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# EXPONENTIAL DECAY TO EQUILIBRIUM FOR A FIBRE LAY-DOWN PROCESS ON A MOVING CONVEYOR BELT

ÉMERIC BOUIN, FRANCA HOFFMANN, AND CLÉMENT MOUHOT

ABSTRACT. We show existence and uniqueness of a stationary state for a kinetic Fokker-Planck equation modelling the fibre lay-down process in the production of non-woven textiles. Following a micro-macro decomposition, we use hypocoercivity techniques to show exponential convergence to equilibrium with an explicit rate assuming the conveyor belt moves slowly enough. This work is an extension of (Dolbeault et al., 2013), where the authors consider the case of a stationary conveyor belt. Adding the movement of the belt, the global Gibbs state is not known explicitly. We thus derive a more general hypocoercivity estimate from which existence, uniqueness and exponential convergence can be derived. To be able to consider potentials growing faster than linearly at infinity, we make use of an additional weight function following the Lyapunov functional approach in (Kolb et al., 2013).

## 1. INTRODUCTION

The mathematical analysis of the fibre lay-down process in the production of non-woven textiles has seen a lot of interest in recent years [11, 12, 6, 8, 9, 4, 10]. Non-woven materials are produced in melt-spinning operations: hundreds of individual endless fibres are obtained by continuous extrusion through nozzles of a melted polymer. The nozzles are densely and equidistantly placed in a row at a spinning beam. The visco-elastic, slender and in-extensible fibres lay down on a moving conveyor belt to form a web, where they solidify due to cooling air streams. Before touching the conveyor belt, the fibres become entangled and form loops due to highly turbulent air flow. In [11] a general mathematical model for the fibre dynamics is presented which enables the full simulation of the process. Due to the huge amount of physical details, these simulations of the fibre spinning and lay-down usually require an extremely large computational effort and high memory storage, see [12]. Thus, a simplified two-dimensional stochastic model for the fibre lay-down process is introduced in [6]. Generalisations of the two-dimensional stochastic model [6] to three dimensions have been developed by Klar et al. in [8] and to any dimension  $d \geq 2$  by Grothaus et al. in [7].

We now describe the model we are interested in, which comes from [6]. We track the position  $x(t) \in \mathbb{R}^2$  and the angle  $\alpha(t) \in \mathbb{S}^1$  of the fibre at the lay-down point where it touches the conveyor belt. Interactions of neighbouring fibres are neglected. If  $x_0(t)$  is the lay-down point in the coordinate system following the conveyor belt, then the tangent vector of the fibre is denoted by  $\tau(\alpha(t))$  with  $\tau(\alpha) = (\cos \alpha, \sin \alpha)$ . Since the extrusion of fibres happens at a constant speed, and the fibres are in-extensible, the lay-down process can be assumed to happen at constant normalised speed  $\|x'_0(t)\| = 1$ . If the conveyor belt moves with constant speed  $\kappa$  in direction  $e_1 = (1, 0)$ , then

$$\frac{dx}{dt} = \tau(\alpha) + \kappa e_1.$$

Note that the speed of the conveyor belt cannot exceed the lay-down speed:  $0 \leq \kappa \leq 1$ . The fibre dynamics in the deposition region close to the conveyor belt are dominated by the turbulent air flow. Applying this concept, the dynamics of the angle  $\alpha(t)$  can be described by a deterministic force

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moving the lay-down point towards the equilibrium  $x = 0$  and by a Brownian motion modelling the effect of the turbulent air flow. We obtain an Itô stochastic differential equation for the random variable  $X_t = (x_t, \alpha_t)$  on  $\mathbb{R}^2 \times \mathbb{S}^1$ ,

$$(1.1) \quad \begin{cases} dx_t &= (\tau(\alpha_t) + \kappa e_1) dt, \\ d\alpha_t &= [-\tau^\perp(\alpha_t) \cdot \nabla_x V(x_t)] dt + A dW_t, \end{cases}$$

where  $W_t$  denotes a one-dimensional Wiener process,  $\tau^\perp = (-\sin \alpha, \cos \alpha)$ ,  $A > 0$  measures its strength relative to the deterministic forcing, and  $V : \mathbb{R}^2 \rightarrow \mathbb{R}$  is an external potential carrying information on the coiling properties of the fibre. More precisely, since a curved fibre tends back to its starting point, the change of the angle  $\alpha$  is assumed to be proportional to  $\tau^\perp(\alpha) \cdot \nabla_x V(x)$ . It has been shown in [10] that under suitable assumptions on the external potential  $V$ , the fibre lay down process (1.1) has a unique invariant distribution and is even geometrically ergodic (see Remark 2). The stochastic approach yields exponential convergence in total variation norm, however without any explicit rate. We will show here that a stronger result can be obtained with a functional analysis approach. Our argument uses crucially the construction of a Lyapunov functional for the fibre lay-down process in the case of unbounded potential gradients as in the stochastic paper (Proposition 3.7, [10]).

The probability density function  $f(t, x, \alpha)$  corresponding to the stochastic process (1.1) is governed by the Fokker-Planck equation

$$(1.2) \quad \partial_t f + (\tau + \kappa e_1) \cdot \nabla_x f - \partial_\alpha \left( \tau^\perp \cdot \nabla_x V f \right) = D \partial_{\alpha\alpha} f$$

with diffusivity  $D = A^2/2$ . We state below assumptions on the external potential  $V$  that will be used regularly throughout the paper:

**(H1) Regularity:**  $V \in W_{\text{loc}}^{2,\infty}(\mathbb{R}^2) \cap C^2(\mathbb{R}^2)$  and  $V$  is spherically symmetric outside some ball  $B(0, R)$ .

**(H2) Normalisation:**  $\int_{\mathbb{R}^2} e^{-V(x)} dx = 1$ .

**(H3) Spectral gap condition:** there exists a positive constant  $\Lambda$  such that for any  $u \in H^1(e^{-V} dx)$  with  $\int_{\mathbb{R}^2} u e^{-V} dx = 0$ , we have

$$\int_{\mathbb{R}^2} |\nabla_x u|^2 e^{-V} dx \geq \Lambda \int_{\mathbb{R}^2} u^2 e^{-V} dx.$$

**(H4) Pointwise condition:** there exists  $c_1 > 0$  such that for any  $x \in \mathbb{R}^2$ ,

$$|D_x^2 V(x)| \leq c_1 (1 + |\nabla_x V(x)|),$$

where  $D_x^2 V$  denotes the Hessian of  $V(x)$ .

**(H5) Behaviour at infinity:**

$$\lim_{|x| \rightarrow \infty} \frac{|\nabla_x V(x)|}{V(x)} = 0, \quad \lim_{|x| \rightarrow \infty} \frac{|D_x^2 V(x)|}{|\nabla_x V(x)|} = 0.$$

**Remark 1.** Assumptions **(H2)**–**(H4)** are as stated in [4]. Assumption **(H1)** assumes stronger regularity of the potential. Assumption **(H5)** is only necessary if the potential gradient  $|\nabla_x V|$  is unbounded. Both bounded and unbounded potential gradients may appear, depending on the physical context. A typical example for an external potential satisfying assumptions **(H1)**–**(H5)** is given by

$$(1.3) \quad V(x) = K (1 + |x|^2)^{\beta/2}$$

for some constants  $K > 0$  and  $\beta \geq 1$  [5, 10]. The potential (1.3) satisfies **(H3)** since

$$\liminf_{|x| \rightarrow \infty} (|\nabla_x V|^2 - 2\Delta V) > 0,$$

see for instance (A.19. Some criteria for Poincaré inequalities, [13], page 135). The other assumptions are trivially satisfied as can be checked by direct inspection. In this family of potentials, the gradient  $\nabla_x V$  is bounded for  $\beta = 1$  and unbounded for  $\beta > 1$ .

