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# SIGN CHANGES OF THE COEFFICIENTS OF AUTOMORPHIC $L$ -FUNCTIONS

YUK-KAM LAU, JIANYA LIU & JIE WU

## 1. INTRODUCTION

The study of the least quadratic non-residue has an almost century-long history, see for example, [80, 10, 52, 49]. To facilitate the investigation, many useful tools in analytic number theory were developed, such as the estimates on character sums [10, 25] and the large sieve inequalities [52, 63]. Viewing characters as  $GL_1$  objects, one may wonder what the  $GL_2$  (or even  $GL_m$ ) analogues are for this problem (about the least quadratic non-residue). As a result, various interesting problems are recently raised, comprising

- the first negative Hecke eigenvalues or more generally the first negative coefficient of automorphic  $L$ -functions for  $GL_m(\mathbb{A}_{\mathbb{Q}})$  with  $m \geq 2$ ;
- the recognition of primitive forms by the values of Hecke eigenvalues or more generally the multiplicity one theorem on cuspidal automorphic representations of  $GL_m(\mathbb{A}_{\mathbb{K}})$  with  $m \geq 2$ ;
- the recognition of primitive forms by the signs of Hecke eigenvalues and its analogue of the multiplicity one theorem;
- the number of Hecke eigenvalues of same signs;
- the distribution of Hecke eigenvalues.

In the past few years, together with our collaborators W. Kohnen, E. Kowalski, Y. Qu, I. E. Shparlinski, K. Soundararajan, Y.-H. Wang and W.-G. Zhai, we made contributions to these problems [49, 50, 40, 51, 54, 43, 53, 46, 47, 41, 83]. This survey gives an account of background and the recent development, which includes the great contributions of other authors. We are attempting to elucidate interesting viewpoints, ideas and methods which, even unspecified, may not originate from us – the present authors. Moreover, we formulate some questions for future study.

## 2. THE LEAST QUADRATIC NON-RESIDUE

Let  $q \geq 2$  be an integer and  $\chi$  a non-principal Dirichlet character modulo  $q$ . We refer to the evaluation of the least integer  $n_{\chi}$  among all positive integers  $n$  for which  $\chi(n) \neq 0, 1$  as Linnik's problem. In case  $\chi$  coincides with the Legendre symbol,  $n_{\chi}$  is the least quadratic non-residue. Naturally an attack of Linnik's problem is to bound  $n_{\chi}$  in terms of the modulus  $q$ .

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**2.1. Estimates for character sums and individual bounds for  $n_\chi$ .** The estimates for character sums have important applications in analytic number theory. Obviously the well known Pólya-Vinogradov inequality

$$(2.1) \quad \max_{x \geq 1} \left| \sum_{n \leq x} \chi(n) \right| \ll q^{1/2} \log q$$

implies that  $n_\chi \ll q^{1/2} \log q$ . For prime  $q$ , Vinogradov [80] improved it to

$$(2.2) \quad n_\chi \ll q^{1/(2\sqrt{e})} (\log q)^2$$

by combining a simple but elegant argument with (2.1). Burgess ([9], [10], [11]) wrote a series of important papers sharpening (2.1). His well known estimate on character sums is as follows: *For any  $\varepsilon > 0$ , there is  $\delta(\varepsilon) > 0$  such that*

$$(2.3) \quad \left| \sum_{n \leq x} \chi(n) \right| \ll_\varepsilon x q^{-\delta(\varepsilon)}$$

*provided  $x \geq q^{1/3+\varepsilon}$ . The last condition can be improved to  $x \geq q^{1/4+\varepsilon}$  if  $q$  is cubefree.* When  $q$  is prime, Burgess deduced, via (2.3) and Vinogradov's argument,

$$(2.4) \quad n_\chi \ll_\varepsilon q^{1/(4\sqrt{e})+\varepsilon}.$$

More generally, we may extend the argument in a straightforward way to get the following result.

**Theorem 1.** ([49], page 424) *Under the above notation, we have*

$$n_\chi \ll_\varepsilon \begin{cases} q^{1/(4\sqrt{e})+\varepsilon} & \text{if } q \text{ is cubefree,} \\ q^{1/(3\sqrt{e})+\varepsilon} & \text{otherwise.} \end{cases}$$

**2.2. Linnik's large sieve and almost bound for  $n_\chi$ .** The exponents of Theorem 1 are not likely to be optimal. Vinogradov conjectured that  $n_\chi \ll_\varepsilon q^\varepsilon$  for all integers  $q \geq 2$  and any  $\varepsilon > 0$ . Under the Generalized Riemann Hypothesis (GRH), Linnik [52] affirmed the conjecture, and later Ankeny [2] gave a sharper estimate

$$(2.5) \quad n_\chi \ll (\log q)^2$$

(still assuming GRH). In 1942, Linnik [52] created a new powerful tool entitled large sieve today. He proved that (2.5) holds for almost all primitive real characters of prime modulus  $q$ . Let us introduce some notation for the set-up. We denote  $\mathcal{D}$  (resp.  $\mathcal{D}(Q)$ ) to be the set of fundamental discriminants  $d$  (resp. with  $|d| \leq Q$  where  $Q \geq 2$ ). Explicitly,  $\mathcal{D}$  is the set of non-zero integers  $d$  which are products of coprime factors of the form  $-4, 8, -8, p'$  where  $p' := (-1)^{(p-1)/2} p$  ( $p$  odd prime). Also, we write  $\mathcal{K}$  (resp.  $\mathcal{K}(Q)$ ) for the set of real primitive characters (resp. with modulus  $q \leq Q$ ). There is a bijection between  $\mathcal{D}$  and  $\mathcal{K}$  given by

$$d \mapsto \chi_d(\cdot) = \left( \frac{\cdot}{d} \right)_K$$

where  $\left( \frac{\cdot}{d} \right)_K$  is the Kronecker symbol. Note that the modulus of  $\chi_d$  equals  $|d|$  and

$$(2.6) \quad |\mathcal{D}(Q)| = |\mathcal{K}(Q)| = \frac{6}{\pi^2} Q + O(Q^{1/2}).$$

Linnik's large sieve inequality states that the estimate

$$\sum_{\substack{q \leq Q \\ q \text{ prime}}} \left| \sum_{\substack{p \leq P \\ p \text{ prime}}} \chi_q(p) \right|^{2j} \ll Q^2 (jP)^j + (jP^2)^j$$

holds uniformly for  $j \geq 1$ ,  $P \geq 2$  and  $Q \geq 2$ . Taking

$$P = (\log Q)^{2+\varepsilon} \quad \text{and} \quad j = (1 - \varepsilon) \frac{\log P}{\log \log P},$$

we easily see that  $n_{\chi_{p'}} \ll_{\varepsilon} (\log p)^{2+\varepsilon}$  for almost all primes  $p$ . (See [14] for more details.) The exponent  $2 + \varepsilon$  is refined to  $1 + \varepsilon$  by Baier in [3, Theorem 2]. Baier used the large sieve inequality of Heath-Brown [28] and obtained

$$|\{p \leq Q : n_{\chi_{p'}} \geq (\log Q)^{\alpha}\}| \ll_{\varepsilon} Q^{1/\alpha+\varepsilon}$$

for any  $\alpha > 1$  and all  $Q \geq 2$ . His argument does not cover the case  $\alpha = 1$ . One might thus ask whether or not the exceptional set of primes  $p$  for which  $n_{\chi_{p'}} \gg \log p$  is sparse. In [49], Lau & Wu gave an affirmative answer as stated in Theorem 2 below, but we would like to take this occasion to acknowledge the priority of Elliott's work [16], see Remark 1 below. Our key tool is the large sieve inequality of Elliott-Montgomery-Vaughan (cf. [18, 63, 49]):

$$\sum_{d \in \mathcal{D}(Q)} \left| \sum_{P < p \leq 2P} \frac{\chi_d(p)}{p} \right|^{2j} \ll Q \left( \frac{6j}{P \log P} \right)^j + \left( \frac{6P}{(\log P)^2} \right)^j$$

uniformly for  $2 \leq P \leq Q$  and  $j \geq 1$ . Using this, we obtain the following result.

**Theorem 2.** ([49], Theorem 1) *We have*

$$(2.7) \quad n_{\chi_d} \ll \log |d|$$

for all but except  $O(Qe^{-c(\log Q)/\log \log Q})$  characters  $\chi_d \in \mathcal{K}(Q)$ .

*Remark 1.* Very recently K. Gong informed us that Elliott [16, page 841] had obtained such a result for primitive quadratic characters via a similar method. We thank Dr. Gong for this information.

Concerning large  $n_{\chi}$ 's, Fridlender [22], Salié [71] and Chowla & Turán [19] showed independently that there are infinitely many primes  $p$  for which

$$(2.8) \quad n_{\chi_{p'}} \gg \log p,$$

or in other words,  $n_{\chi_{p'}} = \Omega(\log p)$ . Under GRH, Montgomery [62] could improve this to  $n_{\chi_{p'}} = \Omega(\log p \log \log p)$ . Unconditionally Graham & Ringrose [25] obtained  $n_{\chi_{p'}} = \Omega(\log p \log \log \log p)$ . On the other hand, Erdős [20] showed that

$$(2.9) \quad \lim_{x \rightarrow \infty} \frac{1}{\pi(x)} \sum_{p \leq x} n_{\chi_{p'}} = \text{constant},$$

where  $\pi(x)$  denotes the number of primes up to  $x$ . Hence the density of  $p'$  for which  $n_{\chi_{p'}}$  satisfies (2.8) is low. The result (2.9) is extended and refined by Elliott in [15, 17].

From (2.9) or its refinement in [15], it follows that for any fixed constant  $\delta > 0$ ,

$$\sum_{\substack{p \leq x \\ n_{\chi_p} \geq \delta \log p}} 1 \ll_{\delta} \frac{x}{(\log x)^2}.$$

Based on the method of Graham & Ringrose [25], Lau & Wu [49] derived the result below, illustrating the qualitative tightness of Theorem 2.

**Theorem 3.** ([49], Theorem 2) *For any fixed constant  $\delta > 0$ , there are a sequence of positive real numbers  $\{Q_n\}_{n=1}^{\infty}$  with  $Q_n \rightarrow \infty$  and a positive constant  $c$  such that*

$$(2.10) \quad \sum_{\substack{Q_n^{1/2} < p \leq Q_n \\ n_{\chi_p} \geq \delta \log p}} 1 \gg_{\delta} Q_n e^{-c(\log Q_n)/\log \log Q_n}.$$

Further if we assume that there is an absolute positive constant  $C$  such that the  $L$ -function

$$\prod_{\ell=1,4} \prod_{m|P_y} L(s, \chi_{\ell m}),$$

where  $P_y := \prod_{p \leq y} p$  and  $L(s, \chi_{\ell m})$  is the Dirichlet  $L$ -function associated to  $\chi_{\ell m}$ , has no exceptional zeros in the region

$$\sigma \geq 1 - \frac{C_1(\log \log P_y)^{1/2}}{\log P_y} \quad \text{and} \quad |\tau| \leq \log P_y,$$

then (2.10) holds for all  $Q \geq Q_0$ , where  $Q_0$  is a large constant.

To end this section, let us remark the main ingredient of Graham & Ringrose's method – a  $q$ -analogue of van der Corput's result, which can be stated as follows: Suppose that  $q = 2^{\nu} r$ , where  $0 \leq \nu \leq 3$  and  $r$  is an odd squarefree integer, and that  $\chi$  is a non-principal character mod  $q$ . Let  $p$  be the largest prime factor of  $q$ . Suppose that  $k$  is a non-negative integer, and  $K = 2^k$ . Finally, assume that  $N \leq M$ . Then

$$\sum_{M < n \leq M+N} \chi(n) \ll M^{1 - \frac{k+3}{8K-2} p^{-\frac{k^2+3k+4}{32K-8}} q^{-\frac{1}{8K-2}} d(q)^{\frac{32k^2+11k+8}{16K-4}} (\log q)^{\frac{k+3}{8K-2}} \sigma_{-1}(q),$$

where  $\sigma_a(q) := \sum_{d|q} d^a$  and  $\tau(q) := \sigma_0(q)$ . The implied constant is absolute. This result gives non-trivial estimates for very short character sums provided  $r$  is friable, i.e.  $r$  is composed of small prime factors only. For example, if  $p \mid r$  for all  $p \leq \exp\{(\log r)^{1/2}\}$ , then (2.3) holds for a much shorter range of  $x \geq q^{\epsilon}$ .

### 3. AUTOMORPHIC $L$ -FUNCTIONS OF $\mathrm{GL}_m(\mathbb{A}_{\mathbb{K}})$

In Section 2 we dealt with the classical object of Dirichlet characters and saw many nice results. Associated to a character  $\chi$ , one defines

$$L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

for  $\Re s > 1$ , which is an  $L$ -function of  $\mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}})$ . Now we turn to automorphic  $L$ -functions and see what is known in this general setting.

