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# DOUBLY-RESONANT SADDLE-NODES IN $(\mathbb{C}^3, 0)$ AND THE FIXED SINGULARITY AT INFINITY IN THE PAINLEVÉ EQUATIONS (PART II): SECTORIAL NORMALIZATION

AMAURY BITTMANN

ABSTRACT. In this work, following [Bit15], we consider analytic singular vector fields in  $\mathbb{C}^3$  with an isolated and doubly-resonant singularity of saddle-node type at the origin. Such vector fields come from irregular two-dimensional differential systems with two opposite non-zero eigenvalues, and appear for instance when studying the irregular singularity at infinity in Painlevé equations  $(P_j)_{j=I\dots V}$ , for generic values of the parameters. Under suitable assumptions, we prove a theorem of analytic normalization over sectorial domains, analogous to the classical one due to Hukuhara-Kimura-Matuda [HKM61] for saddle-nodes in  $\mathbb{C}^2$ . We also prove that the normalizing map is essentially unique and *weakly Gevrey-1 summable*.

## 1. INTRODUCTION

As in [Bit15], we consider (germs of) singular vector fields  $Y$  in  $\mathbb{C}^3$  which can be written in appropriate coordinates  $(x, \mathbf{y}) := (x, y_1, y_2)$  as

$$(1.1) \quad Y = x^2 \frac{\partial}{\partial x} + \left( -\lambda y_1 + f_1(x, \mathbf{y}) \right) \frac{\partial}{\partial y_1} + \left( \lambda y_2 + f_2(x, \mathbf{y}) \right) \frac{\partial}{\partial y_2} \quad ,$$

where  $\lambda \in \mathbb{C}^*$  and  $f_1, f_2$  are germs of holomorphic functions in  $(\mathbb{C}^3, 0)$  of homogeneous valuation (order) at least two. They represent irregular two-dimensional differential systems having two opposite non-zero eigenvalues and a vanishing third eigenvalue. These we call doubly-resonant vector fields of saddle-node type (or simply **doubly-resonant saddle-nodes**). We will impose more (non-generic) conditions in the sequel.

Our main motivation is the study of the irregular singularity at infinity in Painlevé equations  $(P_j)_{j=I\dots V}$ , for generic values of the parameters. These equations were discovered by Paul Painlevé [Pai02] because the only movable singularities of the solutions are poles (the so-called *Painlevé property*). Their study has become a rich domain of research since the important work of Okamoto [Oka77]. The fixed singularities of the Painlevé equations, and more particularly those at infinity, were notably investigated by Boutroux with his famous *tritronquées* solutions [Bou13]. Recently, several authors provided more complete information about such singularities, studying “quasi-linear Stokes phenomena” and also giving connection formulas; we refer to the following (non-exhaustive) sources [JK92, Kap04, KK93, JK01].

Stokes coefficients are invariant under local changes of analytic coordinates, but do not form a complete invariant of the vector field. To the best of our knowledge there currently does not exist a general analytic classification for doubly-resonant saddle-nodes. Such a classification would provide a new framework allowing to analyze Stokes phenomena in that class of singularities. In this paper we provide a theorem of analytic normalization over sectorial domain (*à la* Hukuhara-Kimura-Matuda [HKM61] for saddle-nodes in  $(\mathbb{C}^2, 0)$ ) for a specific class (to be defined later on) of doubly-resonant saddle-nodes which contains the Painlevé case. In a forthcoming paper we use this theorem in order to provide a complete analytic classification for this class of vector fields, based on the important work of Martinet and Ramis [MR82, MR83] for saddle-nodes and resonant saddles in  $(\mathbb{C}^2, 0)$ .

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*Key words and phrases.* Painlevé equations, singular vector field, irregular singularity, resonant singularity, Gevrey-1 summability, normal form.

In [Yos84, Yos85] Yoshida shows that doubly-resonant saddle-nodes arising from the compactification of Painlevé equations  $(P_j)_{j=I\dots V}$  (for generic values for the parameters) are conjugate to vector fields of the form:

$$(1.2) \quad Z = x^2 \frac{\partial}{\partial x} + \left( -(1 + \gamma y_1 y_2) + a_1 x \right) y_1 \frac{\partial}{\partial y_1} + \left( 1 + \gamma y_1 y_2 + a_2 x \right) y_2 \frac{\partial}{\partial y_2} ,$$

with  $\gamma \in \mathbb{C}^*$  and  $(a_1, a_2) \in \mathbb{C}^2$  such that  $a_1 + a_2 = 1$ . One should notice straight away that this “conjugacy” does not agree with what is traditionally (in particular in this paper) meant by conjugacy, for Yoshida’s transform  $\Psi(x, \mathbf{y}) = (x, \psi_1(x, \mathbf{y}), \psi_2(x, \mathbf{y}))$  takes the form

$$(1.3) \quad \psi_i(x, \mathbf{y}) = y_i \left( 1 + \sum_{\substack{(k_0, k_1, k_2) \in \mathbb{N}^3 \\ k_1 + k_2 \geq 1}} \frac{q_{i, \mathbf{k}}(x)}{x^{k_0}} y_1^{k_1 + k_0} y_2^{k_1 + k_0} \right) ,$$

where each  $q_{i, \mathbf{k}}$  is formal power series although  $x$  appears with negative exponents. This expansion may not even be a formal Laurent series. It is, though, the asymptotic expansion along  $\{x = 0\}$  of a function analytic in a domain

$$\left\{ (x, \mathbf{z}) \in S \times \mathbf{D}(0, \mathbf{r}) \mid |z_1 z_2| < \nu |x| \right\}$$

for some small  $\nu > 0$ , where  $S$  is a sector of opening less than  $\pi$  with vertex at the origin and  $\mathbf{D}(0, \mathbf{r})$  is a polydisc of small poly-radius  $\mathbf{r} = (r_1, r_2) \in (\mathbb{R}_{>0})^2$ . Moreover the  $(q_{i, \mathbf{k}}(x))_{i, \mathbf{k}}$  are actually Gevrey-1 power series. The drawback here is that the transforms are convergent on regions so small that taken together they cannot cover an entire neighborhood of the origin in  $\mathbb{C}^3$ .

Several authors studied the problem of convergence of formal transformations putting vector fields as in (1.1) into “normal forms”. Shimomura, improving on a result of Iwano [Iwa80], shows in [Shi83] that analytic doubly-resonant saddle-nodes satisfying more restrictive conditions are conjugate (formally and over sectors) to vector fields of the form

$$x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x) y_1 \frac{\partial}{\partial y_1} + (\lambda + a_2 x) y_2 \frac{\partial}{\partial y_2}$$

via a diffeomorphism whose coefficients have asymptotic expansions as  $x \rightarrow 0$  in sectors of opening greater than  $\pi$ .

Stolovitch then generalized this result to any dimension in [Sto96]. More precisely, Stolovitch’s work offers an analytic classification of vector fields in  $\mathbb{C}^{n+1}$  with an irregular singular point, without further hypothesis on eventual additional resonance relations between eigenvalues of the linear part. However, as Iwano and Shimomura did, he needed to impose other assumptions, among which the condition that the restriction of the vector field to the invariant hypersurface  $\{x = 0\}$  be a linear vector field. In [BDM08], the authors obtain a *Gevrey-1 summable* “normal form”, though not as simple as Stolovitch’s and not unique *a priori*, but for more general kind of vector field with one zero eigenvalue. However, the same assumption on the restriction to the hypersurface  $\{x = 0\}$  is required (the restriction must be a linear vector field). Yet from [Yos85] stems the fact that this condition is not met in the case of Painlevé equations  $(P_j)_{j=I\dots V}$ .

In comparison, we merely ask here that the restricted vector field be orbitally linearizable (see Definition 1.7), *i.e.* the foliation induced by  $Y$  on  $\{x = 0\}$  (and not the vector field  $Y|_{\{x=0\}}$  itself) be linearizable. The fact that this condition is fulfilled by the singularities of generic Painlevé equations is well-known.

### 1.1. Scope of the paper.

The action of local analytic / formal diffeomorphisms  $\Psi$  fixing the origin on local holomorphic vector fields  $Y$  of type (1.1) by change of coordinates is given by

$$\Psi_* Y := D\Psi(Y) \circ \Psi^{-1} .$$

In [Bit15] we performed the formal classification of such vector fields by exhibiting an explicit universal family of vector fields for the action of (fibered) formal changes of coordinates at 0 (called a family of normal forms). Such a result seems currently out of reach in the analytic category: it is unlikely that an explicit universal family for the action of local analytic changes of coordinates be described anytime soon. If we want to describe the space of equivalent classes (of germs of a doubly-resonant saddle-node under local analytic changes of coordinates) with same formal normal form, we therefore need to find a complete set of invariants which is of a different nature. We call **moduli space** this quotient space and would like to give it a (non-trivial) presentation based on functional invariants *à la* Martinet-Ramis [MR82, MR83].

The main ingredient to obtain such analytic invariants is to prove first the existence of analytic sectorial normalizing maps (over a pair of opposite “wide” sectors of opening greater than  $\pi$  whose union covers a full punctured neighborhood of  $\{x = 0\}$ ). This is the main result of the present paper. We have not been able to perform this normalization in such a generality, and only deal here with  $x$ -fibered local analytic conjugacies acting on vector fields of the form (1.1) with some additional assumptions detailed further down (see Definitions 1.1, 1.3 and 1.7). Importantly, these hypothesis are met in the case of Painlevé equations mentioned above.

Our approach has some geometric flavor, since we avoid the use of fixed-point methods altogether to establish the existence of the normalizing maps, and generalize instead the approach of Teyssier [Tey04, Tey03] relying on path-integration of well-chosen 1-forms (following Arnold’s method of characteristics [Arn74]).

As a by-product of this normalization we deduce that the normalizing sectorial diffeomorphisms are weakly Gevrey-1 asymptotic to the normalizing formal power series of [Bit15], retrospectively proving their *weak 1-summability* (see subsection 2.3 for definitions). When the vector field additionally supports a symplectic transverse structure (which is again the case of Painlevé equations) we prove that the (essentially unique) sectorial normalizing map is realized by a transversally symplectic diffeomorphism.

### 1.2. Definitions and main results.

To state our main results we need to introduce some notations and nomenclature.

- For  $n \in \mathbb{N}_{>0}$ , we denote by  $(\mathbb{C}^n, 0)$  an (arbitrary small) open neighborhood of the origin in  $\mathbb{C}^n$ .
- We denote by  $\mathbb{C}\{x, \mathbf{y}\}$ , with  $\mathbf{y} = (y_1, y_2)$ , the  $\mathbb{C}$ -algebra of germs of holomorphic functions at the origin of  $\mathbb{C}^3$ , and by  $\mathbb{C}\{x, \mathbf{y}\}^\times$  the group of invertible elements for the multiplication (also called units), *i.e.* elements  $U$  such that  $U(0) \neq 0$ .
- $\chi(\mathbb{C}^3, 0)$  is the Lie algebra of germs of singular holomorphic vector fields at the origin  $\mathbb{C}^3$ . Any vector field in  $\chi(\mathbb{C}^3, 0)$  can be written as

$$Y = b(x, y_1, y_2) \frac{\partial}{\partial x} + b_1(x, y_1, y_2) \frac{\partial}{\partial y_1} + b_2(x, y_1, y_2) \frac{\partial}{\partial y_2}$$

with  $b, b_1, b_2 \in \mathbb{C}\{x, y_1, y_2\}$  vanishing at the origin.

- $\text{Diff}(\mathbb{C}^3, 0)$  is the group of germs of a holomorphic diffeomorphism fixing the origin of  $\mathbb{C}^3$ . It acts on  $\chi(\mathbb{C}^3, 0)$  by conjugacy: for all

$$(\Phi, Y) \in \text{Diff}(\mathbb{C}^3, 0) \times \chi(\mathbb{C}^3, 0)$$

we define the push-forward of  $Y$  by  $\Phi$  by

$$(1.4) \quad \Phi_*(Y) := (D\Phi \cdot Y) \circ \Phi^{-1} ,$$

where  $D\Phi$  is the Jacobian matrix of  $\Phi$ .

- $\text{Diff}_{\text{fib}}(\mathbb{C}^3, 0)$  is the subgroup of  $\text{Diff}(\mathbb{C}^3, 0)$  of fibered diffeomorphisms preserving the  $x$ -coordinate, *i.e.* of the form  $(x, \mathbf{y}) \mapsto (x, \phi(x, \mathbf{y}))$ .

- We denote by  $\text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id})$  the subgroup of  $\text{Diff}_{\text{fib}}(\mathbb{C}^3, 0)$  formed by diffeomorphisms tangent to the identity.

All these concepts have *formal* analogues, where we only suppose that the objects are defined with formal power series, not necessarily convergent near the origin. The formal counterpart is written with a hat, as in  $\widehat{\text{Diff}}(\mathbb{C}^3, 0)$ .

**Definition 1.1.** A **diagonal doubly-resonant saddle-node** is a vector field  $Y \in \chi(\mathbb{C}^3, 0)$  of the form

$$Y = x^2 \frac{\partial}{\partial x} + \left( -\lambda y_1 + f_1(x, \mathbf{y}) \right) \frac{\partial}{\partial y_1} + \left( \lambda y_2 + f_2(x, \mathbf{y}) \right) \frac{\partial}{\partial y_2} ,$$

with  $\lambda \in \mathbb{C}^*$  and  $f_1, f_2 \in \mathbb{C}\{x, \mathbf{y}\}$  of order at least two. We denote by  $\mathcal{SN}_{\text{diag}}$  the set of such vector fields.

*Remark 1.2.* One can also define the foliation associated to a diagonal doubly-resonant saddle-node in a geometric way. A vector field  $Y \in \chi(\mathbb{C}^3, 0)$  is orbitally equivalent to a diagonal doubly-resonant saddle-node (*i.e.*  $Y$  is conjugate to some  $VX$ , where  $V \in \mathbb{C}\{x, \mathbf{y}\}^\times$  and  $X \in \mathcal{SN}_{\text{diag}}$ ) if and only if the following conditions hold:

- (1)  $\text{Spec}(D_0 Y) = \{0, -\lambda, \lambda\}$  with  $\lambda \neq 0$ ;
- (2) there exists a germ of irreducible analytic hypersurface  $\mathcal{H} = \{S = 0\}$  which is transverse to the eigenspace  $E_0$  (corresponding to the zero eigenvalue) at the origin, and which is stable under the flow of  $Y$ ;
- (3)  $\mathcal{L}_Y(S) = U.S^2$ , where  $\mathcal{L}_Y$  is the Lie derivative of  $Y$  and  $U \in \mathbb{C}\{x, \mathbf{y}\}^\times$ .

By Taylor expansion up to order 1 with respect to  $\mathbf{y}$ , given a vector field  $Y \in \mathcal{SN}_{\text{diag}}$  written as in (1.1) we can consider the associate 2-dimensional system:

$$(1.5) \quad x^2 \frac{d\mathbf{y}}{dx} = \alpha(x) + \mathbf{A}(x) \mathbf{y}(x) + \mathbf{F}(x, \mathbf{y}(x)) ,$$

with  $\mathbf{y} = (y_1, y_2)$ , such that the following conditions hold:

- $\alpha(x) = \begin{pmatrix} \alpha_1(x) \\ \alpha_2(x) \end{pmatrix}$ , with  $\alpha_1, \alpha_2 \in \mathbb{C}\{x\}$  and  $\alpha_1, \alpha_2 \in O(x^2)$
- $\mathbf{A}(x) \in \text{Mat}_{2,2}(\mathbb{C}\{x\})$  with  $\mathbf{A}(0) = \text{diag}(-\lambda, \lambda)$ ,  $\lambda \in \mathbb{C}^*$
- $\mathbf{F}(x, \mathbf{y}) = \begin{pmatrix} F_1(x, \mathbf{y}) \\ F_2(x, \mathbf{y}) \end{pmatrix}$ , with  $F_1, F_2 \in \mathbb{C}\{x, \mathbf{y}\}$  and  $F_1, F_2 \in O(\|\mathbf{y}\|^2)$ .

Based on this expression, we state:

**Definition 1.3.** The **residue** of  $Y \in \mathcal{SN}_{\text{diag}}$  is the complex number

$$\text{res}(Y) := \left( \frac{\text{Tr}(\mathbf{A}(x))}{x} \right)_{|x=0} .$$

We say that  $Y$  is **non-degenerate** (*resp.* **strictly non-degenerate**) if  $\text{res}(Y) \notin \mathbb{Q}_{\leq 0}$  (*resp.*  $\Re(\text{res}(Y)) > 0$ ).

*Remark 1.4.* It is obvious that there is an action of  $\text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id})$  on  $\mathcal{SN}_{\text{diag}}$ . The residue is an invariant of each orbit of  $\mathcal{SN}_{\text{diag}}$  under the action of  $\text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id})$  by conjugacy (it is actually invariant by formal conjugacies, see [Bit15]).

The main result of [Bit15] can now be stated as follows:

**Theorem 1.5.** [Bit15] *Let  $Y \in \mathcal{SN}_{\text{diag}}$  be non-degenerate. Then there exists a unique formal fibered diffeomorphism  $\hat{\Phi}$  tangent to the identity such that:*

$$(1.6) \quad \begin{aligned} \hat{\Phi}_*(Y) &= x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x + c_1(y_1 y_2)) y_1 \frac{\partial}{\partial y_1} \\ &\quad + (\lambda + a_2 x + c_2(y_1 y_2)) y_2 \frac{\partial}{\partial y_2} , \end{aligned}$$

where  $\lambda \in \mathbb{C}^*$ ,  $c_1, c_2 \in v\mathbb{C}[[v]]$  are formal power series in  $v = y_1 y_2$  without constant term and  $a_1, a_2 \in \mathbb{C}$  are such that  $a_1 + a_2 = \text{res}(Y) \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0}$ .

**Definition 1.6.** The vector field obtained in (1.6) is called the **formal normal form** of  $Y$ . The formal fibered diffeomorphism  $\hat{\Phi}$  is called the **formal normalizing map** of  $Y$ .

The above result is valid for formal objects, regardless of convergence problems. The first main result in this work states that this formal normalizing map is analytic in sectorial domains, under some additional assumptions that we are now going to precise.

**Definition 1.7.**

- We say that a germ of a vector field  $X$  in  $(\mathbb{C}^2, 0)$  is **orbitally linear** if

$$X = U(\mathbf{y}) \left( \lambda_1 y_1 \frac{\partial}{\partial y_1} + \lambda_2 y_2 \frac{\partial}{\partial y_2} \right),$$

for some  $U(\mathbf{y}) \in \mathbb{C}\{\mathbf{y}\}^\times$  and  $(\lambda_1, \lambda_2) \in \mathbb{C}^2$ .

- We say that a germ of vector field  $X$  in  $(\mathbb{C}^2, 0)$  is analytically (*resp. formally*) **orbitally linearizable** if  $X$  is analytically (*resp. formally*) conjugate to an orbitally linear vector field.
- We say that a diagonal doubly-resonant saddle-node  $Y \in \mathcal{SN}_{\text{diag}}$  is **div-integrable** if  $Y|_{\{x=0\}} \in \chi(\mathbb{C}^2, 0)$  is (analytically) orbitally linearizable.

*Remark 1.8.* Indeed, a vector field  $Y \in \mathcal{SN}_{\text{diag}}$  admits a unique germ of an analytic and irreducible invariant hypersurface which is tangent to the eigenspace  $E_0$  (associated to the null eigenvalue): it is called the **divisor** of  $Y$ , and is given by  $\{x=0\}$  in (1.1). The assumption made on the restriction states that  $Y|_{\{x=0\}}$  is integrable. Alternatively we could say that the foliation associated to  $Y|_{\{x=0\}}$  is linearizable. Since  $Y|_{\{x=0\}}$  is analytic at the origin of  $\mathbb{C}^2$  and has two opposite eigenvalues, it follows from an important result of Brjuno (see *e.g.* [Mar81]), that  $Y|_{\{x=0\}}$  is analytically orbitally linearizable if and only if it is formally orbitally linearizable.

**Definition 1.9.** We denote by  $\mathcal{SN}_{\text{diag},0}$  the set of strictly non-degenerate diagonal doubly-resonant saddle-nodes which are div-integrable.

The vector field corresponding to the irregular singularity at infinity in the Painlevé equations  $(P_j)_{j=I\dots V}$  is orbitally equivalent to an element of  $\mathcal{SN}_{\text{diag},0}$ , for generic values of the parameters (see [Yos85]).

We can now state the first main result of our paper (we refer to subsection 2.3 for the precise definition of *weak 1-summability*).

**Theorem 1.10.** *Let  $Y \in \mathcal{SN}_{\text{diag},0}$  and let  $\hat{\Phi}$  (given by Theorem 1.5) be the unique formal fibered diffeomorphism tangent to the identity such that*

$$\begin{aligned} \hat{\Phi}_*(Y) &= x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x + c_1(y_1 y_2)) y_1 \frac{\partial}{\partial y_1} + (\lambda + a_2 x + c_2(y_1 y_2)) y_2 \frac{\partial}{\partial y_2} \\ &=: Y_{\text{norm}}, \end{aligned}$$

where  $\lambda \neq 0$  and  $c_1(v), c_2(v) \in v\mathbb{C}[[v]]$  are formal power series without constant term. Then:

- (1) the normal form  $Y_{\text{norm}}$  is analytic (i.e.  $c_1, c_2 \in \mathbb{C}\{v\}$ ), and its restriction  $(Y_{\text{norm}})|_{x=0}$  is orbitally linear, i.e.  $c_1 + c_2 = 0$ ;
- (2) the formal normalizing map  $\hat{\Phi}$  is weakly 1-summable except for  $\arg(\pm\lambda)$ ;
- (3) there exist analytic sectorial fibered diffeomorphisms  $\Phi_+$  and  $\Phi_-$ , (asymptotically) tangent to the identity, defined in sectorial domains of the form  $S_+ \times (\mathbb{C}^2, 0)$  and  $S_- \times (\mathbb{C}^2, 0)$  respectively, where

$$\begin{aligned} S_+ &:= \left\{ x \in \mathbb{C} \mid 0 < |x| < r \text{ and } \left| \arg\left(\frac{x}{i\lambda}\right) \right| < \frac{\pi}{2} + \epsilon \right\} \\ S_- &:= \left\{ x \in \mathbb{C} \mid 0 < |x| < r \text{ and } \left| \arg\left(\frac{-x}{i\lambda}\right) \right| < \frac{\pi}{2} + \epsilon \right\} \end{aligned}$$

(for any  $\epsilon \in ]0, \frac{\pi}{2}[$  and some  $r > 0$  small enough), which admit  $\hat{\Phi}$  as weak Gevrey-1 asymptotic expansion in these respective domains, and which conjugate  $Y$  to  $Y_{\text{norm}}$ . Moreover  $\Phi_+$  and  $\Phi_-$  are the unique such germs of analytic functions in sectorial domains (see Definition 2.2).

*Remark 1.11.* Item 3. above is a direct consequence of the weak 1-summability of  $\hat{\Phi}$  (see subsection 2.3).

**Definition 1.12.** We call  $\Phi_+$  and  $\Phi_-$  the **sectorial normalizing maps** of  $Y \in \mathcal{SN}_{\text{diag},0}$ .

They are the weak 1-sums of  $\hat{\Phi}$  along the respective directions  $\arg(i\lambda)$  and  $\arg(-i\lambda)$ . Notice that  $\Phi_+$  and  $\Phi_-$  are germs of analytic sectorial fibered diffeomorphisms, i.e. they are of the form

$$\begin{aligned} \Phi_+ : S_+ \times (\mathbb{C}^2, 0) &\longrightarrow S_+ \times (\mathbb{C}^2, 0) \\ (x, \mathbf{y}) &\longmapsto (x, \Phi_{+,1}(x, \mathbf{y}), \Phi_{+,2}(x, \mathbf{y})) \end{aligned}$$

and

$$\begin{aligned} \Phi_- : S_- \times (\mathbb{C}^2, 0) &\longrightarrow S_- \times (\mathbb{C}^2, 0) \\ (x, \mathbf{y}) &\longmapsto (x, \Phi_{-,1}(x, \mathbf{y}), \Phi_{-,2}(x, \mathbf{y})) \end{aligned}$$

(see section 2. for a precise definition of *germ of analytic sectorial fibered diffeomorphism*). The fact that they are also (*asymptotically*) *tangent to the identity* means that we have:

$$\Phi_{\pm}(x, \mathbf{y}) = \text{Id}(x, \mathbf{y}) + O\left(\|(x, \mathbf{y})\|^2\right).$$

In fact, we can prove the uniqueness of the sectorial normalizing maps under weaker assumptions.

**Proposition 1.13.** *Let  $\varphi_+$  and  $\varphi_-$  be two germs of sectorial fibered diffeomorphisms in  $S_+ \times (\mathbb{C}^2, 0)$  and  $S_- \times (\mathbb{C}^2, 0)$  respectively, where  $S_+$  and  $S_-$  are as in Theorem 1.10, which are (*asymptotically*) *tangent to the identity* and such that*

$$(\varphi_{\pm})_*(Y) = Y_{\text{norm}}.$$

*Then, they necessarily coincide with the weak 1-sums  $\Phi_+$  and  $\Phi_-$  defined above.*

It is important to say that we will in fact begin with proving the existence of germs of sectorial fibered diffeomorphisms  $\Phi_+$  and  $\Phi_-$  in  $S_+ \times (\mathbb{C}^2, 0)$  and  $S_- \times (\mathbb{C}^2, 0)$  respectively, which are tangent to the identity and conjugate  $Y \in \mathcal{SN}_{\text{diag},0}$  to its normal form (see Corollary 4.2). The proposition above guarantees the uniqueness of such sectorial transforms. It is proved in a second step that  $\Phi_+$  and  $\Phi_-$  admits the formal normalizing map  $\hat{\Phi}$  as weak Gevrey-1 asymptotic expansion, which is thus weakly 1-summable.

*Remark 1.14.* In this paper we prove a theorem of sectorial normalizing map analogous to the classical one due to Hukuhara-Kimura-Matuda for saddle-nodes in  $(\mathbb{C}^2, 0)$  [HKM61], generalized later by Stolovitch in any dimension in [Sto96]. Unlike the method based on a fixed point theorem used by these authors, we use a more geometric approach (following the works of Teyssier [Tey03, Tey04]) based on the resolution of an homological equation, by integrating a well chosen 1-form along asymptotic paths. This latter approach turned out to be more efficient to deal with the fact that  $Y_{\{x=0\}}$  is not necessarily linearizable. Indeed, if we look at [Sto96] in details, one of the first problem is that in the irregular systems that needs to be solved by a fixed point method (for instance equation (2.7) in the cited paper), the non-linear terms would not be divisible by the “time” variable. This would complicate the different estimates that are done later in the cited work. This was the first main new phenomena we have met.

In a forthcoming paper we prove that the sectorial normalizing maps  $\Phi_+$  and  $\Phi_-$  in Theorem 1.10 actually admit the unique formal normalizing map  $\hat{\Phi}$  given by Theorem 1.5 as “true” Gevrey-1 asymptotic expansion in  $S_+ \times (\mathbb{C}^2, 0)$  and  $S_- \times (\mathbb{C}^2, 0)$  respectively. This is done by studying  $\Phi_+ \circ (\Phi_-)^{-1}$  in  $(S_+ \cap S_-) \times (\mathbb{C}^2, 0)$  (and more generally any germ of sectorial fibered isotropy of  $Y_{\text{norm}}$  in “narrow” sectorial domains  $(S_+ \cap S_-) \times (\mathbb{C}^2, 0)$  which admits the identity as weak Gevrey-1 asymptotic expansion). In the announced paper we also:

- prove that the formal normalizing map  $\hat{\Phi}$  in Theorem 1.10 is in fact 1-summable (and not only *weakly* 1-summable).
- provide a theorem of analytic classification, based on the study over “small” sectors  $S_+ \cap S_-$  of the transition maps  $\Phi_+ \circ \Phi_-^{-1}$  (also called *Stokes diffeomorphisms*): they are sectorial isotropies of the normal form  $Y_{\text{norm}}$  which are exponentially close to the identity.

The main difficulty is to establish that such a sectorial isotropy of  $Y_{\text{norm}}$  over the “narrow” sectors described above is necessarily exponentially close to the identity. This will be done by a detailed analysis of these maps in the space of leaves. In fact, this is the second main new difficulty we have met, which is due to the presence of the “resonant” term

$$\frac{c_m (y_1 y_2)^m \log(x)}{x}$$

in the exponential expression of the first integrals of the vector field in normal form. In [Sto96], similar computations are done in subsection 3.4.1. In this part of the paper, infinitely many irregular differential equations appear when identifying terms of same homogeneous degree. The fact that  $Y_{\{x=0\}}$  is linear implies that these differential equations are all linear and uncoupled. In our situation this is not the case, which yields more complicated computations.

### 1.3. Painlevé equations: the transversally Hamiltonian case.

In [Yos85] Yoshida shows that a vector field in the class  $\mathcal{SN}_{\text{diag},0}$  naturally appears after a suitable compactification (given by the so-called Boutroux coordinates [Bou13]) of the phase space of Painlevé equations  $(P_j)_{j=I\dots V}$ , for generic values of the parameters. In these cases the vector field presents an additional transverse Hamiltonian structure. Let us illustrate these computations in the case of the first Painlevé equation:

$$(P_I) \quad \frac{d^2 z_1}{dt^2} = 6z_1^2 + t \quad .$$

As is well known since Okamoto [Oka80],  $(P_I)$  can be seen as a non-autonomous Hamiltonian system

$$\begin{cases} \frac{\partial z_1}{\partial t} = -\frac{\partial H}{\partial z_2} \\ \frac{\partial z_2}{\partial t} = \frac{\partial H}{\partial z_1} \end{cases}$$

with Hamiltonian

$$H(t, z_1, z_2) := 2z_1^3 + tz_1 - \frac{z_2^2}{2}.$$

More precisely, if we consider the standard symplectic form  $\omega := dz_1 \wedge dz_2$  and the vector field

$$Z := \frac{\partial}{\partial t} - \frac{\partial H}{\partial z_2} \frac{\partial}{\partial z_1} + \frac{\partial H}{\partial z_1} \frac{\partial}{\partial z_2}$$

induced by  $(P_I)$ , then the Lie derivative

$$\mathcal{L}_Z(\omega) = \left( \frac{\partial^2 H}{\partial t \partial z_1} dz_1 + \frac{\partial^2 H}{\partial t \partial z_2} dz_2 \right) \wedge dt = dz_1 \wedge dt$$

belongs to the ideal  $\langle dt \rangle$  generated by  $dt$  in the exterior algebra  $\Omega^*(\mathbb{C}^3)$  of differential forms in variables  $(t, z_1, z_2)$ . Equivalently, for any  $t_1, t_2 \in \mathbb{C}$  the flow of  $Z$  at time  $(t_2 - t_1)$  acts as a *symplectomorphism* between fibers  $\{t = t_1\}$  and  $\{t = t_2\}$ .

The weighted compactification given by the Boutroux coordinates [Bou13] defines a chart near  $\{t = \infty\}$  as follows:

$$\begin{cases} z_2 = y_2 x^{-\frac{3}{5}} \\ z_1 = y_1 x^{-\frac{2}{5}} \\ t = x^{-\frac{4}{5}} \end{cases} .$$



In the coordinates  $(x, y_1, y_2)$ , the vector field  $Z$  is transformed, up to a translation  $y_1 \leftarrow y_1 + \zeta$  with  $\zeta = \frac{i}{\sqrt{6}}$ , to the vector field

$$(1.7) \quad \tilde{Z} = -\frac{5}{4x^{\frac{1}{5}}}Y$$

where

$$Y = x^2 \frac{\partial}{\partial x} + \left( -\frac{4}{5}y_2 + \frac{2}{5}xy_1 + \frac{2\zeta}{5}x \right) \frac{\partial}{\partial y_1} + \left( -\frac{24}{5}y_1^2 - \frac{48\zeta}{5}y_1 + \frac{3}{5}xy_2 \right) \frac{\partial}{\partial y_2} .$$

We observe that  $Y$  is a strictly non-degenerate doubly-resonant saddle-node as in Definitions 1.1 and 1.3 with residue  $\text{res}(Y) = 1$ . Furthermore we have:

$$\begin{cases} dt &= -\frac{4}{5}5^{\frac{4}{5}}x^{-\frac{9}{5}}dx \\ dz_1 \wedge dz_2 &= \frac{1}{x}(dy_1 \wedge dy_2) + \frac{1}{5x^2}(2y_1dy_2 - 3y_2dy_1) \wedge dx \\ &\in \frac{1}{x}(dy_1 \wedge dy_2) + \langle dx \rangle \end{cases} ,$$

where  $\langle dx \rangle$  denotes the ideal generated by  $dx$  in the algebra of holomorphic forms in  $\mathbb{C}^* \times \mathbb{C}^2$ . We finally obtain

$$\begin{cases} \mathcal{L}_Y \left( \frac{dy_1 \wedge dy_2}{x} \right) = \frac{1}{5x}(3y_2dy_1 - (2\zeta + 2y_1)dy_2) \wedge dx \\ \mathcal{L}_Y(dx) = 2x dx \end{cases} .$$

Therefore, both  $\mathcal{L}_Y(\omega)$  and  $\mathcal{L}_Y(dx)$  are differential forms who lie in the ideal  $\langle dx \rangle$ , in the algebra of germs of meromorphic 1-forms in  $(\mathbb{C}^3, 0)$  with poles only in  $\{x = 0\}$ . This motivates the following:

**Definition 1.15.** Consider the rational 1-form

$$\omega := \frac{dy_1 \wedge dy_2}{x} .$$

We say that vector field  $Y$  is **transversally Hamiltonian** (with respect to  $\omega$  and  $dx$ ) if

$$\mathcal{L}_Y(dx) \in \langle dx \rangle \quad \text{and} \quad \mathcal{L}_Y(\omega) \in \langle dx \rangle .$$

For any open sector  $S \subset \mathbb{C}^*$ , we say that a germ of sectorial fibered diffeomorphism  $\Phi$  in  $S \times (\mathbb{C}^2, 0)$  is **transversally symplectic** (with respect to  $\omega$  and  $dx$ ) if

$$\Phi^*(\omega) \in \omega + \langle dx \rangle$$

(Here  $\Phi^*(\omega)$  denotes the pull-back of  $\omega$  by  $\Phi$ ).

We denote by  $\text{Diff}_\omega(\mathbb{C}^3, 0, \text{Id})$  the group of transversally symplectic diffeomorphisms which are tangent to the identity.

*Remark 1.16.*

- (1) The flow of a transversally Hamiltonian vector field  $X$  defines a map between fibers  $\{x = x_1\}$  and  $\{x = x_2\}$  which sends  $\omega|_{x=x_1}$  onto  $\omega|_{x=x_2}$ , since

$$(\exp(X))^*(\omega) \in \omega + \langle dx \rangle .$$

- (2) A fibered sectorial diffeomorphism  $\Phi$  is transversally symplectic if and only if  $\det(D\Phi) = 1$ .

