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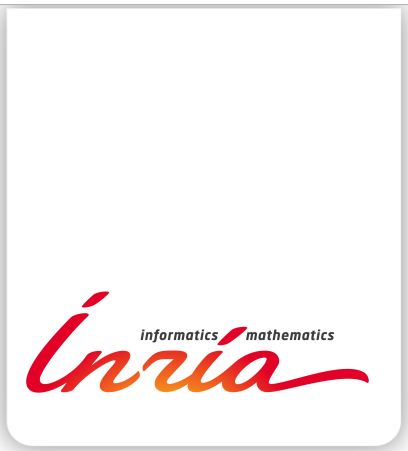

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Abstract: In this research report, an achievability region and a converse region for the two-user Gaussian interference channel with noisy channel-output feedback (G-IC-NOF) are presented. The achievability region is obtained using a random coding argument and three well-known techniques: rate splitting, superposition coding and backward decoding. The converse region is obtained using some of the existing perfect-output feedback outer-bounds as well as a set of new outer-bounds that are obtained by using genie-aided models of the original G-IC-NOF. Finally, it is shown that the achievability region and the converse region approximate the capacity region of the G-IC-NOF to within a constant gap in bits per channel use.

Key-words: Achievable region, converse region, two-user Gaussian interference channel, noisy chanel-output feedback

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Résumé : Ce rapport présente une approximation de la capacité du canal à interférences Gaussien avec rétroalimentation degradée (G-IC-NOF). À cette fin, une région atteignable et une région converse du G-IC-NOF sont introduites, permettant de déterminer que la capacité du G-IC-NOF est approchée à 4.4 bits par utilisation de canal.

Mots-clés : Région atteignable, région converse, canal à interférences Gaussien, rétroalimentation dégradée.

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1 Notation

Throughout this research report, sets are denoted with uppercase calligraphic letters, i.e. $\mathcal{X}$. Random variables are denoted by uppercase letters, e.g., X. The realizations and the set of events from which the random variable X takes values are respectively denoted by x and $\mathcal{X}$. The probability distribution of X over the set $\mathcal{X}$ is denoted P_X. Whenever a second random variable Y is involved, $P_{X\,Y}$ and $P_{Y|X}$ denote respectively the joint probability distribution of (X, Y) and the conditional probability distribution of Y given X. Let N be a fixed natural number. In case a random variable is an N-dimensional vector, it is denoted by $\mathbf{X} = (X_1, X_2, ..., X_N)^\mathsf{T}$ and a corresponding realization is denoted by $\mathbf{x} = (x_1, x_2, ..., x_N)^\mathsf{T} \in \mathcal{X}^N$. Given $\mathbf{X} = (X_1, X_2, ..., X_N)$ and $(a, b) \in \mathbb{N}^2$, with $a < b \leqslant N$, the $(b - a + 1)$-dimensional random variable formed by the components a to b of $\boldsymbol{X}$ is denoted by $\mathbf{X}_{(a:b)} = (X_a, X_{a+1}, \ldots, X_b)^\mathsf{T}$. The notation $(\cdot)^+$ denotes the positive part operator, i.e., $(\cdot)^+ = \max(\cdot, 0)$ and $\mathbb{E}_X[\cdot]$ denotes the expectation with respect to the distribution of the random variable X. The logarithm function log is assumed to be base 2.

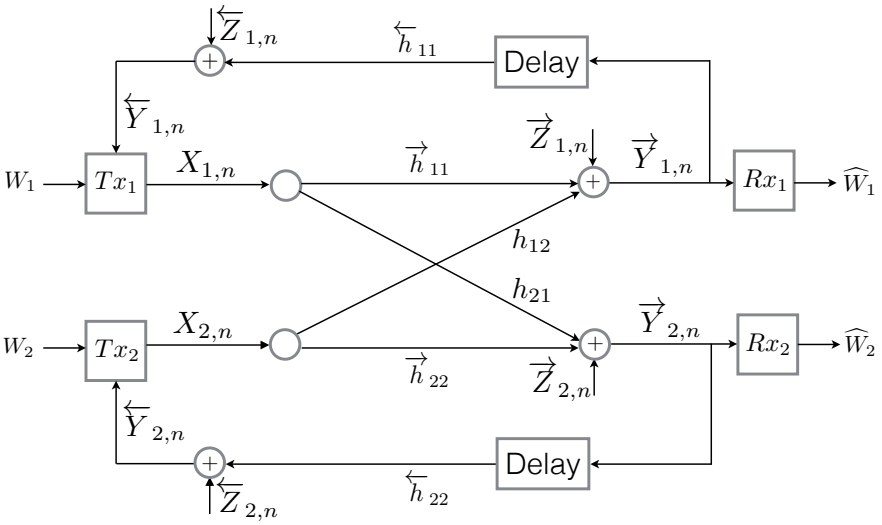

Figure 1: Gaussian interference channel with noisy channel-output feedback at channel use n.

2 Problem Formulation

This section introduces the two-user Gaussian interference channel with noisy channel-output feedback (G-IC-NOF) and defines an approximation to its corresponding capacity region.

Consider the two-user G-IC-NOF in Fig. 1. Transmitter i, with $i \in \{1,2\}$, communicates with receiver i subject to the interference produced by transmitter j, with $j \in \{1,2\}\backslash\{i\}$. There are two independent and uniformly distributed messages, $W_i \in \mathcal{W}_i$, with $\mathcal{W}_i = \{1,2,\ldots,2^{NR_i}\}$, where N denotes the fixed block-length in channel uses and R_i is the transmission rate in bits per channel use. At each block, transmitter i sends the codeword $\boldsymbol{X}_i = (X_{i,1}, X_{i,2}, \ldots, X_{i,N})^{\mathsf{T}} \in \mathcal{X}_i^N$, where $\mathcal{X}_i$ and $\mathcal{X}_i^N$ are respectively the channel-input alphabet and the codebook of transmitter i.

The channel coefficient from transmitter j to receiver i is denoted by h_{ij}; the channel coefficient from transmitter i to receiver i is denoted by $\overrightarrow{h}_{ii}$; and the channel coefficient from channel-output i to transmitter i is denoted by $\overleftarrow{h}_{ii}$. All channel coefficients are assumed to be non-negative real numbers. At a given channel use $n \in \{1,2,\ldots,N\}$, the channel output at receiver i is denoted by $\overrightarrow{Y}_{i,n}$. During channel use n, the input-output relation of the channel model is given by:

$$\overrightarrow{Y}_{i,n} = \overrightarrow{h}_{ii} X_{i,n} + h_{ij} X_{j,n} + \overrightarrow{Z}_{i,n}, \tag{1}$$

where $\overrightarrow{Z}_{i,n}$ is a real Gaussian random variable with zero mean and unit variance that represents the noise at the input of receiver i. Let $d > 0$ be the finite feedback delay in channel uses. At the end of channel use n, transmitter i observes $\overleftarrow{Y}_{i,n}$, which consists of a scaled and noisy version of $\overrightarrow{Y}_{i,n-d}$ (see Fig. 2). More specifically,

$$\overleftarrow{Y}_{i,n} = \begin{cases} \overleftarrow{Z}_{i,n} & \text{for } n \in \{1,2,\ldots,d\} \\ \overleftarrow{h}_{ii}\overrightarrow{Y}_{i,n-d} + \overleftarrow{Z}_{i,n}, & \text{for } n \in \{d+1,d+2,\ldots,N\}, \end{cases} \tag{2}$$

where $\overleftarrow{Z}_{i,n}$ is a real Gaussian random variable with zero mean and unit variance that represents the noise in the feedback link of transmitter-receiver pair i. The random variables $\overrightarrow{Z}_{i,n}$ and

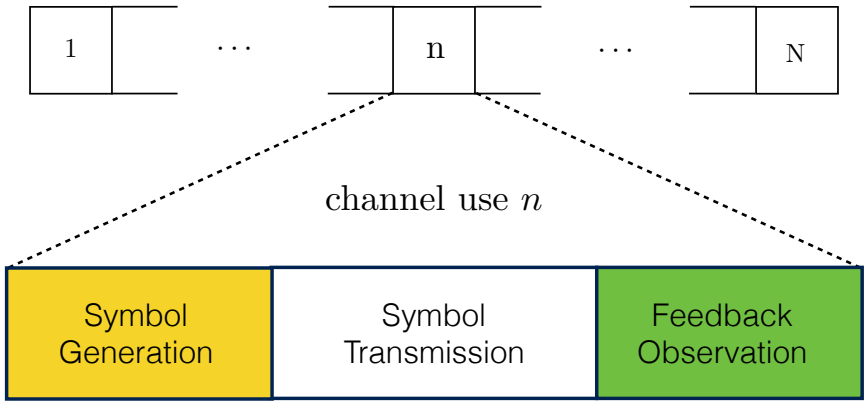

Figure 2: Phases of channel use n at transmitter-receiver pair i: Symbol generation phase occurs following (3); Symbol transmission phase occurs following (1); and feedback observation occurs following (2).

$\overleftarrow{Z}_{i,n}$ are independent and identically distributed. In the following, without loss of generality, the feedback delay is assumed to be one channel use ($d = 1$). The encoder of transmitter i is defined by a set of deterministic functions $f_i^{(1)}, f_i^{(2)}, \ldots, f_i^{(N)}$, with $f_i^{(1)} : \mathcal{W}_i \to \mathcal{X}_i$ and for all $n \in \{2, 3, \ldots, N\}$, $f_i^{(n)} : \mathcal{W}_i \times \mathbb{R}^{n-1} \to \mathcal{X}_i$, such that

$$X_{i,1} = f_i^{(1)}(W_i), \text{ and} \tag{3a}$$
$$X_{i,n} = f_i^{(n)}\left(W_i, \overleftarrow{Y}_{i,1}, \overleftarrow{Y}_{i,2}, \ldots, \overleftarrow{Y}_{i,n-1}\right). \tag{3b}$$

The components of the input vector $\boldsymbol{X}_i$ are real numbers subject to an average power constraint:

$$\frac{1}{N}\sum_{n=1}^{N} \mathbb{E}\left({X_{i,n}}^2\right) \leq 1, \tag{4}$$

where the expectation is taken over the joint distribution of the message indexes W_1, W_2, and the noise terms, i.e., $\overrightarrow{Z}_1$, $\overrightarrow{Z}_2$, $\overleftarrow{Z}_1$, and $\overleftarrow{Z}_2$. The dependence of $X_{i,n}$ on W_1, W_2, and the previously observed noise realizations is due to the effect of feedback as shown in (2) and (3).
Let $T \in \mathbb{N}$ be fixed. Assume that during a given communication, T blocks, each of N channel uses, are transmitted. Hence, the decoder of receiver i is defined by a deterministic function $\psi_i : \mathbb{R}_i^{NT} \to \mathcal{W}_i^T$. At the end of the communication, receiver i uses the vector $\left(\overrightarrow{Y}_{i,1}, \overrightarrow{Y}_{i,2}, \ldots, \overrightarrow{Y}_{i,NT}\right)^{\mathsf{T}}$ to obtain an estimation of the message indices

$$\left(\widehat{W}_i^{(1)}, \widehat{W}_i^{(2)}, \ldots, \widehat{W}_i^{(T)}\right) = \psi_i\left(\overrightarrow{Y}_{i,1}, \overrightarrow{Y}_{i,2}, \ldots, \overrightarrow{Y}_{i,NT}\right), \tag{5}$$

where $\widehat{W}_i^{(t)}$ is an estimation of the message index sent during block $t \in \{1, 2, \ldots, T\}$. The decoding error probability in the two-user G-IC-NOF during block t, denoted by $P_e^{(t)}(N)$, is

given by

$$P_e^{(t)}(N)=\max\left(\Pr\left(\widehat{W_1}^{(t)}\neq W_1^{(t)}\right),\Pr\left(\widehat{W_2}^{(t)}\neq W_2^{(t)}\right)\right). \tag{6}$$

The definition of an achievable rate pair $(R_1,R_2)\in\mathbb{R}_+^2$ is given below.

Definition 1 (Achievable Rate Pairs) A rate pair $(R_1,R_2)\in\mathbb{R}_+^2$ is achievable if there exists at least one pair of codebooks $\mathcal{X}_1^N$ and $\mathcal{X}_2^N$ with codewords of length N, and the corresponding encoding functions $f_1^{(1)},f_1^{(2)},\ldots,f_1^{(N)}$ and $f_2^{(1)},f_2^{(2)},\ldots,f_2^{(N)}$ such that the decoding error probability $P_e^{(t)}(N)$ can be made arbitrarily small by letting the block-length N grow to infinity, for all blocks $t\in\{1,2,\ldots,T\}$.

The two-user G-IC-NOF in Fig. 1 can be fully described by six parameters: $\overrightarrow{\text{SNR}}_i$, $\overleftarrow{\text{SNR}}_i$, and INR_{ij}, with $i\in\{1,2\}$ and $j\in\{1,2\}\backslash\{i\}$, which are defined as follows

$$\overrightarrow{\text{SNR}}_i=\overrightarrow{h}_{ii}^2, \tag{7}$$

$$\text{INR}_{ij}=h_{ij}^2 \text{ and} \tag{8}$$

$$\overleftarrow{\text{SNR}}_i=\overleftarrow{h}_{ii}^2\left(\overrightarrow{h}_{ii}^2+2\overrightarrow{h}_{ii}h_{ij}+h_{ij}^2+1\right). \tag{9}$$

The analysis presented in this report focuses exclusively on the case in which $\text{INR}_{ij}>1$ for all $(i,j)\in\{1,2\}\times\{\{1,2\}\setminus\{i\}\}$. The reason for exclusively considering this case follows from the the fact that when $\text{INR}_{ij}\leqslant 1$, the transmitter-receiver pair i is impaired mainly by noise instead of interference. In this case, treating interference as noise is optimal and feedback does not bring a significant rate improvement.

3 Main Results

This section introduces an achievable region (Theorem 1) and a converse region (Theorem 2), denoted by $\underline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ and $\overline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ respectively, for the two-user G-IC-NOF with fixed parameters $\overrightarrow{\mathrm{SNR}}_1$, $\overrightarrow{\mathrm{SNR}}_2$, INR_{12}, INR_{21}, $\overleftarrow{\mathrm{SNR}}_1$, and $\overleftarrow{\mathrm{SNR}}_2$. In general, the capacity region of a given multi-user channel is said to be approximated to within a constant gap according to the following definition.

Definition 2 (Approximation to within ξ units) A closed and convex set $\mathcal{T} \subset \mathbb{R}_+^m$ is approximated to within ξ units by the sets $\underline{\mathcal{T}}$ and $\overline{\mathcal{T}}$ if $\underline{\mathcal{T}} \subseteq \mathcal{T} \subseteq \overline{\mathcal{T}}$ and for all $\boldsymbol{t} = (t_1, \ldots, t_m) \in \overline{\mathcal{T}}$, $\left((t_1 - \xi)^+, \ldots, (t_m - \xi)^+\right) \in \underline{\mathcal{T}}$.

Denote by $\mathcal{C}_{\mathrm{GIC-NOF}}$ the capacity region of the 2-user G-IC-NOF. The achievable region $\underline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ and the converse region $\overline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ approximate the capacity region $\mathcal{C}_{\mathrm{GIC-NOF}}$ to within 4.4 bits per channel use (Theorem 3).

3.1 An Achievable Region

The description of the achievable region $\underline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ is presented using the constants $a_{1,i}$; the functions $a_{2,i} : [0,1] \to \mathbb{R}_+$, $a_{l,i} : [0,1]^2 \to \mathbb{R}_+$, with $l \in \{3, \ldots, 6\}$; and $a_{7,i} : [0,1]^3 \to \mathbb{R}_+$, which are defined as follows, for all $i \in \{1,2\}$, with $j \in \{1,2\} \setminus \{i\}$:

$$a_{1,i} = \frac{1}{2}\log\left(2 + \frac{\overrightarrow{\mathrm{SNR}}_i}{\mathrm{INR}_{ji}}\right) - \frac{1}{2}, \tag{10a}$$

$$a_{2,i}(\rho) = \frac{1}{2}\log\Big(b_{1,i}(\rho) + 1\Big) - \frac{1}{2}, \tag{10b}$$

$$a_{3,i}(\rho,\mu) = \frac{1}{2}\log\left(\frac{\overleftarrow{\mathrm{SNR}}_i\Big(\mathrm{INR}_{ij} + 1\Big) + b_{1,i}(1) + 1}{\overleftarrow{\mathrm{SNR}}_i\Big((1-\mu)b_{2,i}(\rho) + 2\Big) + b_{1,i}(1) + 1}\right), \tag{10c}$$

$$a_{4,i}(\rho,\mu) = \frac{1}{2}\log\Big(\Big(1-\mu\Big)b_{2,i}(\rho) + 2\Big) - \frac{1}{2}, \tag{10d}$$

$$a_{5,i}(\rho,\mu) = \frac{1}{2}\log\left(2 + \frac{\overrightarrow{\mathrm{SNR}}_i}{\mathrm{INR}_{ji}} + \Big(1-\mu\Big)b_{2,i}(\rho)\right) - \frac{1}{2}, \tag{10e}$$

$$a_{6,i}(\rho,\mu) = \frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_i}{\mathrm{INR}_{ji}}\Big(\Big(1-\mu\Big)b_{2,j}(\rho)+1\Big)+2\right) - \frac{1}{2}, \tag{10f}$$

$$a_{7,i}(\rho,\mu_1,\mu_2) = \frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_i}{\mathrm{INR}_{ji}}\Big(\Big(1-\mu_i\Big)b_{2,j}(\rho)+1\Big) + \Big(1-\mu_j\Big)b_{2,i}(\rho) + 2\right) - \frac{1}{2}. \tag{10g}$$

where the functions $b_{l,i} : [0,1] \to \mathbb{R}_+$, with $(l,i) \in \{1,2\}^2$ are defined as follows

$$b_{1,i}(\rho) = \overrightarrow{\mathrm{SNR}}_i + 2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_i \mathrm{INR}_{ij}} + \mathrm{INR}_{ij} \text{ and} \tag{11a}$$

$$b_{2,i}(\rho) = \Big(1-\rho\Big)\mathrm{INR}_{ij} - 1, \tag{11b}$$

with $j \in \{1,2\} \setminus \{i\}$.
Note that the functions in (10) and (11) depend on $\overrightarrow{\mathrm{SNR}}_1$, $\overrightarrow{\mathrm{SNR}}_2$, INR_{12}, INR_{21}, $\overleftarrow{\mathrm{SNR}}_1$, and $\overleftarrow{\mathrm{SNR}}_2$, however as these parameters are fixed in this analysis, this dependence is not emphasized

Theorem 1 The capacity region $\mathcal{C}_{\mathrm{GIC-NOF}}$ contains the region $\underline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ given by the closure of the set of all possible non-negative achievable rate pairs (R_1, R_2) that satisfy

$$R_1 \leqslant \min\Big(a_{2,1}(\rho), a_{6,1}(\rho,\mu_1) + a_{3,2}(\rho,\mu_1), a_{1,1} + a_{3,2}(\rho,\mu_1) + a_{4,2}(\rho,\mu_1)\Big), \tag{12a}$$

$$R_2 \leqslant \min\Big(a_{2,2}(\rho), a_{3,1}(\rho,\mu_2) + a_{6,2}(\rho,\mu_2), a_{3,1}(\rho,\mu_2) + a_{4,1}(\rho,\mu_2) + a_{1,2}\Big), \tag{12b}$$

$$\begin{aligned} R_1 + R_2 \leqslant \min\Big(&a_{2,1}(\rho) + a_{1,2}, a_{1,1} + a_{2,2}(\rho), a_{3,1}(\rho,\mu_2) + a_{1,1} + a_{3,2}(\rho,\mu_1) + a_{7,2}(\rho,\mu_1,\mu_2),\\ &a_{3,1}(\rho,\mu_2) + a_{5,1}(\rho,\mu_2) + a_{3,2}(\rho,\mu_1) + a_{5,2}(\rho,\mu_1),\\ &a_{3,1}(\rho,\mu_2) + a_{7,1}(\rho,\mu_1,\mu_2) + a_{3,2}(\rho,\mu_1) + a_{1,2}\Big), \end{aligned} \tag{12c}$$

$$\begin{aligned} 2R_1 + R_2 \leqslant \min\Big(&a_{2,1}(\rho) + a_{1,1} + a_{3,2}(\rho,\mu_1) + a_{7,2}(\rho,\mu_1,\mu_2),\\ &a_{3,1}(\rho,\mu_2) + a_{1,1} + a_{7,1}(\rho,\mu_1,\mu_2) + 2a_{3,2}(\rho,\mu_1) + a_{5,2}(\rho,\mu_1),\\ &a_{2,1}(\rho) + a_{1,1} + a_{3,2}(\rho,\mu_1) + a_{5,2}(\rho,\mu_1)\Big), \end{aligned} \tag{12d}$$

$$\begin{aligned} R_1 + 2R_2 \leqslant \min\Big(&a_{3,1}(\rho,\mu_2) + a_{5,1}(\rho,\mu_2) + a_{2,2}(\rho) + a_{1,2},\\ &a_{3,1}(\rho,\mu_2) + a_{7,1}(\rho,\mu_1,\mu_2) + a_{2,2}(\rho) + a_{1,2},\\ &2a_{3,1}(\rho,\mu_2) + a_{5,1}(\rho,\mu_2) + a_{3,2}(\rho,\mu_1) + a_{1,2} + a_{7,2}(\rho,\mu_1,\mu_2)\Big), \end{aligned} \tag{12e}$$

with $(\rho,\mu_1,\mu_2) \in \Big[0, \Big(1 - \max\Big(\frac{1}{\mathrm{INR}_{12}}, \frac{1}{\mathrm{INR}_{21}}\Big)\Big)^+\Big] \times [0,1] \times [0,1]$.

in the definition of these functions. Finally, using this notation, Theorem 1 is presented on the top of this page.

