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HEAT KERNELS, SOLVABLE LIE GROUPS, AND THE MEAN REVERTING SABR STOCHASTIC VOLATILITY MODEL

SIYAN ZHANG, ANNA L. MAZZUCATO, AND VICTOR NISTOR

ABSTRACT. We use commutator techniques and calculations in solvable Lie groups to investigate certain evolution Partial Differential Equations (PDEs for short) that arise in the study of stochastic volatility models for pricing contingent claims on risky assets. In particular, by restricting to domains of bounded volatility, we establish the existence of the semi-groups generated by the spatial part of the operators in these models, concentrating on those arising in the so-called “SABR stochastic volatility model with mean reversion.” The main goal of this work is to approximate the solutions of the Cauchy problem for the SABR PDE with mean reversion, a parabolic problem the generator of which is denoted by L . The fundamental solution for this problem is not known in closed form. We obtain an approximate solution by performing an expansion in the so-called *volvol* or volatility of the volatility, which leads us to study a degenerate elliptic operator L_0 , corresponding to the zero-volvol case of the SABR model with mean reversion, to which the classical results do not apply. However, using Lie algebra techniques we are able to derive an exact formula for the solution operator of the PDE $\partial_t u - L_0 u = 0$. We then compare the semi-group generated by L —the existence of which does follow from standard arguments—to that generated by L_0 , thus establishing a perturbation result that is useful for numerical methods for the SABR PDE with mean reversion. In the process, we are led to study semigroups arising from both a strongly parabolic and a hyperbolic problem.

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1. INTRODUCTION

We study certain parabolic partial differential equations (PDEs for short) that arise in the study of stochastic volatility models for pricing contingent claims on risky assets. More specifically, we consider the PDE

$$(1) \quad \partial_t u - Lu := \partial_t u - \kappa(\theta - \sigma)\partial_\sigma u - \frac{\sigma^2}{2}[(\partial_x^2 u - \partial_x u) - \nu\rho\partial_x\partial_\sigma u - \frac{\nu^2}{2}\partial_\sigma^2 u] = 0$$

for the function $u(t, \sigma, x)$, where $t \geq 0$, $\sigma > 0$ and $x \in \mathbb{R}$. This equation is a forward Kolmogorov equation for the probability density function associated to a two-dimensional stochastic process for the variables σ and x . The parameter $\theta > 0$ represents the mean of the σ process, $\kappa > 0$ is a parameter measuring the strength of the mean reversion, $\nu > 0$ is the variance of the σ process, and ρ measures the correlation between the x and the σ processes. This PDE is often called the λ SABR PDE and has recently received attention in the literature due to its applications in pricing options in mathematical finance and financial applications [24, 25], where it is used as an alternative to the Black-Scholes PDE. In this context, the Green's function is called the *pricing kernel of the economy*, x represents the price of an underlying risky asset such as a stock, and σ is its volatility. Thus σ itself follows a stochastic process, hence the λ SABR model is a stochastic volatility model in which ν represents the volatility of the volatility or *volvol*. Stochastic volatility models are known to perform better in practice than the Black-Scholes model (see e.g. [24, 28, 31]).

Our method consists in the following decomposition of the operator L :

$$(2) \quad L = A + \frac{\sigma^2}{2}B + \nu L_1 + \nu^2 L_2,$$

where

$$(3) \quad \begin{aligned} A &:= \kappa(\theta - \sigma)\partial_\sigma, & B &:= \partial_x^2 - \partial_x, \\ L_1 &:= \rho\sigma^2\partial_x\partial_\sigma, & \text{and } L_2 &:= \frac{1}{2}\sigma^2\partial_\sigma^2, \end{aligned}$$

and then in studying separately these operators and their combinations, based on the commutator identities that they satisfy. We thus establish that L , A , B , and

$$(4) \quad L_0 := A + \frac{\sigma^2}{2}B$$

generate strongly continuous or c_0 semi-groups, provided that we restrict to a domain of bounded volatility $\sigma \in I := (\alpha, \beta)$, where $0 < \alpha < \theta < \beta < \infty$. We stress that L_0 is a degenerate operator, in the sense that the diffusion matrix associated to L_0 is not full rank. Therefore, the existence of the semigroup does not follow from standard arguments.

Throughout, if T is a linear operator that generates a semigroup, we shall denote such semigroup by the usual notation e^{tT} , $t \geq 0$.

We will obtain explicit formulas for the kernel of the semi-groups generated by A , B , and L_0 . While we have no explicit formulas for the kernel of e^{tL} , the solution operator of the PDE (1) of interest in applications, we are nonetheless able to estimate the difference $e^{tL}h - e^{tL_0}h$, provided $h(\sigma, x)$ has enough regularity in σ . The function h represents the initial data for the Cauchy problem associated to (1), and in the specific applications we have in mind, it is actually an analytic, or even constant, function in σ .

The semi-groups investigated in this paper will typically act on exponentially weighted Sobolev spaces. The reason for considering exponentially weighted spaces is that, in the applications of interest, the initial data h for (1) is of the form $h(\sigma, x) := |e^x - K|_+$, where $|y|_+ = (y)^+ := (y + |y|)/2$ denotes the *positive part* of the number $y \in \mathbb{R}$. This particular type of initial data arises in pricing of so-called European call options (we refer to [19, 48] for a more detailed discussion of options). The practical meaning of the initial condition h is the *payoff* of the option at maturity. Similar initial conditions are used for other types of options, such as American and Asian options. From a mathematical point of view, the form of h requires exponential weights and implies low regularity of the initial data in the x direction, but provides analytic regularity in the σ direction, which we indeed exploit in our estimate of $e^{tL}h - e^{tL_0}h$ (see Equation (5) below and the statement of one of our main results, Theorem 5.14).

The semi-groups generated by the operators A and B , and L can be obtained using classical methods, since the operator A gives rise to a transport evolution equation, whereas B and L are uniformly strongly elliptic. In particular, we show that B and L generate *analytic semi-groups*. However, as already mentioned, classical methods do not apply to L_0 , which is degenerate. We will employ a different strategy, which allows us to establish the generation of c_0 semigroup by L_0 and obtain an explicit formula for its kernel. The key observation is that the operators A and $\frac{\sigma^2}{2}B$ generate a solvable, finite-dimensional Lie algebra.

Having an explicit formula is important in obtaining an accurate, yet easily computable, approximation of the solution operator e^{tL} , one of the main goals of this work. To this end, we derive an error estimate of the form:

$$(5) \quad \|e^{tL}h - e^{tL_0}h\|_{L^2} \leq C\nu(\|\partial_\sigma h\|_{L^2} + \|h\|_{L^2}),$$

for $\nu \in (0, 1]$ and with a constant C , possibly dependent on L and κ , but not on h and ν (see Theorem 5.14 for a complete statement). In the process, we also establish several mapping properties for the semi-groups generated by L_0 and L . The method of proof is a perturbative argument based on heat kernels estimates, following the method developed in [9, 10]. This method extends the work on Henry-Labordère on heat kernel asymptotics [26, 27]. A similar method was developed by Pascucci and his collaborators [44, 46]. Heat kernel asymptotics were employed in this context also by Gatheral and his collaborators [20, 21]. See also [11, 32, 40, 43, 37, 16]. We also mention that fundamental solutions for degenerate equations related to $\partial_t u - L_0 u = F$, but in the context of ultraparabolic equations satisfying Hörmander's conditions for hypoellipticity, which does not hold for $\partial_t - L_0$, have been studied by many authors, starting with the seminal work of Kolmogorov [34] (see [14, 15, 45, 36] for some recent, relevant works).

The paper is organized as follows. In Section 2 we review a few needed facts on evolution equations and semi-groups of operators. We also introduced the exponentially weighted spaces used in this paper. In Section 3, we show that the operators L and B , which are both strongly parabolic, generate analytic semi-groups on weighted spaces, using the Lumer–Phillips theorem and the results of Section 2. Section 4 deals with the semi-groups generated by A , which is of transport type, and L_0 , which is degenerate parabolic. An explicit formula for e^{tL_0} is obtained by combining the results for the operators A and B , more specifically by exploiting the commutator identities that A and $f(\sigma)B$ satisfy and Lie group ideas. The last section, Section 5, contains some additional results: a more detailed discussion of Lie group ideas in evolution equations and the proof of the error estimate (5).

Notation: We close this Introduction with some notation used throughout. By $\|\cdot\|$ we denote the functional norm in a Banach space, while the norm of finite-dimensional vectors in \mathbb{R}^n will be simply denoted by $|\cdot|$. Lastly, by (\cdot, \cdot) we mean either the pairing between a Banach spaces and its dual, or the L^2 inner product, depending on the context.

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2. ONE PARAMETER SEMI-GROUPS

This section is devoted to survey general facts about abstract evolution equations and semi-groups of operators. We also review needed facts about the function spaces we employ, in particular *exponentially weighted Sobolev spaces*. As remarked in the Introduction, these spaces are needed to handle initial conditions of the form $h(\sigma, x) := |e^x - K|_+$, $(\sigma, x) \in (0, \infty) \times \mathbb{R}$. Most of the results presented in this section are known. We follow primarily, [2, 41, 47].

2.1. Unbounded operators and c_0 semi-groups. We begin by recalling the notion of a semi-group generated by a linear operator. Throughout, $\mathcal{L}(X)$ will denote the space of bounded linear operators on a Banach space X , which is a Banach algebra using the operator norm.

Definition 2.1. Let X be a Banach space. A *strongly continuous* or *c_0 semi-group of operators on X* is a family of bounded operators $S(t) : X \rightarrow X$, $t \geq 0$, satisfying:

- (i) $S(t_1 + t_2) = S(t_1)S(t_2)$, for all $t_i \geq 0$,
- (ii) $S(0) = I$, where I represent the identity operator on X ,
- (iii) $\lim_{t \rightarrow 0} S(t)x = x$, for all $x \in X$, where the limit is taken with respect to the topology of X .

We recall that a function $T : [a, b] \rightarrow \mathcal{L}(X)$ is *strongly continuous* if the map $[a, b] \ni t \rightarrow T(t)\xi \in X$ is continuous for every $\xi \in X$. It follows from the definition of a c_0 semi-group and the Banach-Steinhaus theorem that, if $S(t)$ is a c_0 semi-group of operators on X , then $S(t)$ is strongly continuous in t , hence the name strongly continuous semigroups.

We shall need also the notion of analytic semi-groups. To this end, for a given $\delta > 0$, we let

$$(6) \quad \Delta_\delta := \{ z = re^{i\theta}, -\delta < \theta < \delta, r > 0 \}.$$

Definition 2.2. Let X be a Banach space. An *analytic semi-group of operators* on X is a function $S : \Delta_\delta \cup \{0\} \rightarrow \mathcal{L}(X)$, $\delta > 0$, with the properties

- (i) S is analytic in Δ_δ ;
- (ii) $S(z_1 + z_2) = S(z_1)S(z_2)$, if $z_i \in \Delta_\delta \cup \{0\}$;
- (iii) $S(0) = I$, the identity operator on X ;
- (iv) $\lim_{z \rightarrow 0} S(z)x = x$, for all $x \in X$.

The limit $\lim_{z \rightarrow 0} S(z)x$ is computed for $z \in \Delta_\delta$. An analytic semi-group is, in particular, a c_0 semi-group.

Definition 2.3. Let X be a normed space. A (possibly unbounded) linear operator on X is a linear map $T : D(T) \rightarrow X$, where $D(T) \subset X$ is a linear subspace, called the *domain* of T . We say that T is *closed* if its graph is closed.

Unbounded linear operators arise naturally as the generators of c_0 semi-groups.

Definition 2.4. The *generator* T of a c_0 semi-group $S(t)$ on X is the operator $T\xi := \lim_{t \searrow 0} t^{-1}(S(t)\xi - \xi)$, with domain the set of vectors $\xi \in X$ for which the limit exists.

