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# Exponential decay to equilibrium for a fibre lay-down process on a moving conveyor belt

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## Abstract

We show existence and uniqueness of a stationary state for a kinetic Fokker-Planck equation modelling the fibre lay-down process in the production of non-woven textiles. Following a micro-macro decomposition, we use hypocoercivity techniques to show exponential convergence to equilibrium with an explicit rate assuming the conveyor belt moves slow enough. This work is an extension of (Dolbeault et al., 2013), where the authors consider the case of a stationary conveyor belt. Adding the movement of the belt, the global Gibbs state is not known explicitly. We thus derive a more general hypocoercivity estimate from which existence, uniqueness and exponential convergence can be derived. To treat the same class of potentials as in (Dolbeault et al., 2013), we make use of an additional weight function following the Lyapunov functional approach in (Kolb et al., 2013).

**Key-words:** Hypocoercivity, rate of convergence, fibre lay-down, existence and uniqueness of a stationary state, perturbation, moving belt,

**AMS Class. No:** 35B20, 35B40, 35B45, 35Q84

## 1 Introduction

The mathematical analysis of the fibre lay-down process in the production of non-woven textiles has seen a lot of interest in recent years [14, 15, 6, 10, 11, 4, 12]. Non-woven materials are produced in melt-spinning operations: hundreds of individual endless fibres are obtained by continuous extrusion through nozzles of a melted polymer. The nozzles are densely and equidistantly placed in a row at a spinning beam. The visco-elastic, slender and in-extensible fibres lay down on a moving conveyor belt to form a web, where they solidify due to cooling air streams. Before touching the conveyor belt, the fibres become entangled and form loops due to highly turbulent air flow. In [14] a general mathematical model for the fibre dynamics is presented which enables the full simulation of the process. Due to the huge amount of physical details, these simulations of the fibre spinning and lay-down usually require an extremely large computational effort and high memory storage, see

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[15]. Thus, a simplified two-dimensional stochastic model for the fibre lay-down process, together with its kinetic limit, is introduced in [6]. Generalisations of the two-dimensional stochastic model [6] to three dimensions have been developed by Klar et al. in [10] and to any dimension  $d \geq 2$  by Grothaus et al. in [7].

We now describe the model we are interested in, which comes from [6]. We track the position  $x(t) \in \mathbb{R}^2$  and the angle  $\alpha(t) \in \mathbb{S}^1$  of the fibre at the lay-down point where it touches the conveyor belt. Interactions of neighbouring fibres are neglected. If  $x_0(t)$  is the lay-down point in the coordinate system following the conveyor belt, then the tangent vector of the fibre is denoted by  $\tau(\alpha(t))$  with  $\tau(\alpha) = (\cos \alpha, \sin \alpha)$ . Since the extrusion of fibres happens at a constant speed, and the fibres are in-extensible, the lay-down process can be assumed to happen at constant normalised speed  $\|x'_0(t)\| = 1$ . If the conveyor belt moves with constant speed  $\kappa$  in direction  $e_1 = (1, 0)$ , then

$$\frac{dx}{dt} = \tau(\alpha) + \kappa e_1.$$

Note that the speed of the conveyor belt cannot exceed the lay-down speed:  $0 \leq \kappa \leq 1$ . The fibre dynamics in the deposition region close to the conveyor belt are dominated by the turbulent air flow. Applying this concept, the dynamics of the angle  $\alpha(t)$  can be described by a deterministic force moving the lay-down point towards the equilibrium  $x = 0$  and by a Brownian motion modelling the effect of the turbulent air flow. We obtain the following stochastic differential equation for the random variable  $X_t = (x_t, \alpha_t)$  on  $\mathbb{R}^2 \times \mathbb{S}^1$ ,

$$\begin{cases} dx_t &= (\tau(\alpha_t) + \kappa e_1) dt, \\ d\alpha_t &= [-\tau^\perp(\alpha_t) \cdot \nabla_x V(x_t)] dt + A dW_t, \end{cases} \quad (1)$$

where  $W_t$  denotes a one-dimensional Wiener process,  $A > 0$  measures its strength relative to the deterministic forcing,  $\tau^\perp = (-\sin \alpha, \cos \alpha)$ , and  $V : \mathbb{R}^2 \rightarrow \mathbb{R}$  is an external potential carrying information on the coiling properties of the fibre. More precisely, since a curved fibre tends back to its starting point, the change of the angle  $\alpha$  is assumed to be proportional to  $\tau^\perp(\alpha) \cdot \nabla_x V(x)$ . It has been shown in [12] that under suitable assumptions on the external potential  $V$ , the fibre lay down process (1) has a unique invariant distribution and is even geometrically ergodic (see Remark 1.2). The stochastic approach yields exponential convergence in total variation norm, however without explicit rate. We will show here that a stronger result can be obtained with a functional analysis approach. Our argument uses crucially the construction of an additional weight functional for the fibre lay-down process in the case of unbounded potential gradients inspired by [12, Proposition 3.7].

The probability density function  $f(t, x, \alpha)$  corresponding to the stochastic process (1) is governed by the Fokker-Planck equation

$$\partial_t f + (\tau + \kappa e_1) \cdot \nabla_x f - \partial_\alpha \left( \tau^\perp \cdot \nabla_x V f \right) = D \partial_{\alpha\alpha} f \quad (2)$$

with diffusivity  $D = A^2/2$ . We state below assumptions on the external potential  $V$  that will be used regularly throughout the paper:

**(H1) Regularity and symmetry:**  $V \in C^2(\mathbb{R}^2)$  and  $V$  is spherically symmetric outside some ball  $B(0, R_V)$ .

