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Abstract: In this research report, the capacity region of the two-user Gaussian interference with noisy channel-output feedback (G-IC-NOF) is approximated to within 4.4 bits per channel use. To this end, a new achievable region and a new converse region are presented.

Key-words: Achievable region, converse region, two-user Gaussian interference channel, noisy chanel-output feedback

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Résumé : Ce rapport présente une approximation de la capacité du canal à interférences Gaussien avec rétroalimentation degradée (G-IC-NOF). À cette fin, une région réalisable et une région converse du G-IC-NOF sont introduites, permettant de déterminer que la capacité du G-IC-NOF est approchée à 4.4 bits par utilisation de canal.

Mots-clés : Région réalisable, région converse, canal à interférences Gaussien, rétroalimentation dégradée.

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1 Problem Formulation

Notation: Throughout this research report, sets are denoted with uppercase calligraphic letters, i.e. $\mathcal{X}$. Random variables are denoted by uppercase letters, e.g., X, except for H, I, N and T. The realizations and the set of events from which the random variable X takes values are respectively denoted by x and $\mathcal{X}$. The probability distribution of X over the set $\mathcal{X}$ is denoted P_X. Whenever a second random variable Y is involved, $P_{X\,Y}$ and $P_{Y|X}$ denote respectively the joint probability distribution of (X, Y) and the conditional probability distribution of Y given X. Let N be a natural. In case the random variable is an N-dimensional vector, it is denoted by $\mathbf{X} = (X_1, X_2, ..., X_N)$ and a corresponding realization is denoted by $\mathbf{x} = (x_1, x_2, ..., x_N) \in \mathcal{X}^N$. Given $\mathbf{X} = (X_1, X_2, ..., X_N)$ and $(a, b) \in \mathbb{N}^2$, with $a < b \leqslant N$, the $(b - a + 1)$-dimensional vector formed by the components a to b of $\boldsymbol{X}$ is denoted by $\mathbf{X}_{(a:b)} = (X_a, \ldots, X_b)$. The notation log represents the logarithmic function with base 2. Finally, $\mathbb{E}_X[\cdot]$ denotes the expected value evaluated according to the distribution P_X of the random variable X.

Consider the two-user Gaussian interference channel with noisy channel-output feedback (G-IC-NOF) in Fig. 1. Transmitter i, $i \in \{1, 2\}$, communicates with receiver i subject to the interference produced by transmitter $j \in \{1, 2\}\backslash\{i\}$. There are two independent and uniformly distributed messages, $W_i \in \mathcal{W}_i$, with $\mathcal{W}_i = \{1, \ldots, 2^{NR_i}\}$ and $i \in \{1, 2\}$, where N denotes the block-length in channel uses and R_i is the transmission rate in bits per channel use. At each block, transmitter i sends the codeword $\boldsymbol{X}_i = (X_{i,1}, \ldots, X_{i,N})^{\mathsf{T}} \in \mathcal{X}_i^N$, where $\mathcal{X}_i$ and $\mathcal{X}_i^N$ are respectively the channel-input alphabet and the codebook of transmitter i. The components of the input vector $\boldsymbol{X}_i$ are reals subject to an average power constraint:

$$\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}\left({X_{i,n}}^2\right) \leq 1. \tag{1}$$

Later in (4), it becomes clear that the expected value of ${X_{i,n}}^2$ is taken over the distributions of the message indexes W_1 and W_2 and the noise variables due to the effect of feedback. The channel coefficient from transmitter j to receiver i is denoted by h_{ij}; the channel coefficient from transmitter i to receiver i is denoted by $\overrightarrow{h}_{ii}$; and the channel coefficient from channel-output i to transmitter i is denoted by $\overleftarrow{h}_{ii}$. All channel coefficients are assumed to be non-negative reals.

At a given channel use n, the channel output at receiver i is denoted by $\overrightarrow{Y}_{i,n}$. During channel use $n \in \{1, \ldots, N\}$, the input-output relation of the channel model is given by the following equality:

$$\overrightarrow{Y}_{i,n} = \overrightarrow{h}_{ii} X_{i,n} + h_{ij} X_{j,n} + \overrightarrow{Z}_{i,n}, \tag{2}$$

where $\overrightarrow{Z}_{i,n}$ is a real Gaussian random variable with zero mean and unit variance and it represents the noise at the input of receiver i. At a given channel use n, the feedback signal at the input of transmitter i is denoted by $\overleftarrow{Y}_{i,n}$. Let $d > 0$ be the finite feedback delay in channel uses. At the end of channel use $n > d$, transmitter i observes a scaled and noisy version of $\overrightarrow{Y}_{i,n-d}$. More specifically

$$\forall n \in \{1, \ldots, d\}, \qquad \overleftarrow{Y}_{i,n} = \overleftarrow{Z}_{i,n} \text{ and} \tag{3a}$$

$$\forall n \in \{d+1, \ldots, N\}, \quad \overleftarrow{Y}_{i,n} = \overleftarrow{h}_{ii}\overrightarrow{Y}_{i,n-d} + \overleftarrow{Z}_{i,n}, \tag{3b}$$

where $\overleftarrow{Z}_{i,n}$ is a real Gaussian random variable with zero mean and unit variance that represents the noise at the input of transmitter i. The random variables $\overrightarrow{Z}_{i,n}$ and $\overleftarrow{Z}_{i,n}$ are independent

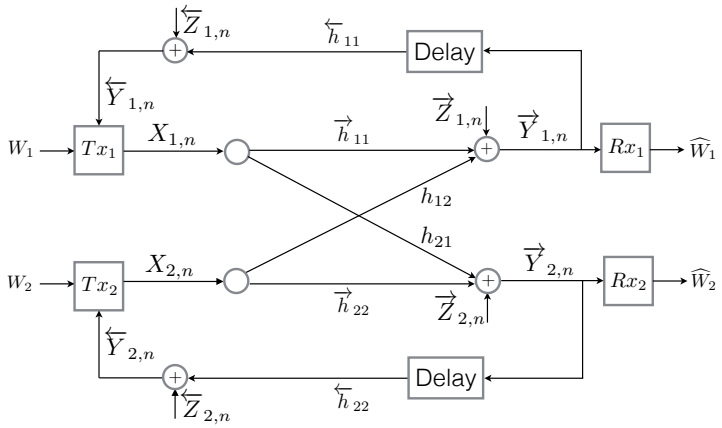

Figure 1: Gaussian interference channel with noisy channel-output feedback at channel use n.

and identically distributed. In the following, without loss of generality, the feedback delay is assumed to be equal to $d = 1$ channel use. The encoder of transmitter i is a set of deterministic functions $f_i^{(1)}, \ldots, f_i^{(N)}$, with $f_i^{(1)} : \mathcal{W}_i \to \mathcal{X}_i$ and $\forall n \in \{2, \ldots, N\}$, $f_i^{(n)} : \mathcal{W}_i \times \mathbb{R}^{n-1} \to \mathcal{X}_i$, such that

$$X_{i,1} = f_i^{(1)}(W_i), \tag{4a}$$

$$X_{i,n} = f_i^{(n)}\left(W_i, \overleftarrow{Y}_{i,1}, \ldots, \overleftarrow{Y}_{i,n-1}\right). \tag{4b}$$

The decoder of receiver i is a deterministic function $\psi_i : \mathbb{R}_i^N \to \mathcal{W}_i$. At the end of the block, receiver i uses the sequence $\overrightarrow{\boldsymbol{Y}} = \left(\overrightarrow{Y}_{i,1}, \ldots, \overrightarrow{Y}_{i,N}\right)$ to obtain an estimation of the index message

$$\widehat{W_i} = \psi_i\left(\overrightarrow{\boldsymbol{Y}}_i\right). \tag{5}$$

The decoding error probability in the two-user G-IC-NOF, denoted by P_e, is

$$P_e = \max\left(\Pr\left(\widehat{W_1} \neq W_1\right), \Pr\left(\widehat{W_2} \neq W_2\right)\right). \tag{6}$$

A rate pair $(R_1, R_2) \in \mathbb{R}_+^2$ is said to be achievable if it satisfies the following definition.

Definition 1 (Achievable Rate Pairs) *A rate pair $(R_1, R_2) \in \mathbb{R}_+^2$ is achievable if there exists at least one pair of codebooks $\mathcal{X}_1^N$ and $\mathcal{X}_2^N$ with codewords of length N, and the corresponding encoding functions $f_1^{(1)}, \ldots, f_1^{(N)}$ and $f_2^{(1)}, \ldots, f_2^{(N)}$ such that the decoding error probability can be made arbitrarily small by letting the block length N grow to infinity.*

When the average power constraint in (1) is saturated, the two-user G-IC-NOF in Fig. 1 can be described by six parameters: $\overrightarrow{\mathrm{SNR}}_i$, $\overleftarrow{\mathrm{SNR}}_i$ and INR_{ij}, with $i \in \{1, 2\}$ and $j \in \{1, 2\} \backslash \{i\}$, that are defined as follows

$$\overrightarrow{\mathrm{SNR}}_i = \overrightarrow{h}_{ii}^2, \tag{7a}$$

$$\mathrm{INR}_{ij} = h_{ij}^2 \text{ and} \tag{7b}$$

$$\overleftarrow{\mathrm{SNR}}_i = \overleftarrow{h}_{ii}^2 \left(\overrightarrow{\mathrm{SNR}}_i + 2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_i \mathrm{INR}_{ij}} + \mathrm{INR}_{ij} + 1\right). \tag{7c}$$

where ρ is the average Pearson correlation coefficient between the channel inputs of transmitter 1 and transmitter 2.

2 Main Results

This section introduces a new achievable region (Theorem 1), denoted by $\underline{\mathcal{C}}_{\text{GIC}-\text{NOF}}$, and a new converse region (Theorem 2), denoted by $\overline{\mathcal{C}}_{\text{GIC}-\text{NOF}}$, for the two-user G-IC-NOF with fixed parameters $\overrightarrow{\text{SNR}}_1$, $\overrightarrow{\text{SNR}}_2$, INR_{12}, INR_{21}, $\overleftarrow{\text{SNR}}_1$ and $\overleftarrow{\text{SNR}}_2$. The capacity region of a given multi-user channel is said to be approximated to within a constant gap if it fits the following definition [1, 2].

Definition 2 (Approximation to within ξ units) *A closed and convex region $\mathcal{T} \subset \mathbb{R}_+^m$ is approximated to within ξ units if there exist two sets $\underline{\mathcal{T}}$ and $\overline{\mathcal{T}}$ such that $\underline{\mathcal{T}} \subseteq \mathcal{T} \subseteq \overline{\mathcal{T}}$ and $\forall\, \boldsymbol{t} = (t_1, \ldots, t_m) \in \overline{\mathcal{T}}$ then $((t_1 - \xi)^+, \ldots, (t_m - \xi)^+) \in \underline{\mathcal{T}}$.*

Denote by $\mathcal{C}_{\text{GIC}-\text{NOF}}$ the capacity region of the 2-user G-IC-NOF. The new achievable region $\underline{\mathcal{C}}_{\text{GIC}-\text{NOF}}$ and the new converse region $\overline{\mathcal{C}}_{\text{GIC}-\text{NOF}}$ approximate the capacity region $\mathcal{C}_{\text{GIC}-\text{NOF}}$ to within 4.4 bits per channel use (Theorem 3).

2.1 A New Achievable Region for the Two-User Gaussian Interference Channel with Noisy Channel-Output Feedback

Theorem 1 presents the new achievable region $\underline{\mathcal{C}}_{\text{GIC}-\text{NOF}}$ for the two-user G-IC-NOF. Consider the variables $a_1, \ldots, a_7$ and $b_1, \ldots, b_7$ in (8) that are defined for a fixed 6-tuple $\left(\overrightarrow{\text{SNR}}_1, \overrightarrow{\text{SNR}}_2, \text{INR}_{12}, \text{INR}_{21}, \overleftarrow{\text{SNR}}_1, \overleftarrow{\text{SNR}}_2\right) \in \mathbb{R}_+^6$, a fixed $(\alpha_1, \alpha_2) \in [0,1]^2$ and a fixed $\rho \in \left[0, \left(1 - \max\left(\frac{1}{\text{INR}_{12}}, \frac{1}{\text{INR}_{21}}\right)\right)^+\right]$:

$$a_1 = \frac{1}{2}\log\left(\frac{\overleftarrow{\text{SNR}}_1(\text{INR}_{12}+1) + \overrightarrow{\text{SNR}}_1 + 2\rho\sqrt{\overrightarrow{\text{SNR}}_1\text{INR}_{12}} + \text{INR}_{12} + 1}{\overleftarrow{\text{SNR}}_1\left((1-\alpha_2)\left((1-\rho)\,\text{INR}_{12} - 1\right) + 2\right) + \overrightarrow{\text{SNR}}_1 + 2\rho\sqrt{\overrightarrow{\text{SNR}}_1\text{INR}_{12}} + \text{INR}_{12} + 1}\right), \tag{8a}$$

$$a_2 = \frac{1}{2}\log\left(\overrightarrow{\text{SNR}}_1 + 2\rho\sqrt{\overrightarrow{\text{SNR}}_1\text{INR}_{12}} + \text{INR}_{12} + 1\right) - \frac{1}{2}, \tag{8b}$$

$$a_3 = \frac{1}{2}\log\left((1-\alpha_2)\left((1-\rho)\,\text{INR}_{12} - 1\right) + 2\right) - \frac{1}{2}, \tag{8c}$$

$$a_4 = \frac{1}{2}\log\left(2 + \frac{\overrightarrow{\text{SNR}}_1}{\text{INR}_{21}}\right) - \frac{1}{2}, \tag{8d}$$

$$a_5 = \frac{1}{2}\log\left(2 + \frac{\overrightarrow{\text{SNR}}_1}{\text{INR}_{21}} + (1-\alpha_2)\left((1-\rho)\,\text{INR}_{12} - 1\right)\right) - \frac{1}{2}, \tag{8e}$$

$$a_6 = \frac{1}{2}\log\left(\frac{\overrightarrow{\text{SNR}}_1}{\text{INR}_{21}}\left((1-\alpha_1)\left((1-\rho)\,\text{INR}_{21} - 1\right) + 1\right) + 2\right) - \frac{1}{2}, \tag{8f}$$

$$a_7 = \frac{1}{2}\log\left(\frac{\overrightarrow{\text{SNR}}_1}{\text{INR}_{21}}\left((1-\alpha_1)\left((1-\rho)\,\text{INR}_{21} - 1\right) + 1\right) + (1-\alpha_2)\left((1-\rho)\,\text{INR}_{12} - 1\right) + 2\right) - \frac{1}{2}, \tag{8g}$$

$$b_1=\frac{1}{2}\log\left(\frac{\overleftarrow{\mathrm{SNR}}_2\left(\mathrm{INR}_{21}+1\right)+\overrightarrow{\mathrm{SNR}}_2+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_2\mathrm{INR}_{21}}+\mathrm{INR}_{21}+1}{\overleftarrow{\mathrm{SNR}}_2\left(\left(1-\alpha_1\right)\left(\left(1-\rho\right)\mathrm{INR}_{21}-1\right)+2\right)+\overrightarrow{\mathrm{SNR}}_2+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_2\mathrm{INR}_{21}}+\mathrm{INR}_{21}+1}\right), \tag{8h}$$

$$b_2=\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_2+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_2\mathrm{INR}_{21}}+\mathrm{INR}_{21}+1\right)-\frac{1}{2}, \tag{8i}$$

$$b_3=\frac{1}{2}\log\left(\left(1-\alpha_1\right)\left(\left(1-\rho\right)\mathrm{INR}_{21}-1\right)+2\right)-\frac{1}{2}, \tag{8j}$$

$$b_4=\frac{1}{2}\log\left(2+\frac{\overrightarrow{\mathrm{SNR}}_2}{\mathrm{INR}_{12}}\right)-\frac{1}{2}, \tag{8k}$$

$$b_5=\frac{1}{2}\log\left(2+\frac{\overrightarrow{\mathrm{SNR}}_2}{\mathrm{INR}_{12}}+\left(1-\alpha_1\right)\left(\left(1-\rho\right)\mathrm{INR}_{21}-1\right)\right)-\frac{1}{2}, \tag{8l}$$

$$b_6=\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_2}{\mathrm{INR}_{12}}\left(\left(1-\alpha_2\right)\left(\left(1-\rho\right)\mathrm{INR}_{12}-1\right)+1\right)+2\right)-\frac{1}{2}, \text{ and} \tag{8m}$$

$$b_7=\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_2}{\mathrm{INR}_{12}}\left(\left(1-\alpha_2\right)\left(\left(1-\rho\right)\mathrm{INR}_{12}-1\right)+1\right)+\left(1-\alpha_1\right)\left(\left(1-\rho\right)\mathrm{INR}_{21}-1\right)+2\right)-\frac{1}{2}. \tag{8n}$$

Using this notation, the new achievable region $\underline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ for the two-user G-IC-NOF is given by the following theorem.

Theorem 1 (A new achievable region for the two-user G-IC-NOF) *The capacity region $\mathcal{C}_{\mathrm{GIC-NOF}}$ contains the region $\underline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ given by the set of non-negative rate pairs (R_1, R_2) that satisfy*

$$R_1\leqslant\min\left(a_2, a_6+b_1, a_4+b_1+b_3\right), \tag{9a}$$

$$R_2\leqslant\min\left(b_2, a_1+b_6, a_1+a_3+b_4\right), \tag{9b}$$

$$R_1+R_2\leqslant\min(a_2+b_4, a_4+b_2, a_1+a_4+b_1+b_7, a_1+a_5+b_1+b_5, a_1+a_7+b_1+b_4), \tag{9c}$$

$$2R_1+R_2\leqslant\min(a_2+a_4+b_1+b_7, a_1+a_4+a_7+2b_1+b_5, a_2+a_4+b_1+b_5) \text{ and} \tag{9d}$$

$$R_1+2R_2\leqslant\min(a_1+a_5+b_2+b_4, a_1+a_7+b_2+b_4, 2a_1+a_5+b_1+b_4+b_7), \tag{9e}$$

for all $(\alpha_1, \alpha_2)\in[0,1]^2$ and for all $\rho\in\left[0, \left(1-\max\left(\frac{1}{\mathrm{INR}_{12}}, \frac{1}{\mathrm{INR}_{21}}\right)\right)^+\right]$.

Proof: The proof of Theorem 1 is presented in Appendix A. ■

The achievable region $\underline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ is obtained using random coding arguments together with three-part message splitting, superposition coding and backward decoding. This coding scheme is fully described in Appendix A. Note that similar coding schemes can also be obtained as special cases from more general schemes, i.e., interference channel with generalized feedback [3]. For instance, the achievability scheme presented in [4] for the two-user G-SIC-NOF uses a particular case of the coding scheme presented in [3].

2.2 A New Converse Region for the Two-User Gaussian Interference Channel with Noisy Channel-Output Feedback

Theorem 2 presents the new converse region $\overline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ for the two-user G-IC-NOF. For all $i\in\{1,2\}$, $j\in\{1,2\}\setminus\{i\}$, consider the following events:

$$S_{1,i}\colon\quad \mathrm{INR}_{ji} \leqslant \overrightarrow{\mathrm{SNR}}_j \wedge \overrightarrow{\mathrm{SNR}}_j \leqslant \mathrm{INR}_{ij}, \tag{10a}$$
$$S_{2,i}\colon\quad \mathrm{INR}_{ji} \leqslant \overrightarrow{\mathrm{SNR}}_j \wedge \overrightarrow{\mathrm{SNR}}_j > \mathrm{INR}_{ij} \wedge \overrightarrow{\mathrm{SNR}}_j \geqslant \mathrm{INR}_{ij}\mathrm{INR}_{ji}, \tag{10b}$$
$$S_{3,i}\colon\quad \mathrm{INR}_{ji} > \overrightarrow{\mathrm{SNR}}_j \wedge \overrightarrow{\mathrm{SNR}}_j \leqslant \mathrm{INR}_{ij}, \tag{10c}$$
$$S_{4,i}\colon\quad \mathrm{INR}_{ji} \leqslant \overrightarrow{\mathrm{SNR}}_j \wedge \overrightarrow{\mathrm{SNR}}_j > \mathrm{INR}_{ij} \wedge \overrightarrow{\mathrm{SNR}}_j < \mathrm{INR}_{ij}\mathrm{INR}_{ji}, \tag{10d}$$
$$S_{5,i}\colon\quad \mathrm{INR}_{ji} > \overrightarrow{\mathrm{SNR}}_j \wedge \overrightarrow{\mathrm{SNR}}_j > \mathrm{INR}_{ij}. \tag{10e}$$

Note that for all $i \in \{1,2\}$ the events $S_{1,i}$, $S_{2,i}$, $S_{3,i}$, $S_{4,i}$ and $S_{5,i}$ exhibit the following property.

Lemma 1 *For all $i \in \{1,2\}$ only one of the events $S_{1,i}$, $S_{2,i}$, $S_{3,i}$, $S_{4,i}$ and $S_{5,i}$ holds true.*

The proof follows from verifying that for $i \in \{1,2\}$, the events $S_{1,i}$, $S_{2,i}$, $S_{3,i}$, $S_{4,i}$ and $S_{5,i}$ are mutually exclusive. For instance consider that the event $S_{1,i}$ is true. Then, $\mathrm{INR}_{ji} \leqslant \overrightarrow{\mathrm{SNR}}_j$ and $\overrightarrow{\mathrm{SNR}}_j \leqslant \mathrm{INR}_{ij}$, and therefore $S_{2,i}$, $S_{3,i}$, $S_{4,i}$ and $S_{5,i}$ are simultaneously false. The same verification can be done for $S_{2,i}$, $S_{3,i}$, $S_{4,i}$ and $S_{5,i}$. Finally, following the same reasoning it can be verified that if any 4-tuple of the events $\{S_{1,i}, S_{2,i}, S_{3,i}, S_{4,i}, S_{5,i}\}$ is false, the remaining event is necessarily true.

The new converse region $\overline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ of the two-user G-IC-NOF in Theorem 2 is described in terms of the events in (10) and the following variables, which are defined for a fixed 6-tuple $\big(\overrightarrow{\mathrm{SNR}}_1, \overrightarrow{\mathrm{SNR}}_2, \mathrm{INR}_{12}, \mathrm{INR}_{21}, \overleftarrow{\mathrm{SNR}}_1, \overleftarrow{\mathrm{SNR}}_2\big) \in \mathbb{R}_+^6$ and a fixed $\rho \in [0,1]$ as follows:

$$\kappa_{1,i}=\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_i + 2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_i\mathrm{INR}_{ij}} + \mathrm{INR}_{ij} + 1\right), \tag{11a}$$
$$\kappa_{2,i}=\frac{1}{2}\log\left(1+\left(1-\rho^2\right)\mathrm{INR}_{ji}\right) + \frac{1}{2}\log\left(1+\frac{\left(1-\rho^2\right)\overrightarrow{\mathrm{SNR}}_i}{1+\left(1-\rho^2\right)\mathrm{INR}_{ji}}\right), \tag{11b}$$
$$\begin{aligned}\kappa_{3,i}=&\frac{1}{2}\log\left(\left(1-\rho^2\right)\overrightarrow{\mathrm{SNR}}_i+1\right)\\ &+\frac{1}{2}\log\left(\frac{\overleftarrow{\mathrm{SNR}}_j\left(\left(1-\rho^2\right)\left(\overrightarrow{\mathrm{SNR}}_i+\mathrm{INR}_{ji}\right)+1\right)}{\left(\overrightarrow{\mathrm{SNR}}_j+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_j\mathrm{INR}_{ji}}+\mathrm{INR}_{ji}+1\right)\left(\left(1-\rho^2\right)\left(\overrightarrow{\mathrm{SNR}}_i+1\right)\right)}+1\right),\end{aligned} \tag{11c}$$
$$\kappa_4=\frac{1}{2}\log\left(1+\frac{\left(1-\rho^2\right)\overrightarrow{\mathrm{SNR}}_1}{1+\left(1-\rho^2\right)\mathrm{INR}_{21}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_2+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_2\mathrm{INR}_{21}}+\mathrm{INR}_{21}+1\right), \tag{11d}$$
$$\kappa_5=\frac{1}{2}\log\left(1+\frac{\left(1-\rho^2\right)\overrightarrow{\mathrm{SNR}}_2}{1+\left(1-\rho^2\right)\mathrm{INR}_{12}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_1\mathrm{INR}_{12}}+\mathrm{INR}_{12}+1\right), \tag{11e}$$
$$\kappa_6=\begin{cases}\kappa_{6,1} & \text{if } (S_{1,2}\vee S_{2,2}\vee S_{3,2})\wedge(S_{1,1}\vee S_{2,1}\vee S_{3,1})=\text{True}\\ \kappa_{6,2} & \text{if } (S_{1,2}\vee S_{2,2}\vee S_{3,2})\wedge(S_{4,1}\vee S_{5,1})=\text{True}\\ \kappa_{6,3} & \text{if } (S_{4,2}\vee S_{5,2})\wedge(S_{1,1}\vee S_{2,1}\vee S_{3,1})=\text{True}\\ \kappa_{6,4} & \text{if } (S_{4,2}\vee S_{5,2})\wedge(S_{4,1}\vee S_{5,1})=\text{True}\end{cases}, \tag{11f}$$
$$\kappa_{7,i}=\begin{cases}\kappa_{7,i,1} & \text{if } (S_{1,i}\vee S_{2,i}\vee S_{3,i})=\text{True}\\ \kappa_{7,i,2} & \text{if } (S_{4,i}\vee S_{5,i})=\text{True}\end{cases}, \tag{11g}$$

where,

$$\begin{aligned}
\kappa_{6,1}=&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_1\mathrm{INR}_{12}}+\mathrm{INR}_{12}+\left(1-\rho^2\right)\mathrm{INR}_{12}\mathrm{INR}_{21}\right)-\frac{1}{2}\log\left(1+\mathrm{INR}_{12}\right)\\
&+\frac{1}{2}\log\left(1+\frac{\left(1-\rho^2\right)\mathrm{INR}_{21}\overleftarrow{\mathrm{SNR}}_2}{\overrightarrow{\mathrm{SNR}}_2+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_2\mathrm{INR}_{21}}+\mathrm{INR}_{21}+1}\right)\\
&+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_2+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_2\mathrm{INR}_{21}}+\mathrm{INR}_{21}+\left(1-\rho^2\right)\mathrm{INR}_{12}\mathrm{INR}_{21}\right)\\
&-\frac{1}{2}\log\left(1+\mathrm{INR}_{21}\right)+\frac{1}{2}\log\left(1+\frac{\left(1-\rho^2\right)\mathrm{INR}_{12}\overleftarrow{\mathrm{SNR}}_1}{\overrightarrow{\mathrm{SNR}}_1+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_1\mathrm{INR}_{12}}+\mathrm{INR}_{12}+1}\right),
\end{aligned}\tag{12a}$$