It is known that the generator of a c_0 semi-group is closed and densely defined. We next review criteria for an *unbounded* operator T to generate a c_0 semi-group $S(t)$. Then $u(t) := S(t)h$ is a (suitable) solution of $u' - Tu = 0$, $u(0) = h$. A useful criterion for T to generate a c_0 semi-group is provided by the Lumer-Phillips theorem, which we discuss next. Since two c_0 semi-groups with the same generator coincide [2, 47], we shall write $S(t) = e^{tT}$ for the semi-group generated by T , if such a semi-group exists.

2.2. Dissipativity. In the following, $\Re(z) = \Re z$ will denote the real part of $z \in \mathbb{C}$. Let X be a Banach space and let X^* denote its dual. If $x \in X$, the Hahn-Banach theorem implies, in particular, that the set

$$\mathcal{F}(x) := \{f \in X^*, f(x) = \|x\|^2 = \|f\|^2\}$$

is not empty.

Definition 2.5. A (possibly unbounded) operator T on a Banach space X is called *quasi-dissipative* if there exists $\mu \geq 0$ such that, for every $x \in D(T)$, there exists an $f \in \mathcal{F}(x) \subset X^*$ with the property that $\Re(f(Tx - \mu x)) \leq 0$.

This definition is simply saying that for some $\mu > 0$, the operator $Tx - \mu x$ is dissipative.

The *numerical range* of T , denoted $\mathfrak{N}(T)$, is the set

$$(7) \quad \mathfrak{N}(T) := \{f(Tx), \|x\| = 1, f \in \mathcal{F}(x)\}.$$

A quasi-dissipative operator T is thus one that has the property that

$$(8) \quad \mathfrak{N}(T) \subset \{z \in \mathbb{C}, \Re(z) \leq \mu\} = \mu + \Delta_{\pi/2}^c$$

with Δ_δ defined in Equation (6) and $\Delta_\delta^c := \mathbb{C} \setminus \Delta_\delta$ its complement.

Quasi-dissipativity, together with some mild conditions on the operator T stated below, is sufficient for the generation of a c_0 semigroup, by the celebrated Lumer-Phillips theorem, which we now recall for the benefit of the reader [2, 47].

Theorem 2.6 (Lumer-Phillips). *Let X be a Banach space and let T be a densely defined, quasi-dissipative operator on X such that $T - \lambda$ is invertible for λ large. Then T generates a c_0 semi-group on X .*

By strengthening the condition (8), we obtain the following similar theorem that yields generators of *analytic semi-groups*. The proof of this theorem is contained in the proof of Theorem 7.2.7 in [47].

Theorem 2.7. *Let X be a Banach space and let T be a densely defined operator on X such that $\Re(T) \subset \mu + \Delta_\vartheta^c$ for some $\mu \in \mathbb{R}$ and some $\vartheta > \pi/2$. Assume also that $T - \lambda$ is invertible for λ large. Then T generates an analytic semi-group.*

We note that the assumption that $T - \lambda$ be invertible in Theorem 2.7 implies that T is closed. The theorem is especially useful when T is a uniformly strongly elliptic operator (see Definition 2.22) in view of the following Lemma, the proof of which is again contained in the proof of Theorem 7.2.7 in [47]. See also [7, 38, 33].

Lemma 2.8. *Let P be an order $2m$ differential operator on some domain $\Omega \subset \mathbb{R}^n$, regarded as an unbounded operator on $L^2(\Omega)$ with domain $D(P) \subset H^{2m}(\Omega)$. We assume that there exists $C > 0$ such that*

$$\Re(Pv, v) \leq -C^{-1}\|v\|_{H^m(\Omega)} \quad \text{and} \quad |(Pv, v)| \leq C\|v\|_{H^m(\Omega)}, \quad (\forall) v \in D(P).$$

Then $\Re(P) \subset \Delta_\vartheta^c$ for some $\vartheta > \pi/2$.

From Theorem 2.7 and Lemma 2.8, we get the following corollary.

Corollary 2.9. *Let P be as in Lemma 2.8 and assume that $D(P)$ is dense in $L^2(\Omega)$ and that $P - \lambda$ is invertible for λ large. Then P generates an analytic semi-group on X .*

2.3. Classical and other types of solutions. Let us consider the initial-value problem for abstract parabolic equations of the form

$$(9) \quad \partial_t u - Pu = F, \quad u(0) = h \in X,$$

where P is a (usually unbounded) operator on a Banach space X and with domain $D(P)$. In our applications, X will be a space of functions on Ω , but first we consider this equation abstractly, from the point of view of semi-groups of operators.

Definition 2.10. We shall say that a function $u : [0, T] \rightarrow X$ is a *strong solution* of the initial value problem (9) for $F \in \mathcal{C}([0, T]; X)$ if

- (i) u is continuous for the norm topology on X and $u(0) = h$;
- (ii) $\partial_t u = u'$ is defined and continuous as a function $(0, T] \rightarrow X$;
- (iii) $u(t) \in D(P)$ for $t \in (0, T]$; and
- (iv) u satisfies the equation $\partial_t u(t) - Pu(t) = F(t) \in X$, for $t \in (0, T]$.

We shall need also the following weaker form of a solution.

Definition 2.11. A function $u : [0, T] \rightarrow X$ is called a *mild solution* of the initial-value problem (9) if $h \in X$, $F \in L^1([0, T], X)$, and

$$u(t) = e^{tP}h + \int_0^t e^{(t-\tau)P} F(\tau) d\tau,$$

with equality as elements of X pointwise in time $t \in (0, T)$.

The following remark recalls the connection between semi-groups and the various types of solutions of the Initial Value Problem (9).

Remark 2.12. For the applications of interest in this work, we can reduce to homogeneous equations, that is $F(0) = 0$, as we assume now. We also assume that the operator P generates a c_0 semi-group e^{tP} on X . Then $u(t) := e^{tP}h$ is a mild solution for any $h \in X$. If, moreover, $h \in D(P)$ or if P generates an analytic semi-group, then $u(t) := e^{tP}h$ is also a strong solution of Equation (9) (see [2, 41, 47], for instance).

We are interested in the case when P is a m -th order partial differential operator defined on a domain $\Omega \subset \mathbb{R}^d$:

$$(10) \quad P := \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha,$$

with coefficients $a_\alpha \in C^\infty(\overline{\Omega})$. We shall occasionally use the convenient notation:

$$u(t)(q) := u(t, q), \quad t \geq 0 \text{ and } q \in \Omega,$$

which is in agreement with (9). When $P = L$, acting on $L^2(\Omega)$, $\Omega = (0, \infty) \times \mathbb{R}$, $F = 0$, and $h(\sigma, x) := |e^x - K|_+$, we recover the initial-value problem (1). In that case, we are interested in *classical* and *weak* solutions. We assume that X is a space of functions on X , that is, $X \subset L^1_{loc}(\Omega)$. We also assume that the domain of P contains the space of smooth functions with compact support in Ω , and hence the same is satisfied by its adjoint.

Definition 2.13. We shall say that a function $u : [0, T] \times \Omega \rightarrow \mathbb{C}$ is a *classical solution* of the initial value problem (9) if

- (i) u is continuous on $[0, T] \times \Omega$ and $u(0, q) = h(q)$, for all $q \in \Omega$;
- (ii) $\partial_t u = u'$ and $\partial^\alpha u$, $|\alpha| \leq m$, are defined and continuous on $(0, T] \times \Omega$; and
- (iii) u satisfies the equation $\partial_t u - Pu = F$ pointwise in $(0, T] \times \Omega$.

If boundary conditions for u on $\partial\Omega$ are given, we require them to be satisfied as equalities of continuous functions.

It follows that if u is a classical solution, then F is continuous. We note that in the abstract setting, strong solutions are often referred to as classical solutions (see e.g. [47]).

Remark 2.14. We recall that, if T is the generator of an analytic semi-group e^{tT} on a Banach space X , then $T^n e^{tT}$ extends to a bounded operator on X and there exists $C > 0$ such that

$$(11) \quad \|T^n e^{tT}\| \leq C t^{-n}, \quad \text{for all } t \in (0, 1].$$

The following lemma follows from known results (cf. [41, 47]).

Lemma 2.15. *Assume that there exists $n \geq 0$ such that $D(P^n) \ni f \rightarrow \partial^\alpha f \in \mathcal{C}(\overline{\Omega})$ is continuous for all $|\alpha| \leq m$. In addition, assume that P generates a c_0 semi-group on X and that $F = 0$. Then $u(t) := e^{tP}h$ is a classical solution of Equation (9) for all $h \in D(P^{n+1})$.*

Proof. For each fixed t , $u(t) \in D(P^{n+1})$ defines a continuous function on $\overline{\Omega}$, since $D(P^n) \subset \mathcal{C}(\overline{\Omega})$ continuously. The same argument shows that the map $[0, T] \ni t \rightarrow u(t) \in \mathcal{C}(\overline{\Omega})$ is continuous, and hence u is continuous on $[0, T] \times \Omega$, that

$\partial_t u$, $\partial^\alpha u$, $|\alpha| \leq m$, are defined and continuous on $(0, T] \times \Omega$, and that $u' = Pu$ (this is where we need the stronger assumption that $h \in D(P^{n+1})$, since we need $\epsilon^{-1}(u(t+\epsilon) - u(t)) \rightarrow Pu(t) \in D(P^n)$, as $\epsilon \rightarrow 0$). \square

Let us denote by

$$(12) \quad P^t v := \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \partial^\alpha (a_\alpha v)$$

be the *transpose* of P (so that $\int_\Omega (Pu)v dx = \int_\Omega u(P^t v) dx$ whenever u and v are compactly supported in Ω). Similarly, we then have the following definition of *weak* or *distributional* solutions.

Definition 2.16. We shall say that $u : [0, T] \times \Omega \rightarrow \mathbb{C}$ is a *weak solution* of the initial value problem (9) if $u, F \in L^1_{loc}([0, T] \times \Omega)$ and, for all $\phi \in \mathcal{C}_c^\infty([0, T] \times \Omega)$,

$$(13) \quad \int_\Omega \left[\phi(0, x) h(x) + \int_0^T (\partial_t \phi + P^t \phi) u dt + \int_0^T \phi F dt \right] dx = 0.$$

If, moreover, v is also a classical solution on $[\delta, T]$ for all $\delta > 0$, we shall say that v is a *classical* solution on $(0, T]$. If v is a classical solution on $(0, T]$, for all $T < R$, then we say that v is a classical solution on $(0, R)$.

Again, the following lemma is well-known (see e.g. [47]).

Lemma 2.17. Assume that P generates a c_0 semi-group on X . Then $u(t) := e^{tP} h$ is a weak solution of the homogeneous Initial-Value Problem (9) with $F = 0$ for all $h \in X$.

Proof. We assume that $h \in D(P)$ and that ϕ is as in Definition 2.16. Then the function $\psi(t) := (e^{tP} h, \phi(t))$ is continuously differentiable on $[0, T]$. The relation $\psi(T) - \psi(0) = \int_0^T \psi'(t) dt$ gives that $u(t) := e^{tP} h$ is a weak solution of the IVP (9). Since the weak form (13) depends continuously on h , we obtain that $u(t) := e^{tP} h$ is a weak solution of (9) by the density of $D(P)$ in X . \square

Combining the two lemmas above we obtain.

Proposition 2.18. Assume that $D(P^n) \ni f \rightarrow \partial^\alpha f \in \mathcal{C}(\overline{\Omega})$ is continuous for all $|\alpha| \leq m$. Assume in addition that P generates an analytic semi-group on X and that $F = 0$. Then, for all $h \in X$, $u(t) := e^{tP} h$ is a classical solution on $(0, \infty)$ of the IVP (9).