Let  $\mathbb{K}$  be an algebraic number field of degree  $\ell := [\mathbb{K} : \mathbb{Q}]$  and  $\mathbb{A}_{\mathbb{K}} = \mathbb{K}_{\infty} \times \mathbb{A}_{\mathbb{K},f}$  be its adèle ring, where  $\mathbb{K}_{\infty}$  is the product of the Archimedean completions of  $\mathbb{K}$ , and the ring  $\mathbb{A}_{\mathbb{K},f}$  of finite adèles is a restricted direct product of the completions  $\mathbb{K}_v$  over non-Archimedean places  $v$ . Suppose that  $\pi$  is an automorphic irreducible cuspidal representation of  $\mathrm{GL}_m(\mathbb{A}_{\mathbb{K}})$ . Then  $\pi$  is a restricted tensor product

$$(3.1) \quad \pi = \otimes_v \pi_v = \pi_{\infty} \otimes \pi_f,$$

where  $v$  runs over all places of  $\mathbb{K}$ , and  $\pi_v$  is unramified for almost all finite places  $v$ . At every finite place  $v$  where  $\pi_v$  is unramified we associate a semisimple conjugacy class

$$\mathbf{A}_{\pi,v} := \begin{pmatrix} \alpha_{\pi,v}(1) & & \\ & \ddots & \\ & & \alpha_{\pi,v}(m) \end{pmatrix}$$

and define the local  $L$ -function for the finite place  $v$  as

$$(3.2) \quad L(s, \pi_v) := \det(\mathbf{I} - q_v^{-s} \mathbf{A}_{\pi,v})^{-1} = \prod_{j=1}^m \left(1 - \frac{\alpha_{\pi,v}(j)}{q_v^s}\right)^{-1},$$

where  $q_v = N(\mathfrak{p}_v) = N(v)$  is the norm in  $\mathbb{K}_v$  and  $\mathbf{I}$  is the unit matrix of degree  $m$ . It is possible to write the local factors at ramified places  $v$  in the form of (3.2) with the convention that some of the  $\alpha_{\pi,v}(j)$ 's may be zero. The finite part  $L$ -function  $L(s, \pi)$  is defined as

$$(3.3) \quad L(s, \pi) := \prod_{v < \infty} L(s, \pi_v).$$

This Euler product is proved to be absolutely convergent for  $\Re s > 1$  and has a Dirichlet series expression of the form

$$(3.4) \quad L(s, \pi) = \sum_{n=1}^{\infty} \frac{\lambda_{\pi}(n)}{n^s}.$$

Also, the Archimedean  $L$ -function is defined as

$$(3.5) \quad L(s, \pi_{\infty}) := \pi^{-\ell m s/2} \prod_{j=1}^{\ell m} \Gamma\left(\frac{s + \mu_{\pi}(j)}{2}\right).$$

The coefficients  $\{\alpha_{\pi,v}(j)\}_{1 \leq j \leq m}$  and  $\{\mu_{\pi}(j)\}_{1 \leq j \leq m}$  are called local parameters of  $\pi$ , respectively, at finite places and at infinite places. They satisfy the following trivial bounds:

$$|\alpha_{\pi,v}(j)| \leq \sqrt{p}, \quad |\Re \mu_{\pi}(j)| \leq \frac{1}{2}.$$

In connection with (3.1), the complete  $L$ -function associated to  $\pi$  is defined by

$$(3.6) \quad \Phi(s, \pi) := L(s, \pi_{\infty}) L(s, \pi).$$

This complete  $L$ -function has an analytic continuation to the whole complex plane, which is entire and satisfies the functional equation

$$\Phi(s, \pi) = W_{\pi} q_{\pi}^{1/2-s} \Phi(1-s, \tilde{\pi})$$

where  $\tilde{\pi}$  is the contragredient of  $\pi$ ,  $W_{\pi}$  a complex number of modulus 1, and  $q_{\pi}$  a positive integer called the arithmetic conductor of  $\pi$ , see [12].

Let  $\pi = \otimes \pi_v$  and  $\pi' = \otimes \pi'_v$  be automorphic irreducible cuspidal representations of  $\mathrm{GL}_m(\mathbb{A}_{\mathbb{K}})$  and  $\mathrm{GL}_{m'}(\mathbb{A}_{\mathbb{K}})$ , respectively. The (finite part of the) Rankin-Selberg  $L$ -function associated to  $\pi$  and  $\pi'$  is defined by the Euler product

$$(3.7) \quad L(s, \pi \times \tilde{\pi}') := \prod_{v < \infty} L(s, \pi_v \times \tilde{\pi}'_v),$$

where

$$(3.8) \quad L(s, \pi_v \times \tilde{\pi}'_v) := \prod_{j=1}^m \prod_{j'=1}^{m'} \left( 1 - \frac{\alpha_{\pi,v}(j) \bar{\alpha}_{\pi',v}(j')}{q_v^s} \right)^{-1}$$

is the local  $L$ -function at the unramified finite place  $v$ , i.e. where  $\pi_v$  and  $\pi'_v$  are both unramified. The local factor can be similarly defined at the place  $v$  where  $\pi_v$  or  $\pi'_v$  are ramified. The Euler product (3.7) is absolutely convergent for  $\Re s > 1$  and admits a Dirichlet series expansion of the form

$$(3.9) \quad L(s, \pi \times \tilde{\pi}') = \sum_{n=1}^{\infty} \frac{\lambda_{\pi \times \tilde{\pi}'}(n)}{n^s}.$$

The complete Rankin-Selberg  $L$ -function is

$$(3.10) \quad \Phi(s, \pi \times \tilde{\pi}') := L(s, \pi_{\infty} \times \tilde{\pi}'_{\infty}) L(s, \pi \times \tilde{\pi}')$$

with

$$(3.11) \quad L(s, \pi_{\infty} \times \tilde{\pi}'_{\infty}) = \pi^{-mm' \ell s / 2} \prod_{j=1}^{\ell m m'} \Gamma \left( \frac{s + \mu_{\pi \times \tilde{\pi}'}(j)}{2} \right).$$

When both  $\pi$  and  $\pi'$  are unramified at the infinite place  $v$ , we have

$$\{\mu_{\pi \times \tilde{\pi}'}(j)\}_{1 \leq j \leq mm'} = \{\mu_{\pi}(j) + \mu_{\tilde{\pi}'}(j')\}_{1 \leq j \leq m, 1 \leq j' \leq m'}.$$

By Shahidi [74, 75, 76, 77], the complete  $L$ -function  $\Phi(s, \pi \times \tilde{\pi}')$  has an analytic continuation to the entire complex plane, and satisfies the functional equation

$$(3.12) \quad \Phi(s, \pi \times \tilde{\pi}') = W_{\pi \times \tilde{\pi}'} q_{\pi \times \tilde{\pi}'}^{1/2-s} \Phi(1-s, \tilde{\pi} \times \pi')$$

where  $W_{\pi \times \tilde{\pi}'} \in \mathbb{C}$  is a constant of modulus 1, and  $q_{\pi \times \tilde{\pi}'} \in \mathbb{N}$  is the arithmetic conductor of  $\pi \times \tilde{\pi}'$ . By Jacquet & Shalika [35] and Mœglin & Waldspurger [61], we know that  $L(s, \pi \times \tilde{\pi}')$  is holomorphic unless  $\pi' = \pi \otimes |\det|^{i\tau_0}$  for some  $\tau_0 \in \mathbb{R}$ , and in this case  $L(s, \pi \times \tilde{\pi}')$  has exactly two simple poles at  $s = i\tau_0$  and  $1 + i\tau_0$  (and  $m = m'$ ). Moreover  $L(s, \pi \times \tilde{\pi}')$  is meromorphic of order one and bounded inside vertical strips but away from its poles, by Gelbart & Shahidi [24].

Following Iwaniec & Sarnak [34], we define the analytic conductors of  $\pi$  and  $\pi \times \tilde{\pi}'$ , respectively, as

$$(3.13) \quad Q_{\pi} := q_{\pi} \prod_{j=1}^{\ell m} (1 + |\mu_{\pi}(j)|),$$

$$(3.14) \quad Q_{\pi \times \tilde{\pi}'} := q_{\pi \times \tilde{\pi}'} \prod_{j=1}^{\ell m m'} (1 + |\mu_{\pi \times \tilde{\pi}'}(j)|).$$

Note that by Bushnell & Henniart [8], the analytic conductors satisfy

$$(3.15) \quad Q_{\pi \times \bar{\pi}'} \leq Q_{\pi}^{m'} Q_{\bar{\pi}'}^m.$$

**3.1. The analytic strong multiplicity one theorem for  $\mathrm{GL}_m(\mathbb{A}_{\mathbb{K}})$ .** Let  $\pi$  and  $\pi'$  be two cuspidal automorphic representations of  $\mathrm{GL}_m(\mathbb{A}_{\mathbb{K}})$  with restricted tensor product decompositions  $\pi = \otimes \pi_v$  and  $\pi' = \otimes \pi'_v$ . The strong multiplicity one theorem states that if  $\pi_v \cong \pi'_v$  for all but finitely many places  $v$ , then  $\pi = \pi'$ . The analytic version of this theorem is to give, in terms of the analytic conductor  $Q_{\pi}$  of  $\pi$ , as precise as possible the number of places needed to decide a cuspidal automorphic representation  $\pi$  of  $\mathrm{GL}_m(\mathbb{A}_{\mathbb{K}})$ . Such an analytic result was first established by Moreno [64]. Let  $\mathcal{B}_m(Q)$  denote the set of all cuspidal automorphic representations  $\pi$  on  $\mathrm{GL}_m(\mathbb{A}_{\mathbb{K}})$  with analytic conductors  $Q_{\pi}$  less than a large real number  $Q$ . Suppose that  $\pi = \otimes \pi_v$  and  $\pi' = \otimes \pi'_v$  are in  $\mathcal{B}_m(Q)$  with  $m \geq 2$ . Then, according to [64], there exist positive constants  $c$  and  $d$  such that, if  $\pi_v \cong \pi'_v$  for all finite places  $v$  with norm

$$(3.16) \quad N(v) \leq \begin{cases} cQ^d & \text{for } m = 2, \\ c \exp \{d(\log Q)^2\} & \text{for } m \geq 3, \end{cases}$$

then  $\pi = \pi'$ . Then Brumley [7] strengthened Moreno's result in (3.16) to require only

$$(3.17) \quad N(v) \leq cQ^{17m/2-4+\varepsilon} \quad \text{for all } m \geq 1,$$

where  $c$  is a positive constant depending on  $m$  and  $\varepsilon$ . Moreno [64] and Brumley [7] made use of the zero-free region for  $L(s, \pi \times \pi')$ . The main new idea of Brumley lies in the nice observation of

$$\lambda_{\pi \times \bar{\pi}'}(\mathfrak{p}^m) \geq 1,$$

where  $\mathfrak{p}$  denotes any unramified places (prime ideals). Wang [81] injected Landau's idea in [45] into the method of Brumley – considering the weighted sum

$$\sum_{n \leq x} \lambda_{\pi \times \bar{\pi}'}(n) w\left(\frac{n}{x}\right),$$

where  $w(t)$  is the non-negative real-valued function of  $C^{\infty}$  given by

$$(3.18) \quad w(t) := \begin{cases} e^{-1/t-1/(1-t)} & \text{if } t \in (0, 1), \\ 0 & \text{otherwise.} \end{cases}$$

This allows one to push the contour of integration to  $-\infty$  and thus reduce the exponent  $17m/2-4+\varepsilon$  of (3.17) to  $d = 4m + \varepsilon$ . Recently Liu & Wang modified effectively the argument in [81] to give a better exponent  $2m + \varepsilon$  in (3.17). To state their result, let  $\mathcal{A}_M(Q)$  denote the set of all irreducible cuspidal automorphic representations  $\pi$  on  $\mathrm{GL}_m(\mathbb{A}_{\mathbb{K}})$ , with  $1 \leq m \leq M$ , whose analytic conductors  $Q_{\pi}$  are less than a large real number  $Q$ . Thus,

$$\mathcal{A}_M(Q) = \bigcup_{m \leq M} \mathcal{B}_m(Q),$$

and the main result of [54] is as follows.

