



**HAL**  
open science

## Exponential decay to equilibrium for a fibre lay-down process on a moving conveyor belt

Emeric Bouin, Franca Hoffmann, Clément Mouhot

► **To cite this version:**

Emeric Bouin, Franca Hoffmann, Clément Mouhot. Exponential decay to equilibrium for a fibre lay-down process on a moving conveyor belt. 2016. hal-01315222v1

**HAL Id: hal-01315222**

**<https://hal.science/hal-01315222v1>**

Preprint submitted on 12 May 2016 (v1), last revised 14 Apr 2017 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Exponential decay to equilibrium for a fibre lay-down process on a moving conveyor belt

Emeric Bouin \*

Franca Hoffmann<sup>†‡</sup>

Clément Mouhot <sup>§</sup>

May 12, 2016

## Abstract

We show existence and uniqueness of a stationary state for a kinetic Fokker-Planck equation modeling the fibre lay-down process in the production of non-woven textiles. Following a micro-macro decomposition, we use hypocoercivity techniques to show exponential convergence to equilibrium with an explicit rate assuming the conveyor belt moves slow enough. This work is an extension of [2], where the authors consider the case of a stationary conveyor belt. Adding the movement of the belt, the global Gibbs state is not known explicitly. We thus derive a more general hypocoercivity estimate from which existence, uniqueness and exponential convergence can be derived. To treat the same class of potentials as in [2], we make use of an additional weight function following the Lyapunov functional approach in [8].

**Keywords**— hypocoercivity, rate of convergence, fibre lay-down, existence and uniqueness of stationary state, perturbation, moving belt

## 1 Introduction

The mathematical analysis of the fibre lay-down process in the production of non-woven textiles has seen a lot of interest in recent years [9, 10, 4, 6, 7, 2, 8]. Non-woven materials are produced in melt-spinning operations: hundreds of individual endless fibres are obtained by continuous extrusion through nozzles of a melted polymer. The nozzles are densely and equidistantly placed in a row at a spinning beam. The viscoelastic, slender and inextensible fibres lay down on a moving conveyor belt to form a web, where they solidify due to cooling air streams. Before touching the conveyor belt, the fibres become entangled and form loops due to highly turbulent air flow. In [9] a general mathematical model for the fibre dynamics is presented which enables the full simulation of the process. Due to the huge amount of physical details these simulations of the fibre spinning and lay-down usually require an extremely large computational effort and high memory storage,

---

\*CEREMADE - Université Paris-Dauphine, UMR CNRS 7534, Place du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16, France. E-mail: [bouin@ceremade.dauphine.fr](mailto:bouin@ceremade.dauphine.fr)

<sup>†</sup>CCA, Centre for Mathematical Sciences, University of Cambridge, Cambridge CB3 0WA, UK. E-mail: [fkoh2@cam.ac.uk](mailto:fkoh2@cam.ac.uk)

<sup>‡</sup>Department of Mathematics, Imperial College London, South Kensington Campus, London SW7 2AZ, UK. E-mail: [fh1009@ic.ac.uk](mailto:fh1009@ic.ac.uk)

<sup>§</sup>DPMMS, Centre for Mathematical Sciences, University of Cambridge, Cambridge CB3 0WA, UK. E-mail: [c.mouhot@dpmms.cam.ac.uk](mailto:c.mouhot@dpmms.cam.ac.uk)

see [10]. Thus, a simplified two-dimensional stochastic model for the fibre lay-down process is introduced in [4]. The density of the stochastic process satisfies an associated Fokker-Planck evolution equation. An analytic investigation of this Fokker-Planck equation has been performed in [4] where asymptotic properties and ergodicity of the process have been proven and explicit rates for the convergence to the stationary solution have been obtained. Generalisations of the two-dimensional stochastic model [4] to three dimensions has been developed by Klar et al. in [6] and to any dimension  $d \geq 2$  by Grothaus et al. in [5].

We now describe the model we are interested in, which comes from [4]. We track the position  $x(t) \in \mathbb{R}^2$  and the angle  $\alpha(t) \in \mathbb{S}^1$  of the fibre at the lay-down point where it touches the conveyor belt. Interactions of neighbouring fibres are neglected. If  $x_0(t)$  is the lay-down point in the coordinate system following the conveyor belt, then the tangent vector of the fibre is denoted by  $\tau(\alpha(t))$  with  $\tau(\alpha) = (\cos \alpha, \sin \alpha)$ . Since the extrusion of fibres happens at a constant speed, and the fibres are inextensible, the lay-down process can be assumed to happen at constant normalised speed  $\|x'_0(t)\| = 1$ . If the conveyor belt moves with constant speed  $\kappa$  in direction  $e_1 = (1, 0)$ , then

$$\frac{dx}{dt} = \tau(\alpha) + \kappa e_1.$$

Note that the speed of the conveyor belt cannot exceed the lay-down speed:  $0 \leq \kappa \leq 1$ . The fibre dynamics in the deposition region close to the conveyor belt are dominated by the turbulent air flow. Applying this concept, the dynamics of the angle  $\alpha(t)$  can be described by a deterministic force moving the lay-down point towards the equilibrium  $x = 0$  and by a Brownian motion modelling the effect of the turbulent air flow. We obtain an Itô stochastic differential equation for the random variable  $X_t = (x_t, \alpha_t)$  on  $\mathbb{R}^2 \times \mathbb{S}^1$ ,

$$\begin{cases} dx_t &= (\tau(\alpha_t) + \kappa e_1) dt, \\ d\alpha_t &= -\tau^\perp(\alpha_t) \cdot \nabla_x V(x_t) dt + A dW_t, \end{cases} \quad (1.1)$$

where  $W_t$  denotes a one-dimensional Wiener process,  $\tau^\perp = (-\sin \alpha, \cos \alpha)$ ,  $A > 0$  measures its strength relative to the deterministic forcing, and  $V : \mathbb{R}^2 \rightarrow \mathbb{R}$  is an external potential carrying information on the coiling properties of the fibre. More precisely, since a curved fibre tends back to its starting point, the change of the angle  $\alpha$  is assumed to be proportional to  $\tau^\perp(\alpha) \cdot x$ . It has been shown in [8] that under suitable assumptions on the external potential  $V$ , the fibre lay down process (1.1) has a unique invariant distribution and is even geometrically ergodic. The stochastic approach yields exponential convergence in total variation norm, however without an explicit rate. We will show here that a stronger result can be obtained with the functional analytic approach. Our argument uses crucially the construction of a Lyapunov functional for the fibre lay-down process in the case of unbounded potential gradients as in the stochastic paper (Proposition 3.7, [8]).

The probability density function  $f(t, x, \alpha)$  corresponding to the stochastic process (1.1) is governed by the Fokker-Planck equation

$$\partial_t f + (\tau + \kappa e_1) \cdot \nabla_x f - \partial_\alpha \left( \tau^\perp \cdot \nabla_x V f \right) = D \partial_{\alpha\alpha} f \quad (1.2)$$

with diffusivity  $D = A^2/2$ . We require the following assumptions on the external potential  $V$ :

**(H1) Regularity:**  $V \in W_{\text{loc}}^{2,\infty}(\mathbb{R}^2) \cap C^2(\mathbb{R}^2)$  and  $V$  is spherically symmetric outside some ball  $B(0, R)$ .

**(H2) Normalisation:**  $\int_{\mathbb{R}^2} e^{-V} dx = 1$ .

**(H3) Spectral gap condition:** there exists a positive constant  $\Lambda$  such that for any  $u \in H^1(e^{-V} dx)$  which  $\int_{\mathbb{R}^2} u e^{-V} dx = 0$ , we have

$$\int_{\mathbb{R}^2} |\nabla_x u|^2 e^{-V} dx \geq \Lambda \int_{\mathbb{R}^2} u^2 e^{-V} dx.$$

**(H4) Pointwise condition:** there exists  $c_1 > 0$  such that for any  $x \in \mathbb{R}^2$ ,

$$|D_x^2 V(x)| \leq c_1(1 + |\nabla_x V(x)|),$$

where  $D_x^2 V$  denotes the Hessian of  $V(x)$ .