2.4. Function spaces. We shall consider various weighted Sobolev spaces as follows. Let $\Omega \subset \mathbb{R}^d$ be an open subset, as in the previous subsection, and let $w \in L^1_{loc}(\Omega)$ satisfy $w \geq 0$. If X is any Banach space of functions on Ω with norm $\|\cdot\|_X$, we define

$$(14) \quad wX := \{ w\xi, \xi \in X \},$$

with the norm $\|w\xi\|_{wX} := \|\xi\|_X$. Thus, if $p < \infty$, if $X = L^p(\Omega, d\mu)$, and if $w > 0$ almost everywhere with respect to μ , $\mu \geq 0$, then $wX = L^p(\Omega, w^{-1/p} d\mu)$. Of course, for any linear operator T we have

$$(15) \quad T : wX \rightarrow wX \text{ is bounded if, and only if } w^{-1}Tw : X \rightarrow X \text{ is bounded.}$$

In fact, these two operators are unitarily equivalent.

In what follows, we choose weights of the form $w := e^{\lambda\langle x \rangle}$, where $\langle \cdot \rangle$ denotes the Japanese bracket:

$$\langle x \rangle := \sqrt{1 + x^2},$$

and $\lambda \in \mathbb{R}$ is a parameter. The weight w will be viewed as acting on functions of $x \in \mathbb{R}$ or of $(\sigma, x) \in I \times \mathbb{R}$, in the later case the weight being independent of σ . For simplicity, we shall usually write

$$(16) \quad H_\lambda^m(\mathbb{R}) := e^{\lambda\langle x \rangle} H^m(\mathbb{R}) = \{ f : \mathbb{R} \rightarrow \mathbb{C}, e^{-\lambda\langle x \rangle} f \in H^m(\mathbb{R}) \} \\ = \{ f : \mathbb{R} \rightarrow \mathbb{C}, e^{-\lambda\langle x \rangle} \partial^i f \in L^2(\mathbb{R}), i \leq m \},$$

where the last equality is valid due to the fact that the weight $w(x) = e^{\lambda\langle x \rangle}$ has the property that $w^{-1}\partial^i w$ forms a bounded family as operators on $H^m(\mathbb{R})$ (by writing $f = wg$, with $g \in H^m(\mathbb{R})$). We also let $L_\lambda^2 = H_\lambda^0$. We recall that this choice of the weight function w is justified by the specific form of the initial data $h(x) := |e^x - K|_+$ for the Cauchy problem for the λ SABR model (1).

Let I be a closed interval in \mathbb{R} . We consider, similarly, the spaces

$$(17) \quad H_\lambda^{i,j}(I \times \mathbb{R}) := wH^i(I; H^j(\mathbb{R})) = \{ u, \partial_\sigma^\alpha \partial_x^\beta u \in L_\lambda^2(I \times \mathbb{R}), \alpha \leq i, \beta \leq j \} \\ = \{ u, \partial_\sigma^\alpha \partial_x^\beta (e^{-\lambda\langle x \rangle} u) \in L^2(I \times \mathbb{R}), \alpha \leq i, \beta \leq j \} = H^i(I; H_\lambda^j(\mathbb{R})).$$

2.5. Operators with totally bounded coefficients. Let $\Omega = \mathbb{R}$ or $\Omega = I \times \mathbb{R}$, with $I \subset \mathbb{R}$ an interval. We shall often use the following class of functions.

Definition 2.19. A function $f : \Omega \rightarrow \mathbb{C}$ is *totally bounded* if it is smooth and bounded and all its derivatives are also bounded.

We have the following simple lemma.

Lemma 2.20. *Let P be an order m differential operator on Ω with totally bounded coefficients. Then P defines continuous a map $H_\lambda^s(\Omega) \rightarrow H_\lambda^{s-m}(\Omega)$, for every $s \geq m$.*

Proof. The proof is a direct calculation. \square

Lemma 2.21. *Let $P := \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha$ be an order m differential operator on Ω with totally bounded coefficients. If $w(\sigma, x) = e^{\lambda\langle x \rangle}$, as before, then $w^{-1}Pw$ also has totally bounded coefficients and the same terms of order m as P .*

Proof. We have $w^{-1}\partial_\sigma w = \partial_\sigma$ and $w^{-1}\partial_x w = \partial_x + w^{-1}\frac{\partial w}{\partial x} = \partial_x + \psi$, where $\psi := w^{-1}\frac{\partial w}{\partial x} = \lambda\frac{\partial \langle x \rangle}{\partial x} = \lambda\langle x \rangle'$. Since $\langle x \rangle'$ is totally bounded, the result follows from the identity $(w^{-1}P_1w)(w^{-1}P_2w) = w^{-1}P_1P_2w$ for any differentiable operators P_1 and P_2 . \square

We formulate the following result in slightly greater generality than needed for the proof of the existence of the semi-group generated by L , for further possible applications. Let us now recall the definition of a second order uniformly strongly elliptic differential operator on $\Omega = \mathbb{R}$ or $\Omega = I \times \mathbb{R}$, with real coefficients, in the form that we will use in this paper.

Definition 2.22. Let $P = a_{xx}(\sigma, x)\partial_x^2 + 2a_{\sigma x}(\sigma, x)\partial_\sigma\partial_x + a_{\sigma\sigma}(\sigma, x)\partial_\sigma^2 + b(\sigma, x)\partial_x + c(\sigma, x)\partial_\sigma + d(\sigma, x)$ be a differential operator with real coefficients on $I \times \mathbb{R}$. We say that P is *uniformly strongly elliptic* if it has bounded coefficients and if there exists $\epsilon > 0$ such that $a_{xx} \geq \epsilon$ and $a_{xx}a_{\sigma\sigma} - a_{\sigma x}^2 \geq \epsilon$.

If $\Omega = \mathbb{R}$, the operator P reduces to $P = a_{xx}(\sigma, x)\partial_x^2 + b(\sigma, x)\partial_x + d(\sigma, x)$ and we have that P is uniformly strongly elliptic if it has bounded coefficients and there exists $\epsilon > 0$ such that $a_{xx} \geq \epsilon$.

We have the following standard regularity results. We continue to assume that $\Omega = I \times \mathbb{R}$ or $\Omega = \mathbb{R}$.

Theorem 2.23. *Let P be second order, uniformly strongly elliptic differential operator with totally bounded coefficients on Ω . Assume $u \in H_\lambda^1(\Omega)$ is such that $Pu \in H_\lambda^{m-1}(\Omega)$. If $\Omega = I \times \mathbb{R}$, we also assume that u vanishes at the endpoints of I . Then $u \in H_\lambda^{m+1}(\Omega)$. Moreover, there exists $C > 0$, independent of u , such that $\|u\|_{H_\lambda^{m+1}(\Omega)} \leq C(\|Pu\|_{H_\lambda^{m-1}(\Omega)} + \|u\|_{H_\lambda^1(\Omega)})$.*

A proof of this result can be obtained by first reducing to the case $\lambda = 0$, that is, $w = 1$, using Lemma 2.21 and then either by using a dyadic partition of unity or by using divided differences (this approach is sometimes called Nirenberg's trick after [1]), which is facilitated in this case since the boundary is straight (see, for instance, [39]). We obtain the following consequence.

Corollary 2.24. *Let P be second order, uniformly strongly elliptic differential operator with totally bounded coefficients on \mathbb{R} . Then $\|u\|_{L_\lambda^2} + \|P^k u\|_{L_\lambda^2}$ defines an equivalent norm on $H_\lambda^{2k}(\mathbb{R})$.*

The above two results hold in the more general framework of manifolds with bounded geometry. See, for example, [42] and the references therein. See also [4, 3, 5, 6, 13, 23, 35] for more recent results on PDEs on manifolds with bounded geometry.

3. THE SEMI-GROUP GENERATED BY L AND B

In this section, we show that L and B generate analytic semi-groups using the Lumer–Phillips theorem and the results of the previous section. Our approach is standard and well known for the case of operators on standard Sobolev spaces. The analysis on exponentially weighted spaces is less developed. For the reader's sake, we work in detail the slightly more complicated case of the operator L and only sketch the proofs of the results for B .

3.1. The differential operator L . Our next goal is to show that the operator L is quasi-dissipative on weighted Sobolev spaces. The space

$$(18) \quad \mathcal{K}_0 := H_\lambda^2(I \times \mathbb{R}) \cap \{u = 0 \text{ on } \partial I \times \mathbb{R}\}$$

will be the common domain of several operators, so it will play an important role in what follows. We formulate the following result in slightly greater generality than needed for the proof of the existence of the semi-group generated by L , for further possible applications.

Definition 3.1. Let \mathcal{P} denote the set of second order differential operators $T = a_{xx}(\sigma, x)\partial_x^2 + 2a_{\sigma x}(\sigma, x)\partial_\sigma\partial_x + a_{\sigma\sigma}(\sigma, x)\partial_\sigma^2 + b(\sigma, x)\partial_x + c(\sigma, x)\partial_\sigma + d(\sigma, x)$ with totally bounded, real coefficients on $I \times \mathbb{R}$ and satisfying

$$a_{xx}, a_{\sigma\sigma}, a_{xx}a_{\sigma\sigma} - a_{\sigma x}^2 \geq 0.$$

For T as in this definition, we shall denote

$$(19) \quad M_T := \begin{bmatrix} a_{xx} & a_{\sigma x} \\ a_{\sigma x} & a_{\sigma\sigma} \end{bmatrix}$$

the matrix determined by its highest order coefficients (the principal symbol) of T .

Proposition 3.2. *If $w(\sigma, x) = e^{\lambda\langle x \rangle}$ and $T \in \mathcal{P}$, then $w^{-1}Tw \in \mathcal{P}$. Let M_T be as in Equation (19), then there exists $C > 0$ such that*

$$(Tu, u)_{L_\lambda^2(I \times \mathbb{R})} \leq - \int_{I \times \mathbb{R}} (M_T \nabla u, \nabla u) e^{-2\lambda\langle x \rangle} d\sigma dx + C \|u\|_{L_\lambda^2(I \times \mathbb{R})}^2, \quad u \in \mathcal{K}_0,$$

and hence, T with domain $\mathcal{K}_0 := H_\lambda^2(I \times \mathbb{R}) \cap \{u = 0 \text{ on } \partial I \times \mathbb{R}\}$ is quasi dissipative on $L_\lambda^2(I \times \mathbb{R})$.