**Theorem 4.** ([54], Theorem) *Let  $\varepsilon > 0$ ,  $\pi = \otimes \pi_v$  and  $\pi' = \otimes \pi'_v$  be in  $\mathcal{A}_M(Q)$ . Then there exists a constant  $c = c(\varepsilon, \mathbb{K}, M)$  depending on  $\varepsilon$ ,  $\mathbb{K}$  and  $M$  only, such that if  $\pi_v \cong \pi'_v$  for all finite places with norm*

$$N(v) < cQ_{\pi \times \pi'}^{1+\varepsilon},$$

*then  $\pi = \pi'$ . In particular the same result holds if  $\pi_v \cong \pi'_v$  for all finite places with norm*

$$N(v) < cQ^{2M+\varepsilon}.$$

**3.2. Sign changes of the coefficients of  $L$ -functions for  $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$ .** In this subsection we take  $\mathbb{K} = \mathbb{Q}$ . For each cuspidal automorphic representations  $\pi$  of  $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$ , the corresponding automorphic  $L$ -function  $L(s, \pi)$  is defined as in (3.3). In view of (3.4), the sequence  $\{\lambda_{\pi}(n)\}_{n=1}^{\infty}$  consists of complex numbers determined by  $\{\alpha_{\pi,p}(j)\}$  and  $\lambda_{\pi}(1) = 1$ . It may happen that  $\lambda_{\pi}(n)$  is real for all  $n \geq 1$ ; for example, it is the case when  $\pi$  is a self-contragredient representation for  $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$  with trivial central character. The sign change problem of  $\lambda_{\pi}(n)$  was firstly studied by Qu [68]. Denoting by  $n_{\pi}$  the smallest integer  $n$  such that  $\lambda_{\pi}(n) < 0$ , the problem is to give a good bound for  $n_{\pi}$  in term of the analytic conductor  $Q_{\pi}$ . To its end, Qu compared the upper and lower bounds of

$$S_{\pi,k}(x) := \sum_{n \leq x} \lambda_{\pi}(n) \left( \log \frac{x}{n} \right)^k.$$

With a suitable choice of  $k$ , it is easy to obtain the upper bound

$$S_{\pi,k}(x) \ll_{k,m,\varepsilon} Q_{\pi}^{1/2+\varepsilon} x^{\varepsilon}$$

by the convexity bound of  $L(s, \pi)$ . In order to obtain a lower bound, she found an elegant inequality

$$(3.19) \quad |\lambda_{\pi}(p)| + |\lambda_{\pi}(p^2)| + \cdots + |\lambda_{\pi}(p^m)| \geq \frac{1}{m}$$

for all unramified primes  $p$  (i.e.  $p \nmid q_{\pi}$ ). This allows her to deduce

$$S_{\pi,k}(n_{\pi}) \geq \sum_{\substack{n_{\pi}^{1/m}/4 < p \leq n_{\pi}^{1/m}/2 \\ p \nmid q_{\pi}}} (|\lambda_{\pi}(p)| + |\lambda_{\pi}(p^2)| + \cdots + |\lambda_{\pi}(p^m)|) \gg_{k,m,\varepsilon} n_{\pi}^{1/m-\varepsilon}$$

under the assumption  $n_{\pi} \gg_{\varepsilon} Q_{\pi}^{\varepsilon}$ . Comparing upper and lower bounds for  $S_{\pi,k}(n_{\pi})$ , she obtained

$$n_{\pi} \ll_{m,\varepsilon} Q_{\pi}^{m/2+\varepsilon}$$

for any  $\varepsilon > 0$ , where the implied constant depends only  $m$  and  $\varepsilon$ . Very recently there is a significant progress made by Liu, Qu & Wu [53].

**Theorem 5.** ([53], Theorem 1) *Let  $m \geq 2$  be an integer and let  $\pi$  be an irreducible unitary cuspidal representation for  $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$ . If  $\lambda_{\pi}(n)$  is real for all  $n \geq 1$ , then*

$$(3.20) \quad n_{\pi} \ll_{m,\varepsilon} Q_{\pi}^{1+\varepsilon}.$$

*The implied constant in (3.20) depends only on  $m$  and  $\varepsilon$ . In particular, the result is true for any self-contragredient irreducible unitary cuspidal representation  $\pi$  for  $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$  with trivial central character.*

Note that the exponent in (3.20) is independent of the rank  $m$  of  $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$ . In order to prove Theorem 5, Liu, Qu & Wu [53] applied the basic idea to introduce Landau's argument in Qu [68] as in [54], i.e. to consider the weighted sum

$$\sum_{n \leq x} \lambda_{\pi}(n) w\left(\frac{n}{x}\right),$$

instead of  $S_{\pi,k}(x)$ . However there are two essential difficulties. First the inequality (3.19) cannot be applied directly. Second Landau's weight function is not applicable. To overcome these difficulties, two new ideas are introduced: construct an integer  $n_0$  such that  $\lambda_{\pi}(n_0)$  is large and propose a new weight function.

In [53], Liu, Qu & Wu also considered sign changes in the sequence  $\{\Lambda(n)a_{\pi}(n)\}_{n=1}^{\infty}$ , which appears naturally in the Dirichlet series expression of the logarithmic derivative of  $L(s, \pi)$  in the half-plane  $\Re s > 1$ :

$$(3.21) \quad \frac{L'}{L}(s, \pi) = - \sum_{n=1}^{\infty} \frac{\Lambda(n)a_{\pi}(n)}{n^s}.$$

Here  $\Lambda(n)$  is the von Mangoldt function.

**Theorem 6.** ([53], Theorem 2) *Let  $m \geq 2$  be an integer and let  $\pi$  be an irreducible unitary cuspidal representation for  $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$ . If  $\Lambda(n)a_{\pi}(n)$  is real for all  $n \geq 1$ , then there is some  $n$  satisfying*

$$(3.22) \quad n \ll_{m,\varepsilon} Q_{\pi}^{1+\varepsilon}$$

*such that  $\Lambda(n)a_{\pi}(n) < 0$ . The constant implied in (3.22) depends only on  $m$  and  $\varepsilon$ . In particular, the result is true for any self-contragredient irreducible unitary cuspidal representation  $\pi$  for  $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$  with trivial central character.*

#### 4. CLASSIC MODULAR FORMS

We move to the case of rank two and restrict to holomorphic primitive forms which are associated with an  $L$ -function of  $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ .

Let  $k \geq 2$  be an even integer and  $N \geq 1$  be an integer. Suppose  $\chi$  is an even Dirichlet character mod  $N$  (i.e.  $\chi(-1) = 1$ ). A holomorphic modular form of weight  $k$  for  $\Gamma_0(N)$  with nebentypus  $\chi$  is a holomorphic function  $f(z)$  at all points of  $\mathbb{H}$  and the cusps of  $\Gamma_0(N)$  that satisfies

$$f(\gamma z) = \chi(d)(cz + d)^k f(z) \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \text{ and } z \in \mathbb{H}.$$

The vector space of all such holomorphic modular forms is denoted by  $M_k(N, \chi)$ . We write  $S_k(N, \chi)$  for the subspace of  $M_k(N, \chi)$  whose elements vanish at each cusp of  $\Gamma_0(N)$ . An  $f \in S_k(N, \chi)$  is called a cusp form, which admits a Fourier series expansion at  $\infty$  of the form

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e(nz)$$

where  $e(z) := e^{2i\pi z}$ . By definition, the  $n$ th Hecke operator  $T_n$  acts on  $S_k(N, \chi)$  by

$$T_n f(z) = \frac{1}{n} \sum_{ad=n} \chi(a) a^k \sum_{b \pmod{d}} f\left(\frac{az+b}{d}\right).$$

When  $n = p$  is a prime, this reduces to

$$(4.1) \quad T_p f(z) = \sum_{n=1}^{\infty} \left( a_f(np) + \chi(p) p^{k-1} a_f\left(\frac{n}{p}\right) \right) e(nz).$$

By the theory of Atkin-Lehner, we may dissect  $S_k(N, \chi)$  and get a special subspace  $S_k^*(N, \chi)$  which has a basis  $H_k^*(N, \chi)$  consisting of some common eigenfunctions of all Hecke operators. Let us normalize each element  $f \in H_k^*(N, \chi)$  so that  $a_f(1) = 1$ . Then  $f$  is called a newform (or primitive form). See [33, Section 14.7] for more details. When  $\chi = \chi_0$  is the trivial nebentypus, we write  $H_k^*(N)$  for  $H_k^*(N, \chi_0)$ . Note that if  $N$  is squarefree, then  $|H_k^*(N)| \asymp k\varphi(N)$ , where  $\varphi(N)$  is the Euler function.

The Fourier coefficient  $a_f(n)$  of a primitive form is the Hecke eigenvalue of  $T_n$  and hence endowed with rich arithmetic properties. Let us write  $a_f(n) = \lambda_f(n)n^{(k-1)/2}$  for  $f \in H_k^*(N)$ . Then  $\lambda_f(n)$ 's satisfy the Hecke relation

$$(4.2) \quad \lambda_f(m)\lambda_f(n) = \sum_{\substack{d|(m,n) \\ (d,N)=1}} \lambda_f\left(\frac{mn}{d^2}\right)$$

for all integers  $m \geq 1$  and  $n \geq 1$ . Moreover  $\lambda_f(n) \in \mathbb{R}$  and  $\lambda_f(1) = 1$ . Furthermore, we may express  $\lambda_f(p)$  in terms of the local parameters  $\alpha_f(p) = \alpha_{\pi,p}(1)$  and  $\beta_f(p) = \alpha_{\pi,p}(2)$  in (3.2):  $\lambda_f(p) = \alpha_f(p) + \beta_f(p)$ . Together with the vital work of Deligne [13], the values of  $\alpha_f(p), \beta_f(p)$  are known as follows:

$$(4.3) \quad \begin{cases} \alpha_f(p) = \beta_f(p) = 0 & \text{if } p^2 | N \\ \alpha_f(p) = \pm p^{-1/2}, \quad \beta_f(p) = 0 & \text{if } p \parallel N \\ |\alpha_f(p)| = \alpha_f(p)\beta_f(p) = 1 & \text{if } p \nmid N \end{cases}$$

for all  $f \in H_k^*(N)$  and all prime  $p$ . This yields, together with (4.2),

$$\lambda_f(p^\nu) = \alpha_f(p)^\nu + \alpha_f(p)^{\nu-1}\beta_f(p) + \cdots + \beta_f(p)^\nu$$

for all integers  $\nu \geq 0$ . In particular we have (Deligne's inequality)

$$(4.4) \quad |\lambda_f(n)| \leq \tau(n) \quad (n \geq 1),$$

where  $\tau(n)$  is the divisor function. By (4.3), we pick a unique angle  $\theta_f(p) \in [0, \pi]$  such that

$$\alpha_f(p) = e^{i\theta_f(p)} \quad \text{and} \quad \beta_f(p) = e^{-i\theta_f(p)} \quad \text{for } p \nmid N.$$

Thus,

$$(4.5) \quad \begin{aligned} \lambda_f(p^\nu) &= e^{\nu i\theta_f(p)} + e^{(\nu-2)i\theta_f(p)} + \cdots + e^{-\nu i\theta_f(p)} \\ &= \frac{\sin((\nu+1)\theta_f(p))}{\sin \theta_f(p)} \quad \text{for } p \nmid N. \end{aligned}$$

Plainly we have  $\lambda_f(p) \in [-2, 2]$ . The Sato-Tate conjecture asserts that if  $f$  is not of CM type (for instance, if  $N$  is squarefree), then the sequence  $\{\lambda_f(p)\}$  is equidistributed on the interval  $[-2, 2]$  with respect to the Sato-Tate measure

$$\mu_{ST} := \frac{\sqrt{4-t^2}}{2\pi} dt$$

on  $[-2, 2]$ . Precisely for any  $[\alpha, \beta] \subset [-2, 2]$ , the conjecture predicts

$$\lim_{x \rightarrow +\infty} \frac{1}{\pi(x)} |\{p \leq x : \alpha \leq \lambda_f(p) \leq \beta\}| = \int_{\alpha}^{\beta} d\mu_{ST},$$

where  $\pi(x)$  is the number of primes up to  $x$  (see Mazur's survey [59]).

Apparently the real primitive Dirichlet characters  $\chi(p)$  and the Hecke eigenvalues  $\lambda_f(p)$  are rather different. The former takes only two non-zero values, i.e.,  $+1$  or  $-1$ , and when  $p$  is of moderate size compared with  $D$ , the values of  $\chi(p)$  distributes like independent random variables taking values  $\pm 1$  equally often. The latter takes values more than two, and thus it is rare for coincidence of  $\lambda_f(p)$ 's in value. Recently Kowalski et al. [43] proposed a way to recover potentially a closer analogy: instead of looking at the values of the Hecke eigenvalues, one considers only their *signs* (where 0 is viewed as being of both signs simultaneously, to increase the possibility of having same sign). Then the work for Dirichlet characters was quite successfully extended to the signs of Hecke eigenvalues.

**4.1. The first negative Hecke eigenvalue.** For  $f \in \mathbf{H}_k^*(N)$ , it is well-known that the coefficients  $\lambda_f(n)$  change sign infinitely often. Similarly to  $n_{\chi}$  in Linnik's problem, we denote by  $n_f$  the smallest integer  $n \geq 1$  such that

$$(4.6) \quad (n, N) = 1 \quad \text{and} \quad \lambda_f(n) < 0.$$

An analogue of the least quadratic non-residue problem is to estimate  $n_f$  in terms of the analytic conductor  $k^2N$  of  $f$ . Iwaniec, Kohnen & Sengupta [32] showed that

$$n_f \ll (k^2N)^{29/60}$$

where the implied constant is absolute. (The standard method with the convexity bound of an  $L$ -function leads to  $n_f \ll_{\varepsilon} (k^2N)^{1/2+\varepsilon}$  only, so it is significant to obtain an exponent  $< 1/2$ , which is called a subconvexity bound.) Recently Kowalski et al. [43] refined the method of Iwaniec et al. to give a better exponent.