**Remark 2.** The ergodicity proof in [10] assumes that the potential satisfies

$$(1.4) \quad \lim_{|x| \rightarrow \infty} \frac{|\nabla_x V(x)|}{V(x)} = 0, \quad \lim_{|x| \rightarrow \infty} \frac{|D_x^2 V(x)|}{|\nabla_x V(x)|} = 0, \quad \lim_{r \rightarrow \infty} V'(r) = \infty.$$

Under these assumptions, there exists an invariant distribution  $\nu$  to the fibre lay-down process (1.1), and some constants  $C(x_0) > 0$ ,  $\lambda > 0$  such that

$$\|\mathcal{P}_{x_0, \alpha_0}(X_t \in \cdot) - \nu\|_{TV} \leq C(x_0)e^{-\lambda t}.$$

The stochastic Lyapunov technique applied in [10] however does not give any information on how the constant  $C(x_0)$  depends on the initial position  $x_0$ , or how the rate of convergence  $\lambda$  depends on the belt speed  $\kappa$ , the potential  $V$  and the noise strength  $A$ . This can be achieved using hypocoercivity techniques, proving convergence in a weighted  $L^2$ -norm, which is slightly stronger than the convergence in total variation norm shown in [10]. Conceptually, conditions (1.4) ensure that the potential  $V$  is driving the process back inside a compact set where the noise can be controlled. Note that conditions (1.4) follow directly from **(H1)**–**(H5)**. They are more general in the sense that we do not require a spectral gap, the proof for exponential convergence to equilibrium done in [10] however makes use of the strong Feller property which can be translated in some cases into a spectral gap. Further, in [10], hypoellipticity of the fibre lay down process allows to deduce the existence of a transition density, which provides the result via an explicit Lyapunov function argument. By making the stronger assumptions **(H1)**–**(H5)**, and adapting the Lyapunov function argument presented in [10], we are able to derive an explicit rate of convergence including its dependence on the initial data  $f_0$ , the relative speed of the conveyor belt  $\kappa$  and the potential  $V$ .

To set up a functional framework associated to this Fokker-Planck equation, we may rewrite (1.2) as an abstract ODE

$$(1.5) \quad \partial_t f = \mathbf{L}_\kappa f = (\mathbf{L} - \mathbf{T})f + \mathbf{P}_\kappa f,$$

where the collision operator  $\mathbf{L} = D\partial_{\alpha\alpha}$  acts as a multiplier in the space variable  $x$ ,  $\mathbf{P}_\kappa$  is the perturbation introduced by the moving belt,

$$\mathbf{P}_\kappa f = -\kappa e_1 \cdot \nabla_x f,$$

and the transport operator  $\mathbf{T}$  is given by

$$\mathbf{T}f = \tau \cdot \nabla_x f - \partial_\alpha \left( \tau^\perp \cdot \nabla_x V f \right).$$

We consider solutions to (1.5) in the space  $L^2(\mathbb{R}^2 \times \mathbb{S}^1, d\mu) = L^2(d\mu)$  with measure

$$d\mu(x, \alpha) = e^{V(x)} \frac{dx d\alpha}{2\pi}.$$

We denote by  $\langle \cdot, \cdot \rangle$  the corresponding scalar product and by  $\|\cdot\|$  the associated norm. We introduce the orthogonal projection  $\Pi$  on the set of local equilibria  $\text{Ker } \mathbf{L}$  consisting of all  $\alpha$ -independent distributions,

$$\Pi f := \frac{1}{2\pi} \int_{\mathbb{S}^1} f d\alpha.$$

We also define the *mass* of a given distribution  $f \in L^2(d\mu)$  through the formula

$$M_f := \frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^1} f \, dx \, d\alpha.$$

We notice after integrating (1.2) over  $\mathbb{R}^2 \times \mathbb{S}^1$  that the mass of any solution of (1.2) is conserved through time. Moreover, any solution of (1.2) remains non-negative as soon as the initial datum is non-negative.

In this functional setting, the operators  $\mathbb{T}$  and  $\mathbb{L}$  have several nice properties that allow us to apply the general theory for linear kinetic equations conserving mass as outlined in [5]. First of all,  $\mathbb{L}$  and  $\mathbb{T}$  are closed operators on  $L^2(d\mu)$  such that  $\mathbb{L} - \mathbb{T}$  generates the  $\mathcal{C}^0$ -semigroup  $e^{(\mathbb{L}-\mathbb{T})t}$  on  $L^2(d\mu)$ . Furthermore,  $\mathbb{L}$  is symmetric and negative semi-definite on  $L^2(d\mu)$ ,

$$\langle \mathbb{L}f, f \rangle = -D \|\partial_\alpha f\|^2 \leq 0,$$

i.e.  $\mathbb{L}$  is dissipative. Further, we have for any  $f \in L^2(d\mu)$ ,

$$\mathbb{T}\Pi f = e^{-V} \tau \cdot \nabla_x u_f$$

with  $u_f := e^V \Pi f$ , which implies  $\Pi \mathbb{T} \Pi = 0$  on  $L^2(d\mu)$ . Since the transport operator  $\mathbb{T}$  is skew-symmetric with respect to  $\langle \cdot, \cdot \rangle$ , we obtain the *energy equality*

$$\frac{1}{2} \frac{d}{dt} \|f\|^2 = \langle \mathbb{L}f, f \rangle + \langle \mathbb{P}_\kappa f, f \rangle,$$

for any  $f$  in  $L^2(d\mu)$ . In the case  $\kappa = 0$ , if the entropy dissipation  $-\langle \mathbb{L}f, f \rangle$  was coercive with respect to the norm  $\|\cdot\|$ , exponential decay to zero would follow as  $t \rightarrow \infty$ . However, such a coercivity property cannot hold since  $\mathbb{L}$  vanishes on the set of local equilibria. Instead, Dolbeault et al. [5] applied a strategy called *hypocoercivity*, first developed by Villani in [13] (see also the previous related papers [2, 3] in a nonlinear context). The full hypocoercivity analysis of the long time behaviour of solutions to this kinetic model in the case of a stationary conveyor belt,  $\kappa = 0$ , is completed in [4]. For technical applications in the production process of non-wovens, one is interested in a model including the movement of the conveyor belt, and our aim is to extend the results in [4] to the case  $\kappa \neq 0$ . Following the idea of a micro-macro decomposition, we shall split our assumptions into two main requirements: *microscopic coercivity*, which assumes that the restriction of  $\mathbb{L}$  to  $\text{Ker}^\perp \mathbb{L}$  is coercive, and *macroscopic coercivity*, which is a spectral gap-like inequality for the operator obtained when taking a parabolic drift-diffusion limit, in other words, the restriction of  $\mathbb{T}$  to  $\text{Ker} \mathbb{L}$  is coercive.

- **Microscopic coercivity:** The operator  $\mathbb{L}$  is symmetric and the Poincaré inequality on  $\mathbb{S}^1$ ,

$$\frac{1}{2\pi} \int_{\mathbb{S}^1} |\partial_\alpha f|^2 \, d\alpha \geq \frac{1}{2\pi} \int_{\mathbb{S}^1} \left( f - \frac{1}{2\pi} \int_{\mathbb{S}^1} f \, d\alpha \right)^2 \, d\alpha,$$

yields that for all  $f \in \mathcal{D}(\mathbb{L})$ ,

$$-\langle \mathbb{L}f, f \rangle \geq D \|(1 - \Pi)f\|^2.$$

- **Macroscopic coercivity:** The operator  $\mathbb{T}$  is skew-symmetric and for any  $g \in L^2(d\mu)$  such that  $u_g \in H^1(e^{-V} dx)$  and  $\int_{\mathbb{R}^2 \times \mathbb{S}^1} g \, d\mu = 0$ , we have

$$\begin{aligned} \|\mathbb{T}\Pi g\|^2 &= \frac{1}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^1} e^{-V} |\nabla_x u_g|^2 \, dx \, d\alpha \\ &\geq \frac{\Lambda}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^1} e^{-V} u_g^2 \, dx \, d\alpha = \frac{\Lambda}{2} \|\Pi g\|^2 \end{aligned}$$

by the spectral gap condition **(H3)**.

In the case  $\kappa = 0$ , we have existence of a unique global normalised equilibrium distribution  $F_0(x) = e^{-V(x)}$  in the intersection of the null spaces of  $\mathbb{T}$  and  $\mathbb{L}$ ,  $\text{Ker } \mathbb{L} \cap \text{Ker } \mathbb{T}$ . For a moving conveyor belt,  $F_0$  is not in the kernel of  $\mathbb{P}_\kappa$  and we are not able to find the global Gibbs state of (1.5) explicitly. However, the hypocoercivity method as applied to the fibre lay-down process in [4] only depends on the first moment which cancels on the solution of the linear equation (1.5). Moreover, the hypocoercivity theory is based on a priori estimates [5], and is therefore quite stable under perturbation. These considerations in mind, we establish existence and uniqueness of a global Gibbs state and determine the rate of convergence of solutions in  $L^2(d\mu)$  towards this equilibrium distribution using hypocoercivity techniques. Namely, we prove the following result, giving an explicit rate of convergence for small enough movement of the conveyor belt:

**Theorem 1.1.** *Under assumptions (H1)–(H5) and for  $0 < \kappa \ll 1$  small enough (with explicit estimate), there exists a unique non-negative stationary state  $F_\kappa \in L^2(d\mu)$  of unit mass:  $M_{F_\kappa} = 1$ . In addition, for any solution  $f(t, \cdot) \in L^2(d\mu)$  of (1.2) with mass  $M_f$  we have*

$$(1.6) \quad \begin{aligned} & \|f(t, \cdot) - M_f F_\kappa\| \\ & \leq C \left( \|f(0) - M_f F_\kappa\| + \sqrt{\kappa \varepsilon_2} \left( \int_{\mathbb{R}^2 \times \mathbb{S}^1} |f(t) - M_f F_\kappa|^2 g \, dx \, d\alpha \right)^{1/2} \right) e^{-\lambda_\kappa t}, \end{aligned}$$

where:

- $\varepsilon_2 = 0$  if  $|\nabla_x V|$  is bounded and  $\varepsilon_2 > 0$  otherwise,
- $g(x, \alpha)$  is a weight explicitly given in terms of  $\kappa$  and  $V$ ,
- the rate of convergence  $\lambda_\kappa = \lambda_\kappa(V) > 0$  and the constant  $C = C(V, D)$  only depend respectively on  $\kappa$ ,  $V$  and  $V$ ,  $D$ .