Proof. The fact that $w^{-1}Tw$ is of the same form as T follows from Lemma 2.21. In view of Equation (15), we can assume that $\lambda = 0$, that is, $w := e^{\lambda\langle x \rangle} = 1$. The rest of the proof is then a well-known direct calculation, which we include for the benefit of the reader. Since we work with Hilbert spaces, we can take $f^*(\xi) = (\xi, f)$ in the condition defining the quasi dissipativity. We first notice that, by changing b , c , and d , we can assume that $Tu = \partial_x(a_{xx}\partial_x u) + \partial_\sigma(a_{\sigma x}\partial_x u) + \partial_x(a_{\sigma x}\partial_\sigma u) + \partial_\sigma(a_{\sigma\sigma}\partial_\sigma u) + b\partial_x u + c\partial_\sigma u + du$. Then, we perform a standard energy estimate, in which the integration by parts is justified by the fact that $u \in \mathcal{K}_0$:

$$(20) \quad \begin{aligned} 2\Re(c\partial_\sigma u, u) &:= 2\Re \int_{I \times \mathbb{R}} c(\partial_\sigma u) \bar{u} d\sigma dx = \int_{I \times \mathbb{R}} c(\partial_\sigma u) \bar{u} d\sigma dx \\ &+ \int_{I \times \mathbb{R}} c(\partial_\sigma \bar{u}) u d\sigma dx = \int_{I \times \mathbb{R}} \partial_\sigma(c|u|^2) d\sigma dx - \int_{I \times \mathbb{R}} (\partial_\sigma c)|u|^2 d\sigma dx \\ &= \int_{\mathbb{R}} (c(\beta, x)|u(\beta, x)|^2 - c(\alpha, x)|u(\alpha, x)|^2) dx - \int_{I \times \mathbb{R}} (\partial_\sigma c)|u|^2 d\sigma dx \\ &= - \int_{I \times \mathbb{R}} (\partial_\sigma c)|u|^2 d\sigma dx, \end{aligned}$$

using that c is real valued and the fact that $u \in \mathcal{K}_0$. Similarly, since b is also real valued,

$$(21) \quad \begin{aligned} 2\Re(b\partial_x u, u) &:= 2\Re \int_{I \times \mathbb{R}} b(\partial_x u) \bar{u} d\sigma dx = \int_{I \times \mathbb{R}} b(\partial_x u) \bar{u} d\sigma dx \\ &+ \int_{I \times \mathbb{R}} b(\partial_x \bar{u}) u d\sigma dx = \int_{I \times \mathbb{R}} \partial_x(b|u|^2) d\sigma dx - \int_{I \times \mathbb{R}} (\partial_x b)|u|^2 d\sigma dx \\ &= - \int_{I \times \mathbb{R}} (\partial_x b)|u|^2 d\sigma dx. \end{aligned}$$

Next, we consider the quadratic terms. By assumption, all the eigenvalues of the matrix M_T of Equation (19) are non-negative. Let δ the smallest of the eigenvalues of M_T . We obtain:

$$(22) \quad \begin{aligned} &-(\partial_x(a_{xx}\partial_x u) + \partial_\sigma(a_{\sigma x}\partial_x u) + \partial_x(a_{\sigma x}\partial_\sigma u) + \partial_\sigma(a_{\sigma\sigma}\partial_\sigma u), u) \\ &= \int_{I \times \mathbb{R}} \left(a_{xx}(\partial_x u)\partial_x \bar{u} + a_{\sigma x}(\partial_\sigma u)\partial_x \bar{u} + a_{\sigma x}(\partial_x u)\partial_\sigma \bar{u} + a_{\sigma\sigma}(\partial_\sigma u)\partial_\sigma \bar{u} \right) d\sigma dx \\ &= \int_{I \times \mathbb{R}} (M_T \nabla u, \nabla u) d\sigma dx \geq \delta \int_{I \times \mathbb{R}} |\nabla u(\sigma, x)|^2 d\sigma dx, \end{aligned}$$

and the last term is positive since the quadratic form defined by a_{xx} , $a_{\sigma x}$, and $a_{\sigma\sigma}$ is positive, by assumption.

Combining Equations (20), (21), and (22), we obtain

(23)

$$\begin{aligned} 2\Re(Tu, u) &\leq - \int_{I \times \mathbb{R}} (M \nabla u, \nabla u) d\sigma dx - \int_{I \times \mathbb{R}} (\partial_x b + \partial_\sigma c + d) |u|^2 d\sigma dx \\ &\leq - \int_{I \times \mathbb{R}} (M_T \nabla u, \nabla u) d\sigma dx + C \|u\|^2 \leq -\delta \int_{I \times \mathbb{R}} \|\nabla u(\sigma, x)\|^2 d\sigma dx + C \|u\|^2, \end{aligned}$$

where $C = \|\partial_x b + \partial_\sigma c + d\|_\infty$. The fact that T is quasi-dissipative follows since $\delta \geq 0$. \square

Let us note also, for further reference, the following consequences of the calculation in the above proof.

Corollary 3.3. *Let T be as in Proposition 3.2. Then there exists a constant $C > 0$ such that $|(Tu, u)| \leq C \|u\|_{H_\lambda^1(I \times \mathbb{R})}$*

Proof. This is a simple calculation, very similar to those in the proof of Proposition 3.2. In particular, we can assume $\lambda = 0$. The main difference is with Equation (25), which is replaced by

$$\begin{aligned} 0 &\leq -(\partial_x(a_{xx}\partial_x u) + \partial_\sigma(a_{\sigma x}\partial_x u) + \partial_x(a_{\sigma x}\partial_\sigma u) + \partial_\sigma(a_{\sigma\sigma}\partial_\sigma u), u) \\ &= \int_{I \times \mathbb{R}} \left(a_{xx}(\partial_x u) \partial_x \bar{u} + a_{\sigma x}(\partial_\sigma u) \partial_x \bar{u} + a_{\sigma x}(\partial_x u) \partial_\sigma \bar{u} + a_{\sigma\sigma}(\partial_\sigma u) \partial_\sigma \bar{u} \right) d\sigma dx \\ &\leq \mu \int_{I \times \mathbb{R}} \|\nabla u(\sigma, x)\|^2 d\sigma dx \leq \mu \|u\|_{H^1(I \times \mathbb{R})}^2, \end{aligned}$$

where μ is the largest of the eigenvalues of the matrix M_T of Equation (19). \square

Garding's inequality also holds in our setting. We have the opposite sign to the one that is typically used, as we work with negative-definite operators.

Corollary 3.4. *Let T be as in the statement of Proposition 3.2. Assume also that there exists $\epsilon > 0$ such that $a_{xx}a_{\sigma\sigma} - a_{\sigma x}^2 \geq \epsilon$. Then there exist $C_1 > 0$ and C_2 such that*

$$\Re(Tu, u) \leq -C_1 \|u\|_{H_\lambda^1(I \times \mathbb{R})}^2 + C_2 \|u\|_{L_\lambda^2(I \times \mathbb{R})}^2.$$

Also, if $u \in H_\lambda^1(I \times \mathbb{R}) \cap \{u|_{\partial I \times \mathbb{R}} = 0\}$ satisfies $Tu \in L_\lambda^2(I \times \mathbb{R})$, then $u \in H_\lambda^2(I \times \mathbb{R})$. Consequently, $T - \mu_0 : \mathcal{K}_0 \rightarrow L_\lambda^2(I \times \mathbb{R})$ is invertible for $\mu_0 > C_2$.

Proof. Garding's inequality is an immediate consequence of Equation (23). The rest is a consequence of this inequality, and we only outline the main steps in the proof.

First of all, by Lemma 2.21, we can assume that $\lambda = 0$. Let $\Omega := I \times \mathbb{R}$. Garding's inequality allows us to invoke the Lax-Milgram Lemma, which gives that $T - \mu_0 : H^1(I \times \mathbb{R}) \cap \{u|_{\partial\Omega} = 0\} \rightarrow H^{-1}(I \times \mathbb{R})$ is invertible for $\mu_0 > C_2$ (that is, $T - \mu_0$ is a continuous bijection with continuous inverse).

By replacing T with $T - \mu_0$, if necessary, we can assume that $T : H^1(I \times \mathbb{R}) \cap \{u|_{\partial\Omega} = 0\} \rightarrow H^{-1}(I \times \mathbb{R})$ is invertible. The assumptions on our coefficients (that they are bounded and that $a_{xx} \geq 0$, $a_{\sigma\sigma} \geq 0$, and $a_{xx}a_{\sigma\sigma} - a_{\sigma x}^2 \geq \epsilon > 0$) imply that T is uniformly strongly elliptic (see Definition 2.22). Therefore, it satisfies elliptic

regularity (Theorem 2.23). In particular, if $u \in H^1(I \times \mathbb{R}) \cap \{u|_{\partial\Omega} = 0\}$ is such that $Tu \in L^2(I \times \mathbb{R})$, then $u \in H^2(I \times \mathbb{R})$ and hence, by taking into account that u vanishes at the boundary, $u \in \mathcal{K}_0$. We finally obtain that

$$T : \mathcal{K}_0 := H_\lambda^2(I \times \mathbb{R}) \cap \{u = 0 \text{ on } \partial\Omega = \partial I \times \mathbb{R}\} \rightarrow L^2(I \times \mathbb{R})$$

is both injective and surjective, and hence it is invertible. (The continuity of the inverse follows either from abstract principles, namely from the Open Mapping Theorem, or, constructively, from Theorem 2.23.) \square

We obtain as a consequence the following theorem.

Theorem 3.5. *Let T be as in the statement of Corollary 3.4. Then T generates an analytic semi-group e^{tL} on $L_\lambda^2(I \times \mathbb{R})$. In particular, if $I = (\alpha, \beta)$ is a bounded interval with $0 < \alpha \leq \beta < \infty$, then L as given in (1) satisfies the hypothesis of Corollary 3.4, and hence it generates an analytic semi-group on $L_\lambda^2(I \times \mathbb{R})$.*

Proof. Corollaries 3.3 and 3.4 show that the T satisfies the assumptions of Lemma 2.8 (that is, T is continuous and satisfies a Garding-type inequality). Since $T - \mu_0$ is invertible for μ_0 large, again by Corollary 3.4, we are in position to use Corollary 2.9 to conclude that T generates an analytic semi-group. If I is bounded, then L has totally bounded coefficients. Since $\alpha > 0$, L is also uniformly strongly elliptic, and the first part of the result applies. \square

Corollary 3.6. *Let T be as in Theorem 3.5 and $h \in L_\lambda^2(I \times \mathbb{R})$, for some $\lambda \in \mathbb{R}$. Then $u(t) := e^{tT}h$ is a strong solution of $\partial_t u - Tu = 0$, $u(0) = h$. It is also a classical solution on $(0, \tau]$, for all $\tau > 0$. Moreover, $u(t)$ does not depend on λ .*

Proof. We have that T generates an analytic semi-group $S(t) = e^{tT}$. Moreover, elliptic regularity gives $D(T^k) \subset H^{2k}(I \times \mathbb{R})$, for all $k \in \mathbb{Z}_+$. The Sobolev embedding theorem then gives us that the assumptions of Lemma 2.15 and Proposition 2.18 are satisfied. This proves the first part of the result.

The independence of u on λ follows from the fact that the map $L_{\lambda'}^2(I \times \mathbb{R}) \rightarrow L_{\lambda''}^2(I \times \mathbb{R})$ is injective and continuous for all $\lambda' < \lambda''$ and from the uniqueness of strong solutions. \square

Remark 3.7. The assumption that I be a bounded interval in the second half of Theorem 3.5, is essential for our method to apply. Our method does not apply, for instance, if $I = (0, \infty)$. The problem lies in the fact that, at $\sigma = 0$, we lose uniform ellipticity and, at $\sigma = \infty$, the coefficient $\theta - \sigma$ becomes unbounded. However, if $\kappa = 0$, we do obtain that L generates an analytic semi-group using the results in [42]. The degeneracy at $\sigma = 0$ and $\sigma = \infty$ could be addressed by introducing appropriate weights in σ . For the applications of interest in this work, it is enough to consider σ in a bounded interval, bounded away from zero.

3.2. The differential operator B . We now consider the operator $B := \partial_x^2 - \partial_x$ (recall Equation 3). The fact that B generates an analytic semigroup is classical. However, since we work with exponentially weighted spaces, we state needed results for clarity and completeness. We start by collecting all the needed technical facts about B in the following proposition, which we state in more generality than actually needed. It can be seen as a special case of the analysis of the operator L (see also Lemma 2.21).

Proposition 3.8. *Let $T = a\partial_x^2 + b\partial_x + c$ be a uniformly strongly elliptic operator with totally bounded coefficients. Then:*

- (i) *wTw^{-1} is also strongly elliptic with totally bounded coefficients.*
- (ii) *There is $C_3 > C_1 > 0$ and $C_2 \in \mathbb{R}$ such that, for any $u \in H_\lambda^2(\mathbb{R})$,*

$$\Re(Tu, u) \leq -C_1 \|u\|_{H_\lambda^1(\mathbb{R})}^2 + C_2 \|u\|_{L_\lambda^2(\mathbb{R})}^2 \quad \text{and} \quad |(Tu, u)| \leq C_3 \|u\|_{H_\lambda^1(\mathbb{R})}^2.$$

- (iii) *$T - \mu_0 : H_\lambda^2(\mathbb{R}) \rightarrow L_\lambda^2(\mathbb{R})$ is invertible for $\mu_0 > C_2$.*

Using the same argument as for Theorem 3.5, we obtain the following result.