$$\begin{aligned}
\kappa_{6,2}=&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_1\mathrm{INR}_{12}}+\mathrm{INR}_{12}+\left(1-\rho^2\right)\mathrm{INR}_{12}\mathrm{INR}_{21}\right)-\frac{1}{2}\log\left(1+\mathrm{INR}_{12}\right)\\
&+\frac{1}{2}\log\left(1+\frac{\left(1-\rho^2\right)\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_2}\left(\mathrm{INR}_{12}+\frac{\overleftarrow{\mathrm{SNR}}_2\left(\overrightarrow{\mathrm{SNR}}_2-2\sqrt{\overrightarrow{\mathrm{SNR}}_2\mathrm{INR}_{12}}+\mathrm{INR}_{12}\right)}{\overrightarrow{\mathrm{SNR}}_2+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_2\mathrm{INR}_{21}}+\mathrm{INR}_{21}+1}\right)\right)\\
&-\frac{1}{2}\log\left(1+\frac{\left(1-\rho^2\right)\mathrm{INR}_{12}\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_2}\right)\\
&+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_2+\mathrm{INR}_{21}+\left(1-\rho^2\right)\frac{\mathrm{INR}_{12}\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_2}\left(2\overrightarrow{\mathrm{SNR}}_2-2\sqrt{\overrightarrow{\mathrm{SNR}}_2\mathrm{INR}_{12}}+\mathrm{INR}_{12}\right)\right.\\
&\left.+\frac{\mathrm{INR}_{21}\sqrt{\mathrm{INR}_{12}}}{\overrightarrow{\mathrm{SNR}}_2}\left(\sqrt{\mathrm{INR}_{12}}-2\sqrt{\overrightarrow{\mathrm{SNR}}_2}\right)+2\rho\sqrt{\mathrm{INR}_{21}}\left(\sqrt{\overrightarrow{\mathrm{SNR}}_2}-\sqrt{\mathrm{INR}_{12}}\right)\right)\\
&-\frac{1}{2}\log\left(1+\mathrm{INR}_{21}\right)+\frac{1}{2}\log\left(1+\frac{\left(1-\rho^2\right)\mathrm{INR}_{12}\overleftarrow{\mathrm{SNR}}_1}{\overrightarrow{\mathrm{SNR}}_1+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_1\mathrm{INR}_{12}}+\mathrm{INR}_{12}+1}\right),
\end{aligned}\tag{12b}$$

$$\begin{aligned}
\kappa_{6,3}=&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{12}+\left(1-\rho^2\right)\frac{\mathrm{INR}_{12}\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_1}\left(2\overrightarrow{\mathrm{SNR}}_1-2\sqrt{\overrightarrow{\mathrm{SNR}}_1\mathrm{INR}_{21}}+\mathrm{INR}_{21}\right)\right.\\
&\left.+\frac{\mathrm{INR}_{12}\sqrt{\mathrm{INR}_{21}}}{\overrightarrow{\mathrm{SNR}}_1}\left(\sqrt{\mathrm{INR}_{21}}-2\sqrt{\overrightarrow{\mathrm{SNR}}_1}\right)+2\rho\sqrt{\mathrm{INR}_{12}}\left(\sqrt{\overrightarrow{\mathrm{SNR}}_1}-\sqrt{\mathrm{INR}_{21}}\right)\right)\\
&-\frac{1}{2}\log\left(1+\mathrm{INR}_{12}\right)+\frac{1}{2}\log\left(1+\frac{\left(1-\rho^2\right)\mathrm{INR}_{21}\overleftarrow{\mathrm{SNR}}_2}{\overrightarrow{\mathrm{SNR}}_2+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_2\mathrm{INR}_{21}}+\mathrm{INR}_{21}+1}\right)\\
&+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_2+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_2\mathrm{INR}_{21}}+\mathrm{INR}_{21}+\left(1-\rho^2\right)\mathrm{INR}_{12}\mathrm{INR}_{21}\right)\\
&-\frac{1}{2}\log\left(1+\mathrm{INR}_{21}\right)-\frac{1}{2}\log\left(1+\frac{\left(1-\rho^2\right)\mathrm{INR}_{12}\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_1}\right)\\
&+\frac{1}{2}\log\left(1+\frac{\left(1-\rho^2\right)\mathrm{INR}_{12}}{\overrightarrow{\mathrm{SNR}}_1}\left(\mathrm{INR}_{21}+\frac{\overleftarrow{\mathrm{SNR}}_1\left(\overrightarrow{\mathrm{SNR}}_1-2\sqrt{\overrightarrow{\mathrm{SNR}}_1\mathrm{INR}_{21}}+\mathrm{INR}_{21}\right)}{\overrightarrow{\mathrm{SNR}}_1+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_1\mathrm{INR}_{12}}+\mathrm{INR}_{12}+1}\right)\right),
\end{aligned}\tag{12c}$$

$$\begin{aligned}
\kappa_{6,4}=&\frac{1}{2}\log\Bigg(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{12}+\left(1-\rho^2\right)\frac{\mathrm{INR}_{12}\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_1}\left(2\overrightarrow{\mathrm{SNR}}_1-2\sqrt{\overrightarrow{\mathrm{SNR}}_1\mathrm{INR}_{21}}+\mathrm{INR}_{21}\right)\\
&+\frac{\mathrm{INR}_{12}\sqrt{\mathrm{INR}_{21}}}{\overrightarrow{\mathrm{SNR}}_1}\left(\sqrt{\mathrm{INR}_{21}}-2\sqrt{\overrightarrow{\mathrm{SNR}}_1}\right)+2\rho\sqrt{\mathrm{INR}_{12}}\left(\sqrt{\overrightarrow{\mathrm{SNR}}_1}-\sqrt{\mathrm{INR}_{21}}\right)\Bigg)\\
&-\frac{1}{2}\log\left(1+\mathrm{INR}_{12}\right)-\frac{1}{2}\log\left(1+\frac{\left(1-\rho^2\right)\mathrm{INR}_{12}\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_2}\right)-\frac{1}{2}\log\left(1+\mathrm{INR}_{21}\right)\\
&+\frac{1}{2}\log\left(1+\frac{\left(1-\rho^2\right)\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_2}\left(\mathrm{INR}_{12}+\frac{\overleftarrow{\mathrm{SNR}}_2\left(\overrightarrow{\mathrm{SNR}}_2-2\sqrt{\overrightarrow{\mathrm{SNR}}_2\mathrm{INR}_{12}}+\mathrm{INR}_{12}\right)}{\overrightarrow{\mathrm{SNR}}_2+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_2\mathrm{INR}_{21}}+\mathrm{INR}_{21}+1}\right)\right)\\
&+\frac{1}{2}\log\Bigg(\overrightarrow{\mathrm{SNR}}_2+\mathrm{INR}_{21}+\left(1-\rho^2\right)\frac{\mathrm{INR}_{12}\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_2}\left(2\overrightarrow{\mathrm{SNR}}_2-2\sqrt{\overrightarrow{\mathrm{SNR}}_2\mathrm{INR}_{12}}+\mathrm{INR}_{12}\right)\\
&+\frac{\mathrm{INR}_{21}\sqrt{\mathrm{INR}_{12}}}{\overrightarrow{\mathrm{SNR}}_2}\left(\sqrt{\mathrm{INR}_{12}}-2\sqrt{\overrightarrow{\mathrm{SNR}}_2}\right)+2\rho\sqrt{\mathrm{INR}_{21}}\left(\sqrt{\overrightarrow{\mathrm{SNR}}_2}-\sqrt{\mathrm{INR}_{12}}\right)\Bigg)\\
&+\frac{1}{2}\log\left(1+\frac{\left(1-\rho^2\right)\mathrm{INR}_{12}}{\overrightarrow{\mathrm{SNR}}_1}\left(\mathrm{INR}_{21}+\frac{\overleftarrow{\mathrm{SNR}}_1\left(\overrightarrow{\mathrm{SNR}}_1-2\sqrt{\overrightarrow{\mathrm{SNR}}_1\mathrm{INR}_{21}}+\mathrm{INR}_{21}\right)}{\overrightarrow{\mathrm{SNR}}_1+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_1\mathrm{INR}_{12}}+\mathrm{INR}_{12}+1}\right)\right)\\
&-\frac{1}{2}\log\left(1+\frac{\left(1-\rho^2\right)\mathrm{INR}_{12}\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_1}\right),
\end{aligned}\tag{12d}$$

$$\begin{aligned}
\kappa_{7,i,1}=&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_i+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_i\mathrm{INR}_{ij}}+\mathrm{INR}_{ij}+1\right)-\frac{1}{2}\log\left(1+\mathrm{INR}_{ij}\right)\\
&+\frac{1}{2}\log\left(1+\frac{\left(1-\rho^2\right)\mathrm{INR}_{ji}\overleftarrow{\mathrm{SNR}}_j}{\overrightarrow{\mathrm{SNR}}_j+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_j\mathrm{INR}_{ji}}+\mathrm{INR}_{ji}+1}\right)\\
&+\frac{1}{2}\log\left(1+\left(1-\rho^2\right)\left(\overrightarrow{\mathrm{SNR}}_i+\mathrm{INR}_{ji}\right)\right)-\frac{1}{2}\log\left(1+\left(1-\rho^2\right)\mathrm{INR}_{ji}\right)\\
&+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_j+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_j\mathrm{INR}_{ji}}+\mathrm{INR}_{ji}+\left(1-\rho^2\right)\mathrm{INR}_{ij}\mathrm{INR}_{ji}\right)
\end{aligned}\tag{12e}$$

$$\begin{aligned}
\kappa_{7,i,2}=&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_i+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_i\mathrm{INR}_{ij}}+\mathrm{INR}_{ij}+1\right)-\frac{1}{2}\log\left(1+\mathrm{INR}_{ij}\right)\\
&+\frac{1}{2}\log\left(1+\left(1-\rho^2\right)\frac{\mathrm{INR}_{ji}}{\overrightarrow{\mathrm{SNR}}_j}\left(\mathrm{INR}_{ij}+\frac{\overleftarrow{\mathrm{SNR}}_j\left(\mathrm{INR}_{ij}-2\sqrt{\mathrm{INR}_{ij}\overrightarrow{\mathrm{SNR}}_j}+\overrightarrow{\mathrm{SNR}}_j\right)}{\overrightarrow{\mathrm{SNR}}_j+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_j\mathrm{INR}_{ji}}+\mathrm{INR}_{ji}+1}\right)\right)\\
&-\frac{1}{2}\log\left(1+\frac{\left(1-\rho^2\right)\mathrm{INR}_{ij}\mathrm{INR}_{ji}}{\overrightarrow{\mathrm{SNR}}_j}\right)+\frac{1}{2}\log\left(1+\left(1-\rho^2\right)\left(\overrightarrow{\mathrm{SNR}}_i+\mathrm{INR}_{ji}\right)\right)\\
&-\frac{1}{2}\log\left(1+\left(1-\rho^2\right)\mathrm{INR}_{ji}\right)\\
&+\frac{1}{2}\log\Bigg(\overrightarrow{\mathrm{SNR}}_j+\mathrm{INR}_{ji}+\frac{\left(1-\rho^2\right)\mathrm{INR}_{ij}\mathrm{INR}_{ji}}{\overrightarrow{\mathrm{SNR}}_j}\left(2\overrightarrow{\mathrm{SNR}}_j-2\sqrt{\mathrm{INR}_{ij}\overrightarrow{\mathrm{SNR}}_j}+\mathrm{INR}_{ij}\right)\\
&+2\rho\sqrt{\mathrm{INR}_{ji}}\left(\sqrt{\overrightarrow{\mathrm{SNR}}_j}-\sqrt{\mathrm{INR}_{ij}}\right)+\frac{\mathrm{INR}_{ji}\sqrt{\mathrm{INR}_{ij}}}{\overrightarrow{\mathrm{SNR}}_j}\left(\sqrt{\mathrm{INR}_{ij}}-2\sqrt{\overrightarrow{\mathrm{SNR}}_j}\right)\Bigg).
\end{aligned}\tag{12f}$$

The following theorem describes the new converse region $\overline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ for the two-user G-IC-NOF.

Theorem 2 (A new converse region for the two-user G-IC-NOF) *The capacity region $\mathcal{C}_{\mathrm{GIC-NOF}}$ is contained into the region $\overline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ given by the set of non-negative rate pairs (R_1, R_2) that satisfy $\forall i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$:*

$$R_i \leqslant \min\left(\kappa_{1,i}, \kappa_{2,i}\right), \tag{13a}$$

$$R_i \leqslant \kappa_{3,i}, \tag{13b}$$

$$R_1 + R_2 \leqslant \min\left(\kappa_4, \kappa_5\right), \tag{13c}$$

$$R_1 + R_2 \leqslant \kappa_6 \text{ and} \tag{13d}$$

$$2R_i + R_j \leqslant \kappa_{7,i}, \tag{13e}$$

for all $\rho \in [0,1]$.

Proof: The proof of Theorem 2 is presented in Appendix B. ■

The outer bounds (13a) and (13c) correspond to the outer bounds of the case of perfect channel-output feedback [5]. The bounds (13b), (13d) and (13e) correspond to new outer bounds.

2.3 A Gap Between the Achievable Region and the Converse Region

Theorem 3 describes the gap existing between the achievable region $\underline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ and the converse region $\overline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ (Definition 2).

Theorem 3 *The capacity region of the two-user G-IC-NOF is approximated to within 4.4 bits per channel use by the achievable region $\underline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ and the converse region $\overline{\mathcal{C}}_{\mathrm{GIC-NOF}}$.*

Proof: The proof of Theorem 3 is presented in Appendix C. ■

Fig. 2 presents the exact gap existing between the achievable region $\underline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ and the converse region $\overline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ for the case in which $\overrightarrow{\mathrm{SNR}}_1 = \overrightarrow{\mathrm{SNR}}_2 = \overrightarrow{\mathrm{SNR}}$, $\mathrm{INR}_{12} = \mathrm{INR}_{21} = \mathrm{INR}$ and $\overleftarrow{\mathrm{SNR}}_1 = \overleftarrow{\mathrm{SNR}}_2 = \overleftarrow{\mathrm{SNR}}$ as a function of $\alpha = \frac{\log \mathrm{INR}}{\log \overrightarrow{\mathrm{SNR}}}$ and $\beta = \frac{\log \overleftarrow{\mathrm{SNR}}}{\log \overrightarrow{\mathrm{SNR}}}$. Note that the maximum gap is 1.1 bits per channel use and occurs at $\alpha = \beta = 1.1$.

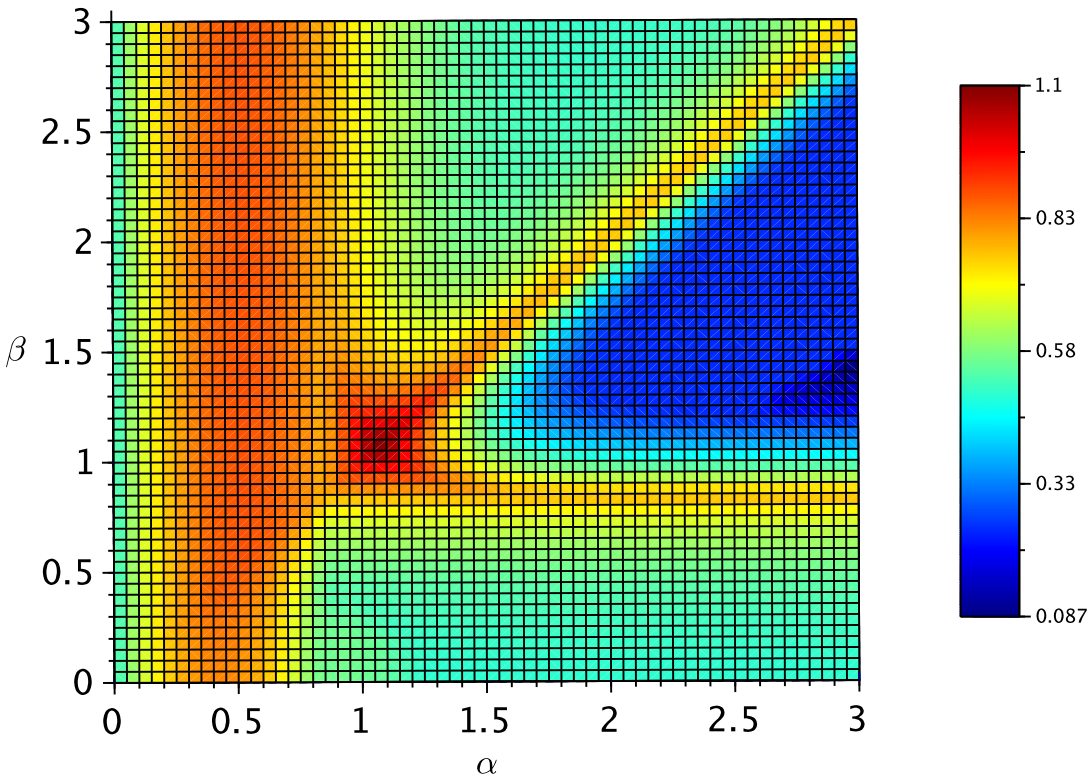

Figure 2: Gap between the converse region $\overline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ and the achievable region $\underline{\mathcal{C}}_{\mathrm{GIC-NOF}}$ of the two-user G-IC-NOF, under symmetric channel conditions, i.e., $\overrightarrow{\mathrm{SNR}}_1 = \overrightarrow{\mathrm{SNR}}_2 = \overrightarrow{\mathrm{SNR}}$, $\mathrm{INR}_{12} = \mathrm{INR}_{21} = \mathrm{INR}$ and $\overleftarrow{\mathrm{SNR}}_1 = \overleftarrow{\mathrm{SNR}}_2 = \overleftarrow{\mathrm{SNR}}$, as a function of $\alpha = \frac{\log \mathrm{INR}}{\log \overrightarrow{\mathrm{SNR}}}$ and $\beta = \frac{\log \overleftarrow{\mathrm{SNR}}}{\log \overrightarrow{\mathrm{SNR}}}$.

3 Conclusions

In this research report, a new achievable region and a new converse region for the two-user G-IC-NOF were presented. These regions approximate the capacity of the two-user G-IC-NOF to within 4.4 bits per channel use.

Appendices

A Proof of Theorem 1

This appendix describes an achievability scheme for the G-IC-NOF based on a three-part message splitting, superposition coding and backward decoding. This coding scheme is general and thus, it holds for the two-user G-IC-NOF and other channel models, e.g, linear deterministic channels [6].

Codebook Generation: fix a joint probability distribution

$$P_{U\,U_1\,U_2\,V_1\,V_2\,X_1\,X_2}(u,u_1,u_2,v_1,v_2,x_1,x_2) = P_U(u)P_{U_1|U}(u_1|u)P_{U_2|U}(u_2|u)P_{V_1|U\,U_1}(v_1|u,u_1)$$
$$P_{V_2|U\,U_2}(v_2|u,u_2)P_{X_1|U\,U_1\,V_1}(x_1|u,u_1,v_1)P_{X_2|U\,U_2\,V_2}(x_2|u,u_2,v_2). \quad (14)$$

Let $R_{1,C1}$, $R_{1,C2}$, $R_{2,C1}$, $R_{2,C2}$, $R_{1,P}$ and $R_{2,P}$ be non-negative reals. Let also $R_{1,C} = R_{1,C1} + R_{1,C2}$, $R_{2,C} = R_{2,C1}+R_{2,C2}$, $R_1 = R_{1,C}+R_{1,P}$ and $R_2 = R_{2,C}+R_{2,P}$. Generate $2^{N(R_{1,C1}+R_{2,C1})}$ i.i.d. N-length codewords $\boldsymbol{u}(s,r) = \big(u_1(s,r),\ldots,u_N(s,r)\big)$ according to

$$P_{\boldsymbol{U}}\big(\boldsymbol{u}(s,r)\big) = \prod_{i=1}^{N} P_U(u_i(s,r)), \quad (15)$$

with $s \in \{1,\ldots,2^{NR_{1,C1}}\}$ and $r \in \{1,\ldots,2^{NR_{2,C1}}\}$.
For encoder 1, generate for each codeword $\boldsymbol{u}(s,r)$, $2^{NR_{1,C1}}$ i.i.d. N-length codewords $\boldsymbol{u}_1(s,r,k) = \big(u_{1,1}(s,r,k),\ldots,u_{1,N}(s,r,k)\big)$ according to

$$P_{\boldsymbol{U}_1|\boldsymbol{U}}\big(\boldsymbol{u}_1(s,r,k)|\boldsymbol{u}(s,r)\big) = \prod_{i=1}^{N} P_{U_1|U}\big(u_{1,i}(s,r,k)|u_i(s,r)\big), \quad (16)$$

with $k \in \{1,\ldots,2^{NR_{1,C1}}\}$. For each pair of codewords $\big(\boldsymbol{u}(s,r),\boldsymbol{u}_1(s,r,k)\big)$, generate $2^{NR_{1,C2}}$ i.i.d. N-length codewords $\boldsymbol{v}_1(s,r,k,l) = \big(v_{1,1}(s,r,k,l),\ldots,v_{1,N}(s,r,k,l)\big)$ according to

$$P_{\boldsymbol{V}_1|\boldsymbol{U}\,\boldsymbol{U}_1}\big(\boldsymbol{v}_1(s,r,k,l)|\boldsymbol{u}(s,r),\boldsymbol{u}_1(s,r,k)\big) = \prod_{i=1}^{N} P_{V_1|U\,U_1}\big(v_{1,i}(s,r,k,l)|u_i(s,r),u_{1,i}(s,r,k)\big), \quad (17)$$

with $l \in \{1,\ldots,2^{NR_{1,C2}}\}$. For each tuple of codewords $\big(\boldsymbol{u}(s,r),\ \boldsymbol{u}_1(s,r,k),\ \boldsymbol{v}_1(s,r,k,l)\big)$, generate $2^{NR_{1,P}}$ i.i.d. N-length codewords $\boldsymbol{x}_1(s,r,k,l,q) = \big(x_{1,1}(s,r,k,l,q),\ldots,\ x_{1,N}(s,r,k,l,q)\big)$ according to

$$P_{\boldsymbol{X}_1|\boldsymbol{U}\,\boldsymbol{U}_1\,\boldsymbol{V}_1}\big(\boldsymbol{x}_1(s,r,k,l,q)|\boldsymbol{u}(s,r),\boldsymbol{u}_1(s,r,k),\boldsymbol{v}_1(s,r,k,l)\big) =$$
$$\prod_{i=1}^{N} P_{X_1|U\,U_1\,V_1}\big(x_{1,i}(s,r,k,l,q)|u_i(s,r),u_{1,i}(s,r,k),v_{1,i}(s,r,k,l)\big), \quad (18)$$

with $q \in \{1,\ldots,2^{NR_{1,P}}\}$.
For encoder 2, generate for each codeword $\boldsymbol{u}(s,r)$, $2^{NR_{2,C1}}$ i.i.d. N-length codewords $\boldsymbol{u}_2(s,r,j) = \big(u_{2,1}(s,r,j),\ldots,u_{2,N}(s,r,j)\big)$ according to

$$P_{\boldsymbol{U}_2|\boldsymbol{U}}\big(\boldsymbol{u}_2(s,r,j)|\boldsymbol{u}(s,r)\big) = \prod_{i=1}^{N} P_{U_2|U}\big(u_{2,i}(s,r,j)|u_i(s,r)\big), \quad (19)$$

with $j \in \{1, \ldots, 2^{NR_{2,C1}}\}$. For each pair of codewords $\big(\boldsymbol{u}(s,r), \boldsymbol{u}_2(s,r,j)\big)$, generate $2^{NR_{2,C2}}$ i.i.d. length-N codewords $\boldsymbol{v}_2(s,r,j,m) = \big(v_{2,1}(s,r,j,m), \ldots, v_{2,N}(s,r,j,m)\big)$ according to

$$P_{\boldsymbol{V}_2|\boldsymbol{U}\,\boldsymbol{U}_2}\big(\boldsymbol{v}_2(s,r,j,m)|\boldsymbol{u}(s,r), \boldsymbol{u}_2(s,r,j)\big) = \prod_{i=1}^{N} P_{V_2|U\,U_2}(v_{2,i}(s,r,j,m)|u_i(s,r), u_{2,i}(s,r,j)), \tag{20}$$

with $m \in \{1, \ldots, 2^{NR_{2,C2}}\}$. For each tuple of codewords $\big(\boldsymbol{u}(s,r),\ \boldsymbol{u}_2(s,r,j), \boldsymbol{v}_2(s,r,j,m)\big)$, generate $2^{NR_{2,P}}$ i.i.d. N-length codewords $\boldsymbol{x}_2(s,r,j,m,b) = \big(x_{2,1}(s,r,j,m,b), \ldots,\ x_{2,N}(s,r,j,m,b)\big)$ according to

$$P_{\boldsymbol{X}_2|\boldsymbol{U}\,\boldsymbol{U}_2\,\boldsymbol{V}_2}\big(\boldsymbol{x}_2(s,r,j,m,b)|\boldsymbol{u}(s,r), \boldsymbol{u}_2(s,r,j), \boldsymbol{v}_2(s,r,j,m)\big) = \prod_{i=1}^{N} P_{X_2|U\,U_2\,V_2}(x_{2,i}(s,r,j,m,b)|u_i(s,r), u_{2,i}(s,r,j), v_{2,i}(s,r,j,m,b)), \tag{21}$$

with $b \in \{1, \ldots, 2^{NR_{2,P}}\}$.

Encoding: denote by $W_i^{(t)} \in \{1, \ldots, 2^{N(R_{i,C}+R_{i,P})}\}$ the message index of transmitter i during block t, respectively. Let $W_i^{(t)} = \big(W_{i,C}^{(t)}, W_{i,P}^{(t)}\big)$ be the message index composed by the message index $W_{i,C}^{(t)} \in \{1, \ldots, 2^{NR_{i,C}}\}$ and message index $W_{i,P}^{(t)} \in \{1, \ldots, 2^{NR_{i,P}}\}$. The message index $W_{i,P}^{(t)}$ must be reliably decoded at receiver i. Let also $W_{i,C}^{(t)} = (W_{i,C1}^{(t)}, W_{i,C2}^{(t)})$ be the message index composed by the message indices $W_{i,C1}^{(t)} \in \{1, \ldots, 2^{NR_{i,C1}}\}$ and $W_{i,C2}^{(t)} \in \{1, \ldots, 2^{NR_{i,C2}}\}$. The message index $W_{i,C1}^{(t)}$ must be reliably decoded at transmitter j (via feedback). The index message $W_{i,C2}^{(t)}$ must be reliably decoded at receiver j.

Consider Markov encoding with a length of T blocks. At encoding step t, with $t \in \{1, \ldots, T\}$, transmitter 1 sends the codeword $\boldsymbol{x}_1^{(t)} = \boldsymbol{x}_1\left(W_{1,C1}^{(t-1)}, W_{2,C1}^{(t-1)}, W_{1,C1}^{(t)}, W_{1,C2}^{(t)}, W_{1,P}^{(t)}\right)$, where $W_{1,C1}^{(0)} = W_{1,C1}^{(T)} = s^*$ and $W_{2,C1}^{(0)} = W_{2,C1}^{(T)} = r^*$. The pair $(s^*, r^*) \in \{1, \ldots, 2^{N\,R_{1,C1}}\} \times \{1, \ldots, 2^{NR_{2,C1}}\}$ is pre-defined and known at both receivers and transmitters. It is worth noting that the message index $W_{2,C1}^{(t-1)}$ is obtained by transmitter 1 from the feedback signal $\overleftarrow{\boldsymbol{y}}_1^{(t-1)}$ at the end of the previous encoding step $t-1$.

Transmitter 2 follows a similar encoding scheme.

