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On optimality of empirical risk minimization in linear aggregation

Adrien Saumard*

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Abstract

In the first part of this paper, we show that the small-ball condition, recently introduced by [Men15], may behave poorly for important classes of localized functions such as wavelets, leading to suboptimal estimates of the rate of convergence of ERM for the linear aggregation problem.

In a second part, we derive optimal upper and lower bounds for the excess risk of ERM when the dictionary is made of trigonometric functions. While the validity of the small-ball condition remains essentially open in the Fourier case, we show strong connection between our results and concentration inequalities recently obtained for the excess risk in [Cha14] and [vdGW16].

Keywords: empirical risk minimization, linear aggregation, small-ball property, concentration inequality, empirical process theory.

1 Introduction

Consider the following general regression framework: $(\mathcal{X}, \mathcal{T}_{\mathcal{X}})$ is a measurable space, $(X, Y) \in \mathcal{X} \times \mathbb{R}$ is a pair of random variables of joint distribution P - the marginal of X being denoted P^X - and it holds

$$Y = s_*(X) + \sigma(X) \varepsilon, \quad (1)$$

where s_* is the regression function of the response variable Y with respect to the *random design* X , $\sigma(X) \geq 0$ is the *heteroscedastic noise* level and ε is the conditionally standardized noise, satisfying $\mathbb{E}[\varepsilon | X] = 0$ and $\mathbb{E}[\varepsilon^2 | X] = 1$. Relation (1) is very general and is indeed satisfied as soon as $\mathbb{E}[Y^2] < +\infty$. In this case $s_* \in L_2(P^X)$ is the orthogonal projection of Y onto the space of X -measurable functions. In particular, no restriction is made on the structure of dependence between Y and X .

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We thus face a typical *learning problem*, where the statistical modelling is minimal, and the goal will be, given a sample $(X_i, Y_i)_{i=1}^n$ of law $P^{\otimes n}$ and a new covariate X_{n+1} , to predict the value of the associated response variable Y_{n+1} . More precisely, we want to construct a function \hat{s} , depending on the data $(X_i, Y_i)_{i=1}^n$, such that the least-squares risk $R(\hat{s}) = \mathbb{E}[(Y_{n+1} - \hat{s}(X_{n+1}))^2]$ is as small as possible, the pair (X_{n+1}, Y_{n+1}) being independent of the sample $(X_i, Y_i)_{i=1}^n$.

In this paper, we focus on the technique of *linear aggregation* via Empirical Risk Minimization (ERM). This means that we are given a dictionary $S = \{s_1, \dots, s_D\}$ and that we produce the least-squares estimator \hat{s} on its linear span $m = \text{Span}(S)$,

$$\hat{s} \in \arg \min_{s \in m} R_n(s) \quad , \quad \text{where } R_n(s) = \frac{1}{n} \sum_{i=1}^n (Y_i - s(X_i))^2 \quad . \quad (2)$$

The quantity $R_n(s)$ is called the empirical risk of the function s . The accuracy of the method is tackled through an *oracle inequality*, where the risk of the estimator $R(\hat{s})$ is compared - on an event of probability close to one - to the risk of the best possible function within the linear model m . The latter function is denoted s_m and is called the oracle, or the (orthogonal) *projection* of the regression function s_* onto m ,

$$s_m \in \arg \min_{s \in m} R(s) \quad .$$

An oracle inequality then writes, on an event Ω_0 of probability close to one,

$$R(\hat{s}) \leq R(s_m) + r_n(D) \quad , \quad (3)$$

for a positive residual term $r_n(D)$. An easy and classical computation gives that the *excess risk* satisfies $R(\hat{s}) - R(s_m) = \|\hat{s} - s_m\|_2^2$, where $\|\cdot\|_2$ is the natural quadratic norm in $L_2(P^X)$. Hence, inequality (3) can be rewritten as $\|\hat{s} - s_m\|_2^2 \leq r_n(D)$ and the quantity $r_n(D)$ thus corresponds to the rate of estimation of the projection s_m by the least-squares estimator \hat{s} in terms of excess risk, corresponding here to the squared quadratic norm.

The linear aggregation problem has been well studied in various settings linked to nonparametric regression ([Nem00], [Tsy03], [BTW07], [AC11]) and density estimation ([RT07]). It has been consequently understood that the *optimal rate* $r_n(D)$ of *linear aggregation* is of the order of D/n , where D is the size of the dictionary. Recently, [LM15] have shown that ERM is suboptimal for the linear aggregation problem in general, in the sense that there exist a dictionary S and a pair (X, Y) of random variables for which the rate of ERM (drastically) deteriorates, even in the case where the response variable Y and the dictionary are uniformly bounded.

On the positive side, [LM15] also made a breakthrough by showing that if a so-called *small-ball condition* is achieved with absolute constants, uniformly over the functions in the linear model m , then the optimal rate is recovered by ERM. We recall and discuss in details the small-ball condition in Section 2, but it is worth mentioning here that one of the main advantages of the small-ball method developed in a series of papers, [Men14b], [Men15], [Men14a], [KM15], [LM14], [LM15] is that it enables to prove sharp bounds under very weak moment conditions

and thus to derive results that were *unachievable* with more standard concentration arguments.

In Section 2, we contribute to the growing understanding of this very recent approach by looking at the behavior of the small-ball condition when the dictionary is made of localized functions such as compactly supported wavelets, histograms or piecewise polynomials. It appears that with such functions, the small-ball condition *can't be satisfied with absolute constants* and the resulting bounds obtained in [LM15] are far from optimal in this case since they are of the order of D^3/n .

The question of the validity of the small-ball approach in the Fourier case, which is a case where the functions are typically unlocalized, remains an open issue, of potentially great consequences in compressed sensing, [LM15]. Despite this lack of understanding, we prove by other means optimal upper and lower bounds for ERM when the dictionary is made of trigonometric functions. Our result, stated in Section 3, also outperforms previously obtained bounds [AC11].

An outline of the proofs related to Section 3 is given in Section 3.2. While detailing our arguments, we show the strong connection of our approach to optimal bounds with recent works of [Cha14] concerning least-squares under convex constraint, extended to the setting of regularized ERM by [MvdG15] and [vdGW16].

Finally, complete proofs are dispatched in the Appendix.

2 The small-ball method for classical functional bases

We recall in Section 2.1 one of the main results of [LM15], linking the small-ball condition to the rate of convergence of ERM in linear aggregation. Then, we show in Section 2.2 that the constants involved in the small-ball condition behave poorly for dictionaries made of localized bases.

2.1 The small-ball condition and the rate of ERM in linear aggregation

Let us first recall the definition of the small-ball condition for a linear span, as exposed in [LM15].

Definition 1 *A linear span $m \subset L_2(P^X)$ is said to satisfy the **small-ball condition** for some positive constants κ_0 and β_0 if for every $s \in m$,*

$$\mathbb{P}(|s(X)| \geq \kappa_0 \|s\|_2) \geq \beta_0 . \tag{4}$$

The small-ball condition thus ensures that the functions of the model m do not put too much weight around zero. From a statistical perspective, it is also explained in [LM15] that the small-ball condition can be viewed as quantified version of identifiability of the model m . A more general small-ball condition - that reduces

to the previous definition for linear models - is also available when the model isn't necessary linear, [Men15].

Under the small-ball condition, [LM15] derive the following result, describing the rate of convergence of ERM in linear aggregation.

Theorem 2 ([LM15]) *Let $S = \{s_1, \dots, s_D\} \subset L_2(P^X)$ be a dictionary and assume that $m = \text{Span}(S)$ satisfies the small-ball condition with constants κ_0 and β_0 (see Definition 1 above). Let $n \geq (400)^2 D/\beta_0^2$ and set $\zeta = Y - s_m(X)$, where s_m is the projection of the regression function s_* onto m . Assume further that one of the following two conditions holds:*

1. ζ is independent of X and $\mathbb{E}\zeta^2 \leq \sigma^2$, or
2. $|\zeta| \leq \sigma$ almost surely.

Then the least-squares estimator \hat{s} on m , defined in (2), satisfies for every $x > 0$, with probability at least $1 - \exp(-\beta_0^2 n/4) - (1/x)$,

$$\|\hat{s} - s_m\|_2^2 \leq \left(\frac{16}{\beta_0 \kappa_0^2}\right)^2 \frac{\sigma^2 D x}{n}. \quad (5)$$

Notice that Alternative 1 in Theorem 2 is equivalent to assuming that the regression function belongs to m - that is $s_* = s_m$ - and that the noise is independent from the design - that is $\sigma(X) \equiv \sigma$ is homoscedastic and ε is independent of X in relation (1).

The main feature of Theorem 2 is that if the small-ball condition is achieved with *absolute constants* κ_0 and β_0 not depending on the dimension D nor the sample size n , then optimal linear aggregation rates of order D/n are recovered by ERM. If moreover the regression function belongs to m (Alternative 1), then the only moment assumption required is that the noise is in L_2 . Otherwise, Alternative 2 asks for a uniformly bounded noise. Some variants of Theorem 2 are also presented in [LM15], showing for instance that optimal rates can be also derived for ERM when the noise as a fourth moment.

In the analysis of optimal rates in linear aggregation, it is thus worth understanding when the small ball condition stated in Definition 1 is achieved with absolute constants.

One typical such situation is for *linear measurements*, that is when the functions of the dictionary are of the form $f_i(x) = \langle x, t_i \rangle$, $t_i \in \mathbb{R}^d$. Indeed, very weak conditions are asked on the design X in this case to ensure the small-ball property: for instance, it suffices to assume that X has independent coordinates that are absolutely continuous with respect to the Lebesgue measure, with a density almost surely bounded (see [LM14] and [Men15], Section 6, for more details). As shown in [LM14] and [LM16], this implies that the small-ball property has important consequences in sparse recovery and analysis of regularized linear regression.

2.2 The constants in the small-ball condition for general linear bases

Besides linear measurements discussed in Section 2.1 above, an important class of dictionaries for the linear aggregation problem consists in expansions along orthonormal bases of $L_2(P^X)$. Our goal in this section is thus to investigate the behavior of the small-ball condition for some classical orthonormal bases such as piecewise polynomial functions, including histograms, wavelets or the Fourier basis.

The following assumption, that states the equivalence between the L_∞ and L_2 norms for functions in the linear model m , is satisfied by many classical functional bases:

(A1) Take $S = \{s_1, \dots, s_D\} \subset L_2(P^X)$ a dictionary and consider its linear span $m = \text{Span}(S)$. Assume that there exists a positive constant L_0 such that, for every $s \in m$,

$$\|s\|_\infty \leq L_0 \sqrt{D} \|s\|_2 .$$

Examples of linear models m satisfying Assumption **(A1)** with an absolute constant L_0 are given for instance in [BBM99].

More precisely, when $\mathcal{X} = [0, 2\pi]$, X is uniform on \mathcal{X} and S consists of the D first elements of the *Fourier basis*, then **(A1)** is verified.

Furthermore, if $\mathcal{X} = [0, 1]^d$ for some $d \geq 1$, X is uniform on \mathcal{X} , Π is a regular partition on \mathcal{X} made of J hyper-rectangles and m is made of the *piecewise polynomial functions* defined on Π , of maximal degrees on each element of Π not larger than $r \in \mathbb{N}_*$, then Assumption **(A1)** is also satisfied with a dimension $D = (r + 1) J$. This example includes in particular for $r = 0$ the case of *histograms* on Π .