Theorem 3.9. *Let T be as in the statement of Proposition 3.8. Then T generates an analytic semi-group on $L_\lambda^2(\mathbb{R})$. In particular, B generates an analytic semi-group on $L_\lambda^2(\mathbb{R})$.*

Since $D(B^k) = H_\lambda^{2k}(\mathbb{R})$, we also obtain the following.

Corollary 3.10. *The operator B generates an analytic semi-group on $H_\lambda^j(\mathbb{R})$, for all j .*

Again using the same argument as in the previous subsection, we have also the following result.

Corollary 3.11. *Let T be as in Proposition 3.2 and $h \in L_\lambda^2(\mathbb{R})$, for some $\lambda \in \mathbb{R}$. Then $u(t) := e^{tT}h$ is a strong solution of $\partial_t u - Tu = 0$, $u(0) = h$. Moreover, it is a classical solution on any interval $(0, \tau]$, $\tau > 0$, and $u(t)$ does not depend on λ .*

Remark 3.12. In view of the independence of λ , we obtain that the semi-group e^{tB} is given by the following explicit formula

$$(24) \quad e^{tB}h(x) = \frac{1}{\sqrt{4\pi t}} \int e^{-\frac{|x-y-t|^2}{4t}} h(y) dy.$$

In particular, if $\lambda = 0$, the semi-group generated by B consists of contractions.

We will need on several occasions the following well-known lemma. In particular, we will need it to treat families of operators. (We note here that this lemma will be generalized to deal with differentiability in the strong sense in Lemma 3.15.)

Lemma 3.13. *Let $\xi \in \mathcal{C}([0, 1]; X)$ and $[0, 1] \ni t \rightarrow V(t) \in \mathcal{L}(X)$ be strongly continuous. Then the map $[0, 1] \ni t \rightarrow V(t)\xi(t) \in X$ is continuous.*

In order to apply our results on analytic semi-groups to Equation (9), we will need to consider families of operators. In particular, we will show that the operator $P = \frac{\sigma^2}{2}B$, acting on functions of σ and x , that appears in the λ SABR PDE, also generates an analytic semigroup. If $p : I \rightarrow [0, \infty)$ is bounded and continuous, then we shall write pB for the operator $(pBv)(\sigma) = p(\sigma)Bv(\sigma) \in L_\lambda^2(\mathbb{R})$ and e^{pB} for the operator $(e^{pB}v)(\sigma) = e^{p(\sigma)B}v(\sigma) \in L_\lambda^2(\mathbb{R})$, where $v : I \rightarrow H_\lambda^2(\mathbb{R})$. We thus regard both pB and e^{pB} as a family of operators parameterized by $\sigma \in I$ and acting on $L_\lambda^2(\mathbb{R})$ -valued functions defined on I .

Proposition 3.14. *Let T be a differential operator as in Proposition 3.8. Let $I \subset \mathbb{R}$ be an interval and $p : I \rightarrow [0, \infty)$ be a bounded continuous function. Then e^{tpT} , defined by the formula $(e^{tpT}h)(\sigma) := e^{tp(\sigma)T}h(\sigma) \in L_\lambda^2(\mathbb{R})$, $\sigma \in I$, defines a c_0 semi-group on $L_\lambda^2(I \times \mathbb{R})$ with generator pT .*

Proof. Since T generates a c_0 semi-group, $e^{tp(\sigma)T}h(s)$ depends continuously on $\sigma \in I$ whenever $h \in L_\lambda^2(I \times \mathbb{R})$ is continuous in σ . Since $\|e^{tT}\|$ is uniformly bounded for t in a bounded interval, we obtain that the family of operators $e^{tp(\sigma)T}$ thus defines a bounded operator on $L_\lambda^2(I \times \mathbb{R})$. \square

We shall need the following extension of Lemma 3.13.

Lemma 3.15. *Let $J := (0, 1)$ and assume that $\xi \in \mathcal{C}^1(J; X)$, that T is the generator of c_0 semi-group $V(t)$ on X , and that one of the following two conditions is satisfied:*

- (i) $\xi(t) \in D(T)$ and the map $J \ni t \rightarrow T\xi(t) \in X$ is continuous;
- (ii) the semi-group $V(t)$ generated by T is an analytic semi-group.

Then $V(t)\xi(t) \in \mathcal{C}^1(J; X)$ with differential $TV(t)\xi(t) + V(t)\xi'(t)$.

Let $\mathcal{K}_1 := H_\lambda^2(I \times \mathbb{R})$, as in Corollary 4.4 in the previous subsection.

Corollary 3.16. *Let $f : [\alpha, \beta] = \bar{I} \rightarrow [\epsilon, \infty)$, $\epsilon > 0$. Assume that f , f' , and f'' are (defined and) continuous. Then e^{fB} maps \mathcal{K}_1 to itself. Moreover, e^{fB} defines a c_0 -semigroup on \mathcal{K}_1 , generated by fB as an operator with domain*

$$\{\xi \in \mathcal{K}_1, B\xi \in \mathcal{K}_1\} \supset H_\lambda^{2,4}(I \times \mathbb{R}) = H^2(I; H_\lambda^4(\mathbb{R})).$$

Proof. The first part is an immediate consequence of Lemma 3.15(ii) and of Remark 2.14. The second part follows using also Corollary 3.10. \square

4. THE SEMI-GROUP GENERATED BY L_0

In this section, we discuss the derivation of an explicit formula for the distributional kernel of the operator e^{tL_0} using Lie algebra techniques. Besides being of independent interest, in this work we utilize the explicit formula for e^{tL_0} to approximate e^{tL} , for which no closed form are available. This is achieved by means of a perturbative expansion in the parameter ν , the so-called volvol or volatility of the volatility. We recall that $L_0 = A + \frac{\sigma^2}{2}B$ and $L = L_0 + \nu L_1 + \nu^2 L_2$, with L_i independent of ν (see Equations (2) and (3)).

There is an added difficulty in our problem, namely, the fact that L_0 is not strongly elliptic, and $\partial_t - L_0$ is not hypoelliptic in the sense of Hörmander [30] (although L_0 is). As a matter of fact, this expansion is only valid under additional regularity assumptions on the initial data h , which will be discussed in Section 5.

The explicit formula for e^{tL-0} is derived from the corresponding formulas for e^{tA} and $e^{\frac{t\sigma^2}{2}B}$, where the later is defined using Proposition 3.14. Our approach can be viewed as akin to an operator splitting argument, where the hyperbolic and parabolic parts of L_0 are treated separately, although we do not explicitly resort to any splitting in the PDE itself.

We thus assume that $I = (\alpha, \beta)$ satisfies $0 < \alpha < \theta < \beta < \infty$, as in Proposition 3.14. We will make the further assumption that $\kappa > 0$.

This last assumption implies that the characteristics of the operator A are incoming at $\sigma = \alpha$ and $\sigma = \beta$, as long as $\alpha < \theta < \beta$ and $\kappa > 0$. Therefore, no boundary conditions need to be imposed at $\sigma = \alpha$ and $\sigma = \beta$ (cf. the seminal work of Feller [17, 18]). The case $\kappa < 0$ is similar provided one imposes suitable boundary conditions. However, this case will not be needed for our purposes.

We now study e^{tA} and its properties. These will be used in deriving an explicit formula for e^{tL_0} .

4.1. The transport equation. Let $I = (\alpha, \beta) \subset \mathbb{R}$ and $A := \kappa(\theta - \sigma)\partial_\sigma$, as before. We consider the transport equation

$$(25) \quad \partial_t v - Av = 0,$$

where v depends on σ and, possibly, on some parameters. Let

$$(26) \quad \delta_t(\sigma) := \theta(1 - e^{-\kappa t}) + \sigma e^{-\kappa t},$$

which satisfies $\delta_t(I) \subset I$, by our assumptions on I , and $\delta_t \circ \delta_s = \delta_{t+s}$.

Most of the results listed below are classical, at least for $\lambda = 0$. We state and prove results in the form needed for our purposes for clarity and completeness.

Lemma 4.1. *Let $h \in L^1_{loc}(I)$, and let v be given by the formula*

$$(27) \quad v(t, \sigma) := h(\delta_t(\sigma)).$$

Then v is a weak solution of (25) on $[0, \infty) \times \mathbb{R}$ with $v(0) = h$ (i.e. $v(0, \sigma) = h(\sigma)$). If $h \in C^1(I)$, then v is also a classical solution this equation.

Proof. The proof that v is a classical solution if $h \in C^1(I)$ is by a direct calculation. To prove that v is a weak solution in general, we can consider the change of coordinates $(t, \sigma) = (t, \delta_{-t}(s))$ and then perform an integration by parts in s , using also Fubini's theorem. \square

In what follows, we consider A as operator acting of functions of σ with values in a Hilbert space \mathcal{H} . For the application at hand, \mathcal{H} will be an exponentially weighted Sobolev space. The following proposition justified.

Proposition 4.2. *Let \mathcal{H} be a Hilbert space. Let $T(t)h = v(t)$, where v is as in Lemma 4.1 and $h \in L^2(I; \mathcal{H})$. Then $\|T(t)h\| \leq e^{\kappa t/2}\|h\|$, where the norm is the one on $L^2(I; \mathcal{H})$. Moreover, $T(t)$ is a c_0 semi-group whose generator coincides with A on $C^1(I; \mathcal{H})$.*

Proof. The relation $\|T(t)h\| \leq e^{\kappa t/2}\|h\|$ follows by a change of variables (note also that, for $I = \mathbb{R}$, we have equality). The identity $T(t_1)T(t_2)h = T(t_1 + t_2)h$ follows from $\delta_{t_2}(\delta_{t_1}(\sigma)) = \delta_{t_1+t_2}(\sigma)$. If $h \in C^1(I; \mathcal{H})$, we obtain from the definition that $t^{-1}(T(t)h - h) \rightarrow Ah$. Since $\|T(t)\|$ is uniformly bounded for $t \leq 1$, this gives that $T(t)h \rightarrow h$ as $t \rightarrow 0$ for all h . This completes the proof. \square

Below, we shall write $T(t) = e^{tA}$, a notation that is justified by Corollary 4.4.

Corollary 4.3. *Using the notation of Proposition 4.2, we have that v is a strong solution of Equation (25) for $h \in C^1(I; L^2_\lambda(\mathbb{R}))$. If $h \in C^1(I; H^1_\lambda(\mathbb{R}))$, it is also a classical solution.*

Proof. This follows from Lemma 4.1, Proposition 4.2 and the definitions of strong and classical solutions. \square

We obtain the following consequences for classical solutions of Equation (25).

Corollary 4.4. *Let $\mathcal{K}_1 := H^2_\lambda(I \times \mathbb{R})$. Then $\mathcal{K}_1 \subset C^1(I; L^2_\lambda(\mathbb{R}))$. Using the notation of Proposition 4.2, we have that v is a strong solution of Equation (25) for $h \in \mathcal{K}_1$. Moreover, $e^{tA}(\mathcal{K}_1) \subset \mathcal{K}_1$, e^{tA} defines a c_0 semi-group on \mathcal{K}_1 , and hence $v(t) \in \mathcal{K}_1$.*

Proof. The inclusion $\mathcal{K}_1 \subset \mathcal{C}^1(I; L_\lambda^2(\mathbb{R}))$ is a consequence of the Sobolev's embedding theorem. The fact that v is a strong solution follows from Corollary 4.3, and from the inclusion $\mathcal{K}_1 \subset \mathcal{C}^1(I; L_\lambda^2(\mathbb{R}))$. The inclusion $e^{tA}(\mathcal{K}_1) \subset \mathcal{K}_1$ and the fact that e^{tA} defines a c_0 semi-group on \mathcal{K}_1 follow from the explicit formula for e^{tA} . \square

4.2. The generation property for L_0 . It seems difficult to apply the Lumer-Philips Theorem directly to a degenerate operator of the form of L_0 . We will therefore adopt a different strategy and directly prove that e^{tL_0} is a semigroup generated by L_0 .