**Definition 1.17.** A **transversally Hamiltonian doubly-resonant saddle-node** is a transversally Hamiltonian vector field which is conjugate, *via* a transversally symplectic diffeomorphism, to one of the form

$$Y = x^2 \frac{\partial}{\partial x} + \left( -\lambda y_1 + f_1(x, \mathbf{y}) \right) \frac{\partial}{\partial y_1} + \left( \lambda y_2 + f_2(x, \mathbf{y}) \right) \frac{\partial}{\partial y_2} ,$$

with  $\lambda \in \mathbb{C}^*$  and  $f_1, f_2$  analytic in  $(\mathbb{C}^3, 0)$  and of order at least 2.

Notice that a transversally Hamiltonian doubly-resonant saddle-node is necessarily strictly non-degenerate (since its residue is always equal to 1), and also div-integrable (see section 3). It follows from Yoshida's work [Yos85] that the doubly-resonant saddle-nodes at infinity in Painlevé equations  $(P_j)_{j=I\dots V}$  (for generic values of the parameters) all are transversally Hamiltonian.

We recall the second main result from [Bit15].

**Theorem 1.18.** [Bit15]

Let  $Y \in \mathcal{SN}_{\text{diag}}$  be a diagonal doubly-resonant saddle-node which is supposed to be transversally Hamiltonian. Then, there exists a unique formal, fibered, transversally symplectic diffeomorphism  $\hat{\Phi}$ , tangent to identity, such that:

$$(1.8) \quad \begin{aligned} \hat{\Phi}_*(Y) &= x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x - c(y_1 y_2)) y_1 \frac{\partial}{\partial y_1} + (\lambda + a_2 x + c(y_1 y_2)) y_2 \frac{\partial}{\partial y_2} \\ &=: Y_{\text{norm}}, \end{aligned}$$

where  $\lambda \in \mathbb{C}^*$ ,  $c(v) \in v\mathbb{C}[[v]]$  a formal power series in  $v = y_1 y_2$  without constant term and  $a_1, a_2 \in \mathbb{C}$  are such that  $a_1 + a_2 = 1$ .

As a consequence of Theorem 1.18 and Theorem 1.10 we have the following:

**Theorem 1.19.** Let  $Y$  be a transversally Hamiltonian doubly-resonant saddle-node and let  $\hat{\Phi}$  be the unique formal normalizing map given by Theorem 1.18. Then the associated sectorial normalizing maps  $\Phi_+$  and  $\Phi_-$  are also transversally symplectic.

*Proof.* Since  $\hat{\Phi}$  is weakly 1-summable in  $S_{\pm} \times (\mathbb{C}^2, 0)$ , the formal power series  $\det(D\hat{\Phi})$  is also weakly 1-summable in  $S_{\pm} \times (\mathbb{C}^2, 0)$ , and its Gevrey-1 asymptotic expansion has to be the constant 1. By uniqueness of the weak 1-sum, we thus have  $\det(D\Phi_{\pm}) = 1$ .  $\square$

#### 1.4. Outline of the paper.

In section 2, we introduce the different tools we need regarding asymptotic expansion, Gevrey-1 series and 1-summability. We will in particular introduce a notion of “**weak**” 1-summability.

In section 3, we prove Proposition 3.1, which states that we can always formally conjugate a non-degenerate doubly-resonant saddle-node which is also div-integrable to its normal form up to remaining terms of order  $O(x^N)$ , for all  $N \in \mathbb{N}_{>0}$ , and that the conjugacy is actually 1-summable.

In section 4, we prove that for all  $Y \in \mathcal{SN}_{\text{diag},0}$ , there exists a unique pair of sectorial normalizing maps  $(\Phi_+, \Phi_-)$  tangent to the identity which conjugates  $Y$  to its normal form  $Y_{\text{norm}}$  over sectors with opening greater than  $\pi$  and arbitrarily close to (but less than)  $2\pi$ . The existence is given by Corollary 4.2, while the uniqueness clause stated in Proposition 1.13 is proved thanks to Proposition 4.16. Moreover, we will see that  $\Phi_+$  and  $\Phi_-$  both admit the unique formal normalizing map  $\hat{\Phi}$  given by Theorem 1.5 as weak Gevrey-1 asymptotic expansion.

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## 2. BACKGROUND

We refer the reader to [MR82], [Mal95] and [RS93] for a detailed introduction to the theory of asymptotic expansion, Gevrey series and summability (see also [Sto96] for a useful discussion of these concepts), where one can find the proofs of the classical results we recall (without proof). We call  $x \in \mathbb{C}$  the *independent* variable and  $\mathbf{y} := (y_1, \dots, y_n) \in \mathbb{C}^n$ ,  $n \in \mathbb{N}$ , the *dependent* variables. As usual we define  $\mathbf{y}^{\mathbf{k}} := y_1^{k_1} \dots y_n^{k_n}$  for  $\mathbf{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$ , and  $|\mathbf{k}| = k_1 + \dots + k_n$ . The notions of asymptotic expansion, Gevrey series and 1-summability presented here are always considered with respect to the independent variable  $x$  living in (open) sectors

$$S(r, \alpha, \beta) = \{x \in \mathbb{C} \mid 0 < |x| < r \text{ and } \alpha < \arg(x) < \beta\} ,$$

the dependent variable  $\mathbf{y}$  belonging to poly-discs

$$\mathbf{D}(\mathbf{0}, \mathbf{r}) := \{\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{C}^n \mid |y_1| < r_1, \dots, |y_n| < r_n\} ,$$

of poly-radius  $\mathbf{r} = (r_1, \dots, r_n) \in (\mathbb{R}_{>0})^n$ . Given an open subset  $\mathcal{U} \subset \mathbb{C}^{n+1}$ , we denote by  $\mathcal{O}(\mathcal{U})$  the algebra of holomorphic function in  $\mathcal{U}$ .

The results recalled here are valid when  $n = 0$ . Some statements for which we do not give a proof can be proved exactly as in the classical case  $n = 0$ , uniformly in the dependent multi-variable  $\mathbf{y}$ . Finally, we will present some classical results not in their original (and more general) form, but in a form more suited to our current needs.

### 2.1. Sectorial germs.

Let  $\theta \in \mathbb{R}$ ,  $\eta \in \mathbb{R}_{\geq 0}$  and  $n \in \mathbb{N}$ .

**Definition 2.1.** (1) An *x-sectorial neighborhood* (or simply *sectorial neighborhood*) attached to the origin (of  $\mathbb{C}^{n+1}$ ) in the direction  $\theta$  with opening  $\eta$  is an open set  $\mathcal{S} \subset \mathbb{C}^{n+1}$  such that

$$\mathcal{S} \supset S\left(r, \theta - \frac{\eta}{2} - \epsilon, \theta + \frac{\eta}{2} + \epsilon\right) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$$

for some  $r > 0$ ,  $\mathbf{r} \in (\mathbb{R}_{>0})^n$  and  $\epsilon > 0$ . We denote by  $(\mathcal{S}_{\theta, \eta}, \leq)$  the directed set formed by all such neighborhoods, equipped with the order

$$\mathcal{S}_1 \leq \mathcal{S}_2 \iff \mathcal{S}_1 \supset \mathcal{S}_2 .$$

(2) The algebra of *germs of holomorphic functions in a sectorial neighborhood attached to the origin in the direction  $\theta$  with opening  $\eta$*  is the direct limit

$$\mathcal{O}(\mathcal{S}_{\theta, \eta}) := \varinjlim \mathcal{O}(\mathcal{S})$$

with respect to the directed system defined by  $\{\mathcal{O}(\mathcal{S}) : \mathcal{S} \in \mathcal{S}_{\theta, \eta}\}$ .

We now give the definition of a (*germ of a*) *sectorial diffeomorphism*.

**Definition 2.2.** (1) Given an element  $\mathcal{S} \in \mathcal{S}_{\theta, \eta}$ , we denote by  $\text{Diff}_{\text{fib}}(\mathcal{S}, \text{Id})$  the set of holomorphic diffeomorphisms of the form

$$\begin{aligned} \Phi : \mathcal{S} &\rightarrow \Phi(\mathcal{S}) \\ (x, \mathbf{y}) &\mapsto (x, \phi_1(x, \mathbf{y}), \phi_2(x, \mathbf{y})) , \end{aligned}$$

such that  $\Phi(x, \mathbf{y}) - \text{Id}(x, \mathbf{y}) = O(\|x, \mathbf{y}\|^2)$ , as  $(x, \mathbf{y}) \rightarrow (0, \mathbf{0})$  in  $\mathcal{S}$ .<sup>1</sup>

(2) The set of *germs of (fibered) sectorial diffeomorphisms in the direction  $\theta$  with opening  $\eta$ , tangent to the identity*, is the direct limit

$$\text{Diff}_{\text{fib}}(\mathcal{S}_{\theta, \eta}, \text{Id}) := \varinjlim \text{Diff}_{\text{fib}}(\mathcal{S}, \text{Id})$$

with respect to the directed system defined by  $\{\text{Diff}_{\text{fib}}(\mathcal{S}, \text{Id}) : \mathcal{S} \in \mathcal{S}_{\theta, \eta}\}$ . We equip  $\text{Diff}_{\text{fib}}(\mathcal{S}_{\theta, \eta}, \text{Id})$  of a group structure as follows: given two germs  $\Phi, \Psi \in \text{Diff}_{\text{fib}}(\mathcal{S}_{\theta, \eta}, \text{Id})$  we chose corresponding representatives  $\Phi_0 \in \text{Diff}_{\text{fib}}(\mathcal{S}, \text{Id})$  and  $\Psi_0 \in \text{Diff}_{\text{fib}}(\mathcal{T}, \text{Id})$  with  $\mathcal{S}, \mathcal{T} \in \mathcal{S}_{\theta, \eta}$  such that  $\mathcal{T} \subset \Phi_0(\mathcal{S})$  and let  $\Psi \circ \Phi$  be the germ defined by  $\Psi_0 \circ \Phi_0$ .<sup>2</sup>

We will also need the notion of *asymptotic sectors*.

**Definition 2.3.** An (*open*) *asymptotic sector of the origin in the direction  $\theta$  and with opening  $\eta$*  is an open set  $S \subset \mathbb{C}$  such that

$$S \in \bigcap_{0 \leq \eta' < \eta} \mathcal{S}_{\theta, \eta'} .$$

We denote by  $\mathcal{AS}_{\theta, \eta}$  the set of all such (open) asymptotic sectors.

## 2.2. Gevrey-1 power series and 1-summability.

### 2.2.1. Gevrey-1 asymptotic expansions.

In this subsection we fix a formal power series

$$\hat{f}(x, \mathbf{y}) = \sum_{k \geq 0} f_k(\mathbf{y}) x^k = \sum_{(j_0, \mathbf{j}) \in \mathbb{N}^{n+1}} f_{j_0, \mathbf{j}} x^{j_0} \mathbf{y}^{\mathbf{j}} \in \mathbb{C}[[x, \mathbf{y}]] .$$

### Definition 2.4.

- A function  $f$  analytic in a domain  $S(r, \alpha, \beta) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$  admits  $\hat{f}$  as asymptotic expansion *in the sense of Gérard-Sibuya* in this domain if for every closed sub-sector  $S' \subset S(r, \alpha, \beta)$  and compact  $\mathbf{K} \subset \mathbf{D}(\mathbf{0}, \mathbf{r})$ , for all  $N \in \mathbb{N}$ , there exists a constant  $C_{S', K, N} > 0$  such that:

$$\left| f(x, \mathbf{y}) - \sum_{j_0 + j_1 + \dots + j_n \leq N} f_{j_0, \mathbf{j}} x^{j_0} \mathbf{y}^{\mathbf{j}} \right| \leq C_{S', K, N} \|(x, \mathbf{y})\|^{N+1}$$

for all  $(x, \mathbf{y}) \in S' \times \mathbf{K}$ ; here,  $\|\cdot\|$  is a norm in  $\mathbb{C}^{n+1}$ .

- A function  $f$  analytic in a domain  $S(r, \alpha, \beta) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$  admits  $\hat{f}$  as *asymptotic expansion (with respect to  $x$ )* if for all  $k \in \mathbb{N}$  the function  $\mathbf{y} \mapsto f_k(\mathbf{y})$  is analytic in  $\mathbf{D}(\mathbf{0}, \mathbf{r})$ , and if for every closed sub-sector  $S' \subset S(r, \alpha, \beta)$ , compact subset  $\mathbf{K} \subset \mathbf{D}(\mathbf{0}, \mathbf{r})$  and  $N \in \mathbb{N}$ , there exists  $A_{S', K, N} > 0$  such that:

$$\left| f(x, \mathbf{y}) - \sum_{k \geq 0} f_k(\mathbf{y}) x^k \right| \leq A_{S', K, N} |x|^{N+1}$$

for all  $(x, \mathbf{y}) \in S' \times \mathbf{K}$ .

<sup>1</sup>This condition implies in particular that  $\Phi(\mathcal{S}) \in \mathcal{S}_{\theta, \eta}$ .

<sup>2</sup>One can prove that this definition is independent of the choice of the representatives.

- An analytic (and bounded) function  $f$  in a sectorial domain  $S(r, \alpha, \beta) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$  admits  $\hat{f}$  as *Gevrey-1 asymptotic expansion* in this domain, if for all  $k \in \mathbb{N}$  the function  $\mathbf{y} \mapsto f_k(\mathbf{y})$  is analytic in  $\mathbf{D}(\mathbf{0}, \mathbf{r})$ , and if for all closed sub-sector  $S' \subset S(r, \alpha, \beta)$ , there exists  $A, C > 0$  such that:

$$\left| f(x, \mathbf{y}) - \sum_{k=0}^{N-1} f_k(\mathbf{y}) x^k \right| \leq AC^N (N!) |x|^N$$

for all  $N \in \mathbb{N}$  and  $(x, \mathbf{y}) \in S' \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ .

*Remark 2.5.*

- (1) If a function admits  $\hat{f}$  as Gevrey-1 asymptotic expansion in  $S(r, \alpha, \beta) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ , then it also admits  $\hat{f}$  as asymptotic expansion.
- (2) If a function admits  $\hat{f}$  as asymptotic expansion in  $S(r, \alpha, \beta) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ , then it also admits  $\hat{f}$  as asymptotic expansion in the the sense of Gérard-Sibuya.
- (3) Any previously defined asymptotic expansion is unique.

2.2.2. *Borel transform and Gevrey-1 power series.*

**Definition 2.6.**

- We define the Borel transform  $\mathcal{B}(\hat{f})$  of  $\hat{f}$  as:

$$\mathcal{B}(\hat{f})(t, \mathbf{y}) := \sum_{k \geq 0} \frac{f_k(\mathbf{y})}{k!} t^k .$$

- We say that  $\hat{f}$  is Gevrey-1 if  $\mathcal{B}(\hat{f})$  is convergent in a neighborhood of the origin in  $\mathbb{C} \times \mathbb{C}^n$ . Notice that in this case every function  $f_k$ ,  $k \geq 0$ , is analytic in a polydisc  $\mathbf{D}(\mathbf{0}, \mathbf{r})$  of poly-radius  $\mathbf{r} = (r_1, \dots, r_n) \in (\mathbb{R}_{>0})^n$  independent on  $k$ , so that  $\mathcal{B}(\hat{f})$  is analytic in  $\mathbf{D}(0, \rho) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ , for some  $\rho > 0$ . Possibly by reducing  $\rho, r_1, \dots, r_n > 0$ , we can assume that  $\mathcal{B}(\hat{f})$  is bounded in  $\mathbf{D}(0, \rho) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ .

*Remark 2.7.*

- (1) If a sectorial function  $f$  admits  $\hat{f}$  for Gevrey-1 asymptotic expansion as in Definition 2.4 then  $\hat{f}$  is a Gevrey-1 formal power series.
- (2) The set of all Gevrey-1 formal power series is an algebra closed under differentiation  $\frac{\partial}{\partial x}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}$ .

*Remark 2.8.* We will need sometimes to use another definition of the Borel transform, that is:

$$\tilde{\mathcal{B}}(\hat{f})(t, \mathbf{y}) := \sum_{k \geq 0} f_{k+1}(\mathbf{y}) \frac{t^k}{k!} .$$

The first definition we gave has the advantage of being “directly” invertible (*via.* the Laplace transform) for all formal power series who are *1-sommable* in a direction  $\theta$ . Its drawback is that is behave not so good with respect to the product. On the contrary, the second definition will be “directly” invertible (*via.* the Laplace transform) only for formal power series in  $x$  with  $f_0 = 0$  (otherwise on have to be careful with the constant term with respect to  $x$ ) The advantage of the second Borel transform is to change a product into a convolution product:

$$\tilde{\mathcal{B}}(\hat{f}\hat{g}) = \left( \tilde{\mathcal{B}}(\hat{f}) * \tilde{\mathcal{B}}(\hat{g}) \right) ,$$

where the convolution product of two functions  $h_1, h_2$  is defined by

$$(h_1 * h_2)(t, \mathbf{y}) := \int_0^t h_1(s) h_2(s-t) ds .$$

The definition of being Gevrey-1 for a formal power series is not modified with the second Borel transform.

### 2.2.3. Directional 1-summability and Borel-Laplace summation.

**Definition 2.9.** Given  $\theta \in \mathbb{R}$  and  $\delta > 0$ , we define the infinite sector in the direction  $\theta$  with opening  $\delta$  as the set

$$\mathcal{A}_{\theta, \delta}^{\infty} := \left\{ t \in \mathbb{C}^* \mid |\arg(t) - \theta| < \frac{\delta}{2} \right\} .$$

We say that  $\hat{f}$  is 1-summable in the direction  $\theta \in \mathbb{R}$ , if the following three conditions holds:

- $\hat{f}$  is a Gevrey-1 formal power series;
- $\mathcal{B}(\hat{f})$  can be analytically continued to a domain of the form  $\mathcal{A}_{\theta, \delta}^{\infty} \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ ;
- there exists  $\lambda > 0, M > 0$  such that:

$$\forall (t, \mathbf{y}) \in \mathcal{A}_{\theta, \delta}^{\infty} \times \mathbf{D}(\mathbf{0}, \mathbf{r}), \quad \left| \mathcal{B}(\hat{f})(t, \mathbf{y}) \right| \leq M \exp(\lambda |t|) .$$

In this case we set  $\Delta_{\theta, \delta, \rho} := \mathcal{A}_{\theta, \delta}^{\infty} \cup \mathbf{D}(0, \rho)$  and

$$\left\| \hat{f} \right\|_{\lambda, \theta, \delta, \rho, \mathbf{r}} := \sup_{(t, \mathbf{y}) \in \Delta_{\theta, \delta, \rho} \times \mathbf{D}(\mathbf{0}, \mathbf{r})} \left| \mathcal{B}(\hat{f})(t, \mathbf{y}) \exp(-\lambda |t|) \right| .$$

Most of the time the context defines unequivocally the data  $(\theta, \delta, \rho, \mathbf{r})$ , in which case we drop the corresponding indices and simply write  $\left\| \hat{f} \right\|_{\lambda}$ .

*Remark 2.10.* (1) For fixed  $(\lambda, \theta, \delta, \rho, \mathbf{r})$  as above, the set  $\mathfrak{B}_{\lambda, \theta, \delta, \rho, \mathbf{r}}$  of all formal power series  $\hat{f}$ , 1-summable in the direction  $\theta$  and such that  $\left\| \hat{f} \right\|_{\lambda, \theta, \delta, \rho, \mathbf{r}} < +\infty$ , is a Banach vector space for the norm  $\|\cdot\|_{\lambda, \theta, \delta, \rho, \mathbf{r}}$ . We simply write  $(\mathfrak{B}_{\lambda}, \|\cdot\|_{\lambda})$  when there is no ambiguity.

(2) We will also need to use another norm (associated to the second Transform Borel  $\tilde{\mathcal{B}}$ : cf. remark 2.8), that is:

$$\left\| \hat{f} \right\|_{\lambda, \theta, \delta, \rho, \mathbf{r}}^{\text{bis}} := \sup_{(t, \mathbf{y}) \in \Delta_{\theta, \delta, \rho} \times \mathbf{D}(\mathbf{0}, \mathbf{r})} \left| \mathcal{B}(\hat{f})(t, \mathbf{y}) \left(1 + \lambda^2 |t|^2\right) \exp(-\lambda |t|) \right| .$$

This norm “behaves well” with respect to the product (see Proposition below). We denote by  $\mathfrak{B}_{\lambda, \theta, \delta, \rho, \mathbf{r}}^{\text{bis}}$  the set of formal power series  $\hat{f}$  1-sommable in direction  $\theta$  and such that

$$\left\| \hat{f} \right\|_{\lambda, \theta, \delta, \rho, \mathbf{r}}^{\text{bis}} < +\infty .$$

(3) If  $\lambda' \geq \lambda$ , then  $\mathfrak{B}_{\lambda, \theta, \delta, \rho, \mathbf{r}} \subset \mathfrak{B}_{\lambda', \theta, \delta, \rho, \mathbf{r}}$  and  $\mathfrak{B}_{\lambda, \theta, \delta, \rho, \mathbf{r}}^{\text{bis}} \subset \mathfrak{B}_{\lambda', \theta, \delta, \rho, \mathbf{r}}^{\text{bis}}$ .

**Proposition 2.11.** [BDM08, Proposition 4.] *If  $\hat{f}$  and  $\hat{g}$  are elements of  $\mathfrak{B}_{\lambda, \theta, \delta, \rho, \mathbf{r}}^{\text{bis}}$ , then  $\hat{f}\hat{g} \in \mathfrak{B}_{\lambda, \theta, \delta, \rho, \mathbf{r}}^{\text{bis}}$  and we have:*

$$\left\| \hat{f}\hat{g} \right\|_{\lambda, \theta, \delta, \rho, \mathbf{r}}^{\text{bis}} \leq \frac{4\pi}{\lambda} \left\| \hat{f} \right\|_{\lambda, \theta, \delta, \rho, \mathbf{r}}^{\text{bis}} \left\| \hat{g} \right\|_{\lambda, \theta, \delta, \rho, \mathbf{r}}^{\text{bis}} .$$

*Remark 2.12.* If  $\lambda \geq 4\pi$ , then  $\|\cdot\|_{\lambda, \theta, \delta, \rho, \mathbf{r}}^{\text{bis}}$  satisfies:

$$\left\| \hat{f}\hat{g} \right\|_{\lambda, \theta, \delta, \rho, \mathbf{r}}^{\text{bis}} \leq \left\| \hat{f} \right\|_{\lambda, \theta, \delta, \rho, \mathbf{r}}^{\text{bis}} \left\| \hat{g} \right\|_{\lambda, \theta, \delta, \rho, \mathbf{r}}^{\text{bis}} ,$$

and then  $\mathfrak{B}_{\lambda, \theta, \delta, \rho, \mathbf{r}}^{\text{bis}}$  is a Banach algebra.

**Definition 2.13.** Let  $g$  be analytic in a domain  $\mathcal{A}_{\theta, \delta}^{\infty}$  and  $\lambda > 0, M > 0$  such that

$$\forall (t, \mathbf{y}) \in \mathcal{A}_{\theta, \delta}^{\infty} \times \mathbf{D}(\mathbf{0}, \mathbf{r}), \quad |g(t, \mathbf{y})| \leq M \exp(\lambda |t|) .$$

We define the Laplace transform of  $g$  in the direction  $\theta$  as:

$$\mathcal{L}_{\theta}(g)(x, \mathbf{y}) := \int_{e^{i\theta} \mathbb{R}_{>0}} g(t, \mathbf{y}) \exp\left(-\frac{t}{x}\right) \frac{dt}{x} ,$$

which is absolutely convergent for all  $x \in \mathbb{C}$  with  $\Re\left(\frac{e^{i\theta}}{x}\right) > \lambda$  and is analytic with respect to  $(x, \mathbf{y})$ .

*Remark 2.14.* As earlier for the Borel transform, there exists another definition for the Laplace transform, that is:

$$\tilde{\mathcal{L}}_\theta(g)(x, \mathbf{y}) := \int_{e^{i\theta}\mathbb{R}_{>0}} g(t, \mathbf{y}) \exp\left(-\frac{t}{x}\right) dt .$$

**Proposition 2.15.** *A formal power series  $\hat{f} \in \mathbb{C}[[x, \mathbf{y}]]$  is 1-summable in the direction  $\theta$  if and only if there exists a germ of a sectorial holomorphic function  $f_\theta \in \mathcal{O}(\mathcal{S}_{\theta, \pi})$  which admits  $\hat{f}$  as Gevrey-1 asymptotic expansion in some  $\mathcal{S} \in \mathcal{S}_{\theta, \pi}$ . Moreover,  $f_\theta$  is unique (as a germ in  $\mathcal{O}(\mathcal{S}_{\theta, \pi})$ ) and in particular*

$$f_\theta = \mathcal{L}_\theta\left(\mathcal{B}(\hat{f})\right) .$$

The function (germ)  $f_\theta$  is called the 1-sum of  $\hat{f}$  in the direction  $\theta$ .

*Remark 2.16.* With the alternative definitions of the Borel and Laplace transforms, we have have a similar result holds with the formula:

$$f_\theta = \tilde{\mathcal{L}}_\theta\left(\tilde{\mathcal{B}}(\hat{f})\right) + \hat{f}(0, \mathbf{y}) .$$

We recall the following well-known result.

**Lemma 2.17.** *The set  $\Sigma_\theta \subset \mathbb{C}[[x, \mathbf{y}]]$  of 1-summable power series in the direction  $\theta$  is an algebra closed under differentiation. Moreover the map*

$$\begin{aligned} \Sigma_\theta &\longrightarrow \mathcal{O}(\mathcal{S}_{\theta, \pi}) \\ \hat{f} &\longmapsto f_\theta \end{aligned}$$

is an injective morphism of differential algebras.

**Definition 2.18.** A formal power series  $\hat{f} \in \mathbb{C}[[x, \mathbf{y}]]$  is called 1-summable if it is 1-summable in all but a finite number of directions (in  $\mathbb{R}/2\pi\mathbb{Z}$ ), called *Stokes directions*. In this case, if  $\theta_1, \dots, \theta_k \in \mathbb{R}/2\pi\mathbb{Z}$  are possible Stokes directions, we say that  $\hat{f}$  is 1-summable **except for**  $\theta_1, \dots, \theta_k$ .

More generally, we say that an  $m$ -uple  $(f_1, \dots, f_m) \in \mathbb{C}[[x, \mathbf{y}]]^m$  is Gevrey-1 (resp. 1-summable in direction  $\theta$ ) if this property holds for each component  $f_j, j = 1, \dots, m$ . Similarly, a formal vector field (or diffeomorphism) is said to be Gevrey-1 (resp. 1-summable in direction  $\theta$ ) if each one of its components has this property.

#### 2.2.4. Composition of 1-summable power series.

We will use the following particular case of a classical result due to Ramis and Sibuya.

**Theorem 2.19.** (*Ramis-Sibuya*) *Let  $\mathcal{S}_+ \in \mathcal{S}_{\theta, \pi}$ ,  $\mathcal{S}_- \in \mathcal{S}_{\pi-\theta, \pi}$  and  $(f_+, f_-) \in \mathcal{O}(\mathcal{S}_+) \times \mathcal{O}(\mathcal{S}_-)$ . Assume that for all compact  $\mathbf{K} \subset \mathbb{C}^n$  and all sector  $S \subset \mathbb{C}$  such that  $S \times \mathbf{K} \subset \mathcal{S}_+ \cap \mathcal{S}_-$ , there exists  $A, B > 0$  such that:*

$$|f_+(x, \mathbf{y}) - f_-(x, \mathbf{y})| \leq A \exp\left(-\frac{B}{|x|}\right)$$

for all  $(x, \mathbf{y}) \in S \times \mathbf{K}$ . Then, there exists a formal power series  $\hat{f}(x, \mathbf{y}) \in \mathbb{C}[[x, \mathbf{y}]]$  such that  $f_+$  (resp.  $f_-$ ) admits  $\hat{f}$  as Gevrey-1 asymptotic expansion in  $\mathcal{S}_+$  (resp.  $\mathcal{S}_-$ ).

An important consequence of this theorem is the following.

**Proposition 2.20.** *Let  $\hat{\Phi}(x, \mathbf{y}) \in \mathbb{C}[[x, \mathbf{y}]]$  be 1-summable in directions  $\theta$  and  $\theta - \pi$ , and let  $\Phi_+(x, \mathbf{y})$  and  $\Phi_-(x, \mathbf{y})$  be its 1-sums directions  $\theta$  and  $\theta - \pi$  respectively. Let also  $\hat{f}_1(x, \mathbf{z}), \dots, \hat{f}_n(x, \mathbf{z})$  be 1-summable in directions  $\theta, \theta - \pi$ , and  $f_{1,+}, \dots, f_{n,+}$ , and  $f_{1,-}, \dots, f_{n,-}$  be their 1-sums in directions  $\theta$  and  $\theta - \pi$  respectively. Assume that*

$$(2.1) \quad \hat{f}_j(0, \mathbf{0}) = 0, \text{ for all } j = 1, \dots, n .$$

Then

$$\hat{\Psi}(x, \mathbf{z}) := \hat{\Phi}\left(x, \hat{f}_1(x, \mathbf{z}), \dots, \hat{f}_n(x, \mathbf{z})\right)$$

is 1-summable in directions  $\theta, \theta - \pi$ , and its 1-sum in the corresponding direction is

$$\Psi_{\pm}(x, \mathbf{z}) := \Phi_{\pm}(x, f_{1,\pm}(x, \mathbf{z}), \dots, f_{n,\pm}(x, \mathbf{z})) \quad ,$$

which is a germ of a sectorial holomorphic function in this direction.

*Proof.* Since  $\hat{f}_j(0, \mathbf{0}) = 0$  for all  $j = 1, \dots, n$ ,

$$\hat{\Psi}(x, \mathbf{z}) := \hat{\Phi}\left(x, \hat{f}_1(x, \mathbf{z}), \dots, \hat{f}_n(x, \mathbf{z})\right)$$

is well defined as formal power series. Since  $\hat{f}_j(0, \mathbf{0}) = 0$  for all  $j = 1, \dots, n$ , there exists  $(\mathcal{S}_+, \mathcal{S}_-) \in \mathcal{S}_{\theta, \pi} \times \mathcal{S}_{\theta - \pi, \pi}$  and  $(\mathcal{S}'_+, \mathcal{S}'_-) \in \mathcal{S}_{\theta, \pi} \times \mathcal{S}_{\theta - \pi, \pi}$  such that:

- $f_{1,+}, \dots, f_{n,+}$  (*resp.*  $f_{1,-}, \dots, f_{n,-}$ ) are analytic and admit respectively  $\hat{f}_1, \dots, \hat{f}_n$  as Gevrey-1 asymptotic expansion in  $\mathcal{S}_+$  (*resp.*  $\mathcal{S}_-$ );
- the image of  $\mathcal{S}_{\pm}$  by the map

$$(x, \mathbf{z}) \mapsto (x, f_{1,\pm}(x, \mathbf{z}), \dots, f_{n,\pm}(x, \mathbf{z}))$$

is in  $\mathcal{S}'_{\pm}$ ,

- $\Phi_+$  (*resp.*  $\Phi_-$ ) is analytic and admits  $\hat{\Phi}$  as Gevrey-1 asymptotic expansion in  $\mathcal{S}'_+$  (*resp.*  $\mathcal{S}'_-$ ).

Consequently, the function defined by

$$\Psi_{\pm}(x, \mathbf{z}) := \Phi_{\pm}(x, f_{1,\pm}(x, \mathbf{z}), \dots, f_{n,\pm}(x, \mathbf{z}))$$

is analytic in  $\mathcal{S}_{\pm}$ . Moreover, it is clear that  $\Psi_{\pm}(x, \mathbf{z})$  admits  $\hat{\Psi}(x, \mathbf{z})$  as asymptotic expansion as  $(x, \mathbf{z}) \rightarrow (0, \mathbf{0})$  in  $\mathcal{S}_{\pm}$  (in the sense of Gérard-Sibuya). It remains to prove that  $|\Psi_+ - \Psi_-|$  is exponentially small in all subset of  $\mathcal{S}_+ \cap \mathcal{S}_-$  of the form  $S \times \mathbf{K}$ , for some sector  $S \subset \mathbb{C}$  and some compact  $\mathbf{K} \subset \mathbb{C}^n$  (then the theorem of Ramis-Sibuya allows us to conclude).

Let us fix a sector  $S \subset \mathbb{C}$  and a compact  $\mathbf{K} \subset \mathbb{C}^n$  such that  $S \times \mathbf{K} \subset \mathcal{S}_+ \cap \mathcal{S}_-$ . There exist  $A, B > 0$  such that for all  $j = 1, \dots, n$ , for all  $(x, \mathbf{z}) \in S \times \mathbf{K}$

$$|f_{j,+}(x, \mathbf{z}) - f_{j,-}(x, \mathbf{z})| \leq A \exp\left(-\frac{B}{|x|}\right) \quad .$$

The image of  $S \times \mathbf{K}$  by the map

$$(x, \mathbf{z}) \mapsto (x, f_{1,\pm}(x, \mathbf{z}), \dots, f_{n,\pm}(x, \mathbf{z}))$$

is included in  $S \times \mathbf{K}' \subset \mathcal{S}'_+ \cap \mathcal{S}'_-$  for some compact  $\mathbf{K}' \subset \mathbb{C}^n$ . By assumption, there exist  $A', B' > 0$  such that for all  $(x, \mathbf{y}) \in S \times \mathbf{K}'$

$$|\Phi_+(x, \mathbf{y}) - \Phi_-(x, \mathbf{y})| \leq A' \exp\left(-\frac{B'}{|x|}\right) \quad .$$

Indeed, two functions admitting the same Gevrey-1 asymptotic expansion are exponentially close. Finally, there exists  $C > 0$  such that for all  $(x, \mathbf{y}, \mathbf{y}')$  such that  $((x, \mathbf{y}), (x, \mathbf{y}')) \in S \times \mathbf{K}'$ ,

$$|\Phi_+(x, \mathbf{y}) - \Phi_+(x, \mathbf{y}')| \leq C \max_{j=1, \dots, n} (|y_j - y'_j|) \quad .$$



Hence, for all  $(x, \mathbf{z}) \in S \times \mathbf{K}$ :

$$\begin{aligned}
& |\Psi_+(x, \mathbf{z}) - \Psi_-(x, \mathbf{z})| \\
&= |\Phi_+(x, f_{1,+}(x, \mathbf{z}), \dots, f_{n,+}(x, \mathbf{z})) - \Phi_-(x, f_{1,-}(x, \mathbf{z}), \dots, f_{n,-}(x, \mathbf{z}))| \\
&\leq |\Phi_+(x, f_{1,+}(x, \mathbf{z}), \dots, f_{n,+}(x, \mathbf{z})) - \Phi_+(x, f_{1,-}(x, \mathbf{z}), \dots, f_{n,-}(x, \mathbf{z}))| \\
&\quad + |\Phi_+(x, f_{1,-}(x, \mathbf{z}), \dots, f_{n,-}(x, \mathbf{z})) - \Phi_-(x, f_{1,-}(x, \mathbf{z}), \dots, f_{n,-}(x, \mathbf{z}))| \\
&\leq C \max_{j=1, \dots, n} (|f_{j,+}(x, \mathbf{z}) - f_{j,-}(x, \mathbf{z})|) + A' \exp\left(-\frac{B'}{|x|}\right) \\
&\leq CA \exp\left(-\frac{B}{|x|}\right) + A' \exp\left(-\frac{B'}{|x|}\right) \\
&\leq A'' \exp\left(-\frac{B''}{|x|}\right)
\end{aligned}$$

for a convenient choice of  $A'', B'' > 0$ .  $\square$

Consider  $\hat{Y}$  a formal singular vector field at the origin and a formal fibered diffeomorphism  $\hat{\varphi} : (x, \mathbf{y}) \mapsto (x, \hat{\varphi}(x, \mathbf{y}))$ . Assume that both  $\hat{Y}$  and  $\hat{\varphi}$  are 1-summable in directions  $\theta$  and  $\theta - \pi$ , for some  $\theta \in \mathbb{R}$ , and denote by  $Y_+, Y_-$  (resp.  $\varphi_+, \varphi_-$ ) their 1-sums in directions  $\theta$  and  $\theta - \pi$  respectively. As a consequence of Proposition 2.20 and Lemma 2.17, we can state the following:

**Corollary 2.21.** *Under the assumptions above,  $\hat{\varphi}_*(\hat{Y})$  is 1-summable in both directions  $\theta$  and  $\theta - \pi$ , and its 1-sums in these directions are  $(\varphi_+)_*(Y_+)$  and  $(\varphi_-)_*(Y_-)$  respectively.*

### 2.3. Weak Gevrey-1 power series and weak 1-summability.

We present here a weaker notion of 1-summability that we also need. Let us fix a formal power series which we write as

$$\hat{f}(x, \mathbf{y}) = \sum_{\mathbf{j} \in \mathbb{N}^n} \hat{F}_{\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}} \in \mathbb{C}[[x, \mathbf{y}]].$$

#### Definition 2.22.

- The formal power series  $\hat{f}$  is said to be **weakly Gevrey-1** if for all  $\mathbf{j} \in \mathbb{N}^n$ ,  $\hat{F}_{\mathbf{j}}(x) \in \mathbb{C}[[x]]$  is a Gevrey-1 formal power series.
- A function

$$f(x, \mathbf{y}) = \sum_{\mathbf{j} \in \mathbb{N}^n} F_{\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}}$$

analytic and bounded in a domain  $S(r, \alpha, \beta) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ , admits  $\hat{f}$  as **weak Gevrey-1 asymptotic expansion** in  $x \in S(r, \alpha, \beta)$ , if for all  $\mathbf{j} \in \mathbb{N}^n$ ,  $F_{\mathbf{j}}$  admits  $\hat{F}_{\mathbf{j}}$  as Gevrey-1 asymptotic expansion in  $S(r, \alpha, \beta)$ .

