

The Volterra operator is finitely strictly singular from L^1 to L^∞

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Abstract. *We show that the Volterra operator viewed from $L^1([0, 1])$ to $C([0, 1])$ is finitely strictly singular. Actually we estimate the Bernstein numbers and show that their value is $1/(2n - 1)$ in the case of real valued functions. The same ideas apply to the summation operator.*

Key-words. Volterra operator, Bernstein numbers, finitely strictly singular, summation operator.

1 Introduction

In this paper, the spaces $C([0, 1])$ is the space of continuous functions on $[0, 1]$, equipped with the uniform norm. The spaces $L^p([0, 1])$ (where $p \geq 1$) are the usual Lebesgue spaces of integrable functions over $[0, 1]$ relatively to the Lebesgue measure. We shall work with real valued functions. Nevertheless, the conclusion of Theorem 2.1 remains valid in the complex case (see remarks at the end of section 1).

We are interested in the Volterra operator V :

$$\begin{array}{ccc} L^1([0, 1]) & \longrightarrow & C([0, 1]) \\ f & \longmapsto & V(f) \end{array}$$

with

$$V(f)(x) = \int_0^x f(t) dt.$$

The operator V is clearly well defined and bounded with norm 1.

As soon as the Volterra operator is viewed from $L^p([0, 1])$ to $C([0, 1])$ with $p \in (1, +\infty]$, it is very easy to see (with Ascoli's theorem) that we have a compact operator (the same from $L^1([0, 1])$ to $L^p([0, 1])$ with any finite $p \geq 1$). This obviously implies that it is compact when it is viewed from $L^p([0, 1])$ to itself, for any $p \in [1, +\infty]$. There are many papers interested in estimating different kind of approximation numbers, especially Bernstein numbers (see the definition below) for the Volterra operator (or variants like

Hardy type operators or Sobolev embeddings) viewed from L^p to L^q with $1 < p, q < +\infty$ (see for example [BMN], [BG], [EL1], [EL2]).

In our extreme case $L^1 - C$, it is easy to see that V is not compact, not even weakly compact: consider the normalized sequence $\{(n+1)x^n\}$ (for $n \in \mathbb{N}$) in $L^1([0, 1])$, its range (under V) is the sequence $\{x^{n+1}\}$ and admits no weakly convergent subsequence in $C([0, 1])$. Moreover, we point out that V acts as an isometry on positive functions.

There is another notion weakening compactness for operators, namely the notion of completely continuous operator: we recall that an operator is completely continuous if it maps weakly convergent sequences to norm convergent sequences. We can notice that

Proposition 1.1 *The Volterra operator V is completely continuous.*

Proof. Let (f_n) be a sequence in the unit ball of $L^1([0, 1])$, weakly convergent to 0. It is uniformly integrable: given $\varepsilon > 0$, there exists some positive α such that $\int_A |f_n| dt \leq \varepsilon$ for every $n \geq 1$, as soon as the Lebesgue measure of A is less than α . There exists some $x_n \in [0, 1]$ such that $|V(f_n)(x_n)| = \|V(f_n)\|_\infty$.

For some $\varepsilon_0 > 0$, there exists a subsequence of $\{f_n\}$ (again denoted $\{f_n\}$) such that $\|V(f_n)\|_\infty > \varepsilon_0$. By compactness of $[0, 1]$, there exist some $x \in [0, 1]$ and a subsequence of $\{x_n\}$ (again denoted $\{x_n\}$) converging to x .

Choosing $\varepsilon < \varepsilon_0/2$ and n large enough so that $|x_n - x| < \alpha$, we get

$$\varepsilon_0 \leq |V(f_n)(x_n)| \leq \left| \int_0^x f_n(t) dt \right| + \int_x^{x_n} |f_n(t)| dt \leq \left| \int_0^1 f_n(t) \mathbb{I}_{[0, x]} dt \right| + \varepsilon.$$

But, since (f_n) weakly converges to 0, we have $\int_0^1 f_n(t) \mathbb{I}_{[0, x]} dt \rightarrow 0$ and we get a contradiction for n large enough. ■

Another weak form of compactness is the strict singularity, or even the finite strict singularity: let us recall the definitions.