Decoding: both receivers decode their message indices at the end of block T in a backward decoding fashion. At each decoding step t, with $t \in \{1, \ldots, T\}$, receiver 1 obtains the message indices $\big(\widehat{W}_{1,C1}^{(T-t)}, \widehat{W}_{2,C1}^{(T-t)},\ \widehat{W}_{1,C2}^{(T-(t-1))},\ \widehat{W}_{1,P}^{(T-(t-1))}, \widehat{W}_{2,C2}^{(T-(t-1))}\big) \in \{1, \ldots, 2^{NR_{1,C1}}\} \times \{1, \ldots, 2^{NR_{2,C1}}\} \times \{1, \ldots, 2^{NR_{1,C2}}\} \times \{1, \ldots, 2^{NR_{1,P}}\} \times \{1, \ldots, 2^{NR_{2,C2}}\}$. The tuple $\Big(\widehat{W}_{1,C1}^{(T-t)}$, $\widehat{W}_{2,C1}^{(T-t)}, \widehat{W}_{1,C2}^{(T-(t-1))}, \widehat{W}_{1,P}^{(T-(t-1))}, \widehat{W}_{2,C2}^{(T-(t-1))}\Big)$ is the unique tuple that satisfies

$$\begin{aligned}
&\Big(\boldsymbol{u}\left(\widehat{W}_{1,C1}^{(T-t)}, \widehat{W}_{2,C1}^{(T-t)}\right), \boldsymbol{u}_1\left(\widehat{W}_{1,C1}^{(T-t)}, \widehat{W}_{2,C1}^{(T-t)}, W_{1,C1}^{(T-(t-1))}\right),\\
&\boldsymbol{v}_1\left(\widehat{W}_{1,C1}^{(T-t)}, \widehat{W}_{2,C1}^{(T-t)}, W_{1,C1}^{(T-(t-1))}, \widehat{W}_{1,C2}^{(T-(t-1))}\right),\\
&\boldsymbol{x}_1\left(\widehat{W}_{1,C1}^{(T-t)}, \widehat{W}_{2,C1}^{(T-t)}, W_{1,C1}^{(T-(t-1))}, \widehat{W}_{1,C2}^{(T-(t-1))}, \widehat{W}_{1,P}^{(T-(t-1))}\right),\\
&\boldsymbol{u}_2\left(\widehat{W}_{1,C1}^{(T-t)}, \widehat{W}_{2,C1}^{(T-t)}, W_{2,C1}^{(T-(t-1))}\right),\\
&\boldsymbol{v}_2\left(\widehat{W}_{1,C1}^{(T-t)}, \widehat{W}_{2,C1}^{(T-t)}, W_{2,C1}^{(T-(t-1))}, \widehat{W}_{2,C2}^{(T-(t-1))}\right), \overrightarrow{\boldsymbol{y}}_1^{(T-(t-1))}\Big) \in \mathcal{A}_e^{(N)},
\end{aligned} \tag{22}$$

where $W_{1,C1}^{(T-(t-1))}$ and $W_{2,C1}^{(T-(t-1))}$ are assumed to be perfectly decoded in the previous decoding step $t-1$. The set $\mathcal{A}_e^{(N)}$ represents the set of jointly typical sequences. Finally, receiver 2 follows

a similar decoding scheme.

Probability of Error Analysis: an error might occur during encoding step t if the message index $W_{2,C1}^{(t-1)}$ is not correctly decoded at transmitter 1. From the asymptotic equipartion property (AEP) [7], it follows that the message index $W_{2,C1}^{(t-1)}$ can be reliably decoded at transmitter 1 during encoding step t, under the condition:

$$\begin{aligned} R_{2,C1} &\leqslant I\left(\overleftarrow{Y}_1; U_2 | U, U_1, V_1, X_1\right) \\ &= I\left(\overleftarrow{Y}_1; U_2 | U, X_1\right). \end{aligned} \tag{23}$$

An error might occur during the (backward) decoding step t if the message indices $W_{1,C1}^{(T-t)}$, $W_{2,C1}^{(T-t)}$, $W_{1,C2}^{(T-(t-1))}$, $W_{1,P}^{(T-(t-1))}$ and $W_{2,C2}^{(T-(t-1))}$ are not decoded correctly given that the message indices $W_{1,C1}^{(T-(t-1))}$ and $W_{2,C1}^{(T-(t-1))}$ were correctly decoded in the previous decoding step $t-1$. These errors might arise for two reasons: (i) there does not exist a tuple $\left(\widehat{W}_{1,C1}^{(T-t)}, \widehat{W}_{2,C1}^{(T-t)}, \widehat{W}_{1,C2}^{(T-(t-1))}, \widehat{W}_{1,P}^{(T-(t-1))}, \widehat{W}_{2,C2}^{(T-(t-1))}\right)$ that satisfies (22), or (ii) there exist several tuples $\left(\widehat{W}_{1,C1}^{(T-t)}, \widehat{W}_{2,C1}^{(T-t)}, \widehat{W}_{1,C2}^{(T-(t-1))}, \widehat{W}_{1,P}^{(T-(t-1))}, \widehat{W}_{2,C2}^{(T-(t-1))}\right)$ that simultaneously satisfy (22). From the asymptotic equipartion property (AEP) [7], the probability of an error due to (i) tends to zero when N grows to infinity. Consider the error due to (ii) and define the event $E_{(s,r,l,q,m)}$ that describes the case in which the codewords $\big(\boldsymbol{u}(s,r)$, $\boldsymbol{u}_1(s,r,W_{1,C1}^{(T-(t-1))})$, $\boldsymbol{v}_1(s,r,W_{1,C1}^{(T-(t-1))},l)$, $\boldsymbol{x}_1(s,r,W_{1,C1}^{(T-(t-1))},l,q)$, $\boldsymbol{u}_2(s,r,W_{2,C1}^{(T-(t-1))})$ and $\boldsymbol{v}_2(s,r,W_{2,C1}^{(T-(t-1))},m)\big)$ are jointly typical with $\overrightarrow{\boldsymbol{y}}_1^{(T-(t-1))}$ during decoding step t. Assume now that the codeword to be decoded at decoding step t corresponds to the indices $(s,r,l,q,m) = (1,1,1,1,1)$, this is without loss of generality due to the symmetry of the code. Then, the probability of error due to (ii) during decoding step t, can be bounded as follows

$$\begin{aligned} P_e = &\Pr\left(\bigcup_{(s,r,l,q,m)\neq(1,1,1,1,1)} E_{(s,r,l,q,m)}\right) \\ \leqslant &\sum_{s=1,r=1,l=1,q=1,m\neq1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s=1,r=1,l=1,q\neq1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right) \\ &+ \sum_{s=1,r=1,l=1,q\neq1,m\neq1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s=1,r=1,l\neq1,q=1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right) \\ &+ \sum_{s=1,r=1,l\neq1,q=1,m\neq1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s=1,r=1,l\neq1,q\neq1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right) \\ &+ \sum_{s=1,r=1,l\neq1,q\neq1,m\neq1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s=1,r\neq1,l=1,q=1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right) \\ &+ \sum_{s=1,r\neq1,l=1,q=1,m\neq1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s=1,r\neq1,l=1,q\neq1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right) \\ &+ \sum_{s=1,r\neq1,l=1,q\neq1,m\neq1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s=1,r\neq1,l\neq1,q=1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right) \\ &+ \sum_{s=1,r\neq1,l\neq1,q=1,m\neq1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s=1,r\neq1,l\neq1,q\neq1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right) \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{s=1,r\neq 1,l\neq 1,q\neq 1,m\neq 1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s\neq 1,r=1,l=1,q=1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right)\\
&+ \sum_{s\neq 1,r=1,l=1,q=1,m\neq 1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s\neq 1,r=1,l=1,q\neq 1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right)\\
&+ \sum_{s\neq 1,r=1,l=1,q\neq 1,m\neq 1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s\neq 1,r=1,l\neq 1,q=1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right)\\
&+ \sum_{s\neq 1,r=1,l\neq 1,q=1,m\neq 1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s\neq 1,r=1,l\neq 1,q\neq 1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right)\\
&+ \sum_{s\neq 1,r=1,l\neq 1,q\neq 1,m\neq 1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s\neq 1,r\neq 1,l=1,q=1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right)\\
&+ \sum_{s\neq 1,r\neq 1,l=1,q=1,m\neq 1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s\neq 1,r\neq 1,l=1,q\neq 1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right)\\
&+ \sum_{s\neq 1,r\neq 1,l=1,q\neq 1,m\neq 1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s\neq 1,r\neq 1,l\neq 1,q=1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right)\\
&+ \sum_{s\neq 1,r\neq 1,l\neq 1,q=1,m\neq 1} \Pr\left(E_{(s,r,l,q,m)}\right) + \sum_{s\neq 1,r\neq 1,l\neq 1,q\neq 1,m=1} \Pr\left(E_{(s,r,l,q,m)}\right)\\
&+ \sum_{s\neq 1,r\neq 1,l\neq 1,q\neq 1,m\neq 1} \Pr\left(E_{(s,r,l,q,m)}\right). \qquad (24)
\end{aligned}
$$

From the asymptotic equipartion property (AEP) [7], it follows that:

$$
\begin{aligned}
P_e \leqslant\; & 2^{N(R_{2,C2}-I(\overrightarrow{Y}_1;V_2|U,U_1,U_2,V_1,X_1)+2\epsilon)}\\
&+2^{N(R_{1,P}-I(\overrightarrow{Y}_1;X_1|U,U_1,U_2,V_1,V_2)+2\epsilon)}\\
&+2^{N(R_{2,C2}+R_{1,P}-I(\overrightarrow{Y}_1;V_2,X_1|U,U_1,U_2,V_1)+2\epsilon)}\\
&+2^{N(R_{1,C2}-I(\overrightarrow{Y}_1;V_1,X_1|U,U_1,U_2,V_2)+2\epsilon)}\\
&+2^{N(R_{1,C2}+R_{2,C2}-I(\overrightarrow{Y}_1;V_1,V_2,X_1|U,U_1,U_2)+2\epsilon)}\\
&+2^{N(R_{1,C2}+R_{1,P}-I(\overrightarrow{Y}_1;V_1,X_1|U,U_1,U_2,V_2)+2\epsilon)}\\
&+2^{N(R_{1,C2}+R_{1,P}+R_{2,C2}-I(\overrightarrow{Y}_1;V_1,V_2,X_1|U,U_1,U_2)+2\epsilon)}\\
&+2^{N(R_{2,C1}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{2,C1}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{2,C1}+R_{1,P}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{2,C1}+R_{1,P}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{2,C1}+R_{1,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{2,C1}+R_{1,C2}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{2,C1}+R_{1,C2}+R_{1,P}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{2,C1}+R_{1,C2}+R_{1,P}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{1,C1}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{1,C1}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{1,C1}+R_{1,P}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{1,C1}+R_{1,P}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{1,C1}+R_{1,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}
\end{aligned}
$$

$$
\begin{aligned}
&+2^{N(R_{1,C1}+R_{1,C2}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{1,C1}+R_{1,C2}+R_{1,P}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{1,C1}+R_{1,C2}+R_{1,P}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{1,C1}+R_{2,C1}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{1,C1}+R_{2,C1}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{1,C1}+R_{2,C1}+R_{1,P}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{1,C1}+R_{2,C1}+R_{1,P}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{1,C1}+R_{2,C1}+R_{1,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{1,C1}+R_{2,C1}+R_{1,C2}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{1,C1}+R_{2,C1}+R_{1,C2}+R_{1,P}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}\\
&+2^{N(R_{1,C1}+R_{2,C1}+R_{1,C2}+R_{1,P}+R_{2,C2}-I(\overrightarrow{Y}_1;U,U_1,U_2,V_1,V_2,X_1)+2\epsilon)}. \qquad (25)
\end{aligned}
$$

The same analysis of the probability of error holds for transmitter-receiver pair 2. Hence, in general, from (23) and (25), reliable decoding holds under the following conditions for transmitter $i \in \{1,2\}$, with $j \in \{1,2\} \setminus \{i\}$:

$$
\begin{aligned}
R_{j,C1} &\leqslant I\left(\overleftarrow{Y}_i; U_j|U,U_i,V_i,X_i\right)\\
&= I\left(\overleftarrow{Y}_i; U_j|U,X_i\right), && (26a)\\
R_{i,C1}+R_{i,C2}+R_{i,P}+R_{j,C1}+R_{j,C2} &\leqslant I(\overrightarrow{Y}_i; U,U_i,U_j,V_i,V_j,X_i)\\
&= I(\overrightarrow{Y}_i; U,U_j,V_j,X_i), && (26b)\\
R_{j,C2} &\leqslant I(\overrightarrow{Y}_i; V_j|U,U_i,U_j,V_i,X_i)\\
&= I(\overrightarrow{Y}_i; V_j|U,U_j,X_i), && (26c)\\
R_{i,P} &\leqslant I(\overrightarrow{Y}_i; X_i|U,U_i,U_j,V_i,V_j), && (26d)\\
R_{i,P}+R_{j,C2} &\leqslant I(\overrightarrow{Y}_i; V_j,X_i|U,U_i,U_j,V_i), && (26e)\\
R_{i,C2}+R_{i,P} &\leqslant I(\overrightarrow{Y}_i; V_i,X_i|U,U_i,U_j,V_j)\\
&= I(\overrightarrow{Y}_i; X_i|U,U_i,U_j,V_j), && (26f)\\
R_{i,C2}+R_{i,P}+R_{j,C2} &\leqslant I(\overrightarrow{Y}_i; V_i,V_j,X_i|U,U_i,U_j)\\
&= I(\overrightarrow{Y}_i; V_j,X_i|U,U_i,U_j). && (26g)
\end{aligned}
$$

More explicitly, the inequalities in (26) can be written as follows

$$
\begin{aligned}
R_{2,C1} &\leqslant I\left(\overleftarrow{Y}_1; U_2|U,X_1\right) &&= a_1, && (27a)\\
R_{1,C1}+R_{1,C2}+R_{1,P}+R_{2,C1}+R_{2,C2} &\leqslant I(\overrightarrow{Y}_1; U,U_2,V_2,X_1) &&= a_2, && (27b)\\
R_{2,C2} &\leqslant I(\overrightarrow{Y}_1; V_2|U,U_2,X_1) &&= a_3, && (27c)\\
R_{1,P} &\leqslant I(\overrightarrow{Y}_1; X_1|U,U_1,U_2,V_1,V_2) &&= a_4, && (27d)\\
R_{1,P}+R_{2,C2} &\leqslant I(\overrightarrow{Y}_1; V_2,X_1|U,U_1,U_2,V_1) &&= a_5, && (27e)\\
R_{1,C2}+R_{1,P} &\leqslant I(\overrightarrow{Y}_1; X_1|U,U_1,U_2,V_2) &&= a_6, && (27f)\\
R_{1,C2}+R_{1,P}+R_{2,C2} &\leqslant I(\overrightarrow{Y}_1; V_2,X_1|U,U_1,U_2) &&= a_7, && (27g)
\end{aligned}
$$

$$
\begin{aligned}
R_{1,C1} &\leqslant I\left(\overleftarrow{Y}_2; U_1|U, X_2\right) &&= b_1, && (27h)\\
R_{2,C1}+R_{2,C2}+R_{2,P}+R_{1,C1}+R_{1,C2} &\leqslant I(\overrightarrow{Y}_2; U, U_1, V_1, X_2) &&= b_2, && (27i)\\
R_{1,C2} &\leqslant I(\overrightarrow{Y}_2; V_1|U, U_1, X_2) &&= b_3, && (27j)\\
R_{2,P} &\leqslant I(\overrightarrow{Y}_2; X_2|U, U_1, U_2, V_1, V_2) &&= b_4, && (27k)\\
R_{2,P}+R_{1,C2} &\leqslant I(\overrightarrow{Y}_2; V_1, X_2|U, U_1, U_2, V_2) &&= b_5, && (27l)\\
R_{2,C2}+R_{2,P} &\leqslant I(\overrightarrow{Y}_2; X_2|U, U_1, U_2, V_1) &&= b_6, && (27m)\\
R_{2,C2}+R_{2,P}+R_{1,C2} &\leqslant I(\overrightarrow{Y}_2; V_1, X_2|U, U_1, U_2) &&= b_7. && (27n)
\end{aligned}
$$

Taking into account that $R_1 = R_{1,C1} + R_{1,C2} + R_{1,P}$ and $R_2 = R_{2,C1} + R_{2,C2} + R_{2,P}$, a Fourier-Motzkin elimination process in (27) yields

$$
\begin{aligned}
R_1 &\leqslant \min\left(a_2, a_6+b_1, a_4+b_1+b_3\right), && (28a)\\
R_2 &\leqslant \min\left(b_2, a_1+b_6, a_1+a_3+b_4\right), && (28b)\\
R_1+R_2 &\leqslant \min(a_2+b_4, a_2+b_6, a_4+b_2, a_6+b_2, a_1+a_3+a_4+b_1+b_5,\\
&\quad a_1+a_7+b_1+b_5, a_1+a_4+b_1+b_7, a_1+a_5+b_1+b_3+b_4, a_1+a_5+b_1+b_5,\\
&\quad a_1+a_7+b_1+b_4), && (28c)\\
2R_1+R_2 &\leqslant \min(a_2+a_4+b_1+b_7, a_1+a_4+a_7+2b_1+b_5, a_2+a_4+b_1+b_5), && (28d)\\
R_1+2R_2 &\leqslant \min(a_1+a_5+b_2+b_4, a_1+a_7+b_2+b_4, 2a_1+a_5+b_1+b_4+b_7), && (28e)
\end{aligned}
$$

where $a_1, \ldots a_7$ and $b_1, \ldots b_7$ are defined in (27).
Consider that transmitter i uses the following Gaussian input distribution:

$$X_i = U + U_i + V_i + X_{i,P}, \tag{29}$$

where U, U_1, U_2, V_1, V_2, $X_{1,P}$ and $X_{2,P}$ are mutually independent and distributed as follows:

$$
\begin{aligned}
U &\sim \mathcal{N}\left(0, \rho\right), && (30a)\\
U_i &\sim \mathcal{N}\left(0, \alpha_i \cdot \lambda_{i,C}\right), && (30b)\\
V_i &\sim \mathcal{N}\left(0, (1-\alpha_i) \cdot \lambda_{i,C}\right), && (30c)\\
X_{i,P} &\sim \mathcal{N}\left(0, \lambda_{i,P}\right), && (30d)
\end{aligned}
$$

with

$$\rho + \lambda_{i,C} + \lambda_{i,P} = 1 \text{ and} \tag{31a}$$

$$\lambda_{i,P} = \min\left(\frac{1}{\mathrm{INR}_{ji}}, 1\right), \tag{31b}$$

where $\alpha_i \in [0,1]$ and $\rho \in \left[0, \left(1 - \max\left(\frac{1}{\mathrm{INR}_{12}}, \frac{1}{\mathrm{INR}_{21}}\right)\right)^+\right]$. The random variables U, U_1, U_2, V_1 and V_2 can be interpreted as components of the signals X_1 and X_2. For instance, the random variable U_i represents the signal component reliably carrying a message index from transmitter i to transmitter j, via the corresponding feedback link; the random variable V_i represents the signal component reliably carrying a message index from transmitter i to receiver j; the random variable $X_{i,P}$ represents the signal component reliably carrying a message index from transmitter i to receiver i. Finally, the random variable U represents the common component of the channel inputs of transmitter 1 and transmitter 2. Note that such a common signal component can be

constructed using previous observations of the random variables U_1 and U_2, which are obtained at each transmitter thanks to the use of feedback.
The parameters ρ, α_i, $\lambda_{i,C}$, and $\lambda_{i,P}$ define a particular power allocation for transmitter i. The assignment in (31b) is based on the intuition obtained from the linear deterministic model, in which the power of the private signal from transmitter i to receiver j most be observed at the noise level. From (2), (3a), (3b) and (29), the right-hand side of the inequalities in (26) can be written in terms of $\overrightarrow{\mathrm{SNR}}_1$, $\overrightarrow{\mathrm{SNR}}_2$, INR_{12}, INR_{21}, $\overleftarrow{\mathrm{SNR}}_1$, $\overleftarrow{\mathrm{SNR}}_2$, ρ, α_1 and α_2 as follows

$$
\begin{aligned}
&I\left(\overleftarrow{Y}_i; U_j | U, X_i\right)\\
=&\frac{1}{2}\log\left(\frac{\overleftarrow{\mathrm{SNR}}_i\left(\mathrm{INR}_{ij}+1\right)+\overrightarrow{\mathrm{SNR}}_i+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_i\mathrm{INR}_{ij}}+\mathrm{INR}_{ij}+1}{\overleftarrow{\mathrm{SNR}}_i\left(\left(1-\alpha_j\right)\left(\left(1-\rho\right)\mathrm{INR}_{ij}-1\right)+2\right)+\overrightarrow{\mathrm{SNR}}_i+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_i\mathrm{INR}_{ij}}+\mathrm{INR}_{ij}+1}\right),
\end{aligned}
\tag{32a}
$$

$$
I\left(\overrightarrow{Y}_i; U, U_j, V_j, X_i\right)=\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_i+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_i\mathrm{INR}_{ij}}+\mathrm{INR}_{ij}+1\right)-\frac{1}{2}, \tag{32b}
$$

$$
I\left(\overrightarrow{Y}_i; V_j | U, U_j, X_i\right)=\frac{1}{2}\log\left(\left(1-\alpha_j\right)\left(\left(1-\rho\right)\mathrm{INR}_{ij}-1\right)+2\right)-\frac{1}{2}, \tag{32c}
$$

$$
I\left(\overrightarrow{Y}_i; X_i | U, U_i, U_j, V_i, V_j\right)=\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_i}{\mathrm{INR}_{ji}}+2\right)-\frac{1}{2}, \tag{32d}
$$

$$
I\left(\overrightarrow{Y}_i; V_j, X_i | U, U_i, U_j, V_i\right)=\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_i}{\mathrm{INR}_{ji}}+\left(1-\alpha_j\right)\left(\left(1-\rho\right)\mathrm{INR}_{ij}-1\right)+2\right)-\frac{1}{2}, \tag{32e}
$$

$$
I\left(\overrightarrow{Y}_i; X_i | U, U_i, U_j, V_j\right)=\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_i}{\mathrm{INR}_{ji}}\left(\left(1-\alpha_i\right)\left(\left(1-\rho\right)\mathrm{INR}_{ji}-1\right)+1\right)+2\right)-\frac{1}{2}, \tag{32f}
$$

$$
\begin{aligned}
&I\left(\overrightarrow{Y}_i; V_j, X_i | U, U_i, U_j\right)\\
=&\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_i}{\mathrm{INR}_{ji}}\left(\left(1-\alpha_i\right)\left(\left(1-\rho\right)\mathrm{INR}_{ji}-1\right)+1\right)+\left(1-\alpha_j\right)\left(\left(1-\rho\right)\mathrm{INR}_{ij}-1\right)+2\right)-\frac{1}{2}.
\end{aligned}
\tag{32g}
$$

Using (32), with $i \in \{1,2\}$ and $j \in \{1,2\}\backslash\{i\}$, the variables $a_1, \ldots, a_7$ and $b_1, \ldots, b_7$ in (27) can be written as in (8). Note that the sum-rate bound in (28c) can be simplified as follows

$$
R_1 + R_2 \leqslant \min(a_2+b_4, a_4+b_2, a_1+a_4+b_1+b_7, a_1+a_5+b_1+b_5, a_1+a_7+b_1+b_4), \tag{33}
$$

given that $\max(a_2+b_4, a_4+b_2, a_1+a_4+b_1+b_7, a_1+a_5+b_1+b_5, a_1+a_7+b_1+b_4) \leqslant \min(a_2+b_6, a_6+b_2, a_1+a_3+a_4+b_1+b_5, a_1+a_7+b_1+b_5, a_1+a_5+b_1+b_3+b_4)$. Therefore, the inequalities in (28) simplified into (9), which completes the proof of Theorem 1.

B Proof of Theorem 2

This appendix provides a proof of the Theorem 2. The outer bounds (13a) and (13c) in Theorem 2 correspond to the outer bounds of the case of perfect channel-output feedback [5], so this does not need a proof. The bounds (13b), (13d) and (13e) correspond to new outer bounds and are given by Lemma 2 . Before presenting the proof, consider the parameter $h_{ji,U}$, with $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$, defined as follows

$$h_{ji,U} = \begin{cases} 0 & \text{if } (S_{1,i} \vee S_{2,i} \vee S_{3,i}) = \text{True} \\ \sqrt{\frac{\text{INR}_{ij}\text{INR}_{ji}}{\overrightarrow{\text{SNR}}_j}} & \text{if } (S_{4,i} \vee S_{5,i}) = \text{True}, \end{cases} \tag{34}$$

where the events $S_{1,i}$, $S_{2,i}$, $S_{3,i}$, $S_{4,i}$ and $S_{5,i}$ are defined in (10). Note that the parameter $h_{ji,U}$ is well defined for any $(\overrightarrow{\text{SNR}}_1, \overrightarrow{\text{SNR}}_2, \text{INR}_{12}, \text{INR}_{21}) \in \mathbb{R}_+^4$ (Lemma 1). Consider also the following signals:

$$X_{i,C} = \sqrt{INR_{ji}} X_i + \overrightarrow{Z}_j \text{ and} \tag{35}$$

$$X_{i,U} = h_{ji,U} X_i + \overrightarrow{Z}_j, \tag{36}$$

where, X_i and $\overrightarrow{Z}_j$ are the channel input of transmitter i and the noise observed at the receiver j at channel use $n \in \{1, \ldots, N\}$, as described by (2). Note that the channel use index n has been dropped by the ease of notation.

Lemma 2 *Let a non-negative rate pair (R_1, R_2) be in the capacity region of the two-user G-IC-NOF. Then, the pair (R_1, R_2) satisfies the following conditions for all $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$:*

$$R_i \leqslant h\left(\overrightarrow{Y}_i, \overleftarrow{Y}_j | X_j\right) - \log(2\pi e), \tag{37a}$$

$$\begin{aligned} R_1 + R_2 \leqslant & h\left(\overrightarrow{Y}_1 | X_{1,C}, X_{2,U}\right) + h\left(X_{1,U} | X_{2,C}\right) + h\left(\overleftarrow{Y}_2 | X_{1,U}, X_2\right) + h\left(\overrightarrow{Y}_2 | X_{1,U}, X_{2,C}\right) \\ & + h\left(X_{2,U} | X_{1,C}\right) + h\left(\overleftarrow{Y}_1 | X_{2,U}, X_1\right) - 3\log(2\pi e) \end{aligned} \tag{37b}$$

$$\begin{aligned} 2R_i + R_j \leqslant & h\left(\overrightarrow{Y}_i\right) + h\left(X_{i,U} | X_{j,C}\right) + h\left(\overleftarrow{Y}_j | X_{i,U}, X_j\right) + h\left(\overrightarrow{Y}_j | X_{i,U}, X_{j,C}\right) + h\left(\overrightarrow{Y}_i | X_{i,C}, X_j\right) \\ & - \frac{5}{2}\log(2\pi e). \end{aligned} \tag{37c}$$

Proof: The proof of Lemma 2 is presented in Sec. B.1 ■

Using Lemma 2, the proof of Theorem 2 continues as follows.

Proof of (13b): for all $i \in \{1,2\}$ and $j \in \{1,2\} \setminus \{i\}$, from (37a) the following holds

$$\begin{aligned} R_i \leqslant & h\left(\overrightarrow{Y}_i, \overleftarrow{Y}_j | X_j\right) - \log(2\pi e) \\ = & h\left(\overrightarrow{Y}_i, \overleftarrow{Y}_j, X_j\right) - h\left(X_j\right) - \log(2\pi e) \\ \leqslant & \frac{1}{2}\log\left(\left(1-\rho^2\right)\overrightarrow{\text{SNR}}_i + 1\right) \\ & + \frac{1}{2}\log\left(\frac{\overleftarrow{\text{SNR}}_j\left(\left(1-\rho^2\right)\left(\overrightarrow{\text{SNR}}_i + \text{INR}_{ji}\right) + 1\right)}{\left(\overrightarrow{\text{SNR}}_j + 2\rho\sqrt{\overrightarrow{\text{SNR}}_j \text{INR}_{ji}} + \text{INR}_{ji} + 1\right)\left(\left(1-\rho^2\right)\left(\overrightarrow{\text{SNR}}_i + 1\right)\right)} + 1\right). \end{aligned} \tag{38}$$

This completes the proof of (13b).