**(H2)** *Normalisation*:  $\int_{\mathbb{R}^2} e^{-V(x)} dx = 1$ .

**(H3)** *Spectral gap condition* (Poincaré inequality): there exists a positive constant  $\Lambda$  such that for any  $u \in H^1(e^{-V} dx)$  with  $\int_{\mathbb{R}^2} u e^{-V} dx = 0$ ,

$$\int_{\mathbb{R}^2} |\nabla_x u|^2 e^{-V} dx \geq \Lambda \int_{\mathbb{R}^2} u^2 e^{-V} dx.$$

**(H4)** *Pointwise regularity condition on the potential*: there exists  $c_1 > 0$  such that for any  $x \in \mathbb{R}^2$ , the Hessian  $\nabla_x^2 V$  of  $V(x)$  satisfies

$$|\nabla_x^2 V(x)| \leq c_1(1 + |\nabla_x V(x)|).$$

**(H5)** *Behaviour at infinity*:

$$\lim_{|x| \rightarrow \infty} \frac{|\nabla_x V(x)|}{V(x)} = 0, \quad \lim_{|x| \rightarrow \infty} \frac{|\nabla_x^2 V(x)|}{|\nabla_x V(x)|} = 0.$$

**Remark 1.1.** Assumptions **(H2-3-4)** are as stated in [4]. Assumption **(H1)** assumes regularity of the potential that is stronger and included in that discussed in [4] since **(H1)** implies  $V \in W_{loc}^{2,\infty}(\mathbb{R}^2)$ . Assumption **(H5)** is only necessary if the potential gradient  $|\nabla_x V|$  is unbounded. Both bounded and unbounded potential gradients may appear depending on the physical context, and we will treat these two cases separately where necessary. A typical example for an external potential satisfying assumptions **(H1-2-3-4-5)** is given by

$$V(x) = K(1 + |x|^2)^{s/2} \tag{3}$$

for some constants  $K > 0$  and  $s \geq 1$  [5, 12]. The potential (3) satisfies **(H3)** since

$$\liminf_{|x| \rightarrow \infty} (|\nabla_x V|^2 - 2\Delta_x V) > 0,$$

see for instance [17, A.19. Some criteria for Poincaré inequalities, page 135]. The other assumptions are trivially satisfied as can be checked by direct inspection. In this family of potentials, the gradient  $\nabla_x V$  is bounded for  $s = 1$  and unbounded for  $s > 1$ .

**Remark 1.2.** The proof of ergodicity in [12] assumes that the potential satisfies

$$\lim_{|x| \rightarrow \infty} \frac{|\nabla_x V(x)|}{V(x)} = 0, \quad \lim_{|x| \rightarrow \infty} \frac{|\nabla_x^2 V(x)|}{|\nabla_x V(x)|} = 0, \quad \lim_{|x| \rightarrow \infty} |\nabla_x V(x)| = \infty. \tag{4}$$

Under these assumptions, there exists an invariant distribution  $\nu$  to the fibre lay-down process (1), and some constants  $C(x_0) > 0$ ,  $\lambda > 0$  such that

$$\|\mathcal{P}_{x_0, \alpha_0}(X_t \in \cdot) - \nu\|_{TV} \leq C(x_0)e^{-\lambda t},$$

where  $\|\cdot\|_{TV}$  denotes the total variation norm. The stochastic Lyapunov technique applied in [12] however does not give any information on how the constant  $C(x_0)$  depends on the initial position  $x_0$ , or how the rate of convergence  $\lambda$  depends on the conveyor belt speed  $\kappa$ , the potential  $V$  and the

noise strength  $A$ . This can be achieved using hypocoercivity techniques, proving convergence in a weighted  $L^2$ -norm, which is slightly stronger than the convergence in total variation norm shown in [12]. Conceptually, the conditions (4) ensure that the potential  $V$  is driving the process back inside a compact set where the noise can be controlled. Our framework **(H1-2-3-4-5)** is more general than conditions (4) in some aspects (including bounded potential gradient) and more restrictive in others (assuming a Poincaré inequality). The proof in [12] relies on the strong Feller property which can be translated in some cases into a spectral gap; it also uses hypoellipticity to deduce the existence of a transition density, and concludes via an explicit Lyapunov function argument. With our framework **(H1-2-3-4-5)**, and adapting the Lyapunov function argument presented in [12] to control the effect of  $\kappa \partial_{x_1}$ , we derive an explicit rate of convergence in terms of  $\kappa$ ,  $D$  and  $V$ .

To set up a functional framework, rewrite (2) as

$$\partial_t f = \mathbf{L}_\kappa f = (\mathbf{Q} - \mathbf{T}) f + \mathbf{P}_\kappa f, \quad (5)$$

where the collision operator  $\mathbf{Q} := D \partial_{\alpha\alpha}$  acts as a multiplier in the space variable  $x$ ,  $\mathbf{P}_\kappa$  is the perturbation introduced by the moving belt with respect to [4]:

$$\mathbf{P}_\kappa f := -\kappa e_1 \cdot \nabla_x f,$$

and the transport operator  $\mathbf{T}$  is given by

$$\mathbf{T} f := \tau \cdot \nabla_x f - \partial_\alpha \left( \tau^\perp \cdot \nabla_x V f \right).$$

We consider solutions to (5) in the space  $L^2(d\mu_\kappa) := L^2(\mathbb{R}^2 \times \mathbb{S}^1, d\mu_\kappa)$  with measure

$$d\mu_\kappa(x, \alpha) = \left( e^{V(x)} + \zeta \kappa g(x, \alpha) \right) \frac{dx d\alpha}{2\pi}.$$

We denote by  $\langle \cdot, \cdot \rangle_\kappa$  the corresponding scalar product and by  $\| \cdot \|_\kappa$  the associated norm. Here,  $\zeta > 0$  is a free parameter to be chosen later. The construction of the weight  $g$  depends on the boundedness of  $\nabla_x V$ . When it is bounded, no additional weight is needed to control the perturbation, and so we simply set  $g \equiv 0$  in that case. When the gradient is unbounded, the weight is constructed thanks to the following proposition:

**Proposition 1.3.** *Assume that  $V$  satisfies **(H1)** and **(H5)** and that*

$$\lim_{|x| \rightarrow \infty} |\nabla_x V| = +\infty.$$

*If  $\kappa < 1/3$  holds true, then there exists a function  $g(x, \alpha)$ , a constant  $c = c(\kappa, D) > 0$  and a finite radius  $R = R(\kappa, D, V) > 0$  such that*

$$\forall |x| > R, \forall \alpha \in \mathbb{S}^1, \quad \mathcal{L}_\kappa(g)(x, \alpha) \leq -c |\nabla_x V(x)| g(x, \alpha), \quad (6)$$

where  $\mathcal{L}_\kappa$  is defined by

$$\mathcal{L}_\kappa(h) := D \partial_{\alpha\alpha} h + (\tau + \kappa e_1) \cdot \nabla_x h - \left( \tau^\perp \cdot \nabla_x V \right) \partial_\alpha h - (\tau \cdot \nabla_x V) h. \quad (7)$$

The weight  $g$  is of the form

$$g(x, \alpha) := \exp \left( \beta V(x) + |\nabla_x V(x)| \Gamma \left( \tau(\alpha) \cdot \frac{\nabla_x V(x)}{|\nabla_x V(x)|} \right) \right),$$

where the parameter  $\beta > 1$  and the function  $\Gamma \in C^1([-1, 1])$ ,  $\Gamma > 0$  are determined along the proof and only depend on  $\kappa$ .