**(H5) Behaviour at infinity:**  $\lim_{|x| \rightarrow \infty} \frac{|\nabla_x V(x)|}{V(x)} = 0$ .

**Remark 1.** Assumptions **(H2)**–**(H4)** are as stated in [2]. Assumption **(H1)** assumes stronger regularity of the potential. Assumption **(H5)** is only necessary if the potential gradient  $|\nabla_x V|$  is unbounded, and trivial otherwise. Both bounded and unbounded potential gradients may appear, depending on the physical context. A typical example for an external potential satisfying assumptions **(H1)**–**(H5)** is given by

$$V(x) = K(1 + |x|^2)^{\beta/2} \quad (1.3)$$

for some constants  $K > 0$  and  $\beta \geq 1$  [3, 8]. The potential (1.3) satisfies **(H3)** since

$$\liminf_{|x| \rightarrow \infty} (|\nabla_x V|^2 - 2\Delta V) > 0,$$

see for instance (A.19. Some criteria for Poincaré inequalities, [11], page 135). The other assumptions are trivially satisfied as can be checked by direct inspection. The gradient of this choice of potential is bounded for  $\beta = 1$  and unbounded for  $\beta > 1$ .

**Remark 2.** The ergodicity proof in [8] assumes that the potential satisfies

$$\lim_{|x| \rightarrow \infty} \frac{|\nabla_x V(x)|}{V(x)} = 0, \quad \lim_{|x| \rightarrow \infty} \frac{|D_x^2 V(x)|}{|\nabla_x V(x)|} = 0, \quad \lim_{r \rightarrow \infty} V'(r) = \infty. \quad (1.4)$$

Under these assumptions, there exist an invariant distribution  $\nu$  to the fibre lay-down process (1.1), and some constants  $C(x_0) > 0$ ,  $\lambda > 0$  such that

$$\|\mathcal{P}_{x_0, \alpha_0}(X_t \in \cdot) - \nu\|_{TV} \leq C(x_0)e^{-\lambda t}.$$

The stochastic Lyapunov technique applied in [8] however does not give any information on how the constant  $C(x_0)$  depends on the initial position  $x_0$ , or how the rate of convergence  $\lambda$  depends on the belt speed  $\kappa$ , the potential  $V$  and the noise strength  $A$ . This can be achieved using hypocoercivity techniques, proving convergence in a weighted  $L^2$ -norm, which is slightly stronger than the convergence in total variation norm shown in [8]. Conceptually, conditions (1.4) ensure that the potential

$V$  is driving the process back inside a compact set, where the noise can be controlled. Note that conditions (1.4) follow directly from **(H1)**–**(H5)**. They are more general in the sense that we do not require a spectral gap, the proof for exponential convergence to equilibrium done in [8] however makes use of the strong Feller property which can be translated in some cases into a spectral gap. Further, in [8], hypoellipticity of the fiber lay down process allows to deduce the existence of a transition density, which provides the result via an explicit Lyapunov function argument. By making the stronger assumptions **(H1)**–**(H5)**, and adapting the Lyapunov function argument presented in [8], we are able to derive an explicit rate of convergence including its dependence on the initial data  $f_0$ , the relative speed of the conveyor belt  $\kappa$  and the potential  $V$ .

To set up a functional framework associated to this Fokker-Planck equation, we may rewrite (1.2) as an abstract ODE

$$\partial_t f = (\mathbf{L} - \mathbf{T}) f + \mathbf{P}_\kappa f, \quad (1.5)$$

where the collision operator  $\mathbf{L} = D\partial_{\alpha\alpha}$  acts as a multiplier in the position variable  $x$ ,  $\mathbf{P}_\kappa$  is the perturbation introduced by the moving belt,

$$\mathbf{P}_\kappa f = -\kappa e_1 \cdot \nabla_x f,$$

and the transport operator  $\mathbf{T}$  is given by

$$\mathbf{T}f = \tau \cdot \nabla_x f - \partial_\alpha \left( \tau^\perp \cdot \nabla_x V f \right).$$

We consider solutions to (1.5) in the space  $L^2(\mathbb{R}^2 \times \mathbb{S}^1, d\mu) = L^2(d\mu)$  with measure

$$d\mu(x, \alpha) = e^{V(x)} \frac{1}{2\pi} dx d\alpha.$$

We denote by  $\langle \cdot, \cdot \rangle$  the corresponding scalar product and by  $\|\cdot\|$  the associated norm. We introduce the orthogonal projection  $\Pi$  on the set of local equilibria  $\text{Ker } \mathbf{L}$  consisting of all  $\alpha$ -independent distributions,

$$\Pi f := \frac{1}{2\pi} \int_{\mathbb{S}^1} f d\alpha.$$

We also define the *mass* of a given distribution  $f \in L^2(d\mu)$  through the formula

$$M_f := \frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^1} f dx d\alpha$$

so that  $M_f e^{-V}$  is the projection onto the unperturbed equilibrium distribution  $F_0$ . We notice after integrating (1.2) over  $\mathbb{R}^2 \times \mathbb{S}^1$  that the mass of any solution of (1.2) is conserved through time. Moreover, any solution of (1.2) remains non-negative as soon as the initial datum is non-negative.

In this functional setting, the operators  $\mathbf{T}$  and  $\mathbf{L}$  have several nice properties that allow us to apply the general theory for linear kinetic equations conserving mass as outlined in [3]. First of all,  $\mathbf{L}$  and  $\mathbf{T}$  are closed operators on  $L^2(d\mu)$  such that  $\mathbf{L} - \mathbf{T}$  generates the  $\mathcal{C}^0$ -semigroup  $e^{(\mathbf{L}-\mathbf{T})t}$  on  $L^2(d\mu)$ . Furthermore,  $\mathbf{L}$  is symmetric and negative semi-definite on  $L^2(d\mu)$ ,

$$\langle \mathbf{L}f, f \rangle = -D \|\partial_\alpha f\|^2 \leq 0,$$

i.e.  $\mathbf{L}$  is dissipative. Further, we have for any  $f \in L^2(d\mu)$ ,

$$\mathbb{T}\Pi f = e^{-V} \tau \cdot \nabla_x u_f$$

with  $u_f := e^V \Pi f$ , which implies  $\Pi \mathbb{T} \Pi = 0$  on  $L^2(d\mu)$ . Since the transport operator  $\mathbb{T}$  is skew symmetric with respect to  $\langle \cdot, \cdot \rangle$ , we obtain the *entropy equality*

$$\frac{1}{2} \frac{d}{dt} \|f\|^2 = \langle \mathbf{L}f, f \rangle + \langle \mathbf{P}_\kappa f, f \rangle,$$

for any  $f$  in  $L^2(d\mu)$ . In the case  $\kappa = 0$ , if the entropy dissipation  $-\langle \mathbf{L}f, f \rangle$  was coercive with respect to the norm  $\|\cdot\|$ , exponential decay to zero would follow as  $t \rightarrow \infty$ . However, such a coercivity property cannot hold since  $\mathbf{L}$  vanishes on the set of local equilibria. Instead, Dolbeault et al. [3] applied a strategy called *hypocoercivity*, first developed by Villani in [11]. The full hypocoercivity analysis of the long time behaviour of solutions to this kinetic model in the case of a stationary conveyor belt,  $\kappa = 0$ , is completed in [2]. For technical applications in the production process of non-wovens, one is interested in a model including the movement of the conveyor belt, and our aim is to extend the results in [2] to the case  $\kappa \neq 0$ . Following the idea of a micro-macro decomposition, we shall split our assumptions into two main requirements: *microscopic coercivity*, which assumes that the restriction of  $\mathbf{L}$  to  $\text{Ker}^\perp \mathbf{L}$  is coercive, and *macroscopic coercivity*, which is a spectral gap-like inequality for the operator obtained when taking a parabolic drift-diffusion limit, in other words, the restriction of  $\mathbb{T}$  to  $\text{Ker} \mathbf{L}$  is coercive.