Proof of (13d): This proof uses the signals defined in (35) and (36). From (37b), the following holds

$$\begin{aligned}
R_1+R_2 \leqslant & h\left(\overrightarrow{Y}_1|X_{1,C},X_{2,U}\right)+h\left(X_{1,U}|X_{2,C}\right)+h\left(\overleftarrow{Y}_2|X_2,X_{1,U}\right)+h\left(\overrightarrow{Y}_2|X_{2,C},X_{1,U}\right)\\
&+h\left(X_{2,U}|X_{1,C}\right)+h\left(\overleftarrow{Y}_1|X_1,X_{2,U}\right)-3\log\left(2\pi e\right)\\
=&h\left(\overrightarrow{Y}_1,X_{1,C},X_{2,U}\right)-h\left(X_{2,C}\right)+h\left(\overleftarrow{Y}_2,X_2,X_{1,U}\right)-h\left(X_2,X_{1,U}\right)+h\left(\overrightarrow{Y}_2,X_{2,C},X_{1,U}\right)\\
&-h\left(X_{1,C}\right)+h\left(\overleftarrow{Y}_1,X_1,X_{2,U}\right)-h\left(X_1,X_{2,U}\right)-3\log\left(2\pi e\right)\\
\leqslant&\frac{1}{2}\log\left(\det\left(\mathrm{Var}\left(\overrightarrow{Y}_1,X_{1,C},X_{2,U}\right)\right)\right)-\frac{1}{2}\log\left(\mathrm{INR}_{12}+1\right)+\frac{1}{2}\log\left(\det\left(\mathrm{Var}\left(\overleftarrow{Y}_2,X_2,X_{1,U}\right)\right)\right)\\
&-\frac{1}{2}\log\left(\det\left(\mathrm{Var}\left(X_2,X_{1,U}\right)\right)\right)+\frac{1}{2}\log\left(\det\left(\mathrm{Var}\left(\overrightarrow{Y}_2,X_{2,C},X_{1,U}\right)\right)\right)-\frac{1}{2}\log\left(\mathrm{INR}_{21}+1\right)\\
&+\frac{1}{2}\log\left(\det\left(\mathrm{Var}\left(\overleftarrow{Y}_1,X_1,X_{2,U}\right)\right)\right)-\frac{1}{2}\log\left(\det\left(\mathrm{Var}\left(X_1,X_{2,U}\right)\right)\right),
\end{aligned}\tag{39}$$

where, for all $i\in\{1,2\}$ and $j\in\{1,2\}\setminus\{i\}$ the following holds

$$\begin{aligned}
\det\left(\mathrm{Var}\left(\overrightarrow{Y}_j,X_{j,C},X_{i,U}\right)\right)=&\overrightarrow{\mathrm{SNR}}_j+\mathrm{INR}_{ji}+h_{ji,U}^2-2h_{ji,U}\sqrt{\mathrm{INR}_{ji}}+\left(1-\rho^2\right)\Big(\mathrm{INR}_{ij}\mathrm{INR}_{ji}\\
&+h_{ji,U}^2\left(\overrightarrow{\mathrm{SNR}}_j+\mathrm{INR}_{ij}\right)-2h_{ji,U}\mathrm{INR}_{ij}\sqrt{\mathrm{INR}_{ji}}\Big)\\
&+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_j}\left(\sqrt{\mathrm{INR}_{ji}}-h_{ji,U}\right),
\end{aligned}\tag{40a}$$

$$\begin{aligned}
\det\left(\mathrm{Var}\left(\overleftarrow{Y}_j,X_j,X_{i,U}\right)\right)=&1+h_{ji,U}^2\left(1-\rho^2\right)\\
&+\frac{\overleftarrow{\mathrm{SNR}}_j\left(1-\rho^2\right)\left(h_{ji,U}^2-2h_{ji,U}\sqrt{\mathrm{INR}_{ji}}+\mathrm{INR}_{ji}\right)}{\left(\overrightarrow{\mathrm{SNR}}_j+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_j\mathrm{INR}_{ji}}+\mathrm{INR}_{ji}+1\right)}\quad\text{and}
\end{aligned}\tag{40b}$$

$$\det\left(\mathrm{Var}\left(X_j,X_{i,U}\right)\right)=1+\left(1-\rho^2\right)h_{ji,U}^2.\tag{40c}$$

The expressions in (40) depend on $S_{1,i}$, $S_{2,i}$, $S_{3,i}$, $S_{4,i}$ and $S_{5,i}$ via the parameter $h_{ji,U}$ in (34). Hence, the following cases are considered.

Case 1: $(S_{1,2}\vee S_{2,2}\vee S_{3,2})\wedge(S_{1,1}\vee S_{2,1}\vee S_{3,1})=$ True. From (34), it follows that $h_{12,U}=0$ and $h_{21,U}=0$. Then, under these conditions, plugging the expressions (40) into (39) yields (12a).

Case 2: $(S_{1,2}\vee S_{2,2}\vee S_{3,2})\wedge(S_{4,1}\vee S_{5,1})=$ True. From (34), it follows that $h_{12,U}=0$ and $h_{21,U}=\sqrt{\frac{\mathrm{INR}_{12}\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_2}}$. Then, under these conditions, plugging the expressions (40) into (39) yields (12b).

Case 3: $(S_{4,2}\vee S_{5,2})\wedge(S_{1,1}\vee S_{2,1}\vee S_{3,1})=$ True. From (34), it follows that $h_{12,U}=\sqrt{\frac{\mathrm{INR}_{12}\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_1}}$ and $h_{21,U}=0$. Then, under these conditions, plugging the expressions (40) into (39) yields (12c).

Case 4: $(S_{4,2}\vee S_{5,2})\wedge(S_{4,1}\vee S_{5,1})=$ True. From (34), it follows that $h_{12,U}=\sqrt{\frac{\mathrm{INR}_{12}\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_1}}$ and $h_{21,U}=\sqrt{\frac{\mathrm{INR}_{12}\mathrm{INR}_{21}}{\overrightarrow{\mathrm{SNR}}_2}}$. Then, under these conditions, plugging the expressions (40) into (39) yields (12d).

This completes the proof of (13d).

Proof of (13e): This proof uses the signals defined in (35) and (36). From (37c), the following

holds

$$
\begin{aligned}
2R_i + R_j \leqslant & h\left(\overrightarrow{Y}_i\right) + h\left(X_{i,U}|X_{j,C}\right) + h\left(\overleftarrow{Y}_j|X_j, X_{i,U}\right) + h\left(\overrightarrow{Y}_i|X_{i,C}, X_j\right) + h\left(\overrightarrow{Y}_j|X_{j,C}, X_{i,U}\right) \\
& -\frac{5}{2}\log(2\pi e) \\
= & h\left(\overrightarrow{Y}_i\right) + h\left(X_{i,U}, X_{j,C}\right) - h\left(X_{j,C}\right) + h\left(\overleftarrow{Y}_j, X_j, X_{i,U}\right) - h\left(X_j, X_{i,U}\right) \\
& + h\left(\overrightarrow{Y}_i, X_{i,C}, X_j\right) - h\left(X_{i,C}, X_j\right) + h\left(\overrightarrow{Y}_j, X_{j,C}, X_{i,U}\right) - h\left(X_{j,C}, X_{i,U}\right) - \frac{5}{2}\log(2\pi e) \\
= & h\left(\overrightarrow{Y}_i\right) - h\left(X_{j,C}\right) + h\left(\overleftarrow{Y}_j, X_j, X_{i,U}\right) - h\left(X_j, X_{i,U}\right) + h\left(\overrightarrow{Y}_i, X_{i,C}, X_j\right) \\
& - h\left(X_{i,C}, X_j\right) + h\left(\overrightarrow{Y}_j, X_{j,C}, X_{i,U}\right) - \frac{5}{2}\log(2\pi e) \\
\leqslant & \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_i + 2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_i \mathrm{INR}_{ij}} + \mathrm{INR}_{ij} + 1\right) - \frac{1}{2}\log\left(\mathrm{INR}_{ij} + 1\right) \\
& + \frac{1}{2}\log\left(\det\left(\mathrm{Var}\left(\overleftarrow{Y}_j, X_j, X_{i,U}\right)\right)\right) - \frac{1}{2}\log\left(\det\left(\mathrm{Var}\left(X_j, X_{i,U}\right)\right)\right) \\
& + \frac{1}{2}\log\left(1 + \left(1-\rho^2\right)\left(\overrightarrow{\mathrm{SNR}}_i + \mathrm{INR}_{ji}\right)\right) - \frac{1}{2}\log\left(1 + \left(1-\rho^2\right)\mathrm{INR}_{ji}\right) \\
& + \frac{1}{2}\log\left(\det\left(\mathrm{Var}\left(\overrightarrow{Y}_j, X_{j,C}, X_{i,U}\right)\right)\right). \qquad (41)
\end{aligned}
$$

The upper bound on (41) depends on $S_{1,i}$, $S_{2,i}$, $S_{3,i}$, $S_{4,i}$ and $S_{5,i}$ via the parameter $h_{ji,U}$ in (34). Hence, as in the previous part, the following cases are considered.

Case 1: $(S_{1,i} \vee S_{2,i} \vee S_{3,i}) =$ True. From (34), it follows that $h_{ji,U} = 0$. Then, under these conditions, plugging the expressions (40) into (41) yields (12e).

Case 2: $(S_{4,i} \vee S_{5,i}) =$ True. From (34), it follows that $h_{ji,U} = \sqrt{\frac{\mathrm{INR}_{ij}\mathrm{INR}_{ji}}{\overrightarrow{\mathrm{SNR}}_j}}$. Then, under these conditions, plugging the expressions (40) into (41) yields (12f).

This completes the proof of (13e) and the proof of Theorem 2.

B.1 Proof of Lemma 2

The proof of Lemma 2 is presented in three parts.

Part I: Proof of (37a). Given that the message index W_i is uniformly distributed over the set $\mathcal{W}_i$, for all $i \in \{1, 2\}$ the following holds

$$
\begin{aligned}
NR_i = & H\left(W_i\right) \\
\overset{(a)}{=} & H\left(W_i|W_j\right) \\
= & I\left(W_i; \overrightarrow{\boldsymbol{Y}}_i, \overleftarrow{\boldsymbol{Y}}_j|W_j\right) + h\left(W_i|W_j, \overrightarrow{\boldsymbol{Y}}_i, \overleftarrow{\boldsymbol{Y}}_j\right) \\
\overset{(b)}{\leqslant} & I\left(W_i; \overrightarrow{\boldsymbol{Y}}_i, \overleftarrow{\boldsymbol{Y}}_j|W_j\right) + N\delta(N) \\
\overset{(c)}{\leqslant} & h\left(\overrightarrow{\boldsymbol{Y}}_i, \overleftarrow{\boldsymbol{Y}}_j|W_j\right) - h\left(\overrightarrow{\boldsymbol{Y}}_i, \overleftarrow{\boldsymbol{Y}}_j|W_i, W_j, \boldsymbol{X}_i, \boldsymbol{X}_j, \overrightarrow{\boldsymbol{Y}}_j\right) + N\delta(N) \\
= & h\left(\overrightarrow{\boldsymbol{Y}}_i, \overleftarrow{\boldsymbol{Y}}_j|W_j\right) - h\left(\overrightarrow{\boldsymbol{Z}}_i, \overleftarrow{\boldsymbol{Z}}_j|W_i, W_j, \boldsymbol{X}_i, \boldsymbol{X}_j, \overrightarrow{\boldsymbol{Y}}_j\right) + N\delta(N) \\
= & h\left(\overrightarrow{\boldsymbol{Y}}_i, \overleftarrow{\boldsymbol{Y}}_j|W_j\right) - h\left(\overrightarrow{\boldsymbol{Z}}_i\right) - h\left(\overleftarrow{\boldsymbol{Z}}_j\right) + N\delta(N) \\
= & \sum_{n=1}^{N}\left[h\left(\overrightarrow{Y}_{i,n}, \overleftarrow{Y}_{j,n}|W_j, \overrightarrow{\boldsymbol{Y}}_{i,(1:n-1)}, \overleftarrow{\boldsymbol{Y}}_{j,(1:n-1)}\right) - h\left(\overrightarrow{Z}_{i,n}|\overrightarrow{\boldsymbol{Z}}_{i,(1:n-1)}\right) - h\left(\overleftarrow{Z}_{j,n}|\overleftarrow{\boldsymbol{Z}}_{j,(1:n-1)}\right)\right] \\
& + N\delta(N)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(d)}{=}\sum_{n=1}^{N}\Big[h\left(\overrightarrow{Y}_{i,n},\overleftarrow{Y}_{j,n}|W_j,\overrightarrow{\boldsymbol{Y}}_{i,(1:n-1)},\overleftarrow{\boldsymbol{Y}}_{j,(1:n-1)},\boldsymbol{X}_{j,(1:n)}\right)-h\left(\overrightarrow{Z}_{i,n}|\overrightarrow{\boldsymbol{Z}}_{i,(1:n-1)}\right)\\
&\quad -h\left(\overleftarrow{Z}_{j,n}|\overleftarrow{\boldsymbol{Z}}_{j,(1:n-1)}\right)\Big]+N\delta(N)\\
&=\sum_{n=1}^{N}\Big[h\left(\overrightarrow{Y}_{i,n},\overleftarrow{Y}_{j,n}|W_j,\overrightarrow{\boldsymbol{Y}}_{i,(1:n-1)},\overleftarrow{\boldsymbol{Y}}_{j,(1:n-1)},\boldsymbol{X}_{j,(1:n)}\right)-h\left(\overrightarrow{Z}_{i,n}\right)-h\left(\overleftarrow{Z}_{j,n}\right)\Big]+N\delta(N)\\
&\overset{(c)}{\leqslant}\sum_{n=1}^{N}\Big[h\left(\overrightarrow{Y}_{i,n},\overleftarrow{Y}_{j,n}|X_{j,n}\right)-h\left(\overrightarrow{Z}_{i,n}\right)-h\left(\overleftarrow{Z}_{j,n}\right)\Big]+N\delta(N)\\
&=N\left[h\left(\overrightarrow{Y}_{i},\overleftarrow{Y}_{j}|X_j\right)-\log\left(2\pi e\right)\right]+N\delta(N),
\end{aligned}
\tag{42}
$$

where
(a) follows from the fact that W_1 and W_2 are mutually independent;
(b) follows from Fano's inequality;
(c) follows from the fact that conditioning reduces the entropy;
(d) follows from (4).
From (42), in the asymptotic regime, it holds that

$$
R_i\leqslant h\left(\overrightarrow{Y}_{i},\overleftarrow{Y}_{j}|X_j\right)-\log\left(2\pi e\right). \tag{43}
$$

This completes the proof of (37a).
Part II: Proof of (37b). This part uses an intermediate result that is described by the following lemma.

Lemma 3 *For all $i\in\{1,2\}$ and $j\in\{1,2\}\setminus\{i\}$ the following holds*

$$
\begin{aligned}
&I\Big(\boldsymbol{X}_{i,C},\boldsymbol{X}_{j,U},\overleftarrow{\boldsymbol{Y}}_{i},W_i;\overleftarrow{\boldsymbol{Y}}_{j},W_j\Big)\leqslant h\left(\overleftarrow{\boldsymbol{Y}}_{j}|W_j\right)+\sum_{n=1}^{N}\Big[h\left(X_{j,U,n}|X_{i,C,n}\right)\\
&+h\left(\overleftarrow{Y}_{i,n}|X_{i,n},X_{j,U,n}\right)-\frac{3}{2}\log\left(2\pi e\right)\Big].
\end{aligned}
\tag{44}
$$

Proof: The proof of Lemma 3 is presented in Sec. B.2. ■

Given that the message indices W_1 and W_2 are uniformly distributed over the corresponding sets $\mathcal{W}_1$ and $\mathcal{W}_2$, the following holds

$$
\begin{aligned}
N\Big(R_1+R_2\Big)&=H\left(W_1\right)+H\left(W_2\right)\\
&=I\left(W_1;\overrightarrow{\boldsymbol{Y}}_1,\overleftarrow{\boldsymbol{Y}}_1\right)+h\left(W_1|\overrightarrow{\boldsymbol{Y}}_1,\overleftarrow{\boldsymbol{Y}}_1\right)+I\left(W_2;\overrightarrow{\boldsymbol{Y}}_2,\overleftarrow{\boldsymbol{Y}}_2\right)+h\left(W_2|\overrightarrow{\boldsymbol{Y}}_2,\overleftarrow{\boldsymbol{Y}}_2\right)\\
&\overset{(a)}{\leqslant}I\left(W_1;\overrightarrow{\boldsymbol{Y}}_1,\overleftarrow{\boldsymbol{Y}}_1\right)+I\left(W_2;\overrightarrow{\boldsymbol{Y}}_2,\overleftarrow{\boldsymbol{Y}}_2\right)+N\delta(N)\\
&=h\left(\overrightarrow{\boldsymbol{Y}}_1,\overleftarrow{\boldsymbol{Y}}_1\right)-h\left(\overrightarrow{\boldsymbol{Y}}_1,\overleftarrow{\boldsymbol{Y}}_1|W_1\right)+h\left(\overrightarrow{\boldsymbol{Y}}_2,\overleftarrow{\boldsymbol{Y}}_2\right)-h\left(\overrightarrow{\boldsymbol{Y}}_2,\overleftarrow{\boldsymbol{Y}}_2|W_2\right)+N\delta(N)\\
&=h\left(\overrightarrow{\boldsymbol{Y}}_1\right)+h\left(\overleftarrow{\boldsymbol{Y}}_1|\overrightarrow{\boldsymbol{Y}}_1\right)-h\left(\overleftarrow{\boldsymbol{Y}}_1|W_1\right)-h\left(\overrightarrow{\boldsymbol{Y}}_1|W_1,\overleftarrow{\boldsymbol{Y}}_1\right)+h\left(\overrightarrow{\boldsymbol{Y}}_2\right)+h\left(\overleftarrow{\boldsymbol{Y}}_2|\overrightarrow{\boldsymbol{Y}}_2\right)\\
&\quad -h\left(\overleftarrow{\boldsymbol{Y}}_2|W_2\right)-h\left(\overrightarrow{\boldsymbol{Y}}_2|W_2,\overleftarrow{\boldsymbol{Y}}_2\right)+N\delta(N)\\
&\overset{(b)}{=}h\left(\overrightarrow{\boldsymbol{Y}}_1\right)+h\left(\overleftarrow{\boldsymbol{Z}}_1|\overrightarrow{\boldsymbol{Y}}_1\right)-h\left(\overleftarrow{\boldsymbol{Y}}_1|W_1\right)-h\left(\overrightarrow{\boldsymbol{Y}}_1|W_1,\overleftarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_1\right)+h\left(\overrightarrow{\boldsymbol{Y}}_2\right)+h\left(\overleftarrow{\boldsymbol{Z}}_2|\overrightarrow{\boldsymbol{Y}}_2\right)\\
&\quad -h\left(\overleftarrow{\boldsymbol{Y}}_2|W_2\right)-h\left(\overrightarrow{\boldsymbol{Y}}_2|W_2,\overleftarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_2\right)+N\delta(N)
\end{aligned}
$$

$$
\begin{aligned}
&= h\left(\overrightarrow{\boldsymbol{Y}}_1\right) - h\left(\overleftarrow{\boldsymbol{Y}}_1|W_1\right) - h\left(\overrightarrow{\boldsymbol{Y}}_1|W_1, \overleftarrow{\boldsymbol{Y}}_1, \boldsymbol{X}_1\right) + h\left(\overrightarrow{\boldsymbol{Y}}_2\right) - h\left(\overleftarrow{\boldsymbol{Y}}_2|W_2\right) - h\left(\overrightarrow{\boldsymbol{Y}}_2|W_2, \overleftarrow{\boldsymbol{Y}}_2, \boldsymbol{X}_2\right) \\
&\quad + h\left(\overleftarrow{\boldsymbol{Z}}_1\right) + h\left(\overleftarrow{\boldsymbol{Z}}_2\right) + N\delta(N) \\
&\overset{(c)}{=} h\left(\overrightarrow{\boldsymbol{Y}}_1\right) - h\left(\overleftarrow{\boldsymbol{Y}}_1|W_1\right) - h\left(\boldsymbol{X}_{2,C}|W_1, \overleftarrow{\boldsymbol{Y}}_1, \boldsymbol{X}_1\right) + h\left(\overrightarrow{\boldsymbol{Y}}_2\right) - h\left(\overleftarrow{\boldsymbol{Y}}_2|W_2\right) \\
&\quad - h\left(\boldsymbol{X}_{1,C}|W_2, \overleftarrow{\boldsymbol{Y}}_2, \boldsymbol{X}_2\right) + N\log(2\pi e) + N\delta(N) \\
&= h\left(\overrightarrow{\boldsymbol{Y}}_1\right) - h\left(\overleftarrow{\boldsymbol{Y}}_1|W_1\right) - h\left(\boldsymbol{X}_{2,C}, \overrightarrow{\boldsymbol{Z}}_2|W_1, \overleftarrow{\boldsymbol{Y}}_1, \boldsymbol{X}_1\right) + h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1, \overleftarrow{\boldsymbol{Y}}_1, \boldsymbol{X}_1, \boldsymbol{X}_{2,C}\right) + h\left(\overrightarrow{\boldsymbol{Y}}_2\right) \\
&\quad - h\left(\overleftarrow{\boldsymbol{Y}}_2|W_2\right) - h\left(\boldsymbol{X}_{1,C}, \overrightarrow{\boldsymbol{Z}}_1|W_2, \overleftarrow{\boldsymbol{Y}}_2, \boldsymbol{X}_2\right) + h\left(\overrightarrow{\boldsymbol{Z}}_1|W_2, \overleftarrow{\boldsymbol{Y}}_2, \boldsymbol{X}_2, \boldsymbol{X}_{1,C}\right) + N\log(2\pi e) \\
&\quad + N\delta(N) \\
&\overset{(d)}{=} h\left(\overrightarrow{\boldsymbol{Y}}_1\right) - h\left(\overleftarrow{\boldsymbol{Y}}_1|W_1\right) - h\left(\boldsymbol{X}_{2,C}, \boldsymbol{X}_{1,U}|W_1, \overleftarrow{\boldsymbol{Y}}_1, \boldsymbol{X}_1\right) + h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1, \overleftarrow{\boldsymbol{Y}}_1, \boldsymbol{X}_1, \boldsymbol{X}_{2,C}\right) + h\left(\overrightarrow{\boldsymbol{Y}}_2\right) \\
&\quad - h\left(\overleftarrow{\boldsymbol{Y}}_2|W_2\right) - h\left(\boldsymbol{X}_{1,C}, \boldsymbol{X}_{2,U}|W_2, \overleftarrow{\boldsymbol{Y}}_2, \boldsymbol{X}_2\right) + h\left(\overrightarrow{\boldsymbol{Z}}_1|W_2, \overleftarrow{\boldsymbol{Y}}_2, \boldsymbol{X}_2, \boldsymbol{X}_{1,C}\right) + N\log(2\pi e) \\
&\quad + N\delta(N) \\
&= h\left(\overrightarrow{\boldsymbol{Y}}_1\right) - h\left(\overleftarrow{\boldsymbol{Y}}_1|W_1\right) - h\left(\boldsymbol{X}_{2,C}, \boldsymbol{X}_{1,U}|W_1, \overleftarrow{\boldsymbol{Y}}_1\right) + h\left(\overrightarrow{\boldsymbol{Y}}_2\right) - h\left(\overleftarrow{\boldsymbol{Y}}_2|W_2\right) \\
&\quad - h\left(\boldsymbol{X}_{1,C}, \boldsymbol{X}_{2,U}|W_2, \overleftarrow{\boldsymbol{Y}}_2\right) + h\left(\overrightarrow{\boldsymbol{Z}}_1|W_2, \overleftarrow{\boldsymbol{Y}}_2, \boldsymbol{X}_2, \boldsymbol{X}_{1,C}\right) + h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1, \overleftarrow{\boldsymbol{Y}}_1, \boldsymbol{X}_1, \boldsymbol{X}_{2,C}\right) \\
&\quad + N\log(2\pi e) + N\delta(N) \\
&= h\left(\overrightarrow{\boldsymbol{Y}}_1\right) - h\left(\overleftarrow{\boldsymbol{Y}}_1|W_1\right) + \left[I\left(\boldsymbol{X}_{2,C}, \boldsymbol{X}_{1,U}; W_1, \overleftarrow{\boldsymbol{Y}}_1\right) - h\left(\boldsymbol{X}_{2,C}, \boldsymbol{X}_{1,U}\right)\right] + h\left(\overrightarrow{\boldsymbol{Y}}_2\right) - h\left(\overleftarrow{\boldsymbol{Y}}_2|W_2\right) \\
&\quad + \left[I\left(\boldsymbol{X}_{1,C}, \boldsymbol{X}_{2,U}; W_2, \overleftarrow{\boldsymbol{Y}}_2\right) - h\left(\boldsymbol{X}_{1,C}, \boldsymbol{X}_{2,U}\right)\right] + h\left(\overrightarrow{\boldsymbol{Z}}_1|W_2, \overleftarrow{\boldsymbol{Y}}_2, \boldsymbol{X}_2, \boldsymbol{X}_{1,C}\right) \\
&\quad + h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1, \overleftarrow{\boldsymbol{Y}}_1, \boldsymbol{X}_1, \boldsymbol{X}_{2,C}\right) + N\log(2\pi e) + N\delta(N) \\
&\leqslant h\left(\overrightarrow{\boldsymbol{Y}}_1\right) - h\left(\overleftarrow{\boldsymbol{Y}}_1|W_1\right) + \left[I\left(\boldsymbol{X}_{2,C}, \boldsymbol{X}_{1,U}; W_1, \overleftarrow{\boldsymbol{Y}}_1\right) - h\left(\boldsymbol{X}_{2,C}, \boldsymbol{X}_{1,U}\right)\right] + h\left(\overrightarrow{\boldsymbol{Y}}_2\right) - h\left(\overleftarrow{\boldsymbol{Y}}_2|W_2\right) \\
&\quad + \left[I\left(\boldsymbol{X}_{1,C}, \boldsymbol{X}_{2,U}; W_2, \overleftarrow{\boldsymbol{Y}}_2\right) - h\left(\boldsymbol{X}_{1,C}, \boldsymbol{X}_{2,U}\right)\right] \\
&\quad + \left[h\left(\boldsymbol{X}_{2,C}, \boldsymbol{X}_{1,U}|\overrightarrow{\boldsymbol{Y}}_2\right) - h\left(\boldsymbol{X}_{2,C}, \boldsymbol{X}_{1,U}|\overrightarrow{\boldsymbol{Y}}_2, \boldsymbol{X}_1, \boldsymbol{X}_2\right)\right] \\
&\quad + \left[h\left(\boldsymbol{X}_{1,C}, \boldsymbol{X}_{2,U}|\overrightarrow{\boldsymbol{Y}}_1\right) - h\left(\boldsymbol{X}_{1,C}, \boldsymbol{X}_{2,U}|\overrightarrow{\boldsymbol{Y}}_1, \boldsymbol{X}_2, \boldsymbol{X}_1\right)\right] + h\left(\overrightarrow{\boldsymbol{Z}}_1|W_2, \overleftarrow{\boldsymbol{Y}}_2, \boldsymbol{X}_2, \boldsymbol{X}_{1,C}\right) \\
&\quad + h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1, \overleftarrow{\boldsymbol{Y}}_1, \boldsymbol{X}_1, \boldsymbol{X}_{2,C}\right) + N\log(2\pi e) + N\delta(N) \\
&\overset{(e)}{=} h\left(\overrightarrow{\boldsymbol{Y}}_1|\boldsymbol{X}_{1,C}, \boldsymbol{X}_{2,U}\right) - h\left(\overleftarrow{\boldsymbol{Y}}_1|W_1\right) + I\left(\boldsymbol{X}_{2,C}, \boldsymbol{X}_{1,U}; W_1, \overleftarrow{\boldsymbol{Y}}_1\right) + h\left(\overrightarrow{\boldsymbol{Y}}_2|\boldsymbol{X}_{2,C}, \boldsymbol{X}_{1,U}\right) \\
&\quad - h\left(\overleftarrow{\boldsymbol{Y}}_2|W_2\right) + I\left(\boldsymbol{X}_{1,C}, \boldsymbol{X}_{2,U}; W_2, \overleftarrow{\boldsymbol{Y}}_2\right) - h\left(\boldsymbol{X}_{2,C}, \boldsymbol{X}_{1,U}|\overrightarrow{\boldsymbol{Y}}_2, \boldsymbol{X}_1, \boldsymbol{X}_2\right) \\
&\quad - h\left(\boldsymbol{X}_{1,C}, \boldsymbol{X}_{2,U}|\overrightarrow{\boldsymbol{Y}}_1, \boldsymbol{X}_2, \boldsymbol{X}_1\right) + h\left(\overrightarrow{\boldsymbol{Z}}_1|W_2, \overleftarrow{\boldsymbol{Y}}_2, \boldsymbol{X}_2, \boldsymbol{X}_{1,C}\right) + h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1, \overleftarrow{\boldsymbol{Y}}_1, \boldsymbol{X}_1, \boldsymbol{X}_{2,C}\right) \\
&\quad + N\log(2\pi e) + N\delta(N) \\
&\overset{(f)}{=} h\left(\overrightarrow{\boldsymbol{Y}}_1|\boldsymbol{X}_{1,C}, \boldsymbol{X}_{2,U}\right) - h\left(\overleftarrow{\boldsymbol{Y}}_1|W_1\right) + I\left(\boldsymbol{X}_{2,C}, \boldsymbol{X}_{1,U}; W_1, \overleftarrow{\boldsymbol{Y}}_1\right) + h\left(\overrightarrow{\boldsymbol{Y}}_2|\boldsymbol{X}_{2,C}, \boldsymbol{X}_{1,U}\right) \\
&\quad - h\left(\overleftarrow{\boldsymbol{Y}}_2|W_2\right) + I\left(\boldsymbol{X}_{1,C}, \boldsymbol{X}_{2,U}; W_2, \overleftarrow{\boldsymbol{Y}}_2\right) - h\left(\overrightarrow{\boldsymbol{Z}}_1, \overrightarrow{\boldsymbol{Z}}_2|\overrightarrow{\boldsymbol{Y}}_2, \boldsymbol{X}_1, \boldsymbol{X}_2\right) \\
&\quad - h\left(\overrightarrow{\boldsymbol{Z}}_2, \overrightarrow{\boldsymbol{Z}}_1|\overrightarrow{\boldsymbol{Y}}_1, \boldsymbol{X}_2, \boldsymbol{X}_1\right) + h\left(\overrightarrow{\boldsymbol{Z}}_1|W_2, \overleftarrow{\boldsymbol{Y}}_2, \boldsymbol{X}_2, \boldsymbol{X}_{1,C}\right) + h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1, \overleftarrow{\boldsymbol{Y}}_1, \boldsymbol{X}_1, \boldsymbol{X}_{2,C}\right) \\
&\quad + N\log(2\pi e) + N\delta(N) \\
&= h\left(\overrightarrow{\boldsymbol{Y}}_1|\boldsymbol{X}_{1,C}, \boldsymbol{X}_{2,U}\right) - h\left(\overleftarrow{\boldsymbol{Y}}_1|W_1\right) + I\left(\boldsymbol{X}_{2,C}, \boldsymbol{X}_{1,U}; W_1, \overleftarrow{\boldsymbol{Y}}_1\right) + h\left(\overrightarrow{\boldsymbol{Y}}_2|\boldsymbol{X}_{2,C}, \boldsymbol{X}_{1,U}\right) \\
&\quad - h\left(\overleftarrow{\boldsymbol{Y}}_2|W_2\right) + I\left(\boldsymbol{X}_{1,C}, \boldsymbol{X}_{2,U}; W_2, \overleftarrow{\boldsymbol{Y}}_2\right) - h\left(\overrightarrow{\boldsymbol{Z}}_1|\overrightarrow{\boldsymbol{Y}}_2, \boldsymbol{X}_1, \boldsymbol{X}_2\right) - h\left(\overrightarrow{\boldsymbol{Z}}_2|\overrightarrow{\boldsymbol{Y}}_1, \boldsymbol{X}_2, \boldsymbol{X}_1\right) \\
&\quad + h\left(\overrightarrow{\boldsymbol{Z}}_1|W_2, \overleftarrow{\boldsymbol{Y}}_2, \boldsymbol{X}_2, \boldsymbol{X}_{1,C}\right) + h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1, \overleftarrow{\boldsymbol{Y}}_1, \boldsymbol{X}_1, \boldsymbol{X}_{2,C}\right) + N\log(2\pi e) + N\delta(N)
\end{aligned}
$$