In the case of a stationary conveyor belt  $\kappa = 0$ , the stationary state is characterised by the eigenpair  $(\Lambda_0, F_0)$  with  $\Lambda_0 = 0$ ,  $F_0(x) = e^{-V(x)}$ , and so  $\text{Ker}(\mathbb{L}_0) = \langle F_0 \rangle$ . In [4] we estimated the exponential rate of convergence  $\lambda_0$  explicitly under assumptions (H1)–(H4). It means that there is an isolated eigenvalue  $\Lambda_0 = 0$  and a spectral gap of size at least  $[-\lambda_0, 0]$ , with the rest of the spectrum  $Sp(\mathbb{L}_0)$  discrete and to the left of  $-\lambda_0$  in the complex plane. Adding the movement of the conveyor belt, Theorem 1.1 shows that  $\text{Ker}(\mathbb{L}_\kappa) = \langle F_\kappa \rangle$  and the exponential decay to equilibrium with rate  $\lambda_\kappa$  corresponds to a spectral gap of size at least  $[-\lambda_\kappa, 0]$ . Moreover one can easily verify that  $\lambda_\kappa$  converges to  $\lambda_0$  as  $\kappa \rightarrow 0$  recovering the same lower bound on the rate of convergence as in [4] (see **Step 4** in Section 2.2). In general, we are not able to compute the stationary state  $F_\kappa$  for  $\kappa > 0$ , but  $F_\kappa$  converges to  $F_0 = e^{-V}$  weakly as  $\kappa \rightarrow 0$  (see the discussion in Subsection 3.2).

The rest of the paper is organised as follows: in Section 2, we prove the main hypocoercivity estimate, which then allows us to prove the existence and uniqueness of a steady state in Section 3 by a contraction argument.

## 2. HYPOCOERCIVITY ESTIMATE

To follow a hypocoercivity strategy, we need to define a modified entropy functional. For this purpose, let us first introduce the auxiliary operator

$$\mathbb{A} := (1 + (\mathbb{T}\mathbb{P})^*(\mathbb{T}\mathbb{P}))^{-1}(\mathbb{T}\mathbb{P})^*.$$

This operator is now classical after [5] (and the references therein). As in [5], the *modified entropy functional* then reads

$$\mathbb{H}[f] := \frac{1}{2} \|f\|^2 + \varepsilon_1 \langle \mathbb{A}f, f \rangle$$

for some suitably chosen  $\varepsilon_1 \in (0, 1)$  to be determined later. We know from [5] that  $\mathbf{H}[\cdot]$  is equivalent to  $\|\cdot\|^2$  on  $L^2(d\mu)$ ,

$$(2.1) \quad \frac{1 - \varepsilon_1}{2} \|f\|^2 \leq \mathbf{H}[f] \leq \frac{1 + \varepsilon_1}{2} \|f\|^2, \quad f \in L^2(d\mu).$$

With convenient choice of a parameter  $\varepsilon_2$  and a weight  $g$  that will be made clear below, we define the more refined hypocoercivity functional  $\mathbf{G}$ :

$$\mathbf{G}[f] = \mathbf{H}[f] + \kappa\varepsilon_2 \int_{\mathbb{R}^2 \times \mathbb{S}^1} f^2 g \, dx \, d\alpha$$

In this section, we will prove the following hypocoercivity estimate:

**Proposition 2.1.** *Assume that hypothesis **(H1)**–**(H5)** hold and that  $0 < \kappa \ll 1$  is small enough (with a quantitative estimate). Let  $f$  be a solution of (1.2). Define  $\mathbf{G}$  according to the following dichotomy*

- *Case 1:  $|\nabla_x V|$  is bounded. Set  $\varepsilon_2 = 0$ .*
- *Case 2:  $|\nabla_x V|$  is unbounded. Then  $\varepsilon_2 > 0$  and we assume in addition that  $f(t = 0, \cdot) \in L^2(g(x, \alpha) \, dx \, d\alpha)$ , where  $g$  is a suitable Lyapunov function given in Proposition 2.2.*

In both cases,  $f$  satisfies the following Grönwall type estimate:

$$(2.2) \quad \frac{d}{dt} \mathbf{G}[f(t, \cdot)] \leq -\gamma_1 \mathbf{G}[f(t, \cdot)] + \gamma_2 M_f^2,$$

where  $\gamma_1 > 0, \gamma_2 > 0$  are explicit constants only depending on the relative speed of the conveyor belt  $\kappa$  and the potential  $V$ .

In fact, the estimate (2.2) is stronger than what is required for the uniqueness of a global Gibbs state, and represents an extension of the estimate given in [4]. When applied to the difference of two solutions with the same mass, (2.2) gives an estimate on the exponential decay rate towards equilibrium.

**2.1. A coercivity weight at large  $x$ .** In the case of an unbounded potential gradient (see Case 2 above), we introduce the following weight function

$$g(x, \alpha) = \exp \left( \beta V(x) + G \left( \tau(\alpha) \cdot \frac{\nabla_x V}{|\nabla_x V|} \right) \right),$$

where  $\beta > 1$  and  $G \in C^1([-1, 1])$ ,  $G > 0$  are yet to be determined. The idea is to choose the weight  $g$  in a way that it is a Lyapunov functional at large  $x$ , allowing us to control the loss of weight in the perturbation operator  $\mathbf{P}_\kappa$ . Indeed, we can show existence of such a Lyapunov function  $g$  under appropriate conditions following the argument in [10]:

**Proposition 2.2.** *Assume that  $V$  satisfies **(H1)** and **(H5)**. Assume moreover that  $|\nabla_x V| \rightarrow \infty$  as  $|x| \rightarrow \infty$ . If  $\kappa < 1/3$  holds true, then there exists a function  $g(x, \alpha)$ , a constant  $c > 0$  and a finite radius  $R_0 > 0$  such that*

$$(2.3) \quad \forall |x| > R_0, \forall \alpha \in \mathbb{S}^1, \quad \mathcal{L}_\kappa(g)(x, \alpha) \leq -c g(x, \alpha) |\nabla_x V(x)|,$$

where  $\mathcal{L}_\kappa$  is defined by

$$\mathcal{L}_\kappa(g) := D\partial_{\alpha\alpha}g + (\tau + \kappa e_1) \cdot \nabla_x g - \left( \tau^\perp \cdot \nabla_x V \right) \partial_\alpha g - (\tau \cdot \nabla_x V) g.$$

The connection between this proposition and our equation is provided by the identity

$$\int_{\mathbb{R}^2 \times \mathbb{S}^1} \mathcal{L}_\kappa(f) f g \, dx \, d\alpha = \frac{1}{2} \int_{\mathbb{R}^2 \times \mathbb{S}^1} \mathcal{L}_\kappa(g) f^2 \, dx \, d\alpha - \int_{\mathbb{R}^2 \times \mathbb{S}^1} (\partial_\alpha f)^2 g \, dx \, d\alpha.$$

This proposition and its proof are strongly inspired from and a variation of [10]: our weight is different since we work in an  $L^2$ -framework rather than  $L^1$ .