Let $\mathcal{K}_1 := H_\lambda^2(I \times \mathbb{R})$, as in Corollary 4.4 in the previous subsection. Also, we recall the function $\delta_t(\sigma) := \theta(1 - e^{-\kappa t}) + \sigma e^{-\kappa t}$ introduced above.

Lemma 4.5. *Let $g : I \rightarrow [0, \infty)$ be a continuous function. Assume that either g is bounded or that the parameter $\lambda = 0$ in the definition of the weight $w(x) = e^{\lambda(x)}$. Then $e^{tA}e^{gB} = e^{(g \circ \delta_t)B}e^{tA}$.*

Proof. The result follows from $e^{(g \circ \delta_t)B}\xi \circ \delta_t = (e^{gB}\xi) \circ \delta_t = e^{tA}e^{gB}\xi$. \square

Let us now define

$$(28) \quad \mathfrak{D}_\kappa(t) = \mathfrak{D}_\kappa(t, \sigma) := \frac{(\theta - \sigma)^2}{4\kappa}(1 - e^{-2\kappa t}) - \frac{\theta(\theta - \sigma)}{\kappa}(1 - e^{-\kappa t}) + \frac{1}{2}\theta^2 t.$$

Proposition 4.6. *The function $\mathfrak{D}(t, \sigma)$ defined in Equation (28) is analytic in $(\kappa, t, \sigma) \in \mathbb{R}^3$ and satisfies $\mathfrak{D}(0, \sigma) = 0$ and $\mathfrak{D}(t, \sigma) > 0$ for any $t > 0$ and any $\sigma \in \mathbb{R}$.*

Proof. The function $\mathfrak{D}(t, \sigma)$ is analytic since the singularity at zero is removable. We shall regard $\mathfrak{D}(t, \sigma)$ as a second order polynomial in σ with coefficients that are functions of the parameters t and κ . We have that the leading coefficient $\frac{1}{4\kappa}(e^{2\kappa t} - 1)$ is always positive as $t > 0$, so we only need to show that the discriminant of $\mathfrak{D}(t, \sigma)$ is non negative. We let $f(t)$ be the discriminant of $\mathfrak{D}(t, \sigma)$ (regarded as a second-order polynomial in σ , as mentioned above), so that

$$(29) \quad f(t) = \frac{\theta^2}{2\kappa^2}[(2 + \kappa t)e^{-2\kappa t} - 4e^{-\kappa t} + 2 - \kappa t].$$

We then have:

$$f'(t) = \frac{\theta^2}{2\kappa}[(3 - 2\kappa t)e^{2\kappa t} - 4e^{\kappa t} + 1] \quad \text{and} \\ f''(t) = 2\theta^2[(1 - \kappa t)e^{2\kappa t} - e^{\kappa t}] = 2\theta^2e^{2\kappa t}[1 - \kappa t - e^{-\kappa t}] < 0 \quad \text{for } t \neq 0.$$

It follows that $f'(t)$ is decreasing, and hence $f'(t) < f'(0) = 0$ for $t > 0$. Consequently, $f(t)$ is also decreasing, which gives $f(t) < f(0) = 0$ for positive t . \square

This lemma allows us to define $e^{\mathfrak{D}(t)B}$ if I is bounded or if $\lambda = 0$. We let then

$$(30) \quad S(t) := e^{\mathfrak{D}(t)B}e^{tA}.$$

Then $S(t)$ is a bounded operator, since it is the composition of bounded operators.

We will establish that $S(t)$ is a *co-semigroup* generated by L_0 by splitting the proof in a few Lemmas for convenience.

Lemma 4.7. *For all $t, s \geq 0$, the family of operators $S(t)$ satisfies:*

- (1) $S(t)S(s) = S(t + s)$;
- (2) $S(t)\mathcal{K}_1 \subset \mathcal{K}_1$.

Proof. We first notice that $\mathfrak{D}(t) + \mathfrak{D}(s) \circ \delta_t = \mathfrak{D}(t+s)$, which is easy to check by direct calculation. By definition, using also Lemma 4.5, we have

$$(31) \quad S(t)S(s) = e^{\mathfrak{D}(t)B} e^{tA} e^{\mathfrak{D}(s)B} e^{sA} = e^{\mathfrak{D}(t)B} e^{(\mathfrak{D}(s) \circ \delta_t)B} e^{tA} e^{sA} \\ = e^{(\mathfrak{D}(t) + \mathfrak{D}(s) \circ \delta_t)B} e^{(t+s)A} = e^{\mathfrak{D}(t+s)B} e^{(t+s)A} = S(t+s).$$

This calculation completes the proof of the first part. The last part follows from Corollaries 4.4 and 3.16. \square

We recall that we assume σ is in a bounded interval $I \subset (0, \infty)$.

Lemma 4.8. *We have that for all $j \geq 0$,*

$$\|\partial_\sigma^j \mathfrak{D}(t)/t - \sigma^2/2\|_{L^\infty(I)} \rightarrow 0 \quad \text{as } t \rightarrow 0, \quad t > 0.$$

Proof. We observe that the function $\partial_\sigma^j \mathfrak{D}(t)/t$, defined on $I \times (0, 1]$, extends to a continuous function on $\bar{I} \times [0, 1]$. Since I is a bounded interval, this fact is enough to provide the result. \square

Lemma 4.9. *The following limits in \mathcal{H} hold:*

- (i) $\lim_{t \searrow 0} S(t)\xi = \xi$ for all $\xi \in \mathcal{H}$ and, similarly,
- (ii) $\lim_{t \searrow 0} t^{-1}(S(t)\xi - \xi) = L_0\xi$ for all $\xi \in \mathcal{K}_1$.

Proof. By the semigroup property, the operators e^{tB} and e^{tA} are uniformly bounded if $0 \leq t \leq \epsilon$, for any fixed $\epsilon > 0$. Since I is a bounded interval, the functions $\mathfrak{D}(t)$ are uniformly bounded for $t \leq \epsilon$. Moreover, $\|\mathfrak{D}(t)\|_{L^\infty(I)} \rightarrow 0$ as $t \searrow 0$. By the definition of $S(t)$, Equation (30), the first part of the lemma follows.

The second part of the lemma is proved in a similar fashion. Indeed, the relations $S(t)\mathcal{K}_1 \subset \mathcal{K}_1$ (see Lemma 4.7), $\mathfrak{D}'(0) = \sigma^2/2$ (see Lemma 4.8), the fact that e^{tA} is a c_0 semi-group that leaves \mathcal{K}_1 invariant (Corollary 4.4), and Lemma 3.13 give that

$$\begin{aligned} \partial_t(T(t)\xi)|_{t=0} &= \partial_t(e^{\mathfrak{E}(t)B} e^{tA}\xi)|_{t=0} = \lim_{t \rightarrow 0} t^{-1}(e^{\mathfrak{E}(t)B} e^{tA}\xi - \xi) \\ &= \lim_{t \rightarrow 0} t^{-1}(e^{\mathfrak{E}(t)B} e^{tA}\xi - e^{tA}\xi) + \lim_{t \rightarrow 0} t^{-1}(e^{tA}\xi - \xi) = \frac{\partial \mathfrak{E}}{\partial t}(0)B\xi + A\xi = L_0\xi, \end{aligned}$$

whenever $\xi \in \mathcal{K}_1$. \square

We have the following similar result on \mathcal{K}_1 :

Lemma 4.10. *The following limits in \mathcal{K}_1 hold:*

- (i) $\lim_{t \searrow 0} S(t)\xi = \xi$ for all $\xi \in \mathcal{K}_1$ and, similarly,
- (ii) $\lim_{t \searrow 0} t^{-1}(S(t)\xi - \xi) = L_0\xi$ for all $\xi \in \mathcal{K}_1$ such that $L_0\xi \in \mathcal{K}_1$.

These limits are valid also as limits in \mathcal{K}_1 if $\xi \in \mathcal{K}_1$ in the first limit and if $\xi \in H^4(I \times \mathbb{R})$ for the second limit.

Proof. The proof is similar to that of Lemma 4.10, but using also the second part of Corollary 3.16. \square

We finally have that L_0 generates the semigroup $S(t)$.

Theorem 4.11. *Let $\kappa > 0$ and $I = (\alpha, \beta)$, with $0 < \alpha < \theta < \beta < \infty$, as before. Then, $S(t) := e^{\mathfrak{D}(t)B} e^{tA}$ defines a c_0 semi-group on \mathcal{H} , the generator of which coincides with L_0 on \mathcal{K}_1 . Moreover, $S(t)$ defines a c_0 semi-group on \mathcal{K}_1 .*

Proof. The first part is an immediate consequence of Lemmas 4.7 and 4.9. The second part uses Lemma 4.10 instead. \square

Obtaining explicit formulas is important in practice because it allows for very fast methods. This is one of the reasons Heston's method [28] is so popular. Explicit formulas lead also to faster methods in inverse approaches to the determination of implied volatility, see [8], for instance.

Corollary 4.12. *Under the assumptions of Theorem 4.11, let $h = h(\sigma, x) \in L^2_\lambda(I \times \mathbb{R}) := e^{\lambda(x)} L^2(I \times \mathbb{R})$ and set $u(t) := S(t)h$. Then, for almost all $\sigma \in I$:*

$$(32) \quad u(t, \sigma, x) := \frac{1}{\sqrt{4\pi\mathfrak{D}}} \int e^{-\frac{|x-y-\mathfrak{D}|^2}{4\mathfrak{D}}} h(\delta_t(\sigma), y) dy$$

and u is a mild solution of the Initial Value Problem: $\partial_t v - Lv = 0$, $v(0) = h$. If $h \in \mathcal{K}$, then u is a strong solution, and a classical solution provided that $h \in \mathcal{C}^{1,2}(I \times \mathbb{R}) \cap L^2_\lambda(I \times \mathbb{R})$.

5. MAPPING PROPERTIES AND ERROR ESTIMATES

In this section, we prove mapping properties between weighted spaces for e^{tL_0} , by deriving another formula for its distributional kernel. We then use these results to compare the semi-groups $S(t) := e^{tL_0}$ and e^{tL} . We continue to assume that $I = (\alpha, \beta)$, $0 < \alpha < \theta < \beta < \infty$, and that $\kappa > 0$.

5.1. Lie algebra identities and semi-groups. In the previous section we used implicitly commutator estimates between the operators A and B . We collect in the remark below results pertaining to a general class of operators with properties similar to the operators A and B , which, with abuse of notation, we continue to denote by A and B .

Remark 5.1. Let V be a finite dimensional space of (usually unbounded) operators acting on some Banach space X , and let A be a closed operator on X with domain $D(A)$. We make the following assumptions

- (i) All operators in V have the same domain \mathcal{K} , which is endowed with a Banach space norm such that, for any $B \in V$, $B : \mathcal{K} \rightarrow X$ is continuous.
- (ii) The space

$$\mathcal{W} := \{ \xi \in D(A), A\xi \in \mathcal{K} \} \cap \{ \xi \in \mathcal{K}, B\xi \in D(A) \ (\forall) B \in V \}$$

is dense in \mathcal{K} in its induced norm.