- **Microscopic coercivity:** The operator  $\mathbf{L}$  is symmetric and the Poincaré inequality on  $\mathbb{S}^1$ ,

$$\frac{1}{2\pi} \int_{\mathbb{S}^1} |\partial_\alpha f|^2 d\alpha \geq \frac{1}{2\pi} \int_{\mathbb{S}^1} \left( f - \frac{1}{2\pi} \int_{\mathbb{S}^1} f d\alpha \right)^2 d\alpha,$$

yields that for all  $f \in \mathcal{D}(\mathbf{L})$ ,

$$-\langle \mathbf{L}f, f \rangle \geq D \|(1 - \Pi)f\|^2.$$

- **Macroscopic coercivity:** The operator  $\mathbb{T}$  is skew-symmetric and for any  $g \in L^2(d\mu)$  such that  $u_g \in H^1(e^{-V} dx)$  and  $\int_{\mathbb{R}^2 \times \mathbb{S}^1} g d\mu = 0$ ,

$$\|\mathbb{T}\Pi g\|^2 = \frac{1}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^1} e^{-V} |\nabla_x u_g|^2 dx d\alpha \geq \frac{\Lambda}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^1} e^{-V} u_g^2 dx d\alpha = \frac{\Lambda}{2} \|\Pi g\|^2$$

by the spectral gap condition **(H3)**.

In the case  $\kappa = 0$ , we have existence of a unique global normalised equilibrium distribution  $F_0(x) = e^{-V(x)}$  in the intersection of the null spaces of  $\mathbb{T}$  and  $\mathbf{L}$ ,  $\text{Ker} \mathbf{L} \cap \text{Ker} \mathbb{T}$ . For a moving conveyor belt,  $F_0$  is not in the kernel of  $\mathbf{P}_\kappa$  and we are not able to find the global Gibbs state of (1.5) explicitly. However, the hypocoercivity method as applied to the fibre lay-down process in [2] only depends on the first moment which cancels on the solution of the linear equation (1.5). Moreover, the hypocoercivity theory is based on a priori estimates [3], and is therefore quite stable under perturbation. These considerations in mind, we establish existence and uniqueness of a global Gibbs state and determine the rate of convergence of solutions in  $L^2(d\mu)$  towards this equilibrium distribution using hypocoercivity techniques. Namely, we prove the following result, giving an explicit rate of convergence for small enough movement of the conveyor belt:

**Theorem 1.1.** *Under assumptions (H1)–(H5) and for  $0 < \kappa \ll 1$  small enough, there exists a unique stationary state*

$$0 \leq F_\kappa(x, \alpha) \in L^2(d\mu)$$

*of unit mass. In addition, for any solution  $f(t, \cdot) \in L^2(d\mu)$  of (1.2) with mass  $M_f$  we have*

$$\|f(t, \cdot) - M_f F_\kappa\| \leq C e^{-\lambda_\kappa t} \quad (1.6)$$

*with  $\lambda_\kappa > 0$  and  $C > 0$  positive constants which only depend on the initial data  $f(t = 0, \cdot)$ , the relative speed of the conveyor belt  $\kappa$ , and the potential  $V$ .*

In general, we are not able to compute the stationary state explicitly, but from the analysis in [2] for the case  $\kappa = 0$ , we can expect that  $F_\kappa$  converges to  $e^{-V}$  as  $\kappa \rightarrow 0$ .

The rest of the paper is organized as follows. In Section 2, we prove the main hypocoercivity estimate. This allows us to prove existence and uniqueness of a steady state in Section 3 by a contraction argument.

## Acknowledgements

EB is very grateful to the University of Cambridge for its sunny hospitality during the second semester of the academic year 2015-2016. EB and CM acknowledge the support of the ERC Grant MATKIT (ERC-2011-StG). FH acknowledges support from the Engineering and Physical Sciences Research Council (UK) grant number EP/H023348/1 for the University of Cambridge Centre for Doctoral Training, the Cambridge Centre for Analysis.

## 2 Hypocoercivity estimate

To follow an hypocoercivity strategy, we need to define a modified entropy functional. For this purpose, let us first introduce the auxiliary operator

$$A := (1 + (\mathbb{T}\Pi)^*(\mathbb{T}\Pi))^{-1}(\mathbb{T}\Pi)^*.$$

This operator is now classical after [3] (and the references therein). As in [3], the *modified entropy functional* then reads

$$H[f] := \frac{1}{2} \|f\|^2 + \varepsilon_1 \langle Af, f \rangle$$

for some suitably chosen  $\varepsilon_1 \in (0, 1)$  to be determined later. We know crucially from [3] that  $H^{1/2}[\cdot]$  is equivalent to  $\|\cdot\|$  on  $L^2(d\mu)$ ,

$$\frac{1 - \varepsilon_1}{2} \|f\|^2 \leq H[f] \leq \frac{1 + \varepsilon_1}{2} \|f\|^2, \quad f \in L^2(d\mu). \quad (2.1)$$

If the potential gradient is unbounded, we will add an extra term to the energy functional,

$$G[f] = H[f] + \frac{\varepsilon_2}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^1} f^2 g \, dx d\alpha$$

for a convenient choice of weight  $g(x, \alpha)$  to be defined below. If the potential gradient is bounded, we set  $\varepsilon_2 = 0$ . In this section, we will prove the following hypocoercivity estimate:

**Proposition 2.1.** *Assume that hypothesis (H1)–(H5) hold and that  $0 < \kappa \ll 1$  is small enough. Let  $f$  be a solution of (1.2). If  $|\nabla_x V|$  is unbounded, then  $\varepsilon_2 > 0$  and we assume in addition that  $f(t = 0, \cdot) \in L^2\left(g(x, \alpha) \frac{1}{2\pi} dx d\alpha\right)$ , where  $g$  is a suitable Lyapunov function given in Proposition 2.2. Otherwise,  $\varepsilon_2 = 0$ . Then  $f$  satisfies the following Grönwall type estimate:*

$$\frac{d}{dt} \mathbf{G}[f(t, \cdot)] \leq -\gamma_1 \mathbf{G}[f(t, \cdot)] + \gamma_2 M_f^2, \quad (2.2)$$

where  $\gamma_1 > 0, \gamma_2 > 0$  are explicit constants only depending on the initial data  $f(0, \cdot)$ , the relative speed of the conveyor belt  $\kappa$  and the potential  $V$ .

In fact, the estimate (2.2) is stronger than what is required for the uniqueness of a global Gibbs state, and represents an improvement of the estimate given in [2]. If applied to the difference of two solutions with the same mass,  $\tilde{f} = f_1 - f_2$ , we can find an estimate of the exponential decay rate towards equilibrium.

## 2.1 Lyapunov functional approach for weight function $g$

In order to prove the hypocoercivity estimate (2.2), we will distinguish two cases: either the potential gradient is bounded, or it is unbounded. In the case of unbounded potential gradients, we make use of a new weight function

$$g(x, \alpha) = \exp\left(\beta V(x) + G\left(\tau(\alpha) \cdot \frac{\nabla_x V}{|\nabla_x V|}\right)\right),$$

where  $\beta > 1$  and  $G \in C^1([-1, 1])$ ,  $G > 0$  are yet to be determined. The idea is to choose the weight  $g$  in a way that it is a Lyapunov function for the fibre lay down process, allowing us to control the perturbation operator  $\mathbf{P}_\kappa$ . Indeed, we can show existence of such a Lyapunov function  $g$  under appropriate conditions following the argument in [8]:

**Proposition 2.2.** *Assume  $V \in C^2(\mathbb{R}^2)$  is spherically symmetric outside some ball  $B(0, R)$ , and satisfies  $V'(r) \rightarrow \infty$  as  $r \rightarrow \infty$ . If  $\kappa < 1/2$  and (H5) holds true, then there exists a function  $g(x, \alpha)$ , a constant  $c > 0$  and a finite radius  $\rho > 0$  such that*

$$\forall |x| > \rho, \forall \alpha \in \mathbb{S}^1, \quad \mathcal{L}(g) \leq -cg, \quad (2.3)$$

where  $\mathcal{L}$  is the adjoint of  $\mathbf{L}$  in  $L^2(g(x, \alpha) dx d\alpha)$ ,

$$\begin{aligned} \mathcal{L}(g) &:= D\partial_{\alpha\alpha}g + (\tau + \kappa e_1) \cdot \nabla_x g + 2\partial_\alpha\left(\tau^\perp \cdot \nabla_x V\right)g - \partial_\alpha\left(\tau^\perp \cdot \nabla_x Vg\right) \\ &= D\partial_{\alpha\alpha}g + (\tau + \kappa e_1) \cdot \nabla_x g - \left(\tau^\perp \cdot \nabla_x V\right)\partial_\alpha g - (\tau \cdot \nabla_x V)g. \end{aligned}$$

