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# On uniqueness for a rough transport-diffusion equation.

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## Abstract

In this Note, we study a transport-diffusion equation with rough coefficients and we prove that solutions are unique in a low-regularity class. *To cite this article: G. Lévy, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

## Résumé

**Sur l'unicité pour une équation de transport-diffusion irrégulière.** Dans cette Note, nous étudions une équation de transport-diffusion à coefficients irréguliers et nous prouvons l'unicité de sa solution dans une classe de fonctions peu régulières. *Pour citer cet article : G. Lévy, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

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## 1. Introduction

In this note, we address the problem of uniqueness for a transport-diffusion equation with rough coefficients. Our primary interest and motivation is a uniqueness result for an equation obeyed by the vorticity of a Leray-type solution of the Navier-Stokes equation in the full, three dimensional space. The main theorem of this note is the following.

**Theorem 1.1 :** Let  $v$  be a divergence free vector field in  $L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$  and  $a$  a function in  $L^2(\mathbb{R}_+ \times \mathbb{R}^3)$ . Assume that  $a$  is a distributional solution of the Cauchy problem

$$(C) \begin{cases} \partial_t a + \nabla \cdot (av) - \Delta a = 0 \\ a(0) = 0, \end{cases} \quad (1)$$

where the initial condition is understood in the distributional sense. Then  $a$  is identically zero on  $\mathbb{R}_+ \times \mathbb{R}^3$ .

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As a preliminary remark, the assumptions on both  $v$  and  $a$  entail that  $\partial_t a$  belongs to  $L^1_{loc}(\mathbb{R}_+, H^{-2}(\mathbb{R}^3))$  and thus, in particular,  $a$  is also in  $\mathcal{C}(\mathbb{R}_+, \mathcal{D}'(\mathbb{R}^3))$ . In Theorem 1.1,  $a$  is to be thought of as a scalar component of the vorticity of  $v$ , which is in the original problem a Leray solution of the Navier-Stokes equation. In particular, we only know that  $a$  belongs to  $L^2(\mathbb{R}_+ \times \mathbb{R}^3)$  and  $L^\infty(\mathbb{R}_+, \dot{H}^{-1}(\mathbb{R}^3))$ , though we will not use the second assumption. The reader accustomed to three-dimensional fluid mechanics will notice that, comparing the above equation with the actual vorticity equations in  $3D$ , a term of the type  $a\partial_i v$  is missing. In the original problem, where this Theorem first appeared, we actually rely on a double application of Theorem 1.1. For some technical reasons, only the second application of Theorem 1.1 takes in account the abovementioned term.

As opposed to the standard DiPerna-Lions theory, we cannot assume that  $a$  is in  $L^\infty(\mathbb{R}_+, L^p(\mathbb{R}^3))$  for some  $p \geq 1$ . However, our proof does bear a resemblance to the work of DiPerna and Lions; our result may thus be viewed as a generalization of their techniques. Because of the low regularity of both the vector field  $v$  and the scalar field  $a$ , the use of energy-type estimates seems difficult. This is the main reason why we rely instead on a duality argument, embodied by the following theorem.

**Theorem 1.2 :** Given  $v$  a divergence free vector field in  $L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$  and a smooth  $\varphi_0$  in  $\mathcal{D}(\mathbb{R}^3)$ , there exists a distributional solution of the Cauchy problem

$$(C') \begin{cases} \partial_t \varphi - v \cdot \nabla \varphi - \Delta \varphi = 0 \\ \varphi(0) = \varphi_0 \end{cases} \quad (2)$$

with the bounds

$$\|\varphi(t)\|_{L^\infty(\mathbb{R}^3)} \leq \|\varphi_0\|_{L^\infty(\mathbb{R}^3)} \quad (3)$$

and

$$\|\partial_j \varphi(t)\|_{L^2(\mathbb{R}^3)}^2 + \int_0^t \|\nabla \partial_j \varphi(s)\|_{L^2(\mathbb{R}^3)}^2 ds \leq \|\partial_j \varphi_0\|_{L^2(\mathbb{R}^3)}^2 + \|\varphi_0\|_{L^\infty(\mathbb{R}^3)}^2 \|\partial_j v\|_{L^2(\mathbb{R}_+ \times \mathbb{R}^3)}^2 \quad (4)$$

for  $j = 1, 2, 3$  and any positive time  $t$ .