Proof: The proof of Theorem 1 is presented in Appendix A. ∎

3.2 Comments on the Achievability

The achievable region is obtained using a random coding argument and combining three classical tools: rate splitting, superposition coding, and backward decoding. This coding scheme is described in Appendix A and it is specially designed for the two-user IC-NOF. Consequently, only the strictly needed number of superposition code-layers is used. Other achievable schemes, as reported in [1], can also be obtained as special cases of the more general scheme presented in [2]. However, in this more general case, the resulting code for the IC-NOF contains a handful of unnecessary superposing code-layers, which complicates the error probability analysis.

3.3 A Converse Region

The description of the converse region $\overline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ is determined by the ratios $\frac{\mathrm{INR}_{ij}}{\overrightarrow{\mathrm{SNR}}_j}$, and $\frac{\mathrm{INR}_{ji}}{\overrightarrow{\mathrm{SNR}}_j}$, for all $i \in \{1,2\}$, with $j \in \{1,2\} \setminus \{i\}$. All relevant scenarios regarding these ratios are described

by two events denoted by $S_{l_1,1}$ and $S_{l_2,2}$, where $(l_1,l_2)\in\{1,\ldots,5\}^2$:

$$S_{1,i}\colon\quad \overrightarrow{\mathrm{SNR}}_j < \min\left(\mathrm{INR}_{ij},\mathrm{INR}_{ji}\right), \tag{13a}$$

$$S_{2,i}\colon\quad \mathrm{INR}_{ji} \leqslant \overrightarrow{\mathrm{SNR}}_j < \mathrm{INR}_{ij}, \tag{13b}$$

$$S_{3,i}\colon\quad \mathrm{INR}_{ij} \leqslant \overrightarrow{\mathrm{SNR}}_j < \mathrm{INR}_{ji}, \tag{13c}$$

$$S_{4,i}\colon\quad \max\left(\mathrm{INR}_{ij},\mathrm{INR}_{ji}\right) \leqslant \overrightarrow{\mathrm{SNR}}_j < \mathrm{INR}_{ij}\mathrm{INR}_{ji}, \tag{13d}$$

$$S_{5,i}\colon\quad \overrightarrow{\mathrm{SNR}}_j \geqslant \max\left(\mathrm{INR}_{ij},\mathrm{INR}_{ji},\mathrm{INR}_{ij}\mathrm{INR}_{ji}\right). \tag{13e}$$

Note that for all $i\in\{1,2\}$, the events $S_{1,i}$, $S_{2,i}$, $S_{3,i}$, $S_{4,i}$, and $S_{5,i}$ are mutually exclusive. This observation shows that given any 4-tuple $(\overrightarrow{\mathrm{SNR}}_1,\overrightarrow{\mathrm{SNR}}_2,\mathrm{INR}_{12},\mathrm{INR}_{21})$, there always exists one and only one pair of events $(S_{l_1,1},S_{l_2,2})$, with $(l_1,l_2)\in\{1,\ldots,5\}^2$, that identifies a unique scenario. Note also that the pairs of events $(S_{2,1},S_{2,2})$ and $(S_{3,1},S_{3,2})$ are not feasible. In view of this, twenty-three different scenarios can be identified using the events in (13). Once the exact scenario is identified, the converse region is described using the functions $\kappa_{l,i}:[0,1]\to\mathbb{R}_+$, with $l\in\{1,\ldots,3\}$; $\kappa_l:[0,1]\to\mathbb{R}_+$, with $l\in\{4,5\}$; $\kappa_{6,l}:[0,1]\to\mathbb{R}_+$, with $l\in\{1,\ldots,4\}$; and $\kappa_{7,i,l}:[0,1]\to\mathbb{R}_+$, with $l\in\{1,2\}$. These functions are defined as follows for all $i\in\{1,2\}$, with $j\in\{1,2\}\setminus\{i\}$

$$\kappa_{1,i}(\rho)=\frac{1}{2}\log\Big(b_{1,i}(\rho)+1\Big), \tag{14a}$$

$$\kappa_{2,i}(\rho)=\frac{1}{2}\log\Big(1+b_{5,j}(\rho)\Big)+\frac{1}{2}\log\left(1+\frac{b_{4,i}(\rho)}{1+b_{5,j}(\rho)}\right), \tag{14b}$$

$$\kappa_{3,i}(\rho)=\frac{1}{2}\log\left(\frac{\Big(b_{4,i}(\rho)+b_{5,j}(\rho)+1\Big)^{\overleftarrow{\mathrm{SNR}}_j}}{\Big(b_{1,j}(1)+1\Big)\Big(b_{4,i}(\rho)+1\Big)}+1\right)+\frac{1}{2}\log\Big(b_{4,i}(\rho)+1\Big), \tag{14c}$$

$$\kappa_4(\rho)=\frac{1}{2}\log\left(1+\frac{b_{4,1}(\rho)}{1+b_{5,2}(\rho)}\right)+\frac{1}{2}\log\Big(b_{1,2}(\rho)+1\Big), \tag{14d}$$

$$\kappa_5(\rho)=\frac{1}{2}\log\left(1+\frac{b_{4,2}(\rho)}{1+b_{5,1}(\rho)}\right)+\frac{1}{2}\log\Big(b_{1,1}(\rho)+1\Big), \tag{14e}$$

$$\kappa_6(\rho)=\begin{cases}\kappa_{6,1}(\rho) & \text{if } (S_{1,2}\vee S_{2,2}\vee S_{5,2})\wedge(S_{1,1}\vee S_{2,1}\vee S_{5,1})\\ \kappa_{6,2}(\rho) & \text{if } (S_{1,2}\vee S_{2,2}\vee S_{5,2})\wedge(S_{3,1}\vee S_{4,1})\\ \kappa_{6,3}(\rho) & \text{if } (S_{3,2}\vee S_{4,2})\wedge(S_{1,1}\vee S_{2,1}\vee S_{5,1})\\ \kappa_{6,4}(\rho) & \text{if } (S_{3,2}\vee S_{4,2})\wedge(S_{3,1}\vee S_{4,1})\end{cases} \tag{14f}$$

$$\kappa_{7,i}(\rho)=\begin{cases}\kappa_{7,i,1}(\rho) & \text{if } (S_{1,i}\vee S_{2,i}\vee S_{5,i})\\ \kappa_{7,i,2}(\rho) & \text{if } (S_{3,i}\vee S_{4,i})\end{cases} \tag{14g}$$

where

$$\begin{aligned}\kappa_{6,1}(\rho)=&\frac{1}{2}\log\Big(b_{1,1}(\rho)+b_{5,1}(\rho)\mathrm{INR}_{21}\Big)-\frac{1}{2}\log\Big(1+\mathrm{INR}_{12}\Big)+\frac{1}{2}\log\left(1+\frac{b_{5,2}(\rho)^{\overleftarrow{\mathrm{SNR}}_2}}{b_{1,2}(1)+1}\right)\\&+\frac{1}{2}\log\Big(b_{1,2}(\rho)+b_{5,1}(\rho)\mathrm{INR}_{21}\Big)-\frac{1}{2}\log\Big(1+\mathrm{INR}_{21}\Big)+\frac{1}{2}\log\left(1+\frac{b_{5,1}(\rho)^{\overleftarrow{\mathrm{SNR}}_1}}{b_{1,1}(1)+1}\right)\\&+\log(2\pi e),\end{aligned} \tag{15a}$$

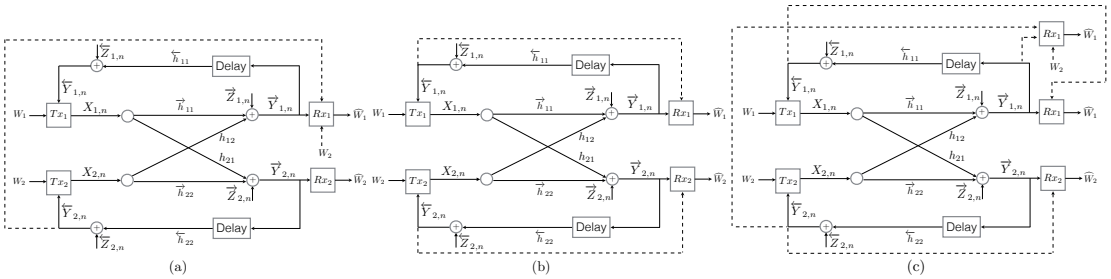

Figure 3: Genie-Aided G-IC-NOF models for channel use n. (a) Model used to calculate the outer-bound on R_1; (b) Model used to calculate the outer-bound on R_1+R_2; and (c) Model used to calculate the outer-bound on $2R_1+R_2$

$$\kappa_{6,2}(\rho)=\frac{1}{2}\log\left(b_{6,2}(\rho)+\frac{b_{5,1}(\rho)\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_2}\left(\overrightarrow{\mathrm{SNR}}_2+b_{3,2}\right)\right)-\frac{1}{2}\log\left(1+\mathrm{INR}_{12}\right) \tag{15b}$$
$$+\frac{1}{2}\log\left(1+\frac{b_{5,1}(\rho)\overleftarrow{\mathrm{SNR}}_1}{b_{1,1}(1)+1}\right)+\frac{1}{2}\log\left(b_{1,1}(\rho)+b_{5,1}(\rho)\mathrm{INR}_{21}\right)-\frac{1}{2}\log\left(1+\mathrm{INR}_{21}\right)$$
$$+\frac{1}{2}\log\left(1+\frac{b_{5,2}(\rho)}{\overrightarrow{\mathrm{SNR}}_2}\left(\mathrm{INR}_{12}+\frac{b_{3,2}\overleftarrow{\mathrm{SNR}}_2}{b_{1,2}(1)+1}\right)\right)-\frac{1}{2}\log\left(1+\frac{b_{5,1}(\rho)\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_2}\right)+\log(2\pi e),$$

$$\kappa_{6,3}(\rho)=\frac{1}{2}\log\left(b_{6,1}(\rho)+\frac{b_{5,1}(\rho)\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_1}\left(\overrightarrow{\mathrm{SNR}}_1+b_{3,1}\right)\right)-\frac{1}{2}\log\left(1+\mathrm{INR}_{12}\right) \tag{15c}$$
$$+\frac{1}{2}\log\left(1+\frac{b_{5,2}(\rho)\overleftarrow{\mathrm{SNR}}_2}{b_{1,2}(1)+1}\right)+\frac{1}{2}\log\left(b_{1,2}(\rho)+b_{5,1}(\rho)\mathrm{INR}_{21}\right)-\frac{1}{2}\log\left(1+\mathrm{INR}_{21}\right)$$
$$+\frac{1}{2}\log\left(1+\frac{b_{5,1}(\rho)}{\overrightarrow{\mathrm{SNR}}_1}\left(\mathrm{INR}_{21}+\frac{b_{3,1}\overleftarrow{\mathrm{SNR}}_1}{b_{1,1}(1)+1}\right)\right)-\frac{1}{2}\log\left(1+\frac{b_{5,1}(\rho)\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_1}\right)+\log(2\pi e),$$

$$\kappa_{6,4}(\rho)=\frac{1}{2}\log\left(b_{6,1}(\rho)+\frac{b_{5,1}(\rho)\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_1}\left(\overrightarrow{\mathrm{SNR}}_1+b_{3,1}\right)\right)-\frac{1}{2}\log\left(1+\mathrm{INR}_{12}\right)-\frac{1}{2}\log\left(1+\mathrm{INR}_{21}\right)$$
$$+\frac{1}{2}\log\left(1+\frac{b_{5,2}(\rho)}{\overrightarrow{\mathrm{SNR}}_2}\left(\mathrm{INR}_{12}+\frac{b_{3,2}\overleftarrow{\mathrm{SNR}}_2}{b_{1,2}(1)+1}\right)\right)-\frac{1}{2}\log\left(1+\frac{b_{5,1}(\rho)\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_2}\right)$$
$$-\frac{1}{2}\log\left(1+\frac{b_{5,1}(\rho)\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_1}\right)+\frac{1}{2}\log\left(b_{6,2}(\rho)+\frac{b_{5,1}(\rho)\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_2}\left(\overrightarrow{\mathrm{SNR}}_2+b_{3,2}\right)\right)$$
$$+\frac{1}{2}\log\left(1+\frac{b_{5,1}(\rho)}{\overrightarrow{\mathrm{SNR}}_1}\left(\mathrm{INR}_{21}+\frac{b_{3,1}\overleftarrow{\mathrm{SNR}}_1}{b_{1,1}(1)+1}\right)\right)+\log(2\pi e), \tag{15d}$$

and

$$\kappa_{7,i,1}(\rho)=\frac{1}{2}\log\left(b_{1,i}(\rho)+1\right)-\frac{1}{2}\log\left(1+\mathrm{INR}_{ij}\right)+\frac{1}{2}\log\left(1+\frac{b_{5,j}(\rho)\overleftarrow{\mathrm{SNR}}_j}{b_{1,j}(1)+1}\right)$$
$$+\frac{1}{2}\log\left(b_{1,j}(\rho)+b_{5,i}(\rho)\mathrm{INR}_{ji}\right)+\frac{1}{2}\log\left(1+b_{4,i}(\rho)+b_{5,j}(\rho)\right)-\frac{1}{2}\log\left(1+b_{5,j}(\rho)\right)$$
$$+2\log(2\pi e), \tag{16a}$$

$$\kappa_{7,i,2}(\rho)=\frac{1}{2}\log\left(b_{1,i}(\rho)+1\right)-\frac{1}{2}\log\left(1+\mathrm{INR}_{ij}\right)-\frac{1}{2}\log\left(1+b_{5,j}(\rho)\right)$$
$$+\frac{1}{2}\log\left(1+b_{4,i}(\rho)+b_{5,j}(\rho)\right)+\frac{1}{2}\log\left(1+\left(1-\rho^2\right)\frac{\mathrm{INR}_{ji}}{\overrightarrow{\mathrm{SNR}}_j}\left(\mathrm{INR}_{ij}+\frac{b_{3,j}\overleftarrow{\mathrm{SNR}}_j}{b_{1,j}(1)+1}\right)\right)$$
$$-\frac{1}{2}\log\left(1+\frac{b_{5,i}(\rho)\mathrm{INR}_{ji}}{\overrightarrow{\mathrm{SNR}}_j}\right)+\frac{1}{2}\log\left(b_{6,j}(\rho)+\frac{b_{5,i}(\rho)\mathrm{INR}_{ji}}{\overrightarrow{\mathrm{SNR}}_j}\left(\overrightarrow{\mathrm{SNR}}_j+b_{3,j}\right)\right)$$
$$+2\log(2\pi e), \tag{16b}$$

where the functions $b_{l,i}$, with $(l,i)\in\{1,2\}^2$ are defined in (11); $b_{3,i}$ are constants; and the functions $b_{l,i}:[0,1]\to\mathbb{R}_+$, with $(l,i)\in\{4,5,6\}\times\{1,2\}$ are defined as follows, with $j\in\{1,2\}\setminus\{i\}$:

$$b_{3,i}=\overrightarrow{\mathrm{SNR}}_i-2\sqrt{\overrightarrow{\mathrm{SNR}}_i\mathrm{INR}_{ji}}+\mathrm{INR}_{ji}, \tag{17a}$$
$$b_{4,i}(\rho)=\left(1-\rho^2\right)\overrightarrow{\mathrm{SNR}}_i, \tag{17b}$$
$$b_{5,i}(\rho)=\left(1-\rho^2\right)\mathrm{INR}_{ij}, \tag{17c}$$
$$b_{6,i}(\rho)=\overrightarrow{\mathrm{SNR}}_i+\mathrm{INR}_{ij}+2\rho\sqrt{\mathrm{INR}_{ij}}\left(\sqrt{\overrightarrow{\mathrm{SNR}}_i}-\sqrt{\mathrm{INR}_{ji}}\right)+\frac{\mathrm{INR}_{ij}\sqrt{\mathrm{INR}_{ji}}}{\overrightarrow{\mathrm{SNR}}_i}\left(\sqrt{\mathrm{INR}_{ji}}-2\sqrt{\overrightarrow{\mathrm{SNR}}_i}\right). \tag{17d}$$

Note that the functions in (14), (15), (16) and (17) depend on $\overrightarrow{\mathrm{SNR}}_1$, $\overrightarrow{\mathrm{SNR}}_2$, INR_{12}, INR_{21}, $\overleftarrow{\mathrm{SNR}}_1$, and $\overleftarrow{\mathrm{SNR}}_2$. However, these parameters are fixed in this analysis, and therefore, this dependence is not emphasized in the definition of these functions. Finally, using this notation, Theorem 2 is presented below.

Theorem 2 The capacity region $\mathcal{C}_{\mathrm{GIC-NOF}}$ is contained within the region $\overline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ given by the closure of the set of non-negative rate pairs (R_1,R_2) that satisfy for all $i\in\{1,2\}$, with $j\in\{1,2\}\setminus\{i\}$:

$$R_i\leqslant\min\left(\kappa_{1,i}(\rho),\kappa_{2,i}(\rho)\right), \tag{18a}$$
$$R_i\leqslant\kappa_{3,i}(\rho), \tag{18b}$$
$$R_1+R_2\leqslant\min\left(\kappa_4(\rho),\kappa_5(\rho)\right), \tag{18c}$$
$$R_1+R_2\leqslant\kappa_6(\rho), \tag{18d}$$
$$2R_i+R_j\leqslant\kappa_{7,i}(\rho), \tag{18e}$$

with $\rho\in[0,1]$.

Proof: The proof of Theorem 2 is presented in Appendix B. ■

3.4 Comments on the Converse Region

The outer bounds (18a) and (18c) correspond to the outer bounds for the case of perfect channel-output feedback [3]. The bounds (18b), (18d) and (18e) correspond to new outer bounds that generalize those presented in [1] for the two-user symmetric G-IC-NOF. These new outer-bounds were obtained using the genie-aided models shown in Fig. 3.

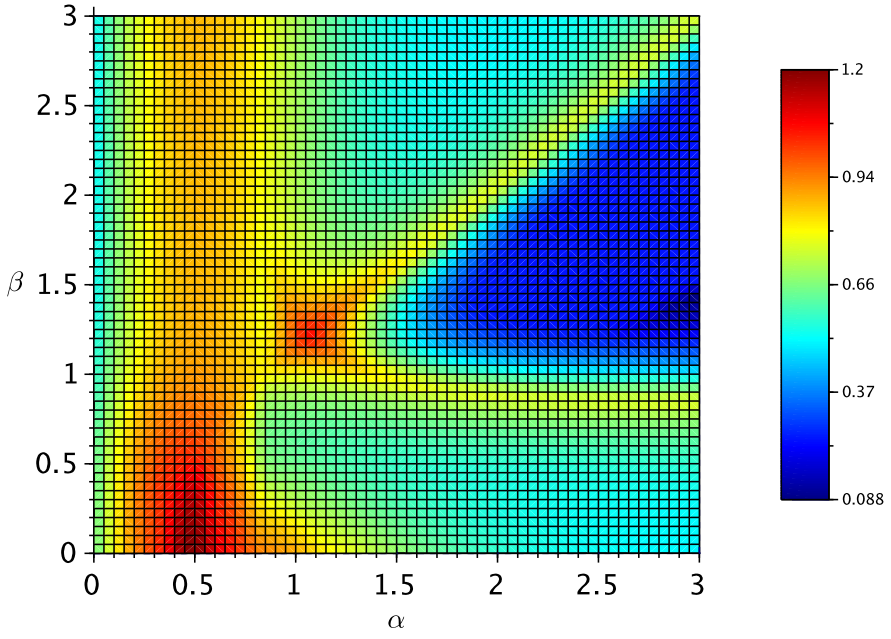

Figure 4: Gap between the converse region $\overline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ and the achievable region $\underline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ of the two-user G-IC-NOF, under symmetric channel conditions, i.e., $\overrightarrow{\mathrm{SNR}}_1 = \overrightarrow{\mathrm{SNR}}_2 = \overrightarrow{\mathrm{SNR}}$, $\mathrm{INR}_{12} = \mathrm{INR}_{21} = \mathrm{INR}$, and $\overleftarrow{\mathrm{SNR}}_1 = \overleftarrow{\mathrm{SNR}}_2 = \overleftarrow{\mathrm{SNR}}$, as a function of $\alpha = \frac{\log \mathrm{INR}}{\log \overrightarrow{\mathrm{SNR}}}$ and $\beta = \frac{\log \overleftarrow{\mathrm{SNR}}}{\log \overrightarrow{\mathrm{SNR}}}$.