$$
\begin{aligned}
&= h\left(\overrightarrow{\boldsymbol{Y}}_1|\boldsymbol{X}_{1,C},\boldsymbol{X}_{2,U}\right) - h\left(\overleftarrow{\boldsymbol{Y}}_1|W_1\right) + I\left(\boldsymbol{X}_{2,C},\boldsymbol{X}_{1,U};W_1,\overleftarrow{\boldsymbol{Y}}_1\right) + h\left(\overrightarrow{\boldsymbol{Y}}_2|\boldsymbol{X}_{2,C},\boldsymbol{X}_{1,U}\right)\\
&\quad -h\left(\overleftarrow{\boldsymbol{Y}}_2|W_2\right) + I\left(\boldsymbol{X}_{1,C},\boldsymbol{X}_{2,U};W_2,\overleftarrow{\boldsymbol{Y}}_2\right) + h\left(\overrightarrow{\boldsymbol{Z}}_1|W_2,\overleftarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_2,\boldsymbol{X}_{1,C}\right)\\
&\quad +h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1,\overleftarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_1,\boldsymbol{X}_{2,C}\right) - h\left(\overrightarrow{\boldsymbol{Z}}_1\right) - h\left(\overrightarrow{\boldsymbol{Z}}_2\right) + N\log\left(2\pi e\right) + N\delta(N)\\
&\overset{(g)}{\leqslant} h\left(\overrightarrow{\boldsymbol{Y}}_1|\boldsymbol{X}_{1,C},\boldsymbol{X}_{2,U}\right) - h\left(\overleftarrow{\boldsymbol{Y}}_1|W_1\right) + I\left(\boldsymbol{X}_{2,C},\boldsymbol{X}_{1,U},W_2,\overleftarrow{\boldsymbol{Y}}_2;W_1,\overleftarrow{\boldsymbol{Y}}_1\right)\\
&\quad +h\left(\overrightarrow{\boldsymbol{Y}}_2|\boldsymbol{X}_{2,C},\boldsymbol{X}_{1,U}\right) - h\left(\overleftarrow{\boldsymbol{Y}}_2|W_2\right) + I\left(\boldsymbol{X}_{1,C},\boldsymbol{X}_{2,U},W_1,\overleftarrow{\boldsymbol{Y}}_1;W_2,\overleftarrow{\boldsymbol{Y}}_2\right)\\
&\quad +h\left(\overrightarrow{\boldsymbol{Z}}_1|W_2,\overleftarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_2,\boldsymbol{X}_{1,C}\right) + h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1,\overleftarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_1,\boldsymbol{X}_{2,C}\right) + N\delta(N)\\
&\overset{(h)}{\leqslant} h\left(\overrightarrow{\boldsymbol{Y}}_1|\boldsymbol{X}_{1,C},\boldsymbol{X}_{2,U}\right) - h\left(\overleftarrow{\boldsymbol{Y}}_1|W_1\right) + h\left(\overleftarrow{\boldsymbol{Y}}_1|W_1\right) + \sum_{n=1}^{N}\Big[h\left(X_{1,U,n}|X_{2,C,n}\right)\\
&\quad +h\left(\overleftarrow{\boldsymbol{Y}}_2^{(n)}|X_{2,n},X_{1,U,n}\right) - \frac{3}{2}\log\left(2\pi e\right)\Big] + h\left(\overrightarrow{\boldsymbol{Y}}_2|\boldsymbol{X}_{2,C},\boldsymbol{X}_{1,U}\right) - h\left(\overleftarrow{\boldsymbol{Y}}_2|W_2\right)\\
&\quad +h\left(\overleftarrow{\boldsymbol{Y}}_2|W_2\right) + \sum_{n=1}^{N}\Big[h\left(X_{2,U,n}|X_{1,C,n}\right) + h\left(\overleftarrow{Y}_{1,n}|X_{1,n},X_{2,U,n}\right) - \frac{3}{2}\log\left(2\pi e\right)\Big]\\
&\quad +h\left(\overrightarrow{\boldsymbol{Z}}_1|W_2,\overleftarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_2,\boldsymbol{X}_{1,C}\right) + h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1,\overleftarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_1,\boldsymbol{X}_{2,C}\right) + N\delta(N)\\
&= h\left(\overrightarrow{\boldsymbol{Y}}_1|\boldsymbol{X}_{1,C},\boldsymbol{X}_{2,U}\right) + \sum_{n=1}^{N}\Big[h\left(X_{1,U,n}|X_{2,C,n}\right) + h\left(\overleftarrow{Y}_{2,n}|X_{2,n},X_{1,U,n}\right) - \frac{3}{2}\log\left(2\pi e\right)\Big]\\
&\quad +h\left(\overrightarrow{\boldsymbol{Y}}_2|\boldsymbol{X}_{2,C},\boldsymbol{X}_{1,U}\right) + \sum_{n=1}^{N}\Big[h\left(X_{2,U,n}|X_{1,C,n}\right) + h\left(\overleftarrow{\boldsymbol{Y}}_1^{(n)}|X_{1,n},X_{2,U,n}\right) - \frac{3}{2}\log\left(2\pi e\right)\Big]\\
&\quad +h\left(\overrightarrow{\boldsymbol{Z}}_1|W_2,\overleftarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_2,\boldsymbol{X}_{1,C}\right) + h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1,\overleftarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_1,\boldsymbol{X}_{2,C}\right) + N\delta(N)\\
&= \sum_{n=1}^{N}\Big[h\left(\overrightarrow{Y}_{1,n}|\boldsymbol{X}_{1,C},\boldsymbol{X}_{2,U},\overrightarrow{\boldsymbol{Y}}_{1,(1:n-1)}\right) + h\left(X_{1,U,n}|X_{2,C,n}\right) + h\left(\overleftarrow{Y}_{2,n}|X_{2,n},X_{1,U,n}\right)\\
&\quad +h\left(\overrightarrow{Y}_{2,n}|\boldsymbol{X}_{2,C},\boldsymbol{X}_{1,U},\overrightarrow{\boldsymbol{Y}}_{2,(1:n-1)}\right) + h\left(X_{2,U,n}|X_{1,C,n}\right) + h\left(\overleftarrow{Y}_{1,n}|X_{1,n},X_{2,U,n}\right)\\
&\quad -3\log\left(2\pi e\right)\Big] + h\left(\overrightarrow{\boldsymbol{Z}}_1|W_2,\overleftarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_2,\boldsymbol{X}_{1,C}\right) + h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1,\overleftarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_1,\boldsymbol{X}_{2,C}\right) + N\delta(N)\\
&= \sum_{n=1}^{N}\Big[h\left(\overrightarrow{Y}_{1,n}|\boldsymbol{X}_{1,C},\boldsymbol{X}_{2,U},\overrightarrow{\boldsymbol{Y}}_{1,(1:n-1)}\right) + h\left(X_{1,U,n}|X_{2,C,n}\right) + h\left(\overleftarrow{Y}_{2,n}|X_{2,n},X_{1,U,n}\right)\\
&\quad +h\left(\overrightarrow{Y}_{2,n}|\boldsymbol{X}_{2,C},\boldsymbol{X}_{1,U},\overrightarrow{\boldsymbol{Y}}_{2,(1:n-1)}\right) + h\left(X_{2,U,n}|X_{1,C,n}\right) + h\left(\overleftarrow{Y}_{1,n}|X_{1,n},X_{2,U,n}\right)\\
&\quad -3\log\left(2\pi e\right)\Big] + h\left(\overrightarrow{\boldsymbol{Z}}_1|W_2,\overleftarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_2,\boldsymbol{X}_{1,C}\right) + h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1,\overleftarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_1,\boldsymbol{X}_{2,C}\right) + N\delta(N)\\
&= \sum_{n=1}^{N}\Big[h\left(\overrightarrow{Y}_{1,n}|\boldsymbol{X}_{1,C},\boldsymbol{X}_{2,U},\overrightarrow{\boldsymbol{Y}}_{1,(1:n-1)}\right) + h\left(X_{1,U,n}|X_{2,C,n}\right) + h\left(\overleftarrow{Y}_{2,n}|X_{2,n},X_{1,U,n}\right)\\
&\quad +h\left(\overrightarrow{Y}_{2,n}|\boldsymbol{X}_{2,C},\boldsymbol{X}_{1,U},\overrightarrow{\boldsymbol{Y}}_{2,(1:n-1)}\right) + h\left(X_{2,U,n}|X_{1,C,n}\right) + h\left(\overleftarrow{Y}_{1,n}|X_{1,n},X_{2,U,n}\right)\\
&\quad -3\log\left(2\pi e\right)\Big] + h\left(\overrightarrow{\boldsymbol{Z}}_1|W_2,\overleftarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_2,\boldsymbol{X}_{1,C}\right) + h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1,\overleftarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_1,\boldsymbol{X}_{2,C}\right) + N\delta(N)\\
&\overset{(i)}{\leqslant} \sum_{n=1}^{N}\Big[h\left(\overrightarrow{Y}_{1,n}|X_{1,C,n},X_{2,U,n}\right) + h\left(X_{1,U,n}|X_{2,C,n}\right) + h\left(\overleftarrow{Y}_{2,n}|X_{2,n},X_{1,U,n}\right)\\
&\quad +h\left(\overrightarrow{Y}_{2,n}|X_{2,C,n},X_{1,U,n}\right) + h\left(X_{2,U,n}|X_{1,C,n}\right) + h\left(\overleftarrow{Y}_{1,n}|X_{1,n},X_{2,U,n}\right) - 3\log\left(2\pi e\right)\Big]\\
&\quad +h\left(\overrightarrow{\boldsymbol{Z}}_1|W_2,\overleftarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_2,\boldsymbol{X}_{1,C}\right) + h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1,\overleftarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_1,\boldsymbol{X}_{2,C}\right) + N\delta(N)
\end{aligned}
$$

$$
\begin{aligned}
&= N\Big[h\left(\overrightarrow{Y}_1|X_{1,C},X_{2,U}\right)+h\left(X_{1,U}|X_{2,C}\right)+h\left(\overleftarrow{Y}_2|X_2,X_{1,U}\right)+h\left(\overrightarrow{Y}_2|X_{2,C},X_{1,U}\right)\\
&\quad +h\left(X_{2,U}|X_{1,C}\right)+h\left(\overleftarrow{Y}_1|X_1,X_{2,U}\right)-3\log\left(2\pi e\right)\Big]+h\left(\overrightarrow{\boldsymbol{Z}}_1|W_2,\overleftarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_2,\boldsymbol{X}_{1,C}\right)\\
&\quad +h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1,\overleftarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_1,\boldsymbol{X}_{2,C}\right)+N\delta(N)\\
&\overset{(c)}{=} N\Big[h\left(\overrightarrow{Y}_1|X_{1,C},X_{2,U}\right)+h\left(X_{1,U}|X_{2,C}\right)+h\left(\overleftarrow{Y}_2|X_2,X_{1,U}\right)+h\left(\overrightarrow{Y}_2|X_{2,C},X_{1,U}\right)\\
&\quad +h\left(X_{2,U}|X_{1,C}\right)+h\left(\overleftarrow{Y}_1|X_1,X_{2,U}\right)-3\log\left(2\pi e\right)\Big]+h\left(\overrightarrow{\boldsymbol{Z}}_1|W_2,\overleftarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_2,\boldsymbol{X}_{1,C},\overrightarrow{\boldsymbol{Y}}_2\right)\\
&\quad +h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1,\overleftarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_1,\boldsymbol{X}_{2,C},\overrightarrow{\boldsymbol{Y}}_1\right)+N\delta(N)\\
&\overset{(j)}{=} N\Big[h\left(\overrightarrow{Y}_1|X_{1,C},X_{2,U}\right)+h\left(X_{1,U}|X_{2,C}\right)+h\left(\overleftarrow{Y}_2|X_2,X_{1,U}\right)+h\left(\overrightarrow{Y}_2|X_{2,C},X_{1,U}\right)\\
&\quad +h\left(X_{2,U}|X_{1,C}\right)+h\left(\overleftarrow{Y}_1|X_1,X_{2,U}\right)-3\log\left(2\pi e\right)\Big]+h\left(\overrightarrow{\boldsymbol{Z}}_1|W_2,\overleftarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_2,\boldsymbol{X}_{1,C},\overrightarrow{\boldsymbol{Y}}_2,\overleftarrow{\boldsymbol{Z}}_2\right)\\
&\quad +h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1,\overleftarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_1,\boldsymbol{X}_{2,C},\overrightarrow{\boldsymbol{Y}}_1,\overleftarrow{\boldsymbol{Z}}_1\right)+N\delta(N)\\
&\overset{(k)}{=} N\Big[h\left(\overrightarrow{Y}_1|X_{1,C},X_{2,U}\right)+h\left(X_{1,U}|X_{2,C}\right)+h\left(\overleftarrow{Y}_2|X_2,X_{1,U}\right)+h\left(\overrightarrow{Y}_2|X_{2,C},X_{1,U}\right)\\
&\quad +h\left(X_{2,U}|X_{1,C}\right)+h\left(\overleftarrow{Y}_1|X_1,X_{2,U}\right)-3\log\left(2\pi e\right)\Big]\\
&\quad +h\left(\overrightarrow{\boldsymbol{Z}}_1|W_1,W_2,\overleftarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_2,\boldsymbol{X}_{1,C},\overrightarrow{\boldsymbol{Y}}_2,\overleftarrow{\boldsymbol{Z}}_2,\overrightarrow{\boldsymbol{Z}}_2,\overleftarrow{\boldsymbol{Z}}_1\right)\\
&\quad +h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1,W_2,\overleftarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_1,\boldsymbol{X}_{2,C},\overrightarrow{\boldsymbol{Y}}_1,\overleftarrow{\boldsymbol{Z}}_1,\overrightarrow{\boldsymbol{Z}}_1,\overleftarrow{\boldsymbol{Z}}_2\right)+N\delta(N)\\
&\overset{(l)}{=} N\Big[h\left(\overrightarrow{Y}_1|X_{1,C},X_{2,U}\right)+h\left(X_{1,U}|X_{2,C}\right)+h\left(\overleftarrow{Y}_2|X_2,X_{1,U}\right)+h\left(\overrightarrow{Y}_2|X_{2,C},X_{1,U}\right)\\
&\quad +h\left(X_{2,U}|X_{1,C}\right)+h\left(\overleftarrow{Y}_1|X_1,X_{2,U}\right)-3\log\left(2\pi e\right)\Big]\\
&\quad +h\left(\overrightarrow{\boldsymbol{Z}}_1|W_1,W_2,\overleftarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_2,\boldsymbol{X}_1,\overrightarrow{\boldsymbol{Y}}_2,\overleftarrow{\boldsymbol{Z}}_2,\overrightarrow{\boldsymbol{Z}}_2,\overleftarrow{\boldsymbol{Z}}_1\right)\\
&\quad +h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1,W_2,\overleftarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_1,\boldsymbol{X}_2,\overrightarrow{\boldsymbol{Y}}_1,\overleftarrow{\boldsymbol{Z}}_1,\overrightarrow{\boldsymbol{Z}}_1,\overleftarrow{\boldsymbol{Z}}_2\right)+N\delta(N)\\
&\overset{(m)}{=} N\Big[h\left(\overrightarrow{Y}_1|X_{1,C},X_{2,U}\right)+h\left(X_{1,U}|X_{2,C}\right)+h\left(\overleftarrow{Y}_2|X_2,X_{1,U}\right)+h\left(\overrightarrow{Y}_2|X_{2,C},X_{1,U}\right)\\
&\quad +h\left(X_{2,U}|X_{1,C}\right)+h\left(\overleftarrow{Y}_1|X_1,X_{2,U}\right)-3\log\left(2\pi e\right)\Big]\\
&\quad +h\left(\overrightarrow{\boldsymbol{Z}}_1|W_1,W_2,\overleftarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_2,\boldsymbol{X}_1,\overrightarrow{\boldsymbol{Y}}_2,\overleftarrow{\boldsymbol{Z}}_2,\overrightarrow{\boldsymbol{Z}}_2,\overleftarrow{\boldsymbol{Z}}_1,\overleftarrow{\boldsymbol{Y}}_{1,(1:N-1)}\right)\\
&\quad +h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1,W_2,\overleftarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_1,\boldsymbol{X}_2,\overrightarrow{\boldsymbol{Y}}_1,\overleftarrow{\boldsymbol{Z}}_1,\overrightarrow{\boldsymbol{Z}}_1,\overleftarrow{\boldsymbol{Z}}_2,\overleftarrow{\boldsymbol{Y}}_{2,(1:N-1)}\right)+N\delta(N)\\
&\overset{(j)}{=} N\Big[h\left(\overrightarrow{Y}_1|X_{1,C},X_{2,U}\right)+h\left(X_{1,U}|X_{2,C}\right)+h\left(\overleftarrow{Y}_2|X_2,X_{1,U}\right)+h\left(\overrightarrow{Y}_2|X_{2,C},X_{1,U}\right)\\
&\quad +h\left(X_{2,U}|X_{1,C}\right)+h\left(\overleftarrow{Y}_1|X_1,X_{2,U}\right)-3\log\left(2\pi e\right)\Big]\\
&\quad +h\left(\overrightarrow{\boldsymbol{Z}}_1|W_1,W_2,\overleftarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_2,\boldsymbol{X}_1,\overrightarrow{\boldsymbol{Y}}_2,\overleftarrow{\boldsymbol{Z}}_2,\overrightarrow{\boldsymbol{Z}}_2,\overleftarrow{\boldsymbol{Z}}_1,\overleftarrow{\boldsymbol{Y}}_{1,(1:N-1)},\overrightarrow{\boldsymbol{Y}}_{1,(1:N-2)}\right)\\
&\quad +h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1,W_2,\overleftarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_1,\boldsymbol{X}_2,\overrightarrow{\boldsymbol{Y}}_1,\overleftarrow{\boldsymbol{Z}}_1,\overrightarrow{\boldsymbol{Z}}_1,\overleftarrow{\boldsymbol{Z}}_2,\overleftarrow{\boldsymbol{Y}}_{2,(1:N-1)},\overrightarrow{\boldsymbol{Y}}_{2,(1:N-2)}\right)+N\delta(N)\\
&\overset{(n)}{=} N\Big[h\left(\overrightarrow{Y}_1|X_{1,C},X_{2,U}\right)+h\left(X_{1,U}|X_{2,C}\right)+h\left(\overleftarrow{Y}_2|X_2,X_{1,U}\right)+h\left(\overrightarrow{Y}_2|X_{2,C},X_{1,U}\right)\\
&\quad +h\left(X_{2,U}|X_{1,C}\right)+h\left(\overleftarrow{Y}_1|X_1,X_{2,U}\right)-3\log\left(2\pi e\right)\Big]\\
&\quad +h\left(\overrightarrow{\boldsymbol{Z}}_1|W_1,W_2,\overleftarrow{\boldsymbol{Y}}_2,\boldsymbol{X}_2,\boldsymbol{X}_1,\overrightarrow{\boldsymbol{Y}}_2,\overleftarrow{\boldsymbol{Z}}_2,\overrightarrow{\boldsymbol{Z}}_2,\overleftarrow{\boldsymbol{Z}}_1,\overleftarrow{\boldsymbol{Y}}_{1,(1:N-1)},\overrightarrow{\boldsymbol{Y}}_{1,(1:N-2)},\overrightarrow{\boldsymbol{Z}}_{1,(1:N-2)}\right)\\
&\quad +h\left(\overrightarrow{\boldsymbol{Z}}_2|W_1,W_2,\overleftarrow{\boldsymbol{Y}}_1,\boldsymbol{X}_1,\boldsymbol{X}_2,\overrightarrow{\boldsymbol{Y}}_1,\overleftarrow{\boldsymbol{Z}}_1,\overrightarrow{\boldsymbol{Z}}_1,\overleftarrow{\boldsymbol{Z}}_2,\overleftarrow{\boldsymbol{Y}}_{2,(1:N-1)},\overrightarrow{\boldsymbol{Y}}_{2,(1:N-2)},\overrightarrow{\boldsymbol{Z}}_{2,(1:N-2)}\right)+N\delta(N)
\end{aligned}
$$

$$
\begin{aligned}
=&N\Big[h\left(\overrightarrow{Y}_1|X_{1,C},X_{2,U}\right)+h\left(X_{1,U}|X_{2,C}\right)+h\left(\overleftarrow{Y}_2|X_2,X_{1,U}\right)+h\left(\overrightarrow{Y}_2|X_{2,C},X_{1,U}\right)\\
&+h\left(X_{2,U}|X_{1,C}\right)+h\left(\overleftarrow{Y}_1|X_1,X_{2,U}\right)-3\log\left(2\pi e\right)\Big]+h\left(\overrightarrow{Z}_{1,N}\right)+h\left(\overrightarrow{Z}_{1,N-1}\right)+h\left(\overrightarrow{Z}_{2,N}\right)\\
&+h\left(\overrightarrow{Z}_{2,N-1}\right)+N\delta(N),
\end{aligned}
\tag{45}
$$

where
(a) follows from Fano's inequality;
(b) follows from (3) and (4);
(c) follows from (2) and (35);
(d) follows from (36);
(e) follows from the fact that $h\left(\overrightarrow{\boldsymbol{Y}}_i\right)-h\left(\boldsymbol{X}_{i,C},\boldsymbol{X}_{j,U}\right)+h\left(\boldsymbol{X}_{i,C},\boldsymbol{X}_{j,U}|\overrightarrow{\boldsymbol{Y}}_i\right)=h\left(\overrightarrow{\boldsymbol{Y}}_i|\boldsymbol{X}_{i,C},\boldsymbol{X}_{j,U}\right)$;
(f) follows from (35) and (36);
(g) follows from the fact that injecting information increases the mutual information;
(h) follows from the Lemma 3;
(i) follows from the fact that conditioning reduces the entropy;
(j) follows from (3);
(k) follows from the fact that W_j, $\overrightarrow{Z}_{i,n}$, $\overleftarrow{Z}_{j,n}$ and $\overrightarrow{Z}_{j,n}$ are mutually independent;
(l) follows from (35);
(m) follows from (4);
(n) follows from (2).

From (45), in the asymptotic regime, it holds that

$$
\begin{aligned}
R_1+R_2\leqslant&h\left(\overrightarrow{Y}_1|X_{1,C},X_{2,U}\right)+h\left(X_{1,U}|X_{2,C}\right)+h\left(\overleftarrow{Y}_2|X_2,X_{1,U}\right)+h\left(\overrightarrow{Y}_2|X_{2,C},X_{1,U}\right)\\
&+h\left(X_{2,U}|X_{1,C}\right)+h\left(\overleftarrow{Y}_1|X_1,X_{2,U}\right)-3\log\left(2\pi e\right).
\end{aligned}
\tag{46}
$$

This completes the proof of (37b).