**Theorem 7.** ([43], Theorem 1) *Let  $k \geq 2$  be an even integer and  $N \geq 1$ . Then for all  $f \in \mathbf{H}_k^*(N)$ , we have*

$$(4.7) \quad n_f \ll (k^2N)^{9/20},$$

where the implied constant is absolute.

The proof of Theorem 7 makes use of some ideas in Iwaniec et al. [32]. Let us simply describe the new ingredients. Let  $y > 0$  be such that  $\lambda_f(n) \geq 0$  for  $n \leq y$  and  $(n, N) = 1$ . The idea to estimate  $y$  (i.e.  $n_f$ ) is a comparison between the upper and lower bounds of the sum

$$S(f, x) := \sum_{\substack{n \leq x \\ (n, N) = 1}}^b \lambda_f(n),$$

where  $\sum^b$  runs over squarefree integers. The upper bound is derived by using the Perron formula and the convexity bound for

$$L(s, f) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} \quad (\Re s > 1).$$

We easily get

$$(4.8) \quad S(f, x) \ll_{\varepsilon} (k^2 N)^{1/4+\varepsilon} x^{1/2+\varepsilon} \quad (x \geq 1).$$

This estimate is independent of any information on  $y$ .

In order to establish a lower bound for  $S(f, x)$ , Kowalski et al. [43] introduced the following auxiliary multiplicative function:

$$h_y(p) = \begin{cases} -2 & \text{if } p > y \text{ and } p \nmid N, \\ 0 & \text{if } \sqrt{y} < p \leq y \text{ or } p \mid N, \\ 1 & \text{if } p \leq \sqrt{y} \text{ and } p \nmid N, \end{cases} \quad h_y(p^\nu) = 0 \quad (\nu \geq 2).$$

They proved that

$$(4.9) \quad \sum_{n \leq y^u} h_y(n) = \frac{6}{\pi^2} \prod_{p \mid N} \left(1 + \frac{1}{p}\right)^{-1} y^u (\rho(2u) - 2 \log u) \left\{1 + O\left(\frac{(\log_2 y)^2}{\log y}\right)\right\}$$

uniformly for  $1 \leq u \leq \frac{3}{2}$  and  $y \geq N^{1/3}$ . Here  $\rho(u)$  is the Dickman function defined as the unique continuous solution of the difference-differential equation

$$u\rho'(u) + \rho(u-1) = 0 \quad (u > 1), \quad \rho(u) = 1 \quad (0 < u \leq 1).$$

Note that  $\rho(2u) - 2 \log u > 0$  for all  $u < \kappa$  where  $\kappa$  is the solution to  $\rho(2\kappa) = 2 \log \kappa$ . From this we deduce

$$S(f, y^u) \geq \sum_{n \leq y^u} h_y(n) \gg \frac{y^u}{\log \log N}$$

for  $u < \kappa$ . Then, a comparison with (4.8) gives the estimate

$$y \leq (k^2 N)^{1/(2\kappa)+o(1)}.$$

Numerical computation shows that  $\kappa > \frac{10}{9}$ . This proves (4.7).

The bound in (4.7) is not yet optimally evaluated by this method. Indeed, an interesting function  $\beta$  arises when one tries to utilize the idea fully and push to its limit. (cf [43, Remark 2.2]). Very recently Matomäki [58] carried this out and sharpened the exponent  $\frac{9}{20} = 0.45$  to  $\frac{3}{8} = 0.375$ . It is worth to notice that these results do not need any subconvexity of the relevant  $L$ -functions. For a Hecke Maass form  $f$ , Qu [69] got a subconvexity bound for  $n_f$  with Michel & Venkatesh's subconvexity bound for  $GL_2$   $L$ -functions (cf. [60]).

Finally let us remark a small difference between  $n_\chi$  and  $n_f$ . Since the Dirichlet character is completely multiplicative,  $n_\chi$  must be a prime. In general, we only know that  $n_f$  is a power of prime. As a random example, for the cusp form  $f$  of weight 2 associated to the elliptic curve  $y^2 = x^3 + x$ , the first negative coefficient is  $\lambda(9) = -3$  (i.e.  $n_f = 9$ ), and the first negative coefficient on primes is  $\lambda(13) = -6$  (cf. [43, page 390]). The following question is thus natural and interesting.

**Question 1.** For  $f \in H_k^*(N)$ , denote  $n_f^*$  the smallest prime  $p$  such that

$$\lambda_f(p) < 0 \quad \text{and} \quad p \nmid N.$$

Find a real number  $\theta$  as small as possible such that

$$n_f^* \ll (k^2 N)^\theta$$

for all  $f \in H_k^*(N)$ , where the implied constant depends on  $\theta$  only.

**4.2. Statistic study of the first sign-change.** As with the problem of the least quadratic non-residue (cf. §2.2), the GRH for  $L(s, f)$  implies that  $n_f \ll (\log(kN))^2$  for  $f \in H_k^*(N)$ , where the implied constant is absolute. Surprisingly one can obtain an almost-all result for  $n_f$  analogous to the case of Dirichlet characters (in Theorem 2 above), when  $N$  is squarefree. Recall  $|H_k^*(N)| \asymp k\varphi(N)$  for squarefree  $N$ .

**Theorem 8.** ([43], Corollary 3) *Let  $k \geq 2$  be an even integer and  $N \geq 1$  be squarefree. There is an absolute positive constant  $c$  such that we have*

$$n_f \leq n_f^* \ll \log(kN),$$

for all  $f \in H_k^*(N)$ , except for  $f$  in an exceptional set with

$$\ll kN \exp\left(-c \frac{\log(kN)}{\log \log(kN)}\right)$$

elements, where the implied constants are absolute.

The exceptional set cannot be too small, as shown in the following result.

**Theorem 9.** ([43], Theorem 4) *Let  $N$  be a squarefree number and  $k \geq 2$  an even integer, and let  $\{\varepsilon_p\}_{p \nmid N}$  be a sequence of signs indexed by prime numbers coprime with  $N$ . For any  $\varepsilon \in (0, \frac{1}{2})$ , there exists a constant  $c = c(\varepsilon) > 0$  such that*

$$|\{f \in H_k^*(N) : \lambda_f(p) \text{ has sign } \varepsilon_p \text{ for } p \leq z \text{ and } p \nmid N\}| \geq \left(\frac{1}{2} - \varepsilon\right)^{\pi(z)} |H_k^*(N)|$$

for  $z = c\sqrt{(\log(kN)) \log \log(kN)}$ , provided  $kN$  is large enough. In particular there are three absolute positive constants  $c, c_1$  and  $c_2$  such that

$$\begin{aligned} \sum_{\substack{f \in H_k^*(N) \\ n_f \geq c\sqrt{(\log(kN)) \log \log(kN)}}} 1 &\geq \sum_{\substack{f \in H_k^*(N) \\ n_f^* \geq c\sqrt{(\log(kN)) \log \log(kN)}}} 1 \\ &\geq c_2 kN \exp\left(-c_1 \sqrt{\frac{\log(kN)}{\log \log(kN)}}\right). \end{aligned}$$

*Remark 2.* One may expect that the same result would be true for  $z \leq c \log(kN)$ . Note that

$$\left(\frac{1}{2}\right)^{\pi(c \log kN)} \geq \exp\left(-c_1 \frac{\log(kN)}{\log \log(kN)}\right),$$

so the expected result would be quite close to the upper estimate in Theorem 8, and essentially best possible. Also it would imply that the signs of  $\lambda_f(p)$  behave almost like independent (and unbiased) random variables in that range of  $p$ .

The key tool for the proof of Theorem 8 is a large sieve inequality of Elliott-Montgomery-Vaughan type [50, Theorem 1]: *Let  $\nu \geq 1$  be a fixed integer and let  $\{b_p\}_p$  be a sequence of real numbers indexed by prime numbers such that  $|b_p| \leq B$  for some constant  $B$  and for all primes  $p$ . Then we have*

$$\sum_{f \in \mathbb{H}_k^*(N)} \left| \sum_{\substack{P < p \leq Q \\ p \nmid N}} b_p \frac{\lambda_f(p^\nu)}{p} \right|^{2j} \ll_{\nu} k \varphi(N) \left( \frac{96B^2(\nu+1)^2j}{P \log P} \right)^j + (kN)^{10/11} \left( \frac{10BQ^{\nu/10}}{\log P} \right)^{2j}$$

uniformly for

$$B > 0, \quad j \geq 1, \quad 2 \mid k, \quad 2 \leq P < Q \leq 2P, \quad N \geq 1 \quad (\text{squarefree}).$$

The implied constant depends on  $\nu$  only.

The proof of Theorem 9 makes an insightful use of the Chebyshev functions – the Chebyshev polynomials form a basis for the space of polynomials and  $\lambda_f(p^\nu)$  equals the value of a Chebyshev function at  $\theta_f(p)$ . With the Petersson trace formula, the following key tool is established [43, Proposition 8]: *Let  $N$  be a squarefree number,  $k \geq 2$  an even integer,  $s \geq 1$  an integer and  $z \geq 2$  a real number. For any prime  $p \leq z$  coprime with  $N$ , let*

$$Y_p(\theta) = \sum_{j=0}^s \hat{y}_p(j) X_j(\theta)$$

be a “polynomial” of degree  $\leq s$  expressed in the basis of Chebychev functions

$$X_j(\theta) := \frac{\sin((j+1)\theta)}{\sin \theta}$$

on  $[0, \pi]$ . Then we have

$$\sum_{f \in \mathbb{H}_k^*(N)} \omega_f \prod_{\substack{p \leq z \\ (p, N) = 1}} Y_p(\theta_f(p)) = \prod_{\substack{p \leq z \\ (p, N) = 1}} \hat{y}_p(0) + O\left(\frac{C^{\pi(z)} D^{sz} (\tau(N) \log(2N))^2}{k^{5/6} N}\right)$$

where  $D \geq 1$ ,  $\langle f, f \rangle$  is the Petersson norm of  $f$ ,

$$\omega_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \frac{N}{\langle f, f \rangle \varphi(N)}, \quad C = \max_{p, j} |\hat{y}_p(j)|,$$

and the implied constant is absolute.

**4.3. Recognition of newforms by signs of Hecke eigenvalues.** Another interesting results of [43] is that a primitive form  $f$  can be determined uniquely by the sequence of signs of its Fourier coefficients  $\lambda_f(p)$ . If  $\lambda_f(p) = 0$ , we may arbitrarily take its sign as positive or negative (which will not affect the result). Let us introduce a weaker concept of density: a set  $E$  of primes has analytic density  $\kappa > 0$  if and only if

$$(4.10) \quad \sum_{p \in E} \frac{1}{p^\sigma} \sim \kappa \sum_p \frac{1}{p^\sigma} \sim -\kappa \log(\sigma - 1) \quad (\sigma \rightarrow 1^+).$$

Note that the existence of the natural density implies that of the analytic density, and that they are equal when both exist.

**Theorem 10.** ([43], Theorem 5) *Let  $k_1, k_2 \geq 2$  be even integers, let  $N_1, N_2 \geq 1$  be integers and  $f_1 \in H_{k_1}^*(N_1)$ ,  $f_2 \in H_{k_2}^*(N_2)$ .*

(i) *If the signs of  $\lambda_{f_1}(p)$  and  $\lambda_{f_2}(p)$  are the same for all  $p$  except those in a set of analytic density 0, then  $f_1 = f_2$ .*

(ii) *Assume that neither of  $f_1$  and  $f_2$  is of CM type, for instance assume that  $N_1$  and  $N_2$  are squarefree. Then, if  $\lambda_{f_1}(p)$  and  $\lambda_{f_2}(p)$  have same sign for every prime  $p$ , except those in a set  $E$  of analytic density  $\kappa$ , with  $\kappa \leq 1/32$ , it follows that  $f_1 = f_2$ .*

Of course, Theorem 10 is also valid for the natural density. Thus we have the following corollary.

**Corollary 1.** ([43], Corollary 6) *For any sequence of signs  $\{\varepsilon_p\}$  indexed by primes, there is at most one pair  $(k, N)$  and one  $f \in H_k^*(N)$  such that  $\lambda_f(p)$  has sign  $\varepsilon_p$  for all primes.*

The proof of Theorem 10(ii) depends crucially on two very deep results:

- Assume that  $f_1$  and  $f_2$  are non-CM cusp forms, and that neither is a quadratic twist of the other (in particular,  $f_1 \neq f_2$ ). Ramakrishnan's Theorem ([70, Theorem M, Section 3]) states that there exists a cuspidal automorphic representation on  $GL_4(\mathbb{A}_{\mathbb{Q}})$  such that  $L(s, \pi) = L(s, f_1 \times f_2)$ ;
- By the Rankin-Selberg theory on  $GL_4 \times GL_4$ ,  $L(s, \pi \times \tilde{\pi})$  has a single pole at  $s = 1$ .