*Proof of Proposition 2.2.* Let us consider the function

$$g(x, \alpha) = \exp\left(\beta V(x) + G\left(\tau(\alpha) \cdot \frac{\nabla_x V}{|\nabla_x V|}\right)\right),$$

where  $G(Y) = |\nabla_x V|\Gamma(Y)$  for a function  $\Gamma(\cdot)$  to be determined, and we define

$$Y := \left(\tau \cdot \frac{\nabla_x V}{|\nabla_x V|}\right), \quad Y^\perp := \left(\tau^\perp \cdot \frac{\nabla_x V}{|\nabla_x V|}\right).$$



Applying  $\mathcal{L}$  to  $g$ , we can compute explicitly

$$\frac{\mathcal{L}(g)}{g} = D(\partial_{\alpha\alpha}G + |\partial_\alpha G|^2) + (\tau(\alpha) + \kappa e_1) \cdot (\beta \nabla_x V + \nabla_x G) - |\nabla_x V| Y^\perp \partial_\alpha G - |\nabla_x V| Y.$$

Since

$$\begin{aligned} \partial_\alpha G &= |\nabla_x V| Y^\perp \Gamma'(Y), \\ \partial_{\alpha\alpha} G &= |\nabla_x V| \partial_\alpha \left( Y^\perp \Gamma'(Y) \right) = |\nabla_x V| \left( -Y \Gamma'(Y) + |Y^\perp|^2 \Gamma''(Y) \right), \end{aligned}$$

we get

$$\begin{aligned} \frac{\mathcal{L}(g)}{g} &= D \left( |\nabla_x V| \left( -Y \Gamma'(Y) + |Y^\perp|^2 \Gamma''(Y) \right) + |\nabla_x V|^2 |Y^\perp|^2 (\Gamma'(Y))^2 \right) \\ &\quad + (\tau(\alpha) + \kappa e_1) \cdot (\beta \nabla_x V + \nabla_x G) - |\nabla_x V|^2 |Y^\perp|^2 \Gamma'(Y) - |\nabla_x V| Y \\ &= (\beta - 1 - D\Gamma'(Y)) |\nabla_x V| Y + \kappa \beta e_1 \cdot \nabla_x V + (\tau(\alpha) + \kappa e_1) \cdot \nabla_x G \\ &\quad + |Y^\perp|^2 \left( D |\nabla_x V| \Gamma''(Y) + |\nabla_x V|^2 \left[ D (\Gamma'(Y))^2 - \Gamma'(Y) \right] \right). \end{aligned}$$

In order to see which  $\Gamma$  to choose, let us divide by  $|\nabla_x V|$  and denote the diffusion and transport part by

$$\text{diff}(x, \alpha) := (\tau(\alpha) + \kappa e_1) \cdot \frac{\nabla_x G}{|\nabla_x V|}, \quad \text{tran}(x) := \frac{e_1 \cdot \nabla_x V}{|\nabla_x V|},$$

so we can rewrite the above and seek a positive constant  $c > 0$  and a radius  $\rho > 0$  such that for any  $\alpha \in \mathbb{S}^1$  and  $|x| > \rho$ ,

$$\begin{aligned} &(\beta - 1 - D\Gamma'(Y))Y + \kappa \beta \text{tran}(x) + \text{diff}(x, \alpha) \\ &+ |Y^\perp|^2 \left( D\Gamma''(Y) + |\nabla_x V| \left[ D (\Gamma'(Y))^2 - \Gamma'(Y) \right] \right) \leq -\frac{c}{|\nabla_x V|}. \end{aligned}$$

To achieve this bound, note first of all that  $|Y| \leq 1$  and  $|\text{tran}| \leq 1$  for all  $(x, \alpha) \in \mathbb{R}^2 \times \mathbb{S}^1$ . Further, the diffusion term  $\text{diff}(\cdot)$  can be made arbitrarily small outside a sufficiently large ball:

$$\begin{aligned} \text{diff}(x, \alpha) &= (\tau + \kappa e_1) \cdot \nabla_x \left( \Gamma \left( \tau \cdot \frac{\nabla_x V}{|\nabla_x V|} \right) \right) + \Gamma \left( \tau \cdot \frac{\nabla_x V}{|\nabla_x V|} \right) (\tau + \kappa e_1) \cdot \frac{\nabla_x (|\nabla_x V|)}{|\nabla_x V|} \\ &= (\tau + \kappa e_1) \cdot \left[ \Gamma'(Y) \nabla_x Y + \Gamma(Y) \frac{\nabla_x (|\nabla_x V|)}{|\nabla_x V|} \right], \end{aligned}$$

and both  $|\nabla_x Y|$  and  $|\nabla_x (|\nabla_x V|)|/|\nabla_x V|$  converge to zero as  $|x| \rightarrow \infty$  since we can deduce

$$\lim_{|x| \rightarrow \infty} \frac{|D_x^2 V(x)|}{|\nabla_x V(x)|} = 0$$

from **(H2)**–**(H4)**–**(H5)**. In other words, using the fact that the potential gradient is unbounded, it remains to show that we can find constants  $\gamma > \kappa \beta > 0$  such that

$$(\beta - 1 - D\Gamma')Y + |Y^\perp|^2 \left( D\Gamma'' + |\nabla_x V| \left[ D (\Gamma')^2 - \Gamma' \right] \right) \leq -\gamma. \quad (2.4)$$

This can be done by an explicit construction. We fix  $\varepsilon_0 \in (0, 1)$  and define  $\Gamma' \in C^0([-1, 1])$  piecewise,

$$\Gamma'(Y) = \begin{cases} \delta^+ & \text{if } Y > \varepsilon_0, \\ \frac{\delta^+ - \delta^-}{2\varepsilon_0} (Y + \varepsilon_0) + \delta^- & \text{if } |Y| \leq \varepsilon_0, \\ \delta^- & \text{if } Y < -\varepsilon_0, \end{cases}$$

where  $0 < \delta^- < \delta^+ < 1/D$  are to be determined. With this choice of  $\Gamma'$ , we can assure that  $\Gamma$  is strictly positive in the interval  $[-1, 1]$ . Now, let us show that there exist suitable choices of  $\gamma$  and  $\beta$  for the bound (2.4) to hold. More precisely, choosing  $\beta$  such that  $(\beta - 1)/D \in (\delta^-, \delta^+)$  and  $\gamma = \varepsilon_0(\beta - 1 - D\delta^-)$ ,  $\tilde{\gamma} = \varepsilon_0(1 + D\delta^+ - \beta)$ , we have  $0 < \gamma < \tilde{\gamma}$ . We split our analysis into cases:

- Assume  $Y > \varepsilon_0$ . Then the LHS of (2.4) can be bounded as follows:

$$(\beta - 1 - D\delta^+)Y + \delta^+ (D\delta^+ - 1) |\nabla_x V| |Y^\perp|^2 < (\beta - 1 - D\delta^+)\varepsilon_0 = -\tilde{\gamma}.$$

- Assume  $Y < -\varepsilon_0$ . Then the LHS of (2.4) can be bounded as follows:

$$(\beta - 1 - D\delta^-)Y + \delta^- (D\delta^- - 1) |\nabla_x V| |Y^\perp|^2 < -(\beta - 1 - D\delta^-)\varepsilon_0 = -\gamma.$$

- Assume  $|Y| \leq \varepsilon_0$ . Since  $1 = |Y|^2 + |Y^\perp|^2$ , we have  $|Y^\perp|^2 \geq 1 - \varepsilon_0^2$ . Further, setting

$$h = aY + b \in (\delta^-, \delta^+), \quad a = \frac{\delta^+ - \delta^-}{2}, \quad b = \frac{\delta^+ + \delta^-}{2},$$

we have  $Dh^2 - h \leq D\delta^-(\delta^+ - 1/D)$ . Now, using the fact that the potential gradient is unbounded, we can find a radius  $\rho > 0$  large enough such that for all  $|x| > \rho$ ,

$$\frac{D(\delta^+ - \delta^-)}{2\varepsilon_0} - D\delta^- \left( \frac{1}{D} - \delta^+ \right) |\nabla_x V| < -2\gamma/(1 - \varepsilon_0^2).$$