Definition 1.2 *An operator T from a Banach space X to a Banach space Y is strictly singular if it never induces an isomorphism on an infinite dimensional (closed) subspace of X : that is for every $\varepsilon > 0$ and every infinite dimensional subspace E of X , there exists v in the unit sphere of E such that $\|T(v)\| \leq \varepsilon$.*

This notion is now very classical and widely studied: see [LT] (p.75) or [LiQ] for instance to know more on this notion.

Definition 1.3 *An operator T from a Banach space X to a Banach space Y is finitely strictly singular if: for every $\varepsilon > 0$, there exists $N_\varepsilon \geq 1$ such that for every subspace E of X with dimension greater than N_ε , there exists v in the unit sphere of E such that $\|T(v)\| \leq \varepsilon$.*

This property is sometimes also denoted as *superstrict singularity* in the literature. This can be also reformulated in terms of Bernstein approximation numbers (see [PI] for instance). Recall that the n^{th} Bernstein number of an operator T is

$$b_n(T) = \sup_{\substack{E \subset X \\ \dim(E)=n}} \inf_{\substack{v \in E \\ \|v\|=1}} \|T(v)\|.$$

Hence, with this terminology, the operator T is finitely strictly singular if and only if $(b_n(T))_{n \geq 1}$ belongs to the space c_0 of null sequences.

This notion appears in the late sixties. For instance, in a paper of V. Milman [Mi], it is proved that the identity from ℓ^p to ℓ^q ($p < q$) is finitely strictly singular. See [CFPPT], [PI], [FHR], [L], [LR] for recent results on this notion. It is well known that

$$\text{compactness} \implies \text{finite strict singularity} \implies \text{strict singularity}$$

and that the reverse implications are not true. Moreover, complete continuity is not comparable to finite strict singularity in general.

We shall show in the next section (see Th.2.1) that the Volterra operator V is finitely strictly singular (hence strictly singular). Actually, in the real case, we can prove that its n^{th} Bernstein number is exactly $\frac{1}{2n-1}$ (see Th. 2.2). Of course the (usual) approximation numbers (the distance to the operators with fixed finite rank) are bounded below since V is not compact.

In section 3, we state and prove the corresponding results in the discrete framework, for the summation operator acting from ℓ^1 to c .

2 Bernstein numbers of V

The main theorem of this section is

Theorem 2.1 *The Volterra operator V is finitely strictly singular.*

The proof is an immediate consequence of the following key theorem.

Theorem 2.2 *We have*

$$b_n(V) = \frac{1}{2n-1}.$$

The proof of this theorem relies on our Lemma 2.4 below, which makes a crucial use of the so-called zigzag lemma:

Lemma 2.3 (*zigzag lemma, see [CFPPT] Cor.2*):

Let X be a subspace of c_0 (or \mathbb{R}^N), viewed as a real space, equipped with the usual infinity norm. We assume that X has dimension $d \geq 1$. Then there exists a zigzag of length d , i.e. some $v \in X$, with norm 1 and indexes $j_1 < \dots < j_d$ such that $v_{j_i} = (-1)^i$ for every $i \in \{1, \dots, d\}$.

Lemma 2.4 Let $\varepsilon_0 > 0$, $d \geq 1$ and E a subspace of $L^1([0, 1])$ with dimension d verifying

$$\forall f \in E, \quad \|V(f)\|_\infty \geq \varepsilon_0 \|f\|_1$$

then

$$d \leq \frac{1}{2} \left(1 + \frac{1}{\varepsilon_0} \right).$$

Proof. We fix $d \geq 1$ and $\varepsilon \in (0, 1)$.

The unit ball of E is compact hence equi-integrable: there exists some $\delta > 0$ such that, given any $x < y$ in $[0, 1]$ with $|x - y| \leq \delta$, we have

$$\sup_{\substack{f \in E \\ \|f\|_1 \leq 1}} \int_x^y |f(t)| dt \leq \varepsilon \varepsilon_0$$

Now, we consider a decreasing sequence $(x_n)_{n \in \mathbb{N}}$ in $(0, 1]$ such that $x_0 = 1$, $x_n - x_{n+1} \leq \delta$ and $x_n \rightarrow 0$. Point out that for every $F \in V(E)$, we have $F(x_n) \rightarrow F(0) = 0$.