- (iii) If $B \in V$, the closure of the operator $[A, B]$ with domain \mathcal{W} is in V .
- (iv) A generates a c_0 semi-group of operators on X that leaves \mathcal{K} invariant and induces a c_0 semi-group on \mathcal{K} .

Then, denoting by $e^{t \text{ad}_A} : V \rightarrow V$ the exponential of the endomorphism $\text{ad}_A : V \rightarrow V$ of the finite dimensional space V , we obtain the following Hadamard type formula

$$(33) \quad e^{tA}B = e^{t \text{ad}_A}(B)e^{tA}, \quad (\forall) B \in V.$$

This relation can be proved by considering the function

$$F(t) := e^{tA}B\xi - e^{t \text{ad}_A}(B)e^{tA}\xi, \quad B \in V \text{ and } \xi \in \mathcal{W}.$$

Our assumptions imply that $F(t) \in D(A)$ for all t , that $F(t)$ is differentiable, and that $F'(t) = AF(t)$. By the uniqueness of strong solutions to this evolution equation [2, 47], it follows that $F(t) = 0$ for all $t \geq 0$, since $F(0) = 0$.

We shall use the above remark in the following setting.

Remark 5.2. Let $V = \mathbb{C}\partial_\sigma$ with domain \mathcal{K}_1 , and let $A := \kappa(\theta - \sigma)\partial_\sigma$. consider the adjoint action of A on V . Since

$$A\partial_\sigma - \partial_\sigma A = [A, \partial_\sigma] = [\kappa(\theta - \sigma)\partial_\sigma, \partial_\sigma] = [\kappa(\theta - \sigma)\partial_\sigma, \partial_\sigma] = \kappa\partial_\sigma,$$

it follows that $e^{tA}\partial_\sigma = e^{\kappa t}\partial_\sigma e^{tA}$.

In the same spirit, we have the following.

Remark 5.3. We keep the same notation and assumptions as in 5.1, but we further assume that $V \simeq \oplus_{a \in \mathbb{R}} V_a$, where

$$(34) \quad [A, B_a] := AB_a - B_a A = aB_a, \quad \text{for any } B_a \in V_a, a \in \mathbb{R}.$$

Of course, $V_a = 0$, except for finitely many values $a \in \mathbb{R}$. Let $B \in V$ and decompose it as $B = \sum_{a \in \mathbb{R}} B_a$, with $B_a \in V_a$. We proceed formally to guess a formula for $e^{t(A+B)}$. We write $e^{t(A+B)} = e^{tA}e^{\sum_a f_a(t)B_a}$. Differentiating this inequality, using the semigroup property $\partial_t e^{t(A+B)} = (A+B)e^{t(A+B)}$, that $e^{tA}B_a = e^{ta}B_a e^{tA}$, and identifying the coefficients, we obtain $f_a(t) = (1 - e^{-at})/a = \mathcal{E}(-at)t$, where $\mathcal{E}(s) = (e^s - 1)/s$. Hence, this procedure gives the (formal!) result

$$(35) \quad e^{t(A+B)} = e^{tA}e^{\sum_a \mathcal{E}(-at)tB_a} = e^{\sum_a \mathcal{E}(at)tB_a}e^{tA}.$$

Of course, this procedure has to be justified independently or one has to make sense of all the steps in its derivation. In this paper, we have chosen to verify independently Formula (30). See also [29].

We close by deriving an equivalent formula for $S(T)$, which, by the smoothing properties of e^{tB} , $t > 0$, in x , can be used to show that, if $\xi \in C^1(I; L_\lambda^2(\mathbb{R}))$, then $u(t) = S(t)\xi$ defines a classical solution of $\partial_t u - L_0 u = 0$ for $t > 0$. This result uses also Corollaries 4.3, 4.4, and 3.16. The method of proof is that of the proof of Lemma 5.6. For this purpose, we introduce the function:

$$(36) \quad \mathfrak{C}(t) := \mathfrak{C}(t, \sigma) := \frac{(\theta - \sigma)^2}{4\kappa}(e^{2\kappa t} - 1) - \frac{\theta(\theta - \sigma)}{\kappa}(e^{\kappa t} - 1) + \frac{1}{2}\theta^2 t.$$

We notice that $\mathfrak{C}(t)$ is obtained from $\mathfrak{D}(t)$ by replacing κ with $-\kappa$, so it is still non negative everywhere (see Proposition 4.6). Applying the reasoning in the previous remark, we obtain the following alternative expression for $S(t)$:

$$(37) \quad S(t) := e^{\mathfrak{D}(t)B}e^{tA} = e^{tA}e^{\mathfrak{C}(t)B}.$$

5.2. Mapping properties. We shall need certain mapping properties for the semi-groups e^{tL} and e^{tL_0} , some of which are standard and some of which we prove in this subsection.

Lemma 5.4. *Assume that $I := (\alpha, \beta)$ is bounded and that $\alpha > 0$. Then there exists $\epsilon > 0$ such that $\mathfrak{D}(t, \sigma) \geq \epsilon t$ for $\sigma \in I$ and $t \in [0, 1]$.*

Proof. Let us consider the function $h(t, \sigma) := \mathfrak{D}(t, \sigma)/t$ for $\sigma \in [\alpha, \beta]$ and $t \in (0, 1]$. By Proposition 4.6, h extends to a continuous function on $[\alpha, \beta] \times [0, 1]$. By the assumption that $\alpha > 0$ and by Proposition 4.6, we have that $h > 0$ on $[\alpha, \beta] \times [0, 1]$. Therefore $\epsilon := \inf h > 0$. \square

We recall also the following general fact.

Remark 5.5. If T generates a c_0 semi-group e^{tT} on a Banach space X , then $(e^{tT})^*$ will also be a semi-group (but the strong continuity property may fail). However, if X is *reflexive*, then $(e^{tT})^*$ is strongly continuous and, in fact, $(e^{tT})^*$ is a c_0 semi-group with generator T^* (see Corollary 1.10.6 in [47]). In other words, $(e^{tT})^* = e^{tT^*}$, if X is reflexive. Moreover, if e^{tT} is an analytic semi-group, then $(e^{tT})^*$ is also analytic since the function $(e^{\bar{z}T})^*$ is holomorphic in a domain of the form Δ_δ , $\delta > 0$.

All the norms $\|\cdot\|$ below refer to the norm of vectors in $\mathcal{H} = L_\lambda^2(I \times \mathbb{R})$ or of bounded operators on that space.

Lemma 5.6. *Let $s \geq 0$. There exists $C_s > 0$ such that, for all $h \in \mathcal{H} := L_\lambda^2(I \times \mathbb{R})$,*

$$t^{s/2} \|e^{\mathfrak{D}(t)B} h\|_{H_\lambda^{0,s}(I \times \mathbb{R})} \leq C_s \|h\| := C_s \|h\|_{L_\lambda^2(I \times \mathbb{R})}, \quad \text{for } t \in (0, 1].$$

Consequently, $\|\partial_x^k e^{tL_0}\| \leq Ct^{-k/2}$, where $t \in (0, 1]$ and C is independent of t . In particular, $\partial_x^k e^{tL_0} \xi$ is continuous in t .

Proof. Let us assume first $s = 2n$, for some positive integer n . We have that the norm $\|g\|_{H_\lambda^{0,2n}(I \times \mathbb{R})}$ is equivalent to the norm $\|g\| + \|B^n g\|$, since B is uniformly strongly elliptic on \mathbb{R} with totally bounded coefficients (see Corollary 2.24). In particular, $\|g\|_{H_\lambda^{0,2n}(I \times \mathbb{R})} \leq C(\|g\| + \|B^n g\|)$. It is therefore enough to show that there exists C'_s such that

$$(38) \quad \|e^{\mathfrak{D}(t)B} h\| + \|B^n e^{\mathfrak{D}(t)B} h\| \leq C'_s t^{-n} \|h\|,$$

since then the desired relation follows with $C_s = CC'_s$. Lemma 5.4 gives

$$\begin{aligned} \|e^{\mathfrak{D}(t)B} h\| + \|B^n e^{\mathfrak{D}(t)B} h\| &= \|e^{\mathfrak{D}(t)B} h\| + \|e^{(\mathfrak{D}(t)-\epsilon t)B} B^n e^{\epsilon t B} h\| \\ &\leq C(\|h\| + \|B^n e^{\epsilon t B} h\|) \leq C(\epsilon t)^{-n} \|h\|, \end{aligned}$$

since e^{gB} is bounded on $L_\lambda^2(I \times \mathbb{R})$, if $g \geq 0$ is bounded measurable, and $t^n B^n e^{tB}$ is also bounded on the same space (by Equation (11) for $T = B$). Here, we have used the assumption that I is bounded. This argument proves Equation (38), and consequently also the result for $s = 2n$. For general $s \geq 0$, the result follows by complex interpolation.

To prove the last part, we write

$$\partial_x^{2k} e^{tL_0} = \partial_x^{2k} (\mu_0 - B)^{-k} (\mu_0 - B)^k e^{\mathfrak{D}(t)B} e^{tA},$$

where μ_0 is large. We have that $\partial_x^{2k} (\mu_0 - B)^{-k}$ is bounded by the uniform strong ellipticity of B and Theorem 2.23. Remark 2.14 and Lemmas 3.13 and 5.4 show that $(\mu_0 - B)^k e^{\mathfrak{D}(t)B}$ depends smoothly on t . Next, Remark 2.14 also gives that $\|(\mu_0 - B)^k e^{\mathfrak{D}(t)B}\| \leq Ct^{-k}$. This implies that $\|\partial_x^{2k} e^{tL_0}\| \leq Ct^{-k}$. Our desired estimate $\|\partial_x^k e^{tL_0}\| \leq Ct^{-k/2}$ is then obtained by interpolation. Finally, using also Lemma 3.13, we obtain that $\partial_x^k e^{tL_0}$ depends continuously on t . \square

In the same way, we obtain the following result.

Lemma 5.7. *If $h \in \mathcal{H} := L_\lambda^2(I \times \mathbb{R})$, then*

$$\|e^{tL} h\|_{H_\lambda^s(I \times \mathbb{R})} \leq Ct^{-s/2} \|h\|.$$

If P is a differential operator of order k with totally bounded coefficients on $I \times \mathbb{R}$, then Pe^{tL} and $e^{tL}P$ extend to bounded operators on \mathcal{H} of norm $\leq Ct^{-k/2}$ that depend smoothly on $t > 0$.

Proof. Since L is uniformly strongly elliptic with totally bounded coefficients, there exists $\mu_0 > 0$ such that

$$L - \mu_0 : H_\lambda^{m+1}(I \times \mathbb{R}) \cap \{u(\alpha, x) = u(\beta, x) = 0\} \rightarrow H_\lambda^{m-1}(I \times \mathbb{R}),$$

is an isomorphism by Theorem 2.23 and Corollary 3.4. Let $(L - \mu_0)^{-1}$ denote the resulting map $L_\lambda^2(I \times \mathbb{R}) \rightarrow H_\lambda^2(I \times \mathbb{R})$. Then $(L - \mu_0)^{-1}$ maps $H_\lambda^{m-1}(I \times \mathbb{R}) \rightarrow H_\lambda^{m+1}(I \times \mathbb{R})$ continuously. In particular, $(L - \mu_0)^{-n} : L_\lambda^2(I \times \mathbb{R}) \rightarrow H_\lambda^{2n}(I \times \mathbb{R})$ is continuous. Let us assume now that $s = 2n$. Then

$$\begin{aligned} \|e^{tL}h\|_{H_\lambda^{2n}(I \times \mathbb{R})} &= \|(L - \mu_0)^{-n}(L - \mu_0)^n e^{tL}h\|_{H_\lambda^{2n}(I \times \mathbb{R})} \\ &\leq C\|(L - \mu_0)^n e^{tL}h\|_{L_\lambda^2(I \times \mathbb{R})} \leq Ct^{-s/2}\|h\|, \end{aligned}$$

since L generates an analytic semi-group. For general s , the inequality follows by interpolation.