Some *wavelet expansions* also satisfy Assumption **(A1)**. As it will be useful in the following, let us more precisely state some notations (for more details about wavelets, see for instance [HKPT98]). We consider in this case that $\mathcal{X} = [0, 1]$ and X is uniformly distributed on \mathcal{X} . Set ϕ_0 the father wavelet and ρ_0 the mother wavelet, two functions defined on \mathbb{R} . Thus, the support of the wavelets may not be contained in $[0, 1]$, but for the estimation, only the wavelets whose support intersects $[0, 1]$ will count. For every integers $j \geq 0$, $1 \leq k \leq 2^j$, define

$$\rho_{j,k} : x \mapsto 2^{j/2} \rho_0(2^j x - k + 1) .$$

We set for every integer $j \geq 0$,

$$\Lambda(j) = \{(j, k) ; 1 \leq k \leq 2^j \ \& \ \text{Support}(\rho_{j,k}) \cap [0, 1] \neq \emptyset\} .$$

Moreover, we set $\rho_{-1,k}(x) = \phi_0(x - k + 1)$ and for any integer $l \geq 0$,

$$\Lambda(-1) = \{(-1, k) ; \text{Support}(\rho_{-1,k}) \cap [0, 1] \neq \emptyset\} \quad \text{and} \quad \Lambda_l = \bigcup_{j=-1}^l \Lambda(j) .$$

Then we consider the model

$$m = \text{Span} \{ \rho_\lambda ; \lambda \in \Lambda_l \} .$$

If ϕ_0 and ρ_0 are compactly supported, then $\{ \rho_\lambda ; \lambda \in \Lambda_l \}$ satisfies Assumption **(A1)** for an absolute constant L_0 and dimension $D = \text{Card}(\Lambda_l)$. It is worth noting that more general multidimensional wavelets could also be considered at the price of more technicalities.

When a model m satisfies Assumption **(A1)**, the small-ball condition is also verified, but with constants that may depend on the dimension of the model.

Proposition 3 *If a linear model m satisfies Assumption **(A1)** then inequality (4) of the small-ball condition given in Definition 1 is verified for any $\kappa_0 \in (0, 1)$ with $\beta_0 = (1 - \kappa_0^2) L_0^{-2} D^{-1}$.*

When applied to Theorem 2, a direct consequence of Proposition 3 is that ERM satisfies the following bound on a model m satisfying Assumption **(A1)**: for every $x > 0$ and $\kappa_0 \in (0, 1)$, with probability at least $1 - \exp(- (1 - \kappa_0^2) n / (4L_0 D^2)) - (1/x)$,

$$\| \widehat{s} - s_m \|_2^2 \leq \left(\frac{16L_0}{(1 - \kappa_0^2) \kappa_0^2} \right)^2 \frac{\sigma^2 D^3 x}{n} ,$$

Hence, in such case Theorem 2 only says something for models of dimension $D \lesssim \sqrt{n}$ (using the condition $\exp(- (1 - \kappa_0^2) n / (4L_0 D^2)) < 1$) and when the latter restriction is achieved, it provides a rate of convergence of the order D^3/n . This is essentially a weakness of the small-ball approach in this case, since considering histograms and piecewise polynomial functions, [Sau12] proved in a bounded setting that the rate of convergence of ERM is actually D/n (whenever $D \lesssim n / (\ln n)^2$), which is the optimal rate of linear aggregation. Furthermore, for some more general linear models with localized bases such as Haar expansions, [Sau15] also proved that the rate of convergence of ERM is still D/n .

The proof of Proposition 3, detailed in the Appendix, is a direct application of Paley-Zygmund's inequality (see [dlPG99]). [LM15] also noticed that more generally, Paley-Zygmund's inequality could be used to prove the small-ball property when for some $p > 2$, the L_p and L_2 norms are equivalent, or also for *subgaussian classes*, where the Orlicz ψ_2 norm is controlled by the L_2 norm, see [LM13].

These conditions are weaker than the control of the L_∞ norm by the L_2 norm, however we will show that the dependence in D for β_0 given in Proposition 3 above is sharp for localized bases such as histograms, piecewise polynomials and wavelets. Hence, the control of the L_∞ norm by the L_2 norm is in some way optimal in these cases, and weaker assumptions could not imply some improvements on the behavior of the small ball property for these models. In conclusion, when applied to histograms and piecewise polynomials on regular partitions, or to compactly supported wavelets, the small-ball method developed in [LM15] enables only to prove suboptimal rates of the order D^3/n .

Consider first a model of histograms on a regular partition Π of $\mathcal{X} = [0, 1]^d$ made of D pieces, X being uniformly distributed on \mathcal{X} . More precisely, for any $I \in \Pi$, set

$$s_I = \frac{\mathbf{1}_I}{\sqrt{P^X(I)}} = \sqrt{D}\mathbf{1}_I$$

and take a dictionary $S = \{s_I; I \in \Pi\}$, associated to the model $m = \text{Span}(S)$. It holds, for every $I \in \Pi$,

$$\|s_I\|_\infty = \sqrt{D} \|s_I\|_2$$

and so, as the s_I 's have disjoint supports, it is easy to see that m satisfies **(A1)** with $L_0 = 1$. Furthermore, for any $\kappa_0 \in (0, 1)$,

$$\mathbb{P}(|s_I(X)| \geq \kappa_0 \|s_I\|_2) = P^X(I) = \frac{1}{D}. \quad (6)$$

This shows that necessarily $\beta_0 \leq D^{-1}$ and so, up to absolute constants, the value of β_0 given in Proposition 3 is optimal in the case of histograms on a regular partition. In particular, Assumption **(A1)** can't be satisfied with absolute constants in this case.

When considering the case of piecewise polynomial functions on a regular partition, identity (6) above still holds for polynomial functions of degree zero supported by one element of the partition. Thus when the degrees of the polynomial functions in the model m are bounded by a constant r , we easily deduce that $\beta_0 \leq rD^{-1}$ for any $\kappa_0 \in (0, 1)$ and the value of β_0 given in Proposition 3 is again optimal in this case.

Finally, when the model m corresponds to a finite expansion in some compactly supported wavelet basis, we have the following property, that again proves that the value of β_0 given in Proposition 3 is optimal. Examples of compactly supported wavelets include Daubechies wavelets and coiflets, see [HKPT98].

Proposition 4 *Assume that $\mathcal{X} = [0, 1]$ and that the design X is uniformly distributed on \mathcal{X} . Take $m = \text{Span}\{\rho_\lambda; \lambda \in \Lambda_l\}$ (using notations above defining the wavelets ρ_λ and the index set Λ_m) a linear model corresponding to some compactly supported wavelet expansion. More precisely, assume that $\text{Supp}(\phi_0) \cup \text{Supp}(\rho_0) \subset [0, R]$ for some $R \geq 1$, where $\text{Supp}(\phi_0)$ and $\text{Supp}(\rho_0)$ are the supports of the father wavelet ϕ_0 and the mother wavelet ρ_0 respectively. Assume that the linear dimension $D = \text{Card}(\Lambda_l)$ of the model m is greater than $[R]$, the integer part of R : $D > [R]$. Then there exists an absolute constant $\kappa_0 > 0$ such that, if m achieves the small-ball condition given in Definition 1 with constants (κ_0, β_0) then $\beta_0 \leq (D - [R])^{-1}$.*

The proof of Proposition 4 consists in basic calculations and can be found in the Appendix.

3 Optimal excess risks bounds for Fourier expansions

3.1 Main theorem

We have shown in Section 2 that the small-ball condition is satisfied for linear models such as histograms, piecewise polynomials or compactly supported wavelets, but with constants that depend on the dimension of the model in such a way that using this condition to analyze the rate of convergence of ERM on these models may lead to suboptimal bounds.

The behavior of the small-ball condition when the model m is spanned by the D first elements of the Fourier basis - considering that $\mathcal{X} = [0, 2\pi]$ - remains however essentially unknown. Indeed, in the Fourier case, the bound obtained in Proposition 3 for the constants involved in the small-ball condition might be suboptimal. In particular, lower bounds such as the one established in Proposition 4 in the case of compactly supported wavelets remains inaccessible for us in the Fourier case. It is worth noting that in the context of sparse recovery, [LM14] already noticed that the small-ball condition for Fourier measurements is hard to check and the authors mention a possible ‘non-uniform’ small-ball property satisfied in this case, but leave this direction open (see Remark 1.5 of [LM14] for more details).

Our aim in this section is to show that optimal rates of linear aggregation are attained by ERM in the Fourier case, that is when the model m is spanned by the D first elements of the Fourier basis. As optimal rates would also be achieved by Theorem 2 if the model m would satisfy the small-ball condition with absolute constants, this supports the conjecture made by [LM14] that the Fourier basis achieves a condition which is close to the small-ball condition with absolute constants.

We only tackle the bounded setting. One of the main reasons for this restriction is that we make a recurrent use along our proofs of classical Talagrand’s type concentration inequalities for suprema of the empirical process with bounded arguments. Indeed, our approach, which is based on [Sau12] and is detailed at a heuristic level in Section 3.2 below, is very different from the small-ball approach. In fact, as we will explain in Section 3.2, it is closely related to recent advances linked to *excess risk concentration* due to [Cha14] in the context of least-squares under convex constraint, extended to regularized ERM by [vdGW16]. Even if concentration inequalities exist for suprema of the empirical process with unbounded arguments, the unbounded case would involve much more technicalities and this would go beyond the scope of this paper.

Let us now precisely detail our assumptions. Assume that the design X is uniformly distributed on $\mathcal{X} = [0, 2\pi]$ and that the regression function s_* satisfies $s_*(0) = s_*(2\pi)$. Then the Fourier basis is orthonormal in $L_2(P^X)$ and we consider a model m of dimension D (assumed to be odd) corresponding to the linear vector space spanned by the first D elements of the Fourier basis. More precisely, if we set $\varphi_0 \equiv 1$, $\varphi_{2k}(x) = \sqrt{2} \cos(kx)$ and $\varphi_{2k+1}(x) = \sqrt{2} \sin(kx)$ for $k \geq 1$, then $(\varphi_j)_{j=0}^{D-1}$

is an orthonormal basis of $(m, \|\cdot\|_2)$, for an integer l satisfying $2l + 1 = D$. Assume also:

- **(H1)** The data and the linear projection of the target onto m are bounded by a positive finite constant A :

$$|Y| \leq A \text{ a.s.} \quad (7)$$

and

$$\|s_m\|_\infty \leq A . \quad (8)$$

Hence, from **(H1)** we deduce that

$$\|s_*\|_\infty = \|\mathbb{E}[Y | X = \cdot]\|_\infty \leq A \quad (9)$$

and that there exists a constant $\sigma_{\max} > 0$ such that

$$\sigma^2(X_i) \leq \sigma_{\max}^2 \leq A^2 \text{ a.s.} \quad (10)$$

- **(H2)** The heteroscedastic noise level σ is not reduced to zero:

$$\|\sigma\|_2 = \sqrt{\mathbb{E}[\sigma^2(X)]} > 0 .$$

We are now in position to state our main result.

Theorem 5 *Let $A_+, A_-, \alpha > 0$ and let m be a linear vector space spanned by a dictionary made of the first D elements of the Fourier basis. Assume **(H1)** and take $\varphi = (\varphi_k)_{k=0}^{D-1}$ the Fourier basis of m . If it holds*

$$A_- (\ln n)^2 \leq D \leq A_+ \frac{n^{1/2}}{(\ln n)^2} \leq n , \quad (11)$$

*then there exists a constant $A_0 > 0$, only depending on α, A_- and on the constants $A, \|\sigma\|_2$ defined in assumptions **(H1)**, **(H2)** respectively, such that by setting*

$$\varepsilon_n = A_0 \max \left\{ \left(\frac{\ln n}{D} \right)^{1/4} , \left(\frac{D^2 \ln n}{n} \right)^{1/4} \right\} , \quad (12)$$

we have for all $n \geq n_0(A_-, A_+, A, \|\sigma\|_2, \alpha)$,

$$\mathbb{P} \left[\|\widehat{s} - s_m\|_2^2 \geq (1 - \varepsilon_n) \frac{D}{n} \mathcal{C}_m^2 \right] \geq 1 - 5n^{-\alpha} , \quad (13)$$

$$\mathbb{P} \left[\|\widehat{s} - s_m\|_2^2 \leq (1 + \varepsilon_n) \frac{D}{n} \mathcal{C}_m^2 \right] \geq 1 - 5n^{-\alpha} , \quad (14)$$

where \widehat{s} is the least-squares estimator on m , defined in (2), and

$$\mathcal{C}_m^2 = \mathbb{E}[\sigma^2(X)] + \|s_* - s_m\|_2^2 .$$

The rate of convergence of ERM for linear aggregation with a Fourier dictionary exhibited by Theorem 5 is thus of the order D/n , which is the optimal rate of linear aggregation. In particular, this outperforms the bounds obtained in Theorem 2.2 of [AC11] under same assumption as Assumption **(A1)**, that is satisfied in the Fourier case, but also under more general moment assumptions on the noise. Indeed, as noticed in [LM15], the bounds obtained by [AC11] are in this case of the order D^3/n , for models of dimension lower than $n^{1/4}$. In Theorem 5, our condition on the permitted dimension which is less restrictive, since models with dimension close to $n^{1/2}$ are allowed.