- The formal power series  $\hat{f}$  is said to be **weakly 1-summable in the direction  $\theta \in \mathbb{R}$** , if the following conditions hold:
  - for all  $\mathbf{j} \in \mathbb{N}^n$ ,  $\hat{F}_{\mathbf{j}}(x) \in \mathbb{C}[[x]]$  is 1-summable in the direction  $\theta$ , whose 1-sum in the direction  $\theta$  is denoted by  $F_{\mathbf{j}, \theta}$ ;
  - the series  $f_{\theta}(x, \mathbf{y}) := \sum_{\mathbf{j} \in \mathbb{N}^n} F_{\mathbf{j}, \theta}(x) \mathbf{y}^{\mathbf{j}}$  defines a germ of a sectorial holomorphic function in a domain of the form

$$S\left(r, \theta - \frac{\pi}{2} - \epsilon, \theta + \frac{\pi}{2} + \epsilon\right) \times \mathbf{D}(\mathbf{0}, \mathbf{r}).$$

In this case,  $f_{\theta}(x, \mathbf{y})$  is called **the weak 1-sum of  $\hat{f}$  in the direction  $\theta$** .

As a consequence to the classical theory of summability and Gevrey asymptotic expansions, we immediately have the following:

- Lemma 2.23.** (1) *The weak Gevrey-1 asymptotic expansion of an analytic function in a domain  $S(r, \alpha, \beta) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$  is unique.*  
(2) *The weak 1-sum of a weak 1-summable formal power series in the direction  $\theta$ , is unique as a germ in  $\mathcal{O}(\mathcal{S}_{\theta, \pi})$ .*  
(3) *The set  $\Sigma_{\theta, \text{weak}} \subset \mathbb{C}[[x, \mathbf{y}]]$  of weakly 1-summable power series in the direction  $\theta$  is an algebra closed under differentiation. Moreover the map*

$$\begin{aligned} \Sigma_{\theta, \text{weak}} &\longrightarrow \mathcal{O}(\mathcal{S}_{\theta, \pi}) \\ \hat{f} &\longmapsto f_{\theta} \end{aligned}$$

*is an injective morphism of differential algebras.*

The following proposition is an analogue of Proposition 2.20 for weak 1-summable formal power series, with the a stronger condition instead of (2.1).

**Proposition 2.24.** *Let*

$$\hat{\Phi}(x, \mathbf{y}) = \sum_{\mathbf{j} \in \mathbb{N}^n} \hat{\Phi}_{\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}} \in \mathbb{C}[[x, \mathbf{y}]]$$

*and*

$$\hat{f}^{(k)}(x, \mathbf{z}) = \sum_{\mathbf{j} \in \mathbb{N}^n} \hat{F}_{\mathbf{j}}^{(k)}(x) \mathbf{z}^{\mathbf{j}} \in \mathbb{C}[[x, \mathbf{z}]],$$

*for  $k = 1, \dots, n$ , be  $n+1$  formal power series which are weakly 1-summable in directions  $\theta$  and  $\theta - \pi$ , and let us denote by  $\Phi_+, f_+^{(1)}, \dots, f_+^{(n)}$  (resp.  $\Phi_-, f_-^{(1)}, \dots, f_-^{(n)}$ ) their respective weak 1-sums in the direction  $\theta$  (resp.  $\theta - \pi$ ). Assume that  $\hat{F}_{\mathbf{0}}^{(k)} = 0$  for all  $k = 1, \dots, n$ . Then,*

$$\hat{\Psi}(x, \mathbf{z}) := \hat{\Phi}\left(x, \hat{f}^{(1)}(x, \mathbf{z}), \dots, \hat{f}^{(n)}(x, \mathbf{z})\right)$$

*is weakly 1-summable directions  $\theta$  and  $\theta - \pi$ , and its 1-sum in the corresponding direction is*

$$\Psi_{\pm}(x, \mathbf{z}) = \Phi_{\pm}\left(x, f_{\pm}^{(1)}(x, \mathbf{z}), \dots, f_{\pm}^{(n)}(x, \mathbf{z})\right),$$

*which is a germ of a sectorial holomorphic function in this direction with opening  $\pi$ .*

*Proof.* First of all,

$$\hat{\Psi}(x, \mathbf{y}) := \hat{\Phi}\left(x, \hat{f}^{(1)}(x, \mathbf{y}), \dots, \hat{f}^{(n)}(x, \mathbf{y})\right)$$

is well-defined as a formal power series since for all  $k = 1, \dots, n$ ,  $\hat{F}_{\mathbf{0}}^{(k)} = 0$ . It is also clear that

$$\Psi_{\pm}(x, \mathbf{y}) := \Phi_{\pm}\left(x, f_{\pm}^{(1)}(x, \mathbf{y}), \dots, f_{\pm}^{(n)}(x, \mathbf{y})\right)$$

is an analytic in a domain  $\mathcal{S}_+ \in \mathcal{S}_{\theta, \pi}$  (resp.  $\mathcal{S}_- \in \mathcal{S}_{\theta - \pi, \pi}$ ), because  $f_{\pm}^{(k)}(x, \mathbf{0}) = 0$  for all  $k = 1, \dots, n$ . Finally, we check that  $\Psi_{\pm}$  admits  $\hat{\Psi}$  as weak Gevrey-1 asymptotic expansion in  $\mathcal{S}_{\pm}$ . Indeed:

$$\begin{aligned} \Psi_{\pm}(x, \mathbf{y}) &= \Phi_{\pm}\left(x, f_{\pm}^{(1)}(x, \mathbf{y}), \dots, f_{\pm}^{(n)}(x, \mathbf{y})\right) \\ &= \sum_{\mathbf{j} \in \mathbb{N}^n} (\Phi_{\mathbf{j}})_{\pm}(x) \left(f_{\pm}^{(1)}(x, \mathbf{y})\right)^{j_1} \dots \left(f_{\pm}^{(n)}(x, \mathbf{y})\right)^{j_n} \\ &= \sum_{\mathbf{j} \in \mathbb{N}^n} (\Phi_{\mathbf{j}})_{\pm}(x) \left(\sum_{|\mathbf{l}| \geq 1} \left(F_{\mathbf{l}}^{(1)}\right)_{\pm}(x) \mathbf{y}^{\mathbf{l}}\right)^{j_1} \dots \\ &\quad \dots \left(\sum_{|\mathbf{l}| \geq 1} \left(F_{\mathbf{l}}^{(n)}\right)_{\pm}(x) \mathbf{y}^{\mathbf{l}}\right)^{j_n} \\ &= \sum_{\mathbf{j} \in \mathbb{N}^n} (\Psi_{\mathbf{j}})_{\pm}(x) \mathbf{y}^{\mathbf{j}} \end{aligned}$$

where for all  $\mathbf{j} \in \mathbb{N}^n$ ,  $(\Psi_{\mathbf{j}})_{\pm}(x)$  is obtained as a finite number of additions and products of the  $(\Phi_{\mathbf{k}})_{\pm}, (F_{\mathbf{k}}^{(1)})_{\pm}, \dots, (F_{\mathbf{k}}^{(n)})_{\pm}$ ,  $|\mathbf{k}| \leq |\mathbf{j}|$ . The same computation is valid for the associated formal power series, and allows us to define the  $\hat{\Psi}_{\mathbf{j}}(x)$ , for all  $\mathbf{j} \in \mathbb{N}^n$ . Then, each  $(\Psi_{\mathbf{j}})_{\pm}$  has  $\hat{\Psi}_{\mathbf{j}}$  as Gevrey-1 asymptotic expansion in  $\mathcal{S}_{\pm}$ .  $\square$

As a consequence of Proposition 2.24 and Lemma 2.23, we have an analogue version of Corollary (2.21) in the weak 1-summable case. Consider  $\hat{Y}$  a formal singular vector field at the origin and a formal fibered diffeomorphism  $\hat{\varphi} : (x, \mathbf{y}) \mapsto (x, \hat{\varphi}(x, \mathbf{y}))$  such that  $\hat{\varphi}(x, \mathbf{0}) = \mathbf{0}$ . Assume that both  $\hat{Y}$  and  $\hat{\varphi}$  are weakly 1-summable in directions  $\theta$  and  $\theta - \pi$ , for some  $\theta \in \mathbb{R}$ , and denote by  $Y_+, Y_-$  (resp.  $\varphi_+, \varphi_-$ ) their weak 1-sums in directions  $\theta$  and  $\theta - \pi$  respectively.

**Corollary 2.25.** *Under the assumptions above,  $\hat{\varphi}_* (\hat{Y})$  is weakly 1-summable in both directions  $\theta$  and  $\theta - \pi$ , and its 1-sums in these directions are  $(\varphi_+)_* (Y_+)$  and  $(\varphi_-)_* (Y_-)$  respectively.*

#### 2.4. 1-summability implies weakly 1-summability.

Any function  $f(x, \mathbf{y})$  analytic in a domain  $\mathcal{U} \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ , where  $\mathcal{U} \subset \mathbb{C}$  is open, and bounded in any domain  $\mathcal{U} \times \overline{\mathbf{D}}(\mathbf{0}, \mathbf{r}')$  with  $r'_1 < r_1, \dots, r'_n < r_n$ , can be written

$$(2.2) \quad f(x, \mathbf{y}) = \sum_{\mathbf{j} \in \mathbb{N}^n} F_{\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}} \quad ,$$

where for all  $\mathbf{j} \in \mathbb{N}^n$ ,  $F_{\mathbf{j}}$  is analytic and bounded on  $\mathcal{U}$ , and defined via the Cauchy formula:

$$F_{\mathbf{j}}(x) = \frac{1}{(2i\pi)^n} \int_{|z_1|=r'_1} \dots \int_{|z_n|=r'_n} \frac{f(x, \mathbf{z})}{(z_1)^{j_1+1} \dots (z_n)^{j_n+1}} dz_n \dots dz_1 \quad .$$

Notice that the convergence of the series (2.2) above is uniform in every compact with respect to  $x$  and  $\mathbf{y}$ .

In the same way, any formal power series  $\hat{f}(x, \mathbf{y}) \in \mathbb{C}[[x, \mathbf{y}]]$  can be written as

$$\hat{f}(x, \mathbf{y}) = \sum_{\mathbf{j} \in \mathbb{N}^n} \hat{F}_{\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}} \quad ,$$

and then:

$$\mathcal{B}(\hat{f})(t, \mathbf{y}) = \sum_{\mathbf{j} \in \mathbb{N}^n} \mathcal{B}(\hat{F}_{\mathbf{j}})(t) \mathbf{y}^{\mathbf{j}} \quad .$$

The next lemma is immediate.

#### Lemma 2.26.

- (1) *The power series  $\mathcal{B}(\hat{f})(t, \mathbf{y})$  is convergent in a neighborhood of the origin in  $\mathbb{C}^{n+1}$  if and only if the  $\mathcal{B}(\hat{F}_{\mathbf{j}})$ ,  $\mathbf{j} \in \mathbb{N}^n$ , are all analytic and bounded in a same disc  $D(0, \rho)$ ,  $\rho > 0$ , and if there exists  $B, L > 0$  such that for all  $\mathbf{j} \in \mathbb{N}^n$ ,  $\sup_{t \in D(0, \rho)} |\mathcal{B}(\hat{F}_{\mathbf{j}})(t)| \leq L.B^{|\mathbf{j}|}$ .*
- (2) *Moreover, in the previous case,  $\mathcal{B}(\hat{f})$  can be analytically continued to a domain  $\mathcal{A}_{\theta, \delta} \times \mathbf{D}(\mathbf{0}, \mathbf{r})$  if and only if for all  $\mathbf{j} \in \mathbb{N}^n$ ,  $\mathcal{B}(\hat{F}_{\mathbf{j}})$  can be analytically continued to  $\mathcal{A}_{\theta, \delta}^{\infty}$  and if for all compact  $K \subset \mathcal{A}_{\theta, \delta}^{\infty}$ , there exists  $B, L > 0$  such that for all  $\mathbf{j} \in \mathbb{N}^n$ ,  $\sup_{t \in K} |\mathcal{B}(\hat{F}_{\mathbf{j}})(t)| \leq L.B^{|\mathbf{j}|}$ .*
- (3) *Moreover, in the latter case, there exists  $\lambda, M > 0$  such that:*

$$\forall (t, \mathbf{y}) \in \mathcal{A}_{\theta, \delta}^{\infty} \times \mathbf{D}(\mathbf{0}, \mathbf{r}), \quad |\mathcal{B}(\hat{f})(t, \mathbf{y})| \leq M. \exp(\lambda |t|)$$

*if and only if there exists  $\lambda, L, B > 0$  such that for all  $\mathbf{j} \in \mathbb{N}^n$ ,*

$$\forall t \in \mathcal{A}_{\theta, \delta}^{\infty}, \quad |\mathcal{B}(\hat{F}_{\mathbf{j}})(t)| \leq L.B^{|\mathbf{j}|} \exp(\lambda |t|) \quad .$$

*Remark 2.27.* The first point in the above lemma states that the formal power series is Gevrey-1. The third point gives a link between weak 1-summability and 1-summability. As usual, there exists an alternative result for the second Borel transform  $\tilde{B}$  and the norm  $\|\cdot\|_{\lambda, \theta, \delta, \pi, \mathbf{r}}^{\text{bis}}$ .

**Corollary 2.28.** *Let*

$$\hat{f}(x, \mathbf{y}) = \sum_{\mathbf{j} \in \mathbb{N}^n} \hat{F}_{\mathbf{j}}(x) \mathbf{y}^{\mathbf{j}} \in \mathbb{C}[[x, \mathbf{y}]]$$

be a formal power series. Then,  $\hat{f}$  is 1-summable in the direction  $\theta \in \mathbb{R}$ , of 1-sum  $f \in \mathcal{O}(\mathcal{S}_{\theta, \pi})$ , if and only if the following two conditions hold:

- $\hat{f}$  is weakly 1-summable in the direction  $\theta$
- there exists  $\lambda, \delta, \rho$  such that  $\forall \mathbf{j} \in \mathbb{N}^n$ ,  $\|\hat{F}_{\mathbf{j}}\|_{\lambda, \theta, \delta, \rho} < \infty$  (resp.  $\|\hat{F}_{\mathbf{j}}\|_{\lambda, \theta, \delta, \rho}^{\text{bis}} < +\infty$ ) and such that the power series  $\sum_{\mathbf{j} \in \mathbb{N}^n} \|\hat{F}_{\mathbf{j}}\|_{\lambda, \theta, \delta, \rho} \mathbf{y}^{\mathbf{j}}$  (resp.  $\sum_{\mathbf{j} \in \mathbb{N}^n} \|\hat{F}_{\mathbf{j}}\|_{\lambda, \theta, \delta, \rho}^{\text{bis}} \mathbf{y}^{\mathbf{j}}$ ) is convergent in some polydisc  $\mathbf{D}(\mathbf{0}, \mathbf{r})$ .

*Proof.* This is an immediate consequence of Lemma 2.26.  $\square$

*Remark 2.29.* Notice that there exist formal power series weakly 1-sommable, but which are not Gevrey-1: for instance, the formal power series

$$\hat{f} := \sum_j \hat{F}_j(x) y^j,$$

where for all  $j \in \mathbb{N}$ ,  $\hat{F}_j(x)$  is such that  $\mathcal{B}(\hat{F}_j)(t) = \frac{1}{t + \frac{1}{j}}$ , is weakly 1-sommable in direction  $0 \in \mathbb{R}/2\pi\mathbb{Z}$ ,

but is not Gevrey-1 since  $\mathcal{B}(\hat{F}_j)$  has a pole in  $-\frac{1}{j} \xrightarrow{j \rightarrow +\infty} 0$ .

## 2.5. Some useful tools on singular linear differential equations.

For future reuse, we give here two results on the 1-summability of formal solutions of some singular linear differential equations with 1-summable right hand side, which generalize a similar result proved in [MR82] (Proposition p. 126): the authors work with a norm  $\|\cdot\|_{\beta}$ , but we will need to use the norm  $\|\cdot\|_{\beta}^{\text{bis}}$  in the proof of Proposition 3.14.

**Proposition 2.30.** *Let  $\hat{b}$  be a formal power series 1-summable in the direction  $\theta$ , consider an associated domain  $\Delta_{\theta, \delta, \rho}$  as in Definition 2.9 for some  $\delta, \rho > 0$  and let  $b_{\theta}$  be its 1-sum. Let also  $\alpha, k \in \mathbb{C}$ .*

- (1) *Assume that  $\|\hat{b}\|_{\beta}^{\text{bis}} < +\infty$ , and let  $k \in \mathbb{C} \setminus \{0\}$  be such that  $d_k := \text{dist}(-k, \Delta_{\theta, \delta, \rho}) > 0$  and*

$$\beta d_k > C |\alpha k|,$$

for some constant  $C > 0$  large enough, independent of  $k, \beta, \theta, \rho$  (for instance, one can take  $C = \frac{2 \exp(2)}{5} + 5$ ). Then the singular (irregular) linear differential equation

$$(2.3) \quad x^2 \frac{da}{dx}(x) + (1 + \alpha x) ka(x) = \hat{b}(x)$$

has a unique formal solution  $\hat{a}$  such that  $\hat{a}(0) = \frac{1}{k} \hat{b}(0)$ . Moreover,  $\hat{a}$  is 1-summable in the direction  $\theta$ , and

$$(2.4) \quad \|\hat{a}\|_{\beta}^{\text{bis}} \leq \frac{\beta}{\beta d_k - C |\alpha k|} \|\hat{b}\|_{\beta}^{\text{bis}}.$$

Finally, the 1-sum  $a_{\theta}$  of  $\hat{a}$  in the direction  $\theta$  is the only solution of

$$x^2 \frac{da_{\theta}}{dx}(x) + (1 + \alpha x) ka_{\theta}(x) = b_{\theta}(x)$$

which is bounded in some  $S_{\theta, \pi} \in \mathcal{S}_{\theta, \pi}$ .

(2) Assume  $\|\hat{b}\|_\beta < +\infty$  and  $\Re(k) > 0$ . Then the regular differential equation

$$(2.5) \quad x \frac{da}{dx}(x) + ka(x) = \hat{b}(x)$$

admits a unique formal solution  $a(x) = \sum_{j \geq 0} a_j x^j \in \mathbb{C}[[x]]$  which is also 1-summable in the direction  $\theta$ , of 1-sum  $a_\theta$ . Moreover,  $a_\theta$  is the only germ of solution of the differential equation

$$x \frac{da}{dx}(x) + ka(x) = b_\theta(x)$$

which is bounded in some  $S_{\theta, \pi} \in \mathcal{S}_{\theta, \pi}$ .

*Proof.*

1. Since  $\hat{b}$  is 1-summable in the direction  $\theta$ , we can choose  $\rho > 0$  and  $\delta$  such that  $\tilde{\mathcal{B}}(\hat{b})$  can be analytically continued to (and is bounded in) any domain of the form  $\Delta_{\theta, \delta, \rho} \cap \overline{\mathbb{D}}(0, R)$ ,  $R > 0$ . Let us apply the Borel transform to equation (2.3): we easily obtain

$$(2.6) \quad (t+k)\tilde{\mathcal{B}}(\hat{a})(t) + \alpha k \int_0^t \tilde{\mathcal{B}}(\hat{a})(s) ds = \tilde{\mathcal{B}}(\hat{b})(t) \quad .$$

The derivative with respect to  $t$  of this equation shows that  $\tilde{\mathcal{B}}(\hat{a})$  is solution of a linear differential equation, with only one (regular) singularity at  $t = -k$  (but this singularity is not in  $\Delta_{\theta, \delta, \rho}$  by assumption):

$$(t+k) \frac{d\tilde{\mathcal{B}}(\hat{a})}{dt}(t) + (\alpha k + 1)\tilde{\mathcal{B}}(\hat{a})(t) = \frac{d\tilde{\mathcal{B}}(\hat{b})}{dt}(t) \quad .$$

Since  $\tilde{\mathcal{B}}(\hat{b})$  can be analytically continued to  $\Delta_{\theta, \delta, \rho}$ , the same goes for  $\frac{d\tilde{\mathcal{B}}(\hat{b})}{dt}(t)$  and also for  $\tilde{\mathcal{B}}(\hat{a})$ .

Since  $\tilde{\mathcal{B}}(\hat{a})(0) = \frac{\tilde{\mathcal{B}}(\hat{b})(0)}{k} = \frac{\hat{b}'(0)}{k}$ , we can write:

$$\begin{aligned} \tilde{\mathcal{B}}(\hat{a})(t) &= (t+k)^{-1-\alpha k} \left( \hat{b}'(0) \cdot k^{\alpha k} + \int_0^t \frac{d\tilde{\mathcal{B}}(\hat{b})}{ds}(s) \cdot (s+k)^{\alpha k} ds \right) \\ &= (t+k)^{-1-\alpha k} \left( \hat{b}'(0) \cdot k^{\alpha k} + \tilde{\mathcal{B}}(\hat{b})(t) \cdot (t+k)^{\alpha k} - \tilde{\mathcal{B}}(\hat{b})(0) \cdot k^{\alpha k} \right. \\ &\quad \left. - \alpha k \int_0^t \tilde{\mathcal{B}}(\hat{b})(s) \cdot (s+k)^{\alpha k-1} ds \right) \\ &= (t+k)^{-1-\alpha k} \left( \tilde{\mathcal{B}}(\hat{b})(t) \cdot (t+k)^{\alpha k} \right. \\ &\quad \left. - \alpha k \int_0^t \tilde{\mathcal{B}}(\hat{b})(s) \cdot (s+k)^{\alpha k-1} ds \right) \\ \tilde{\mathcal{B}}(\hat{a}) &= \frac{\tilde{\mathcal{B}}(\hat{b})(t)}{(t+k)} - \alpha k \cdot (t+k)^{-1-\alpha k} \int_0^t \tilde{\mathcal{B}}(\hat{b})(s) \cdot (s+k)^{\alpha k-1} ds . \end{aligned}$$

The fact that  $\tilde{\mathcal{B}}(\hat{b})$  is bounded in any domain of the form  $\Delta_{\theta, \delta, \rho} \cap \overline{\mathbb{D}}(0, R)$ ,  $R > 0$ , implies that the same goes for  $\tilde{\mathcal{B}}(\hat{a})$ . Let us prove inequality (2.4). For all  $R > 0$ , for all Gevrey-1 series  $\hat{f} \in \mathbb{C}[[x, \mathbf{y}]]$

such that  $\tilde{\mathcal{B}}(\hat{f})$  can be analytically continued to  $\Delta_{\theta, \delta, r}$ , we set:

$$\|\hat{f}\|_{\beta, R}^{\text{bis}} := \sup_{t \in \Delta_{\theta, \delta, \rho} \cap \overline{\mathbb{D}}(0, R)} \left\{ \left| \tilde{\mathcal{B}}(\hat{f})(t) \exp(-\beta|t|) (1 + \beta^2|t|^2) \right| \right\} \in \mathbb{R} \cup \{\infty\} .$$

Notice that  $\|\hat{f}\|_{\beta}^{\text{bis}} = \sup_{R>0} \left\{ \|\hat{f}\|_{\beta, R}^{\text{bis}} \right\}$  for all  $\hat{f}$  as above, and that for all  $R > 0$ ,  $\|\hat{a}\|_{\beta, R}^{\text{bis}} < +\infty$ , since  $\tilde{\mathcal{B}}(\hat{a})$  is bounded in any domain of the form  $\Delta_{\theta, \delta, \rho} \cap \overline{\mathbb{D}}(0, R)$ . Fix some  $R > 0$ , and let  $t \in \Delta_{\theta, \delta, \rho} \cap \overline{\mathbb{D}}(0, R)$ . From equation (2.6) and since  $-k \notin \Delta_{\theta, \delta, \rho}$ , we have

$$\tilde{\mathcal{B}}(\hat{a})(t) = \frac{1}{(t+k)} \left( \tilde{\mathcal{B}}(\hat{b})(t) - \alpha k \int_0^t \tilde{\mathcal{B}}(\hat{a})(s) ds \right)$$

so that

$$\begin{aligned} \left| \tilde{\mathcal{B}}(\hat{a})(t) \right| &\leq \frac{1}{|t+k|} \left[ \|\hat{b}\|_{\beta}^{\text{bis}} \frac{\exp(\beta|t|)}{1 + \beta^2|t|^2} + |\alpha k| \cdot \|\hat{a}\|_{\beta, R}^{\text{bis}} \int_0^{|t|} \frac{\exp(\beta u)}{1 + \beta^2 u^2} du \right] \\ &\leq \frac{1}{d_k} \frac{\exp(\beta|t|)}{1 + \beta^2|t|^2} \left[ \|\hat{b}\|_{\beta}^{\text{bis}} + |\alpha k| \|\hat{a}\|_{\beta, R}^{\text{bis}} \frac{C}{\beta} \right] , \end{aligned}$$

with  $C = \frac{2\exp(2)}{5} + 5$  (for instance). We have used here the following elementary fact.

**Fact 2.31.** *There exists a constant  $C > 0$  (e.g.  $C = \frac{2\exp(2)}{5} + 5$ ) such that for all  $\beta > 0$ , we have:*

$$\forall t \geq 0, \int_0^t \frac{\exp(\beta u)}{1 + \beta^2 u^2} du \leq \frac{C \exp(\beta t)}{\beta (1 + \beta^2 t^2)} .$$

*Proof.* Let us consider the function  $F : u \mapsto \frac{\exp(\beta u)}{1 + \beta^2 u^2}$ , for all  $u \geq 0$ . If  $t \in \left[0, \frac{2}{\beta}\right]$ , we have

$$\int_0^t F(u) du \leq \frac{\exp(2)}{5} \cdot \frac{2}{\beta} ,$$

since  $F$  is increasing in  $\mathbb{R}_+$ :

$$F'(u) = \beta F(u) \cdot \frac{(1 - \beta u)^2}{1 + \beta^2 u^2} \geq 0 .$$

Moreover, for all  $t \geq 0$ ,  $F(t) \geq F(0) = 1$ , so that for all  $t \in \left[0, \frac{2}{\beta}\right]$  we have:

$$\int_0^t F(u) du \leq \frac{\exp(2)}{5} \cdot \frac{2}{\beta} \cdot F(t) .$$

If  $t \geq \frac{2}{\beta}$ , then:

$$\int_0^t F(u) du \leq \frac{\exp(2)}{5} \cdot \frac{2}{\beta} F(t) + \int_{\frac{2}{\beta}}^t F(u) du .$$

Notice that, if  $u \geq \frac{\beta}{2}$ , then

$$\frac{(1 - \beta u)^2}{1 + \beta^2 u^2} \geq \frac{1}{5} ,$$

and therefore we have:

$$F'(u) = \beta F(u) \cdot \frac{(1 - \beta u)^2}{1 + \beta^2 u^2} \geq \frac{\beta}{5} F(u) .$$

Hence:

$$\begin{aligned} \int_0^t F(u) \, du &\leq \int_0^{\frac{2}{\beta}} F(u) \, du + \int_{\frac{2}{\beta}}^t F(u) \, du . \\ &\leq \frac{\exp(2)}{5} \cdot \frac{2}{\beta} F(t) + \frac{5}{\beta} \int_{\frac{2}{\beta}}^t F'(u) \, du \\ &\leq \frac{F(t)}{5\beta} \cdot (2 \exp(2) + 25) . \end{aligned}$$

Conclusion: the fact is proved with  $C = \frac{2 \exp(2)}{5} + 5$ .  $\square$

Let us go back to the proof of the lemma. Finally, we have

$$\|\hat{a}\|_{\beta, R}^{\text{bis}} \leq \frac{1}{d_k} \left[ \|\hat{b}\|_{\beta}^{\text{bis}} + \frac{C |\alpha k| \cdot \|\hat{a}\|_{\beta, R}^{\text{bis}}}{\beta} \right] ,$$

and then:

$$\|\hat{a}\|_{\beta, R}^{\text{bis}} \leq \frac{\beta}{\beta d_k - C |\alpha k|} \|\hat{b}\|_{\beta}^{\text{bis}} ,$$

for all  $R > 0$ . As a consequence we have:

$$\|\hat{a}\|_{\beta}^{\text{bis}} \leq \frac{\beta}{\beta d_k - C |\alpha k|} \|\hat{b}\|_{\beta}^{\text{bis}} .$$

2. Let us write  $\hat{b}(x) = \sum_{j \geq 0} b_j x^j$ . A direct computation shows that the only formal solution of equation (2.5) is  $\hat{a}(x) = \sum_{j \geq 0} a_j x^j$  where for all  $j \in \mathbb{N}$ ,  $a_j = \frac{b_j}{j+k}$ . This formula indeed defines a complex number  $a_j$  for  $k \notin \mathbb{Z}_{\leq 0}$ . In particular, we see immediately that  $\hat{a}$  is Gevrey-1, because the same goes for  $\hat{b}$ . In other words, the Borel transform  $\mathcal{B}(\hat{a})$  is analytic in some disc  $D(0, \rho)$ ,  $\rho > 0$ . In  $D(0, \rho)$ ,  $\mathcal{B}(\hat{a})$  satisfies:

$$t \frac{d\mathcal{B}(\hat{a})}{dt}(t) + k \mathcal{B}(\hat{a})(t) = \mathcal{B}(\hat{b})(t) .$$

The general solution near the origin of this equation is

$$y(t) = \frac{c}{t^k} + \frac{1}{t^k} \int_0^t \mathcal{B}(\hat{b})(s) s^{k-1} ds , \quad c \in \mathbb{C} .$$

In particular, the only solution analytic in  $D(0, \rho)$  is the one with  $c = 0$ , *i.e.*

$$\mathcal{B}(\hat{a})(t) = \frac{1}{t^k} \int_0^t \mathcal{B}(\hat{b})(s) s^{k-1} ds .$$

$\mathcal{B}(\hat{b})$  can be analytically continued to an infinite domain, that we have denoted by  $\Delta_{\theta, \delta, \rho}$ , which is bisected by  $\mathbb{R}_+ e^{i\theta}$  (because  $\hat{b}$  is 1-summable in the direction  $\theta$ ). Therefore  $\mathcal{B}(\hat{a})$  can also be analytically continued to the same domain. Moreover, there exists  $\beta > 0$  such that  $\|\hat{b}\|_{\beta} < +\infty$ , *i.e.* for all  $t \in \Delta_{\theta, \delta, \rho}$ :

$$|\mathcal{B}(\hat{b})(t)| \leq \|\hat{b}\|_{\beta} \exp(\beta |t|) .$$

Thus we have:

$$\begin{aligned}
|\exp(-\beta|t|)\mathcal{B}(\hat{a})(t)| &\leq \frac{1}{|t^k|} \int_0^{|t|} |\exp(-\beta|t|)| \left| \mathcal{B}(\hat{b})\left(se^{i\arg(t)}\right) \right| \left| s^{k-1} e^{i(k-1)\arg(t)} \right| ds \\
&\leq \frac{1}{|t|^{\Re(k)}} \int_0^{|t|} |\exp(-\beta s)| \left| \mathcal{B}(\hat{b})\left(se^{i\arg(t)}\right) \right| s^{\Re(k)-1} ds \\
&\leq \frac{\|\hat{b}\|_\beta}{|t|^{\Re(k)}} \int_0^{|t|} s^{\Re(k)-1} ds \\
&= \frac{\|\hat{b}\|_\beta}{\Re(k)}.
\end{aligned}$$

Hence,  $\hat{a}$  is 1-summable in the direction  $\theta$ . □

### 3. 1-SUMMABLE PREPARATION UP TO ARBITRARY ORDER

The aim of this section is to prove that we can always formally conjugate a non-degenerate doubly-resonant saddle-node, which is also div-integrable, to its normal form up to a remainder of order  $O(x^N)$  for every  $N \in \mathbb{N}_{>0}$ . Moreover, we prove that this conjugacy is in fact 1-summable except for  $\arg(\pm\lambda)$ .

**Proposition 3.1.** *Let  $Y \in \mathcal{SN}_{\text{diag}}$  be a non-degenerate diagonal doubly-resonant saddle-node which is div-integrable, with  $D_0Y = \text{diag}(0, -\lambda, \lambda)$ ,  $\lambda \neq 0$ . Then, for all  $N \in \mathbb{N}_{>0}$ , there exists a formal fibered diffeomorphism  $\Psi^{(N)}$  tangent to the identity and 1-summable except for  $\arg(\pm\lambda)$  such that:*

$$\begin{aligned}
(\Psi^{(N)})_*(Y) &= x^2 \frac{\partial}{\partial x} + \left( \left( -\left( \lambda + d^{(N)}(y_1 y_2) \right) + a_1 x \right) + x^N F_1^{(N)}(x, \mathbf{y}) \right) y_1 \frac{\partial}{\partial y_1} \\
&\quad + \left( \left( \lambda + d^{(N)}(y_1 y_2) + a_2 x \right) + x^N F_2^{(N)}(x, \mathbf{y}) \right) y_2 \frac{\partial}{\partial y_2} \\
&=: Y^{(N)},
\end{aligned}$$

where  $\lambda \in \mathbb{C}^*$ ,  $(a_1 + a_2) = \text{res}(Y) \in \mathbb{C} \setminus \mathbb{Q}_{\leq 0}$ ,  $d^{(N)}(v) \in \mathbb{C}\{v\}$  vanishes at the origin, and  $F_1^{(N)}, F_2^{(N)} \in \mathbb{C}[[x, \mathbf{y}]]$  are 1-summable except for  $\arg(\pm\lambda)$ , and of order at least 1 with respect to  $\mathbf{y}$ . Moreover, one can choose  $d^{(2)} = \dots = d^{(N)}$  for all  $N \geq 2$ .