3.5 A Gap Between the Achievable Region and the Converse Region

Theorem 3 describes the gap between the achievable region $\underline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ and the converse region $\overline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ using the approximation notion described in Definition 2.

Theorem 3 The capacity region of the two-user G-IC-NOF is approximated to within 4.4 bits per channel use by the achievable region $\underline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ and the converse region $\overline{\mathcal{C}}_{\mathrm{GIC-NOF}}$.

Proof: The proof of Theorem 3 is presented in C. ■

Fig. 4 presents the exact gap existing between the achievable region $\underline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ and the converse region $\overline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ for the case in which $\overrightarrow{\mathrm{SNR}}_1 = \overrightarrow{\mathrm{SNR}}_2 = \overrightarrow{\mathrm{SNR}}$, $\mathrm{INR}_{12} = \mathrm{INR}_{21} = \mathrm{INR}$, and $\overleftarrow{\mathrm{SNR}}_1 = \overleftarrow{\mathrm{SNR}}_2 = \overleftarrow{\mathrm{SNR}}$ as a function of $\alpha = \frac{\log \mathrm{INR}}{\log \overrightarrow{\mathrm{SNR}}}$ and $\beta = \frac{\log \overleftarrow{\mathrm{SNR}}}{\log \overrightarrow{\mathrm{SNR}}}$. Note that in this case, the maximum gap is 1.3 bits per channel use and occurs when $\alpha = 0.5$ and $\beta = 0$.

4 Conclusions

An achievable region and a converse region for the two-user G-IC-NOF have been introduced. It has been shown that these regions approximate the capacity region of the two-user G-IC-NOF to within 4.4 bits per channel use.

Appendices

A Proof of Achievability

This appendix describes an achievability scheme for the IC-NOF based on a three-part message splitting, superposition coding, and backward decoding.
Codebook Generation: fix a strictly positive joint probability distribution

$$P_{U\,U_1\,U_2\,V_1\,V_2\,X_{1,P}\,X_{2,P}}(u,u_1,u_2,v_1,v_2,x_{1,P},x_{2,P}) = P_U(u)P_{U_1|U}(u_1|u)P_{U_2|U}(u_2|u)$$
$$P_{V_1|U\,U_1}(v_1|u,u_1)P_{V_2|U\,U_2}(v_2|u,u_2)P_{X_{1,P}|U\,U_1\,V_1}(x_{1,P}|u,u_1,v_1)P_{X_{2,P}|U\,U_2\,V_2}(x_{2,P}|u,u_2,v_2), \quad (19)$$

for all $(u,u_1,u_2,v_1,v_2,x_{1,P},x_{2,P}) \in (\mathcal{X}_1 \cup \mathcal{X}_2) \times \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_1 \times \mathcal{X}_2$.
Let $R_{1,C1}$, $R_{1,C2}$, $R_{2,C1}$, $R_{2,C2}$, $R_{1,P}$, and $R_{2,P}$ be non-negative real numbers. Let also $R_{1,C} = R_{1,C1} + R_{1,C2}$, $R_{2,C} = R_{2,C1} + R_{2,C2}$, $R_1 = R_{1,C} + R_{1,P}$, and $R_2 = R_{2,C} + R_{2,P}$.
Generate $2^{N(R_{1,C1}+R_{2,C1})}$ i.i.d. N-length codewords $\boldsymbol{u}(s,r) = \big(u_1(s,r), u_2(s,r), \ldots, u_N(s,r)\big)$ according to

$$P_{\boldsymbol{U}}\big(\boldsymbol{u}(s,r)\big) = \prod_{i=1}^{N} P_U(u_i(s,r)), \quad (20)$$

with $s \in \{1,2,\ldots,2^{NR_{1,C1}}\}$ and $r \in \{1,2,\ldots,2^{NR_{2,C1}}\}$.
For encoder 1, generate for each codeword $\boldsymbol{u}(s,r)$, $2^{NR_{1,C1}}$ i.i.d. N-length codewords $\boldsymbol{u}_1(s,r,k) = \big(u_{1,1}(s,r,k), u_{1,2}(s,r,k), \ldots, u_{1,N}(s,r,k)\big)$ according to

$$P_{\boldsymbol{U}_1|\boldsymbol{U}}\big(\boldsymbol{u}_1(s,r,k)|\boldsymbol{u}(s,r)\big) = \prod_{i=1}^{N} P_{U_1|U}\big(u_{1,i}(s,r,k)|u_i(s,r)\big), \quad (21)$$

with $k \in \{1,2,\ldots,2^{NR_{1,C1}}\}$. For each pair of codewords $\big(\boldsymbol{u}(s,r), \boldsymbol{u}_1(s,r,k)\big)$, generate $2^{NR_{1,C2}}$ i.i.d. N-length codewords $\boldsymbol{v}_1(s,r,k,l) = \big(v_{1,1}(s,r,k,l), v_{1,2}(s,r,k,l), \ldots, v_{1,N}(s,r,k,l)\big)$ according to

$$P_{\boldsymbol{V}_1|\boldsymbol{U}\,\boldsymbol{U}_1}\big(\boldsymbol{v}_1(s,r,k,l)|\boldsymbol{u}(s,r),\boldsymbol{u}_1(s,r,k)\big) = \prod_{i=1}^{N} P_{V_1|U\,U_1}\big(v_{1,i}(s,r,k,l)|u_i(s,r),u_{1,i}(s,r,k)\big), \quad (22)$$

with $l \in \{1,2,\ldots,2^{NR_{1,C2}}\}$. For each tuple of codewords $\big(\boldsymbol{u}(s,r), \boldsymbol{u}_1(s,r,k), \boldsymbol{v}_1(s,r,k,l)\big)$, generate $2^{NR_{1,P}}$ i.i.d. N-length codewords $\boldsymbol{x}_{1,P}(s,r,k,l,q) = \big(x_{1,P,1}(s,r,k,l,q), x_{1,P,2}(s,r,k,l,q), \ldots, x_{1,P,N}(s,r,k,l,q)\big)$ according to

$$P_{\boldsymbol{X}_{1,P}|\boldsymbol{U}\,\boldsymbol{U}_1\boldsymbol{V}_1}\big(\boldsymbol{x}_{1,P}(s,r,k,l,q)|\boldsymbol{u}(s,r),\boldsymbol{u}_1(s,r,k),\boldsymbol{v}_1(s,r,k,l)\big)$$
$$= \prod_{i=1}^{N} P_{X_{1,P}|U\,U_1\,V_1}\big(x_{1,P,i}(s,r,k,l,q)|u_i(s,r),u_{1,i}(s,r,k),v_{1,i}(s,r,k,l)\big), \quad (23)$$

with $q \in \{1,2,\ldots,2^{NR_{1,P}}\}$.
For encoder 2, generate for each codeword $\boldsymbol{u}(s,r)$, $2^{NR_{2,C1}}$ i.i.d. N-length codewords $\boldsymbol{u}_2(s,r,j) = \big(u_{2,1}(s,r,j), u_{2,2}(s,r,j), \ldots, u_{2,N}(s,r,j)\big)$ according to

$$P_{\boldsymbol{U}_2|\boldsymbol{U}}\big(\boldsymbol{u}_2(s,r,j)|\boldsymbol{u}(s,r)\big) = \prod_{i=1}^{N} P_{U_2|U}\big(u_{2,i}(s,r,j)|u_i(s,r)\big), \quad (24)$$

with $j \in \{1, 2, \ldots, 2^{NR_{2,C1}}\}$. For each pair of codewords $\big(\boldsymbol{u}(s,r), \boldsymbol{u}_2(s,r,j)\big)$, generate $2^{NR_{2,C2}}$ i.i.d. length-N codewords $\boldsymbol{v}_2(s,r,j,m) = \big(v_{2,1}(s,r,j,m), v_{2,2}(s,r,j,m), \ldots, v_{2,N}(s,r,j,m)\big)$ according to

$$P_{\boldsymbol{V}_2|\boldsymbol{U}\,\boldsymbol{U}_2}\big(\boldsymbol{v}_2(s,r,j,m)|\boldsymbol{u}(s,r), \boldsymbol{u}_2(s,r,j)\big) = \prod_{i=1}^{N} P_{V_2|U\,U_2}(v_{2,i}(s,r,j,m)|u_i(s,r), u_{2,i}(s,r,j)), \quad (25)$$

with $m \in \{1, 2, \ldots, 2^{NR_{2,C2}}\}$. For each tuple of codewords $\big(\boldsymbol{u}(s,r), \boldsymbol{u}_2(s,r,j), \boldsymbol{v}_2(s,r,j,m)\big)$, generate $2^{NR_{2,P}}$ i.i.d. N-length codewords $\boldsymbol{x}_{2,P}(s,r,j,m,b) = \big(x_{2,P,1}(s,r,j,m,b), x_{2,P,2}(s,r,j,m,b), \ldots, x_{2,P,N}(s,r,j,m,b)\big)$ according to

$$\begin{aligned} &P_{\boldsymbol{X}_{2,P}|\boldsymbol{U}\,\boldsymbol{U}_2\boldsymbol{V}_2}\big(\boldsymbol{x}_{2,P}(s,r,j,m,b)|\boldsymbol{u}(s,r), \boldsymbol{u}_2(s,r,j), \boldsymbol{v}_2(s,r,j,m)\big) \\ &= \prod_{i=1}^{N} P_{X_{2,P}|U\,U_2\,V_2}\big(x_{2,P,i}(s,r,j,m,b)|u_i(s,r), u_{2,i}(s,r,j), v_{2,i}(s,r,j,m,b)\big), \end{aligned} \quad (26)$$

with $b \in \{1, 2, \ldots, 2^{NR_{2,P}}\}$. The resulting code structure is shown in Fig. 5.

Encoding: denote by $W_i^{(t)} \in \{1, 2, \ldots, 2^{NR_i}\}$ the message index of transmitter $i \in \{1, 2\}$ during block $t \in \{1, 2, \ldots, T\}$, with T the total number of blocks. Let $W_i^{(t)}$ be composed by the message index $W_{i,C}^{(t)} \in \{1, 2, \ldots, 2^{NR_{i,C}}\}$ and message index $W_{i,P}^{(t)} \in \{1,\ 2,\ \ldots, 2^{NR_{i,P}}\}$. That is, $W_i^{(t)} = \big(W_{i,C}^{(t)}, W_{i,P}^{(t)}\big)$. The message index $W_{i,P}^{(t)}$ must be reliably decoded at receiver i. Let also $W_{i,C}^{(t)}$ be composed by the message indices $W_{i,C1}^{(t)} \in \{1, 2, \ldots, 2^{NR_{i,C1}}\}$ and $W_{i,C2}^{(t)} \in \{1, 2, \ldots, 2^{NR_{i,C2}}\}$. That is, $W_{i,C}^{(t)} = \big(W_{i,C1}^{(t)}, W_{i,C2}^{(t)}\big)$. The message index $W_{i,C1}^{(t)}$ must be reliably decoded by both receivers and transmitter j (via feedback). The message index $W_{i,C2}^{(t)}$ must be reliably decoded by both receivers.

Consider Markov encoding over the T blocks. At encoding step t, with $t \in \{1, 2, \ldots, T\}$, transmitter 1 sends the codeword $\boldsymbol{x}_1^{(t)} = \Theta_1\Big(\boldsymbol{u}\big(W_{1,C1}^{(t-1)}, W_{2,C1}^{(t-1)}\big),\ \boldsymbol{u}_1\big(W_{1,C1}^{(t-1)}, W_{2,C1}^{(t-1)}, W_{1,C1}^{(t)}\big),\ \boldsymbol{v}_1\big(W_{1,C1}^{(t-1)}, W_{2,C1}^{(t-1)}, W_{1,C1}^{(t)}, W_{1,C2}^{(t)}\big),\ \boldsymbol{x}_{1,P}\big(W_{1,C1}^{(t-1)}, W_{2,C1}^{(t-1)},\ W_{1,C1}^{(t)}, W_{1,C2}^{(t)}, W_{1,P}^{(t)}\big)\Big)$, where $\Theta_1 : (\mathcal{X}_1 \cup \mathcal{X}_2)^N \times \mathcal{X}_1^N \times \mathcal{X}_1^N \times \mathcal{X}_1^N \to \mathcal{X}_1^N$ is a function that transforms the codewords $\boldsymbol{u}\big(W_{1,C1}^{(t-1)}, W_{2,C1}^{(t-1)}\big)$, $\boldsymbol{u}_1\big(W_{1,C1}^{(t-1)}, W_{2,C1}^{(t-1)}, W_{1,C1}^{(t)}\big)$, $\boldsymbol{v}_1\big(W_{1,C1}^{(t-1)}, W_{2,C1}^{(t-1)}, W_{1,C1}^{(t)}, W_{1,C2}^{(t)}\big)$, and $\boldsymbol{x}_{1,P}\big(W_{1,C1}^{(t-1)}, W_{2,C1}^{(t-1)}, W_{1,C1}^{(t)}, W_{1,C2}^{(t)}, W_{1,P}^{(t)}\big)$ into the N-dimensional vector $\boldsymbol{x}_1^{(t)}$ of channel inputs. The indices $W_{1,C1}^{(0)} = W_{1,C1}^{(T)} = s^*$ and $W_{2,C1}^{(0)} = W_{2,C1}^{(T)} = r^*$, and the pair $(s^*, r^*) \in \{1, 2, \ldots, 2^{NR_{1,C1}}\} \times \{1, 2, \ldots, 2^{NR_{2,C1}}\}$ are pre-defined and known by both receivers and transmitters. It is worth noting that the message index $W_{2,C1}^{(t-1)}$ is obtained by transmitter 1 from the feedback signal $\overleftarrow{\boldsymbol{y}}_1^{(t-1)}$ at the end of the previous encoding step $t-1$ (see Fig. 2).

Transmitter 2 follows a similar encoding scheme.

Decoding: both receivers decode their message indices at the end of block T in a backward decoding fashion. At each decoding step t, with $t \in \{1, 2, \ldots, T\}$, receiver 1 obtains the message indices $\big(\widehat{W}_{1,C1}^{(T-t)}, \widehat{W}_{2,C1}^{(T-t)}, \widehat{W}_{1,C2}^{(T-(t-1))}, \widehat{W}_{1,P}^{(T-(t-1))}, \widehat{W}_{2,C2}^{(T-(t-1))}\big) \in \{1,\ 2, \ldots,\ 2^{NR_{1,C1}}\} \times \{1, 2, \ldots, 2^{NR_{2,C1}}\} \times \{1, 2, \ldots, 2^{NR_{1,C2}}\} \times \{1, 2, \ldots, 2^{NR_{1,P}}\} \times \{1, 2, \ldots, 2^{NR_{2,C2}}\}$ from the channel output $\overrightarrow{\boldsymbol{y}}_1^{(T-(t-1))}$. The tuple $\Big(\widehat{W}_{1,C1}^{(T-t)}, \widehat{W}_{2,C1}^{(T-t)}, \widehat{W}_{1,C2}^{(T-(t-1))}, \widehat{W}_{1,P}^{(T-(t-1))}, \widehat{W}_{2,C2}^{(T-(t-1))}\Big)$ is the

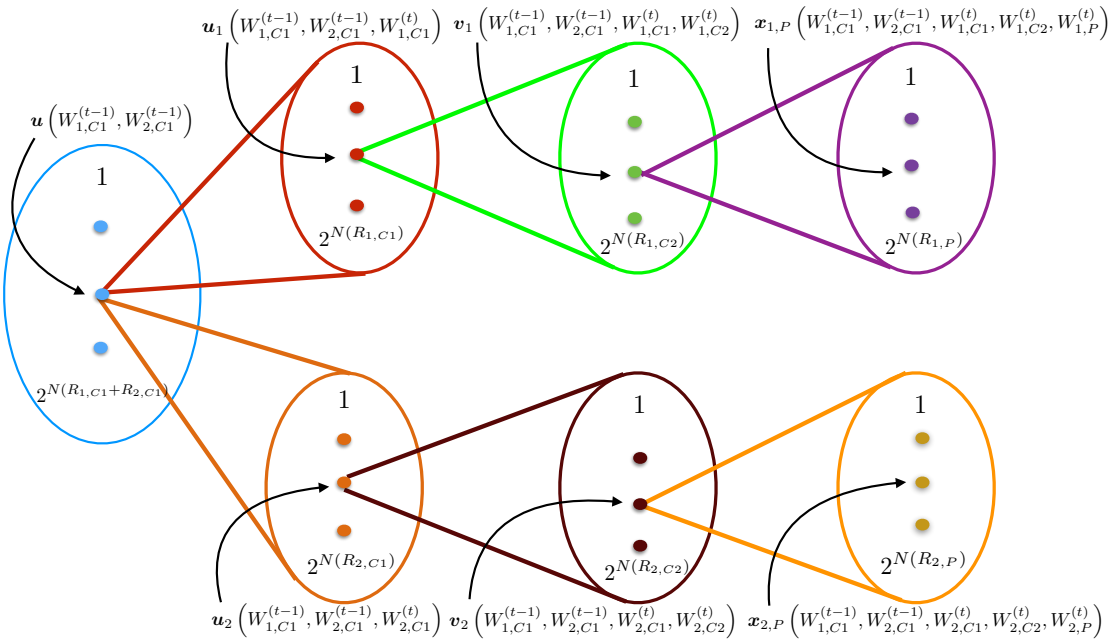

Figure 5: Structure of the superposition code. The codewords corresponding to the message indices $W_{1,C1}^{(t-1)}, W_{2,C1}^{(t-1)}, W_{i,C1}^{(t)}, W_{i,C2}^{(t)}, W_{i,P}^{(t)}$ with $i \in \{1,2\}$ as well as the block index t are both highlighted. The (approximate) number of codewords for each code layer is also highlighted.

unique tuple that satisfies

$$\begin{aligned}
&\Big(\boldsymbol{u}\left(\widehat{W}_{1,C1}^{(T-t)}, \widehat{W}_{2,C1}^{(T-t)}\right), \boldsymbol{u}_1\left(\widehat{W}_{1,C1}^{(T-t)}, \widehat{W}_{2,C1}^{(T-t)}, W_{1,C1}^{(T-(t-1))}\right),\\
&\boldsymbol{v}_1\left(\widehat{W}_{1,C1}^{(T-t)}, \widehat{W}_{2,C1}^{(T-t)}, W_{1,C1}^{(T-(t-1))}, \widehat{W}_{1,C2}^{(T-(t-1))}\right),\\
&\boldsymbol{x}_{1,P}\Big(\widehat{W}_{1,C1}^{(T-t)}, \widehat{W}_{2,C1}^{(T-t)}, W_{1,C1}^{(T-(t-1))}, \widehat{W}_{1,C2}^{(T-(t-1))}, \widehat{W}_{1,P}^{(T-(t-1))}\Big),\\
&\boldsymbol{u}_2\left(\widehat{W}_{1,C1}^{(T-t)}, \widehat{W}_{2,C1}^{(T-t)}, W_{2,C1}^{(T-(t-1))}\right), \boldsymbol{v}_2\left(\widehat{W}_{1,C1}^{(T-t)}, \widehat{W}_{2,C1}^{(T-t)}, W_{2,C1}^{(T-(t-1))}, \widehat{W}_{2,C2}^{(T-(t-1))}\right),\\
&\overrightarrow{\boldsymbol{y}}_1^{(T-(t-1))}\Big) \in \mathcal{T}^{(N,e)}_{\left[U\ U_1\ V_1\ X_{1,P}\ U_2\ V_2\ \overrightarrow{Y}_1\right]},
\end{aligned} \tag{27}$$

where $W_{1,C1}^{(T-(t-1))}$ and $W_{2,C1}^{(T-(t-1))}$ are assumed to be perfectly decoded in the previous decoding step $t-1$. The set $\mathcal{T}^{(N,e)}_{\left[U\ U_1\ V_1\ X_{1,P}\ U_2\ V_2\ \overrightarrow{Y}_1\right]}$ represents the set of jointly typical sequences of the random variables $U, U_1, V_1, X_{1,P}, U_2, V_2$, and $\overrightarrow{Y}_1$, with $e > 0$. Finally, receiver 2 follows a similar decoding scheme.