For every $F \in V(E)$, there exists some $f \in E$ and $c \in (0, 1]$ such that $F = V(f)$ and $|F(c)| = \|F\|_\infty = \|V(f)\|_\infty$. For some $n \in \mathbb{N}$, we have $x_{n+1} < c \leq x_n$ so

$$\|F\|_\infty - |F(x_n)| = |F(c)| - |F(x_n)| \leq \int_c^{x_n} |f(t)| dt \leq \varepsilon \varepsilon_0 \|f\|_1 \leq \varepsilon \|F\|_\infty.$$

Now we can define the map

$$\Phi : \begin{cases} V(E) & \longrightarrow & c_0 \\ F & \longmapsto & (F(x_n))_{n \in \mathbb{N}} \end{cases}$$

which satisfies for every $F \in V(E)$:

$$(1 - \varepsilon) \|F\|_\infty \leq \sup_{k \in \mathbb{N}} |F(x_k)| = \|\Phi(F)\|_\infty \leq \|F\|_\infty.$$

Applying the zigzag lemma to the subspace $\Phi(V(E))$, we get some $u \in c_0$ with norm 1 and integers $j_1 < \dots < j_d$ such that $u_{j_k} = (-1)^k$. There exists $f \in E$ with $\Phi(V(f)) = u$. Moreover

$$\varepsilon_0 \|f\|_1 \leq \|V(f)\|_\infty \leq \frac{1}{1 - \varepsilon}.$$

At last,

$$\begin{aligned} 1 + 2(d - 1) &= |F(0) - F(x_{j_d})| + \sum_{k=1}^{d-1} |F(x_{j_{k+1}}) - F(x_{j_k})| \\ &\leq \int_0^{x_{j_d}} |f(t)| dt + \sum_{k=1}^{d-1} \int_{x_{j_{k+1}}}^{x_{j_k}} |f(t)| dt \\ &\leq \int_0^1 |f(t)| dt \leq \frac{1}{\varepsilon_0(1 - \varepsilon)}. \end{aligned}$$

Since ε is arbitrary, we get the conclusion. ■

Now we finish the proof of the key theorem.

Proof of theorem 2.2. The upper estimate: from the definition of Bernstein numbers, we know that, for every $\varepsilon_0 \in (0, b_n(V))$, there exists some $E \subset L^1([0, 1])$ with dimension n such that

$$\forall f \in E, \quad \|V(f)\|_\infty \geq \varepsilon_0 \|f\|_1.$$

By Lemma 2.4, we get that $\varepsilon_0 \leq \frac{1}{2n-1}$.

For the lower estimate, consider the space E spanned by $\{\varphi_1, \dots, \varphi_n\}$, where the φ_k 's are the normalized characteristic functions of the intervals (a_{k-1}, a_k) with $a_j = \frac{j}{n}$ for $0 \leq j \leq n$. Point out that E has dimension n and is isometric to ℓ_n^1 as a subspace of $L^1([0, 1])$:

$$\forall \alpha_1, \dots, \alpha_n \in \mathbb{R}, \quad \left\| \sum_{j=1}^n \alpha_j \varphi_j \right\|_1 = \sum_{j=1}^n |\alpha_j|.$$

Now for every $f \in E \subset L^1([0, 1])$, we can write $f = \sum_{j=1}^n \alpha_j \varphi_j$.

We point out that $\alpha_1 = V(f)(a_1)$ and for every $k \in \{2, \dots, n\}$, we have

$$\alpha_k = \int_{a_{k-1}}^{a_k} f(t) dt = V(f)(a_k) - V(f)(a_{k-1}).$$

We get $|\alpha_1| \leq \|V(f)\|_\infty$ and $|\alpha_k| \leq 2\|V(f)\|_\infty$ when $k \geq 2$. So that

$$\|f\|_1 = \sum_{j=1}^n |\alpha_j| \leq (2n-1)\|V(f)\|_\infty.$$

Hence

$$b_n(V) \geq \frac{1}{2n-1}. \quad \blacksquare$$

Remarks.