By reversing the arrow of time, this amounts to build, for any strictly positive  $T$ , a solution on  $[0, T] \times \mathbb{R}^3$  of the Cauchy problem

$$(-C') \begin{cases} -\partial_t \varphi - v \cdot \nabla \varphi - \Delta \varphi = 0 \\ \varphi(T) = \varphi_T, \end{cases} \quad (5)$$

where we have set  $\varphi_T := \varphi_0$  for the reader's convenience.

## 2. Proofs

We begin with the dual existence result.

**Proof (of Theorem 1.2.) :** Let us choose some mollifying kernel  $\rho = \rho(t, x)$  and denote  $v^\delta := \rho_\delta * v$ , where  $\rho_\delta(t, x) := \delta^{-4} \rho(\frac{t}{\delta}, \frac{x}{\delta})$ . Let  $(C'_\delta)$  be the Cauchy problem  $(C')$  where we replaced  $v$  by  $v^\delta$ . The existence of a (smooth) solution  $\varphi^\delta$  to  $(C'_\delta)$  is then easily obtained thanks to, for instance, a Friedrichs method combined with heat kernel estimates. We now turn to estimates uniform in the regularization parameter  $\delta$ . The first one is a sequence of energy estimates done in  $L^p$  with  $p \geq 2$ , which yields the maximum principle in the limit. Multiplying the equation on  $\varphi^\delta$  by  $\varphi^\delta |\varphi^\delta|^{p-2}$  and integrating in space and time, we get

$$\frac{1}{p} \|\varphi^\delta(t)\|_{L^p(\mathbb{R}^3)}^p + (p-1) \int_0^t \|\nabla \varphi^\delta(s) |\varphi^\delta(s)|^{\frac{p-2}{2}}\|_{L^2(\mathbb{R}^3)}^2 ds = \frac{1}{p} \|\varphi_0\|_{L^p(\mathbb{R}^3)}^p. \quad (6)$$

Discarding the gradient term, taking  $p$ -th root in both sides and letting  $p$  go to infinity gives

$$\|\varphi^\delta(t)\|_{L^\infty(\mathbb{R}^3)} \leq \|\varphi_0\|_{L^\infty(\mathbb{R}^3)}. \quad (7)$$

To obtain the last estimate, let us derive for  $1 \leq j \leq 3$  the equation satisfied by  $\partial_j \varphi^\delta$ . We have

$$\partial_t \partial_j \varphi^\delta - v^\delta \cdot \nabla \partial_j \varphi^\delta - \Delta \partial_j \varphi^\delta = \partial_j v^\delta \cdot \nabla \varphi^\delta. \quad (8)$$

Multiplying this new equation by  $\partial_j \varphi^\delta$  and integrating in space and time gives

$$\begin{aligned} \frac{1}{2} \|\partial_j \varphi^\delta(t)\|_{L^2(\mathbb{R}^3)}^2 + \int_0^t \|\nabla \partial_j \varphi^\delta(s)\|_{L^2(\mathbb{R}^3)}^2 ds &= \frac{1}{2} \|\partial_j \varphi_0\|_{L^2(\mathbb{R}^3)}^2 \\ &+ \int_0^t \int_{\mathbb{R}^3} \partial_j \varphi^\delta(s, x) \partial_j v^\delta(s, x) \cdot \nabla \varphi^\delta(s, x) dx ds. \end{aligned} \quad (9)$$

Since  $v$  is divergence free, the gradient term in the left-hand side does not contribute to Equation (9). Denote by  $I(t)$  the last integral written above. Integrating by parts and recalling that  $v$  is divergence free, we have

$$\begin{aligned} I(t) &= - \int_0^t \int_{\mathbb{R}^3} \varphi^\delta(s, x) \partial_j v^\delta(s, x) \cdot \nabla \partial_j \varphi^\delta(s, x) dx ds \\ &\leq \|\varphi_0\|_{L^\infty(\mathbb{R}^3)} \int_0^t \|\partial_j v^\delta(s)\|_{L^2(\mathbb{R}^3)} \|\nabla \partial_j \varphi^\delta(s)\|_{L^2(\mathbb{R}^3)} ds \\ &\leq \frac{1}{2} \int_0^t \|\nabla \partial_j \varphi^\delta(s)\|_{L^2(\mathbb{R}^3)}^2 ds + \frac{1}{2} \|\varphi_0\|_{L^\infty(\mathbb{R}^3)}^2 \int_0^t \|\partial_j v^\delta(s)\|_{L^2(\mathbb{R}^3)}^2 ds. \end{aligned}$$