Applying these, one can prove that

$$\sum_{p, \lambda_{f_1}(p)\lambda_{f_2}(p) < 0} \frac{1}{p^\sigma} \geq \frac{1}{32} \sum_p \frac{1}{p^\sigma} + O(1) \quad (\sigma \rightarrow 1^+).$$

The proof of Theorem 10(i) is based on the distribution of the angles  $\theta_f(p)$  for a CM form  $f \in H_k^*(N)$ : there exists a real, non-trivial, primitive Dirichlet character  $\chi_f$  such that  $\lambda_f(p) = 0$  when  $\lambda_f(p) = -1$  (a set of primes  $I_f$  of density  $\frac{1}{2}$ ), and for  $p \notin I_f$ , the  $\theta_f(p) \in [0, \pi]$  for  $p \leq x$  become uniformly distributed as  $x \rightarrow \infty$ , i.e.,

$$\sum_{p \leq x, p \notin I_f} e^{2im\theta_f(p)} = o(\pi(x)) \quad (x \rightarrow \infty)$$

for all non-zero integers  $m \in \mathbb{Z}$  (see, e.g., [66, p.197]).

There is a recent progress in the non-CM case. Matomäki [58] improved the constant  $\frac{1}{32}$  to  $\frac{6}{25}$  using the Sato-Tate conjecture (proved recently by Barnet-Lamb, Geraghty Harris & Taylor [4]) and to  $\frac{3}{20}$  without using it. Her approach utilized linear programming to take full advantage of all the available information about the behavior of the Fourier coefficients (in Section 3). Besides, she presented an alternative approach leading to the proportion  $\frac{1}{10}$ . This latter approach is based on a result of rearrangement inequality type, which is very useful in many other problems.

**4.4. Recognition of newforms by values of Hecke eigenvalues.** Another analogue of the least quadratic non-residue problem is the problem of distinguishing modular forms by their first Fourier coefficients. In this regard, Kowalski, Michel & Vanderkam [44, Corollary 1.3] applied their subconvexity bound for Rankin-Selberg  $L$ -functions to prove the following result: *Let  $g$  be a primitive cusp form,  $k \geq 2$  an even integer, and  $\varepsilon > 0$ . There exists a constant  $C = C(g, k, \varepsilon)$  depending only*

on  $g$ ,  $k$  and  $\varepsilon$  such that for any primitive holomorphic form  $f \in \mathbf{H}_k^*(N)$ , there exists  $n \leq CN^{1-1/40+\varepsilon}$  such that  $\lambda_f(n) \neq \lambda_g(n)$ . In this case, the ‘‘trivial’’ bound is  $n \leq CN^{1+\varepsilon}$ . In [37, Theorem], Kohnen obtained the following result on the aspect of weights: Let  $f_1 \in \mathbf{H}_{k_1}(N)$  and  $f_2 \in \mathbf{H}_{k_2}(N)$ . Suppose that  $k_1, k_2 \geq 2$  and  $k_1 \neq k_2$ . Let  $A \in \mathbb{N}$  with  $N \mid A$ . Then there exists a prime  $p$  with  $p \nmid A$  and  $p \ll_A (k_1 + k_2)|k_1 - k_2|$  such that  $\lambda_{f_1}(p) \neq \lambda_{f_2}(p)$ . The constant implied in  $\ll_A$  is effective and depends on  $A$  only.

Naturally it is desired to establish an explicit bound for all parameters, which can be obtained with the help of Theorem 4. Let  $f_1 \in \mathbf{H}_{k_1}(N_1)$  and  $f_2 \in \mathbf{H}_{k_2}(N_2)$ . The analytic conductor of  $L(s, f_1 \times f_2)$  is equal to

$$(4.11) \quad Q_{f_1 \times f_2} := N_{f_1 \times f_2} (k_1 + k_2) (|k_1 - k_2| + 1)$$

where  $N_{f_1 \times f_2}$  is the arithmetic conductor of  $f_1 \times f_2$ . According to [26], we have

$$(N_1 N_2)^2 / (N_1, N_2)^4 \leq N_{f_1 \times f_2} \leq (N_1 N_2)^2 / (N_1, N_2)^2.$$

In particular  $N_{f_1 \times f_2} = (N_1 N_2)^2$  if  $(N_1, N_2) = 1$ .

The following is an immediate corollary of Theorem 4 in Section 3.

**Theorem 11.** *For any  $\varepsilon > 0$ , there exists*

$$n \ll_\varepsilon (N_{f_1 \times f_2} (k_1 + k_2) (|k_1 - k_2| + 1))^{1+\varepsilon}$$

such that for all  $f_1 \in \mathbf{H}_{k_1}(N_1)$  and  $f_2 \in \mathbf{H}_{k_2}(N_2)$  with  $f_1 \neq f_2$  such that

$$\lambda_{f_1}(n) \neq \lambda_{f_2}(n).$$

**Question 2.** *Improve the exponent  $1 + \varepsilon$  in Theorem 11.*

**4.5. Matching signs of Hecke eigenvalues of two newforms.** Let  $k_1, k_2 \geq 2$  be even integers and  $N_1, N_2 \geq 1$ . Suppose  $f \in \mathbf{H}_{k_1}(N_1)$  and  $f_2 \in \mathbf{H}_{k_2}(N_2)$ . The symbol  $n_{f_1, f_2}$  denotes the smallest positive integer for which the signs of  $\lambda_{f_1}(n)$  and  $\lambda_{f_2}(n)$  are different. We hope to give an estimate to the size of  $n_{f_1, f_2}$  in terms of the weights and levels. By a theorem of Ramakrishnan, there exists a cuspidal automorphic representation  $\pi$  on  $\mathrm{GL}_4(\mathbb{A}_{\mathbb{Q}})$  with the analytic conductor  $Q_{f_1 \times f_2}$  (see (4.11) above) such that  $L(s, f_1 \times f_2) = L(s, \pi)$ . This reduces our question to the evaluation of  $n_\pi$  in Section 3, for  $n_\pi = n_{f_1, f_2}$ . Thus Theorem 5 of Section 3 implies immediately the following.

**Corollary 2.** *Let  $k_1, k_2 \geq 2$  be even integers and  $N_1, N_2 \geq 1$ . Then for any  $\varepsilon > 0$  and all  $f_1 \in \mathbf{H}_{k_1}^*(N_1)$  and  $f_2 \in \mathbf{H}_{k_2}^*(N_2)$ , we have*

$$n_{f_1, f_2} \ll_\varepsilon (N_{f_1 \times f_2} (k_1 + k_2) (|k_1 - k_2| + 1))^{1+\varepsilon},$$

where the implied constant depends on  $\varepsilon$  only.

In [43], Kowalski et al. got an almost-all result in the case of  $n_{f_1, f_2}$  as well.

**Theorem 12.** ([43], Corollary 7) *Let  $k_1, k_2 \geq 2$  be even integers and  $N_1, N_2 \geq 1$  squarefree. For any fixed  $f_2 \in \mathbf{H}_{k_2}^*(N_2)$ , there is an absolute positive constant  $c$  such that*

$$n_{f_1, f_2} \ll_{f_2} \log(k_1 N_1),$$

for all  $f_1 \in H_{k_1}^*(N_1)$  except for those in an exceptional set with

$$\ll_{f_2} k_1 N_1 \exp\left(-c \frac{\log(k_1 N_1)}{\log \log(k_1 N_1)}\right)$$

elements, where the implied constants depend only on  $f_2$ .

**Question 3.** *Improve the exponent  $1 + \varepsilon$  in Corollary 2.*

**4.6. The number of Hecke eigenvalues of same signs.** Kohnen, Lau & Shparlinski [40] evaluated the number of Hecke eigenvalues of the same sign

$$\mathcal{N}_f^\pm(x) := \sum_{\substack{n \leq x, (n, N) = 1 \\ \lambda_f(n) \geq 0}} 1,$$

counting the number of positive and negative eigenvalues respectively. Their result [40, Theorem 1] says that

$$(4.12) \quad \mathcal{N}_f^\sigma(x) \gg_f \frac{x}{(\log x)^{17}} \quad (\sigma = + \text{ or } -)$$

for all  $x \geq x_0(f)$ . The exponent 17 was improved to  $1 - 1/\sqrt{3}$  and  $2 - 16/(3\pi)$  (assuming the Sato-Tate conjecture) in Wu [82, Corollary], as a simple application of his new estimate for the power sums of Hecke eigenvalues. Using the method of  $\mathcal{B}$ -free numbers to be introduced below, Lau & Wu [51] removed all the logarithmic factor – see the following result, which is the best possible in order of magnitude.

**Theorem 13.** ([51], Theorem 1) *Let  $\sigma = +$  or  $-$  and  $f \in H_k^*(N)$ . Then there is a constant  $x_0$  such that the inequalities*

$$(4.13) \quad \mathcal{N}_f^\sigma(x) \gg_f x$$

hold for all  $x \geq x_0$ .

Let us give an outline on  $\mathcal{B}$ -free numbers. Let  $\mathcal{B} = \{b_i\}_{i \geq 1}$  be a strictly increasing sequence of integers such that  $b_1 > 1$ ,

$$(4.14) \quad \sum_{i \geq 1} \frac{1}{b_i} < \infty \quad \text{and} \quad (b_i, b_j) = 1 \quad (i \neq j).$$

Erdős [21] introduced the  $\mathcal{B}$ -free numbers that are integers indivisible by any element in  $\mathcal{B}$ . List the  $\mathcal{B}$ -free numbers as an increasing sequence  $\mathcal{A} := \{a_i\}_{i \geq 1}$ . Erdős [21] showed that  $\mathcal{A}$  is of positive (natural) density

$$(4.15) \quad \lim_{x \rightarrow \infty} \frac{|\mathcal{A} \cap [1, x]|}{x} = \prod_{i=1}^{\infty} \left(1 - \frac{1}{b_i}\right) > 0.$$

In [72, (181)], Serre proved that

$$|\{p \leq x : \lambda_f(p) = 0\}| \ll_{f, \delta} \frac{x}{(\log x)^{1+\delta}}$$

for  $x \geq 2$  and any  $\delta < \frac{1}{2}$ . Using the theory of  $\mathcal{B}$ -free numbers, the sparsity of vanishing  $\lambda_f(p)$  will lead to

$$\mathcal{N}_f^+(x) + \mathcal{N}_f^-(x) \gg x.$$

As  $\lambda_f(n)$  is multiplicative, a switching principle applies and guarantees that both  $\mathcal{N}_f^\pm(x)$  are  $\gg x$ . See [51] for details.

One may also study the same problem in short intervals. It is shown in [40, Theorem 2] (see also [38, Theorem 3.4]) that there are absolute constants  $\eta < 1$  and  $A > 0$  such that for any  $f \in \mathbb{H}_k^*(N)$ , the inequalities

$$(4.16) \quad \mathcal{N}_f^\sigma(x + x^\eta) - \mathcal{N}_f^\sigma(x) > 0 \quad (\sigma = + \text{ or } -)$$

hold for  $x \geq (kN)^A$ . No explicit value of  $\eta$  is calculated out. An application of (4.16) is that  $\lambda_f(n)$  has a sign change in the short interval  $[x, x + x^\eta]$  for all sufficiently large  $x$ . The following result of Lau & Wu [51] gives explicitly an admissible value of  $\eta$  and a much bigger lower bound in (4.16).

**Theorem 14.** ([51], Theorem 2) *Let  $\sigma = +$  or  $-$  and  $f \in \mathbb{H}_k^*(N)$ . There is an absolute constant  $C > 0$  such that for any  $\varepsilon > 0$  and all sufficiently large  $x \geq N^2 x_0(k)$ , we have*

$$(4.17) \quad \mathcal{N}_f^\sigma(x + C_N x^{1/2}) - \mathcal{N}_f^\sigma(x) \gg_\varepsilon (Nx)^{1/4-\varepsilon}$$

where

$$C_N := CN^{1/2}\Psi(N)^3, \quad \Psi(N) := \sum_{d|N} d^{-1/2} \log(2d)$$

and  $x_0(k)$  is a suitably large constant depending on  $k$  and the implied constant in  $\gg_\varepsilon$  depends only on  $\varepsilon$ .

The proof of Theorem 14 is completed by a method in Heath-Brown & Tsang [29]. To this end, we need a truncated Voronoi formula for

$$S_f^*(x) := \sum_{\substack{n \leq x \\ (n, N)=1}} \lambda_f(n),$$

stated as follows (cf. [51, Lemma 3.1]) : *Let  $f \in \mathbb{H}_k^*(N)$ . Then for any  $A > 0$  and  $\varepsilon > 0$ , we have*

$$(4.18) \quad \begin{aligned} S_f^*(x) &= \frac{\eta_f}{\pi\sqrt{2}} (Nx)^{1/4} \sum_{d|N} \frac{(-1)^{\omega(d)} \lambda_f(d)}{d^{1/4}} \sum_{n \leq M} \frac{\lambda_f(n)}{n^{3/4}} \cos\left(4\pi\sqrt{\frac{nx}{dN}} - \frac{\pi}{4}\right) \\ &+ O\left(N^{1/2} \left\{1 + \left(\frac{x}{M}\right)^{1/2} + \left(\frac{N}{x}\right)^{1/4}\right\} (Nx)^\varepsilon\right) \end{aligned}$$

uniformly for  $1 \leq M \leq x^A$  and  $x \geq N^{1+\varepsilon}$ , where  $\eta_f = \pm 1$  depends on  $f$  and the implied  $O$ -constant depends on  $A, \varepsilon$  and  $k$  only. The function  $\omega(d)$  counts the number of all distinct prime factors of  $d$ .