We show in Section 3 the existence of such a weight function  $g$  under appropriate conditions following ideas from [12].

We denote  $\mathcal{C} := C_c^\infty(\mathbb{R}^2 \times \mathbb{S}^1)$ , and define the orthogonal projection  $\Pi$  on the set of local equilibria  $\text{Ker } Q$

$$\Pi f := \int_{\mathbb{S}^1} f \frac{d\alpha}{2\pi},$$

and the *mass*  $M_f$  of a given distribution  $f \in L^2(d\mu_\kappa)$ ,

$$M_f := \int_{\mathbb{R}^2 \times \mathbb{S}^1} f \frac{dx d\alpha}{2\pi}.$$

Integrating (2) over  $\mathbb{R}^2 \times \mathbb{S}^1$  shows that the mass of solutions of (2) is conserved over time, and standard maximum principle arguments show that it remains non-negative for non-negative initial data. The collision operator  $Q$  is symmetric and satisfies

$$\forall f \in \mathcal{C}, \quad \langle Qf, f \rangle_0 = -D \|\partial_\alpha f\|_0^2 \leq 0,$$

i.e.  $Q$  is dissipative in  $L^2(d\mu_0)$ . Further, we have  $T\Pi f = e^{-V} \tau \cdot \nabla_x u_f$  for  $f \in \mathcal{C}$ , with  $u_f := e^V \Pi f$ , which implies  $\Pi T \Pi = 0$  on  $\mathcal{C}$ . Since the transport operator  $T$  is skew-symmetric with respect to  $\langle \cdot, \cdot \rangle_0$ ,

$$\langle L_\kappa f, f \rangle_0 = \langle Qf, f \rangle_0 + \langle P_\kappa f, f \rangle_0$$

for any  $f$  in  $\mathcal{C}$ . In the case  $\kappa = 0$ , if the entropy dissipation  $-\langle Qf, f \rangle_0$  was coercive with respect to the norm  $\|\cdot\|_0$ , exponential decay to zero would follow as  $t \rightarrow \infty$ . However, such a coercivity property cannot hold since  $Q$  vanishes on the set of local equilibria. Instead, Dolbeault et al. [5] applied a strategy called *hypocoercivity* (as theorised in [17]) and developed by several groups in the 2000s, see for instance [9, 8, 13, 2, 3]. The full hypocoercivity analysis of the long time behaviour of solutions to this kinetic model in the case of a stationary conveyor belt,  $\kappa = 0$ , is completed in [4]. For technical applications in the production process of non-wovens, one is interested in a model including the movement of the conveyor belt, and our aim is to extend the results in [4] to small  $\kappa > 0$ .

We follow the approach of hypocoercivity for linear kinetic equations conserving mass developed in [5], with several new difficulties. Considering the case  $\kappa = 0$ ,  $Q$  and  $T$  are closed operators on  $L^2(d\mu_0)$  such that  $Q - T$  generates the  $\mathcal{C}_0$ -semigroup  $e^{(Q-T)t}$  on  $L^2(d\mu_0)$ . When  $\kappa > 0$ , we use the additional weight function  $g > 0$  to control the perturbative term  $P_\kappa$  in the case of unbounded potential gradients; and show the existence of a  $\mathcal{C}_0$ -semigroup for  $L_\kappa = Q - T + P_\kappa$  (see Section 4.1). Unless otherwise specified, all computations are performed on the operator core  $\mathcal{C}$ , and can be extended to  $L^2(d\mu_\kappa)$  by density arguments.

When  $\kappa = 0$ , the hypocoercivity result in [5, 4] is based on: *microscopic coercivity*, which assumes that the restriction of  $Q$  to  $(\text{Ker } Q)^\perp$  is coercive, and *macroscopic coercivity*, which is a spectral gap-like inequality for the operator obtained when taking a parabolic drift-diffusion limit, in other words, the restriction of  $T$  to  $\text{Ker } Q$  is coercive. The two properties are satisfied in the case of a stationary conveyor belt:

- The operator  $Q$  is symmetric and the Poincaré inequality on  $\mathbb{S}^1$ ,

$$\frac{1}{2\pi} \int_{\mathbb{S}^1} |\partial_\alpha f|^2 d\alpha \geq \frac{1}{2\pi} \int_{\mathbb{S}^1} \left( f - \frac{1}{2\pi} \int_{\mathbb{S}^1} f d\alpha \right)^2 d\alpha,$$

implies that  $-\langle Qf, f \rangle_0 \geq D \|(1 - \Pi)f\|_0^2$ .

- The operator  $\mathsf{T}$  is skew-symmetric and for any  $h \in L^2(d\mu_0)$  such that  $u_h = e^V \Pi h \in H^1(e^{-V} dx)$  and  $\int_{\mathbb{R}^2 \times \mathbb{S}^1} h d\mu_0 = 0$ , **(H3)** implies

$$\|\mathsf{T}\Pi h\|_0^2 = \frac{1}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^1} e^{-V} |\nabla_x u_h|^2 dx d\alpha \geq \frac{\Lambda}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^1} e^{-V} u_h^2 dx d\alpha = \frac{\Lambda}{2} \|\Pi h\|_0^2.$$

In the case  $\kappa = 0$ , the unique global normalised equilibrium distribution  $F_0 = e^{-V}$  lies in the intersection of the null spaces of  $\mathsf{T}$  and  $\mathsf{Q}$ . When  $\kappa > 0$ ,  $F_0$  is not in the kernel of  $\mathsf{P}_\kappa$  and we are not able to find the global Gibbs state of (5) explicitly. However, the hypocoercivity theory is based on a priori estimates [5] that are, as we shall prove, to some extent stable under perturbation. Our main result reads:

**Theorem 1.4.** *Let  $f_{\text{in}} \in L^2(d\mu_\kappa)$  and let **(H1-2-3-4-5)** hold. For  $0 < \kappa < 1$  small enough (with a quantitative estimate) and  $\zeta > 0$  large enough (with a quantitative estimate), there exists a unique non-negative stationary state  $F_\kappa \in L^2(d\mu_\kappa)$  with unit mass  $M_{F_\kappa} = 1$ . In addition, for any solution  $f$  of (2) in  $L^2(d\mu_\kappa)$  with mass  $M_f$  and subject to the initial condition  $f(t=0) = f_{\text{in}}$ , we have*

$$\|f(t, \cdot) - M_f F_\kappa\|_\kappa \leq C \|f_{\text{in}} - M_f F_\kappa\|_\kappa e^{-\lambda_\kappa t}, \quad (8)$$

where the rate of convergence  $\lambda_\kappa > 0$  depends only on  $\kappa$ ,  $D$  and  $V$ , and the constant  $C > 0$  depends only on  $D$  and  $V$ .