1. For the lower estimate, another natural idea is to consider the space E spanned by $\{1, \dots, x^{n-1}\}$ and the Bernstein-Markov inequality stating that for every polynomials Q of degree at most d and every $t \in (-1, 1)$, we have

$$|Q'(t)| \leq \frac{d}{\sqrt{1-t^2}} \sup_{|t| \leq 1} |Q(t)|.$$

Obviously, E has dimension n . Now for every normalized $f \in E$, we apply the Bernstein-Markov inequality to $P = V(f)$ which is a

polynomial with degree less than n , and we integrate over $(0, 1)$ (after a change of scale):

$$1 = \|f\|_1 = \|P'\|_1 \leq \pi n \|P\|_\infty = \pi n \|V(f)\|_\infty$$

hence this gives $b_n(V) \geq \frac{1}{\pi n}$, which is a slightly worst estimate than the one of Th. 2.2.

2. We worked until now with real valued functions, because of the zigzag lemma. Nevertheless, Th. 2.2 remains mainly true in the complex case: $b_n(V) \approx \frac{1}{n}$.

Clearly, the same lower estimate holds with the same proof. We are going to show that a similar (a bit less sharp) upper estimate remains true working with complex valued functions: we claim that $b_{2n+1}(V) \leq \frac{\sqrt{2}}{2n+1}$. Indeed, let us mention only the few changes in the previous argument: in the definition of Φ , we could have replaced c_0 by \mathbb{R}^N where N is large enough (compared to $1/\delta$). Then, in the complex case, just map $F = f + ig$ to $(f(x_1), \dots, f(x_N), g(x_1), \dots, g(x_N)) \in \mathbb{R}^{2N}$. The point is that we “lose” a $\sqrt{2}$ since

$$|F(x_j)| \leq \sqrt{2} \max \{|f(x_1)|, \dots, |f(x_N)|, |g(x_1)|, \dots, |g(x_N)|\}.$$

The zigzag lemma applies and from a zigzag of length $2n+1$, we can extract a zigzag of length $n+1$ concerning either the N first coordinates or the N last. The estimation follows. Hence Theorem 2.1 is also still true in the complex case.

3. Let us mention that Lemma 2.4 gives an immediate answer to problem 11, raised by Mokobodzki and Rogalski in [MR]: for any $K > 0$,

$$\sup \{ \dim(E) \mid E \subset C([0, 1]), \text{ closed}; \forall f \in E, \|f'\|_1 \leq K \|f\|_\infty \} \leq \frac{K+1}{2}.$$

4. Actually, after this work was completed, G. Godefroy kindly informed us that point 3. was already known and settled by Voigt in [V]. It turns out too that the zigzag lemma was also already contained in [V]. Let us mention too that a vector valued version of this result is given in [G].

3 The discrete version

The same ideas apply to the discrete Volterra operator acting from the space ℓ^1 of absolutely convergent series to the space c of convergent sequences.

$$\begin{aligned}\ell^1 &\longrightarrow c \\ u &\longmapsto \sigma(u)\end{aligned}$$

with

$$\sigma(u)_n = \sum_{k=0}^n u_k.$$

This operator is obviously bounded with norm 1, completely continuous (thanks to the Schur property for ℓ^1) and not compact. It is not even weakly compact: take $c_m \in (0, 1)$ with $c_m \rightarrow 1$ and the sequence of vectors $v^{(m)}$ defined by $v_n^{(m)} = (1 - c_m)c_m^n$. We have $\|v^{(m)}\|_1 = 1$ and $\sigma(v^{(m)})_n = 1 - c_m^{n+1}$ for every $n \geq 1$. Hence if a subsequence weakly converge to some $y \in c$ then $y = 0$. Testing now the functional $u \rightarrow \lim(u)$ which belongs to c^* , we have $1 = \lim(\sigma(v^{(m)})) \rightarrow \lim(y) = 0$. The contradiction follows.

Nevertheless like in the continuous case, we have

Theorem 3.1 *The discrete Volterra operator σ is finitely strictly singular.*

This immediately follows from

Theorem 3.2 *We have the following estimates for every $n \geq 1$*

- *In the real case, we have* $b_n(\sigma) = \frac{1}{2n-1}$.
- *In the complex case, we have* $\frac{1}{2n-1} \leq b_n(\sigma) \leq \frac{\sqrt{2}}{2\tilde{n}-1}$
where \tilde{n} stands for the integer part of $(n+1)/2$.