Putting these estimates together, we obtain for  $|x| > \rho$ :

$$\begin{aligned} & (\beta - 1 - Dh)Y + |Y^\perp|^2 \left( \frac{D(\delta^+ - \delta^-)}{2\varepsilon_0} + |\nabla_x V| [Dh^2 - h] \right) \\ & \leq (\beta - 1 - D\delta^-)\varepsilon_0 + |Y^\perp|^2 \left( \frac{D(\delta^+ - \delta^-)}{2\varepsilon_0} + |\nabla_x V| \left[ D\delta^- \left( \delta^+ - \frac{1}{D} \right) \right] \right) \\ & \leq \gamma + (1 - \varepsilon_0^2) \left( \frac{D(\delta^+ - \delta^-)}{2\varepsilon_0} + |\nabla_x V| \left[ D\delta^- \left( \delta^+ - \frac{1}{D} \right) \right] \right) \leq -\gamma. \end{aligned}$$

□

**Remark 3.** We point out that it is always possible to choose  $\delta^-, \delta^+, \varepsilon_0, \beta$  such that  $\kappa\beta < \gamma$  holds true. Indeed, recall that  $0 < \gamma < \tilde{\gamma}$  and if  $k < 1/2$ , then we can choose

- $0 < \delta^- < \delta^+ < 1/D$  such that

$$0 < k < 1 - \frac{1 + D\delta^-}{1 + D\delta^+}.$$

- $1 + D\delta^- < \beta < 1 + D\delta^+$  such that

$$\frac{1 + D\delta^-}{1 + D\delta^+} < \frac{1 + D\delta^-}{\beta} < 1 - \kappa.$$

For example, we could choose

$$\beta = \frac{1}{2} \left( \frac{1 + D\delta^-}{1 - k} + 1 + D\delta^+ \right).$$

- $\varepsilon_0 \in (0, 1)$  such that

$$\frac{\kappa\beta}{(\beta - 1 - D\delta^-)} < \varepsilon_0 < 1.$$

For example, we could choose

$$\varepsilon_0 = \frac{1}{2} \left( 1 + \frac{\kappa\beta}{\beta - 1 - D\delta^-} \right).$$

Then we have  $\kappa\beta < \gamma$  as required.

## 2.2 Proof of Proposition 2.1

Differentiating the modified entropy  $G[f]$ , we obtain

$$\frac{d}{dt}G[f] = D_0[f] + D_1[f] + D_2[f],$$

where the *entropy dissipation functionals*  $D_0$ ,  $D_1$  and  $D_2$  are given by

$$D_0[f] = \langle Lf, f \rangle - \varepsilon_1 \langle AT\Pi f, f \rangle - \varepsilon_1 \langle AT(1 - \Pi)f, f \rangle + \varepsilon_1 \langle T Af, f \rangle + \varepsilon_1 \langle ALf, f \rangle, \quad (2.5)$$

$$D_1[f] = \varepsilon_1 \langle AP_\kappa f, f \rangle + \varepsilon_1 \langle P_\kappa^* Af, f \rangle, \quad (2.6)$$

$$D_2[f] = \langle P_\kappa f, f \rangle + \frac{\varepsilon_2}{2\pi} \frac{d}{dt} \int_{\mathbb{R}^2 \times \mathbb{S}^1} f^2 g \, dx d\alpha. \quad (2.7)$$

Note that the term  $\langle LAf, f \rangle$  vanishes since it has been shown in [3] that  $A = \Pi A$  and hence  $Af \in \text{Ker}(L)$ . We shall now estimate the dissipation of the entropy in the same spirit as in [2].

### # Step 1: Estimation of $D_0[f]$ .

We will show the boundedness of  $D_0$ , which is in fact the dissipation functional for a stationary conveyor belt. We thus recall in the following Lemma some results from [2].

**Lemma 2.1** (Dolbeault et al.). *The following estimates hold true :*

$$\langle Lf, f \rangle \leq -D \|(1 - \Pi)f\|^2, \quad \|AT(1 - \Pi)f\| \leq C_V \|(1 - \Pi)f\|.$$

$$\|ALf\| \leq \frac{D}{2} \|(1 - \Pi)f\|, \quad \|T Af\| \leq \|(1 - \Pi)f\|.$$

In the analysis, we just take care of the fact that we work with densities of non-zero mass to control the remaining contribution in  $D_0$ . In what follows, we denote

$$\tilde{g} = f - M_f e^{-V},$$

for any density  $f \in L^2(d\mu)$ . Then  $\tilde{g}$  has zero mass and  $\int_{\mathbb{R}^2} \Pi \tilde{g} dx = 0$  for all  $f \in L^2(d\mu)$ . To control the second term in (2.5), we note that  $\mathbf{A}\mathbf{T}\Pi = (1 + (\mathbf{T}\Pi)^* \mathbf{T}\Pi)^{-1} (\mathbf{T}\Pi)^* \mathbf{T}\Pi$  shares its spectral decomposition with  $(\mathbf{T}\Pi)^* \mathbf{T}\Pi$ , and by *macroscopic coercivity*

$$\langle (\mathbf{T}\Pi)^* \mathbf{T}\Pi f, f \rangle = \|\mathbf{T}\Pi f\|^2 = \|\mathbf{T}\Pi \tilde{g}\|^2 \geq \frac{\Lambda}{2} \|\Pi \tilde{g}\|^2$$

and hence

$$\langle \mathbf{A}\mathbf{T}\Pi f, f \rangle \geq \frac{\Lambda/2}{1 + \Lambda/2} \|\Pi \tilde{g}\|^2.$$

Now recalling Lemma 2.1 and using  $\|\Pi \tilde{g}\|^2 = \|\Pi f\|^2 - M_f^2$ , we can estimate :

$$D_0[f] \leq -D\|(1 - \Pi)f\|^2 + \varepsilon_1 B \|(1 - \Pi)f\| \cdot \|f\| - \varepsilon_1 \frac{\Lambda/2}{1 + \Lambda/2} (\|\Pi f\|^2 - M_f^2),$$

where we defined  $B := C_V + 1 + D/2$ .

### # Step 2: Estimation of $D_1[f]$ .

We now turn to the entropy dissipation functional  $D_1$ , which we will estimate using elliptic regularity. Instead of bounding  $\mathbf{A}\mathbf{P}_\kappa$ , we apply an elliptic regularity strategy to its adjoint, as for  $\mathbf{A}\mathbf{T}(1 - \Pi)$  in [2]. Let  $f \in L^2(d\mu)$  and define  $h := (1 + (\mathbf{T}\Pi)^* \mathbf{T}\Pi)^{-1} f$  so that  $u_h = e^V \Pi h$  satisfies

$$\Pi f = e^{-V} u_h - \Pi \mathbf{T}^2 (e^{-V} u_h) = e^{-V} u_h - \frac{1}{2} \nabla_x \cdot (e^{-V} \nabla_x u_h).$$

Then

$$\mathbf{A}^* f = \mathbf{T}\Pi h = e^{-V} \tau \cdot \nabla_x u_h$$

and it follows that

$$\begin{aligned} \|(\mathbf{A}\mathbf{P}_\kappa)^* f\|^2 &= \|\kappa \tau \cdot \nabla_x (e_1 \cdot \nabla_x u_h) e^{-V}\|^2 \\ &= \frac{\kappa^2}{2} \int_{\mathbb{R}^2} e^{-V} |\nabla_x (e_1 \cdot \nabla_x u_h)|^2 dx \\ &\leq \frac{\kappa^2}{2} \|D_x^2 u_h\|_{L^2(e^{-V} dx)}^2 \\ &\leq \frac{\kappa^2}{2} C_V^2 \|\Pi f\|^2, \end{aligned}$$

where we used the elliptic regularity estimate of Proposition 5 in [2] in the last inequality. Here, the positive constant  $C_V$  can be chosen to be the same as in Lemma 2.1. This concludes the boundedness of  $\mathbf{A}\mathbf{P}_\kappa$ ,

$$\|\mathbf{A}\mathbf{P}_\kappa f\| \leq C_V \frac{\kappa}{\sqrt{2}} \|\Pi f\|. \quad (2.8)$$