Concerning the assumptions, uniform boundedness of the projection of the target onto the model, as described in (8), is guaranteed as soon as the regression belongs to a broad class of functions named the Wiener algebra, that is whenever the Fourier coefficients of the regression function are summable (in other words when the Fourier series of the regression function is absolutely convergent). For instance, functions that are Hölder continuous with index greater than $1/2$ belong to the Wiener algebra, [Kat04].

Furthermore, Theorem 5 gives an information that is far more precise than simply the rate of convergence of the least-squares estimator. Indeed, the conjunction of inequalities (13) and (14) of Theorem 5 actually proves the *concentration of the excess risk* of the least-squares estimator around one precise value, which is DC_m^2/n .

There are only very few and recent such concentration results for the excess risk of a M-estimator in the literature and this question constitutes an exiting new line of research in learning theory. Considering the same regression framework as ours, [Sau12] has shown concentration bounds for the excess risk of the least-squares estimator on models of piecewise polynomial functions. In a slightly different context of least-squares estimation under convex constraint, [Cha14] also proved the concentration in L_2 norm, with fixed design and Gaussian noise. Under the latter assumptions, [MvdG15] have shown the excess risk's concentration for the penalized least-squares estimator. Finally, [vdGW16] recently proved some concentration results for some regularized M-estimators. They also give an application of their results to a linearized regression context with random design and independent Gaussian noise.

3.2 Outline of the approach

The aim of this section is to explain the main ideas leading to the proof of Theorem 5 and to highlight some connections with other works in the literature.

The proof of Theorem 5 is technically very involved and is an adaptation to the Fourier case of the approach developed in [Sau12] concerning the performance of the least-squares estimator on models of piecewise polynomial functions and more general models endowed with a localized orthonormal basis.

An orthonormal basis $(\psi_k)_{k=1}^D$ of a linear model $m \subset L_2(P^X)$ is said to be *localized* if there exists a constant $L > 0$ such that $\left\| \sum_{k=1}^D \beta_k \psi_k \right\|_\infty \leq L\sqrt{D} \sup_k |\beta_k|$

for any $(\beta_k)_{k=1}^D \in \mathbb{R}^D$. This condition, taken into advantage in [Sau12], is typically valid for models of piecewise polynomial functions and wavelets, but it is false in the Fourier case. Indeed, if $(\varphi_k)_{k=0}^{D-1}$ is the collection of $D = 2l + 1$ first elements of the Fourier basis defined in Section 3.1 above, then by taking $(\beta_k)_{k=0}^{D-1} = (1, 1, 0, 1, 0, 1, 0, \dots, 1, 0)$, it holds

$$\left\| \sum_{k=0}^{D-1} \beta_k \varphi_k \right\|_{\infty} \geq \sum_{k=0}^{D-1} \beta_k \varphi_k(0) = 1 + \sqrt{2} \sum_{k=0}^{l-1} \cos(k \times 0) \geq l + 1 \geq \frac{D}{2} \sup_k |\beta_k| .$$

Hence, the Fourier basis has a behavior with respect to the sup-norm which is harder to control from this point of view than localized bases. It is however possible, for models of dimension $D \lesssim \sqrt{n}$, to prove the consistency in sup-norm of the least-squares estimator towards the projection of the regression function s_* onto a model corresponding to finite Fourier expansions. More precisely, we prove the following theorem, which is an essential piece in the proof of Theorem 5.

Theorem 6 *Let $\alpha > 0$. Assume that m is a linear vector space spanned by the first D elements of the Fourier basis. Assume that **(H1)** holds and that there exists $A_+ > 0$ such that*

$$D \leq A_+ \frac{n^{1/2}}{(\ln n)^2} \leq n .$$

Then we have, for all $n \geq n_0(A_+, \alpha)$,

$$\mathbb{P} \left(\|\widehat{s} - s_m\|_{\infty} \geq L_{A,\alpha}^{(1)} D \sqrt{\frac{\ln n}{n}} \right) \leq n^{-\alpha} .$$

The proof of Theorem 6, which uses concentration tools such as Bernstein's inequality (see Theorem 19) can be found in the Appendix, Section 5.1. Using Theorem 6, we can localize with probability close to one (equal to $1 - n^{-\alpha}$) our analysis in a ball in sup-norm $B_{L_{\infty}}(s_m, R_{n,D,\alpha})$, centered on the projection of the target and of radius $R_{n,D,\alpha} = L_{A,\alpha}^{(1)} D \sqrt{\ln n / n}$.

As explained in [Sau12], empirical process theory can be used to derive optimal bounds such as Theorem 5, through the use of a *representation formula* for the excess risk, in terms of local suprema of the underlying empirical process. Indeed, set the least-squares contrast $\gamma : L_2(P^X) \rightarrow L_1(P)$ defined by,

$$\gamma(s) : (x, y) \in \mathcal{X} \times \mathbb{R} \mapsto (y - s(x))^2 .$$

Then, we can write $\widehat{s} \in \arg \min_{s \in m} \{P_n(\gamma(s))\}$, where P_n is the empirical measure associated to the sample and we also have $\|\widehat{s} - s_m\|_2^2 = P(\gamma(\widehat{s}) - \gamma(s_m))$. Furthermore, the following representation formula holds for the excess risk (see identity (3.9) of [Sau12]) with probability close to one,

$$\|\widehat{s} - s_m\|_2^2 \in \arg \max_{C \geq 0} \left\{ \sup_{s \in \mathcal{D}_C} (P - P_n)(\gamma(s) - \gamma(s_m)) - C \right\} , \quad (15)$$

where

$$\mathcal{D}_C := \{s \in m ; \|s - s_m\|_2^2 = C\} \cap B_{L_\infty}(s_m, R_{n,d,\alpha}) .$$

A similar representation formula is also at the core of the approach developed by [Cha14] for least-squares estimation under convex constraints, extended to regularized ERM by [MvdG15] and [vdGW16]. In [Cha14] and [MvdG15], the framework allows to replace the empirical process appearing in (15) by a Gaussian process, while in [vdGW16] the more general framework of M-estimation forces the authors to work with an empirical process, exactly as in (15).

Using concentration inequalities for the supremum of the empirical process, one may show from (15) that with probability close to one,

$$\|\widehat{s} - s_m\|_2^2 \sim \arg \max_{C \geq 0} \left\{ \mathbb{E} \left[\sup_{s \in \mathcal{D}_C} (P - P_n) (\gamma(s) - \gamma(s_m)) \right] - C \right\} . \quad (16)$$

The quantity of interest is thus the expectation of supremum of the empirical process over a slice \mathcal{D}_C of the model m . Moreover, as we want to derive optimal bounds, we are looking for a control to the right constant of the first order of this quantity. To this end, we introduce an argument of contrast expansion into a linear and quadratic part, originally developed in [Sau12] and which is also one of the main features of the small-ball approach first built in [Men15]. More precisely, for every $s \in m$ and $z = (x, y) \in \mathcal{X} \times \mathbb{R}$, it holds

$$\gamma(s)(z) - \gamma(s_m)(z) = \psi_{1,m}(z)(s - s_m)(x) + (s - s_m)^2(x) \quad (17)$$

where $\psi_{1,m}(z) = -2(y - s_m(x))$. Using (17), we then split the quantity of interest in (16) into two parts,

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \in \mathcal{D}_C} (P - P_n) (\gamma(s) - \gamma(s_m)) \right] \\ \leq & \underbrace{\mathbb{E} \left[\sup_{s \in \mathcal{D}_C} (P - P_n) (\psi_{1,m} \cdot (s - s_m)) \right]}_{\text{main part}} + \underbrace{\mathbb{E} \left[\sup_{s \in \mathcal{D}_C} (P - P_n) (s - s_m)^2 \right]}_{\text{remainder term}} . \end{aligned}$$

We finally show that the empirical process corresponding to the linear part of the contrast expansion gives the exact first order of rate of linear aggregation of the least-squares estimator, while the empirical process corresponding to the quadratic part of the contrast expansion contributes only through remainder terms. Some technical lemmas roughly corresponding to the previous observations can be found in the Appendix, Section 5.2.1.

4 Proofs related to Section 2

Proof of Proposition 3. Take $s \in m$ and $\kappa_0 \in (0, 1)$. Set $\Omega_{\kappa_0} = \{|s(X)| \geq \kappa_0 \|s\|_2\}$. By Paley-Zygmund's inequality (Corollary 3.3.2 in [dlPG99]), it holds

$$\mathbb{P}(\Omega_{\kappa_0}) \geq (1 - \kappa_0^2) \frac{\|s\|_2^2}{\|s\|_\infty^2} \geq \frac{1 - \kappa_0^2}{L_0^2} \frac{1}{D} ,$$

which readily proves Proposition 3. ■

Proof of Proposition 4. Take $(j, k) \in \Lambda_m$, $j \geq 0$. It holds

$$\begin{aligned} \|\rho_{j,k}\|_2^2 &= \int_0^1 \rho_{j,k}^2(x) dx \\ &= 2^j \int_0^1 |\rho_0(2^j x - k + 1)|^2 dx \\ &= \int_{-k+1}^{2^j - k + 1} |\rho_0(y)|^2 dy . \end{aligned}$$

Thus, whenever $\text{Supp}(\rho_0) \subset [-k + 1, 2^j - k + 1]$, one has $\|\rho_{j,k}\|_2 = 1$. Take $j_0 \geq 0$ such that $2^{j_0} \geq R$. Then $\|\rho_{j_0,1}\|_2 = 1$ and it is easy to see that for any $j \geq j_0$ there exists at least 2^{j-j_0} values of k such that $\text{Supp}(\rho_0) \subset [-k + 1, 2^j - k + 1]$ and $\|\rho_{j,k}\|_2 = 1$. Now, take $\kappa_0 > 0$ such that

$$\int_0^R \mathbf{1}_{\{|\rho_0(y)| \geq \kappa_0\}} dy \leq \frac{1}{2} .$$

For any $j \geq j_0$ and k such that $\text{Supp}(\rho_0) \subset [-k + 1, 2^j - k + 1]$, we get

$$\begin{aligned} &\mathbb{P}(|\rho_{j,k}(X)| \geq \kappa_0 \|\rho_{j,k}\|_2) \\ &= \mathbb{P}(2^{j/2} |\rho_0(2^j X - k + 1)| \geq \kappa_0) \\ &= 2^{-j} \int_{-k+1}^{2^j - k + 1} |\rho_0(y)| dy = 2^{-j} \int_0^R |\rho_0(y)| dy \leq 2^{-j-1} . \end{aligned} \quad (18)$$

Furthermore, notice that $\text{Card}(\Lambda(-1)) \leq [R] + 1$ and $D = \text{Card}(\Lambda_l) \leq [R] + 2^{l+1}$. Hence, taking $j = l$ in (18), we deduce that,

$$\mathbb{P}(|\rho_{l,k}(X)| \geq \kappa_0 \|\rho_{l,k}\|_2) \leq \frac{1}{D - [R]} ,$$

which gives the result. ■

5 Proofs related to Section 3

5.1 Proof of Theorem 6

Proof of Theorem 6. Let $\alpha, C > 0$. Set

$$\mathcal{F}_C^\infty := \{s \in m ; \|s - s_m\|_\infty \leq C\}$$

and

$$\mathcal{F}_{>C}^\infty := \{s \in m ; \|s - s_m\|_\infty > C\} = m \setminus \mathcal{F}_C^\infty .$$