In the case of a stationary conveyor belt  $\kappa = 0$  considered in [4], the stationary state is characterised by the eigenpair  $(\Lambda_0, F_0)$  with  $\Lambda_0 = 0$ ,  $F_0 = e^{-V}$ , and so  $\text{Ker } \mathsf{L}_0 = \langle F_0 \rangle$ . This means that there is an isolated eigenvalue  $\Lambda_0 = 0$  and a spectral gap of size at least  $[-\lambda_0, 0]$  with the rest of the spectrum  $\Sigma(\mathsf{L}_0)$  to the left of  $-\lambda_0$  in the complex plane. Adding the movement of the conveyor belt, Theorem 1.4 shows that  $\text{Ker } \mathsf{L}_\kappa = \langle F_\kappa \rangle$  and the exponential decay to equilibrium with rate  $\lambda_\kappa$  corresponds to a spectral gap of size at least  $[-\lambda_\kappa, 0]$ . Further, it allows to recover an explicit expression for the rate of convergence  $\lambda_0$  for  $\kappa = 0$  (see Step 5 in Section 2.1). In general, we are not able to compute the stationary state  $F_\kappa$  for  $\kappa > 0$  explicitly, but  $F_\kappa$  converges to  $F_0 = e^{-V}$  weakly as  $\kappa \rightarrow 0$  (see the discussion in Section 5). Let us finally emphasize that a specific contribution of our paper is to introduce *two* (and not one as in [5, 4]) modifications of the entropy: 1) we first modify the *space itself* with the coercivity weight  $g$ , then 2) we change the norm with an auxiliary operator following the hypocoercivity approach.

The rest of the paper deals with the case  $\kappa > 0$  and is organised as follows. In Section 2, we prove the main hypocoercivity estimate. This allows us to establish the existence of solutions to (2) using semigroup theory and to deduce the existence and uniqueness of a steady state in Section 4 by a contraction argument. In Section 3, we give a detailed definition of the weight function  $g$  that is needed for the hypocoercivity estimate in Section 2.

## 2 Hypocoercivity estimate

Following [5] we introduce the auxiliary operator

$$\mathsf{A} := (1 + (\mathsf{T}\Pi)^*(\mathsf{T}\Pi))^{-1}(\mathsf{T}\Pi)^*,$$

and a modified entropy, i.e. a *hypocoercivity functional*  $G$  on  $L^2(d\mu_\kappa)$ :

$$G[f] := \frac{1}{2} \|f\|_\kappa^2 + \varepsilon_1 \langle Af, f \rangle_0, \quad f \in L^2(d\mu_\kappa)$$

for some suitably chosen  $\varepsilon_1 \in (0, 1)$  to be determined later. It follows from [5] that  $|\langle Af, f \rangle_0| \leq \|f\|_0^2$ . Also,  $\|f\|_0^2 \leq \|f\|_\kappa^2$  by construction of  $\mu_\kappa$ , and hence  $G[\cdot]$  is norm-equivalent to  $\|\cdot\|_\kappa^2$ :

$$\forall f \in L^2(d\mu_\kappa), \quad \left( \frac{1 - \varepsilon_1}{2} \right) \|f\|_\kappa^2 \leq G[f] \leq \left( \frac{1 + \varepsilon_1}{2} \right) \|f\|_\kappa^2, \quad (9)$$

In this section, we prove the following hypocoercivity estimate:

**Proposition 2.1.** *Assume that hypothesis **(H1-2-3-4-5)** hold and that  $0 < \kappa < 1$  is small enough (with a quantitative estimate). Let  $f_{\text{in}} \in L^2(d\mu_\kappa)$  and  $f = f(t, x, \alpha)$  be a solution of (2) in  $L^2(d\mu_\kappa)$  subject to the initial condition  $f(t = 0) = f_{\text{in}}$ . Then  $f$  satisfies the following Grönwall type estimate:*

$$\frac{d}{dt} G[f(t, \cdot)] \leq -\gamma_1 G[f(t, \cdot)] + \gamma_2 M_f^2, \quad (10)$$

where  $\gamma_1 > 0, \gamma_2 > 0$  are explicit constants only depending on  $\kappa, D$  and  $V$ .

Note that the estimate (10) is stronger than what is required for the uniqueness of a global Gibbs state, and represents an extension of the estimate given in [4]. When applied to the difference of two solutions with the same mass, (10) gives an estimate on the exponential decay rate towards equilibrium.

## 2.1 Proof of Proposition 2.1

Differentiate in time  $G[f]$  to get

$$\frac{d}{dt} G[f] = D_0[f] + D_1[f] + D_2[f] + D_3[f],$$

where the *entropy dissipation functionals*  $D_0, D_1, D_2$  and  $D_3$  are given by

$$\begin{aligned} D_0[f] &:= \langle Qf, f \rangle_0 - \varepsilon_1 \langle AT\Pi f, \Pi f \rangle_0 - \varepsilon_1 \langle AT(1 - \Pi)f, \Pi f \rangle_0 \\ &\quad + \varepsilon_1 \langle TAf, (1 - \Pi)f \rangle_0 + \varepsilon_1 \langle AQf, \Pi f \rangle_0, \\ D_1[f] &:= \varepsilon_1 \langle AP_\kappa f, \Pi f \rangle_0 + \varepsilon_1 \langle P_\kappa^* Af, \Pi f \rangle_0, \\ D_2[f] &:= \langle P_\kappa f, f \rangle_0, \\ D_3[f] &:= \kappa \zeta \int_{\mathbb{R}^2 \times \mathbb{S}^1} L_\kappa(f) f g \frac{dx d\alpha}{2\pi}. \end{aligned}$$

Note that the term  $\langle LAf, f \rangle_0$  vanishes since it has been shown in [5] that  $A = \Pi A$  and hence  $Af \in \text{Ker } Q$ . Further,  $\langle Tf, f \rangle = 0$  since  $T$  is skew-symmetric. We estimate the entropy dissipation of the case  $\kappa = 0$  as in [4]:

### # Step 1: Estimation of $D_0[f]$ .

We will show the boundedness of  $D_0$ , which is in fact the dissipation functional for a stationary conveyor belt. We thus recall without proof in the following lemma some results from [4].