Using a similar approach for the operator  $P_\kappa^*A$ , we rewrite its adjoint as

$$A^*P_\kappa f = T\Pi h,$$

where we define  $h := (1 + (T\Pi)^*T\Pi)^{-1}P_\kappa f$  for a given  $f \in L^2(d\mu)$ , or equivalently

$$e^{-V}u_h - \frac{1}{2}\nabla_x \cdot (e^{-V}\nabla_x u_h) = \Pi P_\kappa f = P_\kappa \Pi f.$$

Multiplying by  $u_h$  and integrating over  $\mathbb{R}^2$ , we have

$$\begin{aligned} \|u_h\|_{L^2(e^{-V} dx)}^2 + \frac{1}{2}\|\nabla_x u_h\|_{L^2(e^{-V} dx)}^2 &= -\kappa \int_{\mathbb{R}^2} \mathbf{e}_1 \cdot \nabla_x (\Pi f) u_h dx \\ &= \kappa \int_{\mathbb{R}^2} (\Pi f) \mathbf{e}_1 \cdot \nabla_x u_h dx \\ &\leq \kappa \int_{\mathbb{R}^2} |\nabla_x u_h e^{-V/2}| |\Pi f e^{V/2}| dx \\ &\leq \kappa \|\nabla_x u_h\|_{L^2(e^{-V} dx)} \|\Pi f\| \\ &\leq \frac{1}{4}\|\nabla_x u_h\|_{L^2(e^{-V} dx)}^2 + \kappa^2 \|\Pi f\|^2. \end{aligned}$$

This inequality can be understood as a  $H^1(e^{-V} dx) \rightarrow H^{-1}(e^{-V} dx)$  elliptic regularity result. Hence

$$\|A^*P_\kappa f\|^2 = \|T\Pi h\|^2 = \frac{1}{2}\|\nabla_x u_h\|_{L^2(e^{-V} dx)}^2 \leq 2\kappa^2 \|\Pi f\|^2,$$

and so we conclude

$$\|P^*A_\kappa f\| \leq \sqrt{2}\kappa \|\Pi f\|. \quad (2.9)$$

Combining (2.8) and (2.9), the entropy dissipation functional  $D_1$  as given in (2.6) is bounded by

$$D_1[f] \leq \kappa \varepsilon_1 \left( \frac{C_V}{\sqrt{2}} + \sqrt{2} \right) \|\Pi f\| \|f\| \leq 2\kappa \lambda_1 \|f\|^2,$$

where we defined

$$\lambda_1 := \frac{1}{2} \left( \frac{C_V}{\sqrt{2}} + \sqrt{2} \right),$$

and where we used the relation  $\|\Pi f\| \leq \|f\|$ , which follows directly from Jensen's inequality.

### # Step 3: Estimation of $D_2[f]$ .

The adjoint for  $\langle \cdot, \cdot \rangle$  of the perturbation operator  $P_\kappa$  is given by

$$P_\kappa^* = -P_\kappa - P_\kappa V.$$

We deduce

$$\langle P_\kappa f, f \rangle = \langle f, P_\kappa^* f \rangle = \langle -P_\kappa f, f \rangle - \langle (P_\kappa V) f, f \rangle,$$

and thus we can write

$$\langle P_\kappa f, f \rangle = -\frac{1}{2} \langle (P_\kappa V) f, f \rangle = \frac{\kappa}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^1} (\mathbf{e}_1 \cdot \nabla_x V) f^2 e^V dx d\alpha.$$

We will now split our analysis into two cases: bounded and unbounded potential gradients. In the first case, the control of the perturbation operator is trivial:

**Case 1** Assume there exists a constant  $c_2 > 0$  such that  $|\nabla_x V| \leq c_2$  for all  $x \in \mathbb{R}^2$ . Since in this case  $\varepsilon_2 = 0$ , the estimation of the dissipation functional  $D_2$  is trivially

$$D_2[f] = \langle P_\kappa f, f \rangle \leq \frac{\kappa c_2}{2} \|f\|^2 \quad (2.10)$$

and we don't require the construction of the additional weight  $g$ .

**Case 2** Assume  $|\nabla_x V| \rightarrow \infty$  as  $|x| \rightarrow \infty$ . In that case, Proposition 2.3 allows us to control the  $g$ -weighted  $L^2$ -norm outside some fixed ball. More precisely, we have

$$\begin{aligned} \frac{1}{2\pi} \frac{d}{dt} \int_{\mathbb{R}^2 \times \mathbb{S}^1} f^2 g \, dx d\alpha &= \frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^1} (\mathsf{L} - \mathsf{T} + \mathsf{P}_\kappa)(f) f g \, dx d\alpha \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^1} \mathcal{L}(g) f^2 \, dx d\alpha - \frac{1}{\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^1} (\partial_\alpha f)^2 g \, dx d\alpha \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \int_{|x| < \rho} \mathcal{L}(g) f^2 \, dx d\alpha + \frac{1}{2\pi} \int_0^{2\pi} \int_{|x| > \rho} \mathcal{L}(g) f^2 \, dx d\alpha \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \int_{|x| < \rho} ((\mathcal{L}(g) + cg) e^{-V}) f^2 e^V \, dx d\alpha - \frac{c}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^1} f^2 g \, dx d\alpha. \\ &\leq C_3(\rho) \|f\|^2 - \frac{c}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^1} f^2 g \, dx d\alpha, \end{aligned} \quad (2.11)$$

where  $C_3(\rho) := \sup_{|x| \leq \rho} (|\mathcal{L}(g) + cg| e^{-V})$ . Further, thanks to the choice of  $g$ , we have the estimate

$$\int_{\mathbb{R}^2 \times \mathbb{S}^1} |\nabla_x V| f^2 e^V \, dx d\alpha \leq C_4 \int_{\mathbb{R}^2 \times \mathbb{S}^1} f^2 g \, dx d\alpha \quad (2.12)$$

with

$$C_4 := \sup_{x \in \mathbb{R}^2} (|\nabla_x V| e^{(1-\beta)V}) < \infty.$$

Combining estimates (2.11) and (2.12), we have

$$\begin{aligned} D_2[f] &= \frac{\kappa}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^1} (e_1 \cdot \nabla_x V) f^2 e^V \, dx d\alpha + \frac{\varepsilon_2}{2\pi} \frac{d}{dt} \int_{\mathbb{R}^2 \times \mathbb{S}^1} f^2 g \, dx d\alpha \\ &\leq \frac{1}{2\pi} \left( \frac{\kappa C_4}{2} - \varepsilon_2 c \right) \int_{\mathbb{R}^2 \times \mathbb{S}^1} f^2 g \, dx d\alpha + \varepsilon_2 C_3(\rho) \|f\|^2. \end{aligned} \quad (2.13)$$

**# Step 4: Putting the three previous steps together.**

We are now ready to choose a suitable  $\varepsilon_1 > 0$  following the approach in [2]. We write

$$\begin{aligned} D_0[f] + D_1[f] &\leq -D\|(1-\Pi)f\|^2 + \varepsilon_1 B\|(1-\Pi)f\| \cdot \|f\| - \varepsilon_1 \frac{\Lambda/2}{1+\Lambda/2} (\|\Pi f\|^2 - M_f^2) \\ &\quad + 2\kappa\lambda_1 \|f\|^2 \\ &\leq \left(-D + \varepsilon_1 B \left(1 + \frac{1}{2\delta}\right)\right) \|(1-\Pi)f\|^2 + \varepsilon_1 \left(\frac{B\delta}{2} - \frac{\Lambda/2}{1+\Lambda/2}\right) \|\Pi f\|^2 \\ &\quad + 2\kappa\lambda_1 \|f\|^2 + \varepsilon_1 \frac{\Lambda/2}{1+\Lambda/2} M_f^2, \end{aligned}$$

for any choice of  $\delta > 0$ , and where we used the identity

$$\|(1-\Pi)f\| \cdot \|f\| \leq \|(1-\Pi)f\|^2 + \|(1-\Pi)f\| \cdot \|\Pi f\|.$$

Let us choose first  $\delta$  and then  $\varepsilon_1$  following the analysis in the case of a stationary belt [2]: we take

$$\delta = \frac{\Lambda/2}{B(1+\Lambda/2)}$$

and define

$$r(D) := B \left(1 + B \frac{\Lambda/2}{1+\Lambda/2}\right), \quad s := \frac{\Lambda/4}{1+\Lambda/2}.$$