Part III: Proof of (37c). Given that the message indices W_1 and W_2 are uniformly distributed over the corresponding sets $\mathcal{W}_1$ and $\mathcal{W}_2$, for all $i\in\{1,2\}$ and $j\in\{1,2\}\setminus\{i\}$, the following holds

$$
\begin{aligned}
N\Big(2R_i+R_j\Big)&=2H\left(W_i\right)+H\left(W_j\right)\\
&\overset{(a)}{=}H\left(W_i\right)+H\left(W_i|W_j\right)+H\left(W_j\right)\\
&=I\left(W_i;\overrightarrow{\boldsymbol{Y}}_i,\overleftarrow{\boldsymbol{Y}}_i\right)+h\left(W_i|\overrightarrow{\boldsymbol{Y}}_i,\overleftarrow{\boldsymbol{Y}}_i\right)+I\left(W_i;\overrightarrow{\boldsymbol{Y}}_i,\overleftarrow{\boldsymbol{Y}}_j|W_j\right)+h\left(W_i|\overrightarrow{\boldsymbol{Y}}_i,W_j,\overleftarrow{\boldsymbol{Y}}_j\right)\\
&\quad+I\left(W_j;\overrightarrow{\boldsymbol{Y}}_j,\overleftarrow{\boldsymbol{Y}}_j\right)+h\left(W_j|\overrightarrow{\boldsymbol{Y}}_j,\overleftarrow{\boldsymbol{Y}}_j\right)\\
&\overset{(b)}{\leqslant}I\left(W_i;\overrightarrow{\boldsymbol{Y}}_i,\overleftarrow{\boldsymbol{Y}}_i\right)+I\left(W_i;\overrightarrow{\boldsymbol{Y}}_i,\overleftarrow{\boldsymbol{Y}}_j|W_j\right)+I\left(W_j;\overrightarrow{\boldsymbol{Y}}_j,\overleftarrow{\boldsymbol{Y}}_j\right)+N\delta(N)\\
&=h\left(\overrightarrow{\boldsymbol{Y}}_i,\overleftarrow{\boldsymbol{Y}}_i\right)-h\left(\overrightarrow{\boldsymbol{Y}}_i,\overleftarrow{\boldsymbol{Y}}_i|W_i\right)+I\left(W_i;\overrightarrow{\boldsymbol{Y}}_i,\overleftarrow{\boldsymbol{Y}}_j|W_j\right)+h\left(\overrightarrow{\boldsymbol{Y}}_j,\overleftarrow{\boldsymbol{Y}}_j\right)-h\left(\overrightarrow{\boldsymbol{Y}}_j,\overleftarrow{\boldsymbol{Y}}_j|W_j\right)\\
&\quad+N\delta(N)\\
&=h\left(\overrightarrow{\boldsymbol{Y}}_i\right)+h\left(\overleftarrow{\boldsymbol{Y}}_i|\overrightarrow{\boldsymbol{Y}}_i\right)-h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right)-h\left(\overrightarrow{\boldsymbol{Y}}_i|W_i,\overleftarrow{\boldsymbol{Y}}_i\right)+I\left(W_i;\overrightarrow{\boldsymbol{Y}}_i,\overleftarrow{\boldsymbol{Y}}_j|W_j\right)+h\left(\overrightarrow{\boldsymbol{Y}}_j\right)\\
&\quad-h\left(\overleftarrow{\boldsymbol{Y}}_j|W_j\right)-h\left(\overrightarrow{\boldsymbol{Y}}_j|W_j,\overleftarrow{\boldsymbol{Y}}_j\right)+N\delta(N)
\end{aligned}
$$

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\begin{aligned}
&= h\left(\overrightarrow{\boldsymbol{Y}}_i\right) + h\left(\overleftarrow{\boldsymbol{Z}}_i|\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) - h\left(\overrightarrow{\boldsymbol{Y}}_i|W_i,\overleftarrow{\boldsymbol{Y}}_i\right) + I\left(W_i;\overleftarrow{\boldsymbol{Y}}_j|W_j\right)\\
&\quad + I\left(W_i;\overrightarrow{\boldsymbol{Y}}_i|W_j,\overleftarrow{\boldsymbol{Y}}_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j\right) + h\left(\overleftarrow{\boldsymbol{Z}}_j|\overrightarrow{\boldsymbol{Y}}_j\right) - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_j\right) - h\left(\overrightarrow{\boldsymbol{Y}}_j|W_j,\overleftarrow{\boldsymbol{Y}}_j\right)\\
&\quad + N\delta(N)\\
&= h\left(\overrightarrow{\boldsymbol{Y}}_i\right) + h\left(\overleftarrow{\boldsymbol{Z}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) - h\left(\overrightarrow{\boldsymbol{Y}}_i|W_i,\overleftarrow{\boldsymbol{Y}}_i\right) + h\left(\overleftarrow{\boldsymbol{Y}}_j|W_j\right) - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i,W_j\right)\\
&\quad + I\left(W_i;\overrightarrow{\boldsymbol{Y}}_i|W_j,\overleftarrow{\boldsymbol{Y}}_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j\right) + h\left(\overleftarrow{\boldsymbol{Z}}_j\right) - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_j\right) - h\left(\overrightarrow{\boldsymbol{Y}}_j|W_j,\overleftarrow{\boldsymbol{Y}}_j\right) + N\delta(N)\\
&= h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) - h\left(\overrightarrow{\boldsymbol{Y}}_i|W_i,\overleftarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i,W_j\right) + I\left(W_i;\overrightarrow{\boldsymbol{Y}}_i|W_j,\overleftarrow{\boldsymbol{Y}}_j\right)\\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_j\right) - h\left(\overrightarrow{\boldsymbol{Y}}_j|W_j,\overleftarrow{\boldsymbol{Y}}_j\right) + N\log(2\pi e) + N\delta(N)\\
&\overset{(c)}{=} h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) - h\left(\overrightarrow{\boldsymbol{Y}}_i|W_i,\overleftarrow{\boldsymbol{Y}}_i,\boldsymbol{X}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i,W_j\right) + I\left(W_i;\overrightarrow{\boldsymbol{Y}}_i|W_j,\overleftarrow{\boldsymbol{Y}}_j\right)\\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_j\right) - h\left(\overrightarrow{\boldsymbol{Y}}_j|W_j,\overleftarrow{\boldsymbol{Y}}_j,\boldsymbol{X}_j\right) + N\log(2\pi e) + N\delta(N)\\
&\overset{(d)}{=} h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) - h\left(\boldsymbol{X}_{j,C}|W_i,\overleftarrow{\boldsymbol{Y}}_i,\boldsymbol{X}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i,W_j\right) + I\left(W_i;\overrightarrow{\boldsymbol{Y}}_i|W_j,\overleftarrow{\boldsymbol{Y}}_j\right)\\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_j\right) - h\left(\boldsymbol{X}_{i,C}|W_j,\overleftarrow{\boldsymbol{Y}}_j,\boldsymbol{X}_j\right) + N\log(2\pi e) + N\delta(N)\\
&= h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) - h\left(\boldsymbol{X}_{j,C},\overrightarrow{\boldsymbol{Z}}_j|W_i,\overleftarrow{\boldsymbol{Y}}_i,\boldsymbol{X}_i\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i,\overleftarrow{\boldsymbol{Y}}_i,\boldsymbol{X}_i,\boldsymbol{X}_{j,C}\right)\\
&\quad - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i,W_j\right) + I\left(W_i;\overrightarrow{\boldsymbol{Y}}_i|W_j,\overleftarrow{\boldsymbol{Y}}_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j\right) - h\left(\boldsymbol{X}_{i,C}|W_j,\overleftarrow{\boldsymbol{Y}}_j,\boldsymbol{X}_j\right)\\
&\quad + N\log(2\pi e) + N\delta(N)\\
&\overset{(e)}{=} h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) - h\left(\boldsymbol{X}_{j,C},\boldsymbol{X}_{i,U}|W_i,\overleftarrow{\boldsymbol{Y}}_i,\boldsymbol{X}_i\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i,\overleftarrow{\boldsymbol{Y}}_i,\boldsymbol{X}_i,\boldsymbol{X}_{j,C}\right)\\
&\quad - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i,W_j\right) + I\left(W_i;\overrightarrow{\boldsymbol{Y}}_i|W_j,\overleftarrow{\boldsymbol{Y}}_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j\right) - h\left(\boldsymbol{X}_{i,C}|W_j,\overleftarrow{\boldsymbol{Y}}_j,\boldsymbol{X}_j\right)\\
&\quad + N\log(2\pi e) + N\delta(N)\\
&\overset{(f)}{\leqslant} h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) - h\left(\boldsymbol{X}_{j,C},\boldsymbol{X}_{i,U}|W_i,\overleftarrow{\boldsymbol{Y}}_i\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i,\overleftarrow{\boldsymbol{Y}}_i,\boldsymbol{X}_i,\boldsymbol{X}_{j,C}\right)\\
&\quad - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i,W_j\right) + I\left(W_i;\overrightarrow{\boldsymbol{Y}}_i,\boldsymbol{X}_{i,C}|W_j,\overleftarrow{\boldsymbol{Y}}_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j\right) - h\left(\boldsymbol{X}_{i,C}|W_j,\overleftarrow{\boldsymbol{Y}}_j\right)\\
&\quad + N\log(2\pi e) + N\delta(N)\\
&= h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) - h\left(\boldsymbol{X}_{j,C},\boldsymbol{X}_{i,U}|W_i,\overleftarrow{\boldsymbol{Y}}_i\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i,\overleftarrow{\boldsymbol{Y}}_i,\boldsymbol{X}_i,\boldsymbol{X}_{j,C}\right)\\
&\quad - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i,W_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_i,\boldsymbol{X}_{i,C}|W_j,\overleftarrow{\boldsymbol{Y}}_j\right) - h\left(\overrightarrow{\boldsymbol{Y}}_i,\boldsymbol{X}_{i,C}|W_i,W_j,\overleftarrow{\boldsymbol{Y}}_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j\right)\\
&\quad - h\left(\boldsymbol{X}_{i,C}|W_j,\overleftarrow{\boldsymbol{Y}}_j\right) + N\log(2\pi e) + N\delta(N)\\
&= h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) - h\left(\boldsymbol{X}_{j,C},\boldsymbol{X}_{i,U}|W_i,\overleftarrow{\boldsymbol{Y}}_i\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i,\overleftarrow{\boldsymbol{Y}}_i,\boldsymbol{X}_i,\boldsymbol{X}_{j,C}\right) - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i,W_j\right)\\
&\quad + h\left(\boldsymbol{X}_{i,C}|W_j,\overleftarrow{\boldsymbol{Y}}_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_i|W_j,\overleftarrow{\boldsymbol{Y}}_j,\boldsymbol{X}_{i,C}\right) - h\left(\overrightarrow{\boldsymbol{Y}}_i,\boldsymbol{X}_{i,C}|W_i,W_j,\overleftarrow{\boldsymbol{Y}}_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j\right)\\
&\quad - h\left(\boldsymbol{X}_{i,C}|W_j,\overleftarrow{\boldsymbol{Y}}_j\right) + N\log(2\pi e) + N\delta(N)\\
&= h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) - h\left(\boldsymbol{X}_{j,C},\boldsymbol{X}_{i,U}|W_i,\overleftarrow{\boldsymbol{Y}}_i\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i,\overleftarrow{\boldsymbol{Y}}_i,\boldsymbol{X}_i,\boldsymbol{X}_{j,C}\right) - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i,W_j\right)\\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_i|W_j,\overleftarrow{\boldsymbol{Y}}_j,\boldsymbol{X}_{i,C}\right) - h\left(\overrightarrow{\boldsymbol{Y}}_i,\boldsymbol{X}_{i,C}|W_i,W_j,\overleftarrow{\boldsymbol{Y}}_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j\right) + N\log(2\pi e) + N\delta(N)\\
&\overset{(g)}{\leqslant} h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) - h\left(\boldsymbol{X}_{j,C},\boldsymbol{X}_{i,U}|W_i,\overleftarrow{\boldsymbol{Y}}_i\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i,\overleftarrow{\boldsymbol{Y}}_i,\boldsymbol{X}_i,\boldsymbol{X}_{j,C}\right) - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i,W_j\right)\\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_i|W_j,\overleftarrow{\boldsymbol{Y}}_j,\boldsymbol{X}_{i,C}\right) - h\left(\overrightarrow{\boldsymbol{Y}}_i,\boldsymbol{X}_{i,C}|W_i,W_j,\overleftarrow{\boldsymbol{Y}}_j,\boldsymbol{X}_i,\boldsymbol{X}_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j\right) + N\log(2\pi e) + N\delta(N)\\
&= h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) - h\left(\boldsymbol{X}_{j,C},\boldsymbol{X}_{i,U}|W_i,\overleftarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i,W_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_i|W_j,\overleftarrow{\boldsymbol{Y}}_j,\boldsymbol{X}_{i,C}\right)\\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_j\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i,\overleftarrow{\boldsymbol{Y}}_i,\boldsymbol{X}_i,\boldsymbol{X}_{j,C}\right) - h\left(\overrightarrow{\boldsymbol{Z}}_i,\overrightarrow{\boldsymbol{Z}}_j|W_i,W_j,\overleftarrow{\boldsymbol{Y}}_j,\boldsymbol{X}_i,\boldsymbol{X}_j\right) + N\log(2\pi e)\\
&\quad + N\delta(N)
\end{aligned}
$$

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\begin{aligned}
&= h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) - h\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}|W_i, \overleftarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i, W_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_{i,C}\right) \\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_j\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) + N\log(2\pi e) - h\left(\overrightarrow{\boldsymbol{Z}}_i, \overrightarrow{\boldsymbol{Z}}_j\right) + N\delta(N) \\
&= h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) - h\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}|W_i, \overleftarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i, W_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_{i,C}\right) \\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_j\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) + N\delta(N) \\
&\leqslant h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) + I\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}; W_i, \overleftarrow{\boldsymbol{Y}}_i\right) - h\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}\right) - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i, W_j\right) \\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_{i,C}\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j\right) + h\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}|\overrightarrow{\boldsymbol{Y}}_j\right) - h\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}|\overrightarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_i, \boldsymbol{X}_j\right) \\
&\quad + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) + N\delta(N) \\
&\overset{(h)}{=} h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) + I\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}; W_i, \overleftarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i, W_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_{i,C}\right) \\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_j|\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}\right) - h\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}|\overrightarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_i, \boldsymbol{X}_j\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) + N\delta(N) \\
&= h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) + I\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}; W_i, \overleftarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i, W_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_{i,C}\right) \\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_j|\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}\right) - h\left(\overrightarrow{\boldsymbol{Z}}_i, \overrightarrow{\boldsymbol{Z}}_j|\overrightarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_i, \boldsymbol{X}_j\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) + N\delta(N) \\
&= h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) + I\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}; W_i, \overleftarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i, W_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_{i,C}\right) \\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_j|\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}\right) - h\left(\overrightarrow{\boldsymbol{Z}}_i|\overrightarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_i, \boldsymbol{X}_j\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) + N\delta(N) \\
&= h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) + I\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}; W_i, \overleftarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i, W_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_{i,C}\right) \\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_j|\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}\right) - h\left(\overrightarrow{\boldsymbol{Z}}_i\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) + N\delta(N) \\
&= h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) + I\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}; W_i, \overleftarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i, W_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_{i,C}\right) \\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_j|\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) - \frac{N}{2}\log(2\pi e) + N\delta(N) \\
&\overset{(f)}{\leqslant} h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) + I\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}, W_j, \overleftarrow{\boldsymbol{Y}}_j; W_i, \overleftarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i, W_j\right) \\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_{i,C}\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j|\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) - \frac{N}{2}\log(2\pi e) \\
&\quad + N\delta(N) \\
&\overset{(i)}{\leqslant} h\left(\overrightarrow{\boldsymbol{Y}}_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) + h\left(\overleftarrow{\boldsymbol{Y}}_i|W_i\right) + \sum_{n=1}^{N}\Big[h\left(X_{i,U,n}|X_{j,C,n}\right) + h\left(\overleftarrow{\boldsymbol{Y}}_{j,n}|X_{j,n}, X_{i,U,n}\right) \\
&\quad - \frac{3}{2}\log(2\pi e)\Big] - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i, W_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_{i,C}\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j|\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}\right) \\
&\quad + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) - \frac{N}{2}\log(2\pi e) + N\delta(N) \\
&= h\left(\overrightarrow{\boldsymbol{Y}}_i\right) + \sum_{n=1}^{N}\Big[h\left(X_{i,U,n}|X_{j,C,n}\right) + h\left(\overleftarrow{\boldsymbol{Y}}_{j,n}|X_{j,n}, X_{i,U,n}\right) - 2\log(2\pi e)\Big] - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i, W_j\right) \\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_{i,C}\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j|\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) + N\delta(N) \\
&\overset{(j)}{\leqslant} h\left(\overrightarrow{\boldsymbol{Y}}_i\right) + \sum_{n=1}^{N}\Big[h\left(X_{i,U,n}|X_{j,C,n}\right) + h\left(\overleftarrow{\boldsymbol{Y}}_{j,n}|X_{j,n}, X_{i,U,n}\right) - 2\log(2\pi e)\Big] - h\left(\overleftarrow{\boldsymbol{Y}}_j|W_i, W_j, \overrightarrow{\boldsymbol{Y}}_j\right) \\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_{i,C}\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j|\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) + N\delta(N)
\end{aligned}
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\begin{aligned}
&= h\left(\overrightarrow{\boldsymbol{Y}}_i\right) + \sum_{n=1}^{N}\Big[h\left(X_{i,U,n}|X_{j,C,n}\right) + h\left(\overleftarrow{\boldsymbol{Y}}_{j,n}|X_{j,n}, X_{i,U,n}\right) - 2\log\left(2\pi e\right)\Big] \\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_{i,C}\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j|\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) - h\left(\overleftarrow{\boldsymbol{Z}}_j\right) \\
&\quad + N\delta(N) \\
&\overset{(k)}{=} h\left(\overrightarrow{\boldsymbol{Y}}_i\right) + \sum_{n=1}^{N}\Big[h\left(X_{i,U,n}|X_{j,C,n}\right) + h\left(\overleftarrow{\boldsymbol{Y}}_{j,n}|X_{j,n}, X_{i,U,n}\right) - \frac{5}{2}\log\left(2\pi e\right)\Big] \\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_i|W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_{i,C}, \boldsymbol{X}_j\right) + h\left(\overrightarrow{\boldsymbol{Y}}_j|\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) + N\delta(N) \\
&\overset{(j)}{=} h\left(\overrightarrow{\boldsymbol{Y}}_i\right) + \sum_{n=1}^{N}\Big[h\left(X_{i,U,n}|X_{j,C,n}\right) + h\left(\overleftarrow{\boldsymbol{Y}}_{j,n}|X_{j,n}, X_{i,U,n}\right) - \frac{5}{2}\log\left(2\pi e\right)\Big] + h\left(\overrightarrow{\boldsymbol{Y}}_i|\boldsymbol{X}_{i,C}, \boldsymbol{X}_j\right) \\
&\quad + h\left(\overrightarrow{\boldsymbol{Y}}_j|\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}\right) + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) + N\delta(N) \\
&= \sum_{n=1}^{N}\Big[h\left(\overrightarrow{Y}_{i,n}|\overrightarrow{\boldsymbol{Y}}_{i,(1:n-1)}\right) + h\left(X_{i,U,n}|X_{j,C,n}\right) + h\left(\overleftarrow{Y}_{j,n}|X_{j,n}, X_{i,U,n}\right) - \frac{5}{2}\log\left(2\pi e\right) \\
&\quad + h\left(\overrightarrow{Y}_{i,n}|\boldsymbol{X}_{i,C}, \boldsymbol{X}_j, \overrightarrow{\boldsymbol{Y}}_{i,(1:n-1)}\right) + h\left(\overrightarrow{Y}_{j,n}|\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}, \overrightarrow{\boldsymbol{Y}}_{j,(1:n-1)}\right)\Big] \\
&\quad + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) + N\delta(N) \\
&\overset{(h)}{\leqslant} \sum_{n=1}^{N}\Big[h\left(\overrightarrow{Y}_{i,n}\right) + h\left(X_{i,U,n}|X_{j,C,n}\right) + h\left(\overleftarrow{Y}_{j,n}|X_{j,n}, X_{i,U,n}\right) - \frac{5}{2}\log\left(2\pi e\right) \\
&\quad + h\left(\overrightarrow{Y}_{i,n}|X_{i,C,n}, X_{j,n}\right) + h\left(\overrightarrow{Y}_{j,n}|X_{j,C,n}, X_{i,U,n}\right)\Big] + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) + N\delta(N) \\
&= \sum_{n=1}^{N}\Big[h\left(\overrightarrow{Y}_{i,n}\right) + h\left(X_{i,U,n}|X_{j,C,n}\right) + h\left(\overleftarrow{Y}_{j,n}|X_{j,n}, X_{i,U,n}\right) + h\left(\overrightarrow{Y}_{i,n}|X_{i,C,n}, X_{j,n}\right) \\
&\quad + h\left(\overrightarrow{Y}_{j,n}|X_{j,C,n}, X_{i,U,n}\right) - \frac{5}{2}\log\left(2\pi e\right)\Big] + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) + N\delta(N) \\
&= \sum_{n=1}^{N}\Big[h\left(\overrightarrow{Y}_{i,n}\right) + h\left(X_{i,U,n}|X_{j,C,n}\right) + h\left(\overleftarrow{Y}_{j,n}|X_{j,n}, X_{i,U,n}\right) + h\left(\overrightarrow{Y}_{i,n}|X_{i,C,n}, X_{j,n}\right) \\
&\quad + h\left(\overrightarrow{Y}_{j,n}|X_{j,C,n}, X_{i,U,n}\right) - \frac{5}{2}\log\left(2\pi e\right)\Big] + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) + N\delta(N) \\
&= N\Big[h\left(\overrightarrow{Y}_i\right) + h\left(X_{i,U}|X_{j,C}\right) + h\left(\overleftarrow{Y}_j|X_j, X_{i,U}\right) + h\left(\overrightarrow{Y}_i|X_{i,C}, X_j\right) + h\left(\overrightarrow{Y}_j|X_{j,C}, X_{i,U}\right) \\
&\quad - \frac{5}{2}\log\left(2\pi e\right)\Big] + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}\right) + N\delta(N) \\
&\overset{(d)}{=} N\Big[h\left(\overrightarrow{Y}_i\right) + h\left(X_{i,U}|X_{j,C}\right) + h\left(\overleftarrow{Y}_j|X_j, X_{i,U}\right) + h\left(\overrightarrow{Y}_i|X_{i,C}, X_j\right) + h\left(\overrightarrow{Y}_j|X_{j,C}, X_{i,U}\right) \\
&\quad - \frac{5}{2}\log\left(2\pi e\right)\Big] + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}, \overrightarrow{\boldsymbol{Y}}_i\right) + N\delta(N) \\
&\overset{(l)}{=} N\Big[h\left(\overrightarrow{Y}_i\right) + h\left(X_{i,U}|X_{j,C}\right) + h\left(\overleftarrow{Y}_j|X_j, X_{i,U}\right) + h\left(\overrightarrow{Y}_i|X_{i,C}, X_j\right) + h\left(\overrightarrow{Y}_j|X_{j,C}, X_{i,U}\right) \\
&\quad - \frac{5}{2}\log\left(2\pi e\right)\Big] + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}, \overrightarrow{\boldsymbol{Y}}_i, \overleftarrow{\boldsymbol{Z}}_i\right) + N\delta(N) \\
&\overset{(m)}{=} N\Big[h\left(\overrightarrow{Y}_i\right) + h\left(X_{i,U}|X_{j,C}\right) + h\left(\overleftarrow{Y}_j|X_j, X_{i,U}\right) + h\left(\overrightarrow{Y}_i|X_{i,C}, X_j\right) + h\left(\overrightarrow{Y}_j|X_{j,C}, X_{i,U}\right) \\
&\quad - \frac{5}{2}\log\left(2\pi e\right)\Big] + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, W_j, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_{j,C}, \overrightarrow{\boldsymbol{Y}}_i, \overrightarrow{\boldsymbol{Z}}_i, \overleftarrow{\boldsymbol{Z}}_i, \overleftarrow{\boldsymbol{Z}}_j\right) + N\delta(N)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(n)}{=} N\Big[h\left(\overrightarrow{Y}_i\right) + h\left(X_{i,U}|X_{j,C}\right) + h\left(\overleftarrow{Y}_j|X_j, X_{i,U}\right) + h\left(\overrightarrow{Y}_i|X_{i,C}, X_j\right) + h\left(\overrightarrow{Y}_j|X_{j,C}, X_{i,U}\right) \\
&\quad -\frac{5}{2}\log\left(2\pi e\right)\Big] + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, W_j, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_j, \overrightarrow{\boldsymbol{Y}}_i, \overrightarrow{\boldsymbol{Z}}_i, \overleftarrow{\boldsymbol{Z}}_i, \overleftarrow{\boldsymbol{Z}}_j\right) + N\delta(N) \\
&\overset{(k)}{=} N\Big[h\left(\overrightarrow{Y}_i\right) + h\left(X_{i,U}|X_{j,C}\right) + h\left(\overleftarrow{Y}_j|X_j, X_{i,U}\right) + h\left(\overrightarrow{Y}_i|X_{i,C}, X_j\right) + h\left(\overrightarrow{Y}_j|X_{j,C}, X_{i,U}\right) \\
&\quad -\frac{5}{2}\log\left(2\pi e\right)\Big] + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, W_j, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_j, \overrightarrow{\boldsymbol{Y}}_i, \overrightarrow{\boldsymbol{Z}}_i, \overleftarrow{\boldsymbol{Z}}_i, \overleftarrow{\boldsymbol{Z}}_j, \overleftarrow{\boldsymbol{Y}}_{j,(1:N-1)}\right) + N\delta(N) \\
&\overset{(l)}{=} N\Big[h\left(\overrightarrow{Y}_i\right) + h\left(X_{i,U}|X_{j,C}\right) + h\left(\overleftarrow{Y}_j|X_j, X_{i,U}\right) + h\left(\overrightarrow{Y}_i|X_{i,C}, X_j\right) + h\left(\overrightarrow{Y}_j|X_{j,C}, X_{i,U}\right) \\
&\quad -\frac{5}{2}\log\left(2\pi e\right)\Big] + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, W_j, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_j, \overrightarrow{\boldsymbol{Y}}_i, \overrightarrow{\boldsymbol{Z}}_i, \overleftarrow{\boldsymbol{Z}}_i, \overleftarrow{\boldsymbol{Z}}_j, \overleftarrow{\boldsymbol{Y}}_{j,(1:N-1)}, \overrightarrow{\boldsymbol{Y}}_{j,(1:N-2)}\right) + N\delta(N) \\
&\overset{(o)}{=} N\Big[h\left(\overrightarrow{Y}_i\right) + h\left(X_{i,U}|X_{j,C}\right) + h\left(\overleftarrow{Y}_j|X_j, X_{i,U}\right) + h\left(\overrightarrow{Y}_i|X_{i,C}, X_j\right) + h\left(\overrightarrow{Y}_j|X_{j,C}, X_{i,U}\right) \\
&\quad -\frac{5}{2}\log\left(2\pi e\right)\Big] + h\left(\overrightarrow{\boldsymbol{Z}}_j|W_i, W_j, \overleftarrow{\boldsymbol{Y}}_i, \boldsymbol{X}_i, \boldsymbol{X}_j, \overrightarrow{\boldsymbol{Y}}_i, \overrightarrow{\boldsymbol{Z}}_i, \overleftarrow{\boldsymbol{Z}}_i, \overleftarrow{\boldsymbol{Z}}_j, \overleftarrow{\boldsymbol{Y}}_{j,(1:N-1)}, \overrightarrow{\boldsymbol{Y}}_{j,(1:N-2)}, \overrightarrow{\boldsymbol{Z}}_{j,(1:N-2)}\right) \\
&\quad +N\delta(N) \\
&= N\Big[h\left(\overrightarrow{Y}_i\right) + h\left(X_{i,U}|X_{j,C}\right) + h\left(\overleftarrow{Y}_j|X_j, X_{i,U}\right) + h\left(\overrightarrow{Y}_i|X_{i,C}, X_j\right) + h\left(\overrightarrow{Y}_j|X_{j,C}, X_{i,U}\right) \\
&\quad -\frac{5}{2}\log\left(2\pi e\right)\Big] + h\left(\overrightarrow{Z}_{j,N}\right) + h\left(\overrightarrow{Z}_{j,N-1}\right) + N\delta(N), \qquad (47)
\end{aligned}
$$

where
(a) follows from the fact that W_1 and W_2 are mutually independent;
(b) follows from Fano's inequality;
(c) follows from (3) and (4);
(d) follows from (2) and (35);
(e) follows from (36);
(f) follows from the fact that injecting information increases the mutual information;
(g) follows from (4) and the fact that conditioning reduces the entropy;
(h) follows from the fact that $h\left(\overrightarrow{\boldsymbol{Y}}_j\right) - h\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}\right) + h\left(\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}|\overrightarrow{\boldsymbol{Y}}_j\right) = h\left(\overrightarrow{\boldsymbol{Y}}_j|\boldsymbol{X}_{j,C}, \boldsymbol{X}_{i,U}\right)$;
(i) follows from the Lemma 3;
(j) follows from the fact that conditioning reduces the entropy;
(k) follows from (4);
(l) follows from (3);
(m) follows from the fact that W_j, $\overrightarrow{Z}_{i,n}$, $\overleftarrow{Z}_{j,n}$ and $\overrightarrow{Z}_{j,n}$ are mutually independent;
(n) follows from (35);
(o) follows from (2).