It seems rather difficult to reduce the exponent  $1/2$  in the inequalities (4.17), that is equivalently, to show the inequalities for all sufficiently large intervals  $[x, x + U]$  of length  $U < x^{1/2}$  (in order of magnitude). In [83], Wu & Zhai attempted to look for shorter intervals on which one of the inequalities in (4.17) holds. It was discovered that intervals of any short length suffice for the purpose.

**Theorem 15.** ([83], Theorem 4) *Suppose  $f \in \mathbf{H}_k^*(N)$  and let  $0 < \vartheta < 1/2$ . Then for any  $\varepsilon > 0$ , the inequality*

$$(4.19) \quad \mathcal{N}_f^+(x + x^\vartheta) - \mathcal{N}_f^+(x) \gg_{f, \vartheta, \varepsilon} x^{\vartheta/2 - \varepsilon}$$

*holds for infinitely many  $x$ . The same result holds for  $\mathcal{N}_f^-(x)$ .*

When  $\vartheta$  is not small, one may show an abundance of  $x$  for (4.19); see the theorems below. The proof modifies that in [29], and uses the second and fourth moments of  $S_f^*(x)$  in short intervals. For example, the following is shown in [83, Corollary 4]: *For all  $f \in \mathbf{H}_k^*(N)$  and any  $\varepsilon > 0$ , we have*

$$(4.20) \quad \int_T^{2T} |S_f^*(x + U) - S_f^*(x)|^4 dx \ll_{f, \varepsilon} TU^2.$$

*uniformly for  $T \geq 1$  and  $T^{3/7 + \varepsilon} \ll U \ll T^{1/2 - \varepsilon}$ .*

**Theorem 16.** ([83], Theorem 2) *Suppose  $f \in \mathbf{H}_k^*(N)$  and  $3/7 < \vartheta < 1/2$ . Then there exist two positive constants  $C_i = C_i(f, \vartheta)$  ( $i = 1, 2$ ) such that for any large parameter  $T$ , there are at least  $C_1 T^{1 - \vartheta} (\log T)^2$  disjoint subintervals of length  $C_2 T^\vartheta (\log T)^{-2}$  in  $[T, 2T]$ , such that the inequality (4.19) holds, whenever  $x$  lies in any of these subintervals. Moreover, we have*

$$|\{x \in [T, 2T] : (4.19) \text{ holds}\}| \gg_{f, \vartheta, \varepsilon} T$$

*for all sufficiently large  $T$ . These results remain valid if  $\mathcal{N}_f^+(x)$  is replaced by  $\mathcal{N}_f^-(x)$ .*

The value of  $\vartheta$  can be reduced further if we relax the number of subintervals.

**Theorem 17.** ([83], Theorem 3) *Suppose  $f \in \mathbf{H}_k^*(N)$  and  $3/8 \leq \vartheta \leq 3/7$ . Then there exist two positive constants  $C_i = C_i(f, \vartheta)$  ( $i = 3, 4$ ) such that for any large parameter  $T$ , there are at least  $C_3 T^{1 - \vartheta}$  disjoint subintervals of length  $C_4 T^{\vartheta - \varepsilon}$  in  $[T, 2T]$ , such that the inequality (4.19) holds, whenever  $x$  lies in any of these subintervals. Moreover, we have*

$$|\{x \in [T, 2T] : (4.19) \text{ holds}\}| \gg_{f, \vartheta, \varepsilon} T^{1 - \varepsilon}$$

*for all sufficiently large  $T$ . The above results also hold for  $\mathcal{N}_f^-(x)$ .*

Finally we state a conjecture of Wu & Zhai.

**Conjecture 1.** ([83], Conjecture 1) *Let  $f \in \mathbf{H}_k^*(N)$ ,  $0 < \vartheta < 1/2$  and  $\sigma = +$  or  $-$ . Then for any  $\varepsilon > 0$ , the inequalities*

$$\mathcal{N}_f^\sigma(x + x^\vartheta) - \mathcal{N}_f^\sigma(x) \gg_{f, \vartheta, \varepsilon} x^{\vartheta/2 - \varepsilon}$$

*hold for all  $x \geq x_0(f, \vartheta, \varepsilon)$ .*

**Question 4.** *Prove or disprove Conjecture 1.*

5. SYMMETRIC SQUARE  $L$ -FUNCTIONS

In this section we consider a special case of  $\mathrm{GL}_3$  — the symmetric square lift of a  $\mathrm{GL}_2$  form. To each  $f \in \mathrm{H}_k^*(N)$  is associated a symmetric square  $L$ -function, defined as

$$\begin{aligned} L(s, \mathrm{sym}^2 f) &:= \prod_p \left( 1 - \frac{\lambda_f(p^2)}{p^s} + \frac{\psi_N(p)\lambda_f(p^2)}{p^{2s}} - \frac{\psi_N(p)}{p^{3s}} \right)^{-1} \\ &=: \sum_{n \geq 1} \frac{\lambda_{\mathrm{sym}^2 f}(n)}{n^s} \end{aligned}$$

for  $\Re s > 1$ , where  $\psi_N$  denotes the principal character mod  $N$ . According to the work of Gelbart & Jacquet [23], there is an irreducible unitary cuspidal representation  $\pi$  for  $\mathrm{GL}_3(\mathbb{A}_{\mathbb{Q}})$  such that  $L(s, \mathrm{sym}^2 f) = L(s, \pi)$ . Inherited from the construction,  $\lambda_{\mathrm{sym}^2 f}(n)$  is real, multiplicative and satisfies

$$(5.1) \quad \lambda_{\mathrm{sym}^2 f}(n) = \sum_{d^2 m = n} \lambda_f(m^2) \quad \text{for } (n, N) = 1.$$

**5.1. The first negative coefficient of symmetric square  $L$ -functions.** For  $f \in \mathrm{H}_k^*(N)$ , let us write  $n_{\mathrm{sym}^2 f}$  for the least integer  $n$  such that

$$(5.2) \quad \lambda_{\mathrm{sym}^2 f}(n) < 0 \quad \text{and} \quad (n, N) = 1.$$

Since  $\mathrm{sym}^2 f$  is an irreducible unitary cuspidal representation  $\pi$  for  $\mathrm{GL}_3(\mathbb{A}_{\mathbb{Q}})$  with analytic conductor  $k^2 N^2$ , the bound (3.20) of Liu, Qu & Wu now states that

$$(5.3) \quad n_{\mathrm{sym}^2 f} \ll_{\varepsilon} (k^2 N^2)^{1+\varepsilon},$$

where the implied constant depends on  $\varepsilon$  only. Extending the method of [32, 43, 58], we can derive a better subconvexity bound for  $n_{\mathrm{sym}^2 f}$ .

**Theorem 18.** ([47], Theorem 1) *Let  $k \geq 2$  be an even integer and  $N \geq 1$  be an integer. Then for all  $f \in \mathrm{H}_k^*(N)$ , we have*

$$(5.4) \quad n_{\mathrm{sym}^2 f} \ll (k^2 N^2)^{40/113},$$

where the implied constant is absolute.

The method follows closely the approach in Section 4.1 (and of course is similar to that in [32, 43, 58]). Let  $y$  be the greatest integer such that

$$(5.5) \quad \lambda_{\mathrm{sym}^2 f}(n) \geq 0 \quad \text{for } n \leq y \quad \text{and} \quad (n, N) = 1,$$

and consider

$$(5.6) \quad S_{\mathrm{sym}^2 f}(y^u) := \sum_{\substack{n \leq y^u \\ (n, N) = 1}} \mu(n)^2 \lambda_{\mathrm{sym}^2 f}(n),$$

where  $u > 1$  and  $\mu(n)$  is the Möbius function. We shall obtain the required estimate for  $y$  by comparing the upper and lower bounds for  $S_{\mathrm{sym}^2 f}(y^u)$ . However, the situation here is more involved as the coefficients are more complicated:

$$(5.7) \quad \lambda_{\mathrm{sym}^2 f}(p^\nu) = \frac{\sin((\nu + 2)\theta_f(p)) \sin((\nu + 1)\theta_f(p))}{\sin \theta_f(p) \sin(2\theta_f(p))} \quad (p \nmid N, \nu \geq 1).$$

Using merely this identity and the positivity hypothesis (5.5), we cannot derive directly the required lower bound for  $\lambda_{\text{sym}^2 f}(p)$ . We must exclude the “bad” primes  $p$ , all of which are contained in the set

$$(5.8) \quad \begin{aligned} \mathcal{P}_f &:= \bigcup_{1 \leq \nu \leq 4} \{p : |\lambda_f(p)| = 2 \cos(\pi/(\nu + 2))\} \\ &= \bigcup_{1 \leq \nu \leq 4} \{p : \lambda_{\text{sym}^2 f}(p^\nu) = 0\}. \end{aligned}$$

Such primes are few. Indeed, Kowalski [42] observed the sparsity of  $p$  with  $\lambda_f(p) = \pm 1$ . Lemma 2.4 of [47] is a generalization of his observation to suit our purpose, which yields: *Let  $k \geq 2$  be an even integer and  $N \geq 1$  be an integer. There is an absolute constant  $C$  such that the inequality*

$$(5.9) \quad |\mathcal{P}_f| \leq \frac{4}{\log 2} \log(kN) + C$$

holds for all  $f \in \mathbf{H}_k^*(N)$ .

Our upshot is the adjustment below. Let

$$(5.10) \quad N_f := \prod_{p|N} p \times \prod_{p \in \mathcal{P}_f} p.$$

Then for  $p \leq y^{1/\nu}$  and  $p \nmid N_f$ , we have

$$(5.11) \quad \lambda_{\text{sym}^2 f}(p) \geq \kappa_\nu := 3 - 4 \sin^2(\pi/(\nu + 2)).$$

More precisely

$$(5.12) \quad \lambda_{\text{sym}^2 f}(p) \geq \begin{cases} 0 & \text{if } y^{1/2} < p \leq y \text{ and } p \nmid N_f, \\ 1 & \text{if } y^{1/3} < p \leq y^{1/2} \text{ and } p \nmid N_f, \\ (\sqrt{5} + 1)/2 & \text{if } y^{1/4} < p \leq y^{1/3} \text{ and } p \nmid N_f, \\ 2 & \text{if } p \leq y^{1/4} \text{ and } p \nmid N_f. \end{cases}$$

We introduce the auxiliary multiplicative function  $h = h_{N_f, y}$  defined as

$$h_{N_f, y}(p) = \begin{cases} -1 & \text{if } p > y \text{ and } p \nmid N_f, \\ 0 & \text{if } y^{1/2} < p \leq y \text{ or } p | N_f, \\ 1 & \text{if } y^{1/3} < p \leq y^{1/2} \text{ and } p \nmid N_f, \\ (\sqrt{5} + 1)/2 & \text{if } y^{1/4} < p \leq y^{1/3} \text{ and } p \nmid N_f, \\ 2 & \text{if } p \leq y^{1/4} \text{ and } p \nmid N_f, \end{cases}$$

and  $h_{N_f, y}(p^\nu) = 0$  for all primes  $p$  and integers  $\nu \geq 2$ . The key lower bound for  $S_{\text{sym}^2 f}(y^u)$  stems from the mean value of  $h_{N_f, y}(n)$  in [47, Lemma 3.2] : *Let*

$$(5.13) \quad \Pi_{N_f, 2} := \left( \frac{\varphi(N_f)}{N_f} \right)^2 \prod_{p \nmid N_f} \left( 1 - \frac{1}{p} \right)^2 \left( 1 + \frac{2}{p} \right).$$

Then we have

$$(5.14) \quad \sum_{n \leq y^u} h_{N_f, y}(n) \geq \Pi_{N_f, 2} y^u (\log y^{1/4}) \delta(u) \left\{ 1 + O\left( \frac{(\log \log y)^8}{\sqrt{\log y}} \right) \right\}$$

uniformly for

$$(5.15) \quad \frac{4}{3} \leq u \leq \frac{3}{2} \quad \text{and} \quad y \geq (k^2 N^2)^{1/100},$$

where  $\delta(u)$  is an explicit (complicated) function. Numerical calculation shows that  $\delta(u) > 0$  for all  $u < 113/80$ , which will imply (5.4).

Plainly  $n_{\text{sym}^2 f} = p^\nu$  is a prime power due to the multiplicativity of  $\lambda_{\text{sym}^2 f}(n)$ . As discussed before, it is not known whether or not the first negative coefficient of symmetric square  $L$ -function is attained at a prime argument (i.e.  $\nu = 1$ ). Write  $n_{f,2}$  for the least prime number  $p \nmid N$  such that  $\lambda_{\text{sym}^2 f}(p) < 0$ . Then  $n_{\text{sym}^2 f} \leq n_{f,2}$ . The estimate  $n_{f,2} \ll (\log(kN))^2$ , where the implied constant is absolute, can be shown under the Grand Riemann Hypothesis for  $L(s, \text{sym}^2 f)$ . In [43], Kowalski et al. gave the following almost-all result: *Let  $k \geq 2$  be an even integer and  $N \geq 1$  be a squarefree integer. There is a positive absolute constant  $c$  such that*

$$n_{f,2} \ll \log(kN)$$

for all but except  $O(kN e^{-c \log(kN)/\log_2(kN)})$  forms  $f \in H_k^*(N)$ . Here the implied constants in the  $\ll$  and  $O$ -symbols are absolute. These conditional and almost-all bounds for  $n_{f,2}$  also hold for  $n_{\text{sym}^2 f}$ , since  $n_{\text{sym}^2 f} \leq n_{f,2}$ .