And finally, the energy estimate on  $\partial_j \varphi^\delta$  reads

$$\|\partial_j \varphi^\delta(t)\|_{L^2(\mathbb{R}^3)}^2 + \int_0^t \|\nabla \partial_j \varphi^\delta(s)\|_{L^2(\mathbb{R}^3)}^2 ds \leq \|\partial_j \varphi_0\|_{L^2(\mathbb{R}^3)}^2 + \|\varphi_0\|_{L^\infty(\mathbb{R}^3)}^2 \|\partial_j v\|_{L^2(\mathbb{R}_+ \times \mathbb{R}^3)}^2. \quad (10)$$

Thus, the family  $(\varphi^\delta)_\delta$  is bounded in  $L^\infty(\mathbb{R}_+, H^1(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+, \dot{H}^2(\mathbb{R}^3)) \cap L^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$ . Up to some extraction, we have the weak convergence of  $(\varphi^\delta)_\delta$  in  $L^2(\mathbb{R}_+, \dot{H}^2(\mathbb{R}^3))$  and its weak-\* convergence in  $L^\infty(\mathbb{R}_+, H^1(\mathbb{R}^3)) \cap L^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$  to some function  $\varphi$ .

By interpolation, we also have  $\nabla \varphi^\delta \rightharpoonup \nabla \varphi$  weakly in  $L^4(\mathbb{R}_+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))$  as  $\delta \rightarrow 0$ . As a consequence, because  $v^\delta \rightarrow v$  strongly in  $L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$  as  $\delta \rightarrow 0$ , the following convergences hold :

$$\begin{aligned} \Delta \varphi^\delta &\rightharpoonup \Delta \varphi \text{ in } L^2(\mathbb{R}_+ \times \mathbb{R}^3); \\ v^\delta \cdot \nabla \varphi^\delta, \partial_t \varphi^\delta &\rightharpoonup v \cdot \nabla \varphi, \partial_t \varphi \text{ in } L^{\frac{4}{3}}(\mathbb{R}_+, L^2(\mathbb{R}^3)). \end{aligned}$$

In particular, such a  $\varphi$  is a distributional solution of  $(C')$  with the desired regularity.  $\square$

We now state a Lemma which will be useful in the final proof.

**Lemma 2.1 :** Let  $v$  be a fixed, divergence free vector field in  $L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))$ . Let  $(\varphi^\delta)_\delta$  be a bounded family in  $L^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$ . Let  $\rho = \rho(x)$  be some smooth function supported inside the unit ball of  $\mathbb{R}^3$  and define  $\rho_\varepsilon := \varepsilon^{-3} \rho(\frac{\cdot}{\varepsilon})$ . Define the commutator  $C^{\varepsilon, \delta}$  by

$$C^{\varepsilon, \delta}(s, x) := v(s, x) \cdot (\nabla \rho_\varepsilon * \varphi^\delta(s))(x) - (\nabla \rho_\varepsilon * (v(s) \varphi^\delta(s)))(x).$$

Then

$$\|C^{\varepsilon, \delta}\|_{L^2(\mathbb{R}_+ \times \mathbb{R}^3)} \leq \|\nabla \rho\|_{L^1(\mathbb{R}^3)} \|\nabla v\|_{L^2(\mathbb{R}_+, \dot{H}^1(\mathbb{R}^3))} \|\varphi^\delta\|_{L^\infty(\mathbb{R}_+, H^1(\mathbb{R}^3))}. \quad (11)$$

This type of lemma is absolutely not new. Actually, it is strongly reminiscent of Lemma II.1 in [2] and serves the same purpose. We are now in position to prove the main theorem of this note.

**Proof (of Theorem 1.1.) :** Let  $\rho = \rho(x)$  be a radial mollifying kernel and define  $\rho_\varepsilon(x) := \varepsilon^{-3}\rho(\frac{x}{\varepsilon})$ . Convolving the equation on  $a$  by  $\rho_\varepsilon$  gives, denoting  $a_\varepsilon := \rho_\varepsilon * a$ ,

$$(C_\varepsilon) \quad \partial_t a_\varepsilon + \nabla \cdot (a_\varepsilon v) - \Delta a_\varepsilon = \nabla \cdot (a_\varepsilon v) - \rho_\varepsilon * \nabla \cdot (av). \quad (12)$$