This allows us to rewrite the bound on the dissipation functional as

$$\begin{aligned} D_0[f] + D_1[f] &\leq -(D - \varepsilon_1 r(D)) \|(1-\Pi)f\|^2 - \varepsilon_1 s \|\Pi f\|^2 + 2\kappa\lambda_1 \|f\|^2 \\ &\quad + \varepsilon_1 \frac{\Lambda/2}{1+\Lambda/2} M_f^2. \end{aligned}$$

With the same choice of  $\varepsilon_1 \in (0, 1)$  as in [2], we can find  $\lambda_0 > 0$  such that

$$D - \varepsilon_1 r(D) \geq \varepsilon_1 s \geq 2\lambda_0$$

with the constant  $\lambda_0$  only depending on  $\Lambda$  and  $C_V$  and, thus, only on the potential  $V$ . From this analysis, we conclude

$$D_0[f] + D_1[f] \leq -2(\lambda_0 - \kappa\lambda_1) \|f\|^2 + \varepsilon_1 \frac{\Lambda/2}{1+\Lambda/2} M_f^2.$$

Now adding the control of  $D_2$ , we obtain a different estimate depending on the behaviour of the potential gradient at infinity, following the analysis done in **Step 3**.

Case 1 If the potential gradient is bounded, we have  $G[\cdot] = H[\cdot]$ , and we conclude from (2.10),

$$\frac{d}{dt} G[f] = D_0[f] + D_1[f] + D_2[f] \leq -\gamma_1 \|f\|^2 + \gamma_2 M_f^2 \leq -\gamma_1 G[f] + \gamma_2 M_f^2$$

by the norm equivalence (2.1). Here, we defined

$$\gamma_1 = 2\lambda_0 - 2\kappa\lambda_1 + \frac{\kappa c_2}{2} > 0, \quad \gamma_2 = \varepsilon_1 \frac{\Lambda/2}{1+\Lambda/2} > 0.$$

For our analysis to work, we have to impose that the movement of the conveyor belt is slow enough with respect to the speed at which the fibres are produced,  $\kappa \ll 1$ . More precisely, in the case of a bounded potential gradient, let us suppose that

$$0 < \kappa < \frac{2\lambda_0}{2\lambda_1 + c_2/2} < \frac{\lambda_0}{\lambda_1}.$$

**Case 2** We conclude from (2.13),

$$\begin{aligned} \frac{d}{dt} \mathbf{G}[f] &= \mathbf{D}_0[f] + \mathbf{D}_1[f] + \mathbf{D}_2[f] \\ &\leq - (2\lambda_0 - 2\kappa\lambda_1 - \varepsilon_2 C_3(\rho)) \|f\|^2 + \frac{1}{2\pi} \left( \frac{\kappa C_4}{2} - \varepsilon_2 c \right) \int_{\mathbb{R}^2 \times \mathbb{S}^1} f^2 g \, dx d\alpha + \gamma_2 M_f^2 \\ &\leq - \frac{2(2\lambda_0 - 2\kappa\lambda_1 - \varepsilon_2 C_3(\rho))}{1 + \varepsilon_1} \mathbf{H}[f] + \frac{1}{2\pi} \left( \frac{\kappa C_4}{2} - \varepsilon_2 c \right) \int_{\mathbb{R}^2 \times \mathbb{S}^1} f^2 g \, dx d\alpha + \gamma_2 M_f^2 \\ &\leq - \gamma_1 \mathbf{G}[f] + \gamma_2 M_f^2, \end{aligned}$$

again by norm equivalence (2.1), and where we defined

$$\gamma_1 = \min \left\{ \frac{2(2\lambda_0 - 2\kappa\lambda_1 - \varepsilon_2 C_3(\rho))}{1 + \varepsilon_1}, c - \frac{\kappa C_4}{2\varepsilon_2} \right\} > 0, \quad \gamma_2 = \varepsilon_1 \frac{\Lambda/2}{1 + \Lambda/2} > 0.$$

Here, we choose  $\varepsilon_2$  small enough such that

$$0 < \varepsilon_2 < \frac{2(\lambda_0 - \lambda_1)}{C_3(\rho)},$$

and we impose that the relative speed of the conveyor belt  $\kappa$  is small enough such that

$$0 < \kappa < \min \left\{ \frac{2c\varepsilon_2}{C_4}, \frac{1}{2} \right\}.$$

This restriction also implies that  $\kappa < \lambda_0/\lambda_1$  as  $0 < \kappa < 1$ .

### 3 Existence and uniqueness of a steady state

Proposition 2.1 is the key result that allows us to easily deduce Theorem 1.1 from the hypocoercivity estimate (2.1).

#### 3.1 Proof of Theorem 1.1.

We are now ready to prove the existence of a global Gibbs state  $F_\kappa \in L^2(d\mu)$  of mass 1, using the contraction of the modified entropy  $\mathbf{G}[\cdot]$ . Since the  $\mathcal{C}^0$ -semigroup  $(S_t)_{t \geq 0}$  conserves mass and positivity, the set  $\mathcal{C}$  defined by

$$\mathcal{C} := \{f \in L^2(d\mu) : f \geq 0, M_f = 1\}$$

remains invariant under the action of the  $\mathcal{C}^0$ -semigroup  $(S_t)_{t \geq 0}$ ,

$$S_t(\mathcal{C}) \subset \mathcal{C}.$$



Integrating the hypocoercivity estimate (2.2) in Proposition 2.1, we find

$$\mathbf{G}[S_t f - S_t g] \leq e^{-\gamma t} \mathbf{G}[f - g]$$

for any  $t > 0$  and  $f, g \in \mathcal{C}$ . It follows by Banach's fixed point theorem that there exists a unique  $u^t \in \mathcal{C}$  such that  $S_t(u^t) = u^t$  for each  $t \geq 0$ . In fact, there exists a function  $u \in \mathcal{C}$  such that  $S_t(u) = u$  for all  $t \geq 0$ . To see this, let  $t_n = 2^{-n}$ ,  $n \in \mathbb{N}$ ,  $u_n = u^{t_n}$ . Then  $S_{2^{-n}}(u_n) = u_n$ , and by repeatedly applying the semigroup property,

$$\forall k \in \mathbb{N}, \forall n \in \mathbb{N}, \quad S_{k2^{-n}}(u_n) = u_n.$$

Since  $\mathcal{C}$  is bounded in  $L^2(d\mu)$ , it is weakly compact in  $L^2(d\mu)$ , and thus we can find a subsequence  $(n_j)_{j=1}^\infty$  and  $u \in \mathcal{C}$  such that  $u_{n_j}$  converges weakly to  $u$  in  $\mathcal{C}$ . We will now show that

$$\forall n \in \mathbb{N}, \forall k \in \mathbb{N}, \quad S_{k2^{-n}}(u) = u.$$

Fix  $n \in \mathbb{N}$ . Then for all  $M \in \mathbb{N}$ ,

$$S_{2^{-n-M}}(u_{n+M}) = u_{n+M}.$$

Define  $M_j = n_j - n$  for all  $j \in \mathbb{N}$  such that  $M_j > 0$ . We have

$$S_{k2^{-n}}(u_{n+M_j}) = S_{k2^{M_j}2^{-n-M_j}}(u_{n+M_j}) = u_{n+M_j}.$$

By continuity of  $S_t(\cdot)$  in the weak topology,  $u_{n+M_j} \rightharpoonup u$  as  $j \rightarrow \infty$  implies

$$S_{k2^{-n}}(u_{n+M_j}) \rightharpoonup S_{k2^{-n}}(u),$$

which proves  $S_{k2^{-n}}(u) = u$  as claimed. By density of the dyadic rationals  $\{k2^{-n} : k \in \mathbb{N}, n \in \mathbb{N}\}$  in  $\mathbb{R}_{>0}$  and uniform continuity of  $S_t(u)$  in  $t$  for all  $u \in \mathcal{C}$ , we conclude

$$\forall t \geq 0, \quad S_t(u) = u.$$

This shows the existence and uniqueness of a global stationary state  $F_\kappa$  of mass 1.