From (47), in the asymptotic regime, it holds that

$$
\begin{aligned}
2R_i + R_j \leqslant & h\left(\overrightarrow{Y}_i\right) + h\left(X_{i,U}|X_{j,C}\right) + h\left(\overleftarrow{Y}_j|X_j, X_{i,U}\right) + h\left(\overrightarrow{Y}_i|X_{i,C}, X_j\right) + h\left(\overrightarrow{Y}_j|X_{j,C}, X_{i,U}\right) \\
& -\frac{5}{2}\log\left(2\pi e\right). \qquad (48)
\end{aligned}
$$

This completes the proof of (37c).

B.2 Proof of Lemma 3

Lemma 3 is proved as follows

$$
\begin{aligned}
& I\left(\boldsymbol{X}_{i,C}, \boldsymbol{X}_{j,U}, \overleftarrow{\boldsymbol{Y}}_i, W_i; \overleftarrow{\boldsymbol{Y}}_j, W_j\right) \\
&= I\left(W_i; \overleftarrow{\boldsymbol{Y}}_j, W_j\right) + I\left(\boldsymbol{X}_{i,C}, \boldsymbol{X}_{j,U}, \overleftarrow{\boldsymbol{Y}}_i; \overleftarrow{\boldsymbol{Y}}_j, W_j | W_i\right) \\
&= h\left(\overleftarrow{\boldsymbol{Y}}_j, W_j\right) - h\left(\overleftarrow{\boldsymbol{Y}}_j, W_j | W_i\right) + h\left(\boldsymbol{X}_{i,C}, \boldsymbol{X}_{j,U}, \overleftarrow{\boldsymbol{Y}}_i | W_i\right) - h\left(\boldsymbol{X}_{i,C}, \boldsymbol{X}_{j,U}, \overleftarrow{\boldsymbol{Y}}_i | W_i, W_j, \overleftarrow{\boldsymbol{Y}}_j\right) \\
&\overset{(a)}{=} h\left(W_j\right) + h\left(\overleftarrow{\boldsymbol{Y}}_j | W_j\right) - h\left(W_j | W_i\right) - h\left(\overleftarrow{\boldsymbol{Y}}_j | W_i, W_j\right) + h\left(\boldsymbol{X}_{i,C}, \boldsymbol{X}_{j,U}, \overleftarrow{\boldsymbol{Y}}_i | W_i\right) \\
&\quad - h\left(\boldsymbol{X}_{i,C}, \boldsymbol{X}_{j,U}, \overleftarrow{\boldsymbol{Y}}_i | W_i, W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_j\right) \\
&\overset{(b)}{=} h\left(\overleftarrow{\boldsymbol{Y}}_j | W_j\right) - h\left(\overleftarrow{\boldsymbol{Y}}_j | W_i, W_j\right) + h\left(\boldsymbol{X}_{i,C}, \boldsymbol{X}_{j,U}, \overleftarrow{\boldsymbol{Y}}_i | W_i\right) - h\left(\boldsymbol{X}_{i,C}, \overrightarrow{\boldsymbol{Z}}_i, \overleftarrow{\boldsymbol{Y}}_i | W_i, W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_j\right) \\
&\overset{(c)}{\leqslant} h\left(\overleftarrow{\boldsymbol{Y}}_j | W_j\right) - h\left(\overleftarrow{\boldsymbol{Y}}_j | W_i, W_j, \overrightarrow{\boldsymbol{Y}}_j\right) + h\left(\boldsymbol{X}_{i,C}, \boldsymbol{X}_{j,U}, \overleftarrow{\boldsymbol{Y}}_i | W_i\right) \\
&\quad - h\left(\boldsymbol{X}_{i,C}, \overrightarrow{\boldsymbol{Z}}_i, \overleftarrow{\boldsymbol{Y}}_i | W_i, W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_i, \boldsymbol{X}_j\right) \\
&\overset{(d)}{=} h\left(\overleftarrow{\boldsymbol{Y}}_j | W_j\right) - h\left(\overleftarrow{\boldsymbol{Z}}_j | W_i, W_j, \overrightarrow{\boldsymbol{Y}}_j\right) + h\left(\boldsymbol{X}_{i,C}, \boldsymbol{X}_{j,U}, \overleftarrow{\boldsymbol{Y}}_i | W_i\right) \\
&\quad - h\left(\overrightarrow{\boldsymbol{Z}}_i, \overrightarrow{\boldsymbol{Z}}_j, \overleftarrow{\boldsymbol{Z}}_i | W_i, W_j, \overleftarrow{\boldsymbol{Y}}_j, \boldsymbol{X}_i, \boldsymbol{X}_j\right) \\
&= h\left(\overleftarrow{\boldsymbol{Y}}_j | W_j\right) + h\left(\boldsymbol{X}_{i,C}, \boldsymbol{X}_{j,U}, \overleftarrow{\boldsymbol{Y}}_i | W_i\right) - 2N \log\left(2\pi e\right) \\
&= h\left(\overleftarrow{\boldsymbol{Y}}_j | W_j\right) + \sum_{n=1}^{N} \Big[h\left(X_{i,C,n}, X_{j,U,n}, \overleftarrow{Y}_{i,n} | W_i, \boldsymbol{X}_{i,C,(1:n-1)}, \boldsymbol{X}_{j,U,(1:n-1)}, \overleftarrow{\boldsymbol{Y}}_{i,(1:n-1)}\right) \\
&\quad - 2\log\left(2\pi e\right) \Big] \\
&\overset{(a)}{=} h\left(\overleftarrow{\boldsymbol{Y}}_j | W_j\right) + \sum_{n=1}^{N} \Big[h\left(X_{i,C,n}, X_{j,U,n}, \overleftarrow{Y}_{i,n} | W_i, \boldsymbol{X}_{i,C,(1:n-1)}, \boldsymbol{X}_{j,U,(1:n-1)}, \overleftarrow{\boldsymbol{Y}}_{i,(1:n-1)}, \boldsymbol{X}_{i,(1:n)}\right) \\
&\quad - 2\log\left(2\pi e\right) \Big] \\
&\overset{(c)}{\leqslant} h\left(\overleftarrow{\boldsymbol{Y}}_j | W_j\right) + \sum_{n=1}^{N} \Big[h\left(X_{i,C,n}, X_{j,U,n}, \overleftarrow{Y}_{i,n} | X_{i,n}\right) - 2\log\left(2\pi e\right) \Big] \\
&= h\left(\overleftarrow{\boldsymbol{Y}}_j | W_j\right) + \sum_{n=1}^{N} \Big[h\left(X_{i,C,n} | X_{i,n}\right) + h\left(X_{j,U,n} | X_{i,n}, X_{i,C,n}\right) + h\left(\overleftarrow{Y}_{i,n} | X_{i,n}, X_{i,C,n}, X_{j,U,n}\right) \\
&\quad - 2\log\left(2\pi e\right) \Big] \\
&\overset{(e)}{=} h\left(\overleftarrow{\boldsymbol{Y}}_j | W_j\right) + \sum_{n=1}^{N} \Big[h\left(\overrightarrow{Z}_{j,n} | X_{i,n}\right) + h\left(X_{j,U,n} | X_{i,n}, X_{i,C,n}\right) + h\left(\overleftarrow{Y}_{i,n} | X_{i,n}, X_{i,C,n}, X_{j,U,n}\right) \\
&\quad - 2\log\left(2\pi e\right) \Big] \\
&= h\left(\overleftarrow{\boldsymbol{Y}}_j | W_j\right) + \sum_{n=1}^{N} \Big[h\left(\overrightarrow{Z}_{j,n}\right) + h\left(X_{j,U,n} | X_{i,n}, X_{i,C,n}\right) + h\left(\overleftarrow{Y}_{i,n} | X_{i,n}, X_{i,C,n}, X_{j,U,n}\right) \\
&\quad - 2\log\left(2\pi e\right) \Big] \\
&= h\left(\overleftarrow{\boldsymbol{Y}}_j | W_j\right) + \sum_{n=1}^{N} \Big[h\left(X_{j,U,n} | X_{i,n}, X_{i,C,n}\right) + h\left(\overleftarrow{Y}_{i,n} | X_{i,n}, X_{i,C,n}, X_{j,U,n}\right) - \frac{3}{2}\log\left(2\pi e\right) \Big] \\
&\overset{(c)}{\leqslant} h\left(\overleftarrow{\boldsymbol{Y}}_j | W_j\right) + \sum_{n=1}^{N} \Big[h\left(X_{j,U,n} | X_{i,C,n}\right) + h\left(\overleftarrow{Y}_{i,n} | X_{i,n}, X_{j,U,n}\right) - \frac{3}{2}\log\left(2\pi e\right) \Big], \qquad (49)
\end{aligned}
$$

where
(a) follows from (4);
(b) follows from (36) and the fact that W_1 and W_2 are mutually independent;
(c) follows from the fact that conditioning reduces the entropy;
(d) follows from (2), (3) and (36);
(e) follows from (35).
This completes the proof of Lemma 3.

C Proof of Theorem 3

This appendix presents a proof of the Theorem 3. The gap, denoted by δ, between the sets $\overline{\mathcal{C}}_{\text{GIC-NOF}}$ and $\underline{\mathcal{C}}_{\text{GIC-NOF}}$ (Def. 2) is obtained as follows

$$\delta = \max\left(\max(\delta_{R_1}, \delta_{R_2}), \frac{\delta_{2R}}{2}, \frac{\max(\delta_{3R_1}, \delta_{3R_2})}{3}\right), \tag{50}$$

where,

$$\delta_{R_1} = \min(\kappa_{1,1}, \kappa_{2,1}, \kappa_{3,1}) - \min(a_2, a_6 + b_1, a_4 + b_1 + b_3, \vartheta_1), \tag{51a}$$
$$\delta_{R_2} = \min(\kappa_{1,2}, \kappa_{2,2}, \kappa_{3,2}) - \min(b_2, a_1 + b_6, a_1 + a_3 + b_4, \vartheta_2), \tag{51b}$$
$$\delta_{2R} = \min(\kappa_4, \kappa_5, \kappa_6) - \min(a_2 + b_4, a_4 + b_2, a_1 + a_4 + b_1 + b_7, a_1 + a_5 + b_1 + b_5, \tag{51c}$$
$$a_1 + a_7 + b_1 + b_4, \vartheta_1 + \vartheta_2), \tag{51d}$$
$$\delta_{3R_1} = \kappa_{7,1} - \min(a_2 + a_4 + b_1 + b_7, a_1 + a_4 + a_7 + 2b_1 + b_5, a_2 + a_4 + b_1 + b_5, 2\vartheta_1 + \vartheta_2), \tag{51e}$$
$$\delta_{3R_2} = \kappa_{7,2} - \min(a_1 + a_5 + b_2 + b_4, a_1 + a_7 + b_2 + b_4, 2a_1 + a_5 + b_1 + b_4 + b_7, \vartheta_1 + 2\vartheta_2), \tag{51f}$$

and, ϑ_1 and ϑ_2 are information rates that are achievable by transmitter-receiver pair 1 and transmitter-receiver pair 2 using particular power allocations. Note that δ_{R_1} and δ_{R_2} represent the gap between the active achievable single-rate bound and the active converse single-rate bound; δ_{2R} represents the gap between the active achievable sum-rate bound and the active converse sum-rate bound; and, δ_{3R_1} and δ_{3R_2} represent the gap between the active achievable weighted sum-rate bound and the active converse weighted sum-rate bound.
A key point in order to find the gap between the achievable region and the converse region is to choose the right power allocation for the achievable region, i.e., ρ, α_1, α_2, $\lambda_{1,C}$, $\lambda_{1,P}$, $\lambda_{2,C}$ and $\lambda_{2,P}$, according to the definitions in (30) for all $i \in \{1, 2\}$. Such a power allocation is different in each interference regime. The following describes all possible cases and the corresponding power allocations. In order to simplify the expressions in the achievable region, it is chosen that $\rho = 0$ in all power allocations. In the converse region, ρ takes values to obtain the tightest gap. Both interference to noise ratios, i.e., INR_{21} and INR_{12}, are assumed to be bigger than 1.

<u>Case 1</u>: $\text{INR}_{12} > \overrightarrow{\text{SNR}}_1$ and $\text{INR}_{21} > \overrightarrow{\text{SNR}}_2$. This case corresponds to the scenario in which both transmitter-receiver pairs are in high interference regime (HIR). Three sub-cases follow depending whether or not feedback plays a key role.
<u>Case 1.1</u>: $\overleftarrow{\text{SNR}}_2 \leqslant \overrightarrow{\text{SNR}}_1$ and $\overleftarrow{\text{SNR}}_1 \leqslant \overrightarrow{\text{SNR}}_2$. In this case feedback does not play a key role and the power allocation is: $\rho = 0$, $\alpha_1 = 0$ and $\alpha_2 = 0$.
<u>Case 1.2</u>: $\overleftarrow{\text{SNR}}_2 > \overrightarrow{\text{SNR}}_1$ and $\overleftarrow{\text{SNR}}_1 > \overrightarrow{\text{SNR}}_2$. In this case feedback plays a key role and the power allocation is: $\rho = 0$, $\alpha_1 = 1$ and $\alpha_2 = 1$.
<u>Case 1.3</u>: $\overleftarrow{\text{SNR}}_2 \leqslant \overrightarrow{\text{SNR}}_1$ and $\overleftarrow{\text{SNR}}_1 > \overrightarrow{\text{SNR}}_2$. In this case feedback on the transmitter-receiver pair 1 plays a key role and the power allocation is: $\rho = 0$, $\alpha_1 = 0$ and $\alpha_2 = 1$.

<u>Case 2</u>: $\text{INR}_{12} \leqslant \overrightarrow{\text{SNR}}_1$ and $\text{INR}_{21} \leqslant \overrightarrow{\text{SNR}}_2$. This case corresponds to the scenario in which both transmitter-receiver pairs are in low interference regime (LIR). Three sub-cases follow depending whether or not feedback plays a key role.
<u>Case 2.1</u>. In this case feedback does not play a key role and the power allocation is: $\rho = 0$, $\alpha_1 = 0$ and $\alpha_2 = 0$.
<u>Case 2.2</u>. In this case feedback plays a key role and the power allocation is: $\Big(\rho = 0, (1 - \alpha_1)\,\lambda_{1,C} + \lambda_{1,P} = \frac{\overrightarrow{\text{SNR}}_2}{\text{INR}_{21}\overleftarrow{\text{SNR}}_2 + \overrightarrow{\text{SNR}}_2}$ and $(1 - \alpha_2)\,\lambda_{2,C} + \lambda_{2,P} = \frac{\overrightarrow{\text{SNR}}_1}{\text{INR}_{12}\overleftarrow{\text{SNR}}_1 + \overrightarrow{\text{SNR}}_1}\Big)$.
<u>Case 2.3</u>. In this case feedback on the transmitter-receiver pair 1 plays a key role and the power

allocation is: $\left(\rho = 0,\ (1-\alpha_1)\,\lambda_{1,C} + \lambda_{1,P} = 1 \text{ and } (1-\alpha_2)\,\lambda_{2,C} + \lambda_{2,P} = \frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{12}\overleftarrow{\mathrm{SNR}}_1 + \overrightarrow{\mathrm{SNR}}_1}\right)$.

Case 3: $\mathrm{INR}_{12} > \overrightarrow{\mathrm{SNR}}_1$ and $\mathrm{INR}_{21} \leqslant \overrightarrow{\mathrm{SNR}}_2$. This case corresponds to the scenario in which transmitter-receiver pair 1 is in HIR and transmitter-receiver pair 2 is in LIR. Two sub-cases follow depending whether or not feedback in transmitter-receiver pair 2 plays a key role. This is a mixed interference regime (MIR) and in this case feedback on the transmitter-receiver pair 1 does not have any effect on the capacity region.
Case 3.1. In this case feedback on the transmitter-receiver pair 2 does not play a key role and the power allocation is: $\rho = 0$, $\alpha_1 = 0$ and $\alpha_2 = 0$.
Case 3.2. In this case feedback on the transmitter-receiver pair 2 plays a key role and the power allocation is: $\rho = 0$, $\alpha_1 = 1$ and $\alpha_2 = 0$.

The following is the calculation of the gap δ in case 1.1. From the conditions of case 1.1 it follows that:

$$\overleftarrow{\mathrm{SNR}}_2 \leqslant \overrightarrow{\mathrm{SNR}}_1 < \mathrm{INR}_{12} \text{ and} \tag{52a}$$
$$\overleftarrow{\mathrm{SNR}}_1 \leqslant \overrightarrow{\mathrm{SNR}}_2 < \mathrm{INR}_{21}. \tag{52b}$$

Using the corresponding power allocation of case 1.1, the variables $a_1, \ldots, a_7$ and $b_1, \ldots, b_7$ in (8) simplify into:

$$a_1 = 0, \tag{53a}$$
$$a_2 = \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1 + \mathrm{INR}_{12} + 1\right) - \frac{1}{2}, \tag{53b}$$
$$a_3 = \frac{1}{2}\log\left(\mathrm{INR}_{12} + 1\right) - \frac{1}{2}, \tag{53c}$$
$$a_4 = \frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}} + 2\right) - \frac{1}{2}, \tag{53d}$$
$$a_5 = \frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}} + \mathrm{INR}_{12} + 1\right) - \frac{1}{2}, \tag{53e}$$
$$a_6 = \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1 + 2\right) - \frac{1}{2}, \tag{53f}$$
$$a_7 = \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1 + \mathrm{INR}_{12} + 1\right) - \frac{1}{2}, \tag{53g}$$
$$b_1 = 0, \tag{53h}$$
$$b_2 = \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_2 + \mathrm{INR}_{21} + 1\right) - \frac{1}{2}, \tag{53i}$$
$$b_3 = \frac{1}{2}\log\left(\mathrm{INR}_{21} + 1\right) - \frac{1}{2}, \tag{53j}$$
$$b_4 = \frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_2}{\mathrm{INR}_{12}} + 2\right) - \frac{1}{2}, \tag{53k}$$
$$b_5 = \frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_2}{\mathrm{INR}_{12}} + \mathrm{INR}_{21} + 1\right) - \frac{1}{2}, \tag{53l}$$
$$b_6 = \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_2 + 2\right) - \frac{1}{2}, \tag{53m}$$
$$b_7 = \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_2 + \mathrm{INR}_{21} + 1\right) - \frac{1}{2}. \tag{53n}$$

1. Calculation of δ_{R_1}. From (51a), it follows that:

$$\begin{aligned}\delta_{R_1} &= \min(\kappa_{1,1}, \kappa_{2,1}, \kappa_{3,1}) - \min(a_2, a_6 + b_1, a_4 + b_1 + b_3, \vartheta_1)\\ &\overset{(a)}{=} \min(\kappa_{1,1}, \kappa_{2,1}, \kappa_{3,1}) - \min(a_2, a_6, a_4 + b_3, \vartheta_1)\\ &\overset{(b)}{=} \min(\kappa_{1,1}, \kappa_{2,1}, \kappa_{3,1}) - \min(a_6, a_4 + b_3, \vartheta_1),\end{aligned} \tag{54}$$

where,

(a) follows from (53);
(b) follows from the fact that $a_2 \geqslant a_6$ in (53) and $\mathrm{INR}_{12} > 1$.

Let ζ_1 be defined as follows

$$\begin{aligned}\zeta_1 &= \min\left(a_6, a_4 + b_3, \vartheta_1\right)\\ &= \min\left(\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1 + 2\right) - \frac{1}{2}, \frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}} + 2\right) + \frac{1}{2}\log\left(\mathrm{INR}_{21} + 1\right) - 1, \vartheta_1\right).\end{aligned} \tag{55}$$

Denote by $\zeta_{1,1}$, $\zeta_{1,2}$ and $\zeta_{1,3}$ the following parameters:

$$\zeta_{1,1} = \frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1 + 2\right) - \frac{1}{2}, \tag{56a}$$

$$\zeta_{1,2} = \frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}} + 2\right) + \frac{1}{2}\log\left(\mathrm{INR}_{21} + 1\right) - 1, \tag{56b}$$

$$\zeta_{1,3} = \vartheta_1. \tag{56c}$$

Let ξ_1 be defined as follows

$$\begin{aligned}\xi_1 &= \min\left(\kappa_{1,1}, \kappa_{2,1}, \kappa_{3,i}\right),\\ &= \min\Bigg(\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1 + 2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_1\mathrm{INR}_{12}} + \mathrm{INR}_{12} + 1\right), \frac{1}{2}\log\left(1 + \left(1-\rho^2\right)\mathrm{INR}_{21}\right)\\ &\quad + \frac{1}{2}\log\left(1 + \frac{\left(1-\rho^2\right)\overrightarrow{\mathrm{SNR}}_1}{1 + \left(1-\rho^2\right)\mathrm{INR}_{21}}\right), \frac{1}{2}\log\left(\left(1-\rho^2\right)\overrightarrow{\mathrm{SNR}}_1 + 1\right)\\ &\quad + \frac{1}{2}\log\left(\frac{\overleftarrow{\mathrm{SNR}}_2\left(\left(1-\rho^2\right)\left(\overrightarrow{\mathrm{SNR}}_1 + \mathrm{INR}_{21}\right) + 1\right)}{\left(\overrightarrow{\mathrm{SNR}}_2 + 2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_2\mathrm{INR}_{21}} + \mathrm{INR}_{21} + 1\right)\left(\left(1-\rho^2\right)\left(\overrightarrow{\mathrm{SNR}}_1 + 1\right)\right)} + 1\right)\Bigg),\end{aligned} \tag{57}$$

with $0 \leqslant \rho \leqslant 1$.

Denote by $\xi_{1,1}$, $\xi_{1,2}$ and $\xi_{1,3}$ the following parameters:

$$\xi_{1,1}=\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_1\mathrm{INR}_{12}}+\mathrm{INR}_{12}+1\right), \tag{58a}$$

$$\begin{aligned}\xi_{1,2}=&\frac{1}{2}\log\left(1+\left(1-\rho^2\right)\mathrm{INR}_{21}\right)+\frac{1}{2}\log\left(1+\frac{\left(1-\rho^2\right)\overrightarrow{\mathrm{SNR}}_1}{1+\left(1-\rho^2\right)\mathrm{INR}_{21}}\right)\\ =&\frac{1}{2}\log\left(1+\left(1-\rho^2\right)\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{21}\right)\right),\end{aligned} \tag{58b}$$

$$\begin{aligned}\xi_{1,3}=&\frac{1}{2}\log\left(\left(1-\rho^2\right)\overrightarrow{\mathrm{SNR}}_1+1\right)\\ &+\frac{1}{2}\log\left(\frac{\overleftarrow{\mathrm{SNR}}_2\left(\left(1-\rho^2\right)\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{21}\right)+1\right)}{\left(\overrightarrow{\mathrm{SNR}}_2+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_2\mathrm{INR}_{21}}+\mathrm{INR}_{21}+1\right)\left(\left(1-\rho^2\right)\left(\overrightarrow{\mathrm{SNR}}_1+1\right)\right)}+1\right).\end{aligned} \tag{58c}$$

The parameters in (58) can be upper bounded as follows

$$\begin{aligned}\xi_{1,1}&\leqslant\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_1\mathrm{INR}_{12}}+\mathrm{INR}_{12}+1\right),\\ &\overset{(c)}{\leqslant}\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\sqrt{\overrightarrow{\mathrm{SNR}}_1\mathrm{INR}_{12}}+\mathrm{INR}_{12}+1\right)\\ &\overset{(d)}{\leqslant}\frac{1}{2}\log\left(2\overrightarrow{\mathrm{SNR}}_1+2\mathrm{INR}_{12}+1\right)\\ &\leqslant\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{12}+1\right)+\frac{1}{2},\end{aligned} \tag{59a}$$

$$\begin{aligned}\xi_{1,2}&\leqslant\frac{1}{2}\log\left(1+\left(1-\rho^2\right)\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{21}\right)\right)\\ &\leqslant\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{21}+1\right),\end{aligned} \tag{59b}$$

$$\begin{aligned}\xi_{1,3}&\leqslant\frac{1}{2}\log\left(\left(1-\rho^2\right)\overrightarrow{\mathrm{SNR}}_1+1\right)\\ &\quad+\frac{1}{2}\log\left(\frac{\overleftarrow{\mathrm{SNR}}_2\left(\left(1-\rho^2\right)\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{21}\right)+1\right)}{\left(\overrightarrow{\mathrm{SNR}}_2+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_2\mathrm{INR}_{21}}+\mathrm{INR}_{21}+1\right)\left(\left(1-\rho^2\right)\left(\overrightarrow{\mathrm{SNR}}_1+1\right)\right)}+1\right)\\ &\overset{(e)}{\leqslant}\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+1\right)+\frac{1}{2}\log\left(\frac{\overleftarrow{\mathrm{SNR}}_2\left(\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{21}\right)+1\right)}{\left(\overrightarrow{\mathrm{SNR}}_2+\mathrm{INR}_{21}+1\right)\left(\overrightarrow{\mathrm{SNR}}_1+1\right)}+1\right)\\ &=\frac{1}{2}\log\left(\frac{\overleftarrow{\mathrm{SNR}}_2\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{21}+1\right)}{\overrightarrow{\mathrm{SNR}}_2+\mathrm{INR}_{21}+1}+\overrightarrow{\mathrm{SNR}}_1+1\right),\end{aligned} \tag{59c}$$

where,
(c) follows from the fact that $0\leqslant\rho\leqslant1$;
(d) follows from the fact that $\left(\sqrt{\overrightarrow{\mathrm{SNR}}_1}-\sqrt{\mathrm{INR}_{12}}\right)^2\geqslant0$;

(e) follows from the fact that the bound in (58c) is a monotonically decreasing function of ρ.

The key point here is to determine a gap for each parameter in (56) using one of the parameters in (59).