Notice that even without any smoothing in time,  $a_\varepsilon, \partial_t a_\varepsilon$  are in  $L^\infty(\mathbb{R}_+, C^\infty(\mathbb{R}^3))$  and  $L^1(\mathbb{R}_+, C^\infty(\mathbb{R}^3))$  respectively, which is enough to make the upcoming computations rigorous. In what follows, we let  $\varphi^\delta$  be a solution of the Cauchy problem  $(-C'_\delta)$ , with  $(-C'_\delta)$  being  $(-C')$  with  $v$  replaced by  $v^\delta$ . Let us now multiply, for  $\delta, \varepsilon > 0$  the equation  $(C_\varepsilon)$  by  $\varphi^\delta$  and integrate in space and time. After integrating by parts (which is justified by the high regularity of the terms we have written), we get

$$\int_0^T \int_{\mathbb{R}^3} \partial_t a_\varepsilon(s, x) \varphi^\delta(s, x) dx ds = \langle a_\varepsilon(T), \varphi_T \rangle_{\mathcal{D}'(\mathbb{R}^3), \mathcal{D}(\mathbb{R}^3)} - \int_0^T \int_{\mathbb{R}^3} a_\varepsilon(s, x) \partial_t \varphi^\delta(s, x) dx ds$$

and

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^3} [\nabla \cdot (v(s, x) a_\varepsilon(s, x)) - \rho_\varepsilon(x) * \nabla \cdot (v(s, x) a(s, x))] \varphi^\delta(s, x) dx ds \\ = \int_0^T \int_{\mathbb{R}^3} a(s, x) C^{\varepsilon, \delta}(s, x) dx ds, \end{aligned}$$

where the commutator  $C^{\varepsilon, \delta}$  has been defined in the Lemma. From these two identities, it follows that

$$\begin{aligned} \langle a_\varepsilon(T), \varphi_T \rangle_{\mathcal{D}'(\mathbb{R}^3), \mathcal{D}(\mathbb{R}^3)} &= \int_0^T \int_{\mathbb{R}^3} a(s, x) C^{\varepsilon, \delta}(s, x) dx ds \\ &\quad - \int_0^T \int_{\mathbb{R}^3} a_\varepsilon(s, x) (-\partial_t \varphi^\delta(s, x) - v(s, x) \cdot \nabla \varphi^\delta(s, x) - \Delta \varphi^\delta(s, x)) dx ds. \end{aligned}$$

From the Lemma, we know that  $(C^{\varepsilon, \delta})_{\varepsilon, \delta}$  is bounded in  $L^2(\mathbb{R}^+ \times \mathbb{R}^3)$ . Because  $v \cdot \nabla \varphi^\delta \rightarrow v \cdot \nabla \varphi$  in  $L^{\frac{4}{3}}(\mathbb{R}^+, L^2(\mathbb{R}^3))$  as  $\delta \rightarrow 0$ , the only weak limit point in  $L^2(\mathbb{R}_+ \times \mathbb{R}^3)$  of the family  $(C^{\varepsilon, \delta})_{\varepsilon, \delta}$  as  $\delta \rightarrow 0$  is  $C^{\varepsilon, 0}$ . Thanks to the smoothness of  $a_\varepsilon$  for each fixed  $\varepsilon$ , we can take the limit  $\delta \rightarrow 0$  in the last equation, which leads to

$$\langle a_\varepsilon(T), \varphi_T \rangle_{\mathcal{D}'(\mathbb{R}^3), \mathcal{D}(\mathbb{R}^3)} = \int_0^T \int_{\mathbb{R}^3} a(s, x) C^{\varepsilon, 0}(s, x) dx ds. \quad (13)$$

Again, the family  $(C^{\varepsilon, 0})_\varepsilon$  is bounded in  $L^2(\mathbb{R}_+ \times \mathbb{R}^3)$  and its only limit point as  $\varepsilon \rightarrow 0$  is 0, simply because  $v \cdot \nabla \varphi_\varepsilon - \rho_\varepsilon * (v \cdot \nabla \varphi) \rightarrow 0$  in  $L^{\frac{4}{3}}(\mathbb{R}^+, L^2(\mathbb{R}^3))$ . Taking the limit  $\varepsilon \rightarrow 0$ , we finally obtain

$$\langle a(T), \varphi_T \rangle_{\mathcal{D}'(\mathbb{R}^3), \mathcal{D}(\mathbb{R}^3)} = 0. \quad (14)$$

This being true for any test function  $\varphi_T$ ,  $a(T)$  is the zero distribution and finally  $a \equiv 0$ .  $\square$

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