Probability of Error Analysis: an error might occur during encoding step t if the message index $W_{2,C1}^{(t-1)}$ is not correctly decoded at transmitter 1. From the asymptotic equipartion property (AEP) [4], it follows that the message index $W_{2,C1}^{(t-1)}$ can be reliably decoded at transmitter 1

during encoding step t, under the condition:

$$\begin{aligned} R_{2,C1} &\leqslant I\left(\overleftarrow{Y}_1; U_2|U, U_1, V_1, X_1\right) \\ &= I\left(\overleftarrow{Y}_1; U_2|U, X_1\right). \end{aligned} \tag{28}$$

An error might occur during the (backward) decoding step t if the message indices $W_{1,C1}^{(T-t)}$, $W_{2,C1}^{(T-t)}$, $W_{1,C2}^{(T-(t-1))}$, $W_{1,P}^{(T-(t-1))}$, and $W_{2,C2}^{(T-(t-1))}$ are not decoded correctly given that the message indices $W_{1,C1}^{(T-(t-1))}$ and $W_{2,C1}^{(T-(t-1))}$ were correctly decoded in the previous decoding step $t-1$. These errors might arise for two reasons: (i) there does not exist a tuple $\left(\widehat{W}_{1,C1}^{(T-t)}, \widehat{W}_{2,C1}^{(T-t)}, \widehat{W}_{1,C2}^{(T-(t-1))}, \widehat{W}_{1,P}^{(T-(t-1))}, \widehat{W}_{2,C2}^{(T-(t-1))}\right)$ that satisfies (27), or (ii) there exist several tuples $\left(\widehat{W}_{1,C1}^{(T-t)}, \widehat{W}_{2,C1}^{(T-t)}, \widehat{W}_{1,C2}^{(T-(t-1))}, \widehat{W}_{1,P}^{(T-(t-1))}, \widehat{W}_{2,C2}^{(T-(t-1))}\right)$ that simultaneously satisfy (27). From the asymptotic equipartion property (AEP) [4], the probability of an error due to (i) tends to zero when N grows to infinity. Consider the error due to (ii) and define the event $E_{(s,r,l,q,m)}$ that describes the case in which the codewords $\big(\boldsymbol{u}(s,r)$, $\boldsymbol{u}_1(s,r,W_{1,C1}^{(T-(t-1))})$, $\boldsymbol{v}_1(s,r,W_{1,C1}^{(T-(t-1))},l)$, $\boldsymbol{x}_{1,P}(s,r,W_{1,C1}^{(T-(t-1))},l,q)$, $\boldsymbol{u}_2(s,r,W_{2,C1}^{(T-(t-1))})$, and $\boldsymbol{v}_2(s,r,W_{2,C1}^{(T-(t-1))},m)\big)$ are jointly typical with $\overrightarrow{\boldsymbol{y}}_1^{(T-(t-1))}$ during decoding step t. Assume now that the codeword to be decoded at decoding step t corresponds to the indices $(s,r,l,q,m)=(1,1,1,1,1)$, this is without loss of generality due to the symmetry of the code. Then, the probability of error due to (ii) during decoding step t, can be bounded as follows

$$\begin{aligned} P_e &= \Pr\left(\bigcup_{(s,r,l,q,m)\neq(1,1,1,1,1)} E_{(s,r,l,q,m)}\right) \\ &\leqslant \sum_{(s,r,l,q,m)\in\mathcal{T}} \Pr\left(E_{(s,r,l,q,m)}\right), \end{aligned} \tag{29}$$

with $\mathcal{T} = \Big\{\{1,2,\dots 2^{NR_{1,C1}}\} \times \{1,2,\dots 2^{NR_{2,C1}}\} \times \{1,2,\dots 2^{NR_{1,C2}}\} \times \{1,2,\dots 2^{NR_{1,P}}\} \times \{1\ , 2,\dots 2^{NR_{2,C2}}\}\Big\} \setminus \{(1,1,1,1,1)\}$.
From the asymptotic equipartion property (AEP) [4], it follows that:

$$\begin{aligned} P_e \leqslant\ & 2^{N(R_{2,C2}-I(\overrightarrow{Y}_1;V_2|U,U_1,U_2,V_1,X_1)+2\epsilon)} + 2^{N(R_{1,P}-I(\overrightarrow{Y}_1;X_1|U,U_1,U_2,V_1,V_2)+2\epsilon)} \\ &+2^{N(R_{2,C2}+R_{1,P}-I(\overrightarrow{Y}_1;V_2,X_1|U,U_1,U_2,V_1)+2\epsilon)} + 2^{N(R_{1,C2}-I(\overrightarrow{Y}_1;V_1,X_1|U,U_1,U_2,V_2)+2\epsilon)} \\ &+2^{N(R_{1,C2}+R_{2,C2}-I(\overrightarrow{Y}_1;V_1,V_2,X_1|U,U_1,U_2)+2\epsilon)} + 2^{N(R_{1,C2}+R_{1,P}-I(\overrightarrow{Y}_1;V_1,X_1|U,U_1,U_2,V_2)+2\epsilon)} \\ &+2^{N(R_{1,C2}+R_{1,P}+R_{2,C2}-I(\overrightarrow{Y}_1;V_1,V_2,X_1|U,U_1,U_2)+2\epsilon)} + 2^{N(R_{2,C1}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\ &+2^{N(R_{2,C1}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} + 2^{N(R_{2,C1}+R_{1,P}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\ &+2^{N(R_{2,C1}+R_{1,P}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} + 2^{N(R_{2,C1}+R_{1,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\ &+2^{N(R_{2,C1}+R_{1,C2}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\ &+2^{N(R_{2,C1}+R_{1,C2}+R_{1,P}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\ &+2^{N(R_{2,C}+R_{1,C2}+R_{1,P}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} + 2^{N(R_{1,C1}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \\ &+2^{N(R_{1,C1}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} + 2^{N(R_{1,C1}+R_{1,P}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} \end{aligned}$$

$$
\begin{aligned}
&+2^{N(R_{1,C1}+R_{1,P}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} + 2^{N(R_{1,C1}+R_{1,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{1,C1}+R_{1,C2}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{1,C1}+R_{1,C2}+R_{1,P}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{1,C1}+R_{1,C2}+R_{1,P}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{1,C1}+R_{2,C1}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} + 2^{N(R_{1,C1}+R_{2,C1}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{1,C1}+R_{2,C1}+R_{1,P}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{1,C1}+R_{2,C1}+R_{1,P}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{1,C1}+R_{2,C1}+R_{1,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{1,C1}+R_{2,C1}+R_{1,C2}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{1,C1}+R_{2,C1}+R_{1,C2}+R_{1,P}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)} + 2^{N(R_1+R_{2,C}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}.
\end{aligned}
\tag{30}
$$

The same analysis of the probability of error holds for transmitter-receiver pair 2. Hence, in general, from (28) and (30), reliable decoding holds under the following conditions for transmitter $i \in \{1,2\}$, with $j \in \{1,2\} \setminus \{i\}$:

$$
\begin{aligned}
R_{j,C1} &\leqslant I\left(\overleftarrow{Y}_i; U_j | U, U_i, V_i, X_i\right)\\
&= I\left(\overleftarrow{Y}_i; U_j | U, X_i\right)\\
&\triangleq \theta_{1,i}, && \text{(31a)}\\
R_i + R_{j,C} &\leqslant I(\overrightarrow{Y}_i; U, U_i, U_j, V_i, V_j, X_i)\\
&= I(\overrightarrow{Y}_i; U, U_j, V_j, X_i)\\
&\triangleq \theta_{2,i}, && \text{(31b)}\\
R_{j,C2} &\leqslant I(\overrightarrow{Y}_i; V_j | U, U_i, U_j, V_i, X_i)\\
&= I(\overrightarrow{Y}_i; V_j | U, U_j, X_i)\\
&\triangleq \theta_{3,i}, && \text{(31c)}\\
R_{i,P} &\leqslant I(\overrightarrow{Y}_i; X_i | U, U_i, U_j, V_i, V_j)\\
&\triangleq \theta_{4,i}, && \text{(31d)}\\
R_{i,P} + R_{j,C2} &\leqslant I(\overrightarrow{Y}_i; V_j, X_i | U, U_i, U_j, V_i)\\
&\triangleq \theta_{5,i}, && \text{(31e)}\\
R_{i,C2} + R_{i,P} &\leqslant I(\overrightarrow{Y}_i; V_i, X_i | U, U_i, U_j, V_j)\\
&= I(\overrightarrow{Y}_i; X_i | U, U_i, U_j, V_j)\\
&\triangleq \theta_{6,i}, \text{ and} && \text{(31f)}\\
R_{i,C2} + R_{i,P} + R_{j,C2} &\leqslant I(\overrightarrow{Y}_i; V_i, V_j, X_i | U, U_i, U_j)\\
&= I(\overrightarrow{Y}_i; V_j, X_i | U, U_i, U_j)\\
&\triangleq \theta_{7,i}. && \text{(31g)}
\end{aligned}
$$

Taking into account that $R_i = R_{i,C1} + R_{i,C2} + R_{i,P}$, a Fourier-Motzkin elimination process in

(31) yields

$$R_1 \leqslant \min\left(\theta_{2,1}, \theta_{6,1}+\theta_{1,2}, \theta_{4,1}+\theta_{1,2}+\theta_{3,2}\right), \tag{32a}$$

$$R_2 \leqslant \min\left(\theta_{2,2}, \theta_{1,1}+a_{6,2}, \theta_{1,1}+\theta_{3,1}+\theta_{4,2}\right), \tag{32b}$$

$$\begin{aligned} R_1+R_2 \leqslant \min(&\theta_{2,1}+\theta_{4,2}, \theta_{2,1}+a_{6,2}, \theta_{4,1}+\theta_{2,2}, \theta_{6,1}+\theta_{2,2}, \theta_{1,1}+\theta_{3,1}+\theta_{4,1}+\theta_{1,2}+\theta_{5,2}, \\ &\theta_{1,1}+\theta_{7,1}+\theta_{1,2}+\theta_{5,2}, \theta_{1,1}+\theta_{4,1}+\theta_{1,2}+\theta_{7,2}, \theta_{1,1}+\theta_{5,1}+\theta_{1,2}+\theta_{3,2}+\theta_{4,2}, \\ &\theta_{1,1}+\theta_{5,1}+\theta_{1,2}+\theta_{5,2}, \theta_{1,1}+\theta_{7,1}+\theta_{1,2}+\theta_{4,2}), \end{aligned} \tag{32c}$$

$$2R_1+R_2 \leqslant \min(\theta_{2,1}+\theta_{4,1}+\theta_{1,2}+\theta_{7,2}, \theta_{1,1}+\theta_{4,1}+\theta_{7,1}+2\theta_{1,2}+\theta_{5,2}, \theta_{2,1}+\theta_{4,1}+\theta_{1,2}+\theta_{5,2}), \tag{32d}$$

$$R_1+2R_2 \leqslant \min(\theta_{1,1}+\theta_{5,1}+\theta_{2,2}+\theta_{4,2}, \theta_{1,1}+\theta_{7,1}+\theta_{2,2}+\theta_{4,2}, 2\theta_{1,1}+\theta_{5,1}+\theta_{1,2}+\theta_{4,2}+\theta_{7,2}), \tag{32e}$$

where $\theta_{l,i}$ are defined in (31) with $(l,i) \in \{1,\ldots,7\} \times \{1,2\}$.
Consider that transmitter i uses the following Gaussian input distribution:

$$X_i = U + U_i + V_i + X_{i,P}, \tag{33}$$

where U, U_1, U_2, V_1, V_2, $X_{1,P}$, and $X_{2,P}$ in (19) are mutually independent and distributed as follows:

$$U \sim \mathcal{N}(0,\rho), \tag{34a}$$

$$U_i \sim \mathcal{N}(0,\mu_i\lambda_{i,C}), \tag{34b}$$

$$V_i \sim \mathcal{N}(0,(1-\mu_i)\lambda_{i,C}), \tag{34c}$$

$$X_{i,P} \sim \mathcal{N}(0,\lambda_{i,P}), \tag{34d}$$

with

$$\rho + \lambda_{i,C} + \lambda_{i,P} = 1 \text{ and} \tag{35a}$$

$$\lambda_{i,P} = \min\left(\frac{1}{\mathrm{INR}_{ji}}, 1\right), \tag{35b}$$

where $\mu_i \in [0,1]$ and $\rho \in \left[0, \left(1-\max\left(\frac{1}{\mathrm{INR}_{12}}, \frac{1}{\mathrm{INR}_{21}}\right)\right)^+\right]$.
The parameters ρ, μ_i, and $\lambda_{i,P}$ define a particular power allocation for transmitter i. The assignment in (35b) is based on the intuition obtained from the linear deterministic model, in which the power of the signal $X_{i,P}$ from transmitter i to receiver j must be observed at the noise level. From (1), (2), and (33), the right-hand side of the inequalities in (31) can be written in terms of $\overrightarrow{\mathrm{SNR}}_1$, $\overrightarrow{\mathrm{SNR}}_2$, INR_{12}, INR_{21}, $\overleftarrow{\mathrm{SNR}}_1$, $\overleftarrow{\mathrm{SNR}}_2$, ρ, μ_1, and μ_2 as follows

$$\begin{aligned} \theta_{1,i} &= I\left(\overleftarrow{Y}_i; U_j | U, X_i\right) \\ &= \frac{1}{2}\log\left(\frac{\overleftarrow{\mathrm{SNR}}_i\left(\mathrm{INR}_{ij}+1\right)+b_{1,i}(\rho)+1}{\overleftarrow{\mathrm{SNR}}_i\left((1-\mu_j)\,b_{2,i}(\rho)+2\right)+b_{1,i}(\rho)+1}\right) \\ &= a_{3,i}(\rho,\mu_j), \end{aligned} \tag{36a}$$

$$\begin{aligned}
\theta_{2,i}=&I\left(\overrightarrow{Y}_i;U,U_j,V_j,X_i\right)\\
=&\frac{1}{2}\log\Big(b_{1,i}(\rho)+1\Big)-\frac{1}{2}\\
=&a_{2,i}(\rho), &\text{(36b)}\\
\theta_{3,i}=&I\left(\overrightarrow{Y}_i;V_j|U,U_j,X_i\right)\\
=&\frac{1}{2}\log\Big(\Big(1-\mu_j\Big)b_{2,i}(\rho)+2\Big)-\frac{1}{2}\\
=&a_{4,i}(\rho,\mu_j), &\text{(36c)}\\
\theta_{4,i}=&I\left(\overrightarrow{Y}_i;X_i|U,U_i,U_j,V_i,V_j\right)\\
=&\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}_i}}{\mathrm{INR}_{ji}}+2\right)-\frac{1}{2}\\
=&a_{1,i}, &\text{(36d)}\\
\theta_{5,i}=&I\left(\overrightarrow{Y}_i;V_j,X_i|U,U_i,U_j,V_i\right)\\
=&\frac{1}{2}\log\left(2+\frac{\overrightarrow{\mathrm{SNR}_i}}{\mathrm{INR}_{ji}}+\Big(1-\mu_j\Big)b_{2,i}(\rho)\right)-\frac{1}{2}\\
=&a_{5,i}(\rho,\mu_j), &\text{(36e)}\\
\theta_{6,i}=&I\left(\overrightarrow{Y}_i;X_i|U,U_i,U_j,V_j\right)\\
=&\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}_i}}{\mathrm{INR}_{ji}}\Big(\Big(1-\mu_i\Big)b_{2,j}(\rho)+1\Big)+2\right)-\frac{1}{2}\\
=&a_{6,i}(\rho,\mu_i), &\text{(36f)}\\
\theta_{7,i}=&I\left(\overrightarrow{Y}_i;V_j,X_i|U,U_i,U_j\right)\\
=&\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}_i}}{\mathrm{INR}_{ji}}\Big(\Big(1-\mu_i\Big)b_{2,j}(\rho)+1\Big)+\Big(1-\mu_j\Big)b_{2,i}(\rho)+2\right)-\frac{1}{2}\\
=&a_{7,i}(\rho,\mu_1,\mu_2). &\text{(36g)}
\end{aligned}$$

Finally, plugging (36) into (32) (after some trivial manipulations) yields the system of inequalities in Theorem 1. Note that the sum-rate bound in (32c) can be simplified as follows

$$\begin{aligned}
R_1+R_2\leqslant\min\Big(&a_{2,1}(\rho)+a_{1,2},a_{1,1}+a_{2,2}(\rho),a_{3,1}(\rho,\mu_2)+a_{1,1}+a_{3,2}(\rho,\mu_1)+a_{7,2}(\rho,\mu_1,\mu_2),\\
&a_{3,1}(\rho,\mu_2)+a_{5,1}(\rho,\mu_2)+a_{3,2}(\rho,\mu_1)+a_{5,2}(\rho,\mu_1),\\
&a_{3,1}(\rho,\mu_2)+a_{7,1}(\rho,\mu_1,\mu_2)+a_{3,2}(\rho,\mu_1)+a_{1,2}\Big). &\text{(37)}
\end{aligned}$$

This is mainly because $\max(a_{2,1}(\rho)+a_{1,2},a_{1,1}+a_{2,2}(\rho)\ ,\ a_{3,1}(\rho,\mu_2)+a_{1,1}+a_{3,2}(\rho,\mu_1)+a_{7,2}(\rho,\mu_1,\mu_2),\ a_{3,1}(\rho,\mu_2)+a_{5,1}(\rho,\mu_2)+a_{3,2}(\rho,\mu_1)+a_{5,2}(\rho,\mu_1),a_{3,1}(\rho,\mu_2)+a_{7,1}(\rho,\mu_1,\mu_2)+a_{3,2}(\rho,\mu_1)+a_{1,2})\leqslant\min(a_{2,1}+a_{6,2}(\rho,\mu_2),a_{6,1}(\rho,\mu_1)+a_{2,2}(\rho),a_{3,1}(\rho,\mu_2)+a_{4,1}(\rho,\mu_2)+a_{1,1}+a_{3,2}(\rho,\mu_1)+a_{5,2}(\rho,\mu_1),a_{3,1}(\rho,\mu_2)+a_{7,1}(\rho,\mu_1,\mu_2)+a_{3,2}(\rho,\mu_1)+a_{5,2}(\rho,\mu_1),a_{3,1}(\rho,\mu_2)+a_{5,1}(\rho,\mu_2)+a_{3,2}(\rho,\mu_1)+\theta_{3,2}+a_{1,2})$. Therefore, the inequalities in (32) simplify into (12) and this completes the proof of Theorem 1.

B Proof of Converse

This appendix provides a proof of the Theorem 2. The outer bounds (18a) and (18c) correspond to the outer bounds of the case of perfect channel-output feedback [3]. The bounds (18b), (18d) and (18e) correspond to new outer bounds. Before presenting the proof, consider the parameter $h_{ji,U}$, with $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$, defined as follows

$$h_{ji,U} = \begin{cases} 0 & \text{if } (S_{1,i} \vee S_{2,i} \vee S_{3,i}) \\ \sqrt{\frac{\text{INR}_{ij}\text{INR}_{ji}}{\overrightarrow{\text{SNR}}_j}} & \text{if } (S_{4,i} \vee S_{5,i}), \end{cases} \tag{38}$$

where the events $S_{1,i}$, $S_{2,i}$, $S_{3,i}$, $S_{4,i}$, and $S_{5,i}$ are defined in (13). Consider also the following signals:

$$X_{i,C}=\sqrt{\text{INR}_{ji}}X_i + \overrightarrow{Z}_j \text{ and} \tag{39}$$

$$X_{i,U}=h_{ji,U}X_i + \overrightarrow{Z}_j, \tag{40}$$

where X_i and $\overrightarrow{Z}_j$ are the channel input of transmitter i and the noise observed at the receiver j during a given channel use $n \in \{1,2,\ldots,N\}$, as described by (1). Note that the channel use index n has been dropped for the ease of notation. The following lemma is also fundamental in the present proof of Theorem 2.

Lemma 1 For all $i \in \{1,2\}$, with $j \in \{1,2\} \setminus \{i\}$, the following holds

$$I\Big(\boldsymbol{X}_{i,C}, \boldsymbol{X}_{j,U}, \overleftarrow{\boldsymbol{Y}}_i, W_i; \overleftarrow{\boldsymbol{Y}}_j, W_j\Big) \leqslant h\left(\overleftarrow{\boldsymbol{Y}}_j | W_j\right) + \sum_{n=1}^{N} \Big[h\left(X_{j,U,n}|X_{i,C,n}\right) + h\left(\overleftarrow{Y}_{i,n}|X_{i,n}, X_{j,U,n}\right) - \frac{3}{2}\log\left(2\pi e\right)\Big]. \tag{41}$$

Proof: The proof of Lemma 1 is presented at the end of this Appendix. ∎

Proof of (18b): From the assumption that the message index W_i is i.i.d. following a uniform distribution over the set $\mathcal{W}_i$, the following holds

$$\begin{aligned} NR_i &= H\left(W_i\right) \\ &= H\left(W_i|W_j\right) \\ &\overset{(a)}{\leqslant} I\left(W_i; \overrightarrow{\boldsymbol{Y}}_i, \overleftarrow{\boldsymbol{Y}}_j | W_j\right) + N\delta(N) \\ &\leqslant \sum_{n=1}^{N} \Big[h\Big(\overrightarrow{Y}_{i,n}, \overleftarrow{Y}_{j,n} | W_j, \overrightarrow{\boldsymbol{Y}}_{i,(1:n-1)}, \overleftarrow{\boldsymbol{Y}}_{j,(1:n-1)}, \boldsymbol{X}_{j,(1:n)}\Big) - h\left(\overrightarrow{Z}_{i,n}\right) - h\left(\overleftarrow{Z}_{j,n}\right) \Big] + N\delta(N) \\ &\leqslant \sum_{n=1}^{N} \Big[h\left(\overrightarrow{Y}_{i,n}, \overleftarrow{Y}_{j,n} | X_{j,n}\right) - h\left(\overrightarrow{Z}_{i,n}\right) - h\left(\overleftarrow{Z}_{j,n}\right) \Big] + N\delta(N) \\ &= N\Big[h\left(\overrightarrow{Y}_i, \overleftarrow{Y}_j | X_j\right) - \log\left(2\pi e\right)\Big] + N\delta(N) \text{ for any } n \in \{1,2,\ldots,N\}, \end{aligned} \tag{42}$$

where
(a) follows from Fano's inequality (see Fig. 3a).
From (42), the following holds in the asymptotic regime:

$$\begin{aligned} R_i &\leqslant h\left(\overrightarrow{Y}_i, \overleftarrow{Y}_j | X_j\right) - \log\left(2\pi e\right) \\ &\leqslant \frac{1}{2}\log\Big(b_{3,i}+1\Big) + \frac{1}{2}\log\left(\frac{\Big(b_{3,i}+b_{4,j}(\rho)+1\Big)\overleftarrow{\text{SNR}}_j}{\Big(b_{1,j}(\rho)+1\Big)\Big(b_{3,i}+(1-\rho^2)\Big)} + 1 \right). \end{aligned} \tag{43}$$

This completes the proof of (18b).