Proof. Since the proof follows the lines of the ones in section 1, we do not give all the details.

Considering the space ℓ_n^1 spanned by n first vectors of the canonical basis (e_0, \dots, e_{n-1}) of ℓ^1 , we get the lower estimates (for both the real and the complex case):

$$\|x\|_1 = |\sigma(x)_0| + \sum_{k=1}^{n-1} |\sigma(x)_k - \sigma(x)_{k-1}|$$

so that $\|x\|_1 \leq (2n-1)\|\sigma(x)\|_\infty$ and $b_n(\sigma) \geq \frac{1}{2n-1}$.

For the upper estimate. We fix $\varepsilon \in (0, 1)$, take $\varepsilon_0 \in (0, b_n(\sigma))$, there exists a subspace $E \subset \ell^1$ with dimension n such that

$$\forall x \in E, \quad \|\sigma(x)\|_\infty \geq \varepsilon_0 \|x\|_1.$$

By compactness in finite dimension, there exists some $r \geq n$ such that, for every x in the unit ball of E ,

$$\sum_{k \geq r} |x_k| \leq \varepsilon \cdot \varepsilon_0$$

For every $y \in \sigma(E)$, we easily check that $(1 - \varepsilon)\|y\|_\infty \leq \max_{k \leq r} |y_k|$.

Now we specify the two cases

In the real case.

The map

$$\Phi : \begin{cases} \sigma(E) & \longrightarrow \mathbb{R}^{r+1} \\ y & \longmapsto (y_n)_{0 \leq n \leq r} \end{cases}$$

satisfies for every $y \in \sigma(E)$: $(1 - \varepsilon)\|y\|_\infty \leq \|\Phi(y)\|_\infty \leq \|y\|_\infty$.

Applying the zigzag lemma to the subspace $\Phi(\sigma(E))$, we get some $x \in E$ with $\varepsilon_0\|x\|_1 \leq \|\sigma(x)\|_\infty \leq \frac{1}{1 - \varepsilon}$ and integers $j_1 < \dots < j_n$ such that $(\sigma(x))_{j_k} = (-1)^k$.

We can now estimate

$$1 + 2(n - 1) = |(\sigma(x))_{j_1}| + \sum_{k=1}^{n-1} |(\sigma(x))_{j_{k+1}} - (\sigma(x))_{j_k}| \leq \|x\|_1 \leq \frac{1}{\varepsilon_0(1 - \varepsilon)}.$$

we get the conclusion.

In the complex case.

In the sequel, we write $y = a + ib$ where a and b are real valued and define

$$\Psi : \begin{cases} \sigma(E) & \longrightarrow \mathbb{R}^{2(r+1)} \\ y & \longmapsto (a_0, \dots, a_r, b_0, \dots, b_r) \end{cases}$$

For every $y \in \sigma(E)$: $(1 - \varepsilon)\|y\|_\infty \leq \max_{k \leq r} |y_k| \leq \sqrt{2}\|\Psi(y)\|_\infty$.

Applying the zigzag lemma to the subspace $\Psi(\sigma(E))$, we get some $x \in E$ with $\varepsilon_0\|x\|_1 \leq \|\sigma(x)\|_\infty \leq \frac{\sqrt{2}}{1 - \varepsilon}$ and integers $j_1 < \dots < j_n$ such that $(\Psi(\sigma(x)))_{j_k} = (-1)^k$. But among j_1, \dots, j_n , half of them at least are either between 0 and r or between $r + 1$ and $2r + 1$ (assume that it is the first case for the computation below). Writing \tilde{n} for the integer part of $(n + 1)/2$, we obtain

$$1 + 2(\tilde{n} - 1) = |a_{j_1}| + \sum_{k=1}^{\tilde{n}-1} |a_{j_{k+1}} - a_{j_k}| \leq \sum |Re(x)_k| \leq \|x\|_1 \leq \frac{\sqrt{2}}{\varepsilon_0(1 - \varepsilon)}.$$

We get $b_n(\sigma) \leq \frac{\sqrt{2}}{n - 1}$ when n is even and $b_n(\sigma) \leq \frac{\sqrt{2}}{n}$ when n is odd. ■

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