**5.2. Positive and negative values over a positive density of primes.** Let  $f \in H_k^*(N)$  be a non-CM form. As  $\lambda_{\text{sym}^2 f}(p) = \lambda_f(p)^2 - 1 \in [-1, 3]$  for  $p \nmid N$ , the Sato-Tate conjecture implies that for any  $[u, v] \subset [-1, 3]$ ,

$$(5.16) \quad \sum_{\substack{p \leq x, p \nmid N \\ u \leq \lambda_{\text{sym}^2 f}(p) \leq v}} 1 \sim \frac{x}{\log x} \int_u^v \sqrt{\frac{3-t}{t+1}} \frac{dt}{2\pi} \quad (x \rightarrow \infty).$$

In [46], the elegant method of Serre in [73, Appendix] is applied to derive some thresholds of positive and negative values exceeded by  $\lambda_{\text{sym}^2 f}(p)$  for a positive density of primes. One should remark that our result of these thresholds below is superseded by (5.16) on the basis of the (settled) Sato-Tate conjecture.

**Theorem 19.** ([46], Theorem 1) *Let  $k \geq 2$  be an even integer,  $N \geq 1$  squarefree and  $f \in H_k^*(N)$ .*

(i) *For any  $\varepsilon > 0$ , there is a positive density of primes  $p$  such that*

$$(5.17) \quad \lambda_{\text{sym}^2 f}(p) \leq -5/7 + \varepsilon.$$

(ii) *For any  $\varepsilon > 0$ , there is a positive density of primes  $p$  such that*

$$(5.18) \quad \lambda_{\text{sym}^2 f}(p) \geq 2 - \varepsilon.$$

*Remark 3.* The assertions (5.17) and (5.18) were known if we took the weaker constants  $-1/3$  and  $1$  in place of  $-5/7$  and  $2$  respectively. See R. Murty [65, Corollaries 1 and 2]. Subsequently the above constant  $1$  was improved to  $4 \cos^2(2\pi/11) - 1 \approx 1.83$  by Kim & Shahidi [36, Proposition 4.11].

5.3.  $\Omega_{\pm}$ -results over integers. From (5.1), we deduce that

$$(5.19) \quad |\lambda_{\text{sym}^2 f}(n)| \leq \exp \left\{ (\log 4 + o(1)) \frac{\log n}{\log \log n} \right\}$$

for all integers  $n \geq 2$ . The next theorem gives an  $\Omega_{\pm}$ -estimate showing that the factor in the exponent is of the right order of magnitude.

**Theorem 20.** ([46], Theorem 2) *Let  $k \geq 2$  be an even integer and  $N \geq 1$  be square-free. Then for each  $f \in \mathbf{H}_k^*(N)$ , there are two positive constants  $C_{\pm}(f)$  such that*

$$\lambda_{\text{sym}^2 f}(n) = \Omega_{\pm} \left( \exp \left\{ C_{\pm}(f) \frac{\log n}{\log_2 n} \right\} \right)$$

as  $n \rightarrow \infty$ .

The proof of this theorem is based on Theorem 19.

5.4. **The number of coefficients of  $L(s, \text{sym}^2 f)$  of same signs.** Evidently (5.16) implies that the positive density of integers  $n$  where  $\lambda_{\text{sym}^2 f}(n)$  is positive (resp. negative). We shall prove this consequence by a number-theoretic method, together with the fact that the density for those  $p$  satisfying  $\lambda_{\text{sym}^2 f}(p) = a$  is zero for each fixed  $a$ . This density result was already established in [72]. Define

$$(5.20) \quad \mathcal{N}_{\text{sym}^2 f}^{\pm}(x) := \sum_{\substack{n \leq x, (n, N) = 1 \\ \lambda_{\text{sym}^2 f}(n) \gtrless 0}} 1.$$

Our result is as follows.

**Theorem 21.** ([46], Theorem 3) *Let  $k \geq 2$  be an even integer,  $N \geq 1$  be squarefree and  $\sigma = +$  or  $-$ . Then for any  $f \in \mathbf{H}_k^*(N)$ , we have*

$$\mathcal{N}_{\text{sym}^2 f}^{\sigma}(x) \gg_f x$$

for  $x \geq x_0(f)$ .

In [46], we also numerated  $\lambda_{\text{sym}^2 f}(n)$  of the same sign over short intervals to visualize the fluctuation of  $\lambda_{\text{sym}^2 f}(n)$ .

**Theorem 22.** ([46], Theorem 4) *Let  $k \geq 2$  be an even integer,  $N \geq 1$  be squarefree,  $f \in \mathbf{H}_k^*(N)$  and  $\sigma = +$  or  $-$ . Set  $\Psi(N) := \sum_{d|N} d^{-2/3}$  and  $N_1 := N\Psi(N)^{3/4}$ . There is an absolute constant  $C > 0$  such that for any  $\varepsilon > 0$  and all sufficiently large  $x \geq N_1^2 x_0(k)$ , we have*

$$(5.21) \quad \mathcal{N}_{\text{sym}^2 f}^{\sigma}(x + C(N_1 x)^{2/3}) - \mathcal{N}_{\text{sym}^2 f}^{\sigma}(x) \gg_{\varepsilon} (Nx)^{1/3-\varepsilon},$$

where  $x_0(k)$  is a suitably large constant depending on  $k$  and the implied constant in  $\gg_{\varepsilon}$  depends only on  $\varepsilon$ .

The proof goes along the same line as in [51, Theorem 2], but we need some ideas in [27] and [48] to manage the delicacy in this case.

## 6. HALF-INTEGRAL WEIGHT CASE

The notion of half-integral modular forms stems naturally from theta functions, which will be briefed in subsection 6.1. Next we define formally the half-integral weight modular forms and introduce the Hecke operators, mainly following the book [67]. We need them to understand the beautiful theory of Shimura correspondence. Finally we discuss the sign-change problems and results.

**6.1. Theta functions.** Let us look at the classical example

$$\theta(z) := \sum_{m \in \mathbb{Z}} e^{i\pi m^2 z}$$

of theta functions, which is an  $1/2$ -integral weight modular form for  $\Gamma_0(4)$ . That is,  $\theta(z)$  satisfies the automorphy condition

$$\theta(\gamma z) = j_\gamma(z)\theta(z) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4) \text{ and } z \in \mathbb{H}$$

where

$$j_\gamma(z) := \left(\frac{c}{d}\right) \varepsilon_d^{-1} (cz + d)^{1/2}.$$

Here  $\varepsilon_d := 1$  or  $i$  according as  $d \equiv 1$  or  $3 \pmod{4}$ . The Kronecker symbol  $\left(\frac{c}{d}\right)$  is defined for  $d \neq 0$  as follows: if  $(c, d) \neq 1^*$ , then  $\left(\frac{c}{d}\right) := 0$ ; otherwise,

$$\left(\frac{0}{-1}\right) := 1, \quad \left(\frac{c}{-1}\right) := \text{sgn } c, \quad \left(\frac{c}{1}\right) := 1 \quad \forall c \in \mathbb{Z},$$

moreover, let  $2^\mu \parallel d$ ,

$$\left(\frac{c}{d}\right) := (c, d)_\infty \left(\frac{2}{c}\right)^\mu \prod_{\substack{p^\nu \parallel d \\ p \text{ odd prime}}} \left(\frac{c}{p}\right)^\nu$$

where  $(c, d)_\infty := -1$  if both  $c$  and  $d$  are  $< 0$ , and  $(c, d)_\infty := 1$  otherwise. <sup>†</sup> The branch of  $z^{1/2}$  has its argument in  $(-\pi/2, \pi/2]$ .

In general, suppose  $P$  is a homogeneous polynomial in  $\mathbb{R}^r$  of degree  $\nu$  that is orthogonal to any homogeneous polynomial of lower degree.<sup>‡</sup> Let  $\mathbf{A}$  be a  $r \times r$  (symmetric) positive definite integral matrix and  $N \in \mathbb{N}$  such that  $N\mathbf{A}^{-1}$  has integral entries. Write  $P_{\mathbf{A}}(\mathbf{x}) = P(\mathbf{B}\mathbf{x})$  where  $\mathbf{B}^t\mathbf{B} = \mathbf{A}$ . The theta function

$$\Theta(z) = \sum_{\mathbf{m} \in \mathbb{Z}^r} P_{\mathbf{A}}(\mathbf{m}) e\left(\frac{\mathbf{A}[\mathbf{m}]}{2N^2} z\right)$$

is a modular form of weight  $k = \nu + \frac{r}{2}$  for  $\Gamma_0(2N)$ , where  $\mathbf{m}^t$  is the transpose of  $\mathbf{m}$  and  $\mathbf{A}[\mathbf{m}] := \mathbf{m}^t\mathbf{A}\mathbf{m}$ . If  $\nu > 0$ , then  $\Theta(z)$  is a cusp form. (See [31, Theorem

\* $(a, b) = (|a|, |b|)$  and  $(0, a) = |a|$

<sup>†</sup>Here  $\left(\frac{2}{c}\right)$  is the Jacobi symbol whose value is  $(-1)^{(c^2-1)/8}$  for odd  $c$  and  $\left(\frac{c}{p}\right)$  is the usual Legendre symbol. See [33, p.51-53].

<sup>‡</sup>Let  $f$  and  $g$  be two homogeneous polynomials of  $r$  variables. As  $f(\mathbf{x}) = |\mathbf{x}|^\nu f(\mathbf{x}/|\mathbf{x}|)$  for some  $\nu$ ,  $f$  is orthogonal to  $g$  if

$$\int_{S^{r-1}} f(\mathbf{x})\overline{g(\mathbf{x})} d\mathbf{x} = 0$$

where  $d\mathbf{x}$  is the measure on the sphere  $S^{r-1}$ . See [31, Chapter 9].

10.8].) Furthermore, suppose  $\mathbf{A}$  has even (integral) entries on the diagonal and define the quadratic form  $Q(\mathbf{x}) := \frac{1}{2}\mathbf{A}[\mathbf{x}]$ . The number of representation  $r(n, Q)$  of an integer  $n$  by  $Q(\mathbf{x})$  can be encoded in the theta function. Assume  $P \equiv 1$  and let  $\Theta(z, Q) := \Theta(N^2z)$ . Then

$$\Theta(z, Q) = \sum_{n=0}^{\infty} r(n, Q)e(nz).$$

On the other hand,  $\Theta(z, Q)$  can be decomposed into a sum of Eisenstein series and cusp forms. An interesting particular case of level one is described in [31, p.186] and this idea of using the decomposition is applied in [55, 56, 57] to study a divisor problem associated to the Epstein zeta-function.

## 6.2. Half-integral weight forms and Shimura's theory.

**Definition 1.** Let  $k \in \mathbb{N} \cup \{0\}$  and  $N \in \mathbb{N}$ . Suppose  $\chi$  is a Dirichlet character mod  $4N$ . A holomorphic half-integral weight modular form with nebentypus  $\chi$  and weight  $k + 1/2$  is a holomorphic function  $g(z)$  on  $\mathbb{H}$  that is holomorphic at each cusp of  $\Gamma_0(4N)$  and

$$g(\gamma z) = \chi(d)j_\gamma(z)^{2k+1}g(z), \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N) \text{ and } z \in \mathbb{H}.$$

We further call  $g$  a cusp form if  $g$  vanishes at all cusps. The collection of all these holomorphic half-integral weight modular (and resp. cusp) forms is denoted by  $M_{k+1/2}(4N, \chi)$  (and resp.  $S_{k+1/2}(4N, \chi)$ ).

Let  $p$  be a prime. The half-integral weight Hecke operator  $\tilde{T}_{p^2}$  is defined as: for

$$f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in M_{k+1/2}(4N, \chi),$$

we put

$$\tilde{T}_{p^2}f(z) := \sum_{n=0}^{\infty} \left( a(p^2n) + \chi^*(p) \binom{n}{p} p^{k-1}a(n) + \chi(p^2)p^{2k-1}a\left(\frac{n}{p^2}\right) \right) e(nz),$$

where

$$\chi^*(p) := \left( \frac{(-1)^k}{p} \right) \chi(p) = (-1)^{k(p-1)/2} \chi(p)$$

and  $a(n/p^2) = 0$  if  $p^2 \nmid n$ . The common eigenfunctions of all  $\tilde{T}_{p^2}$  with  $p \nmid 4N$  in  $M_{k+1/2}(4N, \chi)$  are known as Hecke eigenforms. We say that  $f$  is a complete Hecke eigenform if  $f$  is an eigenfunction of all  $\tilde{T}_{p^2}$  (including any prime  $p \mid (4N)$ ).