Take the Fourier basis $(\varphi_k)_{k=0}^{D-1}$ of $(m, \|\cdot\|_2)$. By assumption on D , it holds $D \leq n$. Hence, by Lemma 8 below, we get that there exists $L_{A,\alpha}^{(1)} > 0$ such that, by setting

$$\Omega_1 = \left\{ \max_{k \in \{0, \dots, D-1\}} |(P_n - P)(\psi_{1,m} \cdot \varphi_k)| \leq L_{A,\alpha}^{(1)} \sqrt{\frac{\ln n}{n}} \right\},$$

we have for all $n \geq n_0(A_+)$, $\mathbb{P}(\Omega_1) \geq 1 - n^{-\alpha}$. Moreover, we set

$$\Omega_2 = \left\{ \max_{k \in \{0, \dots, D-1\}^2} |(P_n - P)(\varphi_k \cdot \varphi_l)| \leq L_\alpha^{(2)} \sqrt{\frac{\ln n}{n}} \right\},$$

where $L_\alpha^{(2)}$ is defined in Lemma 7 below. By Lemma 7, we have $\mathbb{P}(\Omega_2) \geq 1 - n^{-\alpha}$ and so, for all $n \geq n_0(A_+)$,

$$\mathbb{P}(\Omega_1 \cap \Omega_2) \geq 1 - 2n^{-\alpha}. \quad (19)$$

We thus have for all $n \geq n_0(A_+)$,

$$\begin{aligned} & \mathbb{P}(\|\hat{s} - s_m\|_\infty > C) \\ & \leq \mathbb{P}\left(\inf_{s \in \mathcal{F}_{>C}^\infty} P_n(\gamma(s) - \gamma(s_m)) \leq \inf_{s \in \mathcal{F}_C^\infty} P_n(\gamma(s) - \gamma(s_m))\right) \\ & = \mathbb{P}\left(\sup_{s \in \mathcal{F}_{>C}^\infty} P_n(\gamma(s_m) - \gamma(s)) \geq \sup_{s \in \mathcal{F}_C^\infty} P_n(\gamma(s_m) - \gamma(s))\right) \\ & \leq \mathbb{P}\left(\left\{ \sup_{s \in \mathcal{F}_{>C}^\infty} P_n(\gamma(s_m) - \gamma(s)) \geq \sup_{s \in \mathcal{F}_{C/2}^\infty} P_n(\gamma(s_m) - \gamma(s)) \right\} \cap \Omega_1 \cap \Omega_2\right) + 2n^{-2\alpha} \end{aligned} \quad (20)$$

Now, for any $s \in m$ such that

$$s - s_m = \sum_{k=0}^{D-1} \beta_k \varphi_k, \quad \beta = (\beta_k)_{k=0}^{D-1} \in \mathbb{R}^D,$$

we have

$$\begin{aligned} & P_n(\gamma(s_m) - \gamma(s)) \\ & = (P_n - P)(\psi_{1,m} \cdot (s_m - s)) - (P_n - P)((s - s_m)^2) - \|s - s_m\|_2^2 \\ & = \sum_{k=0}^{D-1} \beta_k (P_n - P)(\psi_{1,m} \cdot \varphi_k) - \sum_{k,l=0}^{D-1} \beta_k \beta_l (P_n - P)(\varphi_k \cdot \varphi_l) - \sum_{k=0}^{D-1} \beta_k^2. \end{aligned}$$

We set for any $(k, l) \in \{0, \dots, D-1\}^2$,

$$R_k^{(1)} = (P_n - P)(\psi_{1,m} \cdot \varphi_k) \quad \text{and} \quad R_{k,l}^{(2)} = (P_n - P)(\varphi_k \cdot \varphi_l).$$

Moreover, we set a function h_n , defined as follows,

$$h_n : \beta = (\beta_k)_{k=0}^{D-1} \longmapsto \sum_{k=0}^{D-1} \beta_k R_k^{(1)} - \sum_{k,l=0}^{D-1} \beta_k \beta_l R_{k,l}^{(2)} - \sum_{k=0}^{D-1} \beta_k^2 .$$

We thus have for any $s \in m$ such that $s - s_m = \sum_{k=0}^{D-1} \beta_k \varphi_k$, $\beta = (\beta_k)_{k=0}^{D-1} \in \mathbb{R}^D$,

$$P_n (\gamma(s_m) - \gamma(s)) = h_n(\beta) . \quad (21)$$

In addition we set for any $\beta = (\beta_k)_{k=0}^{D-1} \in \mathbb{R}^D$,

$$|\beta|_{m,\infty} = \sqrt{2D} |\beta|_\infty . \quad (22)$$

It is straightforward to see that $|\cdot|_{m,\infty}$ is a norm on \mathbb{R}^D , proportional to the sup-norm. We also set for a real $D \times D$ matrix B , its operator norm $\|A\|_m$ associated to the norm $|\cdot|_{m,\infty}$ on the D -dimensional vectors. More explicitly, we set for any $B \in \mathbb{R}^{D \times D}$,

$$\|B\|_m := \sup_{\beta \in \mathbb{R}^D, \beta \neq 0} \frac{|B\beta|_{m,\infty}}{|\beta|_{m,\infty}} = \sup_{\beta \in \mathbb{R}^D, \beta \neq 0} \frac{|B\beta|_\infty}{|\beta|_\infty} .$$

Note that $\|\cdot\|_m$ is an operator norm and so $\|B^k\|_m \leq \|B\|_m^k$ for any $k \in \mathbb{N}$. We also have, for any $B = (B_{k,l})_{k,l=0,\dots,D-1} \in \mathbb{R}^{D \times D}$, the following classical formula

$$\|B\|_m = \max_{k \in \{0,\dots,D-1\}} \left\{ \left\{ \sum_{l \in \{0,\dots,D-1\}} |B_{k,l}| \right\} \right\} . \quad (23)$$

Notice that for any $\beta = (\beta_k)_{k=0}^{D-1} \in \mathbb{R}^D$,

$$\left\| \sum_{k=0}^{D-1} \beta_k \varphi_k \right\|_\infty \leq D |\beta|_\infty \sup_k \|\varphi_k\|_\infty \leq |\beta|_{m,\infty} .$$

Hence, it holds

$$\mathcal{F}_{>C}^\infty \subset \left\{ s \in m ; s - s_m = \sum_{k=0}^{D-1} \beta_k \varphi_k \ \& \ |\beta|_{m,\infty} \geq C \right\} \quad (24)$$

and

$$\mathcal{F}_{C/2}^\infty \supset \left\{ s \in m ; s - s_m = \sum_{k=0}^{D-1} \beta_k \varphi_k \ \& \ |\beta|_{m,\infty} \leq C/2 \right\} . \quad (25)$$

Hence, from (20), (21) (25) and (24) we deduce that if we find on $\Omega_1 \cap \Omega_2$ a value of C such that

$$\sup_{\beta \in \mathbb{R}^D, |\beta|_{m,\infty} \geq C} h_n(\beta) < \sup_{\beta \in \mathbb{R}^D, |\beta|_{m,\infty} \leq C/2} h_n(\beta) ,$$

then we will get

$$\mathbb{P}(\|\widehat{s} - s_m\|_\infty > C) \leq 2n^{-\alpha} .$$

Taking the partial derivatives of h_n with respect to the coordinates of its arguments, it then holds for any $(k, l) \in \{0, \dots, D-1\}^2$ and $\beta = (\beta_i)_{i=0}^{D-1} \in \mathbb{R}^D$,

$$\frac{\partial h_n}{\partial \beta_k}(\beta) = R_k^{(1)} - 2 \sum_{i=0}^{D-1} \beta_i R_{k,i}^{(2)} - 2\beta_k \quad (26)$$

We look now at the set of solutions β of the following system,

$$\frac{\partial h_n}{\partial \beta_k}(\beta) = 0, \quad \forall k \in \{0, \dots, D-1\} . \quad (27)$$

We define the $D \times D$ matrix $R_n^{(2)}$ to be

$$R_n^{(2)} := \left(R_{k,l}^{(2)} \right)_{k,l=0..D-1}$$

and by (26), the system given in (27) can be written

$$2(I_D + R_n^{(2)})\beta = R_n^{(1)}, \quad (\mathbf{S})$$

where $R_n^{(1)}$ is a D -dimensional vector defined by

$$R_n^{(1)} = \left(R_{n,k}^{(1)} \right)_{k=0..D-1} .$$

Let us give an upper bound of the norm $\|R_n^{(2)}\|_m$, in order to show that the matrix $I_D + R_n^{(2)}$ is nonsingular. On Ω_2 we have

$$\begin{aligned} \|R_n^{(2)}\|_m &= \max_{k \in \{0, \dots, D-1\}} \left\{ \left\{ \sum_{l \in \{0, \dots, D-1\}} |(P_n - P)(\varphi_k \cdot \varphi_l)| \right\} \right\} \\ &\leq L_\alpha^{(2)} \max_{k \in \{0, \dots, D-1\}} \left\{ \left\{ \sum_{l \in \{0, \dots, D-1\}} \sqrt{\frac{\ln n}{n}} \right\} \right\} \\ &\leq L_\alpha^{(2)} D \sqrt{\frac{\ln n}{n}} \end{aligned} \quad (28)$$

Hence, from (28) and the fact that $D \leq A_+ \frac{n^{1/2}}{(\ln n)^2}$, we get that for all $n \geq n_0(A_+, \alpha)$, it holds on Ω_2 ,

$$\|R_n^{(2)}\|_m \leq \frac{1}{2}$$

and the matrix $(I_d + R_n^{(2)})$ is nonsingular, of inverse $(I_d + R_n^{(2)})^{-1} = \sum_{u=0}^{+\infty} (-R_n^{(2)})^u$.

Hence, the system (\mathbf{S}) admits a unique solution $\beta^{(n)}$, given by

$$\beta^{(n)} = \frac{1}{2} (I_d + R_n^{(2)})^{-1} R_n^{(1)} .$$

Now, on Ω_1 we have,

$$|R_n^{(1)}|_{m,\infty} \leq \sqrt{2}D \max_{k \in \{0, \dots, D-1\}} |(P_n - P)(\psi_{1,m} \cdot \varphi_k)| \leq L_{A,\alpha}^{(1)} D \sqrt{\frac{2 \ln n}{n}}$$

and we deduce that for all $n_0(A_+, \alpha)$, it holds on $\Omega_2 \cap \Omega_1$,

$$|\beta^{(n)}|_{m,\infty} \leq \frac{1}{2} \left\| (I_d + R_n^{(2)})^{-1} \right\|_m |R_n^{(1)}|_{m,\infty} \leq L_{A,\alpha}^{(1)} D \sqrt{\frac{2 \ln n}{n}}. \quad (29)$$

Moreover, by the formula (21) we have

$$h_n(\beta) = P_n(\gamma(s_m)) - \frac{1}{n} \sum_{i=1}^n \left(Y_i - s_m(X_i) - \sum_{k=0}^{D-1} \beta_k \varphi_k(X_i) \right)^2$$

and we thus see that h_n is concave. Hence, for all $n_0(A_+, \alpha)$, we get that on Ω_2 , $\beta^{(n)}$ is the unique maximum of h_n and on $\Omega_2 \cap \Omega_1$, by (29), concavity of h_n and uniqueness of $\beta^{(n)}$, we get

$$h_n(\beta^{(n)}) = \sup_{\beta \in \mathbb{R}^D, |\beta|_{m,\infty} \leq C/2} h_n(\beta) > \sup_{\beta \in \mathbb{R}^D, |\beta|_{m,\infty} \geq C} h_n(\beta),$$

with $C = 2L_{A,\alpha}^{(1)} D \sqrt{\frac{2 \ln n}{n}}$, which concludes the proof. ■

Lemma 7 *Let $\alpha > 0$. Assume that m is a linear vector space spanned by the first D elements of the Fourier basis, where $D \leq n$. Then there exists $L_\alpha^{(2)} > 0$ such that*

$$\mathbb{P} \left(\max_{k \in \{0, \dots, D-1\}^2} |(P_n - P)(\varphi_k \cdot \varphi_l)| \geq L_\alpha^{(2)} \sqrt{\frac{\ln n}{n}} \right) \leq n^{-\alpha}. \quad (30)$$