Let P now be as in the statement of the lemma. Then $P : H_\lambda^k(I \times \mathbb{R}) \rightarrow L_\lambda^2(I \times \mathbb{R})$ is bounded. This implies the result for Pe^{tL} . The result for $e^{tL}P$ is obtained by taking adjoints, since L^* is uniformly strongly elliptic with totally bounded coefficients and generates an analytic semi-group. \square

Lemma 5.7 gives the following result. All norms of operators are on $L_\lambda^2(I \times \mathbb{R})$.

Lemma 5.8. *The operator $F(s) := e^{(t-s)L}\partial_\sigma e^{sL}$ extends, for each $s \in [0, t]$, to a bounded operator on $L_\lambda^2(I \times \mathbb{R})$, and the resulting function is continuous in $s \in [0, t]$ and differentiable for $s \in (0, t)$. Its derivative is the function*

$$F'(s) = e^{(t-s)L}[\partial_\sigma, L]e^{sL},$$

which satisfies $\|F'(s)\| \leq Ct^{-1}$, with C independent of $0 < s < t \leq 1$.

Proof. Lemma 5.7 gives that both functions $e^{(t-s)L}$ and $\partial_\sigma e^{sL}$ are continuous on $(0, T]$ and infinitely many times differentiable on $(0, t)$ as functions with values in the space of bounded operators. The formula for the derivative follow from the standard formula $(e^{sL})' = Le^{sL}$, which we note to be valid in norm, since L generates an analytic semi-group and $s > 0$. The continuity on $[0, t)$ follows in the same way by considering $e^{(t-s)L}\partial_\sigma$ and e^{sL} .

If $s \leq t/2$, since $[\partial_\sigma, L]$ is a second order differential operator, Lemma 5.7 implies that $e^{(t-s)L}[\partial_\sigma, L]$ is bounded with norm $\leq C(t-s)^{-1} \leq 2Ct^{-1}$. In addition, $\|F'(s)\| \leq Ct^{-1}$ given that e^{sL} is norm bounded. The case $s \geq t/2$ is completely similar, using the bounds for $[\partial_\sigma, L]e^{sL}$ provided by Lemma 5.7. \square

5.3. A comparison of e^{tL} and e^{tL_0} . In this last section, we compare the semi-groups $S(t) := e^{tL_0}$ and e^{tL} . We recall that we set $L = L_0 + V$, where $V = \nu L_1 + \nu^2 L_2 = \nu\rho\sigma^2\partial_x\partial_\sigma + \frac{\nu^2\sigma^2}{2}\partial_\sigma^2$, and we think of L as a perturbation of L_0 for ν sufficiently small. We recall also that $\mathcal{K}_1 := H_\lambda^2(I \times \mathbb{R})$ and $\mathcal{K}_0 := H_\lambda^2(I \times \mathbb{R}) \cap \{u(\alpha, x) = u(\beta, x) = 0\}$, where $I = (\alpha, \beta)$ is a fixed bounded interval containing θ .

The approach presented in this subsection can be iterated to derive higher-order approximate solutions in the parameter ν . These are the focus of current work by the authors.

Lemma 5.9. *Let $\xi \in \mathcal{K}_1$. Then $F(s) := e^{(t-s)L}e^{sL_0}\xi$ is continuous on $[0, t]$ and differentiable on $(0, t)$, with $F'(s) = -e^{(t-s)L}Ve^{sL_0}\xi$.*

Proof. Since ξ is in the domain of L_0 (which contains \mathcal{K}_1 , by Theorem 4.11), the function $\zeta(s) := e^{sL_0}\xi$ is differentiable for $s \geq 0$. But e^{tL} is a c_0 semi-group, therefore Lemma 3.13 gives that $F(s) = e^{(t-s)L}\zeta(s)$ is continuous on $[0, t]$. Since e^{tL} is an analytic semi-group, it follows in addition that $F(s)$ is differentiable for $s \in (0, t)$, by Lemma 3.15, and its derivative is $F'(s) = -e^{(t-s)L}Ve^{sL_0}\xi$. \square

We continue to assume that $\|\cdot\|$ refers to the norm in $\mathcal{H} = L_\lambda^2(I \times \mathbb{R})$ or the operator norm of bounded operators on \mathcal{H} .

Lemma 5.10. *Let $\xi \in \mathcal{K}_1$, then $e^{(t-s)L}L_1e^{sL_0}\xi$ depends continuously on s and*

$$(\rho\nu)^{-1}\|e^{(t-s)L}L_1e^{sL_0}\xi\| = \|e^{(t-s)L}\sigma^2\partial_\sigma\partial_xe^{sL_0}\xi\| \leq C(t-s)^{-1/2}s^{-1/2}\|\xi\|.$$

Consequently, $\left\|\int_0^t e^{(t-s)L}L_1e^{sL_0}ds\right\| \leq C\rho\nu$.

Proof. Lemmas 5.6 and 5.7 show that $e^{(t-s)L}\sigma^2\partial_\sigma$ and $\partial_xe^{s(L_0-\kappa)}\xi$ satisfy the assumptions of Lemma 3.13, so $e^{(t-s)L}\sigma^2\partial_\sigma\partial_xe^{s(L_0-\kappa)}\xi$ is continuous in s . Similarly, Lemmas 5.6 and 5.7 give

$$\|e^{(t-s)L}\sigma^2\partial_\sigma\partial_xe^{s(L_0-\kappa)}\xi\| \leq \|e^{(t-s)L}\sigma^2\partial_\sigma\| \|\partial_xe^{s(L_0-\kappa)}\xi\| \leq C(t-s)^{-1/2}s^{-1/2}\|\xi\|.$$

The integral can be estimated by splitting the interval $[0, t]$ in two halves. \square

To estimate the terms involving L_2 , we exploit the next result.

Lemma 5.11. *Let $\xi \in \mathcal{K}_1$, then $\partial_\sigma e^{tL_0}\xi = e^{t(L_0-\kappa)}\partial_\sigma\xi + \frac{\partial\mathfrak{D}(t,\sigma)}{\partial\sigma}Be^{tL_0}\xi$.*

Proof. The main calculation is contained in Remark 5.2. More precisely, this is a direct calculation using Equation (30), together with Lemma 3.15, with Hadamard's theorem (see Remarks 5.1 and 5.2), and with the fact that $\text{ad}_{L_0}(\partial_\sigma)\text{ad}_A(\partial_\sigma) = \kappa\partial_\sigma$. \square

However, the terms in L_2 present some additional challenges, since L_0 is not elliptic.

Lemma 5.12. *Let $\xi \in \mathcal{K}_1$, then $e^{(t-s)L}L_2e^{sL_0}\xi$ depends continuously on s and the following estimate holds:*

$$\frac{2}{\nu^2}\|e^{(t-s)L}L_2e^{sL_0}\xi\| = \|e^{(t-s)L}\sigma^2\partial_\sigma^2e^{sL_0}\xi\| \leq C(t-s)^{-1/2}(\|\partial_\sigma\xi\| + \|\xi\|).$$

Consequently, $\left\|\int_0^t e^{(t-s)L}L_2e^{sL_0}\xi ds\right\| \leq C\nu^2\sqrt{t}(\|\partial_\sigma\xi\| + \|\xi\|)$.

Proof. Lemma 5.11 gives

$$(39) \quad e^{(t-s)L}\sigma^2\partial_\sigma^2e^{sL_0}\xi = e^{(t-s)L}\sigma^2\partial_\sigma\left(e^{s(L_0-\kappa)}\partial_\sigma\xi + \frac{\partial\mathfrak{D}(s,\sigma)}{\partial\sigma}Be^{sL_0}\xi\right).$$

As in the proof of Lemma 5.10, Lemmas 5.7 and 5.6 give that both $e^{(t-s)L}\sigma^2\partial_\sigma e^{sL_0}$ and $e^{(t-s)L}\sigma^2\partial_\sigma\frac{\partial\mathfrak{D}}{\partial\sigma}Be^{sL_0}$ define bounded operators that depend continuously on $s \in (0, t)$ in the strong operator topology. We estimate separately the norm of each of them. Again from Lemma 5.7, we obtain

$$\|e^{(t-s)L}\sigma^2\partial_\sigma e^{s(L_0-\kappa)}\| \leq \|e^{(t-s)L}\sigma^2\partial_\sigma\| \|e^{s(L_0-\kappa)}\| \leq C(t-s)^{-1/2}.$$

For the estimate of the second term, we first notice that $\|\frac{\partial \mathfrak{D}(t, \sigma)}{\partial \sigma}\|_{L^\infty(I)} \leq Ct$, since the function $\frac{\partial \mathfrak{D}(t, \sigma)}{t \partial \sigma}$ extends to a continuous function on $\bar{I} \times [0, 1]$. Hence, $\|\frac{\partial \mathfrak{D}(s, \sigma)}{\partial \sigma} B e^{sL_0}\| \leq \|s B e^{sL_0}\| \leq C$ by Lemma 5.6, and

$$\left\| e^{(t-s)L} \sigma^2 \frac{\partial \mathfrak{D}(s, \sigma)}{\partial \sigma} B e^{sL_0} \right\| \leq \|e^{(t-s)L} \sigma^2 \partial_\sigma\| \left\| \frac{\partial \mathfrak{D}(s, \sigma)}{\partial \sigma} B e^{sL_0} \right\| \leq C(t-s)^{-1/2}.$$

The last two displayed equations and Equation (39) then combine to give the first part of the statement. The last relation in the statement follows directly by integrating the first one. \square

Combining the previous two lemmas we obtain the following corollary.

Corollary 5.13. *The family $G(s) := e^{(t-s)L} V e^{sL_0}$ consists of bounded operators on \mathcal{H} . Moreover, for any $\xi \in \mathcal{K}_1$, $G(s)\xi$ is continuous and integrable in $s \in (0, t)$ and we have:*

$$\left\| \int_0^t G(s) \xi \, ds \right\| := \left\| \int_0^t e^{(t-s)L} V e^{sL_0} \xi \, ds \right\| \leq C \left(\rho \nu \|\xi\| + \nu^2 \sqrt{t} (\|\partial_\sigma \xi\| + \|\xi\|) \right).$$

Lemma 5.9 and Corollary 5.13 then give:

$$e^{tL} \xi - e^{tL_0} \xi = F(0) - F(t) = \int_0^t e^{(t-s)L} V e^{sL_0} \xi \, ds.$$

The final estimate is for $\xi \in H^1(I, L_\lambda^2(\mathbb{R})) := \{\zeta \in L_\lambda^2(I \times \mathbb{R}), \partial_\sigma \zeta \in L_\lambda^2(I \times \mathbb{R})\}$.

Theorem 5.14. *There is $C > 0$ such that*

$$\|e^{tL} \xi - e^{tL_0} \xi\| \leq C \nu (\|\xi\| + \nu \|\partial_\sigma \xi\|),$$

for $\xi \in H^1(I, L_\lambda^2(\mathbb{R}))$ and $0 \leq t \leq T$. The bound C depends on T , but not on ξ .

Proof. The statement was proved for $\xi \in \mathcal{K}_1$. For general ξ , it follows from the density of $\mathcal{K}_1 := H_\lambda^2(I \times \mathbb{R})$ in $H^1(I, L_\lambda^2(\mathbb{R}))$ and the continuity on $H^1(I, L_\lambda^2(\mathbb{R}))$ of all the operators appearing on the left and right sides of the inequality. \square

We close by observing that similar commutator estimates were obtained in [9, 10, 12, 22]. The main difficulty addressed in this work is that L_0 is not an elliptic operator.

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