**Lemma 2.2** (Dolbeault et al. [4]). *The following estimates hold:*

$$\langle Qf, f \rangle_0 \leq -\|(1 - \Pi)f\|_0^2, \quad \|\mathbf{A}\mathbf{T}(1 - \Pi)f\|_0 \leq C_V \|(1 - \Pi)f\|_0,$$

$$\|\mathbf{A}Qf\|_0 \leq \frac{D}{2} \|(1 - \Pi)f\|_0, \quad \|\mathbf{T}\mathbf{A}f\|_0 \leq \|(1 - \Pi)f\|_0.$$

In order to control the contribution  $\langle \mathbf{A}\mathbf{T}\Pi f, \Pi f \rangle_0$  in  $D_0$ , we note that

$$\mathbf{A}\mathbf{T}\Pi = (1 + (\mathbf{T}\Pi)^*\mathbf{T}\Pi)^{-1} (\mathbf{T}\Pi)^*\mathbf{T}\Pi$$

shares its spectral decomposition with  $(\mathbf{T}\Pi)^*\mathbf{T}\Pi$ , and by *macroscopic coercivity*

$$\langle (\mathbf{T}\Pi)^*\mathbf{T}\Pi f, f \rangle_0 = \|\mathbf{T}\Pi f\|_0^2 = \|\mathbf{T}\Pi(f - M_f e^{-V})\|_0^2 \geq \frac{\Lambda}{2} \|\Pi(f - M_f e^{-V})\|_0^2.$$

Hence,

$$\langle \mathbf{A}\mathbf{T}\Pi f, f \rangle_0 \geq \frac{\Lambda/2}{1 + \Lambda/2} \|\Pi(f - M_f e^{-V})\|_0^2.$$

Now, recalling Lemma 2.2 and using  $\|\Pi(f - M_f e^{-V})\|_0^2 = \|\Pi f\|_0^2 - M_f^2$ , we estimate

$$D_0[f] \leq (\varepsilon_1 - D) \|(1 - \Pi)f\|_0^2 + \varepsilon_1 \lambda_2 \|(1 - \Pi)f\|_0 \|\Pi f\|_0 - \varepsilon_1 \gamma_2 \left( \|\Pi f\|_0^2 - M_f^2 \right),$$

with  $\lambda_2 := C_V + D/2 > 0$  and  $\gamma_2 := \frac{\Lambda/2}{1 + \Lambda/2} > 0$ .

### # Step 2: Estimation of $D_1[f]$ .

We now turn to the entropy dissipation functional  $D_1$ , which we will estimate using elliptic regularity. Instead of bounding  $\mathbf{A}\mathbf{P}_\kappa$ , we apply an elliptic regularity strategy to its adjoint, as for  $\mathbf{A}\mathbf{T}(1 - \Pi)$  in [4]. Let  $f \in L^2(d\mu_0)$  and define  $h := (1 + (\mathbf{T}\Pi)^*\mathbf{T}\Pi)^{-1} f$  so that  $u_h = e^V \Pi h$  satisfies

$$\Pi f = e^{-V} u_h + \Pi \mathbf{T}^* \mathbf{T} (e^{-V} u_h) = e^{-V} u_h - \frac{1}{2} \nabla_x \cdot (e^{-V} \nabla_x u_h).$$

We have used here the fact that in the space  $L^2(d\mu_0)$ :

$$\begin{cases} \mathbf{T} = \tau \cdot \nabla_x - \partial_\alpha [(\tau^\perp \cdot \nabla_x V)] , \\ \mathbf{T}^* = -\tau \cdot \nabla_x + (\tau^\perp \cdot \nabla_x V) \partial_\alpha - (\tau \cdot \nabla_x V) . \end{cases}$$

Then

$$\mathbf{A}^* f = \mathbf{T}\Pi h = e^{-V} \tau \cdot \nabla_x u_h,$$

and since the adjoint for  $\langle \cdot, \cdot \rangle_0$  of the perturbation operator  $\mathbf{P}_\kappa$  is given by

$$\mathbf{P}_\kappa^* = -\mathbf{P}_\kappa - \mathbf{P}_\kappa V,$$

it follows that

$$\begin{aligned}
\|(\mathbf{A}\mathbf{P}_\kappa)^*f\|_0^2 &= \|\kappa \tau \cdot \nabla_x (e_1 \cdot \nabla_x u_h) e^{-V}\|_0^2 \\
&= \frac{\kappa^2}{2} \int_{\mathbb{R}^2 \times \mathbb{S}^1} e^{-V} |\tau \cdot \nabla_x (e_1 \cdot \nabla_x u_h)|^2 d\mu_0 \\
&= \frac{\kappa^2}{2} \int_{\mathbb{R}^2} e^{-V} |\nabla_x (e_1 \cdot \nabla_x u_h)|^2 dx \\
&\leq \frac{\kappa^2}{2} \|\nabla_x^2 u_h\|_{L^2(e^{-V} dx)}^2 \\
&\leq \frac{\kappa^2}{2} C_V^2 \|\Pi f\|_0^2,
\end{aligned}$$

where in the last inequality we have used an elliptic regularity estimate. This estimate turns out to be a particular case of [4, Proposition 5 and Sections 2-3], where the positive constant  $C_V$  is the same as in Lemma 2.2 reproduced from [4]. This concludes the boundedness of  $\mathbf{A}\mathbf{P}_\kappa$ ,

$$\|\mathbf{A}\mathbf{P}_\kappa f\|_0 \leq \kappa \frac{C_V}{\sqrt{2}} \|\Pi f\|_0 \leq \kappa \frac{C_V}{\sqrt{2}} \|f\|_0. \quad (11)$$

Using a similar approach for the operator  $\mathbf{P}_\kappa^* \mathbf{A}$ , we rewrite its adjoint as

$$\mathbf{A}^* \mathbf{P}_\kappa f = \mathbf{T} \Pi \tilde{h},$$

where we define  $\tilde{h} := (1 + (\mathbf{T}\Pi)^* \mathbf{T}\Pi)^{-1} \mathbf{P}_\kappa f$  for a given  $f \in L^2(d\mu_0)$ , or equivalently

$$e^{-V} u_{\tilde{h}} - \frac{1}{2} \nabla_x \cdot (e^{-V} \nabla_x u_{\tilde{h}}) = \Pi \mathbf{P}_\kappa f = \mathbf{P}_\kappa \Pi f.$$