Proof. For any $(k, l) \in \{0, \dots, D-1\}^2$, we have

$$\mathbb{E} [(\varphi_k \cdot \varphi_l)^2] \leq \|\varphi_k \cdot \varphi_l\|_\infty^2 \leq 4.$$

Hence, we apply Bernstein's inequality (see Proposition 2.9 in [Mas07]) and we get, for all $\gamma > 0$,

$$\mathbb{P} \left(|(P_n - P)(\varphi_k \cdot \varphi_l)| \geq 2\sqrt{\frac{2\gamma \ln n}{n}} + \frac{2\gamma \ln n}{3n} \right) \leq 2n^{-\gamma}. \quad (31)$$

We get from (31) that for all $\gamma > 0$,

$$\begin{aligned} & \mathbb{P} \left(\max_{(k,l) \in \{0, \dots, D-1\}^2} |(P_n - P)(\varphi_k \cdot \varphi_l)| \geq \left(2\sqrt{2\gamma} + \frac{2\gamma}{3} \right) \sqrt{\frac{\ln n}{n}} \right) \\ & \leq 2D^2 n^{-\gamma} \leq n^{-\gamma+2}. \end{aligned} \quad (32)$$

To conclude, take $\gamma = \alpha + 2$. ■

Lemma 8 *Let $\alpha > 0$. Assume that m is a linear vector space spanned by the first D elements of the Fourier basis, where $D \leq n$. If **(H1)** holds and $\psi_{1,m}(X, Y) := -2(Y - s_m(X))$ then*

$$\mathbb{P} \left(\max_{k \in \{1, \dots, D\}} |(P_n - P)(\psi_{1,m} \cdot \varphi_k)| \geq L_{A,\alpha}^{(1)} \sqrt{\frac{\ln n}{n}} \right) \leq n^{-\alpha} . \quad (33)$$

Proof. Let $\beta > 0$. By Bernstein's inequality, we get by straightforward computations (of the spirit of the proof of Lemma 7) that there exists $L_{A,\beta} > 0$ such that, for all $k \in \{0, \dots, D - 1\}$,

$$\mathbb{P} \left(|(P_n - P)(\psi_{1,m} \cdot \varphi_k)| \geq L_{A,\beta}^{(1)} \sqrt{\frac{\ln n}{n}} \right) \leq n^{-\beta} .$$

Now the result follows from a simple union bound with $\beta = \alpha + 1$. ■

5.2 Proof of Theorem 5

Aiming at clarifying the proofs, the arguments involved and the connection with the proofs exposed in [Sau12], we generalize a little bit the Fourier framework by invoking along the proofs the three following assumptions, that are satisfied for Fourier expansions. From now on, $m \subset L_2(P^X)$ is considered to be a linear model of dimension D , not necessarily built from the Fourier basis.

Let us define a function Ψ_m on \mathcal{X} , that we call the unit envelope of m , such that

$$\Psi_m(x) = \frac{1}{\sqrt{D}} \sup_{s \in m, \|s\|_2 \leq 1} |s(x)| . \quad (34)$$

As m is a finite dimensional real vector space, the supremum in (34) can also be taken over a countable subset of m , so Ψ_m is a measurable function.

- **(H3)** The unit envelope of m is uniformly bounded on \mathcal{X} : a positive constant $A_{3,m}$ exists such that

$$\|\Psi_m\|_\infty \leq A_{3,m} < \infty .$$

In the Fourier case, **(H3)** is valid by taking $A_{3,m} \leq \sqrt{2}$. In fact, it is easy to see that assumption **(H3)** is equivalent to assumption **(A1)**. Moreover, several technical lemmas derived in [Sau12] only assume the validity of **(H3)** and will thus be used without repeating their proofs.

- **(H4)** Uniformly bounded basis : there exists an orthonormal basis $\varphi = (\varphi_k)_{k=1}^D$ in $(m, \|\cdot\|_2)$ that satisfies, for a positive constant $u_m(\varphi)$

$$\|\varphi_k\|_\infty \leq u_m(\varphi) .$$

Again, in the Fourier case, **(H4)** is valid by taking $u_m(\varphi) \leq \sqrt{2}$.

Remark 3 **(H4)** implies **(H3)** and in that case $A_{3,m} = u_m(\varphi)$.

The assumption of consistency in sup-norm:

We assume that the least squares estimator is consistent for the sup-norm on the space \mathcal{X} . More precisely, this requirement can be stated as follows.

- **(H5)** Assumption of consistency in sup-norm: for any $A_+ > 0$, if m is a model of dimension D satisfying

$$D \leq A_+ \frac{n^{1/2}}{(\ln n)^2} ,$$

then for every $\alpha > 0$, we can find a positive integer n_1 and a positive constant A_{cons} satisfying the following property: there exists $R_{n,D,\alpha} > 0$ depending on D , n and α , such that

$$R_{n,D,\alpha} \leq \frac{A_{cons}}{\sqrt{\ln n}} \quad (35)$$

and by setting

$$\Omega_{\infty,\alpha} = \{ \|\hat{s} - s_m\|_{\infty} \leq R_{n,D,\alpha} \} , \quad (36)$$

it holds for all $n \geq n_1$,

$$\mathbb{P} [\Omega_{\infty,\alpha}] \geq 1 - n^{-\alpha} . \quad (37)$$

By Theorem 6, **(H5)** is verified with $R_{n,D,\alpha} \sim D\sqrt{\ln n/n}$.

In order to express the quantities of interest in the proof of Theorem 5, we need preliminary definitions. Let $\alpha > 0$ be fixed and for $R_{n,D,\alpha}$ defined in **(H5)**, we set

$$\tilde{R}_{n,D,\alpha} = \max \left\{ R_{n,D,\alpha} ; A_{\infty} \sqrt{\frac{D \ln n}{n}} \right\} \quad (38)$$

where A_{∞} is a positive constant to be chosen later. Moreover, we set

$$\nu_n = \max \left\{ \sqrt{\frac{\ln n}{D}} ; \sqrt{\frac{D \ln n}{n}} ; R_{n,D,\alpha} \right\} . \quad (39)$$

Thanks to the assumption of consistency in sup-norm **(H5)**, our analysis will be localized in the subset

$$B_{(m,L_{\infty})} \left(s_m, \tilde{R}_{n,D,\alpha} \right) = \left\{ s \in m, \|s - s_m\|_{\infty} \leq \tilde{R}_{n,D,\alpha} \right\}$$

of m .

Let us define several slices of excess risk on the model m : for any $C \geq 0$,

$$\begin{aligned} \mathcal{F}_C &= \left\{ s \in m, \|s - s_m\|_2^2 \leq C \right\} \cap B_{(m,L_{\infty})} \left(s_m, \tilde{R}_{n,D,\alpha} \right) \\ \mathcal{F}_{>C} &= \left\{ s \in m, \|s - s_m\|_2^2 > C \right\} \cap B_{(m,L_{\infty})} \left(s_m, \tilde{R}_{n,D,\alpha} \right) \end{aligned}$$

and for any interval $J \subset \mathbb{R}$,

$$\mathcal{F}_J = \{s \in m, \|s - s_m\|_2^2 \in J\} \cap B_{(m, L_\infty)}(s_m, \tilde{R}_{n, D, \alpha}) .$$

We also define, for all $L \geq 0$,

$$D_L = \{s \in m, \|s - s_m\|_2^2 = L\} \cap B_{(m, L_\infty)}(s_m, \tilde{R}_{n, D, \alpha}) .$$

Recall that the contrasted functions satisfy, for every $s \in m$ and $z = (x, y) \in \mathcal{X} \times \mathbb{R}$,

$$\gamma(s)(z) - \gamma(s_m)(z) = \psi_{1, m}(z)(s - s_m)(x) + (s - s_m)^2(x) \quad (40)$$

where $\psi_{1, m}(z) = -2(y - s_m(x))$. Note that, for all $s \in m$,

$$P(\psi_{1, m} \cdot s) = 0 \quad (41)$$

and by **(H1)**,

$$\|\psi_{1, m}\|_\infty \leq 4A . \quad (42)$$

Also, for the term \mathcal{C}_m defined in Theorem 5, simple computations give that

$$\mathcal{C}_m^2 = \frac{1}{4D} \sum_{k=0}^{D-1} \text{Var}(\psi_{1, m} \cdot \varphi_k) \quad (43)$$

for the Fourier basis $(\varphi_k)_{k=0}^{D-1}$ of $(m, \|\cdot\|_2)$. Moreover, it is easy to see that under **(H1)** we have,

$$\mathcal{C}_m \leq \sigma_{\max} + 2A \leq 3A . \quad (44)$$

We also have

$$0 < \sqrt{\mathbb{E}[\sigma^2(X)]} = \|\sigma\|_2 \leq \mathcal{C}_m . \quad (45)$$

Finally, when **(H3)** holds (it is the case when **(H4)** holds), we have by (34),

$$\sup_{s \in m, \|s\|_2 \leq 1} \|s\|_\infty \leq A_{3, m} \sqrt{D} \quad (46)$$

and so, for any orthonormal basis $(\varphi_k)_{k=1}^D$ of $(m, \|\cdot\|_2)$, it holds for all $k \in \{1, \dots, D\}$, as $P(\varphi_k^2) = 1$,

$$\|\varphi_k\|_\infty \leq A_{3, m} \sqrt{D} . \quad (47)$$

We are now in position to prove Theorem 5. The proof of Theorem 5 relies on Lemmas 13, 14 and 15 stated in Section 5.2.1, and that give sharp estimates of suprema of the empirical process on the contrasted functions over slices of interest. Recall that in the Fourier case, assumptions **(H3)** and **(H4)** hold, as well as **(H5)** with $R_{n, D, \alpha} \sim D\sqrt{\ln n/n}$.

Proof of inequality (13). Let $\alpha > 0$, $r \in (1, 2]$ to be chosen later and $C > 0$ such that

$$rC = \frac{D}{n} \mathcal{C}_m^2. \quad (48)$$

By **(H5)** there exists a positive integer n_1 such that it holds, for all $n \geq n_1$,

$$\mathbb{P}(\|\hat{s} - s_m\|_2^2 \leq C) \leq \mathbb{P}\left(\{\|\hat{s} - s_m\|_2^2 \leq C\} \cap \Omega_{\infty, \alpha}\right) + n^{-\alpha} \quad (49)$$

and also

$$\begin{aligned} & \mathbb{P}\left(\{\|\hat{s} - s_m\|_2^2 \leq C\} \cap \Omega_{\infty, \alpha}\right) \\ & \leq \mathbb{P}\left(\inf_{s \in \mathcal{F}_C} P_n(\gamma(s) - \gamma(s_m)) \leq \inf_{s \in \mathcal{F}_{>C}} P_n(\gamma(s) - \gamma(s_m))\right) \\ & \leq \mathbb{P}\left(\inf_{s \in \mathcal{F}_C} P_n(\gamma(s) - \gamma(s_m)) \leq \inf_{s \in \mathcal{F}_{(C, rC]}} P_n(\gamma(s) - \gamma(s_m))\right) \\ & = \mathbb{P}\left(\sup_{s \in \mathcal{F}_C} P_n(\gamma(s_m) - \gamma(s)) \geq \sup_{s \in \mathcal{F}_{(C, rC]}} P_n(\gamma(s_m) - \gamma(s))\right). \end{aligned} \quad (50)$$

Now, by (48) and (45) we have

$$\frac{D}{2n} \|\sigma\|_2^2 \leq C \leq (1 + A_4 \nu_n)^2 \frac{D}{n} \mathcal{C}_m^2,$$

where A_4 is defined in Lemma 13. Hence we can apply Lemma 13 with $\alpha = \beta$, $A_l = \|\sigma\|_2^2/2$ and $A_{3,m} = u_m(\varphi)$. Therefore it holds, for all $n \geq n_0(A_\infty, A_{cons}, A_+, \|\sigma\|_2, \alpha)$,

$$\mathbb{P}\left[\sup_{s \in \mathcal{F}_C} P_n(\gamma(s_m) - \gamma(s)) \geq 2(1 + L_{A_\infty, A, u_m(\varphi), \|\sigma\|_2, A_-, \alpha} \times \nu_n) \sqrt{\frac{CD}{n}} \mathcal{C}_m - C\right] \leq 2n^{-\alpha}. \quad (51)$$