*Proof of Proposition 2.2.* Let us consider the function

$$g(x, \alpha) = \exp \left( \beta V(x) + G \left( \tau(\alpha) \cdot \frac{\nabla_x V}{|\nabla_x V|} \right) \right),$$

where  $G(Y) = |\nabla_x V| \Gamma(Y)$  for a function  $\Gamma(\cdot)$  to be determined, and we define

$$Y(x, \alpha) := \tau(\alpha) \cdot \frac{\nabla_x V}{|\nabla_x V|}, \quad Y^\perp(x, \alpha) := \tau^\perp(\alpha) \cdot \frac{\nabla_x V}{|\nabla_x V|}.$$

Applying  $\mathcal{L}_\kappa$  to  $g$ , we can compute explicitly

$$\begin{aligned} \frac{\mathcal{L}_\kappa(g)}{g} &= D \left( \partial_{\alpha\alpha} (G(Y)) + |\partial_\alpha (G(Y))|^2 \right) + (\tau(\alpha) + \kappa e_1) \cdot (\beta \nabla_x V + \nabla_x (G(Y))) \\ &\quad - |\nabla_x V| Y^\perp \partial_\alpha (G(Y)) - |\nabla_x V| Y. \end{aligned}$$

Since

$$\begin{aligned} \partial_\alpha G &= |\nabla_x V| Y^\perp \Gamma'(Y), \\ \partial_{\alpha\alpha} G &= |\nabla_x V| \partial_\alpha \left( Y^\perp \Gamma'(Y) \right) = |\nabla_x V| \left( -Y \Gamma'(Y) + |Y^\perp|^2 \Gamma''(Y) \right), \end{aligned}$$

we get

$$\begin{aligned} \frac{\mathcal{L}_\kappa(g)}{g} &= D \left( |\nabla_x V| \left( -Y \Gamma'(Y) + |Y^\perp|^2 \Gamma''(Y) \right) + |\nabla_x V|^2 |Y^\perp|^2 (\Gamma'(Y))^2 \right) \\ &\quad + (\tau(\alpha) + \kappa e_1) \cdot (\beta \nabla_x V + \nabla_x (G(Y))) - |\nabla_x V|^2 |Y^\perp|^2 \Gamma'(Y) - |\nabla_x V| Y \\ &= (\beta - 1 - D \Gamma'(Y)) |\nabla_x V| Y + \kappa \beta e_1 \cdot \nabla_x V + (\tau(\alpha) + \kappa e_1) \cdot \nabla_x (G(Y)) \\ &\quad + |Y^\perp|^2 \left( D |\nabla_x V| \Gamma''(Y) + |\nabla_x V|^2 \left[ D (\Gamma'(Y))^2 - \Gamma'(Y) \right] \right). \end{aligned}$$

In order to see which  $\Gamma$  to choose, let us divide by  $|\nabla_x V|$  and denote the diffusion and transport part by

$$\text{diff}(x, \alpha) := (\tau(\alpha) + \kappa e_1) \cdot \frac{\nabla_x (G(Y))}{|\nabla_x V|}, \quad \text{tran}(x) := \frac{e_1 \cdot \nabla_x V}{|\nabla_x V|}.$$

Now, we can rewrite the statement of Proposition 2.2: we seek a positive constant  $c > 0$  and a radius  $R_0 > 0$  such that for any  $\alpha \in \mathbb{S}^1$  and  $|x| > R_0$ ,

$$(\beta - 1 - D \Gamma'(Y)) Y + \kappa \beta \text{tran}(x) + \text{diff}(x, \alpha) + |Y^\perp|^2 \left( D \Gamma''(Y) + |\nabla_x V| \left[ D (\Gamma'(Y))^2 - \Gamma'(Y) \right] \right) \leq -c.$$

To achieve this bound, note that  $|Y| \leq 1$  and  $|\text{tran}| \leq 1$  for all  $(x, \alpha) \in \mathbb{R}^2 \times \mathbb{S}^1$ . Further, the diffusion term  $\text{diff}(\cdot)$  can be made arbitrarily small outside a sufficiently large ball:

$$\text{diff}(x, \alpha) = (\tau + \kappa e_1) \cdot \left[ \Gamma'(Y) \nabla_x Y + \Gamma(Y) \frac{\nabla_x (|\nabla_x V|)}{|\nabla_x V|} \right],$$

and both  $|\nabla_x Y|$  and  $|\nabla_x (|\nabla_x V|)|/|\nabla_x V|$  converge to zero as  $|x| \rightarrow \infty$  by assumption **(H5)**. In other words, using the fact that the potential gradient is unbounded, it remains to show that we can find constants  $\gamma > \kappa\beta > 0$  and a radius  $R > 0$  such that

$$(2.4) \quad \forall |x| > R, \quad (\beta - 1 - D \Gamma') Y + |Y^\perp|^2 \left( D \Gamma'' + |\nabla_x V| \left[ D (\Gamma')^2 - \Gamma' \right] \right) \leq -\gamma.$$

Then we can choose  $r > 0$  such that

$$|x| > r \implies \forall \alpha \in \mathbb{S}^1, \quad \text{diff}(x, \alpha) \leq \frac{\gamma - \kappa\beta}{2},$$

and we conclude for the statement of Proposition 2.2 with  $R_0 := \max\{r, R\}$  and  $c := (\gamma - \kappa\beta)/2 > 0$ .



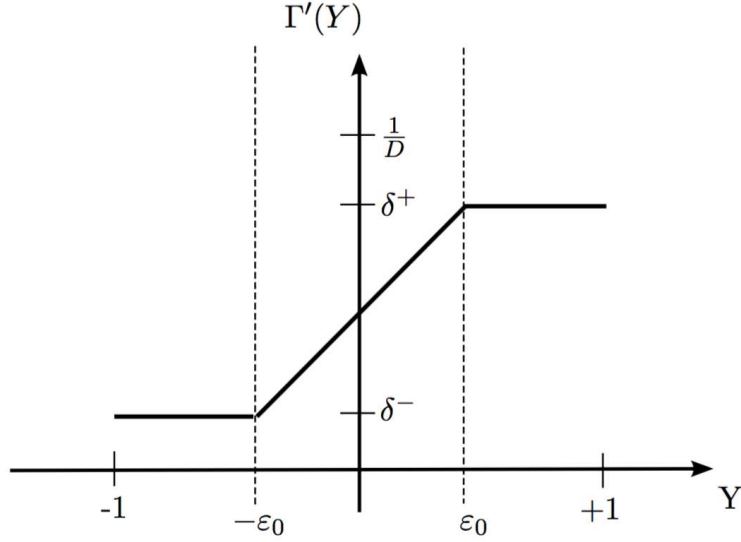


FIGURE 1. Derivative of  $\Gamma$

Proving (2.4) can be done by an explicit construction. We define  $\Gamma' \in C^0([-1, 1])$  piecewise,

$$\Gamma'(Y) = \begin{cases} \delta^+ & \text{if } Y > \varepsilon_0, \\ \frac{\delta^+ - \delta^-}{2\varepsilon_0} (Y + \varepsilon_0) + \delta^- & \text{if } |Y| \leq \varepsilon_0, \\ \delta^- & \text{if } Y < -\varepsilon_0, \end{cases}$$

where  $0 < \delta^- < \delta^+ < 1/D$  and  $\varepsilon_0 \in (0, 1)$  are to be determined. With this choice of  $\Gamma'$ , we can assure that  $\Gamma$  is strictly positive in the interval  $[-1, 1]$ . Now, let us show that there exist suitable choices of  $\gamma$  and  $\beta$  for the bound (2.4) to hold. More precisely, we choose a suitable  $\beta$  such that  $(\beta - 1)/D \in (\delta^-, \delta^+)$  and  $0 < \gamma < \tilde{\gamma}$ , defining  $\gamma := \varepsilon_0(1 + D\delta^+ - \beta)$ ,  $\tilde{\gamma} := \varepsilon_0(\beta - 1 - D\delta^-)$ . We split our analysis into cases:

- Assume  $Y > \varepsilon_0$ . Then the LHS of (2.4) can be bounded as follows:

$$(\beta - 1 - D\delta^+)Y + \delta^+ (D\delta^+ - 1) |\nabla_x V| |Y^\perp|^2 < (\beta - 1 - D\delta^+)\varepsilon_0 = -\gamma.$$

- Assume  $Y < -\varepsilon_0$ . Then the LHS of (2.4) can be bounded as follows:

$$(\beta - 1 - D\delta^-)Y + \delta^- (D\delta^- - 1) |\nabla_x V| |Y^\perp|^2 < -(\beta - 1 - D\delta^-)\varepsilon_0 = -\tilde{\gamma}.$$