The parameter (56b) can be lower bounded as follows

$$\begin{aligned}\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}+2\right)+\frac{1}{2}\log\left(\mathrm{INR}_{21}+1\right)-1 \geqslant & \frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}+2\right)+\frac{1}{2}\log\left(\mathrm{INR}_{21}\right)-1\\ =&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\mathrm{INR}_{21}\right)-1\\ =&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{21}+\mathrm{INR}_{21}\right)-1\\ \geqslant&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{21}+1\right)-1. \end{aligned} \tag{60}$$

From (60), given that ϑ_1 is by definition an achievable single rate, then it can be defined as follows

$$\zeta_{1,3}=\vartheta_1=\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{21}+1\right)-1. \tag{61}$$

The gap that involves the parameters (59b) and (56b) is calculated indirectly using (61) instead of (56b) as follows

$$\begin{aligned}\delta_{R_{1,1}}=&\xi_{1,2}-\zeta_{1,3}\\ \leqslant&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{21}+1\right)-\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{21}+1\right)+1\\ =&1. \end{aligned} \tag{62}$$

The gap involving the parameter (56a) is calculated considering the following 4 subcases, which correspond to the possible conditions between $\overrightarrow{\mathrm{SNR}}_1$ and INR_{21}, and between $\overrightarrow{\mathrm{SNR}}_2$ and INR_{12}, as follows

<u>Case 1.1.1</u>: $\overrightarrow{\mathrm{SNR}}_1 \geqslant \mathrm{INR}_{21} \wedge \overrightarrow{\mathrm{SNR}}_2 \geqslant \mathrm{INR}_{12}$.
<u>Case 1.1.2</u>: $\overrightarrow{\mathrm{SNR}}_1 \geqslant \mathrm{INR}_{21} \wedge \overrightarrow{\mathrm{SNR}}_2 < \mathrm{INR}_{12}$.
<u>Case 1.1.3</u>: $\overrightarrow{\mathrm{SNR}}_1 < \mathrm{INR}_{21} \wedge \overrightarrow{\mathrm{SNR}}_2 \geqslant \mathrm{INR}_{12}$.
<u>Case 1.1.4</u>: $\overrightarrow{\mathrm{SNR}}_1 < \mathrm{INR}_{21} \wedge \overrightarrow{\mathrm{SNR}}_2 < \mathrm{INR}_{12}$.

The case 1.1.1 is not possible by contradiction based on (52).

In case 1.1.2 with (52), the parameter (59b) can be upper bounded as follows

$$\begin{aligned}\xi_{1,2}\leqslant&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{21}+1\right)\\ \leqslant&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\overrightarrow{\mathrm{SNR}}_1+1\right)\\ \leqslant&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\right)+\frac{1}{2}. \end{aligned} \tag{63}$$

Then, the following is the gap for the case 1.1.2 that involves the parameters (56a) and (63):

$$\begin{aligned}\delta_{R_{1,2}}=&\xi_{1,2}-\zeta_{1,1}\\ \leqslant&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\right)+\frac{1}{2}-\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\right)+\frac{1}{2}\\ =&1.\end{aligned}$$

In case 1.1.3 with (52), the parameter (59c) can be upper bounded as follows

$$\begin{aligned}\xi_{1,3}\leqslant&\frac{1}{2}\log\left(\frac{\overleftarrow{\mathrm{SNR}}_2\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{21}+1\right)}{\overrightarrow{\mathrm{SNR}}_2+\mathrm{INR}_{21}+1}+\overrightarrow{\mathrm{SNR}}_1+1\right)\\ \leqslant&\frac{1}{2}\log\left(\overleftarrow{\mathrm{SNR}}_2+\overrightarrow{\mathrm{SNR}}_1+1\right)\\ \leqslant&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\overrightarrow{\mathrm{SNR}}_1+1\right)\\ \leqslant&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\right)+\frac{1}{2}.\end{aligned}\tag{64}$$

Then, the following is the gap for the case 1.1.3 that involves the parameters (56a) and (64):

$$\begin{aligned}\delta_{R_{1,3}}=&\xi_{1,3}-\zeta_{1,1}\\ \leqslant&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\right)+\frac{1}{2}-\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\right)+\frac{1}{2}\\ =&1.\end{aligned}$$

In case 1.1.4 with (52), two additional cases has to be defined:

<u>Case 1.1.4.1</u>: $\overrightarrow{\mathrm{SNR}}_1\geqslant\overrightarrow{\mathrm{SNR}}_2$.
<u>Case 1.1.4.2</u>: $\overrightarrow{\mathrm{SNR}}_1<\overrightarrow{\mathrm{SNR}}_2$.

In case 1.1.4.1 with (52), the parameter (59c) can be upper bounded as follows

$$\begin{aligned}\xi_{1,3}\leqslant&\frac{1}{2}\log\left(\frac{\overleftarrow{\mathrm{SNR}}_2\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{21}+1\right)}{\overrightarrow{\mathrm{SNR}}_2+\mathrm{INR}_{21}+1}+\overrightarrow{\mathrm{SNR}}_1+1\right)\\ \leqslant&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+1\right)+\frac{1}{2}\log\left(\frac{\overleftarrow{\mathrm{SNR}}_2\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{21}+1\right)}{\left(\overrightarrow{\mathrm{SNR}}_2+\mathrm{INR}_{21}+1\right)\left(\overrightarrow{\mathrm{SNR}}_1+1\right)}+1\right)\\ \leqslant&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+1\right)+\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_1\left(\mathrm{INR}_{21}+\mathrm{INR}_{21}+\mathrm{INR}_{21}\right)}{\mathrm{INR}_{21}\overrightarrow{\mathrm{SNR}}_1}+1\right)\\ =&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+1\right)+1\\ \leqslant&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\right)+1.\end{aligned}\tag{65}$$

Then, the following is the gap for the case 1.1.4.1 that involves the parameters (56a) and (65):

$$\begin{aligned}\delta_{R_{1,4}}=&\xi_{1,3}-\zeta_{1,1}\\ \leqslant&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\right)+1-\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\right)+\frac{1}{2}\\ =&\frac{3}{2}.\end{aligned}$$

In case 1.1.4.2 with (52), the parameter (59c) can be upper bounded as follows

$$\begin{aligned}\xi_{1,3}\leqslant&\frac{1}{2}\log\left(\frac{\overleftarrow{\mathrm{SNR}}_2\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{21}+1\right)}{\overrightarrow{\mathrm{SNR}}_2+\mathrm{INR}_{21}+1}+\overrightarrow{\mathrm{SNR}}_1+1\right)\\ \leqslant&\frac{1}{2}\log\left(\overleftarrow{\mathrm{SNR}}_2+\overrightarrow{\mathrm{SNR}}_1+1\right)\\ \leqslant&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\overrightarrow{\mathrm{SNR}}_1+1\right)\\ \leqslant&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\right)+\frac{1}{2}.\end{aligned}\tag{66}$$

Then, the following is the gap for the case 1.1.4.2 that involves the parameters (56a) and (66):

$$\begin{aligned}\delta_{R_{1,5}}=&\xi_{1,3}-\zeta_{1,1}\\ \leqslant&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\right)+\frac{1}{2}-\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\right)+\frac{1}{2}\\ =&1.\end{aligned}$$

Then, the maximum gap for the single-rate bound is obtained as follows

$$\begin{aligned}\delta_{R_1}=&\max\left(\delta_{R_{1,1}},\delta_{R_{1,2}},\delta_{R_{1,3}},\delta_{R_{1,4}},\delta_{R_{1,5}}\right)\\ \leqslant&\frac{3}{2}.\end{aligned}\tag{67}$$

2. <u>Calculation of δ_{2R}</u>. From (51d) it follows that:

$$\begin{aligned}\delta_{2R}=&\min(\kappa_4,\kappa_5,\kappa_6)-\min(a_2+b_4,a_4+b_2,a_1+a_4+b_1+b_7,a_1+a_5+b_1+b_5\\ &a_1+a_7+b_1+b_4,\vartheta_1+\vartheta_2)\\ \overset{(f)}{=}&\min(\kappa_4,\kappa_5,\kappa_6)-\min(a_2+b_4,a_4+b_2,a_4+b_7,a_5+b_5,a_7+b_4,\vartheta_1+\vartheta_2)\\ \overset{(g)}{=}&\min(\kappa_4,\kappa_5,\kappa_6)-\min(a_2+b_4,a_4+b_2,a_5+b_5,\vartheta_1+\vartheta_2),\end{aligned}\tag{68}$$

where,

(f) follows from (53);
(g) follows from the fact that $a_4 + b_2 = a_4 + b_7$ and $a_2 + b_4 = a_7 + b_4$ from (53).

Let ζ_s be defined as follows

$$\begin{aligned}\zeta_s=&\min\left(a_2+b_4, a_4+b_2, a_5+b_5, \vartheta_1+\vartheta_2\right)\\ =&\min\left(\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{12}+1\right)+\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_2}{\mathrm{INR}_{12}}+2\right)-1,\right.\\ &\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}+2\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_2+\mathrm{INR}_{21}+1\right)-1,\\ &\left.\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}+\mathrm{INR}_{12}+1\right)+\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_2}{\mathrm{INR}_{12}}+\mathrm{INR}_{21}+1\right)-1, \vartheta_1+\vartheta_2\right). \quad (69)\end{aligned}$$

Denote by $\zeta_{s,1}$, $\zeta_{s,2}$, $\zeta_{s,3}$ and $\zeta_{s,4}$ the following parameters:

$$\zeta_{s,1}=\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{12}+1\right)+\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_2}{\mathrm{INR}_{12}}+2\right)-1, \quad (70a)$$

$$\zeta_{s,2}=\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}+2\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_2+\mathrm{INR}_{21}+1\right)-1, \quad (70b)$$

$$\zeta_{s,3}=\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}+\mathrm{INR}_{12}+1\right)+\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_2}{\mathrm{INR}_{12}}+\mathrm{INR}_{21}+1\right)-1, \quad (70c)$$

$$\zeta_{s,4}=\vartheta_1+\vartheta_2. \quad (70d)$$

Let ξ_s be defined as follows

$$\begin{aligned}\xi_s=&\min\left(\kappa_4, \kappa_5, \kappa_6\right),\\ =&\min\left(\frac{1}{2}\log\left(1+\frac{\left(1-\rho^2\right)\overrightarrow{\mathrm{SNR}}_1}{1+\left(1-\rho^2\right)\mathrm{INR}_{21}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_2+\right.\right.\\ &\left.2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_2\mathrm{INR}_{21}}+\mathrm{INR}_{21}+1\right), \frac{1}{2}\log\left(1+\frac{\left(1-\rho^2\right)\overrightarrow{\mathrm{SNR}}_2}{1+\left(1-\rho^2\right)\mathrm{INR}_{12}}\right)+\\ &\left.\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_1\mathrm{INR}_{12}}+\mathrm{INR}_{12}+1\right), \kappa_6\right), \quad (71)\end{aligned}$$

with $0 \leqslant \rho \leqslant 1$. Only the bounds corresponding to κ_4 and κ_5, which are coming from the case of perfect channel-output feedback [5], are considered in the calculation of the gap on the sum rate.

Denote by $\xi_{s,1}$, $\xi_{s,2}$ and $\xi_{s,3}$ the following parameters:

$$\xi_{s,1}=\frac{1}{2}\log\left(1+\frac{\left(1-\rho^2\right)\overrightarrow{\mathrm{SNR}}_1}{1+\left(1-\rho^2\right)\mathrm{INR}_{21}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_2+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_2\mathrm{INR}_{21}}+\mathrm{INR}_{21}+1\right), \tag{72a}$$

$$\xi_{s,2}=\frac{1}{2}\log\left(1+\frac{\left(1-\rho^2\right)\overrightarrow{\mathrm{SNR}}_2}{1+\left(1-\rho^2\right)\mathrm{INR}_{12}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_1\mathrm{INR}_{12}}+\mathrm{INR}_{12}+1\right), \tag{72b}$$

$$\xi_{s,3}=\kappa_6. \tag{72c}$$

The parameters in (72) can be upper bounded as follows

$$\begin{aligned}
\xi_{s,1} &= \frac{1}{2}\log\left(1+\frac{\left(1-\rho^2\right)\overrightarrow{\mathrm{SNR}}_1}{1+\left(1-\rho^2\right)\mathrm{INR}_{21}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_2+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_2\mathrm{INR}_{21}}+\mathrm{INR}_{21}+1\right),\\
&\leqslant \frac{1}{2}\log\left(1+\frac{\left(1-\rho^2\right)\overrightarrow{\mathrm{SNR}}_1}{\left(1-\rho^2\right)\mathrm{INR}_{21}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_2+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_2\mathrm{INR}_{21}}+\mathrm{INR}_{21}+1\right),\\
&= \frac{1}{2}\log\left(1+\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_2+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_2\mathrm{INR}_{21}}+\mathrm{INR}_{21}+1\right),\\
&\overset{(h)}{\leqslant} \frac{1}{2}\log\left(1+\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}\right)+\frac{1}{2}\log\left(2\overrightarrow{\mathrm{SNR}}_2+2\mathrm{INR}_{21}+1\right),\\
&\leqslant \frac{1}{2}\log\left(1+\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_2+\mathrm{INR}_{21}+1\right)+\frac{1}{2},\\
&\leqslant \frac{1}{2}\log\left(2+\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_2+\mathrm{INR}_{21}+1\right)+\frac{1}{2},
\end{aligned} \tag{73a}$$

$$\begin{aligned}
\xi_{s,2} &= \frac{1}{2}\log\left(1+\frac{\left(1-\rho^2\right)\overrightarrow{\mathrm{SNR}}_2}{1+\left(1-\rho^2\right)\mathrm{INR}_{12}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_1\mathrm{INR}_{12}}+\mathrm{INR}_{12}+1\right),\\
&\leqslant \frac{1}{2}\log\left(1+\frac{\left(1-\rho^2\right)\overrightarrow{\mathrm{SNR}}_2}{\left(1-\rho^2\right)\mathrm{INR}_{12}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_1\mathrm{INR}_{12}}+\mathrm{INR}_{12}+1\right),\\
&= \frac{1}{2}\log\left(1+\frac{\overrightarrow{\mathrm{SNR}}_2}{\mathrm{INR}_{12}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+2\rho\sqrt{\overrightarrow{\mathrm{SNR}}_1\mathrm{INR}_{12}}+\mathrm{INR}_{12}+1\right),\\
&\overset{(i)}{\leqslant} \frac{1}{2}\log\left(1+\frac{\overrightarrow{\mathrm{SNR}}_2}{\mathrm{INR}_{12}}\right)+\frac{1}{2}\log\left(2\overrightarrow{\mathrm{SNR}}_1+2\mathrm{INR}_{12}+1\right),\\
&\leqslant \frac{1}{2}\log\left(1+\frac{\overrightarrow{\mathrm{SNR}}_2}{\mathrm{INR}_{12}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{12}+1\right)+\frac{1}{2},\\
&\leqslant \frac{1}{2}\log\left(2+\frac{\overrightarrow{\mathrm{SNR}}_2}{\mathrm{INR}_{12}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{12}+1\right)+\frac{1}{2},
\end{aligned} \tag{73b}$$

$$\xi_{s,3}=\kappa_6. \tag{73c}$$

where,

(h) follows from the fact that $\left(\sqrt{\overrightarrow{\mathrm{SNR}}_2}-\sqrt{\mathrm{INR}_{21}}\right)^2 \geqslant 0$;

(i) follows from the fact that $\left(\sqrt{\overrightarrow{\mathrm{SNR}}_1}-\sqrt{\mathrm{INR}_{12}}\right)^2 \geqslant 0$.

The gap that involves the parameters (70a) end (73b) is calculated as follows

$$\begin{aligned}\delta_{(R_1+R_2)_1}=&\xi_{s,2}-\zeta_{s,1}\\ \leqslant&\frac{1}{2}\log\left(2+\frac{\overrightarrow{\mathrm{SNR}}_2}{\mathrm{INR}_{12}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{12}+1\right)+\frac{1}{2}\\ &-\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{12}+1\right)-\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_2}{\mathrm{INR}_{12}}+2\right)+1\\ =&\frac{3}{2}.\end{aligned} \tag{74}$$

The gap that involves the parameters (70b) and (73a) is calculated as follows

$$\begin{aligned}\delta_{(R_1+R_2)_2}=&\xi_{s,1}-\zeta_{s,2}\\ \leqslant&\frac{1}{2}\log\left(2+\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_2+\mathrm{INR}_{21}+1\right)+\frac{1}{2}\\ &-\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_2+\mathrm{INR}_{21}+1\right)-\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}+2\right)+1\\ =&\frac{3}{2}.\end{aligned} \tag{75}$$

The gap involving the parameter (70c) is calculated considering the following 4 subcases, which correspond to the possible conditions between $\overrightarrow{\mathrm{SNR}}_1$ and INR_{21}, and between $\overrightarrow{\mathrm{SNR}}_2$ and INR_{12}, as follows

<u>Case 1.1.1</u>: $\overrightarrow{\mathrm{SNR}}_1 \geqslant \mathrm{INR}_{21} \wedge \overrightarrow{\mathrm{SNR}}_2 \geqslant \mathrm{INR}_{12}$.
<u>Case 1.1.2</u>: $\overrightarrow{\mathrm{SNR}}_1 \geqslant \mathrm{INR}_{21} \wedge \overrightarrow{\mathrm{SNR}}_2 < \mathrm{INR}_{12}$.
<u>Case 1.1.3</u>: $\overrightarrow{\mathrm{SNR}}_1 < \mathrm{INR}_{21} \wedge \overrightarrow{\mathrm{SNR}}_2 \geqslant \mathrm{INR}_{12}$.
<u>Case 1.1.4</u>: $\overrightarrow{\mathrm{SNR}}_1 < \mathrm{INR}_{21} \wedge \overrightarrow{\mathrm{SNR}}_2 < \mathrm{INR}_{12}$.

The case 1.1.1 is not possible by contradiction based on (52).
In case 1.1.2 with (52), the parameter (70c) can be lower bounded as follows

$$\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}+\mathrm{INR}_{12}+1\right)+\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_2}{\mathrm{INR}_{12}}+\mathrm{INR}_{21}+1\right)-1 \geqslant \frac{1}{2}\log\left(\mathrm{INR}_{12}+1\right)-1. \tag{76}$$

From (76), given that $\vartheta_1+\vartheta_2$ is by definition an achievable sum rate, then it can be defined as follows

$$\zeta_{s,4}=\vartheta_1+\vartheta_2=\frac{1}{2}\log\left(\mathrm{INR}_{12}+1\right)-1. \tag{77}$$

The parameter (73b) can be upper bounded as follows

$$\begin{aligned}\xi_{s,2} \leqslant & \frac{1}{2}\log\left(2+\frac{\overrightarrow{\mathrm{SNR}}_2}{\mathrm{INR}_{12}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{12}+1\right)+\frac{1}{2}\\ \leqslant & \frac{1}{2}\log(2+1)+\frac{1}{2}\log(\mathrm{INR}_{12}+\mathrm{INR}_{12}+1)+\frac{1}{2}\\ \leqslant & \frac{1}{2}\log(3)+\frac{1}{2}\log(\mathrm{INR}_{12}+1)+1. \end{aligned} \tag{78}$$

Then, the following is the gap for the case 1.1.2 that involves the parameters (61) and (78):

$$\begin{aligned}\delta_{(R_1+R_2)_3} = & \xi_{s,2}-\zeta_{s,4}\\ \leqslant & \frac{1}{2}\log(3)+\frac{1}{2}\log(\mathrm{INR}_{12}+1)+1-\frac{1}{2}\log(\mathrm{INR}_{12}+1)+1\\ = & 2+\frac{1}{2}\log(3). \end{aligned} \tag{79}$$

In case 1.1.3 with (52), the parameter (70c) can be lower bounded as follows

$$\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}+\mathrm{INR}_{12}+1\right)+\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_2}{\mathrm{INR}_{12}}+\mathrm{INR}_{21}+1\right)-1 \geqslant \frac{1}{2}\log(\mathrm{INR}_{21}+1)-1. \tag{80}$$

From (80), given that $\vartheta_1+\vartheta_2$ is by definition an achievable sum rate, then it can be redefined as follows

$$\zeta_{s,4}=\vartheta_1+\vartheta_2=\frac{1}{2}\log(\mathrm{INR}_{21}+1)-1. \tag{81}$$

The parameter (73a) can be upper bounded as follows

$$\begin{aligned}\xi_{s,1} \leqslant & \frac{1}{2}\log\left(2+\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_2+\mathrm{INR}_{21}+1\right)+\frac{1}{2}\\ \leqslant & \frac{1}{2}\log(2+1)+\frac{1}{2}\log(\mathrm{INR}_{21}+\mathrm{INR}_{21}+1)+\frac{1}{2}\\ \leqslant & \frac{1}{2}\log(3)+\frac{1}{2}\log(\mathrm{INR}_{21}+1)+1. \end{aligned} \tag{82}$$

Then, the following is the gap for the case 1.1.3. that involves the parameters (81) and (82):

$$\begin{aligned}\delta_{(R_1+R_2)_4} = & \xi_{s,3}-\zeta_{s,4}\\ \leqslant & \frac{1}{2}\log(3)+\frac{1}{2}\log(\mathrm{INR}_{21}+1)+1-\frac{1}{2}\log(\mathrm{INR}_{21}+1)+1\\ = & 2+\frac{1}{2}\log(3). \end{aligned} \tag{83}$$

In case 1.1.4 with (52), the parameter (73b) can be upper bounded as follows

$$\begin{aligned}\xi_{s,2}\leqslant&\frac{1}{2}\log\left(2+\frac{\overrightarrow{\mathrm{SNR}}_2}{\mathrm{INR}_{12}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{12}+1\right)+\frac{1}{2},\\ \leqslant&\frac{1}{2}\log\left(2+1\right)+\frac{1}{2}\log\left(\mathrm{INR}_{12}+\mathrm{INR}_{12}+1\right)+\frac{1}{2}\\ \leqslant&\frac{1}{2}\log\left(3\right)+\frac{1}{2}\log\left(\mathrm{INR}_{12}+1\right)+1. \end{aligned}\tag{84}$$

Then, the following is the gap for the case 1.1.4 that involves the parameters (77) and (84):

$$\begin{aligned}\delta_{(R_1+R_2)_5}=&\xi_{s,2}-\zeta_{s,3}\\ \leqslant&\frac{1}{2}\log\left(3\right)+\frac{1}{2}\log\left(\mathrm{INR}_{12}+1\right)+1-\frac{1}{2}\log\left(\mathrm{INR}_{12}+1\right)+1\\ =&2+\frac{1}{2}\log\left(3\right). \end{aligned}\tag{85}$$

Then, the maximum gap for the sum-rate bound is obtained as follows

$$\begin{aligned}\delta_{(R_1+R_2)}=&\max\left(\delta_{(R_1+R_2)_1},\delta_{(R_1+R_2)_2},\delta_{(R_1+R_2)_3},\delta_{(R_1+R_2)_4},\delta_{(R_1+R_2)_5}\right)\\ \leqslant&2+\frac{1}{2}\log\left(3\right). \end{aligned}\tag{86}$$

3. Calculation of δ_{3R_1}

From (51e) it follows that

$$\begin{aligned}\delta_{3R_1}=&\kappa_{7,1}-\min(a_2+a_4+b_1+b_7,a_1+a_4+a_7+2b_1+b_5,a_2+a_4+b_1+b_5,2\vartheta_1+\vartheta_2)\\ \overset{(j)}{=}&\kappa_{7,1}-\min(a_2+a_4+b_7,a_4+a_7+b_5,a_2+a_4+b_5,2\vartheta_1+\vartheta_2)\\ \overset{(k)}{=}&\kappa_{7,1}-\min(a_4+a_7+b_5,a_2+a_4+b_5,2\vartheta_1+\vartheta_2)\\ \overset{(l)}{=}&\kappa_{7,1}-\min(a_4+a_7+b_5,2\vartheta_1+\vartheta_2). \end{aligned}\tag{87}$$

where,
(j) follows from (53);
(k) follows from the fact that $a_2+a_4+b_5\leqslant a_2+a_4+b_7$ given that $b_5\leqslant b_7$ from (53);
(l) follows from the fact that $a_4+a_7+b_5=a_2+a_4+b_5$ given that $a_2=a_7$ from (53).

Let ζ_w be defined as follows

$$\begin{aligned}\zeta_w=&\min(a_4+a_7+b_5,2\vartheta_1+\vartheta_2)\\ =&\min\left(\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}+2\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{12}+1\right)+\right.\\ &\left.\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_2}{\mathrm{INR}_{12}}+\mathrm{INR}_{21}+1\right)-\frac{3}{2},2\vartheta_1+\vartheta_2\right). \end{aligned}\tag{88}$$

Denote by $\zeta_{w,1}$ and $\zeta_{w,2}$ the following parameters:

$$\begin{aligned}\zeta_{w,1}=&\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}+2\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{12}+1\right)+\\ &\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_2}{\mathrm{INR}_{12}}+\mathrm{INR}_{21}+1\right)-\frac{3}{2}, \end{aligned}\tag{89a}$$

$$\zeta_{w,2}=2\vartheta_1+\vartheta_2. \tag{89b}$$

The parameter (89a) can be lower bounded as follows

$$\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}+2\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{12}+1\right)+\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_2}{\mathrm{INR}_{12}}+\mathrm{INR}_{21}+1\right)-\frac{3}{2}\geqslant$$
$$\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}+2\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{12}+1\right)+\frac{1}{2}\log\left(\mathrm{INR}_{21}+1\right)-\frac{3}{2}. \tag{90}$$

From (90), given that $2\vartheta_1+\vartheta_2$ is by definition an achievable weighted sum rate, then it can be redefined as follows

$$\zeta_{w,2}=2\vartheta_1+\vartheta_2=\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}+2\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{12}+1\right)+\frac{1}{2}\log\left(\mathrm{INR}_{21}+1\right)-\frac{3}{2}. \tag{91}$$

Denote by ξ_w the following parameter

$$\xi_w=\kappa_{7,1}. \tag{92}$$

The parameter (92) can be redefined as the sum of the parameters (59a) and (73a) as follows

$$\begin{aligned}\xi_w=&\xi_{1,1}+\xi_{s,1}\\ \leqslant&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{12}+1\right)+\frac{1}{2}\log\left(2+\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_2+\mathrm{INR}_{21}+1\right)+1\end{aligned} \tag{93}$$

The parameter in (93) can be upper bounded as follows

$$\begin{aligned}\xi_w\leqslant&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{12}+1\right)+\frac{1}{2}\log\left(2+\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}\right)+\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_2+\mathrm{INR}_{21}+1\right)+1\\ \leqslant&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{12}+1\right)+\frac{1}{2}\log\left(2+\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}\right)+\frac{1}{2}\log\left(\mathrm{INR}_{21}+\mathrm{INR}_{21}+1\right)+1\\ \leqslant&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{12}+1\right)+\frac{1}{2}\log\left(2+\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}\right)+\frac{1}{2}\log\left(\mathrm{INR}_{21}+1\right)+\frac{3}{2}.\end{aligned}$$

Then, the following is the gap for the parameters (91) and (94):

$$\begin{aligned}\delta_{3R_1}=&\xi_w-\zeta_{w,2}\\ \leqslant&\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{12}+1\right)+\frac{1}{2}\log\left(2+\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}\right)+\frac{1}{2}\log\left(\mathrm{INR}_{21}+1\right)+\frac{3}{2}\\ &-\frac{1}{2}\log\left(\frac{\overrightarrow{\mathrm{SNR}}_1}{\mathrm{INR}_{21}}+2\right)-\frac{1}{2}\log\left(\overrightarrow{\mathrm{SNR}}_1+\mathrm{INR}_{12}+1\right)-\frac{1}{2}\log\left(\mathrm{INR}_{21}+1\right)+\frac{3}{2}\\ =&3.\end{aligned} \tag{94}$$

Therefore, the gap between the converse region $\overline{\mathcal{C}}_{\text{GIC}-\text{NOF}}$ and the achievable region $\underline{\mathcal{C}}_{\text{GIC}-\text{NOF}}$ in case 1.1 is:

$$\begin{aligned}\delta_1 &= \max\left(\delta_{R_1}, \frac{\delta_{(R_1+R_2)}}{2}, \frac{\delta_{3R_1}}{3}\right)\\ &\leqslant \max\left(\frac{3}{2}, \frac{1}{2}\left(2+\frac{1}{2}\log(3)\right), \frac{1}{3}3\right)\\ &= \frac{3}{2}\frac{\text{bits}}{\text{channel use}}.\end{aligned} \tag{95}$$

This completes the calculation of the gap in case 1.1. The same procedure can be applied to all the other cases, and it follows that $\delta \leqslant 4.4$ bits per channel use.

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