Proof of (18d):

From the assumption that the message indices W_1 and W_2 are i.i.d. following a uniform distribution over the sets $\mathcal{W}_1$ and $\mathcal{W}_2$ respectively, the following holds

$$
\begin{aligned}
N\left(R_1+R_2\right) &= H\left(W_1\right)+H\left(W_2\right)\\
&\overset{(a)}{\leqslant} I\left(W_1;\overrightarrow{\boldsymbol{Y}}_1,\overleftarrow{\boldsymbol{Y}}_1\right)+I\left(W_2;\overrightarrow{\boldsymbol{Y}}_2,\overleftarrow{\boldsymbol{Y}}_2\right)+N\delta(N)\\
&= h\left(\overrightarrow{\boldsymbol{Y}}_1\right)+h\left(\overleftarrow{\boldsymbol{Z}}_1|\overrightarrow{\boldsymbol{Y}}_1\right)-h\left(\overleftarrow{\boldsymbol{Y}}_1|W_1\right)-h\left(\overrightarrow{\boldsymbol{Y}}_1|W_1,\overleftarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_1\right)+h\left(\overrightarrow{\boldsymbol{Y}}_2\right)+h\left(\overleftarrow{\boldsymbol{Z}}_2|\overrightarrow{\boldsymbol{Y}}_2\right)\\
&\quad -h\left(\overleftarrow{\boldsymbol{Y}}_2|W_2\right)-h\left(\overrightarrow{\boldsymbol{Y}}_2|W_2,\overleftarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_2\right)+N\delta(N)\\
&\leqslant h\left(\overrightarrow{\boldsymbol{Y}}_1\right)+h\left(\overleftarrow{\boldsymbol{Z}}_1\right)-h\left(\overleftarrow{\boldsymbol{Y}}_1|W_1\right)-h\left(\boldsymbol{X}_{2,C}|W_1,\overleftarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_1\right)+h\left(\overrightarrow{\boldsymbol{Y}}_2\right)+h\left(\overleftarrow{\boldsymbol{Z}}_2\right)-h\left(\overleftarrow{\boldsymbol{Y}}_2|W_2\right)\\
&\quad -h\left(\boldsymbol{X}_{1,C}|W_2,\overleftarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_2\right)+N\delta(N)\\
&= h\left(\overrightarrow{\boldsymbol{Y}}_1\right)-h\left(\overleftarrow{\boldsymbol{Y}}_1|W_1\right)-h\left(\boldsymbol{X}_{2,C},\overrightarrow{\boldsymbol{Z}}_2|W_1,\overleftarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_1\right)+h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1,\overleftarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_1,\boldsymbol{X}_{2,C}\right)+h\left(\overrightarrow{\boldsymbol{Y}}_2\right)\\
&\quad -h\left(\overleftarrow{\boldsymbol{Y}}_2|W_2\right)-h\left(\boldsymbol{X}_{1,C},\overrightarrow{\boldsymbol{Z}}_1|W_2,\overleftarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_2\right)+h\left(\overrightarrow{\boldsymbol{Z}}_1|W_2,\overleftarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_2,\boldsymbol{X}_{1,C}\right)+N\log\left(2\pi e\right)\\
&\quad +N\delta(N)\\
&= h\left(\overrightarrow{\boldsymbol{Y}}_1\right)-h\left(\overleftarrow{\boldsymbol{Y}}_1|W_1\right)-h\left(\boldsymbol{X}_{2,C},\boldsymbol{X}_{1,U}|W_1,\overleftarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_1\right)+h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1,\overleftarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_1,\boldsymbol{X}_{2,C}\right)\\
&\quad +h\left(\overrightarrow{\boldsymbol{Y}}_2\right)-h\left(\overleftarrow{\boldsymbol{Y}}_2|W_2\right)-h\left(\boldsymbol{X}_{1,C},\boldsymbol{X}_{2,U}|W_2,\overleftarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_2\right)+h\left(\overrightarrow{\boldsymbol{Z}}_1|W_2,\overleftarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_2,\boldsymbol{X}_{1,C}\right)\\
&\quad +N\log\left(2\pi e\right)+N\delta(N)\\
&= h\left(\overrightarrow{\boldsymbol{Y}}_1\right)-h\left(\overleftarrow{\boldsymbol{Y}}_1|W_1\right)+\left[I\left(\boldsymbol{X}_{2,C},\boldsymbol{X}_{1,U};W_1,\overleftarrow{\boldsymbol{Y}}_1\right)-h\left(\boldsymbol{X}_{2,C},\boldsymbol{X}_{1,U}\right)\right]+h\left(\overrightarrow{\boldsymbol{Y}}_2\right)\\
&\quad -h\left(\overleftarrow{\boldsymbol{Y}}_2|W_2\right)+\left[I\left(\boldsymbol{X}_{1,C},\boldsymbol{X}_{2,U};W_2,\overleftarrow{\boldsymbol{Y}}_2\right)-h\left(\boldsymbol{X}_{1,C},\boldsymbol{X}_{2,U}\right)\right]\\
&\quad +h\left(\overrightarrow{\boldsymbol{Z}}_1|W_2,\overleftarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_2,\boldsymbol{X}_{1,C}\right)+h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1,\overleftarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_1,\boldsymbol{X}_{2,C}\right)+N\log\left(2\pi e\right)+N\delta(N)\\
&\leqslant h\left(\overrightarrow{\boldsymbol{Y}}_1\right)-h\left(\overleftarrow{\boldsymbol{Y}}_1|W_1\right)+\left[I\left(\boldsymbol{X}_{2,C},\boldsymbol{X}_{1,U};W_1,\overleftarrow{\boldsymbol{Y}}_1\right)-h\left(\boldsymbol{X}_{2,C},\boldsymbol{X}_{1,U}\right)\right]+h\left(\overrightarrow{\boldsymbol{Y}}_2\right)\\
&\quad -h\left(\overleftarrow{\boldsymbol{Y}}_2|W_2\right)+\left[I\left(\boldsymbol{X}_{1,C},\boldsymbol{X}_{2,U};W_2,\overleftarrow{\boldsymbol{Y}}_2\right)-h\left(\boldsymbol{X}_{1,C},\boldsymbol{X}_{2,U}\right)\right]+\left[h\left(\boldsymbol{X}_{2,C},\boldsymbol{X}_{1,U}|\overrightarrow{\boldsymbol{Y}}_2\right)-\right.\\
&\quad \left.h\left(\boldsymbol{X}_{2,C},\boldsymbol{X}_{1,U}|\overrightarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_1,\boldsymbol{X}_2\right)\right]+\left[h\left(\boldsymbol{X}_{1,C},\boldsymbol{X}_{2,U}|\overrightarrow{\boldsymbol{Y}}_1\right)-h\left(\boldsymbol{X}_{1,C},\boldsymbol{X}_{2,U}|\overrightarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_2,\boldsymbol{X}_1\right)\right]\\
&\quad +h\left(\overrightarrow{\boldsymbol{Z}}_1|W_2,\overleftarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_2,\boldsymbol{X}_{1,C}\right)+h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1,\overleftarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_1,\boldsymbol{X}_{2,C}\right)+N\log\left(2\pi e\right)+N\delta(N)\\
&\overset{(b)}{=} h\left(\overrightarrow{\boldsymbol{Y}}_1|\boldsymbol{X}_{1,C},\boldsymbol{X}_{2,U}\right)-h\left(\overleftarrow{\boldsymbol{Y}}_1|W_1\right)+I\left(\boldsymbol{X}_{2,C},\boldsymbol{X}_{1,U};W_1,\overleftarrow{\boldsymbol{Y}}_1\right)+h\left(\overrightarrow{\boldsymbol{Y}}_2|\boldsymbol{X}_{2,C},\boldsymbol{X}_{1,U}\right)\\
&\quad -h\left(\overleftarrow{\boldsymbol{Y}}_2|W_2\right)+I\left(\boldsymbol{X}_{1,C},\boldsymbol{X}_{2,U};W_2,\overleftarrow{\boldsymbol{Y}}_2\right)-h\left(\overrightarrow{\boldsymbol{Z}}_1,\overrightarrow{\boldsymbol{Z}}_2|\overrightarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_1,\boldsymbol{X}_2\right)\\
&\quad -h\left(\overrightarrow{\boldsymbol{Z}}_2,\overrightarrow{\boldsymbol{Z}}_1|\overrightarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_2,\boldsymbol{X}_1\right)+h\left(\overrightarrow{\boldsymbol{Z}}_1|W_2,\overleftarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_2,\boldsymbol{X}_{1,C}\right)+h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1,\overleftarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_1,\boldsymbol{X}_{2,C}\right)\\
&\quad +N\log\left(2\pi e\right)+N\delta(N)\\
&\overset{(c)}{\leqslant} h\left(\overrightarrow{\boldsymbol{Y}}_1|\boldsymbol{X}_{1,C},\boldsymbol{X}_{2,U}\right)-h\left(\overleftarrow{\boldsymbol{Y}}_1|W_1\right)+I\left(\boldsymbol{X}_{2,C},\boldsymbol{X}_{1,U};W_1,\overleftarrow{\boldsymbol{Y}}_1\right)+h\left(\overrightarrow{\boldsymbol{Y}}_2|\boldsymbol{X}_{2,C},\boldsymbol{X}_{1,U}\right)\\
&\quad -h\left(\overleftarrow{\boldsymbol{Y}}_2|W_2\right)+I\left(\boldsymbol{X}_{1,C},\boldsymbol{X}_{2,U};W_2,\overleftarrow{\boldsymbol{Y}}_2\right)+N\log\left(2\pi e\right)+N\delta(N)\\
&\leqslant h\left(\overrightarrow{\boldsymbol{Y}}_1|\boldsymbol{X}_{1,C},\boldsymbol{X}_{2,U}\right)-h\left(\overleftarrow{\boldsymbol{Y}}_1|W_1\right)+I\left(\boldsymbol{X}_{2,C},\boldsymbol{X}_{1,U},W_2,\overleftarrow{\boldsymbol{Y}}_2;W_1,\overleftarrow{\boldsymbol{Y}}_1\right)\\
&\quad +h\left(\overrightarrow{\boldsymbol{Y}}_2|\boldsymbol{X}_{2,C},\boldsymbol{X}_{1,U}\right)-h\left(\overleftarrow{\boldsymbol{Y}}_2|W_2\right)+I\left(\boldsymbol{X}_{1,C},\boldsymbol{X}_{2,U},W_1,\overleftarrow{\boldsymbol{Y}}_1;W_2,\overleftarrow{\boldsymbol{Y}}_2\right)+N\log\left(2\pi e\right)\\
&\quad +N\delta(N)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(d)}{\leqslant}\sum_{n=1}^{N}\Big[h\left(\overrightarrow{Y}_{1,n}|\boldsymbol{X}_{1,C},\boldsymbol{X}_{2,U},\overrightarrow{\boldsymbol{Y}}_{1,(1:n-1)}\right)+h\left(X_{1,U,n}|X_{2,C,n}\right)+h\left(\overleftarrow{Y}_{2,n}|X_{2,n},X_{1,U,n}\right)\\
&\quad+h\left(\overrightarrow{Y}_{2,n}|\boldsymbol{X}_{2,C},\boldsymbol{X}_{1,U}\overrightarrow{\boldsymbol{Y}}_{2,(1:n-1)}\right)+h\left(X_{2,U,n}|X_{1,C,n}\right)+h\left(\overleftarrow{Y}_{1,n}|X_{1,n},X_{2,U,n}\right)\\
&\quad-3\log\left(2\pi e\right)\Big]+N\log\left(2\pi e\right)+N\delta(N)\\
&\leqslant\sum_{n=1}^{N}\Big[h\left(\overrightarrow{Y}_{1,n}|X_{1,C,n},X_{2,U,n}\right)+h\left(X_{1,U,n}|X_{2,C,n}\right)+h\left(\overleftarrow{Y}_{2,n}|X_{2,n},X_{1,U,n}\right)\\
&\quad+h\left(\overrightarrow{Y}_{2,n}|X_{2,C,n},X_{1,U,n}\right)+h\left(X_{2,U,n}|X_{1,C,n}\right)+h\left(\overleftarrow{Y}_{1,n}|X_{1,n},X_{2,U,n}\right)-3\log\left(2\pi e\right)\Big]\\
&\quad+N\log\left(2\pi e\right)+N\delta(N)\\
&=N\Big[h\left(\overrightarrow{Y}_{1}|X_{1,C},X_{2,U}\right)+h\left(X_{1,U}|X_{2,C}\right)+h\left(\overleftarrow{Y}_{2}|X_{2},X_{1,U}\right)+h\left(\overrightarrow{Y}_{2}|X_{2,C},X_{1,U}\right)\\
&\quad+h\left(X_{2,U}|X_{1,C}\right)+h\left(\overleftarrow{Y}_{1}|X_{1},X_{2,U}\right)-3\log\left(2\pi e\right)\Big]+N\log\left(2\pi e\right)+N\delta(N) \qquad (44)\\
&\quad\text{for any } n\in\{1,2,\ldots,N\},
\end{aligned}
$$

where
(a) follows from Fano's inequality (see Fig. 3b);
(b) follows from the fact that $h\left(\overrightarrow{\boldsymbol{Y}}_i\right)-h\left(\boldsymbol{X}_{i,C},\boldsymbol{X}_{j,U}\right)+h\left(\boldsymbol{X}_{i,C},\boldsymbol{X}_{j,U}|\overrightarrow{\boldsymbol{Y}}_i\right)=h\left(\overrightarrow{\boldsymbol{Y}}_i|\boldsymbol{X}_{i,C},\boldsymbol{X}_{j,U}\right)$;
(c) follows from the fact tha $h\left(\overrightarrow{\boldsymbol{Z}}_i|W_j,\overleftarrow{\boldsymbol{Y}}_j,\boldsymbol{X}_j,\boldsymbol{X}_{i,C}\right)-h\left(\overrightarrow{\boldsymbol{Z}}_i,\overrightarrow{\boldsymbol{Z}}_j|\overrightarrow{\boldsymbol{Y}}_j,\boldsymbol{X}_i,\boldsymbol{X}_j\right)\leqslant 0$;
(d) follows from Lemma 1.
From (44), the following holds in the asymptotic regime:

$$
\begin{aligned}
R_1+R_2&\leqslant h\left(\overrightarrow{Y}_{1}|X_{1,C},X_{2,U}\right)+h\left(X_{1,U}|X_{2,C}\right)+h\left(\overleftarrow{Y}_{2}|X_{2},X_{1,U}\right)+h\left(\overrightarrow{Y}_{2}|X_{2,C},X_{1,U}\right)\\
&\quad+h\left(X_{2,U}|X_{1,C}\right)+h\left(\overleftarrow{Y}_{1}|X_{1},X_{2,U}\right)-2\log\left(2\pi e\right)\\
&\leqslant\frac{1}{2}\log\left(\det\left(\mathrm{Var}\left(\overrightarrow{Y}_{1},X_{1,C},X_{2,U}\right)\right)\right)-\frac{1}{2}\log\left(\mathrm{INR}_{12}+1\right)+\frac{1}{2}\log\left(\det\left(\mathrm{Var}\left(\overleftarrow{Y}_{2},X_{2},X_{1,U}\right)\right)\right)\\
&\quad-\frac{1}{2}\log\left(\det\left(\mathrm{Var}\left(X_2,X_{1,U}\right)\right)\right)+\frac{1}{2}\log\left(\det\left(\mathrm{Var}\left(\overrightarrow{Y}_{2},X_{2,C},X_{1,U}\right)\right)\right)-\frac{1}{2}\log\left(\mathrm{INR}_{21}+1\right)\\
&\quad+\frac{1}{2}\log\left(\det\left(\mathrm{Var}\left(\overleftarrow{Y}_{1},X_{1},X_{2,U}\right)\right)\right)-\frac{1}{2}\log\left(\det\left(\mathrm{Var}\left(X_1,X_{2,U}\right)\right)\right)+\log\left(2\pi e\right), \qquad (45)
\end{aligned}
$$

where for all $i\in\{1,2\}$, with $j\in\{1,2\}\setminus\{i\}$ the following holds

$$
\begin{aligned}
\det\left(\mathrm{Var}\left(\overrightarrow{Y}_{j},X_{j,C},X_{i,U}\right)\right)=&\overrightarrow{\mathrm{SNR}}_j+\mathrm{INR}_{ji}+h_{ji,U}^2-2h_{ji,U}\sqrt{\mathrm{INR}_{ji}}+\left(1-\rho^2\right)\Big(\mathrm{INR}_{ij}\mathrm{INR}_{ji}\\
&+h_{ji,U}^2\left(\overrightarrow{\mathrm{SNR}}_j+\mathrm{INR}_{ij}\right)-2h_{ji,U}\mathrm{INR}_{ij}\sqrt{\mathrm{INR}_{ji}}\Big)\\
&+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_j}\left(\sqrt{\mathrm{INR}_{ji}}-h_{ji,U}\right), \qquad (46a)\\
\det\left(\mathrm{Var}\left(\overleftarrow{Y}_{j},X_{j},X_{i,U}\right)\right)=&1+h_{ji,U}^2\left(1-\rho^2\right)\\
&+\frac{\overleftarrow{\mathrm{SNR}}_j\left(1-\rho^2\right)\left(h_{ji,U}^2-2h_{ji,U}\sqrt{\mathrm{INR}_{ji}}+\mathrm{INR}_{ji}\right)}{\left(\overrightarrow{\mathrm{SNR}}_j+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_j\mathrm{INR}_{ji}}+\mathrm{INR}_{ji}+1\right)} \text{ and} \qquad (46b)\\
\det\left(\mathrm{Var}\left(X_j,X_{i,U}\right)\right)=&1+\left(1-\rho^2\right)h_{ji,U}^2. \qquad (46c)
\end{aligned}
$$

The expressions in (46) depend on $S_{1,i}$, $S_{2,i}$, $S_{3,i}$, $S_{4,i}$, and $S_{5,i}$ via the parameter $h_{ji,U}$ in (38). Hence, the following cases are considered.