*Remark 4.* (i) One may define the Hecke operator  $\tilde{T}_{m^2}$  for  $m \in \mathbb{N}$ , as in [78, p.450]. Hecke operators for non-squares are not defined as they vanish on  $M_{k+1/2}(4N, \chi)$ .

(ii) By Proposition 1.6 in [78], we have  $\tilde{T}_{m^2n^2} = \tilde{T}_{m^2}\tilde{T}_{n^2}$  whenever  $(m, n) = 1$  or  $m \mid (4N)^\infty$  (or  $n \mid (4N)^\infty$ ).

The Shimura correspondence maps a half-integral weight cusp form to an integral weight cusp form. Let us summarize the profound results in [78] and [67] in the following.

**Theorem 23.** *Let  $k \geq 1$ ,  $N \geq 1$  and*

$$f(z) = \sum_{n=1}^{\infty} a(n)e(nz) \in S_{k+1/2}(4N, \chi).$$

For any given positive squarefree integer  $t$ , define

$$\sum_{n=1}^{\infty} \frac{A_t(n)}{n^s} := L(s - k + 1, \psi_t) \sum_{n=1}^{\infty} \frac{a(tn^2)}{n^s},$$

where

$$\psi_t(n) := \chi(n) \left( \frac{-1}{n} \right)^k \left( \frac{t}{n} \right)$$

is a Dirichlet character mod  $4tN$ .

(i) We have

$$S_{t,k}f(z) := \sum_{n=1}^{\infty} A_t(n)e(nz) \in M_{2k}(2N, \chi^2),$$

i.e.  $S_{t,k}f$  is a modular form of integral weight  $2k$  and of nebentypus  $\chi^2$  for  $\Gamma_0(2N)$ .

(ii) Moreover, for  $k \geq 2$ ,  $S_{t,k}f(z) \in S_{2k}(2N, \chi^2)$ , i.e. a cusp form, and for  $k = 1$ ,

$$S_{t,1}f(z) \in S_{2k}(2N, \chi^2)$$

if  $f(z) \in \langle \theta \in S_{3/2}(4N, \chi) \rangle^\perp$ , where  $\langle \theta \in S_{3/2}(4N, \chi) \rangle^\perp$  denotes the orthogonal complement of the subspace  $\langle \theta \in S_{3/2}(4N, \chi) \rangle$  spanned by the theta functions in  $S_{3/2}(4N, \chi)$ .

(iii) If  $f$  is a Hecke eigenform (i.e.,  $\tilde{T}_{p^2}f = \omega_p f$  for all primes  $p \nmid 4N$ ), then

$$\sum_{n=1}^{\infty} \frac{a(tn^2)}{n^s} = \sum_{n|(4N)^\infty} \frac{a(tn^2)}{n^s} \prod_{p \nmid 4N} \left( 1 - \frac{\psi_t(p)}{p^{s-k+1}} \right) \left( 1 - \frac{\omega_p}{p^s} + \frac{\chi(p)^2}{p^{2s-2k+1}} \right)^{-1}.$$

Hence we have

$$a(tm^2)a(tn^2) = a(t)a(tm^2n^2) \quad \text{if } (m, n) = (mn, 4N) = 1,$$

and

$$\sum_{n=1}^{\infty} \frac{A_t(n)}{n^s} = \sum_{n|(4N)^\infty} \frac{a(tn^2)}{n^s} \prod_{p \nmid 4N} \left( 1 - \frac{\omega_p}{p^s} + \frac{\chi(p)^2}{p^{2s-2k+1}} \right)^{-1}.$$

*Remark 5.* (i) The subspace  $\langle \theta \in S_{3/2}(4N, \chi) \rangle$  is described more explicitly in [5]: let  $\psi$  be an odd primitive Dirichlet character modulo  $r \geq 1$  and define for  $m \geq 1$ , the unary theta function

$$\theta_{\psi,m}(z) = \sum_{n \in \mathbb{Z}} \psi(n)ne(mn^2z)$$

which lies in  $S_{3/2}(4r^2m, \left(\frac{-4m}{\cdot}\right)\psi)$ . Let  $W$  be the span of all these  $\theta_{\psi,m}$ . Then

$$\langle \theta \in S_{3/2}(4N, \chi) \rangle = W \cap S_{3/2}(4N, \chi).$$

(ii) The map  $S_{t,k} : S_{k+1/2}(4N, \chi) \rightarrow M_{2k}(2N, \chi^2)$  commutes with Hecke operators  $\tilde{T}_{p^2}$  where  $(p, 4N) = 1$ :

$$S_{t,k} \circ \tilde{T}_{p^2} f = T_p \circ S_{t,k} f \quad \text{for any prime } p \nmid 4N.^\S$$

(iii) There is a cute sufficient condition for eigenforms of  $\tilde{T}_{p^2}$  shown in Bruinier [5]: Let  $f(z) = \sum_{n \geq 1} a(n)e(nz) \in S_{k+1/2}(4N, \chi)$ ,  $p \nmid 4N$  be a prime and  $\varepsilon \in \{\pm 1\}$ . If

$$a(n) = 0 \quad \text{whenever} \quad \left(\frac{n}{p}\right) = -\varepsilon,$$

then  $f$  is an eigenform of  $\tilde{T}_{p^2}$  with eigenvalue  $\omega_p = \varepsilon \chi^*(p)(p^k + p^{k-1})$ .

(iv) Theorem 23(iii) is obtained along the argument in the proof of [78, Corollary 1.8]. If furthermore  $f$  is a complete Hecke eigenform, then we may split the sum over  $n|(4N)^\infty$  into an Euler product (see Remark 4(ii)). Hence,

$$\sum_{n=1}^{\infty} \frac{A_t(n)}{n^s} = a(t) \prod_p \left(1 - \frac{\omega_p}{p^s} + \frac{\chi(p)^2}{p^{2s-2k+1}}\right)^{-1}$$

and

$$a(tm^2)a(tn^2) = a(t)a(tm^2n^2)$$

for any  $(m, n) = 1$ . This is in Theorem 1.9 and (1.18) of [78].

**6.3. Sign-change problem of half-integral weight cusp forms.** In [79], Vigneras proved that every non-zero

$$f(z) = \sum_{n \geq 1} a(n)e(nz) \in S_{k+1/2}(4N, \chi)$$

has infinitely many non-vanishing Fourier coefficients  $a(tm_t^2)$  where  $t$  is squarefree and  $m_t \in \mathbb{N}$ . Later Bruinier showed that  $a(t) \neq 0$  for infinitely many squarefree  $t$  if  $f \in S_{k+1/2}(4N, \chi)$  is a Hecke eigenform.<sup>¶</sup> In case the Fourier coefficients are real, one may further ask how their signs distribute. Suppose  $f \in S_{k+1/2}(4N, \chi_0)$  where  $\chi_0$  is the trivial character and the Fourier coefficients of  $f$  are real. Bruinier & Kohnen [6] conjectured

$$\lim_{x \rightarrow \infty} \frac{|\{n \leq x : a(n) \geq 0\}|}{|\{n \leq x : a(n) \neq 0\}|} = \frac{1}{2}$$

and

$$\lim_{x \rightarrow \infty} \frac{|\{d \leq x : d \text{ fundamental discriminant, } a(|d|) \geq 0\}|}{|\{d \leq x : d \text{ fundamental discriminant, } a(|d|) \neq 0\}|} = \frac{1}{2}$$

with empirical evidence. In addition, they obtained the following results in [6].

**Theorem 24.** (Bruinier & Kohnen) *Suppose  $f \in S_{k+1/2}(4N, \chi)$ .*

(i) *Let*

$$\mathbb{S} := \{t \geq 1 \text{ squarefree} : a(t) \neq 0\}.$$

<sup>§</sup>See (4.1) for  $T_p g(z)$ . The condition  $p \nmid t$  in [67, Corollary 3.16] is not needed in our checking of the equality.

<sup>¶</sup>The results of Vigneras and Bruinier are indeed stronger, for instance, Bruinier can pick square-free  $t$  to be quadratic residues or non-residues of some given primes so that  $a(t) \neq 0$ . We state the weak form for simplicity.

Assume the Dirichlet  $L$ -function  $L(s, \chi_{t,N})$  has no real zero in the interval  $(0, 1)$  where  $\chi_{t,N} = \left(\frac{(-1)^{k_t}}{\cdot}\right)\chi(\cdot)$  and  $t \in \mathbb{S}$ . Then the sequence  $\{a(tn^2)\}_{n \in \mathbb{N}}$  has infinitely many sign changes.

(ii) Given distinct primes  $p_1, \dots, p_r$  coprime to  $4N$  and  $\varepsilon_1, \dots, \varepsilon_r \in \{\pm 1\}$ . Let

$$\mathbb{S}' := \left\{ t \geq 1 \text{ squarefree} : \left(\frac{t}{p_j}\right) = \varepsilon_j, \forall 1 \leq j \leq r \right\}.$$

There exist  $n_t \in \mathbb{N}$  and  $t \in \mathbb{S}'$ , such that (unconditionally) the sequence  $\{a(tn_t^2)\}_{t \in \mathbb{S}'}$  has infinitely many sign changes.

(iii) In case  $\chi$  is real and  $f$  is a Hecke eigenform, the sequence  $\{a(tp^m)\}_{m \in \mathbb{N}}$  has infinitely many sign changes where  $p$  is any prime coprime to  $N$  and  $t \in \mathbb{S}$  defined as in (i).

*Remark 6.* The proof relies heavily on the Shimura lift in Theorem 23. Let us explain the idea of the proof of part (i) for example. Now we have

$$(6.1) \quad \sum_{n \geq 1} a(tn^2)n^{-s} = L(s - k + 1, \psi_t)^{-1} \sum_{n \geq 1} A_t(n)n^{-s}.$$

Part (i) follows by scrutinizing the singularities on both sides. On one hand, a classical result of Landau says that the positivity of all but finitely many of  $a(tn^2)$  will imply either (a) the existence of a singularity at some point  $x \in (\sigma_a, \infty)$  or (b)  $\sigma_a = -\infty$  where  $\sigma_a$  denotes the abscissa of absolute convergence of the series on the left-hand side. On the other hand, neither (a) nor (b) can happen under the condition. The condition in the assertion (i) is expected to be true and is known as Chowla's conjecture. In [39], Kohnen could remove the assumption of Chowla's conjecture.

Recently, Kohnen, Lau & Wu derive some quantitative results in [41] and let us state two as examples. Define  $\mathbb{S}$  as in Case (i) of Theorem 24.

**Theorem 25.** *Let  $f \in S_{k+1/2}(N, \chi)$  and  $t \in \mathbb{S}$ . Assume that the sequence  $\{a(tn^2)\}_{n \in \mathbb{N}}$  is real. Then there is a small constant  $\alpha = \alpha(f, t) > 0$  such that for all sufficiently large  $x \geq x_0(f, t)$ ,  $a(tp^2)$  has at least one sign-change when  $p$  runs through primes in the interval  $[x^\alpha, x]$ .*

**Theorem 26.** *Let  $f \in S_{k+1/2}(4N, \chi)$  be a Hecke eigenform and  $t \in \mathbb{S}$ . Assume that its Shimura lift  $S_{t,k}f$  is not of CM type. Then both*

$$\sum_{\substack{n \leq x, n \text{ squarefree} \\ (n, Nt)=1, a(tn^2) > 0}} 1 \quad \text{and} \quad \sum_{\substack{n \leq x, n \text{ squarefree} \\ (n, Nt)=1, a(tn^2) < 0}} 1 \gg_{f,t} x$$

for  $x \geq x_0(f, t)$ . If  $N/2$  is squarefree, the assumption of a non-CM Shimura lift will automatically hold and hence can be omitted.

The methods of proof for Theorems 25 and 26 also make use of the Shimura lifting. Indeed, (6.1) is equivalent to the Möbius inversion:

$$a(tn^2) = \sum_{d|n} \mu(d)\psi_t(d)d^{k-1}A_t(n/d).$$

When  $n = p$  is a prime, it is simply  $a(tp^2)p^{1/2-k} = A_t(p) - a(t)\psi_t(p)p^{-1/2}$ . The work in Section 4, together with Atkin-Lehner's theory, will lead to the results.

*Remark 7.* The results above do not apply to the sequence  $\{a(t)\}$  where  $t$  runs over squarefree integers, which is of particular interest. This is because the Shimura lift satisfies

$$S_{t,k}f = a(t)S_{1,k}f$$

if  $f$  is a complete Hecke eigenform, by Remark 5 (iv). Very recently, Hulse et al. [30] made a significant progress:  $a(t)$  changes sign infinitely often if  $f \in S_{k+1/2}(4, \chi_0)$  is a complete Hecke eigenform where  $\chi_0$  is the trivial character.

**Question 5.** *Prove a quantitative version of Theorem 1.1 in [30], i.e. give lower-bound estimates for*

$$\sum_{\substack{t \leq x \\ a(t) > 0, t \text{ squarefree}}} 1 \quad \text{and} \quad \sum_{\substack{t \leq x \\ a(t) < 0, t \text{ squarefree}}} 1.$$

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