*Remark 3.2.*

- (1) Observe that this result does not require the more restrictive assumptions of being “strictly non-degenerate” (i.e.  $\Re(a_1 + a_2) > 0$ ) defining the set  $\mathcal{SN}_0 \subset \mathcal{SN}_{\text{diag}}$ .
- (2) As a consequence of Corollary 2.21, the 1-sum  $\Psi_\theta^{(N)}$  of  $\Psi^{(N)}$  in a direction  $\theta \neq \arg(\pm\lambda)$  is a germ of sectorial fibered diffeomorphism tangent to the identity, i.e.  $\Psi_\theta^{(N)} \in \text{Diff}_{\text{fib}}(\mathcal{S}_{\theta, \pi}, \text{Id})$ , which conjugates  $Y$  to the 1-sum  $Y_\theta^{(N)}$  of  $Y^{(N)}$  in the direction  $\theta$ .

In order to prove this result we will proceed in several steps and use after each step Proposition 2.20 and Corollary 2.21 in order to prove the 1-summability in every direction  $\theta \neq \arg(\pm\lambda)$  of the different objects. First, we will normalize analytically the vector field restricted to  $\{x=0\}$ . Then, we will straighten the formal separatrix to  $\{y_1 = y_2 = 0\}$  in suitable coordinates. Next, we will simplify the linear terms with respect to  $\mathbf{y}$ . After that, we will straighten two invariant hypersurfaces to  $\{y_1 = 0\}$  and  $\{y_2 = 0\}$ . Finally, we will conjugate the vector field to its final normal form up to remaining terms of order  $O(x^N)$ .

#### 3.1. Analytic normalization on the hyperplane $\{x=0\}$ .



### 3.1.1. The transversally Hamiltonian case.

**Proposition 3.3.** *If  $Y \in \mathcal{SN}_{\text{diag}}$  is transversally Hamiltonian, then  $Y$  is div-integrable.*

*Proof.* Let us consider more generally a diagonal doubly-resonant saddle-node  $Y \in \mathcal{SN}_{\text{diag}}$  such that  $Y|_{\{x=0\}}$  is a Hamiltonian vector field with respect to  $dy_1 \wedge dy_2$  (this is the case if  $Y$  is transversally Hamiltonian): there exists a Hamiltonian  $H(\mathbf{y}) = \lambda y_1 y_2 + O(\|\mathbf{y}\|^3) \in \mathbb{C}\{\mathbf{y}\}$ , such that

$$Y = x^2 \frac{\partial}{\partial x} + \left( \left( -\frac{\partial H}{\partial y_2} + xF_1(x, \mathbf{y}) \right) \frac{\partial}{\partial y_1} + \left( \frac{\partial H}{\partial y_1} + xF_2(x, \mathbf{y}) \right) \frac{\partial}{\partial y_2} \right),$$

where  $F_1, F_2 \in \mathbb{C}\{x, \mathbf{y}\}$  are vanishing at the origin. If we define  $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{C})$  and  $\nabla H := {}^t(DH)$ , then  $Y|_{\{x=0\}} = J\nabla H$ . According to the Morse lemma for holomorphic functions, there exists a germ of an analytic change of coordinates  $\varphi \in \text{Diff}(\mathbb{C}^2, 0)$  given by

$$\mathbf{y} = (y_1, y_2) \mapsto \varphi(\mathbf{y}) = \left( y_1 + O(\|\mathbf{y}\|^2), y_2 + O(\|\mathbf{y}\|^2) \right),$$

such that  $\tilde{H}(\mathbf{y}) := H(\varphi^{-1}(\mathbf{y})) = y_1 y_2$ . Let us now recall a trivial result from linear algebra.

**Fact.** *Let  $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{C})$ , and  $P \in M_2(\mathbb{C})$ . Then,  $PJ^t P = \det(P) J$ .*

As a consequence we have:

**Corollary 3.4.** *Let  $H \in \mathbb{C}\{\mathbf{y}\}$  be a germ of analytic function at 0,  $Y_0 := J\nabla H$  the associated Hamiltonian vector field in  $\mathbb{C}^2$  (for the usual symplectic form  $dy_1 \wedge dy_2$ ), and an analytic diffeomorphism near the origin denoted by  $\varphi$ . Then:*

$$\varphi_*(Y_0) := (D\varphi \circ \varphi^{-1}) \cdot (Y_0 \circ \varphi^{-1}) = \det(D\varphi \circ \varphi^{-1}) J\nabla \tilde{H},$$

where  $\tilde{H} := H \circ \varphi^{-1}$ .

As a conclusion we have proved that  $Y$  is div-integrable. □

### 3.1.2. General case.

Now, we consider a vector field of the form:

$$Y = x^2 \frac{\partial}{\partial x} + (- (\lambda + h(\mathbf{y})) y_1 + xF_1(x, \mathbf{y})) \frac{\partial}{\partial y_1} + ((\lambda + h(\mathbf{y})) y_2 + xF_2(x, \mathbf{y})) \frac{\partial}{\partial y_2},$$

with  $h(\mathbf{y}) \in \mathbb{C}\{\mathbf{y}\}$  vanishing at the origin of  $\mathbb{C}^2$ , and  $F_1, F_2 \in \mathbb{C}\{x, \mathbf{y}\}$  also vanishing at the origin of  $\mathbb{C}^3$ .

**Proposition 3.5.** *Let  $Y_0$  be a germ of analytic vector field in  $(\mathbb{C}^2, 0)$  of the form*

$$Y_0 = (\lambda + h(\mathbf{y})) \left( -y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} \right),$$

with  $h \in \mathbb{C}\{\mathbf{y}\}$  vanishing at the origin. Then there exists  $\phi \in \text{Diff}(\mathbb{C}^2, 0, \text{Id})$  such that

$$\phi_*(Y_0) = (\lambda + d(v)) \left( -y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} \right),$$

with  $v := y_1 y_2$  and  $d \in v\mathbb{C}\{v\}$ .

*Proof.* We claim that  $\phi$  can be taken in the form

$$\phi(\mathbf{y}) = \left( y_1 e^{-\gamma(\mathbf{y})}, y_2 e^{\gamma(\mathbf{y})} \right),$$

for a conveniently chosen  $\gamma \in \mathbb{C}\{\mathbf{y}\}$ . Indeed, let us study how such a diffeomorphism acts on  $Y_0$ . Let us write  $U := (\lambda + h(v))$  and  $L := \left(-y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}\right)$ , such that  $Y_0 = UL$ . An easy computation shows that

$$\begin{aligned}\phi_*(Y_0) &= \phi_*(UL) \\ &= ([U \cdot (1 - \mathcal{L}_L(\gamma))] \circ \phi^{-1}) L ,\end{aligned}$$

where  $\mathcal{L}_L$  is the Lie derivative of associated to  $L$ . We want to find  $\gamma$  such that the unit

$$D := [U(1 - \mathcal{L}_L(\gamma))] \circ \phi^{-1}$$

is free from *non-resonant* terms, *i.e.* is of the form

$$D = \lambda + d(y_1 y_2) = \lambda + \sum_{k \geq 1} d_k (y_1 y_2)^k .$$

Notice that if a unit  $W = \sum_{k \geq 0} W_k (y_1 y_2)^k \in \mathbb{C}\{\mathbf{y}\}^\times$  is free from non-resonant terms, then:

$$\begin{cases} W \circ \phi^{-1} = W \\ \mathcal{L}_L(W) = 0 \end{cases} .$$

Thus, let us split both  $U$  and  $\gamma$  in a “resonant” and a “non-resonant” part:

$$\begin{cases} U = U_{\text{res}} + U_{\text{n-res}} \\ \gamma = \gamma_{\text{res}} + \gamma_{\text{n-res}} \end{cases}$$

where

$$\begin{cases} U_{\text{n-res}} = \sum_{k_1 \neq k_2} U_{k_1, k_2} y_1^{k_1} y_2^{k_2} \\ U_{\text{res}} = \sum_k U_{k, k} (y_1 y_2)^k \\ \gamma_{\text{n-res}} = \sum_{k_1 \neq k_2} \gamma_{k_1, k_2} y_1^{k_1} y_2^{k_2} \\ \gamma_{\text{res}} = \sum_k \gamma_{k, k} (y_1 y_2)^k \end{cases} .$$

Then the non-resonant terms of  $U(1 - \mathcal{L}_L(\gamma))$  are

$$(U_{\text{n-res}} - (U_{\text{n-res}} + U_{\text{res}}) \mathcal{L}_L(\gamma_{\text{n-res}})) \circ \phi^{-1} .$$

Hence, the partial differential equation we want to solve is:

$$\mathcal{L}_L(\gamma) = \frac{U_{\text{n-res}}}{U_{\text{res}} + U_{\text{n-res}}} .$$

One sees immediately that this equation admit an analytic solution (and even infinitely many solutions)  $\gamma \in \mathbb{C}\{\mathbf{y}\}$ , since the unit  $U \in \mathbb{C}\{\mathbf{y}\}$  is analytic.  $\square$

Finally, we can assume that  $Y$  is actually of the form:

$$Y = x^2 \frac{\partial}{\partial x} + (- (\lambda + d(v)) y_1 + x T_1(x, \mathbf{y})) \frac{\partial}{\partial y_1} + ((\lambda + d(v)) y_2 + x T_2(x, \mathbf{y})) \frac{\partial}{\partial y_2} ,$$

with  $v := y_1 y_2$ ,  $d(v) \in v\mathbb{C}\{v\}$  vanishing at the origin of  $\mathbb{C}$ , and  $T_1, T_2 \in \mathbb{C}\{x, \mathbf{y}\}$  both vanishing at the origin of  $\mathbb{C}^3$ .

### 3.2. 1-summable simplification of the “dependent” affine part.

### 3.2.1. Statement.

We are studying vector fields of the form

$$(3.1) \quad Y = x^2 \frac{\partial}{\partial x} + (-\lambda y_1 + f_1(x, \mathbf{y})) \frac{\partial}{\partial y_1} + (\lambda y_2 + f_2(x, \mathbf{y})) \frac{\partial}{\partial y_2} ,$$

with

$$\begin{cases} f_1(x, \mathbf{y}) = -d(y_1 y_2) y_1 + x T_1(x, \mathbf{y}) \\ f_2(x, \mathbf{y}) = d(y_1 y_2) y_2 + x T_2(x, \mathbf{y}) \end{cases} ,$$

where  $d(v) \in v\mathbb{C}\{v\}$  and  $T_1, T_2 \in \mathbb{C}\{x, \mathbf{y}\}$  are of order at least 1.

**Proposition 3.6.** *Let  $Y \in \mathcal{SN}_{\text{diag}}$  be a doubly-resonant saddle-node of the form*

$$Y = x^2 \frac{\partial}{\partial x} + (-\lambda y_1 + f_1(x, \mathbf{y})) \frac{\partial}{\partial y_1} + (\lambda y_2 + f_2(x, \mathbf{y})) \frac{\partial}{\partial y_2} ,$$

where  $f_1, f_2 \in \mathbb{C}\{x, \mathbf{y}\}$  are such that  $f_1(x, \mathbf{y}), f_2(x, \mathbf{y}) = O(\|(x, \mathbf{y})\|^2)$ . Then there exists  $\hat{y}_1, \hat{y}_2, \hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2 \in x\mathbb{C}[[x]]$  which are 1-summable except for  $\arg(\pm\lambda)$ , such that the formal fibered diffeomorphism

$$\hat{\Phi} : (x, y_1, y_2) \mapsto \left( x, \hat{y}_1(x) + (1 + \hat{\alpha}_1(x)) y_1 + \hat{\beta}_1(x) y_2, \hat{y}_2(x) + \hat{\alpha}_2(x) y_1 + (1 + \hat{\beta}_1(x)) y_2 \right) ,$$

(which is tangent to the identity and 1-summable except for  $\arg(\pm\lambda)$ ) conjugates  $Y$  to

$$\hat{\Phi}_*(Y) = x^2 \frac{\partial}{\partial x} + \left( (-\lambda + a_1 x) y_1 + \hat{F}_1(x, \mathbf{y}) \right) \frac{\partial}{\partial y_1} + \left( (\lambda + a_2 x) y_2 + \hat{F}_2(x, \mathbf{y}) \right) \frac{\partial}{\partial y_2} ,$$

where  $a_1, a_2 \in \mathbb{C}$  and  $\hat{F}_1, \hat{F}_2 \in \mathbb{C}[[x, \mathbf{y}]]$  are of order at least 2 with respect to  $\mathbf{y}$ .

*Remark 3.7.* Notice that  $\hat{F}_1, \hat{F}_2$  are 1-summable except for  $\arg(\pm\lambda)$ . Moreover,  $\hat{\Phi}|_{\{x=0\}} = \text{Id}$ , so that  $\hat{F}_i(0, \mathbf{y}) = f_i(0, \mathbf{y})$  for  $i = 1, 2$ . Finally, the residue of  $\hat{\Phi}_*(Y)$  is  $a_1 + a_2$ .

We postpone the proof till subsection 3.2.3. It is based on the next technical lemmas.

### 3.2.2. Technical lemmas.

**Lemma 3.8.** *There exists a pair of formal power series  $(\hat{y}_1(x), \hat{y}_2(x)) \in (x\mathbb{C}[[x]])^2$  which are 1-summable except for  $\arg(\pm\lambda)$ , such that the formal diffeomorphism given by*

$$\hat{\Phi}_1(x, y_1, y_2) = (x, y_1 - \hat{y}_1(x), y_2 - \hat{y}_2(x)),$$

(which is 1-summable except for  $\arg(\pm\lambda)$ ) conjugates  $Y$  in (3.1) to

$$(3.2) \quad \hat{Y}_1(x, \mathbf{y}) = x^2 \frac{\partial}{\partial x} + (-\lambda y_1 + \hat{g}_1(x, \mathbf{y})) \frac{\partial}{\partial y_1} + (\lambda y_2 + \hat{g}_2(x, \mathbf{y})) \frac{\partial}{\partial y_2} ,$$

where  $\hat{g}_1, \hat{g}_2$  are formal power series of order at least 2 such that  $\hat{g}_1(x, \mathbf{0}) = \hat{g}_2(x, \mathbf{0}) = 0$ .

*Remark 3.9.* Notice that  $\hat{g}_1, \hat{g}_2$  are 1-summable except for  $\arg(\pm\lambda)$ .

In other words, in the new coordinates, the curve given by  $(y_1, y_2) = (0, 0)$  is invariant by the flow of the vector field.

*Proof.* This is an immediate consequence of an important theorem by Sibuya on the summability of formal solutions of irregular differential systems [Sib90, p233]. This theorem asserts the existence and the 1-summability except for  $\arg(\pm\lambda)$ , of  $\hat{y}_1$  and  $\hat{y}_2$ , defined as the unique formal solution of

$$\begin{cases} x^2 \frac{dy_1}{dx} = -\lambda y_1(x) + f_1(x, y_1(x), y_2(x)) \\ x^2 \frac{dy_2}{dx} = \lambda y_2(x) + f_2(x, y_1(x), y_2(x)) \end{cases} ,$$

such that  $(\hat{y}_1(0), \hat{y}_2(0)) = (0, 0)$ . The 1-summability of  $\hat{g}_1$  and  $\hat{g}_2$  comes from Proposition 2.20.  $\square$

The next step consists in putting linear terms with respect to  $\mathbf{y}$  in ‘‘diagonal’’ form.

**Lemma 3.10.** *There exists a pair of formal power series  $(\hat{p}_1, \hat{p}_2) \in (\mathbb{C}[[x]])^2$  which are 1-summable except for  $\arg(\pm\lambda)$ , such that the formal fibered diffeomorphism given by*

$$\hat{\Phi}_2(x, y_1, y_2) = (x, y_1 + x\hat{p}_2(x)y_2, y_2 + x\hat{p}_1(x)y_1) ,$$

(which is tangent to the identity and 1-summable except for  $\arg(\pm\lambda)$ ) conjugates  $\hat{Y}_1$  in (3.2), to

$$(3.3) \quad \begin{aligned} \hat{Y}_2(x, \mathbf{y}) &= x^2 \frac{\partial}{\partial x} + \left( (-\lambda + x\hat{a}_1(x))y_1 + \hat{H}_1(x, \mathbf{y}) \right) \frac{\partial}{\partial y_1} \\ &+ \left( (\lambda + x\hat{a}_2(x))y_2 + \hat{H}_2(x, \mathbf{y}) \right) \frac{\partial}{\partial y_2} , \end{aligned}$$

where  $\hat{a}_1, \hat{a}_2, \hat{H}_1, \hat{H}_2$  are formal power series and  $\hat{H}_1, \hat{H}_2$  are of order at least 2 with respect to  $\mathbf{y}$ .

*Remark 3.11.* Notice that  $\hat{a}_1, \hat{a}_2, \hat{H}_1, \hat{H}_2$  are 1-summable except for  $\arg(\pm\lambda)$ .

*Proof.* Let us write

$$\begin{cases} \hat{g}_1(x, \mathbf{z}) = x\hat{b}_1(x)z_1 + x\hat{c}_1(x)z_2 + \hat{G}_1(x, \mathbf{z}) \\ \hat{g}_2(x, \mathbf{z}) = x\hat{c}_2(x)z_1 + x\hat{b}_2(x)z_2 + \hat{G}_2(x, \mathbf{z}) \end{cases} ,$$

where  $\hat{b}_1, \hat{b}_2, \hat{c}_1, \hat{c}_2, \hat{G}_1, \hat{G}_2$  are formal power series 1-summable except for  $\arg(\pm\lambda)$ , such that  $\hat{G}_1$  and  $\hat{G}_2$  are of order at least 2 with respect to  $\mathbf{z}$ . Let us consider the following irregular differential system naturally associated to  $\hat{Y}_1$ :

$$(3.4) \quad x^2 \frac{d\mathbf{z}}{dx}(x) = \hat{\mathbf{B}}(x) \mathbf{z}(x) + \hat{\mathbf{G}}(x, \mathbf{z}(x)) ,$$

where

$$\hat{\mathbf{B}}(x) = \begin{pmatrix} -\lambda + x\hat{b}_1(x) & x\hat{c}_1(x) \\ x\hat{c}_2(x) & \lambda + x\hat{b}_2(x) \end{pmatrix}, \quad \hat{\mathbf{G}}(x, \mathbf{z}(x)) = \begin{pmatrix} \hat{G}_1(x, \mathbf{z}(x)) \\ \hat{G}_2(x, \mathbf{z}(x)) \end{pmatrix} .$$

We are looking for

$$\hat{\mathbf{P}}(x) = \begin{pmatrix} 1 & x\hat{p}_2(x) \\ x\hat{p}_1(x) & 1 \end{pmatrix} \in \text{GL}_2(\mathbb{C}[[x]]) ,$$

where  $\hat{p}_1, \hat{p}_2$  are formal power series in  $x$ , 1-summable except for  $\arg(\pm\lambda)$ , such that the linear transformation given by  $\mathbf{z}(x) = \hat{\mathbf{P}}(x) \mathbf{y}(x)$  changes equation (3.4) to

$$x^2 \frac{d\mathbf{y}}{dx}(x) = \hat{\mathbf{A}}(x) \mathbf{y}(x) + \hat{\mathbf{H}}(x, \mathbf{y}(x)) ,$$

with

$$\hat{\mathbf{A}}(x) = \begin{pmatrix} -\lambda + x\hat{a}_1(x) & 0 \\ 0 & \lambda + x\hat{a}_2(x) \end{pmatrix}, \quad \hat{\mathbf{H}}(x, \mathbf{y}(x)) = \begin{pmatrix} \hat{H}_1(x, \mathbf{y}(x)) \\ \hat{H}_2(x, \mathbf{y}(x)) \end{pmatrix} ,$$

and  $\hat{a}_1, \hat{a}_2, \hat{H}_1, \hat{H}_2$  are formal power series in  $x$ , 1-summable except for  $\arg(\pm\lambda)$ .

We have

$$x^2 \frac{d\mathbf{y}}{dx}(x) = \hat{\mathbf{P}}(x)^{-1} \left( \hat{\mathbf{B}}(x) \hat{\mathbf{P}}(x) - x^2 \frac{d\hat{\mathbf{P}}}{dx}(x) \right) \mathbf{y}(x) + \hat{\mathbf{P}}(x)^{-1} \hat{\mathbf{G}}(x, \hat{\mathbf{P}}(x) \mathbf{y}(x))$$

and we want to determine  $\hat{\mathbf{A}}(x)$  and  $\hat{\mathbf{P}}(x)$  as above so that

$$\hat{\mathbf{B}}(x) \hat{\mathbf{P}}(x) - x^2 \frac{d\hat{\mathbf{P}}}{dx}(x) = \hat{\mathbf{A}}(x) .$$

This gives four equations:

$$(3.5) \quad \begin{cases} \hat{a}_1(x) = \hat{b}_1(x) + x\hat{c}_1(x)\hat{p}_1(x) \\ \hat{a}_2(x) = \hat{b}_2(x) + x\hat{c}_2(x)\hat{p}_2(x) \\ x^2 \frac{d\hat{p}_1}{dx}(x) = \left(2\lambda + x\hat{b}_2(x) - x - x\hat{b}_1(x)\right)\hat{p}_1(x) + \hat{c}_2(x) - x^2\hat{c}_1(x)\hat{p}_1(x)^2 \\ x^2 \frac{d\hat{p}_2}{dx}(x) = \left(-2\lambda + x\hat{b}_1(x) - x - x\hat{b}_2(x)\right)\hat{p}_2(x) + \hat{c}_1(x) - x^2\hat{c}_2(x)\hat{p}_2(x)^2 \end{cases} .$$

Thanks to the theorem by Sibuya on the summability of formal solutions of irregular systems [Sib90], we have the existence and the 1-summability except for  $\arg(\pm\lambda)$ , of  $\hat{p}_1$  and  $\hat{p}_2$ :  $(\hat{p}_1(x), \hat{p}_2(x))$  is defined as the unique formal solution of

$$\begin{cases} x^2 \frac{d\hat{p}_1}{dx}(x) = \left(2\lambda + x\hat{b}_2(x) - x - x\hat{b}_1(x)\right)\hat{p}_1(x) + \hat{c}_2(x) - x^2\hat{c}_1(x)\hat{p}_1(x)^2 \\ x^2 \frac{d\hat{p}_2}{dx}(x) = \left(-2\lambda + x\hat{b}_1(x) - x - x\hat{b}_2(x)\right)\hat{p}_2(x) + \hat{c}_1(x) - x^2\hat{c}_2(x)\hat{p}_2(x)^2 \end{cases}$$

such that

$$(\hat{p}_1(0), \hat{p}_2(0)) = \left( \frac{-\hat{c}_2(0)}{2\lambda}, \frac{\hat{c}_1(0)}{2\lambda} \right) .$$

Notice that  $\hat{a}_1$  and  $\hat{a}_2$  are defined by the first two equations in (3.5). Finally,  $\hat{\mathbf{H}}$  is defined by

$$\hat{\mathbf{H}}(x, \mathbf{y}) := \hat{\mathbf{P}}(x)^{-1} \hat{\mathbf{G}}\left(x, \hat{\mathbf{P}}(x) \mathbf{y}\right) ,$$

and it is 1-summable except for  $\arg(\pm\lambda)$ , by Proposition 2.20.  $\square$

The goal of the last lemma is to transform  $\hat{a}_1(x)$  and  $\hat{a}_2(x)$  in (3.3) into constant terms.

**Lemma 3.12.** *There exists a pair of formal power series  $(\hat{q}_1, \hat{q}_2) \in (\mathbb{C}[[x]])^2$  with  $\hat{q}_1(0) = \hat{q}_2(0) = 1$ , which are 1-summable except for  $\arg(\pm\lambda)$ , such that the formal fibered diffeomorphism of the form*

$$\hat{\Phi}_3(x, y_1, y_2) = (x, \hat{q}_1(x)y_1, \hat{q}_2(x)y_2) ,$$

(which is tangent to the identity and 1-summable except for  $\arg(\pm\lambda)$ ) conjugates  $\hat{Y}_2$  in (3.3), to

$$\begin{aligned} \hat{Y}_3(x, \mathbf{y}) &= x^2 \frac{\partial}{\partial x} + \left( (-\lambda + a_1x)y_1 + \hat{F}_1(x, \mathbf{y}) \right) \frac{\partial}{\partial y_1} \\ &\quad + \left( (\lambda + a_2x)y_2 + \hat{F}_2(x, \mathbf{y}) \right) \frac{\partial}{\partial y_2} , \end{aligned}$$

where  $\hat{F}_1, \hat{F}_2$  are formal power series of order at least 2 with respect to  $\mathbf{y}$  and  $(a_1, a_2) = (\hat{a}_1(0), \hat{a}_2(0))$ .

*Remark 3.13.* Notice that  $\hat{F}_1, \hat{F}_2$  are 1-summable except for  $\arg(\pm\lambda)$ .

*Proof.* We can associate to  $\hat{Y}_2$  the following irregular differential system:

$$x^2 \frac{d\mathbf{z}}{dx}(x) = \hat{\mathbf{A}}(x) \mathbf{z}(x) + \hat{\mathbf{H}}(x, \mathbf{z}(x)) ,$$

and we are looking for a change of coordinates of the form  $\mathbf{z}(x) = \hat{\mathbf{Q}}(x) \mathbf{y}(x)$ , where

$$\hat{\mathbf{Q}}(x) = \begin{pmatrix} \hat{q}_1(x) & 0 \\ 0 & \hat{q}_2(x) \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}[[x]])$$

with  $\hat{q}_1(0) = \hat{q}_2(0) = 1$ , such that the new system is

$$x^2 \frac{d\mathbf{y}}{dx}(x) = \mathbf{A}(x) \mathbf{y}(x) + \hat{\mathbf{F}}(x, \mathbf{y}(x)) ,$$

with

$$\mathbf{A}(x) = \begin{pmatrix} -\lambda + a_1x & 0 \\ 0 & \lambda + a_2x \end{pmatrix}, \quad \hat{\mathbf{F}}(x, \mathbf{y}(x)) = \begin{pmatrix} \hat{F}_1(x, \mathbf{y}(x)) \\ \hat{F}_2(x, \mathbf{y}(x)) \end{pmatrix} ,$$

and  $(a_1, a_2) = (\hat{a}_1(0), \hat{a}_2(0))$ .

We have

$$x^2 \frac{d\mathbf{y}}{dx}(x) = \underbrace{\hat{\mathbf{Q}}(x)^{-1} \left( \hat{\mathbf{A}}(x) \hat{\mathbf{Q}}(x) - x^2 \frac{d\hat{\mathbf{Q}}}{dx}(x) \right)}_{=} \mathbf{y}(x) + \hat{\mathbf{Q}}(x)^{-1} \hat{\mathbf{H}}(x, \hat{\mathbf{Q}}(x)\mathbf{y}(x))$$

$$\begin{pmatrix} -\lambda + a_1 x & 0 \\ 0 & \lambda + a_2 x \end{pmatrix}$$

so that

$$x^2 \frac{d\hat{\mathbf{Q}}}{dx}(x) = \hat{\mathbf{A}}(x) \hat{\mathbf{Q}}(x) - \hat{\mathbf{Q}}(x) \begin{pmatrix} -\lambda + a_1 x & 0 \\ 0 & \lambda + a_2 x \end{pmatrix}$$

and we obtain:

$$\begin{cases} x^2 \frac{d\hat{q}_1}{dx}(x) = x\hat{q}_1(x) (\hat{a}_1(x) - a_1) \\ x^2 \frac{d\hat{q}_2}{dx}(x) = x\hat{q}_2(x) (\hat{a}_2(x) - a_2) \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{d\hat{q}_1}{dx}(x) = \hat{q}_1(x) \left( \frac{\hat{a}_1(x) - a_1}{x} \right) \\ \frac{d\hat{q}_2}{dx}(x) = \hat{q}_2(x) \left( \frac{\hat{a}_2(x) - a_2}{x} \right) \end{cases}$$

$$\Leftrightarrow \begin{cases} \hat{q}_1(x) = \exp \left( \int_0^x \frac{\hat{a}_1(s) - a_1}{s} ds \right) \\ \hat{q}_2(x) = \exp \left( \int_0^x \frac{\hat{a}_2(s) - a_2}{s} ds \right) \end{cases}, \text{ if we set } \hat{q}_1(0) = \hat{q}_2(0) = 1,$$

and the expression  $\int_0^x \frac{\hat{a}_j(s) - a_j}{s} ds$ , for  $j = 1, 2$ , means the unique anti-derivative of  $\frac{\hat{a}_j(s) - a_j}{s}$  vanishing at 0. Since  $\hat{a}_1$  and  $\hat{a}_2$  are 1-summable except for  $\arg(\pm\lambda)$ , the same goes for  $\hat{q}_1$  and  $\hat{q}_2$ , and then for  $\hat{F}_1$  and  $\hat{F}_2$  by Proposition 2.20.  $\square$

### 3.2.3. Proof of Proposition 3.6.

*Proof.* We start by using Lemma 3.8 in order to straighten the center variety to  $\{y_1 = y_2 = 0\}$ , i.e. to remove every monomial term independent of  $\mathbf{y}$  in the component corresponding to  $\frac{\partial}{\partial y_1}$  and  $\frac{\partial}{\partial y_2}$ . Then, we apply Lemma 3.10 in order to put the linear part with respect to  $\mathbf{y}$  into a diagonal form. Finally, we use Lemma 3.12 in order to obtain the final desired form for our vector field. Corollary 2.21 finally provides the 1-summability.  $\square$

### 3.3. 1-summable straightening of two invariant hypersurfaces.

For any  $\theta \in \mathbb{R}$ , we recall that we denote by  $F_\theta$  the 1-sum of a 1-summable series  $\hat{F}$  in direction  $\theta$ . Let  $\theta \in \mathbb{R}$  with  $\theta \neq \arg(\pm\lambda)$  and consider a formal vector field  $\hat{Y}$ , 1-summable in the direction  $\theta$  of 1-sum  $Y_\theta$ , of the form

$$(3.6) \quad \hat{Y} = x^2 \frac{\partial}{\partial x} + \left( \lambda_1(x) y_1 + \hat{F}_1(x, \mathbf{y}) \right) \frac{\partial}{\partial y_1} + \left( \lambda_2(x) y_2 + \hat{F}_2(x, \mathbf{y}) \right) \frac{\partial}{\partial y_2},$$

where:

- $\lambda_1(x) = -\lambda + a_1 x$
- $\lambda_2(x) = \lambda + a_2 x$
- $\lambda \neq 0$
- $a_1, a_2 \in \mathbb{C}$

- for  $j = 1, 2$ ,

$$\hat{F}_j(x, \mathbf{y}) = \sum_{\substack{\mathbf{n} \in \mathbb{N}^2 \\ |\mathbf{n}| \geq 2}} \hat{F}_{\mathbf{n}}^{(j)}(x) \mathbf{y}^{\mathbf{n}} \in \mathbb{C}[[x, \mathbf{y}]]$$

is 1-summable in the direction  $\theta$ , with 1-sum

$$F_{j,\theta}(x, \mathbf{y}) = \sum_{\substack{\mathbf{n} \in \mathbb{N}^2 \\ |\mathbf{n}| \geq 2}} F_{j,\mathbf{n},\theta}(x) \mathbf{y}^{\mathbf{n}} .$$

In particular, there exists  $A, B, \mu > 0$  such that for all  $\mathbf{n} \in \mathbb{N}^2$ ,  $|\mathbf{n}| \geq 2$ , for  $j = 1, 2$ :

$$\forall t \in \Delta_{\theta, \epsilon, \rho}, \left| \tilde{\mathcal{B}}_x \left( \hat{F}_{j,\mathbf{n}} \right) (t) \right| \leq A.B^{|\mathbf{n}|} \frac{\exp(\mu|t|)}{1 + \mu^2|t|^2} ,$$

for some  $\rho > 0$  and  $\epsilon > 0$  such that  $(\mathbb{R}.\lambda) \cap \mathcal{A}_{\theta, \epsilon} = \emptyset$  (see Definition 2.9 for the notations). Notice that  $F_{j,\theta}$  is analytic and bounded in some sectorial neighborhood  $\mathcal{S} \in \mathcal{S}_{\theta, \pi}$  attached to the origin. For technical reasons, we use in this subsection the second Borel transform  $\tilde{\mathcal{B}}$ , and the associated norm  $\|\cdot\|_{\mu}^{\text{bis}}$  (see Remarks 2.8 and 2.10 and Proposition 2.11).

**Proposition 3.14.** *Under the assumptions above, there exists a pair of formal power series  $(\hat{\phi}_1, \hat{\phi}_2) \in (\mathbb{C}[[x, \mathbf{y}]])^2$ , of order at least two with respect to  $\mathbf{y}$ , which are 1-summable except for  $\arg(\pm\lambda)$ , and such that the formal fibered diffeomorphism*

$$\hat{\Phi}(x, \mathbf{y}) = \left( x, y_1 + \hat{\phi}_1(x, \mathbf{y}), y_2 + \hat{\phi}_2(x, \mathbf{y}) \right) ,$$

(which is tangent to the identity and 1-summable except for  $\arg(\pm\lambda)$ ) conjugates  $\hat{Y}$  in (3.6) to

$$\hat{Y}_{\text{prep}} = x^2 \frac{\partial}{\partial x} + \left( (-\lambda + a_1 x) + y_2 \hat{R}_1(x, \mathbf{y}) \right) y_1 \frac{\partial}{\partial y_1} + \left( (\lambda + a_2 x) + y_1 \hat{R}_2(x, \mathbf{y}) \right) y_2 \frac{\partial}{\partial y_2} ,$$

where  $\hat{R}_1, \hat{R}_2 \in \mathbb{C}[[x, \mathbf{y}]]$ .

*Remark 3.15.* Notice that  $\hat{R}_1, \hat{R}_2$  are 1-summable except for  $\arg(\pm\lambda)$ .