- Assume  $|Y| \leq \varepsilon_0$ . Since  $1 = |Y|^2 + |Y^\perp|^2$ , we have  $|Y^\perp|^2 \geq 1 - \varepsilon_0^2$ . Further, setting

$$h = aY + b \in (\delta^-, \delta^+), \quad a := \frac{\delta^+ - \delta^-}{2\varepsilon_0}, \quad b := \frac{\delta^+ + \delta^-}{2},$$

we have  $\Gamma' = h$  and  $Dh^2 - h \leq D\delta^-(\delta^+ - 1/D)$ . Now, using the fact that the potential gradient is unbounded, we can find a radius  $R > 0$  large enough such that for all  $|x| > R$ ,

$$\frac{D(\delta^+ - \delta^-)}{2\varepsilon_0} - D\delta^- \left( \frac{1}{D} - \delta^+ \right) |\nabla_x V| < -\frac{2\tilde{\gamma}}{1 - \varepsilon_0^2}.$$

Putting these estimates together, we obtain for  $|x| > R$ :

$$\begin{aligned}
& (\beta - 1 - Dh)Y + |Y^\perp|^2 \left( \frac{D(\delta^+ - \delta^-)}{2\varepsilon_0} + |\nabla_x V| [Dh^2 - h] \right) \\
& \leq (\beta - 1 - D\delta^-)\varepsilon_0 + |Y^\perp|^2 \left( \frac{D(\delta^+ - \delta^-)}{2\varepsilon_0} + |\nabla_x V| \left[ D\delta^- \left( \delta^+ - \frac{1}{D} \right) \right] \right) \\
& \leq \tilde{\gamma} + (1 - \varepsilon_0^2) \left( \frac{D(\delta^+ - \delta^-)}{2\varepsilon_0} + |\nabla_x V| \left[ D\delta^- \left( \delta^+ - \frac{1}{D} \right) \right] \right) \leq -\tilde{\gamma}.
\end{aligned}$$

We now come back to the choice of  $\delta^-, \delta^+, \varepsilon_0, \beta$  such that  $\kappa\beta < \gamma$  and  $0 < \gamma < \tilde{\gamma}$  hold true. More precisely, these two constraints translate into the following bound on  $\beta$ :

$$(2.5) \quad 1 + D \left( \frac{\delta^+ + \delta^-}{2} \right) < \beta < \left( \frac{\varepsilon_0}{\kappa + \varepsilon_0} \right) (1 + D\delta^+).$$

It is easy to see that this bound also implies  $1 + D\delta^- < \beta < 1 + D\delta^+$  as required. However, for this to be possible we need to choose  $\varepsilon_0$  such that  $LHS < RHS$ , in other words,

$$(2.6) \quad \kappa \left( \frac{2 + D(\delta^+ + \delta^-)}{D(\delta^+ - \delta^-)} \right) < \varepsilon_0.$$

Since  $\varepsilon_0$  has to be less than 1 and  $D(\delta^+ - \delta^-)/(2 + D(\delta^+ + \delta^-)) < 1/3$ , this bound is only possible if  $\kappa \in (0, 1/3)$ ; then it remains to choose  $0 < \delta^- < \delta^+ < 1/D$  such that

$$(2.7) \quad \kappa < \frac{D(\delta^+ - \delta^-)}{2 + D(\delta^+ + \delta^-)} \in \left( 0, \frac{1}{3} \right).$$

To satisfy all these constraints, we make the choice of parameters (for  $\kappa < 1/3$ ):

$$\delta^+ := \frac{3(1 + \kappa)}{4D}, \quad \delta^- := \frac{(1 - 3\kappa)}{4D}.$$

Then (2.7) holds true, and we can fix  $\varepsilon_0 \in (0, 1)$  to satisfy (2.6):

$$\varepsilon_0 := \frac{1}{2} \left( 1 + \kappa \left( \frac{2 + D(\delta^+ + \delta^-)}{D(\delta^+ - \delta^-)} \right) \right) = \frac{1}{2} \left( \frac{1 + 9\kappa}{1 + 3\kappa} \right).$$

Finally, we choose  $\beta$  satisfying (2.5) as follows:

$$\beta := \frac{1}{2} \left[ 1 + D \left( \frac{\delta^+ + \delta^-}{2} \right) + \left( \frac{\varepsilon_0}{\kappa + \varepsilon_0} \right) (1 + D\delta^+) \right] = \frac{3}{4} + \frac{(1 + 9\kappa)(7 + 3\kappa)}{8(6\kappa^2 + 11\kappa + 1)} \in (1, 2).$$

□

**2.2. Proof of Proposition 2.1.** Differentiating the modified entropy  $G[f]$ , we obtain

$$\frac{d}{dt} G[f] = D_0[f] + D_1[f] + D_2[f],$$

where the *entropy dissipation functionals*  $D_0, D_1$  and  $D_2$  are given by

$$(2.8) \quad D_0[f] = \langle Lf, f \rangle - \varepsilon_1 \langle AT\Pi f, f \rangle - \varepsilon_1 \langle AT(1 - \Pi)f, f \rangle + \varepsilon_1 \langle T Af, f \rangle + \varepsilon_1 \langle ALf, f \rangle,$$

$$(2.9) \quad D_1[f] = \varepsilon_1 \langle AP_\kappa f, f \rangle + \varepsilon_1 \langle P_\kappa^* Af, f \rangle,$$

$$(2.10) \quad D_2[f] = \langle P_\kappa f, f \rangle + \kappa\varepsilon_2 \frac{d}{dt} \int_{\mathbb{R}^2 \times \mathbb{S}^1} f^2 g \, dx \, d\alpha.$$

Note that the term  $\langle LAf, f \rangle$  vanishes since it has been shown in [5] that  $A = \Pi A$  and hence  $Af \in \text{Ker}(L)$ . Further,  $\langle Tf, f \rangle = 0$  since  $T$  is skew-symmetric. We shall now estimate the dissipation of the entropy in the same spirit as in [4].

**# Step 1: Estimation of  $D_0[f]$ .**

We will show the boundedness of  $D_0$ , which is in fact the dissipation functional for a stationary conveyor belt. We thus recall without proof in the following Lemma some results from [4].

**Lemma 2.1** (Dolbeault et al.). *The following estimates hold true :*

$$\begin{aligned} \langle \mathbf{L}f, f \rangle &\leq -D \|(1 - \Pi)f\|^2, & \|\mathbf{A}\mathbf{T}(1 - \Pi)f\| &\leq C_V \|(1 - \Pi)f\|. \\ \|\mathbf{A}\mathbf{L}f\| &\leq \frac{D}{2} \|(1 - \Pi)f\|, & \|\mathbf{T}\mathbf{A}f\| &\leq \|(1 - \Pi)f\|. \end{aligned}$$

In order to control the remaining contribution in  $D_0$ , we need to take care of the fact that we work with densities with non-zero mass. In what follows, we denote

$$\tilde{g} = f - M_f e^{-V},$$

for any density  $f \in L^2(d\mu)$ . Then  $\int_{\mathbb{R}^2} \Pi \tilde{g} dx = 0$  and  $\tilde{g}$  has zero mass. To control the second term in (2.8), we note that

$$\mathbf{A}\mathbf{T}\Pi = (1 + (\mathbf{T}\Pi)^* \mathbf{T}\Pi)^{-1} (\mathbf{T}\Pi)^* \mathbf{T}\Pi$$

shares its spectral decomposition with  $(\mathbf{T}\Pi)^* \mathbf{T}\Pi$ , and by *macroscopic coercivity*

$$\langle (\mathbf{T}\Pi)^* \mathbf{T}\Pi f, f \rangle = \|\mathbf{T}\Pi f\|^2 = \|\mathbf{T}\Pi \tilde{g}\|^2 \geq \frac{\Lambda}{2} \|\Pi \tilde{g}\|^2$$

and hence

$$\langle \mathbf{A}\mathbf{T}\Pi f, f \rangle \geq \frac{\Lambda/2}{1 + \Lambda/2} \|\Pi \tilde{g}\|^2.$$

Now recalling Lemma 2.1 and using  $\|\Pi \tilde{g}\|^2 = \|\Pi f\|^2 - M_f^2$ , we can estimate :

$$D_0[f] \leq -D \|(1 - \Pi)f\|^2 + \varepsilon_1 B \|(1 - \Pi)f\| \cdot \|f\| - \varepsilon_1 \frac{\Lambda/2}{1 + \Lambda/2} \left( \|\Pi f\|^2 - M_f^2 \right),$$

where we defined  $B := C_V + 1 + D/2$ .