Multiplying by  $u_{\tilde{h}}$  and integrating over  $\mathbb{R}^2$ , we have

$$\begin{aligned}
\|u_{\tilde{h}}\|_{L^2(e^{-V} dx)}^2 + \frac{1}{2} \|\nabla_x u_{\tilde{h}}\|_{L^2(e^{-V} dx)}^2 &= -\kappa \int_{\mathbb{R}^2} \mathbf{e}_1 \cdot \nabla_x (\Pi f) u_{\tilde{h}} dx \\
&= \kappa \int_{\mathbb{R}^2} (\Pi f) \mathbf{e}_1 \cdot \nabla_x u_{\tilde{h}} dx \\
&\leq \kappa \int_{\mathbb{R}^2} |\nabla_x u_{\tilde{h}} e^{-V/2}| |\Pi f e^{V/2}| dx \\
&\leq \kappa \|\nabla_x u_{\tilde{h}}\|_{L^2(e^{-V} dx)} \|\Pi f\|_0 \\
&\leq \frac{1}{4} \|\nabla_x u_{\tilde{h}}\|_{L^2(e^{-V} dx)}^2 + \kappa^2 \|\Pi f\|_0^2.
\end{aligned}$$

This inequality is a  $H^1(e^{-V} dx) \rightarrow H^{-1}(e^{-V} dx)$  elliptic regularity result. Hence,

$$\|\mathbf{A}^* \mathbf{P}_\kappa f\|_0^2 = \|\mathbf{T} \Pi \tilde{h}\|_0^2 = \frac{1}{2} \|\nabla_x u_{\tilde{h}}\|_{L^2(e^{-V} dx)}^2 \leq 2\kappa^2 \|\Pi f\|_0^2,$$

and so we conclude

$$\|\mathbf{P}_\kappa^* \mathbf{A} f\|_0 \leq \sqrt{2} \kappa \|(1 - \Pi) f\|_0 \leq \sqrt{2} \kappa \|f\|_0. \quad (12)$$

Combining (11) and (12), the entropy dissipation functional  $D_1$  is bounded by

$$D_1[f] \leq \kappa \varepsilon_1 \left( \frac{C_V}{\sqrt{2}} + \sqrt{2} \right) \|f\|_0^2 = 2\kappa\lambda_1 \|f\|_0^2,$$

where we defined  $\lambda_1 := \frac{1}{2} \left( \frac{C_V}{\sqrt{2}} + \sqrt{2} \right)$ .

**# Step 3: Estimation of  $D_2[f]$ .**

Using integration by parts, we have

$$\langle P_\kappa f, f \rangle_0 = \frac{\kappa}{2} \int_{\mathbb{R}^2 \times \mathbb{S}^1} (e_1 \cdot \nabla_x V) f^2 e^V \frac{dx d\alpha}{2\pi}.$$

The estimation of this term goes differently depending on the boundedness of  $\nabla_x V$ .

If  $\nabla_x V$  is bounded, we write

$$D_2[f] \leq |\langle P_\kappa f, f \rangle_0| \leq \frac{\kappa}{2} \|\nabla_x V\|_\infty \|f\|_0^2 = \frac{\kappa}{2} \|\nabla_x V\|_\infty \|f\|_\kappa^2,$$

where we have used  $\|f\|_\kappa = \|f\|_0$ , since  $g \equiv 0$ .

Assume now that  $|\nabla_x V| \rightarrow \infty$  as  $|x| \rightarrow \infty$ . Thanks to the choice of  $g$ , we have the estimate

$$D_2[f] \leq |\langle P_\kappa f, f \rangle_0| \leq \frac{\kappa}{2} \int_{\mathbb{R}^2 \times \mathbb{S}^1} |\nabla_x V| f^2 e^V \frac{dx d\alpha}{2\pi} \leq \frac{\kappa}{2} C_3 \int_{\mathbb{R}^2 \times \mathbb{S}^1} f^2 g \frac{dx d\alpha}{2\pi}, \quad (13)$$

with

$$C_3 := \sup_{x \in \mathbb{R}^2} (|\nabla_x V| e^V g^{-1}),$$

which is finite by **(H5)**.

**# Step 4: Estimation of  $D_3[f]$ .**

We start by recalling that this estimate is only relevant when  $\nabla_x V$  is unbounded. Indeed, in the opposite case,  $D_3[f] = 0$  since  $g \equiv 0$  by definition. By the identity

$$\int_{\mathbb{R}^2 \times \mathbb{S}^1} \mathcal{L}_\kappa(f) f g dx d\alpha = \frac{1}{2} \int_{\mathbb{R}^2 \times \mathbb{S}^1} \mathcal{L}_\kappa(g) f^2 dx d\alpha - D \int_{\mathbb{R}^2 \times \mathbb{S}^1} |\partial_\alpha f|^2 g dx d\alpha$$

with  $\mathcal{L}_\kappa$  as defined in (7), we have

$$D_3[f] \leq \kappa \zeta \left( \frac{1}{2} \int_{\mathbb{R}^2 \times \mathbb{S}^1} \mathcal{L}_\kappa(g) f^2 \frac{dx d\alpha}{2\pi} \right). \quad (14)$$

Proposition 1.3 allows us to control the  $g$ -weighted  $L^2$ -norm outside some fixed ball. More precisely, take  $R > 0$  in (6) large enough s.t.  $|\nabla_x V| \geq 1$  for all  $|x| > R$ , then

$$\begin{aligned} & \int_{\mathbb{R}^2 \times \mathbb{S}^1} \mathcal{L}_\kappa(g) f^2 \frac{dx d\alpha}{2\pi} \\ & \leq \int_{\mathbb{S}^1} \int_{|x| < R} \mathcal{L}_\kappa(g) f^2 \frac{dx d\alpha}{2\pi} - c \int_{\mathbb{S}^1} \int_{|x| > R} |\nabla_x V| f^2 g \frac{dx d\alpha}{2\pi} \\ & \leq \int_{\mathbb{S}^1} \int_{|x| < R} ((\mathcal{L}_\kappa(g) + cg) e^{-V}) f^2 e^V \frac{dx d\alpha}{2\pi} - c \int_{\mathbb{R}^2 \times \mathbb{S}^1} f^2 g \frac{dx d\alpha}{2\pi} \\ & \leq C_4(R) \|f\|_0^2 - c \int_{\mathbb{R}^2 \times \mathbb{S}^1} f^2 g \frac{dx d\alpha}{2\pi}, \end{aligned} \quad (15)$$

where  $C_4(R) := \sup_{|x| \leq R} (|\mathcal{L}_\kappa(g) + cg|e^{-V})$ .