*Proof.* We follow and adapt the proof of analytic straightening of invariant curves for resonant saddles in two dimensions in [MM80]. We are looking for

$$\hat{\Psi}(x, \mathbf{y}) = \left( x, y_1 + \hat{\psi}_1(x, \mathbf{y}), y_2 + \hat{\psi}_2(x, \mathbf{y}) \right) ,$$

with  $\hat{\psi}_1, \hat{\psi}_2$  of order at least 2, and  $\hat{R}_1, \hat{R}_2$  as above such that:

$$\hat{\Psi}_* \left( \hat{Y}_{\text{prep}} \right) = \hat{Y} ,$$

*i.e.*

$$(3.7) \quad D\hat{\Psi} \cdot \hat{Y}_{\text{prep}} = \hat{Y} \circ \hat{\Psi} .$$

Then, we will set  $\Phi := \Psi^{-1}$ . Let us write

$$\hat{T}_1 := y_1 y_2 \hat{R}_1 = \sum_{|\mathbf{n}| \geq 2} \hat{T}_{1,\mathbf{n}}(x) \mathbf{y}^{\mathbf{n}}$$

$$\hat{T}_2 := y_1 y_2 \hat{R}_2 = \sum_{|\mathbf{n}| \geq 2} \hat{T}_{2,\mathbf{n}}(x) \mathbf{y}^{\mathbf{n}}$$

$$\hat{\psi}_1 = \sum_{|\mathbf{n}| \geq 2} \hat{\psi}_{1,\mathbf{n}}(x) \mathbf{y}^{\mathbf{n}}$$

$$\hat{\psi}_2 = \sum_{|\mathbf{n}| \geq 2} \hat{\psi}_{2,\mathbf{n}}(x) \mathbf{y}^{\mathbf{n}} ,$$

so that equation (3.7) becomes:

$$\begin{aligned} & x^2 \frac{\partial \hat{\psi}_1}{\partial x^2} + \left(1 + \frac{\partial \hat{\psi}_1}{\partial y_1}\right) (\lambda_1(x) y_1 + \hat{T}_1) + \frac{\partial \hat{\psi}_1}{\partial y_2} (\lambda_2(x) y_2 + \hat{T}_2) \\ &= \lambda_1(x) (y_1 + \hat{\psi}_1) + \hat{F}_1(x, y_1 + \hat{\psi}_1, y_2 + \hat{\psi}_2) \end{aligned}$$

and

$$\begin{aligned} & x^2 \frac{\partial \hat{\psi}_2}{\partial x^2} + \frac{\partial \hat{\psi}_2}{\partial y_1} (\lambda_1(x) y_1 + \hat{T}_1) + \left(1 + \frac{\partial \hat{\psi}_2}{\partial y_2}\right) (\lambda_2(x) y_2 + \hat{T}_2) \\ &= \lambda_2(x) (y_2 + \hat{\psi}_2) + \hat{F}_2(x, y_1 + \hat{\psi}_1, y_2 + \hat{\psi}_2) . \end{aligned}$$

These equations can be written as:

$$(3.8) \quad \left\{ \begin{array}{l} \sum_{|\mathbf{n}| \geq 2} \left( \delta_{1,\mathbf{n}}(x) \hat{\psi}_{1,\mathbf{n}}(x) + x^2 \frac{d\hat{\psi}_{1,\mathbf{n}}}{dx}(x) + \hat{T}_{1,\mathbf{n}}(x) \right) \mathbf{y}^{\mathbf{n}} \\ = \hat{F}_1(x, y_1 + \hat{\psi}_1(x, \mathbf{y}), y_2 + \hat{\psi}_2(x, \mathbf{y})) - \hat{T}_1(x) \frac{\partial \hat{\psi}_1}{\partial y_1}(x, \mathbf{y}) - \hat{T}_2(x) \frac{\partial \hat{\psi}_1}{\partial y_2}(x, \mathbf{y}) \\ =: \sum_{|\mathbf{n}| \geq 2} \zeta_{1,\mathbf{n}}(x) \mathbf{y}^{\mathbf{n}} \\ \sum_{|\mathbf{n}| \geq 2} \left( \delta_{2,\mathbf{n}}(x) \hat{\psi}_{2,\mathbf{n}}(x) + x^2 \frac{d\hat{\psi}_{2,\mathbf{n}}}{dx}(x) + \hat{T}_{2,\mathbf{n}}(x) \right) \mathbf{y}^{\mathbf{n}} \\ = \hat{F}_2(x, y_1 + \hat{\psi}_1(x, \mathbf{y}), y_2 + \hat{\psi}_2(x, \mathbf{y})) - \hat{T}_1(x) \frac{\partial \hat{\psi}_2}{\partial y_1}(x, \mathbf{y}) - \hat{T}_2(x) \frac{\partial \hat{\psi}_2}{\partial y_2}(x, \mathbf{y}) \\ =: \sum_{|\mathbf{n}| \geq 2} \zeta_{2,\mathbf{n}}(x) \mathbf{y}^{\mathbf{n}} \end{array} \right.$$

where  $\delta_{j,\mathbf{n}}(x) = \lambda_1(x)n_1 + \lambda_2(x)n_2 - \lambda_j(x)$ ,  $j = 1, 2$ . Notice that we are looking for  $\hat{T}_1, \hat{T}_2$  such that

$$\left\{ \begin{array}{l} \hat{T}_{1,\mathbf{n}} = 0 \quad , \text{ if } n_1 = 0 \text{ or } n_2 = 0 \\ \hat{T}_{2,\mathbf{n}} = 0 \quad , \text{ if } n_1 = 0 \text{ or } n_2 = 0 \end{array} \right. ,$$

and that  $\zeta_{j,\mathbf{n}}$ , for  $j = 1, 2$  and  $|\mathbf{n}| \geq 2$ , depends only on the  $\hat{\psi}_{i,\mathbf{k}}$ 's and the  $\hat{F}_{i,\mathbf{k}}$ 's, for  $i = 1, 2$ ,  $|\mathbf{k}| < \mathbf{n}$ . We can then determine the coefficients  $\hat{\psi}_{j,\mathbf{n}}$  and  $\hat{T}_{j,\mathbf{n}}$ ,  $j = 1, 2$ ,  $|\mathbf{n}| \geq 2$ , by induction on  $|\mathbf{n}|$ , setting

$$\left\{ \begin{array}{l} \hat{T}_{1,\mathbf{n}} = 0 \quad , \text{ if } n_1 = 0 \text{ or } n_2 = 0 \\ \hat{T}_{2,\mathbf{n}} = 0 \quad , \text{ if } n_1 = 0 \text{ or } n_2 = 0 \\ \hat{\psi}_{1,\mathbf{n}} = 0 \quad , \text{ if } n_1 \geq 1 \text{ and } n_2 \geq 1 \\ \hat{\psi}_{2,\mathbf{n}} = 0 \quad , \text{ if } n_1 \geq 1 \text{ and } n_2 \geq 1 \end{array} \right. ,$$

and solving for each  $\mathbf{n} = (n_1, n_2) \in \mathbb{N}^2$  with  $|\mathbf{n}| \geq 2$ , the equations

$$\left\{ \begin{array}{l} \delta_{1,\mathbf{n}}(x) \hat{\psi}_{1,\mathbf{n}}(x) + x^2 \frac{d\hat{\psi}_{1,\mathbf{n}}}{dx}(x) = \zeta_{1,\mathbf{n}}(x) \quad , \text{ if } n_1 = 0 \text{ or } n_2 = 0 \\ \delta_{2,\mathbf{n}}(x) \hat{\psi}_{2,\mathbf{n}}(x) + x^2 \frac{d\hat{\psi}_{2,\mathbf{n}}}{dx}(x) = \zeta_{2,\mathbf{n}}(x) \quad , \text{ if } n_2 = 0 \text{ or } n_2 = 0 \end{array} \right. .$$

**Lemma 3.16.** *There exists  $\beta > 4\pi$ ,  $M > 0$  such that for all  $\mathbf{n} \in \mathbb{N}^2$  with  $|\mathbf{n}| \geq 2$ , and for  $j = 1, 2$ ,  $\|\zeta_{j,\mathbf{n}}\|_{\beta}^{\text{bis}} < +\infty$  and:*

$$\left\| \hat{\psi}_{j,\mathbf{n}} \right\|_{\beta}^{\text{bis}} \leq M \cdot \|\zeta_{j,\mathbf{n}}\|_{\beta}^{\text{bis}} \quad ,$$

where the norm corresponds to the domain  $\Delta_{\theta,\epsilon,\rho}$ .



*Proof.* For  $\mathbf{n} = (n_1, n_2) \in \mathbb{N}^2$  with  $n_1 + n_2 \geq 2$  we want to solve:

$$\begin{cases} \delta_{1,\mathbf{n}}(x) = \lambda_1(x)(n_1 - 1) + \lambda_2(x)n_2 = \lambda(n_2 + 1) + x(-a_1 + a_2n_2) & , \text{ if } n_1 = 0 \text{ or } n_2 = 0 \\ \delta_{2,\mathbf{n}}(x) = \lambda_2(x)(n_2 - 1) + \lambda_1(x)n_1 = -\lambda(n_1 + 1) + x(-a_2 + a_1n_1) & , \text{ if } n_1 = 0 \text{ or } n_2 = 0 \end{cases} .$$

Let us deal with the case of  $\delta_{1,\mathbf{n}}$  (the case of  $\delta_{2,\mathbf{n}}$  is similar). Notice that we are exactly in the situation of Proposition 2.30. In particular, using notations of said proposition, we respectively have:

$$\begin{cases} k = \lambda(n_2 + 1), \alpha = \frac{(-a_1 + a_2n_2)}{\lambda(n_2 + 1)}, \\ d_k = \min \{ |\lambda(n_2 + 1)| - \rho, |\lambda(n_2 + 1)| |\sin(\theta + \epsilon)|, |\lambda(n_2 + 1)| |\sin(\theta - \epsilon)| \} \\ \text{(if } n_1 = 0 \text{)} \end{cases}$$

and

$$\begin{cases} k = -\lambda(n_1 + 1), \alpha = \frac{(-a_2 + a_1n_1)}{-\lambda(n_1 + 1)}, \\ d_k = \min \{ |\lambda(n_1 + 1)| - \rho, |\lambda(n_1 + 1)| |\sin(\theta + \epsilon)|, |\lambda(n_1 + 1)| |\sin(\theta - \epsilon)| \} \\ \text{(if } n_2 = 0 \text{)}. \end{cases}$$

We can chose the domain  $\Delta_{\theta,\epsilon,\rho}$  corresponding to the 1-summability of  $\hat{F}_1$  and  $\hat{F}_2$  with  $0 < \rho < |\lambda|$ , so that  $d_k > 0$ , since  $\epsilon > 0$  is such that  $(\mathbb{R}, \lambda) \cap \mathcal{A}_{\theta,\epsilon} = \emptyset$ . Finally, we chose

$$\beta > \frac{C(|a_1| + |a_2|)}{\min \{ |\lambda| - \rho, |\lambda \sin(\theta + \epsilon)|, |\lambda \sin(\theta - \epsilon)| \}} > 0$$

such that  $\|\hat{F}_1\|_{\beta}^{\text{bis}} < +\infty$  (here  $C = \frac{2\exp(2)}{5} + 5$ ). This choice of  $\beta$  implies that  $\beta d_k > C|\alpha k|$  as required in Proposition 2.30 in both situations  $n_1 = 0$  or  $n_2 = 0$ . Since  $\zeta_{j,\mathbf{n}}$ , for  $j = 1, 2$  and  $|\mathbf{n}| \geq 2$ , depends only on the  $\hat{\psi}_{i,\mathbf{k}}$  and the  $\hat{F}_{i,\mathbf{k}}$ , for  $i = 1, 2$ ,  $|\mathbf{k}| < \mathbf{n}$ , we deduce by induction that

$$\begin{cases} \|\zeta_{1,\mathbf{n}}\|_{\beta}^{\text{bis}} < +\infty & , \text{ if } n_1 = 0 \text{ or } n_2 = 0 \\ \|\zeta_{2,\mathbf{n}}\|_{\beta}^{\text{bis}} < +\infty & , \text{ if } n_1 = 0 \text{ or } n_2 = 0 \end{cases}$$

and then, thanks to Proposition 2.30:

$$\|\hat{\psi}_{1,\mathbf{n}}\|_{\beta}^{\text{bis}} \leq \left( \frac{\beta}{\beta(|\lambda| - \rho) - C(|a_1| + |a_2|)} \right) \cdot \|\zeta_{1,\mathbf{n}}\|_{\beta}^{\text{bis}} .$$

The lemma is proved, with

$$M = \left( \frac{\beta}{\beta \min \{ |\lambda| - \rho, |\lambda \sin(\theta + \epsilon)|, |\lambda \sin(\theta - \epsilon)| \} - C(|a_1| + |a_2|)} \right) .$$

□

In order to finish the proof of Proposition 3.14, we have to prove that for  $j = 1, 2$ , the series  $\overline{\hat{\psi}}_j := \sum_{\mathbf{n} \in \mathbb{N}^2} \|\hat{\psi}_{j,\mathbf{n}}\|_{\beta} \mathbf{y}^{\mathbf{n}}$  is convergent in a poly-disc  $\mathbf{D}(\mathbf{0}, \mathbf{r})$ , with  $\mathbf{r} = (r_1, r_2) \in (\mathbb{R}_{>0})^2$ . We will prove

this by using a method of dominant series. Let us introduce some useful notations. Let  $(\mathfrak{B}, \|\cdot\|)$  be a Banach algebra, such that for all  $f, g \in \mathfrak{B}$ , we have:

$$\|fg\| \leq \|f\| \|g\| .$$

For all formal power series  $f(\mathbf{y}) = \sum_{\mathbf{n}} f_{\mathbf{n}} \mathbf{y}^{\mathbf{n}} \in \mathfrak{B}[[\mathbf{y}]]$ , we define  $\bar{f} := \sum_{\mathbf{n}} \|f_{\mathbf{n}}\| \mathbf{y}^{\mathbf{n}}$ , and  $\bar{f}(y) := \bar{f}(y, y)$ .

If  $g = \sum_{\mathbf{n}} g_{\mathbf{n}} \mathbf{y}^{\mathbf{n}} \in \mathfrak{B}[[\mathbf{y}]]$  is another formal power series, we write  $\bar{f} \prec \bar{g}$  if for all  $\mathbf{n} \in \mathbb{N}^2$ , we have  $\|f_{\mathbf{n}}\| \leq \|g_{\mathbf{n}}\|$ . The main result needed is the following lemma (see *e.g.* [RR11]).

**Lemma 3.17.** [MM80] For  $j = 1, 2$ , let  $f_j = \sum_{|\mathbf{n}| \geq 2} f_{j, \mathbf{n}} \mathbf{y}^{\mathbf{n}} \in \mathfrak{B}[[\mathbf{y}]]$  be two formal power series with coefficients in a Banach algebra  $(\mathfrak{B}, \|\cdot\|)$  as above, and of order at least two. Consider also two other series  $g_j = \sum_{|\mathbf{n}| \geq 2} g_{j, \mathbf{n}} \mathbf{y}^{\mathbf{n}} \in \mathfrak{B}\{\mathbf{y}\}$ ,  $j = 1, 2$ , of order at least two, which have a non-zero radius of convergence at the origin. Assume that there exists  $\sigma > 0$  such that for  $j = 1, 2$ :

$$\sigma \overline{f_j} \prec \overline{g_j}(y_1 + \overline{f_1}, y_2 + \overline{f_2}) \quad .$$

Then,  $f_1$  and  $f_2$  have a non-zero radius of convergence.

*Proof.* The inequality above means implies that

$$\overline{f_j} \prec \frac{1}{\sigma} \overline{g_j}(y + \overline{f_1} + \overline{f_2}, y + \overline{f_1} + \overline{f_2})$$

and then

$$\begin{aligned} \overline{f_1} + \overline{f_2} &\prec \frac{1}{\sigma} \left( \overline{g_1}(y + \overline{f_1} + \overline{f_2}, y + \overline{f_1} + \overline{f_2}) + \overline{g_2}(y + \overline{f_1} + \overline{f_2}, y + \overline{f_1} + \overline{f_2}) \right) \\ &= \frac{1}{\sigma} \left( \overline{g_1}(y + \overline{f_1} + \overline{f_2}) + \overline{g_2}(y + \overline{f_1} + \overline{f_2}) \right) \quad . \end{aligned}$$

Since  $\overline{g_1}, \overline{g_2}$  are convergent at the origin, and of order at least two, there exists  $A, B > 0$  such that for  $j = 1, 2$ :

$$\frac{1}{\sigma} (\overline{g_1} + \overline{g_2}) \prec \frac{Ay^2}{1 - By} \quad ,$$

and therefore

$$\begin{aligned} \frac{1}{y} (\overline{f_1} + \overline{f_2}) &\prec \frac{1}{y} \left( \frac{A(y + \overline{f_1} + \overline{f_2})^2}{1 - B(y + \overline{f_1} + \overline{f_2})} \right) \\ &\prec \frac{Ay \left( \frac{1}{y} (y + \overline{f_1} + \overline{f_2}) \right)^2}{1 - By \left( \frac{1}{y} (y + \overline{f_1} + \overline{f_2}) \right)} \quad . \end{aligned}$$

If we define  $u = \sum_{n \geq 1} u_n y^n := \frac{1}{y} (\overline{f_1} + \overline{f_2})$ , we have just proved that

$$(3.9) \quad u \prec \frac{Ay(1+u)^2}{1 - By(1+u)} \quad .$$

The idea is now to compare  $u$  with the solution  $v = \sum_{n \geq 1} v_n y^n$  of

$$v = \frac{Ay(1+v)^2}{1 - By(1+v)} \quad .$$

It is easy to see that  $v$  is convergent at the origin, for it is solution of a polynomial equation of degree two:

$$-(A+B)v^2 + (1 - (2A+B)y)v - Ay = 0 \quad .$$

Moreover,  $v_1 = A$ , and for each  $n > 1$ ,  $v_n = P_n(v_1, \dots, v_{n-1})$ , where

$$P_n(v_1, \dots, v_{n-1}) = (2A+B)v_{n-1} + (A+B) \sum_{k=1}^{n-1} v_k v_{n-k}$$

is a polynomial in the  $n-1$  variables  $(v_1, \dots, v_{n-1})$  with positive coefficients. If we choose  $A$  such that  $A > u_1$ , it follows from (3.9) that for all  $n > 1$ :

$$u_n \leq P_n(u_1, \dots, u_{n-1})$$

and then,  $v$  by induction on  $n$  and using the fact that  $P_n$  has positive coefficients, that:

$$u_n \leq P_n(u_1, \dots, u_{n-1}) \leq P_n(v_1, \dots, v_{n-1}) = v_n \quad ,$$

and finally  $u \prec v$ . Since  $v$  has a positive radius of convergence, the same goes for  $u$ , and for  $\overline{f}_1$  and  $\overline{f}_2$  as well. The lemma is proved.  $\square$

Taking  $\beta > 4\pi$ , it follows from Proposition 2.11, that for all  $\hat{f}, \hat{g} \in \mathfrak{B}_\beta^{\text{bis}}$ , we have:

$$\|\hat{f}\hat{g}\|_\beta^{\text{bis}} \leq \|\hat{f}\|_\beta^{\text{bis}} \|\hat{g}\|_\beta^{\text{bis}} \quad .$$

If we prove that there exists  $\sigma > 0$  such that for  $j = 1, 2$  we have

$$\sigma \overline{\hat{\psi}}_j \prec \overline{\hat{F}}_j \left( y_1 + \overline{\hat{\psi}}_1, y_2 + \overline{\hat{\psi}}_2 \right) \quad ,$$

the the above lemma allows to conclude. We have already proved that there exists  $M > 0$  such that for  $j = 1, 2$  we have:

$$\|\hat{\psi}_{j,\mathbf{n}}\|_\beta^{\text{bis}} \leq M \cdot \|\zeta_{j,\mathbf{n}}\|_\beta^{\text{bis}} \quad ,$$

where

$$\begin{cases} \zeta_1 := \sum_{|\mathbf{n}| \geq 2} \zeta_{1,\mathbf{n}}(x) \mathbf{y}^{\mathbf{n}} \\ = \hat{F}_1 \left( x, y_1 + \hat{\psi}_1(x, \mathbf{y}), y_2 + \hat{\psi}_2(x, \mathbf{y}) \right) - \hat{T}_1(x) \frac{\partial \hat{\psi}_1}{\partial y_1}(x, \mathbf{y}) - \hat{T}_2(x) \frac{\partial \hat{\psi}_1}{\partial y_2}(x, \mathbf{y}) \\ \zeta_2 := \sum_{|\mathbf{n}| \geq 2} \zeta_{2,\mathbf{n}}(x) \mathbf{y}^{\mathbf{n}} \\ = \hat{F}_2 \left( x, y_1 + \hat{\psi}_1(x, \mathbf{y}), y_2 + \hat{\psi}_2(x, \mathbf{y}) \right) - \hat{T}_1(x) \frac{\partial \hat{\psi}_2}{\partial y_1}(x, \mathbf{y}) - \hat{T}_2(x) \frac{\partial \hat{\psi}_2}{\partial y_2}(x, \mathbf{y}) \end{cases} \quad .$$

If we set  $\sigma := \frac{1}{M}$ , then we have

$$\begin{cases} \sigma \overline{\hat{\psi}}_1 \prec \overline{\zeta}_1 \prec \overline{\hat{F}}_1 \left( x, y_1 + \overline{\hat{\psi}}_1(x, \mathbf{y}), y_2 + \overline{\hat{\psi}}_2(x, \mathbf{y}) \right) + \overline{\hat{T}}_1(x) \frac{\partial \overline{\hat{\psi}}_1}{\partial y_1}(x, \mathbf{y}) + \overline{\hat{T}}_2(x) \frac{\partial \overline{\hat{\psi}}_1}{\partial y_2}(x, \mathbf{y}) \\ \sigma \overline{\hat{\psi}}_2 \prec \overline{\zeta}_2 \prec \overline{\hat{F}}_2 \left( x, y_1 + \overline{\hat{\psi}}_1(x, \mathbf{y}), y_2 + \overline{\hat{\psi}}_2(x, \mathbf{y}) \right) + \overline{\hat{T}}_1(x) \frac{\partial \overline{\hat{\psi}}_2}{\partial y_1}(x, \mathbf{y}) + \overline{\hat{T}}_2(x) \frac{\partial \overline{\hat{\psi}}_2}{\partial y_2}(x, \mathbf{y}) \end{cases} \quad .$$

Moreover, we recall that

$$\begin{cases} \hat{T}_{1,\mathbf{n}} = 0 & , \text{ if } n_1 = 0 \text{ or } n_2 = 0 \\ \hat{T}_{2,\mathbf{n}} = 0 & , \text{ if } n_1 = 0 \text{ or } n_2 = 0 \\ \hat{\psi}_{1,\mathbf{n}} = 0 & , \text{ if } n_1 \geq 1 \text{ and } n_2 \geq 1 \\ \hat{\psi}_{2,\mathbf{n}} = 0 & , \text{ if } n_1 \geq 1 \text{ and } n_2 \geq 1 \end{cases} \quad ,$$

so that we have in fact more precise dominance relations:

$$\begin{cases} \sigma \overline{\hat{\psi}}_1 \prec \overline{\zeta}_1 \prec \overline{\hat{F}}_1 \left( x, y_1 + \overline{\hat{\psi}}_1(x, \mathbf{y}), y_2 + \overline{\hat{\psi}}_2(x, \mathbf{y}) \right) \\ \sigma \overline{\hat{\psi}}_2 \prec \overline{\zeta}_2 \prec \overline{\hat{F}}_2 \left( x, y_1 + \overline{\hat{\psi}}_1(x, \mathbf{y}), y_2 + \overline{\hat{\psi}}_2(x, \mathbf{y}) \right) \end{cases} \quad .$$

It remains to apply the lemma above to conclude.  $\square$

*Remark 3.18.* In the previous proposition, assume that for  $j = 1, 2$ ,

$$\hat{F}_j(x, \mathbf{y}) = \sum_{\mathbf{n} \in \mathbb{N}^2, |\mathbf{n}| \geq 2} \hat{F}_{\mathbf{n}}^{(j)}(x) \mathbf{y}^{\mathbf{n}}$$

in the expression of  $\hat{Y}$  satisfies

$$\begin{cases} \hat{F}_{\mathbf{n}}^{(1)}(0) = 0 & , \forall \mathbf{n} = (n_1, n_2) \mid n_1 + n_2 \geq 2 \text{ and } \begin{pmatrix} n_1 = 0 \text{ or } n_2 = 0 \end{pmatrix} \\ \hat{F}_{\mathbf{n}}^{(2)}(0) = 0 & , \forall \mathbf{n} = (n_1, n_2) \mid n_1 + n_2 \geq 2 \text{ and } \begin{pmatrix} n_1 = 0 \text{ or } n_2 = 0 \end{pmatrix} \end{cases} .$$

Then, the diffeomorphism  $\hat{\Phi}$  in the proposition can be chosen to be the identity on  $\{x = 0\}$ , so that

$$\begin{cases} y_1 y_2 \hat{R}_1(x, \mathbf{y}) & = \hat{F}_1(0, \mathbf{y}) + x \hat{S}_1(x, \mathbf{y}) \\ y_1 y_2 \hat{R}_2(x, \mathbf{y}) & = \hat{F}_2(0, \mathbf{y}) + x \hat{S}_2(x, \mathbf{y}) \end{cases} ,$$

where  $\hat{S}_1, \hat{S}_2$  are 1-summable except for  $\arg(\pm\lambda)$  and  $\hat{F}_1(0, \mathbf{y}), \hat{F}_2(0, \mathbf{y}) \in \mathbb{C}\{\mathbf{y}\}$  are convergent in neighborhood of the origin in  $\mathbb{C}^2$ . Indeed, we easily see by induction on  $|\mathbf{n}| = n_1 + n_2 \geq 2$  that  $\hat{\psi}_1$  and  $\hat{\psi}_2$  can be chosen “divisible” by  $x$ , and that  $\zeta_1, \zeta_2$  are such that  $\zeta_{j,\mathbf{n}}(x)$  is also “divisible” by  $x$  if  $n_1 = 0$  or  $n_2 = 0$ .

### 3.4. 1-summable normal form up to arbitrary order $N$ .

We consider now a formal non-degenerate diagonal doubly-resonant saddle node div-integrable, 1-summable except for  $\arg(\pm\lambda)$ , of the form

$$\begin{aligned} \hat{Y}_{\text{prep}} &= x^2 \frac{\partial}{\partial x} + \left( -\lambda + a_1 x - d(y_1 y_2) + x \hat{S}_1(x, \mathbf{y}) \right) y_1 \frac{\partial}{\partial y_1} \\ &\quad + \left( \lambda + a_2 x + d(y_1 y_2) + x \hat{S}_2(x, \mathbf{y}) \right) y_2 \frac{\partial}{\partial y_2} \quad , \end{aligned}$$

where:

- $\lambda \in \mathbb{C} \setminus \{0\}$ ;
- $\hat{S}_1, \hat{S}_2 \in \mathbb{C}[[x, \mathbf{y}]]$  are of order at least one with respect to  $\mathbf{y}$  and 1-summable except for  $\arg(\pm\lambda)$ ;
- $\text{res}\left(\hat{Y}_{\text{prep}}\right) = a_1 + a_2 \notin \mathbb{Q}_{\leq 0}$ ;
- $d(v) \in v\mathbb{C}\{v\}$  is the germ of an analytic function in  $v := y_1 y_2$  vanishing at the origin.

As usual, we denote by  $Y_{\text{prep},\theta}, S_{1,\theta}, S_{2,\theta}$  the respective 1-sums of  $\hat{Y}, \hat{S}_1, \hat{S}_2$  in the direction  $\theta \neq \arg(\pm\lambda)$ . Let us introduce some useful notations:

$$\hat{Y}_{\text{prep}} = Y_0 + DC + RR \quad ,$$

where

- $\mathcal{C} := -y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}$
- $\mathcal{R} := y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}$
- $Y_0 := \lambda \mathcal{C} + x \left( x \frac{\partial}{\partial x} + a_1 y_1 \frac{\partial}{\partial y_1} + a_2 y_2 \frac{\partial}{\partial y_2} \right)$
- $D(x, \mathbf{y}) = d(y_1 y_2) + x D^{(1)}(x, \mathbf{y}) = d(y_1 y_2) + x \left( \frac{\hat{S}_2 - \hat{S}_1}{2} \right)$  is 1-summable in the direction  $\theta \neq \arg(\pm\lambda)$  of 1-sum  $D_\theta$ : it is called the “*tangential*” part.  $D_\theta$  is also dominated by  $\|\mathbf{y}\| = \max(|y_1|, |y_2|)$  ( $D$  is of order at least 1 with respect to  $\mathbf{y}$ ).
- $R(x, \mathbf{y}) = x R^{(1)}(x, \mathbf{y}) = x \left( \frac{\hat{S}_2 + \hat{S}_1}{2} \right)$  is 1-summable in the direction  $\theta$  of 1-sum  $R_\theta$ : it is called the “*radial*” part.  $R_\theta$  is also dominated by  $\|\mathbf{y}\|_\infty = \max(|y_1|, |y_2|)$  ( $R$  is of order at least one with respect to  $\mathbf{y}$ ).

The following proposition gives a 1-summable normalization up to any order  $N$  with respect to  $x$ .

**Proposition 3.19.** *Let*

$$\hat{Y}_{\text{prep}} = Y_0 + DC + RR$$

*be as above.*

Then for all  $N \in \mathbb{N}_{>0}$  there exist  $d^{(N)}(v) \in \mathbb{C}\{v\}$  of order at least one and  $\Phi^{(N)} \in \text{Diff}_{\text{fib}}(\mathbb{C}^3, 0, \text{Id})$  which conjugates  $\hat{Y}_{\text{prep}}$  (resp. its 1-sums  $Y_{\text{prep}, \theta}$  in the direction  $\theta$ ) to

$$\begin{aligned} Y^{(N)} &= Y_0 + \left( d^{(N)}(y_1 y_2) + x^N D^{(N)}(x, \mathbf{y}) \right) \mathcal{C} + x^N R^{(N)}(x, \mathbf{y}) \mathcal{R} \\ \left( \text{resp. } Y_\theta^{(N)} &= Y_0 + \left( d^{(N)}(y_1 y_2) + x^N D_\theta^{(N)}(x, \mathbf{y}) \right) \mathcal{C} + x^N R_\theta^{(N)}(x, \mathbf{y}) \mathcal{R} \right) \end{aligned} \quad ,$$

where  $D^{(N)}, R^{(N)}$  are 1-summable in the direction  $\theta$ , of order at least one with respect to  $\mathbf{y}$ , of respective 1-sums  $D_\theta^{(N)}, R_\theta^{(N)}$ . Moreover, one can choose  $d^{(2)} = \dots = d^{(N)}$  for all  $N \geq 2$ , and  $d^{(1)} = d$ .

*Proof.* The proof is performed by induction on  $N$ .

- The case  $N = 1$  is the initial situation here, and is already proved with  $\hat{Y}_{\text{prep}} = Y^{(1)}$ .
- Assume that the result holds for  $N \in \mathbb{N}_{>0}$ .
  - (1) We start with the radial part. Let us write

$$R^{(N)}(x, \mathbf{y}) = \sum_{n_1 + n_2 \geq 1} R_{n_1, n_2}^{(N)}(x) y_1^{n_1} y_2^{n_2}$$

and

$$R_{\text{res}}^{(N)}(0, u) = \sum_{k \geq 1} R_{k, k}^{(N)}(0) u^k \quad .$$

We are looking for an analytic solution  $\tau$  of the homological equations:

$$(3.10) \quad \begin{aligned} \mathcal{L}_{Y^{(N)}}(\tau) &= -x^N R^{(N)} + \left( x^{N+1} \tilde{R}^{(N+1)} \right) \circ \Lambda_\tau \\ \mathcal{L}_{Y_\theta^{(N)}}(\tau) &= -x^N R_\theta^{(N)} + \left( x^{N+1} \tilde{R}_\theta^{(N+1)} \right) \circ \Lambda_\tau \end{aligned} \quad ,$$

for a convenient choice of  $\tilde{R}^{(N+1)}, \tilde{R}_\theta^{(N+1)}$ , with

$$\Lambda_\tau(x, \mathbf{y}) := \Phi_{\mathcal{R}}^\tau(x, \mathbf{y}) = (x, y_1 \exp(\tau(x, \mathbf{y})), y_2 \exp(\tau(x, \mathbf{y}))) \quad ,$$

and

$$\tau(x, \mathbf{y}) = x^{N-1} \tau_0(y_1 y_2) + x^N \tau_1(\mathbf{y}) \quad ,$$

where  $\tau_1(\mathbf{y}) = \sum_{j_1 \neq j_2} \tau_{1, j_1 j_2} y_1^{j_1} y_2^{j_2}$ . Notice that  $\Lambda_\tau$  is the formal flow of  $\mathcal{R}$  at formal time  $\tau(x, \mathbf{y})$ .

If we momentarily admit that such an analytic solution  $\tau$  exists, then  $\Lambda_\tau$  is a germ of analytic diffeomorphism fixing the origin, and since it is fibered, its inverse  $\Lambda_\tau^{-1}$  is also fibered. If we consider  $d^{(N)}$  and  $\tilde{D}^{(N)}$  such that

$$\begin{aligned} d^{(N+1)}(z_1 z_2) + x^N \tilde{D}^{(N)}(x, \mathbf{z}) &:= \left( d^{(N)}(y_1 y_2) + x^N D^{(N)}(x, \mathbf{y}) \right) \circ \Lambda_\tau^{-1}(x, \mathbf{z}) \\ d^{(N+1)}(z_1 z_2) + x^N \tilde{D}_\theta^{(N)}(x, \mathbf{z}) &:= \left( d^{(N)}(y_1 y_2) + x^N D_\theta^{(N)}(x, \mathbf{y}) \right) \circ \Lambda_\tau^{-1}(x, \mathbf{z}) \end{aligned} \quad ,$$

then the two equations given in (3.10) become

$$\begin{aligned} (\Lambda_\tau)_* \left( Y^{(N)} \right) &= Y_0 + \left( d^{(N+1)}(z_1 z_2) + x^N \tilde{D}^{(N)}(x, \mathbf{z}) \right) \mathcal{C} \\ &\quad + x^{N+1} \tilde{R}^{(N+1)}(x, \mathbf{z}) \mathcal{R} \\ (\Lambda_\tau)_* \left( Y_\theta^{(N)} \right) &= Y_0 + \left( d^{(N+1)}(z_1 z_2) + x^N \tilde{D}_\theta^{(N)}(x, \mathbf{z}) \right) \mathcal{C} \\ &\quad + x^{N+1} \tilde{R}_\theta^{(N+1)}(x, \mathbf{z}) \mathcal{R} \quad . \end{aligned}$$

Indeed:

$$\begin{aligned} \text{DL}_{\tau} \cdot Y^{(N)} &= \begin{pmatrix} \mathcal{L}_{Y^{(N)}}(x) \\ \mathcal{L}_{Y^{(N)}}(y_1 \exp(\tau(x, \mathbf{y}))) \\ \mathcal{L}_{Y^{(N)}}(y_2 \exp(\tau(x, \mathbf{y}))) \end{pmatrix} \\ &= \begin{pmatrix} x^2 \\ (\mathcal{L}_{Y^{(N)}}(y_1) + y_1 (\mathcal{L}_{Y^{(N)}}(\tau))) \exp(\tau(x, \mathbf{y})) \\ (\mathcal{L}_{Y^{(N)}}(y_2) + y_2 (\mathcal{L}_{Y^{(N)}}(\tau))) \exp(\tau(x, \mathbf{y})) \end{pmatrix} \\ &= \left( Y_0 + \left( d^{(N+1)} + x^N \tilde{D}^{(N)} \right) \mathcal{C} + x^{N+1} \tilde{R}^{(N+1)} \mathcal{R} \right) \circ \Lambda_{\tau}(x, \mathbf{y}) . \end{aligned}$$

These computations also hold with the corresponding 1-sums of formal objects considered here, *i.e.* with  $Y_{\theta}^{(N)}, D_{\theta}^{(N)}, \tilde{D}_{\theta}^{(N)}, \tilde{R}_{\theta}^{(N+1)}$  instead of  $Y^{(N)}, D^{(N)}, \tilde{D}^{(N)}, \tilde{R}^{(N+1)}$  respectively. We use Proposition 2.20 to obtain the 1-summability of the objects defined by compositions.