**# Step 2: Estimation of  $D_1[f]$ .**

We now turn to the entropy dissipation functional  $D_1$ , which we will estimate using elliptic regularity. Instead of bounding  $\mathbf{A}\mathbf{P}_\kappa$ , we apply an elliptic regularity strategy to its adjoint, as for  $\mathbf{A}\mathbf{T}(1 - \Pi)$  in [4]. Let  $f \in L^2(d\mu)$  and define  $h := (1 + (\mathbf{T}\Pi)^* \mathbf{T}\Pi)^{-1} f$  so that  $u_h = e^V \Pi h$  satisfies

$$(2.11) \quad \Pi f = e^{-V} u_h + \Pi \mathbf{T}^* \mathbf{T} (e^{-V} u_h) = e^{-V} u_h - \frac{1}{2} \nabla_x \cdot (e^{-V} \nabla_x u_h).$$

We have used here in the space  $L^2(d\mu)$ :

$$\begin{cases} \mathbf{T} = \tau \cdot \nabla_x - \partial_\alpha [(\tau^\perp \cdot \nabla_x V)] \\ \mathbf{T}^* = -\tau \cdot \nabla_x + (\tau^\perp \cdot \nabla_x V) \partial_\alpha - (\tau \cdot \nabla_x V). \end{cases}$$

Then

$$\mathbf{A}^* f = \mathbf{T}\Pi h = e^{-V} \tau \cdot \nabla_x u_h,$$

and since the adjoint for  $\langle \cdot, \cdot \rangle$  of the perturbation operator  $\mathbf{P}_\kappa$  is given by

$$\mathbf{P}_\kappa^* = -\mathbf{P}_\kappa - \mathbf{P}_\kappa V,$$

it follows that

$$\begin{aligned}
\|(\mathbf{AP}_\kappa)^* f\|^2 &= \|\kappa \tau \cdot \nabla_x (e_1 \cdot \nabla_x u_h) e^{-V}\|^2 \\
&= \frac{\kappa^2}{2} \int_{\mathbb{R}^2 \times \mathbb{S}^1} e^{-V} |\tau \cdot \nabla_x (e_1 \cdot \nabla_x u_h)|^2 d\mu \\
&= \frac{\kappa^2}{2} \int_{\mathbb{R}^2} e^{-V} |\nabla_x (e_1 \cdot \nabla_x u_h)|^2 dx \\
&\leq \frac{\kappa^2}{2} \|D_x^2 u_h\|_{L^2(e^{-V} dx)}^2 \\
&\leq \frac{\kappa^2}{2} C_V^2 \|\Pi f\|^2,
\end{aligned}$$

where in the last inequality we have used an elliptic regularity estimate. This estimate turns out to be a particular case of [4, Proposition 5 and Sections 2-3], where the positive constant  $C_V$  is the same as in Lemma 2.1, **Step 1** reproduced from [4]. This concludes the boundedness of  $\mathbf{AP}_\kappa$ ,

$$(2.12) \quad \|\mathbf{AP}_\kappa f\| \leq C_V \frac{\kappa}{\sqrt{2}} \|\Pi f\|.$$

Using a similar approach for the operator  $\mathbf{P}_\kappa^* \mathbf{A}$ , we rewrite its adjoint as

$$\mathbf{A}^* \mathbf{P}_\kappa f = \mathbf{T} \Pi h,$$

where we define  $h := (1 + (\mathbf{T}\Pi)^* \mathbf{T}\Pi)^{-1} \mathbf{P}_\kappa f$  for a given  $f \in L^2(d\mu)$ , or equivalently

$$e^{-V} u_h - \frac{1}{2} \nabla_x \cdot (e^{-V} \nabla_x u_h) = \Pi \mathbf{P}_\kappa f = \mathbf{P}_\kappa \Pi f.$$

Multiplying by  $u_h$  and integrating over  $\mathbb{R}^2$ , we have

$$\begin{aligned}
\|u_h\|_{L^2(e^{-V} dx)}^2 + \frac{1}{2} \|\nabla_x u_h\|_{L^2(e^{-V} dx)}^2 &= -\kappa \int_{\mathbb{R}^2} \mathbf{e}_1 \cdot \nabla_x (\Pi f) u_h dx \\
&= \kappa \int_{\mathbb{R}^2} (\Pi f) \mathbf{e}_1 \cdot \nabla_x u_h dx \\
&\leq \kappa \int_{\mathbb{R}^2} |\nabla_x u_h e^{-V/2}| |\Pi f e^{V/2}| dx \\
&\leq \kappa \|\nabla_x u_h\|_{L^2(e^{-V} dx)} \|\Pi f\| \\
&\leq \frac{1}{4} \|\nabla_x u_h\|_{L^2(e^{-V} dx)}^2 + \kappa^2 \|\Pi f\|^2.
\end{aligned}$$

This inequality can be understood as a  $H^1(e^{-V} dx) \rightarrow H^{-1}(e^{-V} dx)$  elliptic regularity result. Hence

$$\|\mathbf{A}^* \mathbf{P}_\kappa f\|^2 = \|\mathbf{T} \Pi h\|^2 = \frac{1}{2} \|\nabla_x u_h\|_{L^2(e^{-V} dx)}^2 \leq 2\kappa^2 \|\Pi f\|^2,$$

and so we conclude

$$(2.13) \quad \|\mathbf{P}_\kappa^* \mathbf{A} f\| \leq \sqrt{2} \kappa \|\Pi f\|.$$

Combining (2.12) and (2.13), the entropy dissipation functional  $\mathbf{D}_1$  as given in (2.9) is bounded by

$$\mathbf{D}_1[f] \leq \kappa \varepsilon_1 \left( \frac{C_V}{\sqrt{2}} + \sqrt{2} \right) \|\Pi f\| \|f\| \leq 2\kappa \lambda_1 \|f\|^2,$$

where we defined

$$\lambda_1 := \frac{1}{2} \left( \frac{C_V}{\sqrt{2}} + \sqrt{2} \right),$$

and where we used the relation  $\|\Pi f\| \leq \|f\|$ , which follows from Jensen's inequality.

**# Step 3: Estimation of  $D_2[f]$ .**

By integration by parts, we have

$$\langle \mathbf{P}_\kappa f, f \rangle = \frac{\kappa}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^1} (e_1 \cdot \nabla_x V) f^2 e^V dx d\alpha.$$

We will now split our analysis into two cases: bounded and unbounded potential gradients. In the first case, the control of the perturbation operator is trivial:

**Case 1.** Assume there exists a constant  $c_2 > 0$  such that  $|\nabla_x V| \leq c_2$  for all  $x \in \mathbb{R}^2$ . In this case we simply set  $\varepsilon_2 = 0$ , and the estimation of the dissipation functional  $D_2$  is

$$(2.14) \quad D_2[f] = \langle \mathbf{P}_\kappa f, f \rangle \leq \frac{\kappa c_2}{2} \|f\|^2$$

and we don't require the construction of the additional weight  $g$ .

**Case 2.** Assume  $|\nabla_x V| \rightarrow \infty$  as  $|x| \rightarrow \infty$ . In that case, Proposition 2.2 allows us to control the  $g$ -weighted  $L^2$ -norm outside some fixed ball. More precisely, taking  $R_0 > 0$  in (2.3) large enough such that  $|\nabla_x V| \geq 1$  for all  $|x| > R_0$ , we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2 \times \mathbb{S}^1} f^2 g dx d\alpha &= \int_{\mathbb{R}^2 \times \mathbb{S}^1} 2\mathcal{L}_\kappa(f) f g dx d\alpha \\ &= \int_{\mathbb{R}^2 \times \mathbb{S}^1} \mathcal{L}_\kappa(g) f^2 dx d\alpha - 2 \int_{\mathbb{R}^2 \times \mathbb{S}^1} (\partial_\alpha f)^2 g dx d\alpha \\ &\leq \int_0^{2\pi} \int_{|x| < R_0} \mathcal{L}_\kappa(g) f^2 dx d\alpha + \int_0^{2\pi} \int_{|x| > R_0} \mathcal{L}_\kappa(g) f^2 dx d\alpha \\ &\leq \int_0^{2\pi} \int_{|x| < R_0} \mathcal{L}_\kappa(g) f^2 dx d\alpha - c \int_0^{2\pi} \int_{|x| > R_0} |\nabla_x V| f^2 g dx d\alpha \\ &\leq \int_0^{2\pi} \int_{|x| < R_0} ((\mathcal{L}_\kappa(g) + cg) e^{-V}) f^2 e^V dx d\alpha - c \int_{\mathbb{R}^2 \times \mathbb{S}^1} f^2 g dx d\alpha. \\ (2.15) \quad &\leq C_3(R_0) \|f\|^2 - c \int_{\mathbb{R}^2 \times \mathbb{S}^1} f^2 g dx d\alpha, \end{aligned}$$