**Remark 2.3.** *Observe here that one could take advantage of the growth of  $\nabla_x V$  by playing with the cut-off parameter  $R$  and keeping track of  $\min_{|x| \geq R} |\nabla_x V|$  in the negative term. It could lead to more optimal constants but we chose instead to vary the parameter  $\zeta$  in front of the coercivity weight  $g$  in the measure  $\mu_\kappa$  for simplicity.*

### # Step 5: Putting the four previous steps together.

Combine the previous steps into

$$\begin{aligned}
D_0[f] + D_1[f] &\leq (\varepsilon_1 - D) \|(1 - \Pi)f\|_0^2 + \varepsilon_1 \lambda_2 \|(1 - \Pi)f\|_0 \|\Pi f\|_0 \\
&\quad - \varepsilon_1 \gamma_2 \left( \|\Pi f\|_0^2 - M_f^2 \right) + 2\kappa \lambda_1 \|f\|_0^2 \\
&= - (D - \varepsilon_1 - 2\kappa \lambda_1) \|(1 - \Pi)f\|_0^2 + \varepsilon_1 \lambda_2 \|(1 - \Pi)f\|_0 \|\Pi f\|_0 \\
&\quad - (\varepsilon_1 \gamma_2 - 2\kappa \lambda_1) \|\Pi f\|_0^2 + \varepsilon_1 \gamma_2 M_f^2 \\
&\leq - \left( D - \varepsilon_1 - 2\kappa \lambda_1 - \frac{\varepsilon_1 \lambda_2 b}{2} \right) \|(1 - \Pi)f\|_0^2 \\
&\quad - \left( \varepsilon_1 \gamma_2 - 2\kappa \lambda_1 - \frac{\varepsilon_1 \lambda_2}{2b} \right) \|\Pi f\|_0^2 + \varepsilon_1 \gamma_2 M_f^2 \\
&\leq - 2\xi(\kappa) \|f\|_0^2 + \varepsilon_1 \gamma_2 M_f^2,
\end{aligned}$$

by Young's inequality with the choice  $b = \lambda_2/\gamma_2$ , and where we used the fact that  $\|(1 - \Pi)f\|_0^2 + \|\Pi f\|_0^2 = \|f\|_0^2$ . Here,  $\xi(\kappa)$  is explicit, and given by

$$\begin{aligned}
\xi(\kappa) &:= \frac{1}{2} \min \left\{ D - \varepsilon_1 \left( 1 + \frac{\lambda_2^2}{2\gamma_2} \right), \frac{\varepsilon_1 \gamma_2}{2} \right\} - \kappa \lambda_1 \\
&= \frac{D\gamma_2^2}{2(\gamma_2^2 + 2\gamma_2 + \lambda_2^2)} - \kappa \lambda_1,
\end{aligned}$$

since the minimum in the first term is realised when the two arguments are equal, fixing  $\varepsilon_1 = 2D\gamma_2/(\gamma_2^2 + 2\gamma_2 + \lambda_2^2)$ . Note that this choice of  $\varepsilon_1$  satisfies  $\varepsilon_1 < D$  and  $\varepsilon_1 < 1$ . Choosing  $\kappa$  small enough ensures  $\xi(\kappa) > 0$ . From this analysis we conclude

$$D_0[f] + D_1[f] \leq -2\xi(\kappa) \|f\|_0^2 + \varepsilon_1 \gamma_2 M_f^2. \quad (16)$$

Let us now add the control of  $D_2 + D_3$ . If  $\nabla_x V$  is bounded,  $g \equiv 0$  and  $D_3 = 0$ :

$$\begin{aligned}
\frac{d}{dt} G[f] &= D_0[f] + D_1[f] + D_2[f] \\
&\leq - (4\xi(\kappa) - \kappa \|\nabla_x V\|_\infty) \frac{1}{2} \|f\|_\kappa^2 + \varepsilon_1 \gamma_2 M_f^2 \\
&\leq -\gamma_1 G[f] + \varepsilon_1 \gamma_2 M_f^2
\end{aligned}$$

by the norm equivalence (9). Here, we defined

$$\gamma_1 := \frac{4\xi(\kappa) - \kappa \|\nabla_x V\|_\infty}{1 + \varepsilon_1} > 0.$$

When  $\nabla_x V$  is unbounded, (13)-(14)-(15)-(16) imply

$$\begin{aligned}
\frac{d}{dt} \mathbf{G}[f] &= \mathbf{D}_0[f] + \mathbf{D}_1[f] + \mathbf{D}_2[f] + \mathbf{D}_3[f] \\
&\leq -2\xi(\kappa) \|f\|_0^2 + \varepsilon_1 \gamma_2 M_f^2 + \frac{\kappa}{2} C_3 \int_{\mathbb{R}^2 \times \mathbb{S}^1} f^2 g \frac{dx d\alpha}{2\pi} \\
&\quad + \frac{\kappa \zeta}{2} \left( C_4(R) \|f\|_0^2 - c \int_{\mathbb{R}^2 \times \mathbb{S}^1} f^2 g \frac{dx d\alpha}{2\pi} \right) \\
&= -\frac{1}{2} (4\xi(\kappa) - \kappa \zeta C_4(R)) \|f\|_0^2 - \frac{\kappa \zeta}{2} \left( c - \frac{C_3}{\zeta} \right) \int_{\mathbb{R}^2 \times \mathbb{S}^1} f^2 g \frac{dx d\alpha}{2\pi} + \varepsilon_1 \gamma_2 M_f^2 \\
&\leq -\frac{1}{2} \min \left\{ 4\xi(\kappa) - \kappa \zeta C_4(R), c - \frac{C_3}{\zeta} \right\} \|f\|_\kappa^2 + \varepsilon_1 \gamma_2 M_f^2 \\
&\leq -\gamma_1 \mathbf{G}[f] + \varepsilon_1 \gamma_2 M_f^2
\end{aligned}$$

again by norm equivalence (9), and where we defined

$$\gamma_1 := \frac{1}{1 + \varepsilon_1} \min \left\{ 4\xi(\kappa) - \kappa \zeta C_4(R), c - \frac{C_3}{\zeta} \right\} > 0.$$

This requires  $\zeta > 0$  to be large enough, and the upper bound for  $\kappa$  should be chosen accordingly:

$$\zeta > \frac{C_3}{c}, \quad 4\xi(\kappa) - \kappa \zeta C_4(R) > 0.$$

In order to maximise the rate of convergence to equilibrium given  $\kappa$ ,  $D$  and  $V$ , one can optimise  $\gamma_1$  over  $\zeta$  whilst respecting the above constraints.