Let us prove that there exists a germ of analytic function of the form

$$\tau(x, \mathbf{y}) = x^{N-1} \tau_0(y_1 y_2) + x^N \tau_1(\mathbf{y}) \quad ,$$

of order at least 1 with respect to  $\mathbf{y}$  at the origin, with

$$\tau_1(\mathbf{y}) = \sum_{j_1 \neq j_2} \tau_{1, j_1 j_2} y_1^{j_1} y_2^{j_2}$$

satisfying equation (3.10). This equation can be rewritten

$$\begin{aligned} x^2 \frac{\partial \tau}{\partial x} + \left( -\lambda + a_1 x - d^{(N)}(y_1 y_2) - x^N D^{(N)}(x, \mathbf{y}) + x^N R^{(N)}(x, \mathbf{y}) \right) y_1 \frac{\partial \tau}{\partial y_1} \\ + \left( \lambda + a_2 x + d^{(N)}(y_1 y_2) + x^N D^{(N)}(x, \mathbf{y}) + x^N R^{(N)}(x, \mathbf{y}) \right) y_2 \frac{\partial \tau}{\partial y_2} \\ = -x^N R^{(N)} + \left( x^{N+1} \tilde{R}^{(N+1)} \right) \circ \Lambda_{\tau} \quad , \end{aligned}$$

or equivalently

$$\begin{aligned} x^2 \frac{\partial \tau}{\partial x} + a_1 x y_1 \frac{\partial \tau}{\partial y_1} + a_2 x y_2 \frac{\partial \tau}{\partial y_2} + \left( \lambda + d^{(N)}(y_1 y_2) + x^N D^{(N)}(x, \mathbf{y}) \right) \mathcal{L}_{\mathcal{C}}(\tau) \\ + \left( x^N R^{(N)}(x, \mathbf{y}) \right) \mathcal{L}_{\mathcal{R}}(\tau) = -x^N R^{(N)} + \left( x^{N+1} \tilde{R}^{(N+1)} \right) \circ \Lambda_{\tau} \quad . \end{aligned}$$

Let us consider terms of degree  $N$  with respect to  $x$ :

$$(3.11) \quad \begin{aligned} (N-1) \tau_0(y_1 y_2) + (a_1 + a_2 + 2\delta_{N,1} R^{(N)}(0, \mathbf{y})) y_1 y_2 \frac{\partial \tau_0}{\partial u}(y_1 y_2) \\ + (\lambda + d^{(N)}(y_1 y_2)) \mathcal{L}_{\mathcal{C}}(\tau_1) = -R^{(N)}(0, \mathbf{y}) \end{aligned}$$

(here  $\delta_{N,1}$  is the Kronecker notation). So, considering separately the terms of the forms  $y_1^{j_1} y_2^{j_2}$  with  $j_1 = j_2$  on the one hand, and with  $j_1 \neq j_2$  on the other hand, we have

$$\begin{cases} v \left( a_1 + a_2 + 2\delta_{N,1} R_{\text{res}}^{(N)}(0, v) \right) \frac{d\tau_0}{dv}(v) + (N-1) \tau_0(v) = -R_{\text{res}}^{(N)}(0, v) \\ \tau_0(0) = 0 \end{cases}$$

and

$$\begin{cases} \mathcal{L}_{\mathcal{C}}(\tau_1) = \frac{-1}{\lambda + d^{(N)}(y_1 y_2)} \left( (a_1 + a_2 + 2\delta_{N,1} R^{(N)}(0, \mathbf{y})) y_1 y_2 \frac{d\tau_0}{du}(y_1 y_2) \right. \\ \quad \left. + R^{(N)}(0, \mathbf{y}) + (N-1) \tau_0(y_1 y_2) \right) \\ \tau_1(0) = 0 . \end{cases}$$

Since  $R^{(N)}$  is analytic with respect to  $\mathbf{y}$ ,  $R_{\text{res}}^{(N)}(0, v)$  is analytic near  $v = 0$ . Furthermore, as  $R_{\text{res}}^{(N)}(0, 0) = 0$  and  $a_1 + a_2 \notin \mathbb{Q}_{\leq 0}$ , the first of the two equations above has a unique formal solution  $\tau_0$  with  $\tau_0(0)$ , and this solution is convergent in a neighborhood of the origin. Once  $\tau_0$  is determined, there exists a unique formal solution  $\tau_1$  of the second equation satisfying  $\tau_1(\mathbf{y}) = \sum_{j_1 \neq j_2} \tau_{1, j_1 j_2} y_1^{j_1} y_2^{j_2}$ , which is moreover convergent in a neighborhood of

the origin of  $\mathbb{C}^2$ .

Therefore  $\Lambda_\tau$  is a germ of analytic diffeomorphism fixing the origin, fibered, and conjugating  $Y^{(N)}$  (resp.  $Y_\theta^{(N)}$ ) to  $\tilde{Y}^{(N)} := (\Lambda_\tau)_* (Y^{(N)})$  (resp.  $\tilde{Y}_\theta^{(N)} := (\Lambda_\tau)_* (Y_\theta^{(N)})$ ).

Equation (3.11) implies that  $(\mathcal{L}_{Y^{(N)}}(\tau) + x^N R^{(N)})$  and  $(\mathcal{L}_{Y_\theta^{(N)}}(\tau) + x^N R_\theta^{(N)})$  are divisible by  $x^{N+1}$ , so that we can define:

$$\begin{aligned} \tilde{R}^{(N+1)}(x, \mathbf{z}) &:= \left( \frac{\mathcal{L}_{Y^{(N)}}(\tau) + x^N R^{(N)}}{x^2} \right) \circ \Lambda_\tau^{-1}(x, \mathbf{z}) \\ \tilde{R}_\theta^{(N+1)}(x, \mathbf{z}) &:= \left( \frac{\mathcal{L}_{Y_\theta^{(N)}}(\tau) + x^N R_\theta^{(N)}}{x^2} \right) \circ \Lambda_\tau^{-1}(x, \mathbf{z}) \quad . \end{aligned}$$

By Proposition 2.20,  $\tilde{R}^{(N+1)}$  (resp.  $\tilde{D}^{(N)}$ ) is 1-summable in the direction  $\theta$ , of 1-sum  $\tilde{R}_\theta^{(N+1)}$  (resp.  $\tilde{D}_\theta^{(N)}$ ).

Finally, notice that  $d^{(N+1)} \circ \Lambda_\tau(0, \mathbf{y}) = d^{(N)}(y_1, y_2)$ ,  $\tau(0, \mathbf{y}) = 0$  and then  $\Lambda_\tau(0, \mathbf{y}) = (0, y_1, y_2)$  if  $N > 1$ , so that  $d^{(N+1)} = d^{(N)}$  when  $N > 1$ .

(2) Now we deal with the tangential part. Let us write

$$\tilde{D}^{(N)}(x, \mathbf{z}) = \sum_{n_1 + n_2 \geq 1} \tilde{D}_{n_1, n_2}^{(N)}(x) z_1^{n_1} z_2^{n_2}$$

and

$$\tilde{D}_{\text{res}}^{(N)}(0, v) = \sum_{k \geq 1} \tilde{D}_{k, k}^{(N)}(0) v^k \quad .$$

Exactly as in case 1. dealing with the radial part (the computations are actually easier here), we can prove the existence of a germ of an analytic solution  $\sigma$  of the homological equations:

$$(3.12) \quad \begin{aligned} \mathcal{L}_{\tilde{Y}^{(N)}}(\sigma) &= -x^N \tilde{D}^{(N)} + (x^{N+1} D^{(N+1)}) \circ \Gamma_\sigma \\ \mathcal{L}_{\tilde{Y}_\theta^{(N)}}(\sigma) &= -x^N \tilde{D}_\theta^{(N)} + (x^{N+1} D_\theta^{(N+1)}) \circ \Gamma_\sigma \quad , \end{aligned}$$

for a convenient choice of  $D^{(N+1)}, D_\theta^{(N+1)}$ , with

$$\Gamma_\sigma(x, \mathbf{z}) := (x, y_1 \exp(-\sigma(x, \mathbf{z})), y_2 \exp(\sigma(x, \mathbf{z})))$$

and

$$\sigma(x, \mathbf{z}) = x^{N-1} \sigma_0(z_1 z_2) + x^N \sigma_1(\mathbf{z}) \quad ,$$

where  $\sigma_1(\mathbf{z}) = \sum_{j_1 \neq j_2} \sigma_{1, j_1 j_2} z_1^{j_1} z_2^{j_2}$ . Notice that  $\Gamma_\sigma$  is the formal flow of  $\mathcal{C}$  at formal time

$\sigma(x, \mathbf{z})$ .

Again, as in the first case,  $\Gamma_\sigma$  is a germ of analytic diffeomorphism fixing the origin, and since it is fibered, its inverse  $\Gamma_\sigma^{-1}$  is also fibered. If we consider  $R^{(N+1)}$  and  $R_\theta^{(N+1)}$  such that

$$\begin{aligned} R^{(N+1)}(x, \mathbf{y}) &:= \tilde{R}^{(N+1)} \circ \Gamma_\sigma^{-1}(x, \mathbf{y}) \\ R_\theta^{(N+1)}(x, \mathbf{z}) &:= \tilde{R}_\theta^{(N+1)} \circ \Gamma_\sigma^{-1}(x, \mathbf{y}) \quad , \end{aligned}$$

then the two equations given in (3.12) imply that

$$\begin{aligned} (\Gamma_\sigma)_* \left( \tilde{Y}^{(N)} \right) &= Y_0 + \left( d^{(N+1)}(y_1 y_2) + x^{N+1} D^{(N+1)}(x, \mathbf{y}) \right) \mathcal{C} \\ &\quad + x^{N+1} R^{(N+1)}(x, \mathbf{y}) \mathcal{R} \\ (\Gamma_\theta)_* \left( \tilde{Y}_\theta^{(N)} \right) &= Y_0 + \left( d^{(N+1)}(y_1 y_2) + x^{N+1} D_\theta^{(N+1)}(x, \mathbf{y}) \right) \mathcal{C} \\ &\quad + x^{N+1} R_\theta^{(N+1)}(x, \mathbf{y}) \mathcal{R} \quad . \end{aligned}$$

In fact, we have to chose

$$\begin{aligned} D^{(N+1)}(x, \mathbf{y}) &:= \left( \frac{\mathcal{L}_{\tilde{Y}^{(N)}}(\sigma) + x^N \tilde{D}^{(N)}}{x^2} \right) \circ \Gamma_\sigma^{-1}(x, \mathbf{y}) \\ D_\theta^{(N+1)}(x, \mathbf{y}) &:= \left( \frac{\mathcal{L}_{\tilde{Y}_\theta^{(N)}}(\sigma) + x^N \tilde{D}_\theta^{(N)}}{x^2} \right) \circ \Gamma_\sigma^{-1}(x, \mathbf{y}) \quad . \end{aligned}$$

We use Proposition 2.20 to obtain the 1-summability of the objects defined by compositions. □

### 3.5. Proof of Proposition 3.1.

We now give a short proof of Proposition 3.1, using the different results provided in this section.

*Proof.* (of Proposition 3.1)

We just have to use consecutively Proposition 3.5 (applied to  $Y_0 := Y|_{\{x=0\}}$ ), Proposition 3.6, Proposition 3.14 and finally Proposition 3.19, using at each time Corollary 2.21 in order to obtain the directional 1-summability. □

## 4. SECTORIAL ANALYTIC NORMALIZATION

The aim of this section is to prove that for any  $Y \in \mathcal{SN}_{\text{diag},0}$  and for any  $\eta \in [\pi, 2\pi[$ , there exists a pair

$$(\Phi_+, \Phi_-) \in \text{Diff}_{\text{fib}}(\mathcal{S}_{\arg(i\lambda), \eta}, \text{Id}) \times \text{Diff}_{\text{fib}}(\mathcal{S}_{\arg(-i\lambda), \eta}, \text{Id})$$

whose elements analytically conjugate  $Y$  to its normal form  $Y_{\text{norm}}$  (given by Theorem 1.5) in sectorial neighborhoods attached to the origin with wide opening. The existence of sectorial normalizing maps  $\Phi_+$  and  $\Phi_-$  in domains of the form  $\mathcal{S}_+ \in \mathcal{S}_{\arg(i\lambda), \eta}$  and  $\mathcal{S}_- \in \mathcal{S}_{\arg(-i\lambda), \eta}$  for all  $\eta \in [\pi, 2\pi[$ , is equivalent to the existence of a sectorial normalizing map  $\Phi_\theta$  in domains  $\mathcal{S} \in \mathcal{S}_{\theta, \pi}$ , for all  $\theta \in \mathbb{R}$  such that  $\theta \neq \arg(\pm\lambda)$ . At the end of this section we will also prove that  $\Phi_+$  and  $\Phi_-$  both admit the unique formal normalizing map  $\hat{\Phi}$  (given by Theorem 1.5) as weak Gevrey-1 asymptotic expansion in domains  $\mathcal{S}_+ \in \mathcal{S}_{\arg(i\lambda), \eta}$  and  $\mathcal{S}_- \in \mathcal{S}_{\arg(-i\lambda), \eta}$  respectively. In particular, this will prove that  $\hat{\Phi}$  is weakly 1-summable except for  $\arg(\pm\lambda)$ .

We start with a vector field  $Y^{(N)}$  normalized up to order  $N \geq 2$  as in Proposition 3.1. First of all, we prove the existence of germs of sectorial analytic functions  $\alpha_+ \in \mathcal{O}(\mathcal{S}_+)$ ,  $\alpha_- \in \mathcal{O}(\mathcal{S}_-)$ , which are solutions of homological equations of the form:

$$\mathcal{L}_{Y^{(N)}}(\alpha_\pm) = x^{M+1} A_\pm(x, \mathbf{y}) \quad ,$$

where  $M \in \mathbb{N}$  and  $A_\pm \in \mathcal{O}(\mathcal{S}_\pm)$  is the weak 1-sum of a weakly 1-summable formal power series (see Lemma 4.6). In order to construct such solutions, we will integrate some appropriate meromorphic 1-form on asymptotic paths (see subsection 4.4). Once we have these solutions  $\alpha_+, \alpha_-$ , we will construct the desired germs of sectorial diffeomorphisms as the flows of some elementary linear vector fields at “time”  $\alpha_\pm(x, \mathbf{y})$ . After that, we will prove in subsection 4.5 that there exist unique germs of sectorial fibered diffeomorphisms tangent to the identity which conjugate  $Y \in \mathcal{SN}_{\text{diag},0}$  to its normal form, by studying the sectorial isotropies in sectorial domains with wide opening. Finally, we show in subsection



4.6 that these unique germs of sectorial fibered diffeomorphisms tangent to the identity admits the unique formal normalizing map  $\hat{\Phi}$  given by Theorem 1.5 as weak Gevrey-1 asymptotic expansion.

We go on using the notations introduced in subsection 3.4, *i.e.*

- $\lambda \in \mathbb{C}^*$
- $a_1 + a_2 \notin \mathbb{Q}_{\leq 0}$
- $\mathcal{C} := -y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}$
- $\mathcal{R} := y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}$
- $Y_0 := \lambda \mathcal{C} + x \left( x \frac{\partial}{\partial x} + a_1 y_1 \frac{\partial}{\partial y_1} + a_2 y_2 \frac{\partial}{\partial y_2} \right)$ .

We fix, for  $\epsilon \in ]0, \frac{\pi}{2}[$  and  $r > 0$ , two sectors:

$$S_+(r, \epsilon) := S \left( r, \arg(i\lambda) - \frac{\pi}{2} - \epsilon, \arg(i\lambda) + \frac{\pi}{2} + \epsilon \right)$$

and

$$S_-(r, \epsilon) = S \left( r, \arg(-i\lambda) - \frac{\pi}{2} - \epsilon, \arg(-i\lambda) + \frac{\pi}{2} + \epsilon \right).$$

Let us consider a 1-summable non-degenerate div-integrable doubly-resonant saddle-node normalized up to an order  $N + 2 \geq 2$ :

$$\begin{aligned} Y^{(N+2)} &= Y_0 + \left( c(y_1 y_2) + x^{N+2} D^{(N+2)}(x, \mathbf{y}) \right) \mathcal{C} + x^{N+2} R^{(N+2)}(x, \mathbf{y}) \mathcal{R} \\ &\quad \text{(formal)} \\ Y_{\pm}^{(N+2)} &= Y_0 + \left( c(y_1 y_2) + x^{N+2} D_{\pm}^{(N+2)}(x, \mathbf{y}) \right) \mathcal{C} + x^{N+2} R_{\pm}^{(N+2)}(x, \mathbf{y}) \mathcal{R} \\ &\quad \text{(analytic in } S_{\pm}(r, \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r}) \text{)} \end{aligned}$$

where  $D^{(N+2)}, R^{(N+2)}$  are of order at least one with respect to  $\mathbf{y}$ , and 1-summable except for  $\arg(\pm\lambda)$ : their respective 1-sums in the direction  $\arg(\pm i\lambda)$  are  $D_{\pm}^{(N+2)}, R_{\pm}^{(N+2)}$ , which can be analytically extended in  $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$ . In order to have the complete sectorial normalizing map, we have to assume now that our vector field is **strictly non-degenerate**, *i.e.*

$$\boxed{\Re(a_1 + a_2) > 0}.$$

**Proposition 4.1.** *Under the assumptions above, for all  $\eta \in ]\pi, 2\pi[$ , there exist two germs of sectorial fibered diffeomorphisms*

$$\begin{cases} \Psi_+ \in \text{Diff}_{\text{fib}}(S_{\arg(i\lambda), \eta}, \text{Id}) \\ \Psi_- \in \text{Diff}_{\text{fib}}(S_{\arg(-i\lambda), \eta}, \text{Id}) \end{cases}$$

of the form

$$\Psi_{\pm} : (x, \mathbf{y}) \mapsto \left( x, \mathbf{y} + O(\|\mathbf{y}\|^2) \right),$$

which conjugate  $Y_{\pm}^{(N+2)}$  to its formal normal form

$$Y_{\text{norm}} = x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x - c(y_1 y_2)) y_1 \frac{\partial}{\partial y_1} + (\lambda + a_2 x + c(y_1 y_2)) y_2 \frac{\partial}{\partial y_2},$$

where  $c(v) \in v\mathbb{C}\{v\}$  is the germ of an analytic function in  $v := y_1 y_2$  vanishing at the origin. Moreover, we can choose  $\Psi_{\pm}$  above such that

$$\Psi_{\pm}(x, \mathbf{y}) = \text{Id}(x, \mathbf{y}) + x^N \mathbf{P}_{\pm}^{(N)}(x, \mathbf{y}),$$

where  $\mathbf{P}_{\pm}^{(N)} = (0, P_{1,\pm}, P_{2,\pm})$  is analytic in  $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$  (for some  $r > 0$  and  $\epsilon > \frac{\eta}{2}$ ) and of order at least two with respect to  $\mathbf{y}$ .

By combining Propositions 3.1 (with  $N \geq 2$ ) and 4.1 we immediately obtain the following:

**Corollary 4.2.** *Let  $Y \in \mathcal{SN}_{\text{diag},0}$  be a strictly non-degenerate diagonal doubly-resonant saddle-node which is div-integrable. Then, for all  $\eta \in ]\pi, 2\pi[$ , there exist two germs of sectorial fibered diffeomorphisms*

$$\begin{cases} \Phi_+ \in \text{Diff}_{\text{fib}}(\mathcal{S}_{\arg(i\lambda),\eta}, \text{Id}) \\ \Phi_- \in \text{Diff}_{\text{fib}}(\mathcal{S}_{\arg(-i\lambda),\eta}, \text{Id}) \end{cases}$$

tangent to the identity such that:

$$\begin{aligned} (\Phi_{\pm})_*(Y) &= x^2 \frac{\partial}{\partial x} + (-\lambda + a_1 x - c(y_1 y_2)) y_1 \frac{\partial}{\partial y_1} + (\lambda + a_2 x + c(y_1 y_2)) y_2 \frac{\partial}{\partial y_2} \\ &=: Y_{\text{norm}} , \end{aligned}$$

where  $\lambda \in \mathbb{C}^*$ ,  $\Re(a_1 + a_2) > 0$ , and  $c(v) \in v\mathbb{C}\{v\}$  is the germ of an analytic function in  $v := y_1 y_2$  vanishing at the origin.

As already mentioned, we prove later in this section that  $\Phi_+$  and  $\Phi_-$  are unique as germs (see Proposition 1.13), and that they are the weak 1-sums of the unique formal normalizing map  $\hat{\Phi}$  given by Theorem 1.5.

#### 4.1. Proof of Proposition 4.1.

We give here two consecutive propositions which allow to prove Proposition 4.1 as an immediate consequence.

**Proposition 4.3.** *Let  $Y_{\pm}^{(N+2)} = Y_0 + D_{\pm}\mathcal{C} + R_{\pm}\mathcal{R}$ , where*

$$\begin{cases} D_{\pm}(x, \mathbf{y}) = c(y_1 y_2) + x^{N+2} D_{\pm}^{(N+2)}(x, \mathbf{y}) \\ R_{\pm}(x, \mathbf{y}) = x^{N+2} R_{\pm}^{(N+2)}(x, \mathbf{y}) \end{cases} ,$$

with  $N \in \mathbb{N}_{>0}$ ,  $c(v) \in v\mathbb{C}\{v\}$  of order at least one, and  $D_{\pm}^{(N+2)}, R_{\pm}^{(N+2)}$  analytic in  $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$  and dominated by  $\|\mathbf{y}\|_{\infty}$ . Assume that  $\Re(a_1 + a_2) > 0$ .

Then, possibly by reducing  $r > 0$  and the neighborhood  $(\mathbb{C}^2, 0)$ , there exist two sectorial fibered diffeomorphisms  $\varphi_+$  and  $\varphi_-$  in  $S_+(r, \epsilon) \times (\mathbb{C}^2, 0)$  and  $S_-(r, \epsilon) \times (\mathbb{C}^2, 0)$  respectively, which conjugate  $Y_{\pm}^{(N+2)}$  to

$$Y_{\mathcal{C},\pm} := Y_0 + C_{\pm}\mathcal{C} ,$$

where  $C_{\pm}(x, \mathbf{y}) = D_{\pm} \circ \varphi_{\pm}^{-1}(x, \mathbf{z})$ . Moreover we can chose  $\varphi_{\pm}$  to be of the form

$$\varphi_{\pm}(x, \mathbf{y}) = (x, y_1 \exp(\rho_{\pm}(x, \mathbf{y})), y_2 \exp(\rho_{\pm}(x, \mathbf{y}))) ,$$

where  $\rho_{\pm}(x, \mathbf{y}) = x^{N+1} \tilde{\rho}_{\pm}(x, \mathbf{y})$  and  $\tilde{\rho}_{\pm}$  is analytic in  $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$  and dominated by  $\|\mathbf{y}\|_{\infty}$ .

*Remark 4.4.* Notice that  $\varphi_{\pm}^{-1}$  is of the form

$$\varphi_{\pm}^{-1}(x, \mathbf{z}) = (x, z_1 (1 + x^{N+1} \vartheta(x, \mathbf{z})), z_2 (1 + x^{N+1} \vartheta(x, \mathbf{z}))) ,$$

where  $\vartheta$  is analytic in  $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$  and dominated by  $\|\mathbf{z}\|_{\infty}$ . Consequently:

$$C_{\pm}(x, \mathbf{z}) = c(z_1 z_2) + x^{N+1} C_{\pm}^{(N+1)}(x, \mathbf{z}) ,$$

where  $c$  is the same as above and  $C_{\pm}$  is analytic in  $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$  and dominated by  $\|\mathbf{z}\|_{\infty}$ .

**Proposition 4.5.** *Let  $Y_{\mathcal{C},\pm} := Y_0 + C_{\pm}\mathcal{C}$ , where*

$$C_{\pm}(x, \mathbf{z}) = c(z_1 z_2) + x^{N+1} C_{\pm}^{(N+1)}(x, \mathbf{z}) ,$$

with  $N \in \mathbb{N}_{>0}$ ,  $c(v) \in v\mathbb{C}\{v\}$  of order at least one, and  $C_{\pm}^{(N+1)}$  analytic in  $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$  and dominated by  $\|\mathbf{z}\|_{\infty}$ . Assume  $\Re(a_1 + a_2) > 0$ .

Then, possibly by reducing  $r > 0$  and the neighborhood  $(\mathbb{C}^2, 0)$ , there exist two sectorial fibered diffeomorphisms  $\psi_+$  and  $\psi_-$  in  $S_+(r, \epsilon) \times (\mathbb{C}^2, 0)$  and  $S_-(r, \epsilon) \times (\mathbb{C}^2, 0)$  respectively, which conjugate  $Y_{\mathcal{C}, \pm}$  to

$$Y_{\text{norm}} := Y_0 + c(v)\mathcal{C} \quad .$$

Moreover, we can chose  $\psi_{\pm}$  to be of the form

$$\psi_{\pm}(x, \mathbf{z}) = (x, z_1 \exp(-\chi_{\pm}(x, \mathbf{z})), z_2 \exp(\chi_{\pm}(x, \mathbf{z}))) \quad ,$$

where  $\chi_{\pm}(x, \mathbf{z}) = x^N \tilde{\chi}_{\pm}(x, \mathbf{z})$  and  $\tilde{\chi}$  is analytic in  $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$  and dominated by  $\|\mathbf{z}\|_{\infty}$ .

If we assume for a moment the two propositions above, the proof of Proposition 4.1 becomes obvious.

*Proof.* (Proposition 4.1)

It is an immediate consequence of the successive application of the previous two propositions, by simply taking  $\Psi_{\pm} = \psi_{\pm} \circ \varphi_{\pm}$  (with the notations above).  $\square$

#### 4.2. Proof of Propositions 4.3 and 4.5.

In order to prove Propositions 4.3 and 4.5, we will need the following lemmas. The first one gives the existence of analytic solutions (in sectorial domains) of a homological equations we need to solve.

**Lemma 4.6.** *Let  $Z_{\pm} := Y_0 + C_{\pm}(x, \mathbf{y})\mathcal{C} + xR_{\pm}^{(1)}(x, \mathbf{y})\mathcal{R}$ , with  $C_{\pm}, R_{\pm}^{(1)}$  analytic in  $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$  and dominated by  $\|\mathbf{y}\|_{\infty}$  and let also  $A_{\pm}(x, \mathbf{y})$  be analytic in  $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$  and dominated by  $\|\mathbf{y}\|_{\infty}$ . Then for all  $M \in \mathbb{N}_{>0}$ , possibly by reducing  $r > 0$  and the neighborhood  $(\mathbb{C}^2, 0)$ , there exists a solution  $\alpha_{\pm}$  of the homological equation*

$$(4.1) \quad \mathcal{L}_{Z_{\pm}}(\alpha_{\pm}) = x^{M+1}A_{\pm}(x, \mathbf{y}) \quad ,$$

such that  $\alpha_{\pm}(x, \mathbf{y}) = x^M \tilde{\alpha}_{\pm}(x, \mathbf{y})$ , where  $\tilde{\alpha}_{\pm}$  is a germ of analytic function in  $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$  and dominated by  $\|\mathbf{y}\|_{\infty}$ .

We will prove this lemma in subsection 4.4. The following lemma proves that  $\varphi_{\pm}$  and  $\psi_{\pm}$  constructed in Propositions 4.3 and 4.5 are indeed germs of sectorial fibered diffeomorphisms in domains of the form  $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$ .

**Lemma 4.7.** *Let  $f_{\pm}, g_{\pm}$  be two germs of analytic and bounded functions in  $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$ , which are moreover dominated by  $\|\mathbf{y}\|_{\infty}$ . Then*

$$\phi_{\pm} : (x, \mathbf{y}) \mapsto (x, y_1 \exp(f_{\pm}(x, \mathbf{y})), y_2 \exp(g_{\pm}(x, \mathbf{y})))$$

defines a sectorial fibered diffeomorphism analytic in  $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$  (possibly by reducing  $r > 0$  and the neighborhood  $(\mathbb{C}^2, 0)$ ).

Let us explain why these lemmas imply Propositions 4.3 and 4.5.

*Proof.* (Propositions 4.3 and 4.5)

It is sufficient to apply Lemma 4.6 with  $M = N + 1$ ,  $A_{\pm} = -R_{\pm}^{(N+2)}$ ,  $\alpha_{\pm} = \rho_{\pm}$  and  $Z_{\pm} = Y_{\pm}^{(N+2)}$  for Proposition 4.3, and with  $M = N$ ,  $A_{\pm} = -C_{\pm}^{(N+1)}$ ,  $\alpha_{\pm} = \chi_{\pm}$  and  $Z_{\pm} = Y_{\mathcal{C}, \pm}$  for Proposition 4.5. Then we use Lemma 4.7 to see that  $\varphi_{\pm}$  and  $\psi_{\pm}$  are germs of sectorial fibered diffeomorphisms on the considered domains, and we finally check that they are the conjugacy we seek. With the notations

above:

$$\begin{aligned}
D\varphi_{\pm} \cdot Y_{\pm}^{(N+2)} &= \begin{pmatrix} \mathcal{L}_{Y_{\pm}^{(N+2)}}(x) \\ \mathcal{L}_{Y_{\pm}^{(N+2)}}(y_1 \exp(\rho_{\pm}(x, \mathbf{y}))) \\ \mathcal{L}_{Y_{\pm}^{(N+2)}}(y_2 \exp(\rho_{\pm}(x, \mathbf{y}))) \end{pmatrix} \\
&= \begin{pmatrix} x^2 \\ \left( \mathcal{L}_{Y_{\pm}^{(N+2)}}(y_1) + y_1 \left( \mathcal{L}_{Y_{\pm}^{(N+2)}}(\rho_{\pm}) \right) \right) \exp(\rho_{\pm}(x, \mathbf{y})) \\ \left( \mathcal{L}_{Y_{\pm}^{(N+2)}}(y_2) + y_2 \left( \mathcal{L}_{Y_{\pm}^{(N+2)}}(\rho_{\pm}) \right) \right) \exp(\rho_{\pm}(x, \mathbf{y})) \end{pmatrix} \\
&= \begin{pmatrix} x^2 \\ (-\lambda + a_1 x - D_{\pm}(x, \mathbf{y})) y_1 \exp(\rho_{\pm}(x, \mathbf{y})) \\ (\lambda + a_2 x + D_{\pm}(x, \mathbf{y})) y_2 \exp(\rho_{\pm}(x, \mathbf{y})) \end{pmatrix} \\
&\quad (\text{we have used } \mathcal{L}_{Y_{\pm}^{(N+2)}}(\rho_{\pm}) = -x^{N+2} R_{\pm}^{(N+2)}) \\
&= (Y_0 + C_{\pm} \mathcal{C}) \circ \varphi_{\pm}(x, \mathbf{y}) \\
&= Y_{C, \pm} \circ \varphi_{\pm}(x, \mathbf{y}),
\end{aligned}$$

so that  $(\varphi_{\pm})_* (Y_{\pm}^{(N+2)}) = Y_{C, \pm}$  and then

$$\begin{aligned}
D\psi_{\pm} \cdot Y_{C, \pm} &= \begin{pmatrix} \mathcal{L}_{Y_{C, \pm}}(x) \\ \mathcal{L}_{Y_{C, \pm}}(z_1 \exp(-\chi(x, \mathbf{z}))) \\ \mathcal{L}_{Y_{C, \pm}}(z_2 \exp(\chi(x, \mathbf{z}))) \end{pmatrix} \\
&= \begin{pmatrix} x^2 \\ \left( \mathcal{L}_{Y_{C, \pm}}(z_1) + z_1 \left( \mathcal{L}_{Y_{C, \pm}}(\chi) \right) \right) \exp(-\chi(x, \mathbf{z})) \\ \left( \mathcal{L}_{Y_{C, \pm}}(z_2) + z_2 \left( \mathcal{L}_{Y_{C, \pm}}(\chi) \right) \right) \exp(\chi(x, \mathbf{z})) \end{pmatrix} \\
&= \begin{pmatrix} x^2 \\ (-\lambda + a_1 x - c(z_1 z_2)) z_1 \exp(-\chi(x, \mathbf{z})) \\ (\lambda + a_2 x + c(z_1 z_2)) z_2 \exp(\chi(x, \mathbf{z})) \end{pmatrix} \\
&\quad (\text{we have used } \mathcal{L}_{Y_{C, \pm}}(\chi_{\pm}) = -x^{N+1} C_{\pm}^{(N+1)}) \\
&= (Y_0 + c(u) \mathcal{C}) \circ \psi_{\pm}(x, \mathbf{z}) \\
&= Y_{\text{norm}} \circ \psi_{\pm}(x, \mathbf{z}),
\end{aligned}$$

so that  $(\psi_{\pm})_* (Y_{C, \pm}) = Y_{\text{norm}}$ . □

### 4.3. Proof of Lemma 4.7.

*Proof.* (Lemma 4.7)

We consider two germs of analytic functions  $f_{\pm}, g_{\pm}$  in  $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$  which are dominated there by  $\|\mathbf{y}\|_{\infty}$ , and we define

$$\phi_{\pm} : (x, \mathbf{y}) \mapsto (x, y_1 \exp(f_{\pm}(x, \mathbf{y})), y_2 \exp(g_{\pm}(x, \mathbf{y}))) .$$

Let us first prove that  $\phi_{\pm}$  is into. Let  $\mathbf{x} = (x, y_1, y_2)$  and  $\mathbf{x}' = (x', y'_1, y'_2)$  in  $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$  such that  $\phi_{\pm}(\mathbf{x}) = \phi_{\pm}(\mathbf{x}')$ . Since  $\phi_{\pm}$  is fibered, necessarily  $x = x'$ . Then assume that  $(y_1, y_2) \neq (y'_1, y'_2)$ , such that

$$\|(y_1 - y'_1, y_2 - y'_2)\|_{\infty} > 0$$

and for instance  $\|(y_1 - y'_1, y_2 - y'_2)\|_{\infty} = |y_1 - y'_1| > 0$  (the other case can be done similarly). We denote by  $D_{\mathbf{y}}$  the derivative with respect to variables  $(y_1, y_2)$ . According to the mean value theorem:

$$\left| \frac{e^{f_{\pm}(\mathbf{x})} - e^{f_{\pm}(\mathbf{x}')}}{y_1 - y'_1} \right| \leq \sup_{(z_1, z_2) \in [(y_1, y_2), (y'_1, y'_2)]} \|D_{\mathbf{y}}(e^{f_{\pm}})(x, z_1, z_2)\|_{\infty} .$$

Consequently we have:

$$\begin{aligned}
0 &= \left| y_1 e^{f_{\pm}(\mathbf{x})} - y'_1 e^{f_{\pm}(\mathbf{x}')} \right| \\
&= \left| e^{f_{\pm}(\mathbf{x})} \cdot |y_1 - y'_1| \cdot \left| 1 + \frac{y'_1}{e^{f_{\pm}(\mathbf{x})}} \cdot \frac{e^{f_{\pm}(\mathbf{x})} - e^{f_{\pm}(\mathbf{x}')}}{y_1 - y'_1} \right| \right| \\
&\geq \left| e^{f_{\pm}(\mathbf{x})} \cdot |y_1 - y'_1| \cdot \left( 1 - \left| \frac{y'_1}{e^{f_{\pm}(\mathbf{x})}} \right| \cdot \left| \frac{e^{f_{\pm}(\mathbf{x})} - e^{f_{\pm}(\mathbf{x}')}}{y_1 - y'_1} \right| \right) \right| \\
&\geq \left| e^{f_{\pm}(\mathbf{x})} \cdot |y_1 - y'_1| \cdot \left( 1 - \left| \frac{y'_1}{e^{f_{\pm}(\mathbf{x})}} \right| \cdot \sup_{(z_1, z_2) \in [(y_1, y_2), (y'_1, y'_2)]} \|D_{\mathbf{y}}(e^{f_{\pm}})(x, z_1, z_2)\|_{\infty} \right) \right|
\end{aligned}$$

Assume that we chose  $(\mathbb{C}^2, 0) = \mathbf{D}(\mathbf{0}, \mathbf{r})$  small enough such that  $f_{\pm}$  is analytic in

$$S_{\pm}(r, \epsilon) \times D(0, 3r_1 + \delta) \times D(0, 3r_2 + \delta)$$

with  $\delta > 0$  small. Without loss of generality we can take  $r_1 = r_2$ . We apply Cauchy's integral formula to  $z_1 \mapsto e^{f_{\pm}(x, z_1, z_2)}$ , for all fixed  $z_2$ , integrating on the circle of center 0 and radius  $3r_1 = 3r_2$ . Similarly we also apply Cauchy's integral formula to  $z_2 \mapsto e^{f_{\pm}(x, z_1, z_2)}$ , for all fixed  $z_1$ , integrating on the circle of center 0 and radius  $3r_2 = 3r_1$ . Then we obtain

$$\sup_{(z_1, z_2) \in [(y_1, y_2), (y'_1, y'_2)]} \|D_{\mathbf{y}}(e^{f_{\pm}})(x, z_1, z_2)\|_{\infty} \leq \frac{3}{4r_1} \cdot \exp \left( \sup_{\mathbf{x} \in S_{\pm}(r, \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r})} (|f_{\pm}(\mathbf{x})|) \right),$$

so that:

$$\begin{aligned}
0 &= \left| y_1 e^{f_{\pm}(\mathbf{x})} - y'_1 e^{f_{\pm}(\mathbf{x}')} \right| \\
&\geq \left| e^{f_{\pm}(\mathbf{x})} \cdot |y_1 - y'_1| \cdot \left( 1 - \frac{3}{4} \exp \left( \sup_{\mathbf{x} \in S_{\pm}(r, \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r})} (2|f_{\pm}(\mathbf{x})|) \right) \right) \right|.
\end{aligned}$$

Since  $f_{\pm}(\mathbf{x}) \xrightarrow{\mathbf{x} \rightarrow \mathbf{0}} 0$ , we can choose  $r, r_1$  and  $r_2$  small enough in order that:

$$\exp \left( \sup_{\mathbf{x} \in S_{\pm}(r, \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r})} (2|f_{\pm}(\mathbf{x})|) \right) \leq \frac{5}{4} < \frac{4}{3}.$$

Finally we obtain:

$$\begin{aligned}
0 &= \left| y_1 e^{f_{\pm}(\mathbf{x})} - y'_1 e^{f_{\pm}(\mathbf{x}')} \right| \\
&\geq \left| e^{f_{\pm}(\mathbf{x})} \right| \frac{|y_1 - y'_1|}{16} > 0,
\end{aligned}$$

and so, if  $y_1 \neq y'_1$ ,  $0 = \left| y_1 e^{f_{\pm}(\mathbf{x})} - y'_1 e^{f_{\pm}(\mathbf{x}')} \right| > 0$ , which is a contradiction.