where  $C_3(R_0) := 2\pi \sup_{|x| \leq R_0} (|\mathcal{L}_\kappa(g) + cg| e^{-V})$ . Further, thanks to the choice of  $g$ , we have the estimate

$$(2.16) \quad \int_{\mathbb{R}^2 \times \mathbb{S}^1} |\nabla_x V| f^2 e^V dx d\alpha \leq C_4 \int_{\mathbb{R}^2 \times \mathbb{S}^1} f^2 g dx d\alpha$$

with

$$C_4 := \sup_{x \in \mathbb{R}^2} (|\nabla_x V| e^{(1-\beta)V}) < \infty$$

which is finite by **(H5)** since  $\beta > 1$ . Combining estimates (2.15) and (2.16), we have

$$\begin{aligned} D_2[f] &= \frac{\kappa}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^1} (e_1 \cdot \nabla_x V) f^2 e^V dx d\alpha + \kappa \varepsilon_2 \frac{d}{dt} \int_{\mathbb{R}^2 \times \mathbb{S}^1} f^2 g dx d\alpha \\ (2.17) \quad &\leq \kappa \left( \left( \frac{C_4}{4\pi} - \varepsilon_2 c \right) \int_{\mathbb{R}^2 \times \mathbb{S}^1} f^2 g dx d\alpha + \varepsilon_2 C_3(R) \|f\|^2 \right). \end{aligned}$$

**# Step 4: Putting the three previous steps together.**

We are now ready to choose a suitable  $\varepsilon_1 > 0$  following the approach in [4]. We write

$$\begin{aligned} \mathsf{D}_0[f] + \mathsf{D}_1[f] &\leq -D \|(1 - \Pi)f\|^2 + \varepsilon_1 B \|(1 - \Pi)f\| \cdot \|f\| \\ &\quad - \varepsilon_1 \frac{\Lambda/2}{1 + \Lambda/2} \left( \|\Pi f\|^2 - M_f^2 \right) + 2\kappa\lambda_1 \|f\|^2 \\ &\leq \left( -D + \varepsilon_1 B \left( 1 + \frac{1}{2\delta} \right) \right) \|(1 - \Pi)f\|^2 \\ &\quad + \varepsilon_1 \left( \frac{B\delta}{2} - \frac{\Lambda/2}{1 + \Lambda/2} \right) \|\Pi f\|^2 + 2\kappa\lambda_1 \|f\|^2 + \varepsilon_1 \frac{\Lambda/2}{1 + \Lambda/2} M_f^2 \end{aligned}$$

for any choice of  $\delta > 0$ , and where we used the identity

$$\|(1 - \Pi)f\| \cdot \|f\| = \|(1 - \Pi)f\|^2 + \|(1 - \Pi)f\| \cdot \|\Pi f\|.$$

Let us choose first  $\delta$  and then  $\varepsilon_1$  following the analysis in the case of a stationary belt [4]: we take

$$\delta = \frac{\Lambda/2}{B(1 + \Lambda/2)}$$

and define

$$r(D) := B \left( 1 + B \left( \frac{1 + \Lambda/2}{\Lambda} \right) \right), \quad s := \frac{\Lambda/4}{1 + \Lambda/2}.$$

This allows us to rewrite the bound on the dissipation functional as

$$\begin{aligned} \mathsf{D}_0[f] + \mathsf{D}_1[f] &\leq - (D - \varepsilon_1 r(D)) \|(1 - \Pi)f\|^2 - \varepsilon_1 s \|\Pi f\|^2 + 2\kappa\lambda_1 \|f\|^2 \\ &\quad + \varepsilon_1 \frac{\Lambda/2}{1 + \Lambda/2} M_f^2. \end{aligned}$$

With the same choice of  $\varepsilon_1 \in (0, 1)$  as in [4], we can find  $\lambda_0 > 0$  such that

$$D - \varepsilon_1 r(D) \geq \varepsilon_1 s \geq 2\lambda_0$$

with the constant  $\lambda_0$  only depending on  $\Lambda$  and  $C_V$  and thus only on the potential  $V$ . From this analysis we conclude

$$\mathsf{D}_0[f] + \mathsf{D}_1[f] \leq -2(\lambda_0 - \kappa\lambda_1) \|f\|^2 + \varepsilon_1 \frac{\Lambda/2}{1 + \Lambda/2} M_f^2.$$

Now adding the control of  $\mathsf{D}_2$ , we obtain a different estimate depending on the behaviour of the potential gradient at infinity following the analysis done in **Step 3**.

**Case 1.** If the potential gradient is bounded, we have  $\mathsf{G}[\cdot] = \mathsf{H}[\cdot]$ , and we conclude from (2.14),

$$\frac{d}{dt} \mathsf{G}[f] = \mathsf{D}_0[f] + \mathsf{D}_1[f] + \mathsf{D}_2[f] \leq -\gamma_1 \|f\|^2 + \gamma_2 M_f^2 \leq -\gamma_1 \mathsf{G}[f] + \gamma_2 M_f^2$$

by the norm equivalence (2.1). Here, we defined

$$\gamma_1 := 2\lambda_0 - 2\kappa \left( \lambda_1 + \frac{c_2}{4} \right) > 0, \quad \gamma_2 := \varepsilon_1 \frac{\Lambda/2}{1 + \Lambda/2} > 0.$$

For our analysis to work, we have to impose that the movement of the conveyor belt is slow enough with respect to the speed at which the fibres are produced,  $\kappa \ll 1$ . More precisely, in the case of a bounded potential gradient, let us suppose that

$$0 < \kappa < \frac{\lambda_0}{\lambda_1 + c_2/4} < \frac{\lambda_0}{\lambda_1}.$$

**Case 2.** We conclude from (2.17),

$$\begin{aligned}
\frac{d}{dt} \mathbf{G}[f] &= \mathbf{D}_0[f] + \mathbf{D}_1[f] + \mathbf{D}_2[f] \\
&\leq -(2\lambda_0 - 2\kappa\lambda_1 - \kappa\varepsilon_2 C_3(R_0)) \|f\|^2 + \kappa \left( \frac{C_4}{4\pi} - \varepsilon_2 c \right) \int_{\mathbb{R}^2 \times \mathbb{S}^1} f^2 g \, dx \, d\alpha + \gamma_2 M_f^2 \\
&\leq -\frac{2(2\lambda_0 - \kappa(2\lambda_1 + \varepsilon_2 C_3(R_0)))}{1 + \varepsilon_1} \mathbf{H}[f] - \kappa\varepsilon_2 \left( c - \frac{C_4}{4\pi\varepsilon_2} \right) \int_{\mathbb{R}^2 \times \mathbb{S}^1} f^2 g \, dx \, d\alpha + \gamma_2 M_f^2 \\
&\leq -\gamma_1 \mathbf{G}[f] + \gamma_2 M_f^2
\end{aligned}$$

again by norm equivalence (2.1), and where we defined

$$\gamma_1 := \min \left\{ \frac{2(2\lambda_0 - \kappa(2\lambda_1 + \varepsilon_2 C_3(R_0)))}{1 + \varepsilon_1}, c - \frac{C_4}{4\pi\varepsilon_2} \right\} > 0, \quad \gamma_2 := \varepsilon_1 \frac{\Lambda/2}{1 + \Lambda/2} > 0.$$

So in this case,  $\varepsilon_2$  should be large enough, and the upper bound for  $\kappa$  should be chosen accordingly

$$\varepsilon_2 > \frac{C_4}{4\pi c}, \quad \kappa < \min \left\{ \frac{\lambda_0}{\lambda_1 + \frac{\varepsilon_2}{2} C_3(R_0)}, \frac{1}{3} \right\}.$$

In order to maximise the rate of convergence to equilibrium given a relative speed  $\kappa$  and a potential  $V$ , one can optimise  $\gamma_1$  over  $\varepsilon_2$  whilst respecting the above constraints.

