

$$I_1(k) \leq c_1\delta^\theta \tag{6}$$

for all $k \in \mathbb{N}$. Concerning $I_2(k)$, we have that

$$I_2(k) = c\int_{\Omega\setminus B(0,\delta)} \frac{|x|^\theta v_k^2}{|x|^2}dx \leq c_2\delta^{\theta-2}\int_{\Omega\setminus B(0,\delta)} v_k^2 dx$$

for all $k \in \mathbb{N}$. Since $v_k \to 0$ strongly in L^2, we get that

$$\lim_{k\to\infty} I_2(k) = 0. \tag{7}$$

Let $\epsilon > 0$ be a positive number. Since $\lim_{\delta\to 0} c_1\delta^\theta = 0$, there exists $\delta > 0$ such that $c_1\delta^\theta < \frac{\epsilon}{2}$, and then (6) yields $I_1(k) < \frac{\epsilon}{2}$ for all $k \in \mathbb{N}$. Moreover, it follows from (7) that there exists k_0 such that $I_2(k) < \frac{\epsilon}{2}$ for all $k > k_0$. Then,

$$\left|\int_\Omega \frac{a(x)v_k^2}{|x|^2}dx - \int_\Omega \frac{a(0)v_k^2}{|x|^2}dx\right| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $k > k_0$. Therefore,

$$\lim_{k\to\infty}\left[\int_\Omega \frac{a(x)v_k^2}{|x|^2}\,dx - \int_\Omega \frac{a(0)v_k^2}{|x|^2}\,dx\right] = 0.$$

This proves the claim. □

Step 3: We claim that $\|u\|_{2^*}^2 = 1$.

Proof. We have

$$\|u\|_{2^*}^2\mu_a(\Omega) \leq \int_\Omega |\nabla u|^2 dx - \int_\Omega \frac{a(x)u^2}{|x|^2}dx. \tag{8}$$

It follows from Ghoussoub-Robert [[2], Proposition 4.3] that for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\|v\|_{2^*}^2 \leq (K(n,\gamma)+\epsilon)\int_\Omega \left(|\nabla v|^2 - \frac{\gamma v^2}{|x|^2}\right)dx + C_\epsilon\int_\Omega v^2 dx$$

for all $v \in H_0^1(\Omega)$. When $k \to +\infty$, $v_k \to 0$ in $L^2(\Omega)$, we then get that

$$\|v_k\|_{2^*}^2 \leq (K(n,\gamma)+\epsilon)\int_\Omega \left(|\nabla v_k|^2 - \frac{\gamma v_k^2}{|x|^2}\right)dx + o(1). \tag{9}$$

We combine (4), (5), (8) and (9) and to get

$$\int_\Omega |\nabla v_k|^2 dx + \int_\Omega |\nabla u|^2 dx - \int_\Omega \frac{a(x)v_k^2}{|x|^2}dx - \int_\Omega \frac{a(x)u^2}{|x|^2}dx = \mu_a(\Omega) + o(1)$$

$$\begin{aligned}
\int_\Omega |\nabla u|^2 dx - \int_\Omega \frac{a(x)u^2}{|x|^2}dx &= \mu_a(\Omega) - \int_\Omega |\nabla v_k|^2 dx + \int_\Omega \frac{a(x)v_k^2}{|x|^2}dx + o(1) \\
\|u\|_{2^*}^2 \mu_a(\Omega) &\leq \mu_a(\Omega) - \int_\Omega |\nabla v_k|^2 dx + \int_\Omega \frac{a(x)v_k^2}{|x|^2}dx + o(1) \\
(\|u\|_{2^*}^2 - 1)\mu_a(\Omega) &\leq -\int_\Omega |\nabla v_k|^2 dx + \int_\Omega \frac{a(x)v_k^2}{|x|^2}dx + o(1) \\
(1 - \|u\|_{2^*}^2)\mu_a(\Omega) &\geq \int_\Omega |\nabla v_k|^2 dx - \int_\Omega \frac{a(x)v_k^2}{|x|^2}dx + o(1) \\
&\geq \int_\Omega |\nabla v_k|^2 dx - \int_\Omega \frac{a(0)v_k^2}{|x|^2}dx + o(1) \\
(1 - \|u\|_{2^*}^2)\mu_a(\Omega) &\geq \frac{1}{(K(n,\gamma)+\epsilon)}\|v_k\|_{2^*}^2 + o(1) \qquad (10)
\end{aligned}$$

as $k \to +\infty$, since $a(0) = \gamma$. It follows from Brezis-Lieb that

$$\|u_k\|_{2^*}^2 = \|v_k\|_{2^*}^2 + \|u\|_{2^*}^2 + o(1)$$

as $k \to +\infty$. Since $\|u_k\|_{2^*}^2 = 1$, we get

$$1 = \|v_k\|_{2^*}^2 + \|u\|_{2^*}^2 + o(1). \qquad (11)$$

Inequality (10) and (11) yield

$$\begin{aligned}
(1 - \|u\|_{2^*}^2)\mu_a(\Omega) &\geq \frac{1}{(K(n,\gamma)+\epsilon)}(1 - \|u\|_{2^*}^2) \\
(1 - \|u\|_{2^*}^2)(\mu_a(\Omega) - \frac{1}{(K(n,\gamma)+\epsilon)}) &\geq 0.
\end{aligned}$$