Conclusion:  $(y_1, y_2) = (y'_1, y'_2)$  and then  $\phi_{\pm}$  is into in  $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$ .

Since  $\phi_{\pm}$  is into and analytic in  $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$ , it is a biholomorphism between  $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$  and its image, which is necessarily open (and of the same form).  $\square$

#### 4.4. Proof of Lemma 4.6.

The goal of this subsection is to prove Lemma 4.6 by studying the existence of paths asymptotic to the singularity and tangent to the foliation, and then to use them to construct the solution of the homological equation (4.1).

For convenience and without lost of generality we assume  $\lambda = 1$  during this subsection (otherwise we can divide our vector field by  $\lambda \neq 0$ , make  $x \mapsto \lambda x$  and finally consider  $\exp(-i \arg(\lambda)) \cdot S_{\pm}(r, \epsilon)$  instead of  $S_{\pm}(r, \epsilon)$ : these modifications do not change  $a_1$  and  $a_2$ ).

4.4.1. *Domain of stability and asymptotic paths.*

We consider

$$\begin{aligned} Z_{\pm} &= Y_0 + C_{\pm}(x, \mathbf{y}) \mathcal{C} + xR_{\pm}^{(1)}(x, \mathbf{y}) \mathcal{R} \\ &= \begin{pmatrix} x^2 \\ y_1 \left( -(1 + C_{\pm}(x, \mathbf{y})) + a_1x + xR_{\pm}^{(1)}(x, \mathbf{y}) \right) \\ y_2 \left( 1 + C_{\pm}(x, \mathbf{y}) + a_2x + xR_{\pm}^{(1)}(x, \mathbf{y}) \right) \end{pmatrix} \end{aligned}$$

with  $\Re(a_1 + a_2) > 0$ , and  $C_{\pm}, R_{\pm}^{(1)}$  analytic in  $S_{\pm}(r, \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$  and dominated by  $\|\mathbf{y}\|_{\infty}$ . More precisely, we consider the Cauchy problem of unknown  $\mathbf{x}(t) := (x(t), y_1(t), y_2(t))$ , with real and positive time  $t \geq 0$ , associated to

$$X_{\pm} := \frac{\pm i}{1 + \left(\frac{a_2 - a_1}{2}\right)x + C_{\pm}} Z_{\pm},$$

i.e.

$$(4.2) \quad \begin{cases} \frac{dx}{dt} = \frac{\pm ix^2}{1 + \left(\frac{a_2 - a_1}{2}\right)x + C_{\pm}} \\ \frac{dy_1}{dt} = \frac{\pm iy_1}{1 + \left(\frac{a_2 - a_1}{2}\right)x + C_{\pm}} \left( -(1 + C_{\pm}(x, \mathbf{y})) + a_1x + xR_{\pm}^{(1)}(x, \mathbf{y}) \right) \\ \frac{dy_2}{dt} = \frac{\pm iy_2}{1 + \left(\frac{a_2 - a_1}{2}\right)x + C_{\pm}} \left( 1 + C_{\pm}(x, \mathbf{y}) + a_2x + xR_{\pm}^{(1)}(x, \mathbf{y}) \right) \\ \mathbf{x}(t) = \mathbf{x}_0 = (x_0, y_{1,0}, y_{2,0}) \in S_{\pm}(r, \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r}) \end{cases} .$$

We denote by  $(t, \mathbf{x}_0) \mapsto \Phi_{X_{\pm}}^t(\mathbf{x}_0)$  the flow of  $X_{\pm}$  at positive time  $t \geq 0$  and with initial point  $\mathbf{x}_0$ :  $\Phi_{X_{\pm}}^0(\mathbf{x}_0) = \mathbf{x}_0$ .

We will prove the following:

**Proposition 4.8.** *For all  $\epsilon \in \left]0, \frac{\pi}{2}\right[$ , there exists finite sectors  $S_{\pm}(r, \epsilon), S_{\pm}(r', \epsilon)$  with  $r, r' > 0$  and an open domain  $\Omega_{\pm}$  stable by the flow of (4.2) at positive time  $t \geq 0$  such that*

$$S_{\pm}(r', \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r}') \subset \Omega_{\pm} \subset S_{\pm}(r, \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r}),$$

for some  $0 < r'_1 < r_1$  and  $0 < r'_2 < r_2$ . Moreover, if  $\mathbf{x}_0 \in \Omega_{\pm}$  then the corresponding solution  $\mathbf{x}(t) := \Phi_{X_{\pm}}^t(\mathbf{x}_0)$  exists for all  $t \geq 0$  and  $\mathbf{x}(t) \rightarrow \mathbf{0}$  as  $t \rightarrow +\infty$ .

*Remark 4.9.* This will prove that the solution  $\mathbf{x}(t)$  of (4.2) exists for all  $t \geq 0$  and accumulates on the origin: it defines a path tangent to the foliation and asymptotic to the origin.

**Definition 4.10.** We define the *asymptotic path* with base point  $\mathbf{x}_0 \in \Omega_{\pm}$  associated to  $X_{\pm}$  as the path  $\gamma_{\pm, \mathbf{x}_0} := \left\{ \Phi_{X_{\pm}}^t(\mathbf{x}_0), t \geq 0 \right\}$ .

For convenience and without loss of generality we only detail the case where “ $\pm = +$ ” (the case where “ $\pm = -$ ” is completely similar).

If we write  $a := a_1 + a_2$  and  $b := \frac{a_2 - a_1}{2}$ , in the case “ $\pm = +$ ” we have:

$$\begin{cases} \frac{dx}{dt} = \frac{ix^2}{1 + bx + C_+} \\ \frac{dy_1}{dt} = iy_1 \left( -1 + \left( \frac{\frac{a}{2} + R_+^{(1)}(x, \mathbf{y})}{1 + bx + C_+(x, \mathbf{y})} \right) x \right) \\ \frac{dy_2}{dt} = \pm iy_2 \left( 1 + \left( \frac{\frac{a}{2} + R_+^{(1)}(x, \mathbf{y})}{1 + bx + C_+(x, \mathbf{y})} \right) x \right) \\ \mathbf{x}(t) = \mathbf{x}_0 = (x_0, y_{1,0}, y_{2,0}) \in S_+(r, \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r}). \end{cases}$$

We also consider the differential equations satisfied by  $|x(t)|$ ,  $|y_1(t)|$ ,  $|y_2(t)|$  and  $\theta(t) := \arg(x(t))$ :

$$\begin{cases} \frac{d|x(t)|}{dt} = |x(t)| \Re \left( \frac{ix(t)}{1+bx(t)+C_+(\mathbf{x}(t))} \right) \\ \frac{d|y_1(t)|}{dt} = |y_1(t)| \Re \left( ix(t) \left( \frac{\frac{a}{2} + R_+^{(1)}(\mathbf{x}(t))}{1+bx(t)+C_+(\mathbf{x}(t))} \right) \right) \\ \frac{d|y_2(t)|}{dt} = |y_2(t)| \Re \left( ix(t) \left( \frac{\frac{a}{2} + R_+^{(1)}(\mathbf{x}(t))}{1+bx(t)+C_+(\mathbf{x}(t))} \right) \right) \\ \frac{d\theta(t)}{dt} = \Im \left( \frac{ix(t)}{1+bx(t)+C_+(\mathbf{x}(t))} \right). \end{cases}$$

For any non-zero complex number  $\zeta$  and positive numbers  $R, B > 0$ , we denote by  $\Sigma_+(\zeta, R, B)$  the sector of radius  $R$  bisected by  $i\zeta\mathbb{R}_+$  and of opening  $\pi - 2 \arcsin\left(\frac{B}{|\zeta|}\right) = 2 \arccos\left(\frac{B}{|\zeta|}\right)$ :

$$\Sigma_+(\zeta, R, B) := \{x \in D(0, R) \mid \Im(\zeta x) > B|x|\} \quad .$$

For  $M, R > 0$ , we also denote by  $\Theta_+(R, M)$  (*resp.*  $\Theta_-(R, M)$ ) the sector of radius  $R$  bisected by  $\mathbb{R}_+$  (*resp.*  $\mathbb{R}_-$ ) and of opening  $2 \arccos(M)$ :

$$\begin{aligned} \Theta_+(R, M) &:= \{x \in D(0, R) \mid \Re(x) > M|x|\} \\ \Theta_-(R, M) &:= \{x \in D(0, R) \mid \Re(x) < -M|x|\} \quad . \end{aligned}$$

Since  $\Re(a) > 0$  by, we can chose  $\omega' \in ]0, \Re\left(\frac{a}{2}\right)[$ , so that  $\Sigma_+\left(\frac{a}{2}, r, \omega'\right)$  contains  $i\mathbb{R}_{>0}$ . We also take

$$\omega \in \left] \cos\left(\left|\arg\left(\frac{a}{2}\right)\right| + \arcsin\left(\frac{2\omega'}{|a|}\right)\right), 1 \right[ \quad ,$$

so that  $\Sigma_+(1, r, \omega) \subset \Sigma_+\left(\frac{a}{2}, r, \omega'\right)$ .

The fact that  $0 < \omega' < \Re\left(\frac{a}{2}\right)$  guaranties that  $\left|\arg\left(\frac{a}{2}\right)\right| + \arcsin\left(\frac{2\omega'}{|a|}\right) < \frac{\pi}{2}$ , so that

$$\cos\left(\left|\arg\left(\frac{a}{2}\right)\right| + \arcsin\left(\frac{2\omega'}{|a|}\right)\right) > 0$$

and then  $\omega > 0$ . We also take  $\mu \in ]0, \sqrt{1 - \omega^2}[$  small enough so that

$$\begin{aligned} \Theta_+(r, \mu) \cap \Sigma_+(1, r, \omega) &\neq \emptyset \\ \Theta_-(r, \mu) \cap \Sigma_+(1, r, \omega) &\neq \emptyset \end{aligned}$$

and  $S_+(r, \epsilon) \subset \Sigma_+(1, r, \omega) \cup \Theta_+(r, \mu) \cup \Theta_-(r, \mu)$ .

More precisely, we need to have  $0 < \epsilon < \arccos(\mu)$ . The idea is now to study the behavior of the solution  $t \mapsto x(t)$  on each of the domains  $\Sigma_+(1, r, \omega)$ ,  $\Theta_+(r, \mu)$ ,  $\Theta_-(r, \mu)$ .

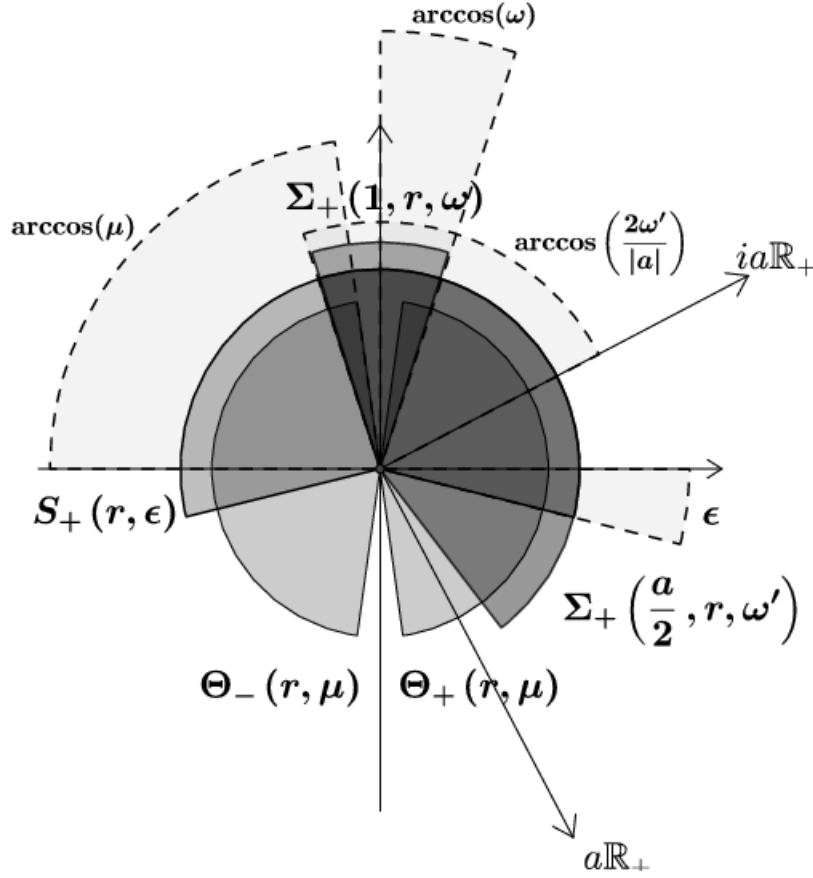


Figure 1: representation of domains  $\Sigma_+(1, r, \omega)$ ,  $\Sigma_+(\frac{a}{2}, r, \omega')$ ,  $\Theta_+(r, \mu)$ ,  $\Theta_-(r, \mu)$ ,  $S_+(r, \epsilon)$ . The radii of the sectors drawn on the figure have been slightly altered as to offer a better understanding. We prove now the following result.

**Lemma 4.11.**

- (1) *There exists  $r, r_1, r_2 > 0$  such that  $\Sigma_+(1, r, \omega) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$  is stable by the flow of (4.2) at positive time  $t \geq 0$ . Moreover in this region  $|x(t)|$ ,  $|y_1(t)|$  and  $|y_2(t)|$  decrease and go to 0 as  $t \rightarrow +\infty$ .*
- (2) *There exists  $0 < r' < r$ ,  $0 < r'_1 < r_1$ ,  $0 < r'_2 < r_2$  and an open domain  $\Omega_+$  stable under the action flow of (4.2) at positive time  $t \geq 0$  such that*

$$S_+(r', \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r}') \subset \Omega_+ \subset S_+(r, \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r}) .$$

*Moreover, if  $x_0 \in \Theta_+(r', \mu)$  (resp.  $x_0 \in \Theta_-(r', \mu)$ ), then  $\theta(t) = \arg(x(t))$ ,  $t \geq 0$  is increasing (resp. decreasing) as long as  $x(t)$  remains in  $\Theta_+(r', \mu)$  (resp.  $\Theta_-(r', \mu)$ ). Finally, there exists  $t_0 \geq 0$  such that for all  $t \geq t_0$ ,  $x(t) \in \Sigma_+(1, r, \omega)$ .*

*Proof.* 1. We fix  $\delta \in ]0, \min(\omega, \mu)[$ ,  $\delta' \in ]0, \omega'[$  and we take  $r > 0$  small enough such that for all  $\mathbf{x} = (x, \mathbf{y}) \in S_+(r, \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ , we have

$$\left\{ \begin{array}{l} \left| \frac{1}{1+bx+C_+(\mathbf{x})} - 1 \right| < \delta \\ \left| \frac{\frac{a}{2} + R_+^{(1)}(\mathbf{x})}{1+bx+C_+(\mathbf{x})} - \frac{a}{2} \right| < \delta' \end{array} \right. .$$



Consequently for all  $\mathbf{x} \in S_+ \times \mathbf{D}(\mathbf{0}, \mathbf{r})$  we have the following estimates:

$$\begin{cases} -|x|(1+\delta) < \Re\left(\frac{ix}{1+bx+C_+(\mathbf{x})}\right) < |x|(1+\delta) \\ -|x|\left(\left|\frac{a}{2}\right|+\delta'\right) < \Re\left(ix\left(\frac{\frac{a}{2}+R_+^{(1)}(x,\mathbf{y})}{1+bx+C_+(x,\mathbf{y})}\right)\right) < |x|\left(\left|\frac{a}{2}\right|+\delta'\right) \end{cases} .$$

Moreover:

- if  $x \in \Sigma_+(1, r, \omega)$  then

$$\Re\left(\frac{ix}{1+bx+C_+(\mathbf{x})}\right) < -|x|(\omega-\delta) \quad ;$$

- if  $x \in \Sigma_+(\frac{a}{2}, r, \omega')$  (in particular if  $x \in \Sigma_+(1, r, \omega)$ ) then

$$\Re\left(ix\left(\frac{\frac{a}{2}+R_+^{(1)}(x,\mathbf{y})}{1+bx+C_+(x,\mathbf{y})}\right)\right) < -|x|(\omega'-\delta') \quad ;$$

- if  $x \in \Theta_-(r, \mu)$  (*resp.*  $\Theta_+(r, \mu)$ ) then

$$\begin{aligned} \Im\left(\frac{ix}{1+bx+C_+(\mathbf{x})}\right) &< -|x|(\mu-\delta) \\ \left(\text{resp. } \Im\left(\frac{ix}{1+bx+C_+(\mathbf{x})}\right) > |x|(\mu-\delta)\right) \end{aligned} .$$

Hence:

- for all  $t \geq 0$

$$-(1+\delta)|x(t)|^2 < \frac{d|x(t)|}{dt} < -(1+\delta)|x(t)|^2$$

and then, as long as  $\mathbf{x}(t) \in S_+(r, \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ , we have

$$|x(t)| > \frac{|x_0|}{1+(1+\delta)|x_0|t} \quad ;$$

- for all  $t \geq 0$ , if  $x(t) \in \Sigma_+(1, r, \omega)$ , then

$$(4.3) \quad \frac{d|x(t)|}{dt} < -(\omega-\delta)|x(t)|^2$$

and

$$(4.4) \quad \begin{cases} \frac{d|y_1(t)|}{dt} < -(\mu-\delta)|y_1(t)||x(t)| \\ \frac{d|y_2(t)|}{dt} < -(\mu-\delta)|y_2(t)||x(t)| \end{cases}$$

so that  $|x(t)|$ ,  $|y_1(t)|$  and  $|y_2(t)|$  are decreasing as long as  $x(t) \in \Sigma_+(1, r, \omega)$ ;

- for all  $t \geq 0$ , if  $x(t) \in \Theta_-(r, \mu)$  (*resp.*  $\Theta_+(r, \mu)$ ) then

$$\begin{aligned} \frac{d\theta}{dt}(t) &< -(\mu-\delta)|x(t)| < \frac{-(\mu-\delta)|x_0|}{1+(1+\delta)|x_0|t} \\ \left(\text{resp. } \frac{d\theta}{dt}(t) > (\mu-\delta)|x(t)| > \frac{(\mu-\delta)|x_0|}{1+(1+\delta)|x_0|t}\right) \end{aligned}$$

so that  $t \mapsto \theta(t)$  is strictly decreasing (*resp.* increasing) as long as  $x(t) \in \Theta_-(r, \mu)$  (*resp.*  $\Theta_+(r, \mu)$ ). Moreover, if  $\theta_0 = \theta(0)$  is such that  $x_0 = x(0) \in \Theta_-(t, \mu) \setminus \Sigma_+(1, r, \omega)$  (*resp.*

$\Theta_+(r, \mu) \setminus \Sigma_+(1, r, \omega)$ , then as long as  $x(t) \in \Theta_-(r, \mu)$  (*resp.*  $\Theta_+(r, \mu)$ ) we have:

$$\begin{aligned} \theta(t) &< \theta_0 - \left(\frac{\mu - \delta}{1 + \delta}\right) \ln(1 + (1 + \delta)|x_0|t) \\ \left(\text{resp. } \theta(t) &> \theta_0 + \left(\frac{\mu - \delta}{1 + \delta}\right) \ln(1 + (1 + \delta)|x_0|t)\right) \end{aligned} .$$

We see that  $x(t) \in \Sigma_+(1, r, \omega)$  for all

$$\begin{aligned} t \geq t_0 &:= \frac{\left(\exp\left(\frac{1+\delta}{\mu-\delta}\left(\theta_0 - \frac{\pi}{2} - \arccos(\omega)\right)\right) - 1\right)}{(1 + \delta)|x_0|} \\ \left(\text{resp. } t_0 &:= \frac{\left(\exp\left(\frac{1+\delta}{\mu-\delta}\left(\frac{\pi}{2} - \arccos(\omega) - \theta_0\right)\right) - 1\right)}{(1 + \delta)|x_0|}\right) \end{aligned} .$$

Notice that

$$(4.5) \quad t_0 \leq \frac{\exp\left(\left(\frac{1+\delta}{\mu-\delta}\right)(\epsilon + \arcsin(\omega))\right)}{(1 + \delta)|x_0|} .$$

On the one hand  $\Sigma_+(1, r, \omega) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$  is stable by the flow of (4.2) in forward time  $t \geq 0$ . Indeed in this region  $|x(t)|, |y_1(t)|$  and  $|y_2(t)|$  are decreasing, and as soon as  $x(t)$  enters  $\Sigma_+(1, r, \omega) \cap \Theta_-(r, \mu)$  (*resp.*  $\Sigma_+(1, r, \omega) \cap \Theta_+(r, \mu)$ ), which is non-empty and contain a part of the boundary of  $\Sigma_+(1, r, \omega)$  with constant argument, the direction  $\theta(t)$  is decreasing (*resp.* increasing). Then,  $x(t)$  stays in  $\Sigma_+(1, r, \omega)$ .

On the other hand, as long as we are in  $\Theta_-(r, \mu)$  (*resp.*  $\Theta_+(r, \mu)$ ) we can re-parametrize the solutions by  $(-\theta)$  (*resp.*  $\theta$ ):

$$\left\{ \begin{aligned} \frac{d|x|}{d(-\theta)} &= -|x| \frac{\Re\left(\frac{ix}{1+bx+C_+(\mathbf{x})}\right)}{\Im\left(\frac{ix}{1+bx+C_+(\mathbf{x})}\right)} \leq |x| \cdot \frac{1+\delta}{\mu-\delta} \\ \left(\text{resp. } \frac{d|x|}{d\theta} &= |x| \frac{\Re\left(\frac{ix}{1+bx+C_+(\mathbf{x})}\right)}{\Im\left(\frac{ix}{1+bx+C_+(\mathbf{x})}\right)} \leq |x| \cdot \frac{1+\delta}{\mu-\delta}\right) \\ \frac{d|y_1|}{d(-\theta)} &= -|y_1| \frac{\Re\left(ix \frac{\left(\frac{a}{2} + R_+^{(1)}(x, \mathbf{y})\right)}{1+bx+C_+(x, \mathbf{y})}\right)}{\Im\left(\frac{ix}{1+bx+C_+(\mathbf{x})}\right)} \leq |y_1| \cdot \frac{\frac{a}{2} + \delta'}{\mu-\delta} \\ \left(\text{resp. } \frac{d|y_1|}{d\theta} &= |y_1| \frac{\Re\left(ix \frac{\left(\frac{a}{2} + R_+^{(1)}(x, \mathbf{y})\right)}{1+bx+C_+(x, \mathbf{y})}\right)}{\Im\left(\frac{ix}{1+bx+C_+(\mathbf{x})}\right)} \leq |y_1| \cdot \frac{\frac{a}{2} + \delta'}{\mu-\delta}\right) \\ \frac{d|y_2|}{d(-\theta)} &= -|y_2| \frac{\Re\left(ix \frac{\left(\frac{a}{2} + R_+^{(1)}(x, \mathbf{y})\right)}{1+bx+C_+(x, \mathbf{y})}\right)}{\Im\left(\frac{ix}{1+bx+C_+(\mathbf{x})}\right)} \leq |y_2| \cdot \frac{\frac{a}{2} + \delta'}{\mu-\delta} \\ \left(\text{resp. } \frac{d|y_2|}{d\theta} &= |y_2| \frac{\Re\left(ix \frac{\left(\frac{a}{2} + R_+^{(1)}(x, \mathbf{y})\right)}{1+bx+C_+(x, \mathbf{y})}\right)}{\Im\left(\frac{ix}{1+bx+C_+(\mathbf{x})}\right)} \leq |y_2| \cdot \frac{\frac{a}{2} + \delta'}{\mu-\delta}\right) \end{aligned} \right. .$$

Hence, if  $\theta_0 := \theta(0)$  is such that  $x_0 := x(0) \in \Theta_-(r, \mu)$  (*resp.*  $\Theta_+(r, \mu)$ ), for  $t \leq t_0$  we have:

$$(4.6) \quad \left\{ \begin{array}{l} |x(t)| \leq |x_0| \exp\left(\frac{1+\delta}{\mu-\delta}(\theta_0 - \theta(t))\right) \\ \left( \text{resp. } |x(t)| \leq |x_0| \exp\left(\frac{1+\delta}{\mu-\delta}(\theta(t) - \theta_0)\right) \right) \\ |y_1(t)| \leq |y_{1,0}| \exp\left(\frac{|\frac{\alpha}{2}|+\delta'}{\mu-\delta}(\theta_0 - \theta(t))\right) \\ \left( \text{resp. } |y_1(t)| \leq |y_{1,0}| \exp\left(\frac{|\frac{\alpha}{2}|+\delta'}{\mu-\delta}(\theta(t) - \theta_0)\right) \right) \\ |y_2(t)| \leq |y_{2,0}| \exp\left(\frac{|\frac{\alpha}{2}|+\delta'}{\mu-\delta}(\theta_0 - \theta(t))\right) \\ \left( \text{resp. } |y_2(t)| \leq |y_{2,0}| \exp\left(\frac{|\frac{\alpha}{2}|+\delta'}{\mu-\delta}(\theta(t) - \theta_0)\right) \right). \end{array} \right.$$

**Definition 4.12.** We define the domain  $\Omega_+$  as the set of all  $\mathbf{x} = (x, y_1, y_2) \in S_+(r, \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$  such that:

$$\begin{array}{l} \bullet \text{ if } \Im(x) \geq \omega|x| \text{ then } \left\{ \begin{array}{l} |x| \leq r \exp\left(\frac{1+\delta}{\mu-\delta}(\arg(x) - \arcsin(\omega))\right) \\ |y_1| \leq r_1 \exp\left(\frac{|\frac{\alpha}{2}|+\delta'}{\mu-\delta}(\arg(x) - \arcsin(\omega))\right) \\ |y_2| \leq r_2 \exp\left(\frac{|\frac{\alpha}{2}|+\delta'}{\mu-\delta}(\arg(x) - \arcsin(\omega))\right) \end{array} \right\} ; \\ \bullet \text{ if } \Im(x) \leq -\omega|x| \text{ then } \left\{ \begin{array}{l} |x| \leq r \exp\left(\frac{1+\delta}{\mu-\delta}(\pi - \arcsin(\omega) - \arg(x))\right) \\ |y_1| \leq r_1 \exp\left(\frac{|\frac{\alpha}{2}|+\delta'}{\mu-\delta}(\pi - \arcsin(\omega) - \arg(x))\right) \\ |y_2| \leq r_2 \exp\left(\frac{|\frac{\alpha}{2}|+\delta'}{\mu-\delta}(\pi - \arcsin(\omega) - \arg(x))\right) \end{array} \right\} . \end{array}$$

We see that  $\Omega_+$  is stable by the flow of (4.2) at positive time  $t \geq 0$ . We have seen that for any initial condition in  $\Omega_+$ , the solution exists for any  $t \geq 0$ , stays in  $\Omega_+$ , and after a finite time  $t_0 \geq 0$  enters and remains in  $\Sigma_+(1, r, \omega)$ . Finally, we have:

$$S_+(r', \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r}') \subset \Omega_+ \subset S_+(r, \epsilon) \times \mathbf{D}(\mathbf{0}, \mathbf{r}) ,$$

where

$$\left\{ \begin{array}{l} r' = r \exp\left(-\left(\frac{1+\delta}{\mu-\delta}\right)(\epsilon + \arcsin(\omega))\right) < r \\ r'_1 = r_1 \exp\left(-\left(\frac{|\frac{\alpha}{2}|+\delta'}{\mu-\delta}\right)(\epsilon + \arcsin(\omega))\right) < r_1 \\ r'_2 = r_2 \exp\left(-\left(\frac{|\frac{\alpha}{2}|+\delta'}{\mu-\delta}\right)(\epsilon + \arcsin(\omega))\right) < r_2 . \end{array} \right.$$

2. Let  $\mathbf{x}_0 = (x_0, \mathbf{y}_0) \in \Sigma_+(1, r, \omega) \times \mathbf{D}(\mathbf{0}, \mathbf{r})$ . From (4.3) and (4.4) we have for all  $t \geq 0$ :

$$(4.7) \quad \left\{ \begin{array}{l} |x(t)| \leq \frac{|x_0|}{1+(\omega-\delta)|x_0|t} \\ |y_1(t)| \leq \frac{|y_{1,0}|}{(1+(1+\delta)|x_0|t)^{\frac{\mu-\delta}{1+\delta}}} \\ |y_2(t)| \leq \frac{|y_{2,0}|}{(1+(1+\delta)|x_0|t)^{\frac{\mu-\delta}{1+\delta}}} , \end{array} \right.$$

which proves that the solutions goes to  $\mathbf{0}$  as  $t \rightarrow +\infty$ .  $\square$

*Remark 4.13.* Another stable domain  $\Omega_-$  is defined similarly when dealing with the case “ $\pm = -$ ”

#### 4.4.2. Construction of a sectorial analytic solution of the homological equation.

We consider the meromorphic 1-form  $\tau := \frac{dx}{x^2}$ , which satisfies  $\tau \cdot (Z_{\pm}) = 1$ . Let also  $A_{\pm}(x, \mathbf{y})$  be analytic in  $S_{\pm}(r, \epsilon) \times (\mathbb{C}^2, 0)$  and dominated by  $\|\mathbf{y}\|_{\infty}$ , and  $M \in \mathbb{N}_{>0}$ . The following proposition is a refinement of Lemma 4.6.

**Proposition 4.14.** *For all  $\mathbf{x}_0 \in \Omega_{\pm}$  (see Definition 4.12), the integral defined by*

$$\alpha_{\pm}(\mathbf{x}_0) := - \int_{\gamma_{\pm, \mathbf{x}_0}} x^{M+1} A_{\pm}(\mathbf{x}) \tau$$

*is absolutely convergent. Moreover, the function  $\mathbf{x}_0 \mapsto \alpha_{\pm}(\mathbf{x}_0)$  is analytic in  $\Omega_{\pm}$ , satisfies*

$$\mathcal{L}_{Z_{\pm}}(\alpha_{\pm}) = x^{M+1} A_{\pm}(\mathbf{x})$$

*and  $\alpha_{\pm}(x, \mathbf{y}) = x^M \tilde{\alpha}_{\pm}(x, \mathbf{y})$ , where  $\tilde{\alpha}_{\pm}$  is analytic on  $\Omega_{\pm}$  and dominated by  $\|\mathbf{y}\|_{\infty}$ .*

*Proof.* We are going to use the estimates obtained in the previous paragraph.

- Let us start by proving that the integral above is convergent. We begin with:

$$\begin{aligned} \alpha_{\pm}(\mathbf{x}_0) &= - \int_0^{+\infty} \frac{x(t)^{M+1} A_{\pm}(\mathbf{x}(t))}{x(t)^2} \frac{ix(t)^2}{1 + bx(t) + C_+(\mathbf{x}(t))} dt \\ &= -i \int_0^{+\infty} \frac{x(t)^{M+1} A_{\pm}(\mathbf{x}(t))}{1 + bx(t) + C_+(\mathbf{x}(t))} dt. \end{aligned}$$

Since  $\mathbf{x}(t) \in \Omega_{\pm}$  for all  $t \geq 0$  and  $A_{\pm}(x, \mathbf{y})$  is dominated by  $\|\mathbf{y}\|_{\infty}$ , we have:

$$\left| \frac{x(t)^{M+1} A_{\pm}(\mathbf{x}(t))}{1 + bx(t) + C_+(\mathbf{x}(t))} \right| \leq C |x(t)|^{M+1} \|\mathbf{y}(t)\|_{\infty}$$

where  $C > 0$  is some constant, independent on  $\mathbf{x}_0$  and  $t$ . For  $t \geq 0$  big enough, we deduce from paragraph 4.4.1 that:

$$\begin{aligned} \left| \frac{x(t)^{M+1} A_{\pm}(\mathbf{x}(t))}{1 + bx(t) + C_+(\mathbf{x}(t))} \right| &\leq C \|\mathbf{y}_0\| \left( \frac{|x_0|}{1 + (\omega - \delta)|x_0|t} \right)^{M+1} \frac{1}{(1 + (1 + \delta)|x_0|t)^{\frac{\mu - \delta}{1 + \delta}}} \\ &= O_{t \rightarrow +\infty} \left( \frac{1}{t^{M+1}} \right) \end{aligned}$$

and then the integral is absolutely convergent.