**Remark 2.4.** The condition  $\gamma_1 > 0$  translates into an explicit upper bound on  $\kappa$ . More precisely, we require  $\xi(\kappa) > \kappa u/4$  where  $u := \|\nabla_x V\|_\infty$  in the case of a bounded potential gradient, and  $u := \zeta C_4(R)$  otherwise. This condition is satisfied for small enough  $\kappa$ :

$$0 \leq \kappa < \frac{\varepsilon_1 \gamma_2}{(4\lambda_1 + u)} = \frac{2D\gamma_2^2}{(4\lambda_1 + u)(\gamma_2^2 + 2\gamma_2 + \lambda_2^2)}$$

which also implies  $\xi(\kappa) > 0$ . Recall that Proposition 1.3 requires  $\kappa < 1/3$  in the case of unbounded potential gradients. These conditions provide a range of  $\kappa$  for which Proposition 2.1 holds.

### 3 The coercivity weight $g$

In this section, we define the function  $g$  in such a way that it allows us to control the loss of weight in the perturbation operator  $\mathbf{P}_\kappa$ . When  $\nabla_x V$  is bounded, we do not need any extra weight since then we may control the perturbation thanks to the stationary weight  $e^V$ , and so we set  $g \equiv 0$  in that case. When it is not, Proposition 1.3 provides a suitable weight function  $g$  by constructive methods.

### 3.1 Proof of Proposition 1.3

The proof is strongly inspired from [12], however our weight is different since we work in an  $L^2$ -framework rather than in an  $L^1$  one. Assuming  $\nabla_x V$  is unbounded, we seek a weight  $g$  of the form

$$g(x, \alpha) = \exp \left( \beta V(x) + |\nabla_x V(x)| \Gamma \left( \tau(\alpha) \cdot \frac{\nabla_x V(x)}{|\nabla_x V(x)|} \right) \right),$$

where the parameter  $\beta > 1$  and the function  $\Gamma \in C^1([-1, 1])$ ,  $\Gamma > 0$  are to be determined. We define

$$Y(x, \alpha) := \tau(\alpha) \cdot \frac{\nabla_x V(x)}{|\nabla_x V(x)|}, \quad Y^\perp(x, \alpha) := \tau^\perp(\alpha) \cdot \frac{\nabla_x V(x)}{|\nabla_x V(x)|},$$

and split the proof into four steps: 1) we rewrite statement (6) using the explicit expression of the weight  $g$ , 2) we simplify the obtained expression using assumption **(H5)**, 3) we prove the equivalent statement obtained in Step 2 by defining a suitable choice of  $\Gamma(\cdot)$  and  $\beta$ , and 4) we demonstrate that it is indeed possible to choose suitable parameters for the calculations in Step 3 to hold, fixing explicit expressions where possible.

#### # Step 1: Rewriting the weight estimate (6).

Applying the operator  $\mathcal{L}_\kappa$  defined in (7) to  $g$ , we can compute explicitly

$$\begin{aligned} \frac{\mathcal{L}_\kappa(g)}{g} = & D \left( |\nabla_x V| \partial_{\alpha\alpha} \Gamma(Y) + |\nabla_x V|^2 |\partial_\alpha \Gamma(Y)|^2 \right) \\ & + (\tau(\alpha) + \kappa e_1) \cdot (\beta \nabla_x V + \nabla_x (|\nabla_x V| \Gamma(Y))) \\ & - |\nabla_x V|^2 Y^\perp \partial_\alpha \Gamma(Y) - |\nabla_x V| Y. \end{aligned}$$

Since

$$\partial_\alpha \Gamma = Y^\perp \Gamma'(Y) \quad \text{and} \quad \partial_{\alpha\alpha} \Gamma = \partial_\alpha \left( Y^\perp \Gamma'(Y) \right) = -Y \Gamma'(Y) + |Y^\perp|^2 \Gamma''(Y),$$

we get

$$\begin{aligned} \frac{\mathcal{L}_\kappa(g)}{g} = & D \left( |\nabla_x V| \left( -Y \Gamma'(Y) + |Y^\perp|^2 \Gamma''(Y) \right) + |\nabla_x V|^2 |Y^\perp|^2 (\Gamma'(Y))^2 \right) \\ & + (\tau(\alpha) + \kappa e_1) \cdot (\beta \nabla_x V + \nabla_x (|\nabla_x V| \Gamma(Y))) \\ & - |\nabla_x V|^2 |Y^\perp|^2 \Gamma'(Y) - |\nabla_x V| Y \\ = & (\beta - 1 - D \Gamma'(Y)) |\nabla_x V| Y + \kappa \beta e_1 \cdot \nabla_x V + (\tau(\alpha) + \kappa e_1) \cdot \nabla_x (|\nabla_x V| \Gamma(Y)) \\ & + |Y^\perp|^2 \left( D |\nabla_x V| \Gamma''(Y) + |\nabla_x V|^2 \left[ D (\Gamma'(Y))^2 - \Gamma'(Y) \right] \right). \end{aligned}$$

In order to see which  $\Gamma$  to choose, let us divide by  $|\nabla_x V|$  and denote the diffusion and transport part by

$$\text{diff}(x, \alpha) := (\tau(\alpha) + \kappa e_1) \cdot \frac{\nabla_x (|\nabla_x V| \Gamma(Y))}{|\nabla_x V|}, \quad \text{tran}(x) := \frac{e_1 \cdot \nabla_x V}{|\nabla_x V|}.$$

Now, we can rewrite the statement of Proposition 1.3: we seek a positive constant  $c > 0$  and a radius  $R > 0$  such that for any  $\alpha \in \mathbb{S}^1$  and  $|x| > R$ ,

$$\begin{aligned} & (\beta - 1 - D \Gamma'(Y)) Y + \kappa \beta \text{tran}(x) + \text{diff}(x, \alpha) \\ & + |Y^\perp|^2 \left( D \Gamma''(Y) + |\nabla_x V| \left[ D (\Gamma'(Y))^2 - \Gamma'(Y) \right] \right) \leq -c. \end{aligned}$$