Since $\mu_a(\Omega) < \frac{1}{K(n,\gamma)}$, then there exists $\epsilon > 0$ such that $\mu_a(\Omega) - \frac{1}{K(n,\gamma)+\epsilon} < 0$, and therefore

$$\|u\|_{2^*}^2 \geq 1.$$

Since $u_k \rightharpoonup u$ converges weakly in $H_0^1(\Omega)$ as $k \to +\infty$, we have that $\|u\|_{2^*}^2 \leq \liminf \|u_k\|_{2^*}^2 = 1$. Therefore, we get that

$$\|u\|_{2^*}^2 = 1.$$

This proves the claim. □

Step 4: We claim that u is a minimizer for $\mu_a(\Omega)$.

Proof. Equation (4) rewrites

$$\int_\Omega |\nabla u|^2 dx - \int_\Omega \frac{a(x)u^2}{|x|^2}dx = \mu_a(\Omega) - \int_\Omega |\nabla v_k|^2 dx + \int_\Omega \frac{a(x)v_k^2}{|x|^2}dx + o(1). \qquad (12)$$

Equation (5) gives

$$\int_\Omega |\nabla v_k|^2 dx - \int_\Omega \frac{a(x)v_k^2}{|x|^2}dx = \int_\Omega |\nabla v_k|^2 dx - \int_\Omega \frac{a(0)v_k^2}{|x|^2}dx + o(1)$$

as $k \to +\infty$. Then, combining (9) with (12), we get that

$$\begin{aligned}
\int_\Omega |\nabla u|^2 dx - \int_\Omega \frac{a(x)u^2}{|x|^2}dx &\leq \mu_a(\Omega) - \frac{1}{K(n,\gamma)+\epsilon}\|v_k\|_{2^*}^2 + o(1) \\
&\leq \mu_a(\Omega) + o(1)
\end{aligned}$$

as $k \to +\infty$. Therefore, since $\|u\|_{2^*}^2 = 1$, we get

$$\frac{\int_\Omega |\nabla u|^2 dx - \int_\Omega \frac{a(x)u^2}{|x|^2} dx}{\|u\|_{2^*}^2} \leq \mu_a(\Omega). \tag{13}$$

Combining (13) with the definition of $\mu_a(\Omega)$, we get that

$$\mu_a(\Omega) = \frac{\int_\Omega |\nabla u|^2 dx - \int_\Omega \frac{a(x)u^2}{|x|^2} dx}{\|u\|_{2^\star}^2}.$$

Therefore, u is a minimizer of $\mu_a(\Omega)$ and the claim is proved. □

Step 5: We claim that, up to multiplying by a positive constant, u is a weak solution to (H).

Proof. Indeed, since u is a minimum for $\mu_a(\Omega)$, it is a critical point of J_a^Ω and then, using that $\|u\|_{2^\star} = 1$, we may as well assume that $u \geq 0$ on Ω (otherwise we replace u by $|u|$ in the minimization process) and we get that

$$\int_\Omega \nabla u \nabla v dx - \int_\Omega \frac{a(x)}{|x|^2} uv dx = \mu_a(\Omega) \int_\Omega |u|^{2^*-2} uv dx \text{ for all } v \in H_0^1(\Omega),$$

and u is a weak solution to equation

$$-\Delta u - \frac{a(x)}{|x|^2} u = \mu_a(\Omega) u^{2^\star - 1} \text{ in } H_0^1(\Omega).$$

We define $u = k\tilde{u}$ where $k := \mu_a(\Omega)^{\frac{-1}{(2^\star-2)}}$. As one checks, $\tilde{u}$ is still a minimizer of J_a^Ω and $\tilde{u}$ is a weak solution to (H). This proves the claim and Step 5. □

Remark 2. *It follows from the above analysis that $u_k \to u$ strongly in $H_0^1(\Omega)$. Indeed, we use equation* (4)*, the coercivity of the operator and that u is minimun to conclude that*

$$\int_\Omega |\nabla v_k|^2 dx - \int_\Omega \frac{a(x) v_k^2 dx}{|x|^2} dx = o(1)$$

and then

$$\int_\Omega |\nabla v_k|^2 dx \leq o(1)$$

as $k \to +\infty$. Since $v_k \to 0$ strongly in $L^2(\Omega)$, we get that $v_k \to 0$ strongly in $H_0^1(\Omega)$. Therefore $u_k \to u$ strongly in $H_0^1(\Omega)$.

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Institut Élie Cartan, Université de Lorraine, BP 70239, F-54506 Vandœuvre-lès-Nancy, France. Département de Mathmatiques, Université Libre de Bruxelles, Campus de la Plaine CP213, Bd. du Triomphe, 1050 Bruxelles, Belgium

E-mail address: hussein.cheikh-ali@univ-lorraine.fr, Hussein.Cheikh-Ali@ulb.ac.be