- Let us prove the analyticity of  $\alpha_{\pm}$  in  $\Omega_{\pm}$ : it is sufficient to prove that it is analytic in every compact  $K \subset \Omega_{\pm}$ . Let  $K$  be such a compact subset. Let  $L > 0$  such that for all  $\mathbf{x} \in K$ , we have:

$$\left| \frac{A_{\pm}(\mathbf{x})}{1 + bx + C_+(\mathbf{x})} \right| \leq L.$$

There exists  $\delta > 0$  such that for all  $\mathbf{x} = (x, y_1, y_2) \in K$ , we have  $\delta < |x| < r$ . Finally, according to the several estimates in paragraph 4.4.1, there exists  $B > 0$  such that for all  $\mathbf{x}_0 \in K$  and  $t \geq 0$ , we have:

$$|x(t)| \leq B \frac{|x_0|}{1 + (\omega - \delta)|x_0|t}.$$

Hence:

$$\begin{aligned} \left| \frac{x(t)^{M+1} A_{\pm}(\mathbf{x}(t))}{1 + bx(t) + C_+(\mathbf{x}(t))} \right| &\leq LB^{M+1} \frac{|x_0|^{M+1}}{(1 + (\omega - \delta)|x_0|t)^{M+1}} \\ &\leq \frac{LB^{M+1} r^{M+1}}{(1 + (\omega - \delta)\delta t)^{M+1}}, \end{aligned}$$

and the classical theorem concerning the analyticity of parametric integrals proves that  $\alpha_{\pm}$  is analytic on any compact  $K \subset \Omega_{\pm}$ , and consequently on  $\Omega_{\pm}$ .

- Let us write  $F(\mathbf{x}) := \frac{\pm ix^{M+1}A_{\pm}(\mathbf{x})}{1+bx+C_{\pm}(\mathbf{x})}$ , so that

$$\alpha_{\pm}(\mathbf{x}_0) = -i \int_0^{+\infty} F\left(\Phi_{X_{\pm}}^t(\mathbf{x}_0)\right) dt .$$

For all  $\mathbf{x}_0 \in \Omega_{\pm}$ , the function  $t \mapsto \mathbf{x}(t) = \Phi_{X_{\pm}}^t(\mathbf{x}_0)$  satisfies:

$$\frac{\partial}{\partial t} \left( \Phi_{X_{\pm}}^t(\mathbf{x}_0) \right) = \frac{\pm i}{1 + bx\left(\Phi_{X_{\pm}}^t(\mathbf{x}_0)\right) + C_{\pm}\left(\Phi_{X_{\pm}}^t(\mathbf{x}_0)\right)} Z_{\pm}\left(\Phi_{\pm}^t(\mathbf{x})\right) .$$

The classical theorem about the analyticity of parametric integrals tells us that we can compute the derivatives inside the integral symbol:

$$\begin{aligned} (\mathcal{L}_{Z_{\pm}}\alpha_{\pm})(\mathbf{x}_0) &= - \int_0^{+\infty} \mathcal{L}_{Z_{\pm}}(F \circ \Phi^s)(\mathbf{x}_0) ds \\ &= - \int_0^{+\infty} DF\left(\Phi_{X_{\pm}}^s(\mathbf{x}_0)\right) \cdot D\Phi_{X_{\pm}}^s(\mathbf{x}_0) \cdot Z_{\pm}(\mathbf{x}_0) ds \\ &= - \int_0^{+\infty} DF\left(\Phi_{X_{\pm}}^s(\mathbf{x})\right) \cdot \frac{\partial}{\partial t} \left( \Phi_{X_{\pm}}^{s+t}(\mathbf{x}_0) \right) \Big|_{t=0} \left( \pm \frac{1+bx_0+C_{\pm}(\mathbf{x}_0)}{i} \right) ds \\ &= - \left( \pm \frac{1+bx_0+C_{\pm}(\mathbf{x}_0)}{i} \right) \cdot \int_0^{+\infty} DF\left(\Phi_{X_{\pm}}^s(\mathbf{x}_0)\right) \cdot \frac{\partial}{\partial t} \left( \Phi_{X_{\pm}}^t(\mathbf{x}_0) \right) \Big|_{t=s} ds \\ &= - \left( \pm \frac{1+bx_0+C_{\pm}(\mathbf{x}_0)}{i} \right) \cdot \int_0^{+\infty} \frac{\partial}{\partial s} \left( F \circ \Phi_{X_{\pm}}^s(\mathbf{x}_0) \right) ds \\ &= - \left( \pm \frac{1+bx_0+C_{\pm}(\mathbf{x}_0)}{i} \right) \cdot \left[ F \circ \Phi_{X_{\pm}}^s(\mathbf{x}_0) \right]_{s=0}^{s=+\infty} \\ &= - \left( \pm \frac{1+bx_0+C_{\pm}(\mathbf{x}_0)}{i} \right) \cdot (-F(\mathbf{x}_0)) \\ &= x_0^{M+1} A_{\pm}(\mathbf{x}_0) . \end{aligned}$$

- Let us prove that  $\tilde{\alpha}_{\pm}(x, \mathbf{y}) := \frac{\alpha_{\pm}(x, \mathbf{y})}{x^M}$  is bounded and dominated by  $\|\mathbf{y}\|_{\infty}$  in  $\Omega_{\pm}$ . The fact that it is analytic in  $\Omega_{\pm}$  is clear because  $\alpha_{\pm}$  is analytic there and  $0 \notin \Omega_{\pm}$ . As above, there exists  $C > 0$  such that for all  $\mathbf{x}_0 := (x_0, \mathbf{y}_0) \in \Omega_{\pm}$  and for all  $t \geq 0$ :

$$\left| \frac{x \left( \Phi_{X_{\pm}}^t(\mathbf{x}_0) \right)^{M+1} A_{\pm} \left( \Phi_{X_{\pm}}^t(\mathbf{x}_0) \right)}{\left( 1 + bx \left( \Phi_{X_{\pm}}^t(\mathbf{x}_0) \right) + C_{\pm} \left( \Phi_{X_{\pm}}^t(\mathbf{x}_0) \right) \right)} \right| \leq C \left| x \left( \Phi_{X_{\pm}}^t(\mathbf{x}_0) \right) \right|^{M+1} \left\| \mathbf{y} \left( \Phi_{X_{\pm}}^t(\mathbf{x}_0) \right) \right\|_{\infty} .$$

We will only deal with the case where  $x_0 \in \Theta_{\pm}(r, \mu)$  (the case where  $\Sigma_{\pm}(1, r, \omega)$  is easier and can be deduced from that one). On the one hand from (4.6) we have for all  $t \leq t_0$ :

$$\begin{cases} \left| x \left( \Phi_{X_{\pm}}^t(\mathbf{x}_0) \right) \right| \leq D |x_0| & , \text{ where } D := \exp \left( \frac{1+\delta}{\mu-\delta} (\arccos(\mu) + \epsilon) \right) \\ \left\| \mathbf{y} \left( \Phi_{X_{\pm}}^t(\mathbf{x}_0) \right) \right\|_{\infty} \leq D' \|\mathbf{y}_0\|_{\infty} & , \text{ where } D' := \exp \left( \frac{\frac{\epsilon}{2} + \delta'}{\mu-\delta} (\arccos(\mu) + \epsilon) \right) . \end{cases}$$

On the other hand we have seen in (4.7) that for all  $t \geq t_0$ :

$$\begin{cases} \left| x \left( \Phi_{X_{\pm}}^t(\mathbf{x}_0) \right) \right| \leq \frac{\left| x \left( \Phi_{X_{\pm}}^{t_0}(\mathbf{x}_0) \right) \right|}{1 + (\omega - \eta) \left| x \left( \Phi_{X_{\pm}}^{t_0}(\mathbf{x}_0) \right) \right|^{(t-t_0)}} \\ \left\| \mathbf{y} \left( \Phi_{X_{\pm}}^t(\mathbf{x}_0) \right) \right\|_{\infty} \leq \|\mathbf{y}_0\|_{\infty} \end{cases} .$$

Hence, we use the Chasles relation and the estimates above to obtain:

$$\begin{aligned}
|\tilde{\alpha}_{\pm}(x_0, \mathbf{y}_0)| &\leq \frac{|\alpha_{\pm}(x_0, \mathbf{y}_0)|}{|x_0|^M} \\
&\leq \frac{CD^{M+1}D' \|\mathbf{y}_0\|_{\infty} |x_0|^{M+1} |t_0|}{|x_0|^M} \\
&\quad + \frac{C \|\mathbf{y}_0\|_{\infty}}{|x_0|^M} \int_{t_0}^{+\infty} \frac{dt}{\left(1 + (\omega - \eta) \left| x \left( \Phi_{X_{\pm}}^t(\mathbf{x}_0) \right) \right| (t - t_0) \right)} \\
&\leq CD^{M+1}D' \|\mathbf{y}_0\|_{\infty} |x_0| |t_0| + \frac{C \|\mathbf{y}_0\|_{\infty} \left| x \left( \Phi_{X_{\pm}}^{t_0}(\mathbf{x}_0) \right) \right|^{M+1}}{M(\omega - \delta) |x_0|^M \left| x \left( \Phi_{X_{\pm}}^{t_0}(\mathbf{x}_0) \right) \right|};
\end{aligned}$$

and according to (4.5) we conclude

$$|\tilde{\alpha}_{\pm}(x_0, \mathbf{y}_0)| \leq \left( \frac{D^2 D'}{(1 + \delta)} + \frac{1}{M(\omega - \delta)} \right) CD^M \|\mathbf{y}_0\|_{\infty}.$$

□

#### 4.5. Sectorial isotropies in “wide” sectors and uniqueness of the normalizing maps.

We consider a normal form  $Y_{\text{norm}}$  as given by Theorem 1.10. We study here the germs of sectorial isotropies of the normal form  $Y_{\text{norm}}$  in a domain  $\mathcal{S}_{\pm} \in \mathcal{S}_{\arg(\pm i\lambda), \eta}$  (sectorial neighborhood attached to the origin with opening  $\eta \in ]\pi, 2\pi[$  in the direction  $\arg(\pm i\lambda)$ ). As a consequence, we establish in Corollary 4.2 the uniqueness of the germs  $\Phi_{\pm}$  (see Proposition 1.13).

**Definition 4.15.** A germ of sectorial fibered diffeomorphism  $\Phi_{\theta, \eta}$  in the direction  $\theta \in \mathbb{R}$  with opening  $\eta \geq 0$ , is a *germ of fibered sectorial isotropy* of  $Y_{\text{norm}}$  (in the direction  $\theta \in \mathbb{R}$  with opening  $\eta \geq 0$ ) if  $(\Phi_{\theta, \eta})_*(Y_{\text{norm}}) = Y_{\text{norm}}$  in  $\mathcal{S} \in \mathcal{S}_{\theta, \eta}$ . We denote by  $\text{Isot}_{\theta, \eta}(Y_{\text{norm}})$  the set of germs of sectorial fibered isotropies of  $Y_{\text{norm}}$  in the direction  $\theta \in \mathbb{R}$  with opening  $\eta \geq 0$ .

Proposition 1.13 is an immediate consequence of the following one.

**Proposition 4.16.** *For all  $\eta \in ]\pi, 2\pi[$ :*

$$\text{Isot}_{\arg(\pm i\lambda), \eta}(Y_{\text{norm}}) = \left\{ \begin{array}{l} (x, y_1, y_2) \mapsto (x, b_1 y_1, b_2 y_2) \mid (b_1, b_2) \in (\mathbb{C} \setminus \{0\})^2 \\ \text{such that } c(b_1 b_2 v) = c(v) \end{array} \right\}.$$

*In particular, the only germ of sectorial fibered isotropy of  $Y_{\text{norm}}$  in the direction  $\arg(\pm i\lambda)$  with opening  $\eta \in ]\pi, 2\pi[$ , which is moreover tangent to the identity, is the identity itself.*

*Proof.* Let  $\phi : (x, \mathbf{y}) \mapsto (x, \phi_1(x, \mathbf{y}), \phi_2(x, \mathbf{y}))$  be a sectorial fibered isotropy of  $Y_{\text{norm}}$  in  $\mathcal{S}_{\pm} \in \mathcal{S}_{\arg(\pm i\lambda), \eta}$  with  $\eta \in ]\pi, 2\pi[$ . Possibly by reducing our domain, we can assume that  $\mathcal{S}_{\pm}$  is bounded and of the form  $S_{\pm} \times \mathbf{D}(\mathbf{0}, \mathbf{r})$  (for some sector  $S_{\pm}$  and some polydisc  $\mathbf{D}(\mathbf{0}, \mathbf{r})$ ) and that  $\phi$  is bounded in this domain. We have

$$\phi_*(Y_{\text{norm}}) = Y_{\text{norm}}$$

*i.e.*

$$D\phi \cdot Y_{\text{norm}} = Y_{\text{norm}} \circ \phi$$

which is also equivalent to:

$$(4.8) \quad \begin{cases} x^2 \frac{\partial \phi_1}{\partial x} + (-1 - c(y_1 y_2) + a_1 x) y_1 \frac{\partial \phi_1}{\partial y_1} + (1 + c(y_1 y_2) + a_2 x) y_2 \frac{\partial \phi_1}{\partial y_2} \\ = \phi_1 (-1 - c(\phi_1 \phi_2) + a_1 x) \\ x^2 \frac{\partial \phi_2}{\partial x} + (-1 - c(y_1 y_2) + a_1 x) y_1 \frac{\partial \phi_2}{\partial y_1} + (1 + c(y_1 y_2) + a_2 x) y_2 \frac{\partial \phi_2}{\partial y_2} \\ = \phi_2 (1 + c(\phi_1 \phi_2) + a_2 x) \end{cases}.$$

Let us consider  $\psi := \phi_1 \phi_2$ . Then

$$x^2 \frac{\partial \psi}{\partial x} + (-1 - c(y_1 y_2) + a_1 x) y_1 \frac{\partial \psi}{\partial y_1} + (1 + c(y_1 y_2) + a_2 x) y_2 \frac{\partial \psi}{\partial y_2} = (a_1 + a_2) x \psi \quad .$$

By assumption we can write

$$\psi(x, \mathbf{y}) = \sum_{j_1 + j_2 \geq 2} \psi_{j_1, j_2}(x) y_1^{j_1} y_2^{j_2} \quad ,$$

where  $\psi_{j_1, j_2}(x)$  is analytic and bounded in  $S_{\pm}$  for all  $j_1, j_2 \geq 0$  and such that

$$\sum_{j_1 + j_2 \geq 1} \left( \sup_{x \in S_{\pm}} (|\psi_{j_1, j_2}(x)|) \right) y_1^{j_1} y_2^{j_2}$$

is a convergent entire series in  $\mathbf{D}(\mathbf{0}, \mathbf{r})$  (possibly by reducing the radius of the polydisc). Consequently, uniform convergence yields that for all  $j_1, j_2 \geq 0$ :

$$\begin{aligned} & x^2 \frac{d\psi_{j_1, j_2}}{dx}(x) + (j_2 - j_1 + (a_1(j_1 - 1) + a_2(j_2 - 1))x) \psi_{j_1, j_2}(x) \\ &= (j_1 - j_2) \sum_{l=1}^{\min(j_1, j_2)} \psi_{j_1-l, j_2-l}(x) c_l \quad . \end{aligned}$$

For  $j_1 = j_2 = j \geq 1$ , we have

$$\psi_{j, j}(x) = b_{j, j} x^{-(j-1)(a_1 + a_2)}, \quad b_{j, j} \in \mathbb{C},$$

and since  $\Re(a_1 + a_2) > 0$ ,  $\psi_{j, j}(x)$  is bounded near the origin if and only if  $b_{j, j} = 0$  or  $j = 1$ . For  $j_1 > j_2$ , we see recursively that  $\psi_{j_1, j_2}(x) = 0$ . Indeed, we obtain by induction that

$$\psi_{j_1, j_2}(x) = b_{j_1, j_2} \exp\left(\frac{j_2 - j_1}{x}\right) x^{-(a_1(j_1-1) + a_2(j_2-1))} \quad ,$$

and since it has to be bounded on  $S_{\pm}$ , we necessarily have  $b_{j_1, j_2} = 0$ . Similarly, for  $j_1 < j_2$ , we see recursively that  $\psi_{j_1, j_2}(x) = 0$ . As a conclusion,  $\psi(x, \mathbf{y}) = b_{1,1} y_1 y_2$ .

We can now solve separately each equation in (4.8):

$$\begin{cases} x^2 \frac{\partial \phi_1}{\partial x} + (-1 - c(y_1 y_2) + a_1 x) y_1 \frac{\partial \phi_1}{\partial y_1} + (1 + c(y_1 y_2) + a_2 x) y_2 \frac{\partial \phi_1}{\partial y_2} \\ = \phi_1 (-1 - c(b_{1,1} y_1 y_2) + a_1 x) \\ x^2 \frac{\partial \phi_2}{\partial x} + (-1 - c(y_1 y_2) + a_1 x) y_1 \frac{\partial \phi_2}{\partial y_1} + (1 + c(y_1 y_2) + a_2 x) y_2 \frac{\partial \phi_2}{\partial y_2} \\ = \phi_2 (1 + c(b_{1,1} y_1 y_2) + a_2 x) \end{cases} \quad .$$

As above for  $i = 1, 2$  we can write  $\phi_i(x, \mathbf{y}) = \sum_{j_1 + j_2 \geq 1} \phi_{i, j_1, j_2}(x) y_1^{j_1} y_2^{j_2}$ , where  $\phi_{i, j_1, j_2}(x)$  is analytic and bounded in  $S_{\pm}$  for all  $j_1, j_2 \geq 0$  and such that

$$\sum_{j_1 + j_2 \geq 1} \left( \sup_{x \in S_{\pm}} (|\phi_{i, j_1, j_2}(x)|) \right) y_1^{j_1} y_2^{j_2}$$

is a convergent entire series in  $\mathbf{D}(\mathbf{0}, \mathbf{r})$  (possibly by reducing the radius of the polydisc). We obtain:

$$\begin{cases} x^2 \frac{d\phi_{1, j_1, j_2}}{dx}(x) + (j_2 - j_1 + 1 + (a_1(j_1 - 1) + a_2 j_2)x) \phi_{1, j_1, j_2}(x) \\ = \sum_{l=1}^{\min(j_1, j_2)} \phi_{1, j_1-l, j_2-l}(x) (j_1 - j_2 - (b_{1,1})^l) c_l \\ x^2 \frac{d\phi_{2, j_1, j_2}}{dx}(x) + (j_2 - j_1 - 1 + (a_1 j_1 + a_2(j_2 - 1))x) \phi_{2, j_1, j_2}(x) \\ = \sum_{l=1}^{\min(j_1, j_2)} \phi_{2, j_1-l, j_2-l}(x) (j_1 - j_2 + (b_{1,1})^l) c_l \end{cases} \quad .$$

From this we deduce:

$$\begin{cases} \phi_{1,1,0}(x) = p_{1,0} \in \mathbb{C} \setminus \{0\} \\ \phi_{2,0,1}(x) = q_{0,1} \in \mathbb{C} \setminus \{0\} \end{cases}$$

with  $p_{1,0}q_{0,1} = b_{1,1}$ . Then, using the assumption that  $\phi_{i,j_1,j_2}(x)$  is analytic and bounded in  $S_{\pm}$  for all  $j_1, j_2 \geq 0$ , one checks easily by induction on  $j \geq 1$  that

$$\forall j \geq 1 \quad \begin{cases} \phi_{1,j+1,j} = 0 \\ \phi_{2,j,j+1} = 0 \\ c_j \left(1 - (b_{1,1})^j\right) = 0 \end{cases} .$$

Indeed, if the result holds for an integer  $j \geq 1$ , then we have:

$$x^2 \frac{d\phi_{1,j+2,j+1}}{dx}(x) + (j+1)(a_1 + a_2)x\phi_{1,j+2,j+1}(x) = p_{1,0}c_{j+1} \left(1 - (b_{1,1})^{j+1}\right)$$

and the general solution of this equation is:

$$\phi_{1,j+2,j+1}(x) = \begin{cases} p_{j+2,j}x^{-(j+1)(a_1+a_2)} + \frac{p_{1,0}c_{j+1} \left(1 - (b_{1,1})^{j+1}\right)}{(j+1)(a_1+a_2) - 1}x^{-1} \\ \left(\text{if } (j+1)(a_1+a_2) \neq 1\right) \\ p_{j+2,j+1}x^{-(j+1)(a_1+a_2)} + p_{1,0}c_{j+1} \left(1 - (b_{1,1})^{j+1}\right)x^{-1} \ln(x) \\ \left(\text{if } (j+1)(a_1+a_2) = 1\right) \end{cases}$$

which is bounded near the origin if and only if  $p_{j+2,j+1} = 0$  and  $c_{j+1} \left(1 - (b_{1,1})^{j+1}\right) = 0$ , since  $\Re(a_1 + a_2) > 0$ . The fact that  $c_j \left(1 - (b_{1,1})^j\right) = 0$  for all  $j \geq 1$  means that  $c(b_{1,1}y_1y_2) = c(y_1y_2)$  for all  $(y_1, y_2)$  near the origin. The same arguments work for  $\phi_{2,j,j+1}$ ,  $j \geq 1$ . Consequently:

$$\begin{cases} x^2 \frac{d\phi_{1,j_1,j_2}}{dx}(x) + (j_2 - j_1 + 1 + (a_1(j_1 - 1) + a_2j_2)x)\phi_{1,j_1,j_2}(x) \\ = (j_1 - j_2 - 1) \sum_{l=1}^{\min(j_1,j_2)} \phi_{1,j_1-l,j_2-l}(x) c_l \\ x^2 \frac{d\phi_{2,j_1,j_2}}{dx}(x) + (j_2 - j_1 - 1 + (a_1j_1 + a_2(j_2 - 1))x)\phi_{2,j_1,j_2}(x) \\ = (j_1 - j_2 + 1) \sum_{l=1}^{\min(j_1,j_2)} \phi_{2,j_1-l,j_2-l}(x) c_l \end{cases} .$$

We find recursively that for  $j_1 > j_2 + 1$ , we have  $\phi_{1,j_1,j_2}(x) = 0$ . Indeed, we obtain by induction that

$$\phi_{1,j_1,j_2}(x) = p_{j_1,j_2} \exp\left(\frac{j_2 - j_1 + 1}{x}\right) x^{-(a_1(j_1-1) + a_2j_2)} ,$$

and since it has to be bounded on  $S_{\pm}$ , we necessarily have  $p_{j_1,j_2} = 0$ . Similarly, for  $j_1 < j_2 + 1$ , we see recursively that  $\phi_{j_1,j_2}(x) = 0$ . As a conclusion,  $\phi_1(x, \mathbf{y}) = p_{1,0}y_1$ ,  $p_{1,0} \neq 0$ .

By exactly the same kind of arguments we have  $\phi_2(x, \mathbf{y}) = q_{0,1}y_2$ ,  $q_{0,1} \neq 0$ . Moreover, we have proved that  $p_{1,0}q_{0,1} = b_{1,1}$  is such that  $c(b_{1,1}v) = c(v)$  for all  $v$  near the origin.  $\square$

#### 4.6. Weak 1-summability of the normalization.

Let us consider the same data as in Lemma 4.6. The following lemma states that an analytic solution of the homological equation considered in  $\mathcal{S}_{\pm} \in \mathcal{S}_{\arg(\pm i\lambda), \eta}$  with  $\eta \in [\pi, 2\pi[$ , admits a weak Gevrey-1 asymptotic expansion in this sector. In other words, it is the weak 1-sum of a formal solution of the homological equation.



**Lemma 4.17.** *Let*

$$Z := Y_0 + C(x, \mathbf{y})\mathcal{C} + xR^{(1)}(x, \mathbf{y})\mathcal{R}$$

*be a formal vector field weakly 1-summable in  $\mathcal{S}_\pm \in \mathcal{S}_{\arg(\pm i\lambda), \eta}$ , with  $\eta \in [\pi, 2\pi[$  and  $C, R^{(1)}$  of order at least one with respect to  $\mathbf{y}$ . We denote by*

$$Z_\pm := Y_0 + C_\pm(x, \mathbf{y})\mathcal{C} + xR_\pm^{(1)}(x, \mathbf{y})\mathcal{R}$$

*the associate weak 1-sums in  $\mathcal{S}_\pm$ . Let also  $A \in \mathbb{C}\llbracket x, \mathbf{y} \rrbracket$  be weakly 1-summable in  $\mathcal{S}_\pm$ , of 1-sum  $A_\pm$  and of order at least one with respect to  $\mathbf{y}$ . Then, any analytic function  $\alpha_\pm(x, \mathbf{y}) = x^M \tilde{\alpha}_\pm(x, \mathbf{y})$ , with  $M \in \mathbb{N}_{>0}$ ,  $\tilde{\alpha}_\pm$  analytic in  $\mathcal{S}_\pm$  and dominated by  $\|\mathbf{y}\|_\infty$ , which satisfies*

$$\mathcal{L}_{Z_\pm}(\alpha_\pm) = x^{M+1}A_\pm(x, \mathbf{y}) \quad ,$$

*has a weak Gevrey-1 asymptotic expansion in  $\mathcal{S}_\pm$ , denoted by  $\alpha$ . Moreover,  $\alpha$  is a formal solution of*

$$\mathcal{L}_Z(\alpha) = x^{M+1}A(x, \mathbf{y}) \quad .$$

*Proof.* Let us write  $Z$  as follows:

$$\begin{aligned} Z &= x^2 \frac{\partial}{\partial x} + (-\lambda + d(y_1 y_2)) + a_1 x + F_1(x, \mathbf{y}) y_1 \frac{\partial}{\partial y_1} \\ &\quad + (\lambda + d(y_1 y_2) + a_2 x + F_2(x, \mathbf{y})) y_2 \frac{\partial}{\partial y_2} \quad , \end{aligned}$$

with  $F_1, F_2$  weakly 1-summable in  $\mathcal{S}_\pm \in \mathcal{S}_{\arg(\pm i\lambda), \eta}$ , with  $\eta \in [\pi, 2\pi[$ , of weak 1-sums  $F_{1,\pm}, F_{2,\pm}$  respectively, which are dominated by  $\|\mathbf{y}\|$ , and with  $d(v) \in v\mathbb{C}\{v\}$ . Consider the Taylor expansion with respect to  $\mathbf{y}$  of  $d, F_1, F_2, A$  and  $\alpha$ :

$$\begin{cases} d(y_1 y_2) = \sum_{k \geq 1} d_k y_1^k y_2^k \\ F_1(x, \mathbf{y}) = \sum_{j_1 + j_2 \geq 1} F_{1,j}(x) \mathbf{y}^j \\ F_2(x, \mathbf{y}) = \sum_{j_1 + j_2 \geq 1} F_{2,j}(x) \mathbf{y}^j \\ A(x, \mathbf{y}) = \sum_{j_1 + j_2 \geq 1} A_j(x) \mathbf{y}^j \\ \alpha(x, \mathbf{y}) = \sum_{j_1 + j_2 \geq 1} \alpha_j(x) \mathbf{y}^j \end{cases}$$

(same expansions are valid in  $\mathcal{S}_\pm$  for the corresponding weak 1-sums). As usual, possibly by reducing  $\mathcal{S}_\pm$ , we can assume that  $\mathcal{S}_\pm = S_\pm \times \mathbf{D}(\mathbf{0}, \mathbf{r})$  for some sector  $S_\pm$  and some polydisc  $\mathbf{D}(\mathbf{0}, \mathbf{r})$ . The homological equation

$$\mathcal{L}_Z(\alpha) = x^{M+1}A(x, \mathbf{y})$$

can be rewritten:

$$\begin{aligned} &wise x^2 \frac{\partial \alpha}{\partial x} + (-\lambda + d(y_1 y_2)) + a_1 x + F_1(x, \mathbf{y}) y_1 \frac{\partial \alpha}{\partial y_1} \\ &\quad + (\lambda + d(y_1 y_2) + a_2 x + F_2(x, \mathbf{y})) y_2 \frac{\partial \alpha}{\partial y_2} = x^{M+1}A(x, \mathbf{y}) \quad . \end{aligned}$$

Invoking the normal convergence on any compact subset of  $\mathcal{S}_\pm$ , we can compute the derivative of

$$\alpha(x, \mathbf{y}) = \sum_{j_1 + j_2 \geq 1} \alpha_j(x) \mathbf{y}^j$$

with respect to  $x, y_1$  or  $y_2$  by term-wise differentiation, in order to obtain after identification:

$$x^2 \frac{d\alpha_{j,\pm}}{dx}(x) + (\lambda(j_2 - j_1) + (a_1 j_1 + a_2 j_2)x) \alpha_{j,\pm}(x) = G_{j,\pm}(x)$$

for all  $\mathbf{j} = (j_1, j_2) \in \mathbb{N}^2$ , where  $G_{\mathbf{j}, \pm}$  depends only on  $d_k, F_{1, \mathbf{k}, \pm}, F_{2, \mathbf{k}, \pm}, \alpha_{\mathbf{k}, \pm}$  and  $A_{1, \pm}$ , for  $k \leq \min(j_1, j_2)$ ,  $|\mathbf{k}| \leq |\mathbf{j}| - 1$  and  $|\mathbf{l}| \leq |\mathbf{j}|$ . We obtain a similar differential equation for the associated formal power series. Let us prove by induction on  $|\mathbf{j}| \geq 0$  that:

- (1)  $G_{\mathbf{j}, \pm}$  is the 1-sum of  $G_{\mathbf{j}}$  in  $S_{\pm}$
- (2)  $G_{j, j}(0) = 0$  if  $\mathbf{j} = (j, j)$
- (3)  $\alpha_{\mathbf{j}, \pm}$  is the 1-sum  $\alpha_{\mathbf{j}}$  in  $S_{\pm}$ .

What will be important is the fact that for all  $\mathbf{j} \in \mathbb{N}^2$ ,  $\alpha_{\mathbf{j}, \pm}$  is bounded in  $S_{\pm}$ .

- For  $\mathbf{j} = (0, 0)$ , we obviously have  $G_{(0,0)} = 0$  and then  $\alpha_{(0,0)} = 0$ .
- Let  $\mathbf{j} = (j_1, j_2) \in \mathbb{N}^2$  with  $|\mathbf{j}| = j_1 + j_2 \geq 1$ . Assume the property holds for all  $\mathbf{k} \in \mathbb{N}^2$  with  $|\mathbf{k}| \leq |\mathbf{j}| - 1$ .
  - (1) Since  $G_{\mathbf{j}}$  depends only on  $d_k, F_{1, \mathbf{k}}, F_{2, \mathbf{k}}, \alpha_{\mathbf{k}}$  and  $A_1$ , for  $k \leq \min(j_1, j_2)$ ,  $|\mathbf{k}| \leq |\mathbf{j}| - 1$  and  $|\mathbf{l}| \leq |\mathbf{j}|$ , then  $G_{\mathbf{j}}$  is 1-summable in  $S_{\pm}$ , of 1-sum  $G_{\mathbf{j}, \pm}$ .
  - (2) We also see that  $G_{j, j}(0) = 0$  if  $\mathbf{j} = (j, j)$ .
  - (3) If  $j_1 \neq j_2$ , then point 1. in Proposition 2.30 tells us that there exists a unique formal solution  $\alpha_{\mathbf{j}}(x)$  of the irregular differential equation we are looking at such that  $\alpha_{\mathbf{j}}(0) = \frac{1}{\lambda(j_2 - j_1)} G_{\mathbf{j}}(0)$ . Moreover, this solution is 1-summable in  $S_{\pm}$  since the same goes for  $G_{\mathbf{j}}$ .
  - (4) If  $j_1 = j_2 = j \geq 1$ , since  $G_{(j,j)}(0) = 0$ , then we can write  $G_{(j,j)}(x) = x \tilde{G}_{(j,j)}(x)$  with  $\tilde{G}_{(j,j)}(x)$  1-summable in  $S_{\pm}$ , and then the differential equation becomes regular:

$$x \frac{d\alpha_{(j,j), \pm}}{dx}(x) + (a_1 + a_2) j \alpha_{(j,j), \pm}(x) = \tilde{G}_{(j,j), \pm}(x) .$$

Since  $a_1 + a_2 \notin \mathbb{Q}_{\leq 0}$ , according to point 2. in Proposition 2.30, the latter equation has a unique formal solution  $\alpha_{(j,j)}(x)$ , which is moreover 1-summable in  $S_{\pm}$ , and its 1-sum is the only solution of this equation bounded in  $S_{\pm}$ . Thus, it is necessarily  $\alpha_{(j,j), \pm}$ .  $\square$

As a consequence, we have the following result.

**Corollary 4.18.** *The sectorial normalizing maps  $\Phi_{\pm}$  in Corollary 4.2 are the weak 1-sums in  $S_{\pm} \in \mathcal{S}_{\arg(\pm\lambda), \eta}$  of the formal normalizing map given by Theorem 1.5, for all  $\eta \in [\pi, 2\pi[$ .*

*Proof.* Let us recall the construction of the sectorial normalizing maps  $\Phi_{\pm}$  obtained in Corollary 4.2. Each sectorial diffeomorphism  $\Phi_{\pm}$  is obtained as the composition of three germs of sectorial diffeomorphisms, using successively Propositions 3.1 and 4.1. The sectorial map obtained thanks to Proposition 3.1 is 1-summable except for  $\arg(\pm\lambda)$ . The sectorial diffeomorphism from Proposition 4.1 is built as the composition of two germs of sectorial diffeomorphisms, using successively Propositions 4.3 and 4.5. These two latter sectorial maps are obtained thanks to the use of Lemma 4.6 (see subsection 4.2). Lemma 4.17 above allows to state that these sectorial diffeomorphisms (obtained through the homological equations in Lemmas 4.6 and 4.17) admit a weak Gevrey-1 asymptotic expansion in domains  $S_{\pm} \in \mathcal{S}_{\arg(\pm\lambda), \eta}$ , for all  $\eta \in [\pi, 2\pi[$ . Consequently, the same goes for the sectorial diffeomorphism of Proposition 4.1, and then for the one of Corollary 4.2 (we have used here Proposition 2.24 for the composition).

Using point 3 in Lemma 2.23, we deduce that the weak Gevrey-1 asymptotic expansion of the sectorial normalizing map of Corollary 4.2 is a formal normalizing map, such as the one of Theorem 1.5. By uniqueness, it is the unique formal normalizing map.  $\square$

#### 4.7. Proof of Theorem 1.10.

*Proof.* (Theorem 1.10)

Theorem 1.10 is a summary of Corollaries 4.2 and 4.18.  $\square$

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