Case 1: $(S_{1,2} \vee S_{2,2} \vee S_{5,2}) \wedge (S_{1,1} \vee S_{2,1} \vee S_{5,1})$. From (38), it follows that $h_{12,U} = 0$ and $h_{21,U} = 0$. Then, under these conditions, plugging the expressions (46) into (45) yields (15a).
Case 2: $(S_{1,2} \vee S_{2,2} \vee S_{5,2}) \wedge (S_{3,1} \vee S_{4,1})$. From (38), it follows that $h_{12,U} = 0$ and $h_{21,U} = \sqrt{\frac{\mathrm{INR}_{12}\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_2}}$. Then, under these conditions, plugging the expressions (46) into (45) yields (15b).
Case 3: $(S_{3,2} \vee S_{4,2}) \wedge (S_{1,1} \vee S_{2,1} \vee S_{5,1})$. From (38), it follows that $h_{12,U} = \sqrt{\frac{\mathrm{INR}_{12}\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_1}}$ and $h_{21,U} = 0$. Then, under these conditions, plugging the expressions (46) into (45) yields (15c).
Case 4: $(S_{3,2} \vee S_{4,2}) \wedge (S_{3,1} \vee S_{4,1})$. From (38), it follows that $h_{12,U} = \sqrt{\frac{\mathrm{INR}_{12}\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_1}}$ and $h_{21,U} = \sqrt{\frac{\mathrm{INR}_{12}\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_2}}$. Then, under these conditions, plugging the expressions (46) into (45) yields (15d).
This completes the proof of (18d).
Proof of (18e): From the assumption that the message indices W_i and W_j are i.i.d. following a uniform distribution over the sets $\mathcal{W}_i$ and $\mathcal{W}_j$ respectively, for all $i \in \{1,2\}$, with $j \in \{1,2\} \setminus \{i\}$, the following holds

$$
\begin{aligned}
N\left(2R_i + R_j\right) &= 2H\left(W_i\right) + H\left(W_j\right) \\
&\overset{(a)}{=} H\left(W_i\right) + H\left(W_i|W_j\right) + H\left(W_j\right) \\
&\overset{(b)}{\leqslant} I\left(W_i; \overrightarrow{\boldsymbol{Y}}_i, \overleftarrow{\boldsymbol{Y}}_i\right) + I\left(W_i; \overrightarrow{\boldsymbol{Y}}_i, \overleftarrow{\boldsymbol{Y}}_j|W_j\right) + I\left(W_j; \overrightarrow{\boldsymbol{Y}}_j, \overleftarrow{\boldsymbol{Y}}_j\right) + N\delta(N) \\
&\leqslant h\left(\overrightarrow{\boldsymbol{Y}}_i\right) + h\left(\overleftarrow{\boldsymbol{Z}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) - h\left(\overrightarrow{\boldsymbol{Y}}_i|W_i, \overleftarrow{\boldsymbol{Y}}_i\right) + h\left(\overleftarrow{\boldsymbol{Y}}_j|W_j\right) - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i, W_j\right) \\
&\quad + I\left(W_i; \overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j\right) + h\left(\overleftarrow{\boldsymbol{Z}}_j\right) - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_j\right) - h\left(\overrightarrow{\boldsymbol{Y}}_j|W_j, \overleftarrow{\boldsymbol{Y}}_j\right) + N\delta(N) \\
&= h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) - h\left(\overrightarrow{\boldsymbol{Y}}_i|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i, W_j\right) + I\left(W_i; \overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j\right) \\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_j\right) - h\left(\overrightarrow{\boldsymbol{Y}}_j|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_j\right) + N\log\left(2\pi e\right) + N\delta(N) \\
&\leqslant h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) - h\left(\overrightarrow{\boldsymbol{Y}}_i|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i\right) + I\left(W_i; \overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j\right) \\
&\quad - h\left(\overrightarrow{\boldsymbol{Y}}_j|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_j\right) + N\log\left(2\pi e\right) + N\delta(N) \\
&\overset{(c)}{=} h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) - h\left(\boldsymbol{X}_{j,C}|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i\right) + I\left(W_i; \overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j\right) \\
&\quad - h\left(\boldsymbol{X}_{i,C}|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_j\right) + N\log\left(2\pi e\right) + N\delta(N) \\
&= h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) - h\left(\boldsymbol{X}_{j,C}, \overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) \\
&\quad + I\left(W_i; \overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j\right) - h\left(\boldsymbol{X}_{i,C}|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_j\right) + N\log\left(2\pi e\right) + N\delta(N) \\
&\overset{(d)}{=} h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) - h\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) \\
&\quad + I\left(W_i; \overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j\right) - h\left(\boldsymbol{X}_{i,C}|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_j\right) + N\log\left(2\pi e\right) + N\delta(N) \\
&\leqslant h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) - h\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}|W_i, \overleftarrow{\boldsymbol{Y}}_i\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) \\
&\quad + I\left(W_i; \overrightarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_{i,C}|W_j, \overleftarrow{\boldsymbol{Y}}_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j\right) - h\left(\boldsymbol{X}_{i,C}|W_j, \overleftarrow{\boldsymbol{Y}}_j\right) + N\log\left(2\pi e\right) + N\delta(N) \\
&= h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) - h\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}|W_i, \overleftarrow{\boldsymbol{Y}}_i\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) \\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_{i,C}\right) - h\left(\overrightarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_{i,C}|W_i, W_j, \overleftarrow{\boldsymbol{Y}}_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j\right) + N\log\left(2\pi e\right) + N\delta(N)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(e)}{\leqslant} h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) - h\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}|W_i, \overleftarrow{\boldsymbol{Y}}_i\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) \\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_{i,C}\right) - h\left(\overrightarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_{i,C}|W_i, W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_i, \boldsymbol{X}_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j\right) + N\log(2\pi e) \\
&\quad + N\delta(N) \\
&= h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) - h\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}|W_i, \overleftarrow{\boldsymbol{Y}}_i\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) \\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_{i,C}\right) - h\left(\overrightarrow{\boldsymbol{Z}}_i, \overrightarrow{\boldsymbol{Z}}_j|W_i, W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_i, \boldsymbol{X}_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j\right) + N\log(2\pi e) \\
&\quad + N\delta(N) \\
&\overset{(f)}{\leqslant} h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) - h\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}|W_i, \overleftarrow{\boldsymbol{Y}}_i\right) + h\left(\overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_{i,C}\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j\right) \\
&\quad + N\log(2\pi e) + N\delta(N) \\
&\leqslant h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) + I\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}; W_i, \overleftarrow{\boldsymbol{Y}}_i\right) - h\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}\right) + h\left(\overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_{i,C}\right) \\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_j\right) + h\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}|\overrightarrow{\boldsymbol{Y}}_j\right) + N\log(2\pi e) + N\delta(N) \\
&\overset{(g)}{=} h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) + I\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}; W_i, \overleftarrow{\boldsymbol{Y}}_i\right) + h\left(\overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_{i,C}\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j|\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}\right) \\
&\quad + N\log(2\pi e) + N\delta(N) \\
&\leqslant h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) + I\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}, W_j, \overleftarrow{\boldsymbol{Y}}_j; W_i, \overleftarrow{\boldsymbol{Y}}_i\right) + h\left(\overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_{i,C}\right) \\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_j|\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}\right) + N\log(2\pi e) + N\delta(N) \\
&\overset{(h)}{\leqslant} h\left(\overrightarrow{\boldsymbol{Y}}_i\right) + \sum_{n=1}^{N}\left[h\left(X_{i,U,n}|X_{j,C,n}\right) + h\left(\overleftarrow{\boldsymbol{Y}}_{j,n}|X_{j,n}, X_{i,U,n}\right) - \frac{3}{2}\log(2\pi e)\right] \\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_{i,C}\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j|\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}\right) + N\log(2\pi e) + N\delta(N) \\
&\overset{(i)}{\leqslant} h\left(\overrightarrow{\boldsymbol{Y}}_i\right) + \sum_{n=1}^{N}\left[h\left(X_{i,U,n}|X_{j,C,n}\right) + h\left(\overleftarrow{\boldsymbol{Y}}_{j,n}|X_{j,n}, X_{i,U,n}\right) - \frac{3}{2}\log(2\pi e)\right] + h\left(\overrightarrow{\boldsymbol{Y}}_i|\boldsymbol{X}_{i,C}, \boldsymbol{X}_j\right) \\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_j|\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}\right) + N\log(2\pi e) + N\delta(N) \\
&\leqslant \sum_{n=1}^{N}\Big[h\left(\overrightarrow{Y}_{i,n}\right) + h\left(X_{i,U,n}|X_{j,C,n}\right) + h\left(\overleftarrow{Y}_{j,n}|X_{j,n}, X_{i,U,n}\right) - \frac{3}{2}\log(2\pi e) \\
&\quad + h\left(\overrightarrow{Y}_{i,n}|X_{i,C,n}, X_{j,n}\right) + h\left(\overrightarrow{Y}_{j,n}|X_{j,C,n}, X_{i,U,n}\right)\Big] + N\log(2\pi e) + N\delta(N) \\
&= N\Big[h\left(\overrightarrow{Y}_i\right) + h\left(X_{i,U}|X_{j,C}\right) + h\left(\overleftarrow{Y}_j|X_j, X_{i,U}\right) - \frac{5}{2}\log(2\pi e) + h\left(\overrightarrow{Y}_i|X_{i,C}, X_j\right) \\
&\quad + h\left(\overrightarrow{Y}_j|X_{j,C}, X_{i,U}\right) + 2\log(2\pi e) + \delta(N)\Big] \text{ for any } n \in \{1, 2, \ldots, N\}
\end{aligned} \tag{47}
$$

where,
(a) follows from the fact that W_1 and W_2 are mutually independent;
(b) follows from Fano's inequality (see Fig. 3c);
(c) follows from (1) and (39);
(d) follows from (40);
(e) follows from (3) and the fact that conditioning reduces the entropy;
(f) follows from the fact that $h\left(\overrightarrow{\boldsymbol{Z}}_j|W_j, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) - h\left(\overrightarrow{\boldsymbol{Z}}_i, \overrightarrow{\boldsymbol{Z}}_j|W_i, W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_i, \boldsymbol{X}_j\right) \leqslant 0$;
(g) follows from the fact that $h\left(\overrightarrow{\boldsymbol{Y}}_j\right) - h\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}\right) + h\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}|\overrightarrow{\boldsymbol{Y}}_j\right) = h\left(\overrightarrow{\boldsymbol{Y}}_j|\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}\right)$;
(h) follows from Lemma 1;
(i) follows from the fact that conditioning reduces the entropy.

From (47), the following holds in the asymptotic regime:

$$\begin{aligned}
2R_i+R_j \leqslant & h\left(\overrightarrow{Y}_i\right)+h\left(X_{i,U}|X_{j,C}\right)+h\left(\overleftarrow{Y}_j|X_j,X_{i,U}\right)+h\left(\overrightarrow{Y}_i|X_{i,C},X_j\right)+h\left(\overrightarrow{Y}_j|X_{j,C},X_{i,U}\right)\\
&-\frac{1}{2}\log\left(2\pi e\right)\\
\leqslant & \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_i+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_i\mathrm{INR}_{ij}}+\mathrm{INR}_{ij}+1\right)-\frac{1}{2}\log\left(\mathrm{INR}_{ij}+1\right)\\
&+\frac{1}{2}\log\left(\det\left(\mathrm{Var}\left(\overleftarrow{Y}_j,X_j,X_{i,U}\right)\right)\right)-\frac{1}{2}\log\left(\det\left(\mathrm{Var}\left(X_j,X_{i,U}\right)\right)\right)\\
&+\frac{1}{2}\log\left(1+\left(1-\rho^2\right)\left(\overrightarrow{\mathrm{SNR}}_i+\mathrm{INR}_{ji}\right)\right)-\frac{1}{2}\log\left(1+\left(1-\rho^2\right)\mathrm{INR}_{ji}\right)\\
&+\frac{1}{2}\log\Big(\det\Big(\mathrm{Var}\left(\overrightarrow{Y}_j,X_{j,C},X_{i,U}\right)\Big)\Big)+2\log\left(2\pi e\right). \qquad (48)
\end{aligned}$$

The outer bound on (48) depends on $S_{1,i}$, $S_{2,i}$, $S_{3,i}$, $S_{4,i}$, and $S_{5,i}$ via the parameter $h_{ji,U}$ in (38). Hence, as in the previous part, the following cases are considered.

Case 1: $(S_{1,i} \vee S_{2,i} \vee S_{5,i})$. From (38), it follows that $h_{ji,U}=0$. Then, under these conditions, plugging the expressions (46) into (48) yields (16a).

Case 2: $(S_{3,i} \vee S_{4,i})$. From (38), it follows that $h_{ji,U}=\sqrt{\frac{\mathrm{INR}_{ij}\mathrm{INR}_{ji}}{\overrightarrow{\mathrm{SNR}}_j}}$. Then, under these conditions, plugging the expressions (46) into (48) yields (16b).

This completes the proof of (18e) and the proof of Theorem 2.

Proof of Lemma 1

Lemma 1 is proved as follows

$$\begin{aligned}
& I\left(\boldsymbol{X}_{i,C},\boldsymbol{X}_{j,U},\overleftarrow{\boldsymbol{Y}}_i,W_i;\overleftarrow{\boldsymbol{Y}}_j,W_j\right)\\
=& I\left(W_i;\overleftarrow{\boldsymbol{Y}}_j,W_j\right)+I\left(\boldsymbol{X}_{i,C},\boldsymbol{X}_{j,U},\overleftarrow{\boldsymbol{Y}}_i;\overleftarrow{\boldsymbol{Y}}_j,W_j|W_i\right)\\
=& h\left(\overleftarrow{\boldsymbol{Y}}_j,W_j\right)-h\left(\overleftarrow{\boldsymbol{Y}}_j,W_j|W_i\right)+h\left(\boldsymbol{X}_{i,C},\boldsymbol{X}_{j,U},\overleftarrow{\boldsymbol{Y}}_i|W_i\right)-h\left(\boldsymbol{X}_{i,C},\boldsymbol{X}_{j,U},\overleftarrow{\boldsymbol{Y}}_i|W_i,W_j,\overleftarrow{\boldsymbol{Y}}_j\right)\\
=& h\left(\overleftarrow{\boldsymbol{Y}}_j|W_j\right)-h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i,W_j\right)+h\left(\boldsymbol{X}_{i,C},\boldsymbol{X}_{j,U},\overleftarrow{\boldsymbol{Y}}_i|W_i\right)-h\left(\boldsymbol{X}_{i,C},\boldsymbol{X}_{j,U},\overleftarrow{\boldsymbol{Y}}_i|W_i,W_j,\overleftarrow{\boldsymbol{Y}}_j\right)\\
=& h\left(\overleftarrow{\boldsymbol{Y}}_j|W_j\right)+h\left(\boldsymbol{X}_{i,C},\boldsymbol{X}_{j,U},\overleftarrow{\boldsymbol{Y}}_i|W_i\right)-h\left(\boldsymbol{X}_{i,C},\boldsymbol{X}_{j,U},\overleftarrow{\boldsymbol{Y}}_i,\overleftarrow{\boldsymbol{Y}}_j|W_i,W_j\right)\\
=& h\left(\overleftarrow{\boldsymbol{Y}}_j|W_j\right)+\sum_{n=1}^{N}\Bigg[h\Big(X_{i,C,n},X_{j,U,n},\overleftarrow{Y}_{i,n}|W_i,\boldsymbol{X}_{i,C,(1:n-1)},\boldsymbol{X}_{j,U,(1:n-1)},\overleftarrow{\boldsymbol{Y}}_{i,(1:n-1)},\boldsymbol{X}_{i,(1:n)}\Big)\\
& -h\Big(X_{i,C,n},X_{j,U,n},\overleftarrow{Y}_{i,n},\overleftarrow{Y}_{j,n}|W_i,W_j,\boldsymbol{X}_{i,C,(1:n-1)},\boldsymbol{X}_{j,U,(1:n-1)},\overleftarrow{\boldsymbol{Y}}_{i,(1:n-1)},\overleftarrow{\boldsymbol{Y}}_{j,(1:n-1)},\\
& \boldsymbol{X}_{i,(1:n)},\boldsymbol{X}_{j,(1:n)}\Big)\Bigg]\\
\leqslant & h\left(\overleftarrow{\boldsymbol{Y}}_j|W_j\right)+\sum_{n=1}^{N}\Bigg[h\Big(X_{i,C,n},X_{j,U,n},\overleftarrow{Y}_{i,n}|X_{i,n}\Big)-h\Big(\overrightarrow{Z}_{j,n},\overrightarrow{Z}_{i,n},\overleftarrow{Y}_{i,n},\overleftarrow{Y}_{j,n}|W_i,W_j,\\
& \boldsymbol{X}_{i,C,(1:n-1)},\boldsymbol{X}_{j,U,(1:n-1)},\overleftarrow{\boldsymbol{Y}}_{i,(1:n-1)},\overleftarrow{\boldsymbol{Y}}_{j,(1:n-1)},\boldsymbol{X}_{i,(1:n)},\boldsymbol{X}_{j,(1:n)}\Big)\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= h\left(\overleftarrow{\boldsymbol{Y}}_j|W_j\right) + \sum_{n=1}^{N}\Bigg[h\Big(X_{i,C,n}|X_{i,n}\Big) + h\Big(X_{j,U,n}|X_{i,n},X_{i,C,n}\Big) + h\Big(\overleftarrow{Y}_{i,n}|X_{i,n},X_{i,C,n},X_{j,U,n}\Big)\\
&\quad -h\left(\overrightarrow{Z}_{j,n}\right) - h\left(\overrightarrow{Z}_{i,n}\right) - h\Big(\overleftarrow{Y}_{i,n},\overleftarrow{Y}_{j,n}|W_i,W_j,\boldsymbol{X}_{i,C,(1:n-1)},\boldsymbol{X}_{j,U,(1:n-1)},\overleftarrow{\boldsymbol{Y}}_{i,(1:n-1)},\\
&\quad \overleftarrow{\boldsymbol{Y}}_{j,(1:n-1)},\boldsymbol{X}_{i,(1:n)},\boldsymbol{X}_{j,(1:n)},\overrightarrow{Z}_{j,n},\overrightarrow{Z}_{i,n}\Big)\Bigg]\\
&\leqslant h\left(\overleftarrow{\boldsymbol{Y}}_j|W_j\right) + \sum_{n=1}^{N}\Bigg[h\Big(\overrightarrow{Z}_{j,n}|X_{i,n}\Big) + h\Big(X_{j,U,n}|X_{i,C,n}\Big) + h\Big(\overleftarrow{Y}_{i,n}|X_{i,n},X_{j,U,n}\Big) - h\left(\overrightarrow{Z}_{j,n}\right)\\
&\quad -h\left(\overrightarrow{Z}_{i,n}\right) - h\Big(\overleftarrow{Z}_{i,n},\overleftarrow{Z}_{j,n}|W_i,W_j,\boldsymbol{X}_{i,C,(1:n-1)},\boldsymbol{X}_{j,U,(1:n-1)},\overleftarrow{\boldsymbol{Y}}_{i,(1:n-1)},\overleftarrow{\boldsymbol{Y}}_{j,(1:n-1)},\boldsymbol{X}_{i,(1:n)},\\
&\quad \boldsymbol{X}_{j,(1:n)},\overrightarrow{Z}_{j,n},\overrightarrow{Z}_{i,n}\Big)\Bigg]\\
&\overset{(a)}{=} h\left(\overleftarrow{\boldsymbol{Y}}_j|W_j\right) + \sum_{n=1}^{N}\Bigg[h\Big(X_{j,U,n}|X_{i,C,n}\Big) + h\Big(\overleftarrow{Y}_{i,n}|X_{i,n},X_{j,U,n}\Big) - h\left(\overrightarrow{Z}_{i,n}\right) - h\Big(\overleftarrow{Z}_{i,n}\Big)\\
&\quad -h\Big(\overleftarrow{Z}_{j,n}\Big)\Bigg]\\
&= h\left(\overleftarrow{\boldsymbol{Y}}_j|W_j\right) + \sum_{n=1}^{N}\left[h\left(X_{j,U,n}|X_{i,C,n}\right) + h\left(\overleftarrow{Y}_{i,n}|X_{i,n},X_{j,U,n}\right) - \frac{3}{2}\log\left(2\pi e\right)\right],
\end{aligned}
$$

where
(a) follows from the fact that $\overleftarrow{Z}_{i,n}$ and $\overleftarrow{Z}_{j,n}$ are independent of W_i, W_j, $\boldsymbol{X}_{i,C,(1:n-1)}$, $\boldsymbol{X}_{j,U,(1:n-1)}$, $\overleftarrow{\boldsymbol{Y}}_{i,(1:n-1)}$, $\overleftarrow{\boldsymbol{Y}}_{j,(1:n-1)}$, $\boldsymbol{X}_{i,(1:n)}$, $\boldsymbol{X}_{j,(1:n)}$, $\overrightarrow{Z}_{j,n}$, and $\overrightarrow{Z}_{i,n}$.
This completes the proof of Lemma 1.