To complete the proof of Theorem 1.1, we apply the hypocoercivity estimate (2.1) to the difference between a solution  $f \in L^2(d\mu)$  and the unique stationary state of the same mass,  $M_f F_\kappa$ , to show exponential convergence to equilibrium in  $\|\cdot\|$ :

$$\begin{aligned} \|f(t) - M_f F_\kappa\|^2 &\leq \frac{2}{1-\varepsilon_1} \left( \mathbf{G}[f(t) - M_f F_\kappa] - \frac{\varepsilon_2}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{S}^1} (f(t) - M_f F_\kappa)^2 g \, dx d\alpha \right) \\ &\leq \frac{2}{1-\varepsilon_1} \mathbf{G}[f(t) - M_f F_\kappa] \\ &\leq \frac{2}{1-\varepsilon_1} \mathbf{G}[f(0) - M_f F_\kappa] e^{-\gamma_1 t}, \end{aligned}$$

which proves (1.6) with  $C^2 = \frac{2}{1-\varepsilon_1} \mathbf{G}[f(0) - M_f F_\kappa]$  and rate of convergence  $\lambda_\kappa := \gamma_1/2$ .

**Remark 4.** *Let us investigate Theorem 1.1 from a spectral point of view. We rewrite the Fokker-Planck equation (1.5) as*

$$\partial_t f = \mathbf{L}_\kappa f = \mathbf{L}_0 f + \mathbf{P}_\kappa f,$$

where  $\mathsf{L}_0 = \mathsf{L} - \mathsf{T}$ . In the case of a stationary conveyor belt  $\kappa = 0$ , the stationary state is characterised by the eigenpair  $(\Lambda_0, F_0)$  with  $\Lambda_0 = 0$ ,  $F_0(x) = e^{-V(x)}$ , and so  $\text{Ker}(\mathsf{L}_0) = \langle F_0 \rangle$ . By hypocoercivity, the exponential rate of convergence  $\lambda_0$  can be calculated explicitly under assumptions **(H1)**-**(H4)**, [2]. In other words, we have an isolated eigenvalue  $\Lambda_0 = 0$  and a spectral gap  $[-\lambda_0, 0]$ , with the rest of the spectrum  $\text{Sp}(\mathsf{L}_0)$  discrete and to the left of  $-\lambda_0$  in the complex plane. Adding the movement of a conveyor belt, it follows from the existence and uniqueness of the equilibrium distribution  $F_\kappa$  that  $\dim(\text{Ker}(\mathsf{L}_\kappa)) = 1$ . More precisely, the exponential decay to equilibrium with rate  $\lambda_\kappa$  corresponds to a spectral gap  $[-\lambda_\kappa, 0]$ .

**Remark 5.** Working in  $L^2(d\mu)$  we are treating the operator  $\mathsf{L}_\kappa$  as a small perturbation of the case  $\kappa = 0$  with stationary conveyor belt. The space that is well-adapted to investigate the convergence to  $F_\kappa$  in the case  $\kappa > 0$  however is  $L^2(F_\kappa^{-1} \frac{1}{2\pi} dx d\alpha)$ . In this  $L^2$ -space the transport operator  $\mathsf{T} - \mathsf{P}_\kappa$  is not skew-symmetric and the collision operator  $\mathsf{L}$  is not self-adjoint, so the hypocoercivity method [3] cannot be applied. To get around this, one can split the operator  $\mathsf{L}_\kappa$  differently into a transport and a collision part following the approach in [1]. More precisely, we can write

$$\mathsf{L}_\kappa = \tilde{\mathsf{L}} - \tilde{\mathsf{T}},$$

where

$$\begin{aligned} \tilde{\mathsf{L}}f &= \partial_\alpha \left( \partial_\alpha F_\kappa - \frac{\partial_\alpha F_\kappa}{F_\kappa} f \right), \\ \tilde{\mathsf{T}}f &= (\tau + \kappa e_1) \cdot \nabla_x f - \partial_\alpha \left[ \left( \tau^\perp \cdot \nabla_x V + \frac{\partial_\alpha F_\kappa}{F_\kappa} \right) f \right]. \end{aligned}$$

It is easily seen that  $\tilde{\mathsf{L}}$  is symmetric and negative semidefinite, and that  $\tilde{\mathsf{T}}$  is skew-symmetric in  $L^2(F_\kappa^{-1} \frac{1}{2\pi} dx d\alpha)$ . Further, the stationary state  $F_\kappa$  lies in the intersection of the kernels of the collision and transport operators,  $F_\kappa \in \text{Ker}(\tilde{\mathsf{L}}) \cap \text{Ker}(\tilde{\mathsf{T}})$ . In order to apply the hypocoercivity approach with this definition of operators, we will need to show microscopic and macroscopic coercivity of  $\tilde{\mathsf{L}}$  and  $\tilde{\mathsf{T}}$ . This requires as in [1] that we are able to control the behavior of the stationary state at infinity:

$$e^{-\mu_1 V(x)} \leq F_\kappa(x, \alpha) \leq e^{-\mu_2 V(x)}$$

for some constants  $\mu_1, \mu_2 > 0$ . This would be a very strong physical information about the behaviour of the stationary state that we don't know how to prove at the moment. Even with this information at hand, this approach requires that the existence of the stationary state is known a priori. The rate of convergence one obtains in this case may be different from the rate obtained here, and it is not clear which method yields the better rate as both are most likely not optimal.

## References

- [1] Vincent Calvez, Gaël Raoul, and Christian Schmeiser. Confinement by biased velocity jumps: aggregation of *escherichia coli*. *Kinet. Relat. Models*, 8(4):651–666, 2015.
- [2] Jean Dolbeault, Axel Klar, Clément Mouhot, and Christian Schmeiser. Exponential rate of convergence to equilibrium for a model describing fiber lay-down processes. *Appl. Math. Res. Express. AMRX*, (2):165–175, 2013.

- [3] Jean Dolbeault, Clément Mouhot, and Christian Schmeiser. Hypocoercivity for linear kinetic equations conserving mass. *Trans. Amer. Math. Soc.*, 367(6):3807–3828, 2015.
- [4] T. Götz, A. Klar, N. Marheineke, and R. Wegener. A stochastic model and associated Fokker-Planck equation for the fiber lay-down process in nonwoven production processes. *SIAM J. Appl. Math.*, 67(6):1704–1717, 2007.
- [5] Martin Grothaus, Axel Klar, Johannes Maringer, and Patrik Stilgenbauer. Geometry, mixing properties and hypocoercivity of a degenerate diffusion arising in technical textile industry. *arXiv/1203.4502*, 2012.
- [6] Axel Klar, Johannes Maringer, and Raimund Wegener. A 3D model for fiber lay-down in nonwoven production processes. *Math. Models Methods Appl. Sci.*, 22(9):1250020, 18, 2012.
- [7] Martin Kolb, Mladen Savov, and Achim Wubker. Geometric ergodicity of a hypoelliptic diffusion modelling the melt-spinning process of nonwoven materials. *arXiv/1112.6159*, 2011.
- [8] Martin Kolb, Mladen Savov, and Achim Wübker. (Non-)ergodicity of a degenerate diffusion modeling the fiber lay down process. *SIAM J. Math. Anal.*, 45(1):1–13, 2013.
- [9] Nicole Marheineke and Raimund Wegener. Fiber dynamics in turbulent flows: general modeling framework. *SIAM J. Appl. Math.*, 66(5):1703–1726 (electronic), 2006.
- [10] Nicole Marheineke and Raimund Wegener. Fiber dynamics in turbulent flows: specific Taylor drag. *SIAM J. Appl. Math.*, 68(1):1–23 (electronic), 2007.
- [11] Cédric Villani. Hypocoercivity. *Mem. Amer. Math. Soc.*, 202(950):iv+141, 2009.