Moreover, by using (45) and (44) in (48) we get

$$\frac{D}{n} \|\sigma\|_2^2 \leq rC \leq \frac{D}{n} (\sigma_{\max} + 2A)^2.$$

We then apply Lemma 15 with

$$\alpha = \beta, \quad A_l = \|\sigma\|_2^2, \quad A_u = (\sigma_{\max} + 2A)^2$$

and $A_\infty \geq 32B_2 A \sqrt{2A_u} \|\sigma\|_2^{-1} u_m(\varphi)$

$$A_\infty \geq 32\sqrt{2}B_2 A (\sigma_{\max} + 2A) \|\sigma\|_2^{-1} u_m(\varphi), \quad (52)$$

so it holds for all $n \geq n_0(A_-, A_+, A, A_\infty, A_{cons}, B_2, u_m(\varphi), \sigma_{\max}, \|\sigma\|_2, \alpha)$,

$$\mathbb{P}\left(\sup_{s \in \mathcal{F}_{(C, rC]}} P_n(\gamma(s_m) - \gamma(s)) \leq 2(1 - L_{A_-, A, A_\infty, \sigma_{\max}, \|\sigma\|_2, u_m(\varphi), \alpha} \times \nu_n) \sqrt{\frac{rCD}{n}} \mathcal{C}_m - rC\right) \leq 2n^{-\alpha}. \quad (53)$$

Now, from (51) and (53) we can find a positive constant \tilde{A}_0 , only depending on $A_-, A, A_\infty, \sigma_{\max}, \|\sigma\|_2, u_m(\varphi)$ and α , such that

for all $n \geq n_0(A_-, A_+, A, A_\infty, A_{\text{cons}}, B_2, u_m(\varphi), \sigma_{\max}, \|\sigma\|_2, \alpha)$, there exists an event of probability at least $1 - 4n^{-\alpha}$ on which

$$\sup_{s \in \mathcal{F}_C} P_n(\gamma(s_m) - \gamma(s)) \leq 2 \left(1 + \tilde{A}_0 \nu_n\right) \sqrt{\frac{CD}{n}} \mathcal{C}_m - C \quad (54)$$

and

$$\sup_{s \in \mathcal{F}_{(C, rC)}} P_n(\gamma(s_m) - \gamma(s)) \geq 2 \left(1 - \tilde{A}_0 \nu_n\right) \sqrt{\frac{rCD}{n}} \mathcal{C}_m - rC. \quad (55)$$

Hence, from (54) and (55) we deduce, using (49) and (50), that if we choose $r \in (1, 2]$ such that

$$2 \left(1 + \tilde{A}_0 \nu_n\right) \sqrt{\frac{CD}{n}} \mathcal{C}_m - C < 2 \left(1 - \tilde{A}_0 \nu_n\right) \sqrt{\frac{rCD}{n}} \mathcal{C}_m - rC \quad (56)$$

then, for all $n \geq n_0(A_-, A_+, A, A_\infty, A_{\text{cons}}, B_2, u_m(\varphi), \sigma_{\max}, \|\sigma\|_2, n_1, \alpha)$ we have

$$\|\hat{s} - s_m\|_2^2 \geq C$$

with probability at least $1 - 5n^{-\alpha}$. Now, by (48) it holds

$$\sqrt{\frac{rCD}{n}} \mathcal{C}_m = rC = \frac{D}{n} \mathcal{C}_m^2,$$

and as a consequence Inequality (56) is equivalent to

$$\left(1 - 2\tilde{A}_0 \nu_n\right) r - 2 \left(1 + \tilde{A}_0 \nu_n\right) \sqrt{r} + 1 > 0. \quad (57)$$

Moreover, we have by (39) and **(H5)**, for all $n \geq n_0(A_+, A_-, A_{\text{cons}}, \tilde{A}_0, \alpha)$,

$$\tilde{A}_0 \nu_n \leq \frac{1}{4} \quad (58)$$

and so, for all $n \geq n_0(A_+, A_-, A_{\text{cons}}, \tilde{A}_0, \alpha)$, simple computations involving (58) show that by taking

$$r = 1 + 48\sqrt{\tilde{A}_0 \nu_n} \quad (59)$$

inequality (57) is satisfied. Notice that, for all $n \geq n_0(A_+, A_-, A_{\text{cons}}, \tilde{A}_0, \alpha)$ we have $0 < 48\sqrt{\tilde{A}_0 \nu_n} < 1$, so that $r \in (1, 2)$. Finally, we compute C by (48) and (59), in such a way that for all $n \geq n_0(A_+, A_-, A_{\text{cons}}, \tilde{A}_0, \alpha)$,

$$C = \frac{rC}{r} = \frac{1}{1 + 48\sqrt{\tilde{A}_0 \nu_n}} \frac{D}{n} \mathcal{C}_m^2 \geq \left(1 - 48\sqrt{\tilde{A}_0 \nu_n}\right) \frac{D}{n} \mathcal{C}_m^2 > 0 \quad (60)$$

which yields the result by noticing that the dependence on σ_{\max} can be released in n_0 and \tilde{A}_0 since by **(H1)** we have $\sigma_{\max} \leq A$. ■

Proof of Inequality (14). Let $\alpha, C > 0$ and $\delta \in (0, \frac{1}{2})$ to be chosen later in such a way that

$$(1 - \delta) C = \frac{D}{n} \mathcal{C}_m^2 \quad (61)$$

and

$$C \geq (1 + A_5 \nu_n)^2 \frac{D}{n} \mathcal{C}_m^2, \quad (62)$$

where A_5 is defined in Lemma 14. We have by **(H5)**, for all $n \geq n_1$,

$$\mathbb{P}(\|\hat{s} - s_m\|_2^2 > C) \leq \mathbb{P}\left(\{\|\hat{s} - s_m\|_2^2 > C\} \cap \Omega_{\infty, \alpha}\right) + n^{-\alpha} \quad (63)$$

and also

$$\begin{aligned} & \mathbb{P}\left(\{\|\hat{s} - s_m\|_2^2 > C\} \cap \Omega_{\infty, \alpha}\right) \\ & \leq \mathbb{P}\left(\inf_{s \in \mathcal{F}_C} P_n(\gamma(s) - \gamma(s_m)) \geq \inf_{s \in \mathcal{F}_{>C}} P_n(\gamma(s) - \gamma(s_m))\right) \\ & = \mathbb{P}\left(\sup_{s \in \mathcal{F}_C} P_n(\gamma(s_m) - \gamma(s)) \leq \sup_{s \in \mathcal{F}_{>C}} P_n(\gamma(s_m) - \gamma(s))\right) \\ & \leq \mathbb{P}\left(\sup_{s \in \mathcal{F}_{\left(\frac{C}{2}, (1-\delta)C\right]}} P_n(\gamma(s_m) - \gamma(s)) \leq \sup_{s \in \mathcal{F}_{>C}} P_n(\gamma(s_m) - \gamma(s))\right). \end{aligned} \quad (64)$$

Now by (62) we can apply Lemma 14 with $\alpha = \beta$ and we obtain, for all $n \geq n_0(A_\infty, A_{cons}, A_+, \alpha)$,

$$\mathbb{P}\left[\sup_{s \in \mathcal{F}_{>C}} P_n(\gamma(s_m) - \gamma(s)) \geq (1 + A_5 \nu_n) \sqrt{\frac{CD}{n}} \mathcal{K}_{1,m} - C\right] \leq 2n^{-\alpha} \quad (65)$$

where A_5 only depends on $A, A_{3,m}, A_\infty, \sigma_{\min}, A_-$ and α . Moreover, we can take $A_{3,m} = u_m(\varphi)$ by Remark 3. Also, by (61), (45) and (44) we can apply Lemma 15 with the quantity C in Lemma 15 replaced by $C/2$, $\alpha = \beta$, $r = 2(1 - \delta)$, $A_u = (\sigma_{\max} + 2A)^2$, $A_l = \|\sigma\|_2^2$ and the constant A_∞ satisfying

$$A_\infty \geq 32\sqrt{2}B_2A(\sigma_{\max} + 2A)\sigma_{\min}^{-1}u_m(\varphi), \quad (66)$$

and so it holds, for all $n \geq n_0(A_-, A_+, A, A_\infty, A_{cons}, B_2, u_m(\varphi), \sigma_{\max}, \|\sigma\|_2, \alpha)$,

$$\mathbb{P}\left(\begin{aligned} & \sup_{s \in \mathcal{F}_{\left(\frac{C}{2}, (1-\delta)C\right]}} P_n(\gamma(s_m) - \gamma(s)) \\ & \leq 2(1 - L_{A_-, A, A_\infty, \sigma_{\max}, \|\sigma\|_2, u_m(\varphi), \alpha} \times \nu_n) \sqrt{\frac{(1-\delta)CD}{n}} \mathcal{C}_m - (1 - \delta)C \end{aligned}\right) \leq 2n^{-\alpha}. \quad (67)$$

Hence from (65) and (67), we deduce that a positive constant \check{A}_0 exists, only depending on $A_-, A, A_\infty, \sigma_{\max}, \|\sigma\|_2, u_m(\varphi)$ and α , such that for all $n \geq n_0(A_-, A_+, A, A_\infty, A_{\text{cons}}, B_2, u_m(\varphi), \sigma_{\max}, \|\sigma\|_2, \alpha)$ it holds on an event of probability at least $1 - 4n^{-\alpha}$,

$$\sup_{s \in \mathcal{F}_{\left(\frac{C}{2}, (1-\delta)C\right]}} P_n(\gamma(s_m) - \gamma(s)) \geq 2(1 - \check{A}_0 \nu_n) \sqrt{\frac{(1-\delta)CD}{n}} \mathcal{C}_m - (1-\delta)C \quad (68)$$

and

$$\sup_{s \in \mathcal{F}_{>C}} P_n(\gamma(s_m) - \gamma(s)) \leq 2(1 + \check{A}_0 \nu_n) \sqrt{\frac{CD}{n}} \mathcal{C}_m - C. \quad (69)$$

Now, from (68) and (69) we deduce, using (63) and (64), that if we choose $\delta \in (0, \frac{1}{2})$ such that (62) and

$$2(1 + \check{A}_0 \nu_n) \sqrt{\frac{CD}{n}} \mathcal{C}_m - C < 2(1 - \check{A}_0 \nu_n) \sqrt{\frac{(1-\delta)CD}{n}} \mathcal{C}_m - (1-\delta)C \quad (70)$$

are satisfied then, for all $n \geq n_0(A_-, A_+, A, A_\infty, A_{\text{cons}}, B_2, u_m(\varphi), \sigma_{\max}, \|\sigma\|_2, n_1, \alpha)$,

$$\|\hat{s} - s_m\|_2^2 \leq C,$$

with probability at least $1 - 5n^{-\alpha}$. By (61) it holds

$$\sqrt{\frac{(1-\delta)CD}{n}} \mathcal{C}_m = (1-\delta)C = \frac{1}{2} \frac{D}{n} \mathcal{C}_m^2,$$

and by consequence, inequality (70) is equivalent to

$$(1 - 2\check{A}_0 \nu_n)(1 - \delta) - 2(1 + \check{A}_0 \nu_n) \sqrt{1 - \delta} + 1 > 0. \quad (71)$$

Moreover, we have by (39) and **(H5)**, for all $n \geq n_0(A_+, A_-, A_{\text{cons}}, \check{A}_0, A_5, \alpha)$,

$$(\check{A}_0 \vee A_5) \nu_n < \frac{1}{72} \quad (72)$$

and so, for all $n \geq n_0(A_+, A_-, A_{\text{cons}}, \check{A}_0, \alpha)$, simple computations involving (72) show that by taking

$$\delta = 6 \left(\sqrt{\check{A}_0} \vee \sqrt{A_5} \right) \sqrt{\nu_n}, \quad (73)$$

inequalities (71) and (62) are satisfied and $\delta \in (0, \frac{1}{2})$. Finally, we can compute C by (61) and (73), in such a way that for all $n \geq n_0(A_+, A_-, A_{\text{cons}}, \check{A}_0, \alpha)$

$$0 < C = \frac{(1-\delta)C}{(1-\delta)} = \frac{1}{(1-\delta)} \frac{D}{n} \mathcal{C}_m^2 \leq \left(1 + 12 \left(\sqrt{\check{A}_0} \vee \sqrt{A_5} \right) \sqrt{\nu_n} \right) \frac{D}{n} \mathcal{C}_m^2, \quad (74)$$

which yields the result by noticing that the dependence on σ_{\max} can be released from n_0 and \check{A}_0 since by **(H1)** we have $\sigma_{\max} \leq A$. ■

5.2.1 Technical Lemmas

We state here some lemmas needed in the proofs of Theorem 5. First, in Lemmas 9, 10 and 11, we derive some controls, from above and from below, of the empirical process indexed by the “linear parts” of the contrasted functions over slices of interest. Secondly, we give in Lemma 12 an upper bound for the empirical process indexed by the “quadratic parts” of the contrasted functions over slices of interest. And finally, we use all these results in Lemmas 13, 14 and 15 to derive upper and lower bounds for the empirical process indexed by the contrasted functions over slices of interest.