C Proof of the Gap between the Converse Region and the Achievable Region

This appendix presents a proof of the Theorem 3. The gap, denoted by δ, between the sets $\overline{\mathcal{C}}_{\text{GIC-NOF}}$ and $\underline{\mathcal{C}}_{\text{GIC-NOF}}$ (Def. 2) is approximated as follows

$$
\delta = \max\left(\delta_{R_1},\delta_{R_2},\frac{\delta_{2R}}{2},\frac{\delta_{3R_1}}{3},\frac{\delta_{3R_2}}{3}\right), \tag{49}
$$

where

$$
\begin{aligned}
\delta_{R_1} &= \min\Big(\kappa_{1,1}(\rho'),\kappa_{2,1}(\rho'),\kappa_{3,1}(\rho')\Big) - \min\Big(a_{2,1}(\rho),a_{6,1}(\rho,\mu_1)+a_{3,2}(\rho,\mu_1),\\
&\quad a_{1,1}+a_{3,2}(\rho,\mu_1)+a_{4,2}(\rho,\mu_1)\Big), &\text{(50a)}\\
\delta_{R_2} &= \min\Big(\kappa_{1,2}(\rho'),\kappa_{2,2}(\rho'),\kappa_{3,2}(\rho')\Big) - \min\Big(a_{2,2}(\rho),a_{3,1}(\rho,\mu_2)+a_{6,2}(\rho,\mu_2),\\
&\quad a_{3,1}(\rho,\mu_2)+a_{4,1}(\rho,\mu_2)+a_{1,2}\Big), &\text{(50b)}
\end{aligned}
$$

$$\begin{aligned}\delta_{2R} = &\min\Big(\kappa_4(\rho'),\kappa_5(\rho'),\kappa_6(\rho')\Big) - \min\Big(a_{2,1}(\rho)+a_{1,2}, a_{1,1}+a_{2,2}(\rho),\\ &a_{3,1}(\rho,\mu_2)+a_{1,1}+a_{3,2}(\rho,\mu_1)+a_{7,2}(\rho,\mu_1,\mu_2),\\ &a_{3,1}(\rho,\mu_2)+a_{5,1}(\rho,\mu_2)+a_{3,2}(\rho,\mu_1)+a_{5,2}(\rho,\mu_1),\\ &a_{3,1}(\rho,\mu_2)+a_{7,1}(\rho,\mu_1,\mu_2)+a_{3,2}(\rho,\mu_1)+a_{1,2}\Big), \qquad (50c)\\ \delta_{3R_1} = &\kappa_{7,1}(\rho') - \min\Big(a_{2,1}(\rho)+a_{1,1}+a_{3,2}(\rho,\mu_1)+a_{7,2}(\rho,\mu_1,\mu_2),\\ &a_{3,1}(\rho,\mu_2)+a_{1,1}+a_{7,1}(\rho,\mu_1,\mu_2)+2a_{3,2}(\rho,\mu_1)+a_{5,2}(\rho,\mu_1),\\ &a_{2,1}(\rho)+a_{1,1}+a_{3,2}(\rho,\mu_1)+a_{5,2}(\rho,\mu_1)\Big), \qquad (50d)\\ \delta_{3R_2} = &\kappa_{7,2}(\rho') - \min\Big(a_{3,1}(\rho,\mu_2)+a_{5,1}(\rho,\mu_2)+a_{2,2}(\rho)+a_{1,2},\\ &a_{3,1}(\rho,\mu_2)+a_{7,1}(\rho,\mu_1,\mu_2)+a_{2,2}(\rho)+a_{1,2},\\ &2a_{3,1}(\rho,\mu_2)+a_{5,1}(\rho,\mu_2)+a_{3,2}(\rho,\mu_1)+a_{1,2}+a_{7,2}(\rho,\mu_1,\mu_2)\Big), \qquad (50e)\end{aligned}$$

where, $\rho' \in [0,1]$ and $(\rho,\mu_1,\mu_2) \in \big[0,\ \big(1-\max\big(\frac{1}{\mathrm{INR}_{12}},\frac{1}{\mathrm{INR}_{21}}\big)\big)^+\big] \times [0,1]\times[0,1]$.
Note that δ_{R_1} and δ_{R_2} represent the gap between the active achievable single-rate bound and the active converse single-rate bound; δ_{2R} represents the gap between the active achievable sum-rate bound and the active converse sum-rate bound; and, δ_{3R_1} and δ_{3R_2} represent the gap between the active achievable weighted sum-rate bound and the active converse weighted sum-rate bound. It is important to highlight that, as suggested in [1, 3], and [5], the gap between $\underline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ and $\overline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ can be calculated more precisely. However, the choice in (49) eases the calculations at the expense of less precision. Note also that the active (achievable or converse) bounds in either of the equalities in (50) depend on the exact values of $\overrightarrow{\mathrm{SNR}}_1$, $\overrightarrow{\mathrm{SNR}}_2$, INR_{12}, INR_{21}, $\overleftarrow{\mathrm{SNR}}_1$, and $\overleftarrow{\mathrm{SNR}}_2$. Hence a key point in order to find the gap between the achievable region and the converse region is to choose a convenient power allocation for the achievable region, i.e., the values of ρ, μ_1, and μ_2, according to the definitions in (50) for all $i \in \{1,2\}$. This particular power allocation is chosen such that the expressions in (50) become simpler to upper bound at the expense of a looser outer bound. Such a power allocation is different in each interference regime. The following describes all the key cases and the corresponding power allocations.

Case 1: $\mathrm{INR}_{12} > \overrightarrow{\mathrm{SNR}}_1$ and $\mathrm{INR}_{21} > \overrightarrow{\mathrm{SNR}}_2$. This case corresponds to the scenario in which both transmitter-receiver pairs are in high interference regime (HIR). Three subcases follow considering the SNR in the feedback links.

- Case 1.1: $\overleftarrow{\mathrm{SNR}}_2 \leqslant \overrightarrow{\mathrm{SNR}}_1$ and $\overleftarrow{\mathrm{SNR}}_1 \leqslant \overrightarrow{\mathrm{SNR}}_2$. In this case the power allocation is: $\rho = 0$, $\mu_1 = 0$ and $\mu_2 = 0$.
- Case 1.2: $\overleftarrow{\mathrm{SNR}}_2 > \overrightarrow{\mathrm{SNR}}_1$ and $\overleftarrow{\mathrm{SNR}}_1 > \overrightarrow{\mathrm{SNR}}_2$. In this case the power allocation is: $\rho = 0$, $\mu_1 = 1$, and $\mu_2 = 1$.
- Case 1.3: $\overleftarrow{\mathrm{SNR}}_2 \leqslant \overrightarrow{\mathrm{SNR}}_1$ and $\overleftarrow{\mathrm{SNR}}_1 > \overrightarrow{\mathrm{SNR}}_2$. In this case the power allocation is: $\rho = 0$, $\mu_1 = 0$, and $\mu_2 = 1$.

Case 2: $\mathrm{INR}_{12} \leqslant \overrightarrow{\mathrm{SNR}}_1$ and $\mathrm{INR}_{21} \leqslant \overrightarrow{\mathrm{SNR}}_2$. This case corresponds to the scenario in which both transmitter-receiver pairs are in low interference regime (LIR). There are twelve subcases that must be studied separately.
In the following four subcases, the achievability scheme presented above is used considering the following power allocation: $\rho = 0$, $\mu_1 = 0$, and $\mu_2 = 0$.

- Case 2.1: $\overleftarrow{\mathrm{SNR}}_1 \leqslant \mathrm{INR}_{21}$, $\overleftarrow{\mathrm{SNR}}_2 \leqslant \mathrm{INR}_{12}$, $\mathrm{INR}_{12}\mathrm{INR}_{21} > \overrightarrow{\mathrm{SNR}}_1$ and $\mathrm{INR}_{12}\mathrm{INR}_{21} > \overrightarrow{\mathrm{SNR}}_2$.
- Case 2.2: $\overleftarrow{\mathrm{SNR}}_1 \leqslant \mathrm{INR}_{21}$, $\overleftarrow{\mathrm{SNR}}_2\mathrm{INR}_{21} \leqslant \overrightarrow{\mathrm{SNR}}_2$, $\mathrm{INR}_{12}\mathrm{INR}_{21} > \overrightarrow{\mathrm{SNR}}_1$ and $\mathrm{INR}_{12}\mathrm{INR}_{21} < \overrightarrow{\mathrm{SNR}}_2$.

Case 2.3: $\overleftarrow{\mathrm{SNR}_1}\mathrm{INR}_{12} \leqslant \overrightarrow{\mathrm{SNR}_1}$, $\overleftarrow{\mathrm{SNR}_2} \leqslant \mathrm{INR}_{12}$, $\mathrm{INR}_{12}\mathrm{INR}_{21} < \overrightarrow{\mathrm{SNR}_1}$ and $\mathrm{INR}_{12}\mathrm{INR}_{21} > \overrightarrow{\mathrm{SNR}_2}$.

Case 2.4: $\overleftarrow{\mathrm{SNR}_1}\mathrm{INR}_{12} \leqslant \overrightarrow{\mathrm{SNR}_1}$, $\overleftarrow{\mathrm{SNR}_2}\mathrm{INR}_{21} \leqslant \overrightarrow{\mathrm{SNR}_2}$, $\mathrm{INR}_{12}\mathrm{INR}_{21} < \overrightarrow{\mathrm{SNR}_1}$ and $\mathrm{INR}_{12}\mathrm{INR}_{21} < \overrightarrow{\mathrm{SNR}_2}$.

In the following four subcases, the achievability scheme presented above is used considering the following power allocation: $\rho = 0$, $\mu_1 = \frac{\mathrm{INR}_{21}^2 \overleftarrow{\mathrm{SNR}_2}}{(\mathrm{INR}_{21}-1)\left(\mathrm{INR}_{21}\overleftarrow{\mathrm{SNR}_2}+\overrightarrow{\mathrm{SNR}_2}\right)}$, and $\mu_2 = \frac{\mathrm{INR}_{12}^2 \overleftarrow{\mathrm{SNR}_1}}{(\mathrm{INR}_{12}-1)\left(\mathrm{INR}_{12}\overleftarrow{\mathrm{SNR}_1}+\overrightarrow{\mathrm{SNR}_1}\right)}$.

Case 2.5: $\overleftarrow{\mathrm{SNR}_1} > \mathrm{INR}_{21}$, $\overleftarrow{\mathrm{SNR}_2} > \mathrm{INR}_{12}$, $\mathrm{INR}_{12}\mathrm{INR}_{21} > \overrightarrow{\mathrm{SNR}_1}$ and $\mathrm{INR}_{12}\mathrm{INR}_{21} > \overrightarrow{\mathrm{SNR}_2}$.

Case 2.6: $\overleftarrow{\mathrm{SNR}_1} > \mathrm{INR}_{21}$, $\overleftarrow{\mathrm{SNR}_2}\mathrm{INR}_{21} > \overrightarrow{\mathrm{SNR}_2}$, $\mathrm{INR}_{12}\mathrm{INR}_{21} > \overrightarrow{\mathrm{SNR}_1}$ and $\mathrm{INR}_{12}\mathrm{INR}_{21} < \overrightarrow{\mathrm{SNR}_2}$.

Case 2.7: $\overleftarrow{\mathrm{SNR}_1}\mathrm{INR}_{12} > \overrightarrow{\mathrm{SNR}_1}$, $\overleftarrow{\mathrm{SNR}_2} > \mathrm{INR}_{12}$, $\mathrm{INR}_{12}\mathrm{INR}_{21} < \overrightarrow{\mathrm{SNR}_1}$ and $\mathrm{INR}_{12}\mathrm{INR}_{21} > \overrightarrow{\mathrm{SNR}_2}$.

Case 2.8: $\overleftarrow{\mathrm{SNR}_1}\mathrm{INR}_{12} > \overrightarrow{\mathrm{SNR}_1}$, $\overleftarrow{\mathrm{SNR}_2}\mathrm{INR}_{21} > \overrightarrow{\mathrm{SNR}_2}$, $\mathrm{INR}_{12}\mathrm{INR}_{21} < \overrightarrow{\mathrm{SNR}_1}$ and $\mathrm{INR}_{12}\mathrm{INR}_{21} < \overrightarrow{\mathrm{SNR}_2}$.

In the following four subcases, the achievability scheme presented above is used considering the following power allocation: $\rho = 0$, $\mu_1 = 0$, and $\mu_2 = \frac{\mathrm{INR}_{12}^2 \overleftarrow{\mathrm{SNR}_1}}{(\mathrm{INR}_{12}-1)\left(\mathrm{INR}_{12}\overleftarrow{\mathrm{SNR}_1}+\overrightarrow{\mathrm{SNR}_1}\right)}$.

Case 2.9: $\overleftarrow{\mathrm{SNR}_1} > \mathrm{INR}_{21}$, $\overleftarrow{\mathrm{SNR}_2} \leqslant \mathrm{INR}_{12}$, $\mathrm{INR}_{12}\mathrm{INR}_{21} > \overrightarrow{\mathrm{SNR}_1}$ and $\mathrm{INR}_{12}\mathrm{INR}_{21} > \overrightarrow{\mathrm{SNR}_2}$.

Case 2.10: $\overleftarrow{\mathrm{SNR}_1} > \mathrm{INR}_{21}$, $\overleftarrow{\mathrm{SNR}_2}\mathrm{INR}_{21} \leqslant \overrightarrow{\mathrm{SNR}_2}$, $\mathrm{INR}_{12}\mathrm{INR}_{21} > \overrightarrow{\mathrm{SNR}_1}$ and $\mathrm{INR}_{12}\mathrm{INR}_{21} < \overrightarrow{\mathrm{SNR}_2}$.

Case 2.11: $\overleftarrow{\mathrm{SNR}_1}\mathrm{INR}_{12} > \overrightarrow{\mathrm{SNR}_1}$, $\overleftarrow{\mathrm{SNR}_2} \leqslant \mathrm{INR}_{12}$, $\mathrm{INR}_{12}\mathrm{INR}_{21} < \overrightarrow{\mathrm{SNR}_1}$ and $\mathrm{INR}_{12}\mathrm{INR}_{21} > \overrightarrow{\mathrm{SNR}_2}$.

Case 2.12: $\overleftarrow{\mathrm{SNR}_1}\mathrm{INR}_{12} > \overrightarrow{\mathrm{SNR}_1}$, $\overleftarrow{\mathrm{SNR}_2}\mathrm{INR}_{21} \leqslant \overrightarrow{\mathrm{SNR}_2}$, $\mathrm{INR}_{12}\mathrm{INR}_{21} < \overrightarrow{\mathrm{SNR}_1}$ and $\mathrm{INR}_{12}\mathrm{INR}_{21} < \overrightarrow{\mathrm{SNR}_2}$.

Case 3: $\mathrm{INR}_{12} > \overrightarrow{\mathrm{SNR}_1}$ and $\mathrm{INR}_{21} \leqslant \overrightarrow{\mathrm{SNR}_2}$. This case corresponds to the scenario in which transmitter-receiver pair 1 is in HIR and transmitter-receiver pair 2 is in LIR. There are four subcases that must be studied separately.

In the following two subcases, the achievability scheme presented above is used considering the following power allocation: $\rho = 0$, $\mu_1 = 0$, and $\mu_2 = 0$.

Case 3.1: $\overleftarrow{\mathrm{SNR}_2} \leqslant \mathrm{INR}_{12}$ and $\mathrm{INR}_{12}\mathrm{INR}_{21} > \overrightarrow{\mathrm{SNR}_2}$.

Case 3.2: $\overleftarrow{\mathrm{SNR}_2}\mathrm{INR}_{21} \leqslant \overrightarrow{\mathrm{SNR}_2}$ and $\mathrm{INR}_{12}\mathrm{INR}_{21} < \overrightarrow{\mathrm{SNR}_2}$.

In the following two subcases, the achievability scheme presented above is used considering the following power allocation: $\rho = 0$, $\mu_1 = 1$, and $\mu_2 = 0$.

Case 3.3: $\overleftarrow{\mathrm{SNR}_2} > \mathrm{INR}_{12}$ and $\mathrm{INR}_{12}\mathrm{INR}_{21} > \overrightarrow{\mathrm{SNR}_2}$.

Case 3.4: $\overleftarrow{\mathrm{SNR}_2}\mathrm{INR}_{21} > \overrightarrow{\mathrm{SNR}_2}$ and $\mathrm{INR}_{12}\mathrm{INR}_{21} < \overrightarrow{\mathrm{SNR}_2}$.

The following is the calculation of the gap δ in Case 1.1.

1. Calculation of δ_{R_1}. From (50a) and considering the corresponding power allocation for the achievable region ($\rho = 0$, $\mu_1 = 0$ and $\mu_2 = 0$), it follows that:

$$\delta_{R_1} \leqslant \min\Big(\kappa_{1,1}(\rho'), \kappa_{2,1}(\rho'), \kappa_{3,1}(\rho')\Big) - \min\Big(a_{6,1}(0,0), a_{1,1} + a_{4,2}(0,0)\Big), \tag{51}$$

where the exact value of ρ' is chosen to provide at least an outer bound for (51).

Note that in this case:

$$\begin{aligned}
\kappa_{1,1}(\rho') &= \frac{1}{2}\log\left(b_{1,1}(\rho')+1\right) \\
&\overset{(a)}{\leqslant} \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\sqrt{\overrightarrow{\mathrm{SNR}}_1\mathrm{INR}_{12}}+\mathrm{INR}_{12}+1\right) \\
&\overset{(b)}{\leqslant} \frac{1}{2}\log\left(2\overrightarrow{\mathrm{SNR}}_1+2\mathrm{INR}_{12}+1\right) \\
&\leqslant \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{12}+1\right)+\frac{1}{2}, \qquad (52a)\\
\kappa_{2,1}(\rho') &= \frac{1}{2}\log\left(1+b_{4,1}(\rho')+b_{5,2}(\rho')\right) \\
&\leqslant \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{21}+1\right), \qquad (52b)\\
\kappa_{3,1}(\rho') &= \frac{1}{2}\log\left(b_{4,1}(\rho')+1\right) \\
&\quad +\frac{1}{2}\log\left(\frac{\overleftarrow{\mathrm{SNR}}_2\left(b_{4,1}(\rho')+b_{5,2}(\rho')+1\right)}{\left(b_{1,2}(1)+1\right)\left(b_{4,1}(\rho')+1\right)}+1\right) \\
&\overset{(c)}{\leqslant} \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+1\right) \\
&\quad +\frac{1}{2}\log\left(\frac{\overleftarrow{\mathrm{SNR}}_2\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{21}+1\right)}{\left(\overrightarrow{\mathrm{SNR}}_2+\mathrm{INR}_{21}+1\right)\left(\overrightarrow{\mathrm{SNR}}_1+1\right)}+1\right) \\
&= \frac{1}{2}\log\left(\frac{\overleftarrow{\mathrm{SNR}}_2\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{21}+1\right)}{\overrightarrow{\mathrm{SNR}}_2+\mathrm{INR}_{21}+1}+\overrightarrow{\mathrm{SNR}}_1+1\right), \qquad (52c)
\end{aligned}$$

where

(a) follows from the fact that $0 \leqslant \rho' \leqslant 1$;

(b) follows from the fact that $\left(\sqrt{\overrightarrow{\mathrm{SNR}}_1}-\sqrt{\mathrm{INR}_{12}}\right)^2 \geqslant 0$; and

(c) follows from the fact that $\kappa_{3,1}(\rho')$ is a monotonically decreasing function of ρ'.

Note also that the achievable bound $a_{1,1}+a_{4,2}(0,0)$ can be lower bounded as follows

$$\begin{aligned}
a_{1,1}+a_{4,2}(0,0) &= \frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}+2\right)+\frac{1}{2}\log\left(\mathrm{INR}_{21}+1\right) \\
&\quad -1 \\
&\geqslant \frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}+2\right)+\frac{1}{2}\log\left(\mathrm{INR}_{21}\right)-1 \\
&= \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\mathrm{INR}_{21}\right)-1 \\
&= \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{21}+\mathrm{INR}_{21}\right)-1 \\
&\geqslant \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{21}+1\right)-1. \qquad (53)
\end{aligned}$$

From (51), (52) and (53), assuming that $a_{1,1} + a_{4,2}(0,0) < a_{6,1}(0,0)$, it follows that:

$$\begin{aligned}\delta_{R_1} \leqslant & \min\Big(\kappa_{1,1}(\rho'), \kappa_{2,1}(\rho'), \kappa_{3,1}(\rho')\Big) - \Big(a_{1,1} + a_{4,2}(0,0)\Big) \\ \leqslant & \kappa_{2,1}(\rho') - \Big(a_{1,1} + a_{4,2}(0,0)\Big) \\ \leqslant & 1. \end{aligned} \tag{54}$$

Now, assuming that $a_{6,1}(0,0) < a_{1,1} + a_{4,2}(0,0)$, the following holds

$$\delta_{R_1} \leqslant \min\Big(\kappa_{1,1}(\rho'), \kappa_{2,1}(\rho'), \kappa_{3,1}(\rho')\Big) - a_{6,1}(0,0). \tag{55}$$

To calculate an upper bound for (55), the following cases are considered:
Case 1.1.1: $\overrightarrow{\mathrm{SNR}}_1 \geqslant \mathrm{INR}_{21} \wedge \overrightarrow{\mathrm{SNR}}_2 < \mathrm{INR}_{12}$;
Case 1.1.2: $\overrightarrow{\mathrm{SNR}}_1 < \mathrm{INR}_{21} \wedge \overrightarrow{\mathrm{SNR}}_2 \geqslant \mathrm{INR}_{12}$; and
Case 1.1.3: $\overrightarrow{\mathrm{SNR}}_1 < \mathrm{INR}_{21} \wedge \overrightarrow{\mathrm{SNR}}_2 < \mathrm{INR}_{12}$.
In Case 1.1.1, from (52) and (55), it follows that:

$$\begin{aligned}\delta_{R_1} \leqslant & \kappa_{2,1}(\rho') - a_{6,1}(0,0) \\ \leqslant & \frac{1}{2}\log\Big(\overrightarrow{\mathrm{SNR}}_1 + \mathrm{INR}_{21} + 1\Big) - \frac{1}{2}\log\Big(\overrightarrow{\mathrm{SNR}}_1 + 2\Big) + \frac{1}{2} \\ \leqslant & \frac{1}{2}\log\Big(\overrightarrow{\mathrm{SNR}}_1 + \overrightarrow{\mathrm{SNR}}_1 + 1\Big) - \frac{1}{2}\log\Big(\overrightarrow{\mathrm{SNR}}_1 + 2\Big) + \frac{1}{2} \\ \leqslant & 1. \end{aligned} \tag{56}$$

In Case 1.1.2, from (52) and (55), it follows that:

$$\begin{aligned}\delta_{R_1} \leqslant & \kappa_{3,1}(\rho') - a_{6,1}(0,0) \\ \leqslant & \frac{1}{2}\log\left(\frac{\overleftarrow{\mathrm{SNR}}_2\Big(\overrightarrow{\mathrm{SNR}}_1 + \mathrm{INR}_{21} + 1\Big)}{\overrightarrow{\mathrm{SNR}}_2 + \mathrm{INR}_{21} + 1} + \overrightarrow{\mathrm{SNR}}_1 + 1\right) - \frac{1}{2}\log\Big(\overrightarrow{\mathrm{SNR}}_1 + 2\Big) + \frac{1}{2} \\ \leqslant & \frac{1}{2}\log\Big(\overleftarrow{\mathrm{SNR}}_2 + \overrightarrow{\mathrm{SNR}}_1 + 1\Big) - \frac{1}{2}\log\Big(\overrightarrow{\mathrm{SNR}}_1 + 2\Big) + \frac{1}{2} \\ \leqslant & \frac{1}{2}\log\Big(\overrightarrow{\mathrm{SNR}}_1 + \overrightarrow{\mathrm{SNR}}_1 + 1\Big) - \frac{1}{2}\log\Big(\overrightarrow{\mathrm{SNR}}_1 + 2\Big) + \frac{1}{2} \\ \leqslant & 1. \end{aligned} \tag{57}$$

In Case 1.1.3 two additional cases are considered:
Case 1.1.3.1: $\overrightarrow{\mathrm{SNR}}_1 \geqslant \overrightarrow{\mathrm{SNR}}_2$; and
Case 1.1.3.2: $\overrightarrow{\mathrm{SNR}}_1 < \overrightarrow{\mathrm{SNR}}_2$.