### 3. EXISTENCE AND UNIQUENESS OF A STEADY STATE

**3.1. Proof of Theorem 1.1.** Proposition 2.1 is the key result that allows us to easily deduce Theorem 1.1 from the hypocoercivity estimate (2.1). Let us begin by proving the existence of a global Gibbs state  $F_\kappa \in L^2(d\mu)$  of mass 1, using the contraction of the modified entropy  $\mathbf{G}[\cdot]$ . Since the  $C^0$ -semigroup  $(S_t)_{t \geq 0}$  associated to the Fokker-Planck equation (1.5) conserves mass and positivity, the set  $\mathcal{C}$  defined by

$$\mathcal{C} := \{f \in L^2(d\mu) : f \geq 0, M_f = 1\}$$

remains invariant under the action of the semigroup:

$$\forall t \geq 0, \quad S_t(\mathcal{C}) \subset \mathcal{C}.$$

Integrating the hypocoercivity estimate (2.2) in Proposition 2.1, we find

$$\mathbf{G}[S_t f - S_t g] \leq e^{-\gamma_1 t} \mathbf{G}[f - g]$$

for any  $t > 0$  and  $f, g \in \mathcal{C}$ . It follows by Banach's fixed point theorem that there exists a unique  $u^t \in \mathcal{C}$  such that  $S_t(u^t) = u^t$  for each  $t > 0$ . In fact, there exists a function  $u \in \mathcal{C}$  such that  $S_t(u) = u$  for all  $t \geq 0$ . To see this, let  $t_n = 2^{-n}$ ,  $n \in \mathbb{N}$ ,  $u_n = u^{t_n}$ . Then  $S_{2^{-n}}(u_n) = u_n$ , and by repeatedly applying the semigroup property,

$$\forall k \in \mathbb{N}, \forall n \in \mathbb{N}, \quad S_{k2^{-n}}(u_n) = u_n.$$

Since  $\mathcal{C}$  is bounded in  $L^2(d\mu)$ , it is weakly compact in  $L^2(d\mu)$ , and thus we can find a subsequence  $(n_j)_{j=1}^\infty$  and  $u \in \mathcal{C}$  such that  $u_{n_j}$  converges weakly to  $u$  in  $\mathcal{C}$ . We will now show that

$$\forall n \in \mathbb{N}, \forall k \in \mathbb{N}, \quad S_{k2^{-n}}(u) = u.$$

Fix  $n \in \mathbb{N}$ . Then for all  $M \in \mathbb{N}$ ,

$$S_{2^{-n-M}}(u_{n+M}) = u_{n+M}.$$

Define  $M_j = n_j - n$  for all  $j \in \mathbb{N}$  such that  $M_j > 0$ . We have

$$S_{k2^{-n}}(u_{n+M_j}) = S_{k2^{M_j} 2^{-n-M_j}}(u_{n+M_j}) = u_{n+M_j}.$$

By continuity of  $S_t(\cdot)$  in the weak topology,  $u_{n+M_j} \rightharpoonup u$  as  $j \rightarrow \infty$  implies

$$S_{k2^{-n}}(u_{n+M_j}) \rightharpoonup S_{k2^{-n}}(u),$$

which proves  $S_{k2^{-n}}(u) = u$  as claimed. By density of the dyadic rationals

$$\{k2^{-n} : k \in \mathbb{N}, n \in \mathbb{N}\}$$

in  $\mathbb{R}_{>0}$  and uniform continuity of  $S_t(u)$  in  $t$  for all  $u \in \mathcal{C}$ , we conclude

$$\forall t \geq 0, \quad S_t(u) = u.$$

This shows the existence and uniqueness of a global stationary state  $F_\kappa$  of mass 1.

To complete the proof of Theorem 1.1, we apply the hypocoercivity estimate (2.1) to the difference between a solution  $f \in L^2(d\mu)$  and the unique stationary state of the same mass,  $M_f F_\kappa$ , to show exponential convergence to equilibrium in  $\|\cdot\|$ : we have first the strict contraction estimate

$$\mathbf{G}[f(t) - M_f F_\kappa] \leq \mathbf{G}[f(0) - M_f F_\kappa] e^{-\gamma_1 t},$$

and we deduce for the more standard  $L^2$  norm:

$$\begin{aligned} \|f(t) - M_f F_\kappa\|^2 &\leq \frac{2}{1 - \varepsilon_1} \left( \mathbf{G}[f(t) - M_f F_\kappa] - \varepsilon_2 \int_{\mathbb{R}^2 \times \mathbb{S}^1} |f(t) - M_f F_\kappa|^2 g \, dx \, d\alpha \right) \\ &\leq \frac{2}{1 - \varepsilon_1} \mathbf{G}[f(t) - M_f F_\kappa] \\ &\leq \frac{2}{1 - \varepsilon_1} \mathbf{G}[f(0) - M_f F_\kappa] e^{-\gamma_1 t} \\ &\leq \left( \frac{1 + \varepsilon_1}{1 - \varepsilon_1} \|f(0) - M_f F_\kappa\|^2 + \kappa \varepsilon_2 \frac{2}{1 - \varepsilon_1} \int_{\mathbb{R}^2 \times \mathbb{S}^1} |f(t) - M_f F_\kappa|^2 g \, dx \, d\alpha \right) e^{-\gamma_1 t} \\ &\leq \frac{2}{1 - \varepsilon_1} \left( \|f(0) - M_f F_\kappa\| + \sqrt{\kappa \varepsilon_2} \left( \int_{\mathbb{R}^2 \times \mathbb{S}^1} |f(t) - M_f F_\kappa|^2 g \, dx \, d\alpha \right)^{1/2} \right)^2 e^{-\gamma_1 t}, \end{aligned}$$

which proves (1.6) with  $C^2 = 2/(1 - \varepsilon_1)$  and rate of convergence  $\lambda_\kappa := \gamma_1/2$ .

**3.2. Concluding remarks.** From our previous estimates, we have that  $G(F_\kappa)$  is uniformly bounded in  $\kappa$  for  $\kappa$  sufficiently small. As a consequence,  $(F_\kappa)_{\kappa>0}$  is a relatively weakly compact family in  $L^2(d\mu)$ , and by uniqueness of the stationary state distributional solution with  $\kappa = 0$ , we deduce that  $F_\kappa \rightarrow F_0$  as  $\kappa \rightarrow 0$ . Observe moreover that, as  $\kappa \rightarrow 0$ , the lower bound on the relaxation rate given by our proof converges to the one already obtained in [4]. It could also be proved with further work that the optimal (spectral gap) relaxation rate is continuous as  $\kappa \rightarrow 0$ .

Working in  $L^2(d\mu)$  we are treating the operator  $\mathbf{L}_\kappa$  as a small perturbation of the case  $\kappa = 0$  with stationary conveyor belt. The space that is well-adapted to investigate the convergence to  $F_\kappa$  in the case  $\kappa > 0$  however is  $L^2(F_\kappa^{-1} dx d\alpha)$ . In this  $L^2$ -space the transport operator  $\mathbf{T} - \mathbf{P}_\kappa$  is not skew-symmetric and the collision operator  $\mathbf{L}$  is not self-adjoint, so the hypocoercivity method [5] cannot be applied. To get around this, one can split the operator  $\mathbf{L}_\kappa$  differently into a transport and a collision part following the approach in [1]. More precisely, we can write  $\mathbf{L}_\kappa = \tilde{\mathbf{L}} - \tilde{\mathbf{T}}$  where

$$\begin{cases} \tilde{\mathbf{L}}f = \partial_\alpha \left( D \partial_\alpha f - \frac{\partial_\alpha F_\kappa}{F_\kappa} f \right), \\ \tilde{\mathbf{T}}f = (\tau + \kappa e_1) \cdot \nabla_x f - \partial_\alpha \left[ (\tau^\perp \cdot \nabla_x V + \frac{\partial_\alpha F_\kappa}{F_\kappa}) f \right]. \end{cases}$$

It is easily seen that  $\tilde{\mathbf{L}}$  is symmetric and negative semi-definite, and that  $\tilde{\mathbf{T}}$  is skew-symmetric in  $L^2(F_\kappa^{-1} dx d\alpha)$ . Furthermore, the stationary state  $F_\kappa$  lies in the intersection of the kernels of the collision and transport operators, i.e.  $F_\kappa \in \text{Ker}(\tilde{\mathbf{L}}) \cap \text{Ker}(\tilde{\mathbf{T}})$ . In order to apply the hypocoercivity



approach with these operators, we would need to show microscopic and macroscopic coercivity of  $\tilde{L}$  and  $\tilde{T}$ . This requires as in [1] that we are able to control the behaviour of the stationary state at infinity, i.e. for large enough  $|x|$ ,

$$\forall \alpha \in \mathbb{S}^1, \quad e^{-\mu_1 V(x)} \leq F_\kappa(x, \alpha) \leq e^{-\mu_2 V(x)}$$

for some constants  $\mu_1, \mu_2 > 0$ . If true, this would be a very strong physical information about the behaviour of the stationary state, but we don't know how to prove it at now. Even with this information at hand, this approach requires that the existence of the stationary state is known a priori. The rate of convergence one obtains in this case may be different from the rate obtained here, and it is not clear which method yields the better rate as both are most likely not optimal.

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