The following lemma is a straightforward adaptation in our context of Lemma 11 in [Sau12].

Lemma 9 *Assume that (H1), (H2) and (H3) hold. Then for any $\beta > 0$, by setting*

$$\tau_n = L_{A,A_3,m,\|\sigma\|_2,\beta} \left(\sqrt{\frac{\ln n}{D}} \vee \frac{\sqrt{\ln n}}{n^{1/4}} \right),$$

It holds, for any orthonormal basis $(\varphi_k)_{k=1}^D$ of $(m, \|\cdot\|_2)$,

$$\mathbb{P} \left[\sqrt{\sum_{k=1}^D (P_n - P)^2 (\psi_{1,m} \cdot \varphi_k)} \geq 2(1 + \tau_n) \sqrt{\frac{D}{n}} \mathcal{C}_m \right] \leq n^{-\beta}. \quad (75)$$

If (H1) and (H3) hold, then for any $\beta > 0$, it holds

$$\mathbb{P} \left[\sqrt{\sum_{k=1}^D (P_n - P)^2 (\psi_{1,m} \cdot \varphi_k)} \geq L_{A,A_3,m,\beta} \sqrt{\frac{D \vee \ln n}{n}} \right] \leq n^{-\beta}. \quad (76)$$

In the next lemma, we state sharp lower bounds for the mean of the supremum of the empirical process on the linear parts of contrasted functions of m belonging to a slice of excess risk. This is done for a model of reasonable dimension.

Lemma 10 *Let $r > 1$ and $C > 0$. Assume that (H1), (H2), (H4) and (35) hold and let $\varphi = (\varphi_k)_{k=1}^D$ be an orthonormal basis of $(m, \|\cdot\|_2)$ satisfying (H4). If positive constants A_-, A_+, A_l, A_u exist such that*

$$A_+ \frac{n}{(\ln n)^2} \geq D \geq A_- (\ln n)^2 \quad \text{and} \quad A_l \frac{D}{n} \leq rC \leq A_u \frac{D}{n},$$

and if the constant A_∞ defined in (38) satisfies

$$A_\infty \geq 32B_2A\sqrt{2A_u}\|\sigma\|_2^{-1}u_m(\varphi), \quad (77)$$

then a positive constant $L_{A,A_l,A_u,\|\sigma\|_2}$ exists such that,

for all $n \geq n_0(A_-, A_+, A_u, A_l, A, B_2, u_m(\varphi), \|\sigma\|_2)$,

$$\mathbb{E} \left[\sup_{s \in \mathcal{F}_{(C,rC)}} (P_n - P) (\psi_{1,m} \cdot (s_m - s)) \right] \geq 2 \left(1 - \frac{L_{A,A_l,A_u,\|\sigma\|_2}}{\sqrt{D}} \right) \sqrt{\frac{rCD}{n}} \mathcal{C}_m. \quad (78)$$

Our argument leading to Lemma 10 shows that we have to assume that the constant A_∞ introduced in (38) is large enough. In order to prove Lemma 10 the following result is needed.

Lemma 11 *Let $r > 1$, $\beta > 0$ and $C \geq 0$. Assume that **(H1)**, **(H2)**, **(H4)** and (35) hold and let $\varphi = (\varphi_k)_{k=1}^D$ be an orthonormal basis of $(m, \|\cdot\|_2)$ satisfying **(H4)**. If positive constants A_+ , A_- and A_u exist such that*

$$A_+ \frac{n}{(\ln n)^2} \geq D \geq A_- (\ln n)^2, \quad rC \leq A_u \frac{D}{n},$$

and if

$$A_\infty \geq 16B_2 A \sqrt{2A_u \beta} \|\sigma\|_2^{-1} u_m(\varphi)$$

then for all $n \geq n_0(A_-, A_+, A, B_2, u_m(\varphi), \|\sigma\|_2, \beta)$, it holds

$$\mathbb{P} \left[\max_{k \in \{1, \dots, D\}} \left| \frac{\sqrt{rC} (P_n - P)(\psi_{1,m} \cdot \varphi_k)}{\sqrt{\sum_{j=1}^D (P_n - P)^2(\psi_{1,m} \cdot \varphi_j)}} \right| \geq \frac{\tilde{R}_{n,D,\alpha}}{u_m(\varphi) \sqrt{D}} \right] \leq \frac{2D+1}{n^\beta}.$$

Proof of Lemma 11. By Cauchy-Schwarz inequality, we get

$$\chi_m = \sqrt{\sum_{k=1}^D (P_n - P)^2(\psi_{1,m} \cdot \varphi_k)} = \sup_{s \in S_m} |(P_n - P)(\psi_{1,m} \cdot s)|,$$

where S_m is the unit sphere of m , that is

$$S_m = \left\{ s \in m, s = \sum_{k=1}^D \beta_k \varphi_k \text{ and } \sqrt{\sum_{k=1}^D \beta_k^2} = 1 \right\}.$$

Thus we can apply Klein-Rio's inequality (101) to χ_m by taking $\mathcal{F} = S_m$ and use the fact that

$$\sup_{s \in S_m} \|\psi_{1,m} \cdot s - P(\psi_{1,m} \cdot s)\|_\infty \leq 4A\sqrt{D}u_m(\varphi) \quad \text{by (41), (42) and (H4)}. \quad (79)$$

$$\sup_{s \in S_m} \text{Var}(\psi_{1,m} \cdot s) = \sup_{s \in S_m} P(\psi_{1,m} \cdot s)^2 \leq 16A^2 \quad \text{by (41), (42)}$$

and also, by using (79) in Inequality (96) applied to χ_m , we get that

$$\begin{aligned} \mathbb{E}[\chi_m] &\geq B_2^{-1} \sqrt{\mathbb{E}[\chi_m^2]} - \frac{4A\sqrt{D}u_m(\varphi)}{n} \\ &= 2B_2^{-1} \sqrt{\frac{D}{n} C_m} - \frac{4A\sqrt{D}u_m(\varphi)}{n}. \end{aligned}$$

We thus obtain by (101), for all $\varepsilon, x > 0$,

$$\mathbb{P} \left(\chi_m \leq 2(1-\varepsilon) B_2^{-1} \sqrt{\frac{D}{n}} \mathcal{C}_m - \sqrt{32A^2 \frac{x}{n}} - \left(1-\varepsilon + \left(1 + \frac{1}{\varepsilon}\right)x\right) \frac{4A\sqrt{D}u_m(\varphi)}{n} \right) \leq \exp(-x) . \quad (80)$$

So, by taking $\varepsilon = \frac{1}{2}$ and $x = \beta \ln n$ in (80), and by observing that $D \geq A_- (\ln n)^2$ and $\mathcal{C}_m \geq \|\sigma\|_2$, we conclude that, for all $n \geq n_0(A_-, A, B_2, u_m(\varphi), \|\sigma\|_2, \beta)$,

$$\mathbb{P} \left[\chi_m \leq \frac{B_2^{-1}}{2} \sqrt{\frac{D}{n}} \mathcal{C}_m \right] \leq n^{-\beta} . \quad (81)$$

Furthermore, combining Bernstein's inequality (97), with the observation that we have, for every $k \in \{1, \dots, D\}$,

$$\begin{aligned} \|\psi_{1,m} \cdot \varphi_k\|_\infty &\leq 4Au_m(\varphi) && \text{by (42) and (H4)} \\ P(\psi_{1,m} \cdot \varphi_k)^2 &\leq \|\psi_{1,m}\|_\infty^2 \leq 16A^2 && \text{by (42)} \end{aligned}$$

we get that, for every $x > 0$ and every $k \in \{1, \dots, D\}$,

$$\mathbb{P} \left[|(P_n - P)(\psi_{1,m} \cdot \varphi_k)| \geq \sqrt{32A^2 \frac{x}{n}} + \frac{4Au_m(\varphi)x}{3n} \right] \leq 2\exp(-x)$$

and so

$$\mathbb{P} \left[\max_{k \in \{1, \dots, D\}} |(P_n - P)(\psi_{1,m} \cdot \varphi_k)| \geq \sqrt{32A^2 \frac{x}{n}} + \frac{4Au_m(\varphi)x}{3n} \right] \leq 2D\exp(-x) . \quad (82)$$

Hence, taking $x = \beta \ln n$ in (82), it comes

$$\mathbb{P} \left[\max_{k \in \{1, \dots, D\}} |(P_n - P)(\psi_{1,m} \cdot \varphi_k)| \geq \sqrt{\frac{32A^2\beta \ln n}{n}} + \frac{4Au_m(\varphi)\beta \ln n}{3n} \right] \leq \frac{2D}{n^\beta} , \quad (83)$$

then, by using (81) and (83), we get for all $n \geq n_0(A_-, A, B_2, u_m(\varphi), \|\sigma\|_2, \beta)$,

$$\mathbb{P} \left[\max_{k \in \{1, \dots, D\}} \left| \frac{\sqrt{rC}(P_n - P)(\psi_{1,m} \cdot \varphi_k)}{\chi_m} \right| \geq \frac{2B_2\sqrt{rC}}{\sqrt{\frac{D}{n}}\mathcal{C}_m} \left(\sqrt{\frac{32A^2\beta \ln n}{n}} + \frac{4Au_m(\varphi)\beta \ln n}{3n} \right) \right] \leq \frac{2D+1}{n^\beta} .$$

Finally, as $A_+ \frac{n}{(\ln n)^2} \geq D$ we have, for all $n \geq n_0(A, A_+, u_m(\varphi), \beta)$,

$$\frac{4Au_m(\varphi)\beta \ln n}{3n} \leq \sqrt{\frac{32A^2\beta \ln n}{n}}$$

and we can check that, since $rC \leq A_u \frac{D}{n}$ and $\mathcal{C}_m \geq \|\sigma\|_2$, if

$$A_\infty \geq 16B_2\sqrt{2A_u A^2\beta} \|\sigma\|_2^{-1} u_m(\varphi)$$

then, for all $n \geq n_0(A_-, A_+, A, B_2, u_m(\varphi), \|\sigma\|_2, \beta)$,

$$\mathbb{P} \left[\max_{k \in \{1, \dots, D\}} \left| \frac{\sqrt{rC} (P_n - P) (\psi_{1,m} \cdot \varphi_k)}{\chi_m} \right| \geq \frac{A_\infty}{u_m(\varphi)} \sqrt{\frac{\ln n}{n}} \right] \leq \frac{2D+1}{n^\beta}$$

which readily gives the result. ■

We are now ready to prove the lower bound (78) for the expected value of the largest increment of the empirical process over $\mathcal{F}_{(C, rC]}$.