In Case 1.1.3.1, from (52) and (55), it follows that:

$$\begin{aligned}
\delta_{R_1} \leqslant & \kappa_{3,1}(\rho') - a_{6,1}(0,0) \\
\leqslant & \frac{1}{2}\log\left(\frac{\overleftarrow{\mathrm{SNR}}_2\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{21}+1\right)}{\overrightarrow{\mathrm{SNR}}_2+\mathrm{INR}_{21}+1}+\overrightarrow{\mathrm{SNR}}_1+1\right)-\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\right)+\frac{1}{2} \\
= & \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+1\right)+\frac{1}{2}\log\left(\frac{\overleftarrow{\mathrm{SNR}}_2\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{21}+1\right)}{\left(\overrightarrow{\mathrm{SNR}}_2+\mathrm{INR}_{21}+1\right)\left(\overrightarrow{\mathrm{SNR}}_1+1\right)}+1\right) \\
& -\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\right)+\frac{1}{2} \\
\leqslant & \frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_1\left(\mathrm{INR}_{21}+\mathrm{INR}_{21}+\mathrm{INR}_{21}\right)}{\mathrm{INR}_{21}\overrightarrow{\mathrm{SNR}}_1}+1\right)+\frac{1}{2} \\
= & \frac{3}{2}.
\end{aligned} \tag{58}$$

In Case 1.1.3.2, from (52) and (55), it follows that:

$$\begin{aligned}
\delta_{R_1} \leqslant & \kappa_{3,1}(\rho') - a_{6,1}(0,0) \\
\leqslant & \frac{1}{2}\log\left(\frac{\overleftarrow{\mathrm{SNR}}_2\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{21}+1\right)}{\overrightarrow{\mathrm{SNR}}_2+\mathrm{INR}_{21}+1}+\overrightarrow{\mathrm{SNR}}_1+1\right)-\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\right)+\frac{1}{2} \\
\leqslant & \frac{1}{2}\log\left(\overleftarrow{\mathrm{SNR}}_2+\overrightarrow{\mathrm{SNR}}_1+1\right)-\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\right)+\frac{1}{2} \\
\leqslant & \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\overrightarrow{\mathrm{SNR}}_1+1\right)-\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\right)+\frac{1}{2} \\
\leqslant & 1.
\end{aligned} \tag{59}$$

Then, from (54), (56), (57), (58), and (59), it follows that in Case 1.1:

$$\delta_{R_1} \leqslant \frac{3}{2}. \tag{60}$$

The same procedure holds to calculate δ_{R_2} and it yields

$$\delta_{R_2} \leqslant \frac{3}{2}. \tag{61}$$

2. Calculation of δ_{2R}. From (50c) and considering the corresponding power allocation for the achievable region ($\rho = 0$, $\mu_1 = 0$ and $\mu_2 = 0$), it follows that:

$$\begin{aligned}
\delta_{2R} \leqslant & \min\left(\kappa_4(\rho'), \kappa_5(\rho'), \kappa_6(\rho')\right) - \min\left(a_{2,1}(0)+a_{1,2}, a_{1,1}+a_{2,2}(0), a_{5,1}(0,0)+a_{5,2}(0,0)\right) \\
\leqslant & \min\left(\kappa_4(\rho'), \kappa_5(\rho')\right) - \min\left(a_{2,1}(0)+a_{1,2}, a_{1,1}+a_{2,2}(0), a_{5,1}(0,0)+a_{5,2}(0,0)\right).
\end{aligned} \tag{62}$$

Note that

$$\begin{aligned}
\kappa_4(\rho') &= \frac{1}{2}\log\left(1+\frac{b_{4,1}(\rho')}{1+b_{5,2}(\rho')}\right)+\frac{1}{2}\log\left(b_{1,2}(\rho')+1\right)\\
&\leqslant \frac{1}{2}\log\left(1+\frac{b_{4,1}(\rho')}{b_{5,2}(\rho')}\right)+\frac{1}{2}\log\left(b_{1,2}(\rho')+1\right)\\
&= \frac{1}{2}\log\left(1+\frac{\overrightarrow{\mathrm{SNR}_1}}{\mathrm{INR}_{21}}\right)+\frac{1}{2}\log\left(b_{1,2}(\rho')+1\right)\\
&\overset{(h)}{\leqslant} \frac{1}{2}\log\left(1+\frac{\overrightarrow{\mathrm{SNR}_1}}{\mathrm{INR}_{21}}\right)+\frac{1}{2}\log\left(2\overrightarrow{\mathrm{SNR}_2}+2\mathrm{INR}_{21}+1\right)\\
&\leqslant \frac{1}{2}\log\left(1+\frac{\overrightarrow{\mathrm{SNR}_1}}{\mathrm{INR}_{21}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}_2}+\mathrm{INR}_{21}+1\right)+\frac{1}{2}\\
&\leqslant \frac{1}{2}\log\left(2+\frac{\overrightarrow{\mathrm{SNR}_1}}{\mathrm{INR}_{21}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}_2}+\mathrm{INR}_{21}+1\right)+\frac{1}{2}, \qquad (63a)
\end{aligned}$$

and

$$\begin{aligned}
\kappa_5(\rho') &= \frac{1}{2}\log\left(1+\frac{b_{4,2}(\rho')}{1+b_{5,1}(\rho')}\right)+\frac{1}{2}\log\left(b_{1,1}(\rho')+1\right)\\
&\leqslant \frac{1}{2}\log\left(1+\frac{b_{4,2}(\rho')}{b_{5,1}(\rho')}\right)+\frac{1}{2}\log\left(b_{1,1}(\rho')+1\right)\\
&= \frac{1}{2}\log\left(1+\frac{\overrightarrow{\mathrm{SNR}_2}}{\mathrm{INR}_{12}}\right)+\frac{1}{2}\log\left(b_{1,1}(\rho')+1\right)\\
&\overset{(i)}{\leqslant} \frac{1}{2}\log\left(1+\frac{\overrightarrow{\mathrm{SNR}_2}}{\mathrm{INR}_{12}}\right)+\frac{1}{2}\log\left(2\overrightarrow{\mathrm{SNR}_1}+2\mathrm{INR}_{12}+1\right)\\
&\leqslant \frac{1}{2}\log\left(1+\frac{\overrightarrow{\mathrm{SNR}_2}}{\mathrm{INR}_{12}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}_1}+\mathrm{INR}_{12}+1\right)+\frac{1}{2},\\
&\leqslant \frac{1}{2}\log\left(2+\frac{\overrightarrow{\mathrm{SNR}_2}}{\mathrm{INR}_{12}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}_1}+\mathrm{INR}_{12}+1\right)+\frac{1}{2}, \qquad (63b)
\end{aligned}$$

where

(h) follows from the fact that $\left(\sqrt{\overrightarrow{\mathrm{SNR}_2}}-\sqrt{\mathrm{INR}_{21}}\right)^2 \geqslant 0$; and

(i) follows from the fact that $\left(\sqrt{\overrightarrow{\mathrm{SNR}_1}}-\sqrt{\mathrm{INR}_{12}}\right)^2 \geqslant 0$.

From (62) and (63), assuming that $a_{2,1}(0)+a_{1,2} < \min\Big(a_{1,1}+a_{2,2}(0), a_{5,1}(0,0)+a_{5,2}(0,0)\Big)$,

it follows that:

$$\begin{aligned}\delta_{2R}\leqslant&\min\Big(\kappa_4(\rho'),\kappa_5(\rho')\Big)-\Big(a_{2,1}(0)+a_{1,2}\Big)\\ \leqslant&\kappa_5(\rho')-\Big(a_{2,1}(0)+a_{1,2}\Big)\\ \leqslant&\frac{1}{2}\log\left(2+\frac{\overrightarrow{\mathrm{SNR}_2}}{\mathrm{INR}_{12}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}_1}+\mathrm{INR}_{12}+1\right)+\frac{1}{2}-\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}_1}+\mathrm{INR}_{12}+1\right)\\ &-\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}_2}}{\mathrm{INR}_{12}}+2\right)+1\\ =&\frac{3}{2}.\end{aligned}\tag{64}$$

From (62) and (63), assuming that $a_{1,1}+a_{2,2}(0)<\min\Big(a_{2,1}(0)+a_{1,2},a_{5,1}(0,0)+a_{5,2}(0,0)\Big)$, it follows that:

$$\begin{aligned}\delta_{2R}\leqslant&\min\Big(\kappa_4(\rho'),\kappa_5(\rho')\Big)-\Big(a_{1,1}+a_{2,2}(0)\Big)\\ \leqslant&\kappa_4(\rho')-\Big(a_{1,1}+a_{2,2}(0)\Big)\\ \leqslant&\frac{1}{2}\log\left(2+\frac{\overrightarrow{\mathrm{SNR}_1}}{\mathrm{INR}_{21}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}_2}+\mathrm{INR}_{21}+1\right)+\frac{1}{2}-\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}_2}+\mathrm{INR}_{21}+1\right)\\ &-\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}_1}}{\mathrm{INR}_{21}}+2\right)+1\\ =&\frac{3}{2}.\end{aligned}\tag{65}$$

Now, assume that $a_{5,1}(0,0)+a_{5,2}(0,0)<\min(a_{2,1}(0)+a_{1,2},a_{1,1}+a_{2,2}(0))$. In this case, the following holds

$$\delta_{2R}\leqslant\min\Big(\kappa_4(\rho'),\kappa_5(\rho')\Big)-\Big(a_{5,1}(0,0)+a_{5,2}(0,0)\Big).\tag{66}$$

To calculate an upper bound for (66), the cases 1.1.1 - 1.1.3 defined above are analyzed hereunder.

In Case 1.1.1, $a_{5,1}(0,0)+a_{5,2}(0,0)$ can be lower bounded as follows

$$\begin{aligned}a_{5,1}(0,0)+a_{5,2}(0,0)=&\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}_1}}{\mathrm{INR}_{21}}+\mathrm{INR}_{12}+1\right)+\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}_2}}{\mathrm{INR}_{12}}+\mathrm{INR}_{21}+1\right)-1\\ \geqslant&\frac{1}{2}\log\left(\mathrm{INR}_{12}+1\right)-1.\end{aligned}\tag{67}$$

From (63), (66), and (67), it follows that:

$$\begin{aligned}
\delta_{2R} \leqslant & \min\Big(\kappa_4(\rho'), \kappa_5(\rho')\Big) - \Big(a_{5,1}(0,0) + a_{5,2}(0,0)\Big) \\
\leqslant & \kappa_5(\rho') - \Big(a_{5,1}(0,0) + a_{5,2}(0,0)\Big) \\
\leqslant & \frac{1}{2}\log\left(2 + \frac{\overrightarrow{\mathrm{SNR}_2}}{\mathrm{INR}_{12}}\right) + \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}_1} + \mathrm{INR}_{12} + 1\right) + \frac{1}{2} - \frac{1}{2}\log\left(\mathrm{INR}_{12} + 1\right) + 1 \\
\leqslant & \frac{1}{2}\log\left(2+1\right) + \frac{1}{2}\log\left(\mathrm{INR}_{12} + \mathrm{INR}_{12} + 1\right) - \frac{1}{2}\log\left(\mathrm{INR}_{12} + 1\right) + \frac{3}{2} \\
\leqslant & \frac{1}{2}\log\left(3\right) + 2. \qquad (68)
\end{aligned}$$

In Case 1.1.2, $a_{5,1}(0,0) + a_{5,2}(0,0)$ can be lower bounded as follows

$$\begin{aligned}
a_{5,1}(0,0) + a_{5,2}(0,0) = & \frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}_1}}{\mathrm{INR}_{21}} + \mathrm{INR}_{12} + 1\right) + \frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}_2}}{\mathrm{INR}_{12}} + \mathrm{INR}_{21} + 1\right) - 1 \\
\geqslant & \frac{1}{2}\log\left(\mathrm{INR}_{21} + 1\right) - 1. \qquad (69)
\end{aligned}$$

From (63), (66), and (69), it follows that:

$$\begin{aligned}
\delta_{2R} \leqslant & \min\Big(\kappa_4(\rho'), \kappa_5(\rho')\Big) - \Big(a_{5,1}(0,0) + a_{5,2}(0,0)\Big) \\
\leqslant & \kappa_4(\rho') - \Big(a_{5,1}(0,0) + a_{5,2}(0,0)\Big) \\
\leqslant & \frac{1}{2}\log\left(2 + \frac{\overrightarrow{\mathrm{SNR}_1}}{\mathrm{INR}_{21}}\right) + \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}_2} + \mathrm{INR}_{21} + 1\right) + \frac{1}{2} - \frac{1}{2}\log\left(\mathrm{INR}_{21} + 1\right) + 1 \\
\leqslant & \frac{1}{2}\log\left(2+1\right) + \frac{1}{2}\log\left(\mathrm{INR}_{21} + \mathrm{INR}_{21} + 1\right) - \frac{1}{2}\log\left(\mathrm{INR}_{21} + 1\right) + \frac{3}{2} \\
\leqslant & \frac{1}{2}\log\left(3\right) + 2. \qquad (70)
\end{aligned}$$

In Case 1.1.3, from (63), (66), and (67), it follows that:

$$\begin{aligned}
\delta_{2R} \leqslant & \min\Big(\kappa_4(\rho'), \kappa_5(\rho')\Big) - \Big(a_{5,1}(0,0) + a_{5,2}(0,0)\Big) \\
\leqslant & \kappa_5(\rho') - \Big(a_{5,1}(0,0) + a_{5,2}(0,0)\Big) \\
\leqslant & \frac{1}{2}\log\left(2 + \frac{\overrightarrow{\mathrm{SNR}_2}}{\mathrm{INR}_{12}}\right) + \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}_1} + \mathrm{INR}_{12} + 1\right) + \frac{1}{2} - \frac{1}{2}\log\left(\mathrm{INR}_{12} + 1\right) + 1 \\
\leqslant & \frac{1}{2}\log\left(2+1\right) + \frac{1}{2}\log\left(\mathrm{INR}_{12} + \mathrm{INR}_{12} + 1\right) - \frac{1}{2}\log\left(\mathrm{INR}_{12} + 1\right) + \frac{3}{2} \\
\leqslant & \frac{1}{2}\log\left(3\right) + 2.
\end{aligned}$$

Then, from (64), (65), (68), (70), and (71), it follows that in Case 1.1:

$$\delta_{2R} \leqslant 2 + \frac{1}{2}\log\left(3\right). \qquad (71)$$

3. Calculation of δ_{3R_1}. From (50d) and considering the corresponding power allocation for the achievable region ($\rho = 0$, $\mu_1 = 0$ and $\mu_2 = 0$), it follows that:

$$\delta_{3R_1} \leqslant \kappa_{7,1}(\rho') - \Big(a_{1,1} + a_{7,1}(0,0,0) + a_{5,2}(0,0)\Big). \tag{72}$$

The sum $a_{1,1} + a_{7,1}(0,0,0) + a_{5,2}(0,0)$ can be lower bounded as follows

$$\begin{aligned} a_{1,1} + a_{7,1}(0,0,0) + a_{5,2}(0,0) &= \frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}_1}}{\mathrm{INR}_{21}} + 2\right) + \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}_1} + \mathrm{INR}_{12} + 1\right) \\ &\quad + \frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}_2}}{\mathrm{INR}_{12}} + \mathrm{INR}_{21} + 1\right) - \frac{3}{2} \\ &\geqslant \frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}_1}}{\mathrm{INR}_{21}} + 2\right) + \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}_1} + \mathrm{INR}_{12} + 1\right) \\ &\quad + \frac{1}{2}\log\left(\mathrm{INR}_{21} + 1\right) - \frac{3}{2}. \end{aligned} \tag{73}$$

If the term $\kappa_{7,1}(\rho')$ is active in the converse region, this can be upper bounded by the sum $\kappa_{1,1}(\rho') + \kappa_4(\rho')$, which corresponds to the sum of the single rate and sum-rate outer bounds respectively, and this can be upper bounded as follows

$$\begin{aligned} \kappa_{7,1}(\rho') &\leqslant \kappa_{1,1}(\rho') + \kappa_4(\rho') \\ &\leqslant \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}_1} + \mathrm{INR}_{12} + 1\right) + \frac{1}{2}\log\left(2 + \frac{\overrightarrow{\mathrm{SNR}_1}}{\mathrm{INR}_{21}}\right) + \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}_2} + \mathrm{INR}_{21} + 1\right) + 1 \\ &\leqslant \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}_1} + \mathrm{INR}_{12} + 1\right) + \frac{1}{2}\log\left(2 + \frac{\overrightarrow{\mathrm{SNR}_1}}{\mathrm{INR}_{21}}\right) + \frac{1}{2}\log\left(\mathrm{INR}_{21} + \mathrm{INR}_{21} + 1\right) + 1 \\ &\leqslant \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}_1} + \mathrm{INR}_{12} + 1\right) + \frac{1}{2}\log\left(2 + \frac{\overrightarrow{\mathrm{SNR}_1}}{\mathrm{INR}_{21}}\right) \\ &\quad + \frac{1}{2}\log\left(\mathrm{INR}_{21} + 1\right) + \frac{3}{2}. \end{aligned} \tag{74}$$

From (72), (73) and (74), it follows that in Case 1.1:

$$\begin{aligned} \delta_{3R_1} &\leqslant \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}_1} + \mathrm{INR}_{12} + 1\right) + \frac{1}{2}\log\left(2 + \frac{\overrightarrow{\mathrm{SNR}_1}}{\mathrm{INR}_{21}}\right) + \frac{1}{2}\log\left(\mathrm{INR}_{21} + 1\right) + \frac{3}{2} \\ &\quad - \frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}_1}}{\mathrm{INR}_{21}} + 2\right) - \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}_1} + \mathrm{INR}_{12} + 1\right) - \frac{1}{2}\log(\mathrm{INR}_{21} + 1) + \frac{3}{2} \\ &= 3. \end{aligned} \tag{75}$$

The same procedure holds to calculate δ_{3R_2} and it yields

$$\delta_{3R_2} \leqslant 3. \tag{76}$$

Therefore, in Case 1.1, from (49), (60), (61), (71), (75) and (76) it follows that:

$$\delta = \max\left(\delta_{R_1}, \delta_{R_2}, \frac{\delta_{2R}}{2}, \frac{\delta_{3R_1}}{3}, \frac{\delta_{3R_2}}{3}\right) \leqslant \frac{3}{2}. \tag{77}$$

This completes the calculation of the gap in Case1.1. The same procedure can be applied to all the other cases and it follows that $\delta \leqslant 4.4$ bits per channel use.

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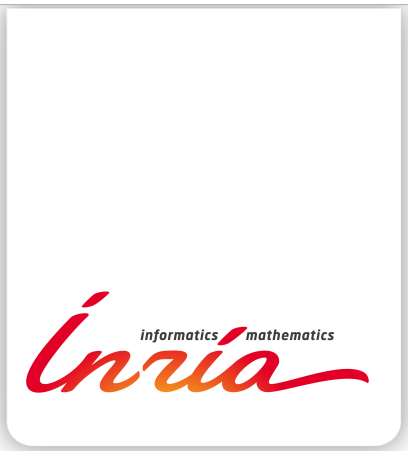

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