Proof of Lemma 10. Let us begin with the lower bound of

$$\mathbb{E}^{\frac{1}{2}} \left(\sup_{s \in \mathcal{F}_{(C, rC]}} (P_n - P) (\psi_{1,m} \cdot (s_m - s)) \right)^2,$$

a result that will be need further in the proof. Introduce for all $k \in \{1, \dots, D\}$,

$$\beta_{k,n} = \frac{\sqrt{rC} (P_n - P) (\psi_{1,m} \cdot \varphi_k)}{\sqrt{\sum_{j=1}^D (P_n - P)^2 (\psi_{1,m} \cdot \varphi_j)}},$$

and observe that the excess risk on m of $\left(\sum_{k=1}^D \beta_{k,n} \varphi_k + s_m \right) \in m$ is equal to rC .

We also set

$$\tilde{\Omega} = \left\{ \max_{k \in \{1, \dots, D\}} |\beta_{k,n}| \leq \frac{\tilde{R}_{n,D,\alpha}}{u_m(\varphi) \sqrt{D}} \right\}.$$

By Lemma 11 we have that for all $\beta > 0$, if $A_\infty \geq 16B_2 \sqrt{2A_u A^2 \beta} \|\sigma\|_2^{-1} u_m(\varphi)$ then,

for all $n \geq n_0(A_-, A_+, A, B_2, u_m(\varphi), \|\sigma\|_2, \beta)$,

$$\mathbb{P}(\tilde{\Omega}) \geq 1 - \frac{2D+1}{n^\beta}. \quad (84)$$

Moreover, by **(H4)**, we get on the event $\tilde{\Omega}$,

$$\left\| \sum_{k=1}^D \beta_{k,n} \varphi_k \right\|_\infty \leq \tilde{R}_{n,D,\alpha},$$

and so, on $\tilde{\Omega}$,

$$\left(s_m + \sum_{k=1}^D \beta_{k,n} \varphi_k \right) \in \mathcal{F}_{(C, rC]}. \quad (85)$$

As a consequence, by (85) it holds

$$\begin{aligned}
& \mathbb{E}^{\frac{1}{2}} \left(\sup_{s \in \mathcal{F}(C, rC]} (P_n - P) (\psi_{1,m} \cdot (s_m - s)) \right)^2 \\
& \geq \mathbb{E}^{\frac{1}{2}} \left[\left((P_n - P) \left(\psi_{1,m} \cdot \left(\sum_{k=1}^D \beta_{k,n} \varphi_k \right) \right) \right)^2 \mathbf{1}_{\tilde{\Omega}} \right] \\
& = \sqrt{rC} \sqrt{\mathbb{E} \left[\left(\sum_{k=1}^D (P_n - P)^2 (\psi_{1,m} \cdot \varphi_k) \right) \mathbf{1}_{\tilde{\Omega}} \right]}. \tag{86}
\end{aligned}$$

Furthermore, since by (41) $P(\psi_{1,m} \cdot \varphi_k) = 0$ and by **(H4)** $\|\varphi_k\|_\infty \leq u_m(\varphi)$ for all $k \in \{1, \dots, D\}$, we have

$$\begin{aligned}
\left| \sum_{k=1}^D (P_n - P)^2 (\psi_{1,m} \cdot \varphi_k) \right| & \leq D \max_{k=1, \dots, D} |(P_n - P)^2 (\psi_{1,m} \cdot \varphi_k)| \\
& = D \max_{k=1, \dots, D} |P_n^2 (\psi_{1,m} \cdot \varphi_k)| \\
& \leq D \max_{k=1, \dots, D} \|\psi_{1,m} \cdot \varphi_k\|_\infty^2 \\
& \leq 16A^2 D u_m^2(\varphi)
\end{aligned}$$

and it ensures

$$\mathbb{E} \left[\left(\sum_{k=1}^D (P_n - P)^2 (\psi_{1,m} \cdot \varphi_k) \right) \mathbf{1}_{\tilde{\Omega}} \right] \geq \mathbb{E} \left[\left(\sum_{k=1}^D (P_n - P)^2 (\psi_{1,m} \cdot \varphi_k) \right) \right] - 16A^2 D u_m^2(\varphi) \mathbb{P} \left[(\tilde{\Omega})^c \right]. \tag{87}$$

Comparing inequality (87) with (86) and using (84), we obtain the following lower bound for all $n \geq n_0(A_-, A_+, A, B_2, u_m(\varphi), \|\sigma\|_2, \beta)$,

$$\begin{aligned}
\mathbb{E}^{\frac{1}{2}} \left(\sup_{s \in \mathcal{F}(C, rC]} (P_n - P) (\psi_{1,m} \cdot (s_m - s)) \right)^2 & \geq \sqrt{rC} \sqrt{\mathbb{E} \left[\left(\sum_{k=1}^D (P_n - P)^2 (\psi_{1,m} \cdot \varphi_k) \right) \right]} \\
& \quad - 4A u_m(\varphi) D \sqrt{rC} \sqrt{\mathbb{P} \left[(\tilde{\Omega})^c \right]} \\
& \geq 2 \sqrt{\frac{rCD}{n}} \mathcal{C}_m - 4A u_m(\varphi) D \sqrt{rC} \sqrt{\frac{2D+1}{n^\beta}}. \tag{88}
\end{aligned}$$

We take $\beta = 4$, and we must have

$$A_\infty \geq 32AB_2 \sqrt{2A_u} \|\sigma\|_2^{-1} u_m(\varphi).$$

Since $D \leq A_+ n (\ln n)^{-2}$ and $\mathcal{C}_m \geq \|\sigma\|_2$ under **(H2)**, we get, for all $n \geq n_0(A, A_+, u_m(\varphi), \|\sigma\|_2)$,

$$4A u_m(\varphi) D \sqrt{rC} \sqrt{\frac{2D+1}{n^\beta}} \leq \frac{2}{\sqrt{D}} \times \sqrt{\frac{rCD}{n}} \mathcal{C}_m \tag{89}$$

and so, by combining (88) and (89), for all $n \geq n_0(A_-, A_+, A, B_2, u_m(\varphi), \|\sigma\|_2)$, it holds

$$\mathbb{E}^{\frac{1}{2}} \left(\sup_{s \in \mathcal{F}_{(C, rC)}} (P_n - P) (\psi_{1,m} \cdot (s_m - s)) \right)^2 \geq 2 \left(1 - \frac{1}{\sqrt{D}} \right) \sqrt{\frac{rCD}{n}} \mathcal{C}_m. \quad (90)$$

Now, as $D \geq A_- (\ln n)^2$ we have for all $n \geq n_0(A_-)$, $D^{-1/2} \leq 1/2$. Moreover, we have $\mathcal{C}_m \geq \|\sigma\|_2 > 0$ by **(H2)** and $rC \geq A_l D n^{-1}$, so we finally deduce from (90) that, for all $n \geq n_0(A_-, A_+, A, B_2, A_l, u_m(\varphi), \|\sigma\|_2)$,

$$\mathbb{E}^{\frac{1}{2}} \left(\sup_{s \in \mathcal{F}_{(C, rC)}} (P_n - P) (\psi_{1,m} \cdot (s_m - s)) \right)^2 \geq 2 \|\sigma\|_2 \sqrt{A_l} \frac{D}{n}. \quad (91)$$

■

We turn now to the lower bound of $\mathbb{E} \left[\sup_{s \in \mathcal{F}_{(C, rC)}} (P_n - P) (\psi_{1,m} \cdot (s_m - s)) \right]$. First observe that $s \in \mathcal{F}_{(C, rC)}$ implies that $(2s_m - s) \in \mathcal{F}_{(C, rC)}$, so that

$$\mathbb{E} \left[\sup_{s \in \mathcal{F}_{(C, rC)}} (P_n - P) (\psi_{1,m} \cdot (s_m - s)) \right] = \mathbb{E} \left[\sup_{s \in \mathcal{F}_{(C, rC)}} |(P_n - P) (\psi_{1,m} \cdot (s_m - s))| \right]. \quad (92)$$

In the next step, we apply Corollary 21. More precisely, using notations of Corollary 21, we set

$$\mathcal{F} = \{ \psi_{1,m} \cdot (s_m - s) ; s \in \mathcal{F}_{(C, rC)} \}$$

and

$$Z = \sup_{s \in \mathcal{F}_{(C, rC)}} |(P_n - P) (\psi_{1,m} \cdot (s_m - s))|.$$

Now, since for all $n \geq n_0(A_+, A_-, A_\infty, A_{cons})$ we have $\tilde{R}_{n,D,\alpha} \leq 1$, we get by (41) and (42), for all $n \geq n_0(A_+, A_-, A_\infty, A_{cons})$,

$$\sup_{f \in \mathcal{F}} \|f - Pf\|_\infty = \sup_{s \in \mathcal{F}_{(C, rC)}} \|\psi_{1,m} \cdot (s_m - s)\|_\infty \leq 4A \tilde{R}_{n,D,\alpha} \leq 4A$$

we set $b = 4A$. Since we assume that $rC \leq A_u \frac{D}{n}$, it moreover holds by (42),

$$\sup_{f \in \mathcal{F}} \text{Var}(f) \leq \sup_{s \in \mathcal{F}_{(C, rC)}} P(\psi_{1,m} \cdot (s_m - s))^2 \leq 16A^2 rC \leq 16A^2 A_u \frac{D}{n}$$

and so we set $\sigma^2 = 16A^2 A_u \frac{D}{n}$. Now, by (91) we have, for all $n \geq n_0(A_-, A_+, A, B_2, A_l, u_m(\varphi), \|\sigma\|_2)$,

$$\sqrt{\mathbb{E}[Z^2]} \geq 2 \|\sigma\|_2 \sqrt{A_l} \frac{D}{n}. \quad (93)$$

Hence, a positive constant $L_{A, A_l, A_u, \|\sigma\|_2} \left(\max \left(2A \sqrt{A_u} A_l^{-1/2} \|\sigma\|_2^{-1} ; \sqrt{2A} A_l^{-1/4} \|\sigma\|_2^{-1/2} \right) \right)$ holds) exists such that, by setting

$$\varkappa_n = \frac{L_{A, A_l, A_u, \|\sigma\|_2}}{\sqrt{D}}$$

we get, using (93), that, for all $n \geq n_0(A_-, A_+, A_l, A_u, A, B_2, u_m(\varphi), A_{cons}, \|\sigma\|_2)$,

$$\begin{aligned}\varkappa_n^2 \mathbb{E}[Z^2] &\geq \frac{\sigma^2}{n}, \\ \varkappa_n^2 \sqrt{\mathbb{E}[Z^2]} &\geq \frac{b}{n}.\end{aligned}$$

Furthermore, since $D \geq A_-(\ln n)^2$, we have for all $n \geq n_0(A_-, A, A_u, A_l, \|\sigma\|_2)$,

$$\varkappa_n \in (0, 1).$$

So, using (92) and Corollary 21, it holds for all $n \geq n_0(A_-, A_+, A_l, A_u, A, B_2, u_m(\varphi), \|\sigma\|_2)$,

$$\begin{aligned}\mathbb{E} \left[\sup_{s \in \mathcal{F}_{(C, rC)}} (P_n - P)(\psi_{1,m} \cdot (s_m - s)) \right] \\ \geq \left(1 - \frac{L_{A, A_l, A_u, \|\sigma\|_2}}{\sqrt{D}} \right) \mathbb{E}^{\frac{1}{2}} \left(\sup_{s \in \mathcal{F}_{(C, rC)}} (P_n - P)(\psi_{1,m} \cdot (s_m - s)) \right)^2.\end{aligned}\quad (94)$$

Finally, by comparing (90) and (94), we deduce that for all $n \geq n_0(A_-, A_+, A_l, A_u, A, B_2, u_m(\varphi), \|\sigma\|_2)$

$$\mathbb{E} \left[\sup_{s \in \mathcal{F}_{(C, rC)}} (P_n - P)(\psi_{1,m} \cdot (s_m - s)) \right] \geq 2 \left(1 - \frac{L_{A, A_l, A_u, \|\sigma\|_2}}{\sqrt{D}} \right) \sqrt{\frac{rCD}{n}} \mathcal{C}_m$$

and so (78) is proved.

In the following Lemma, we formulate uniform upper bounds for the supremum of the empirical process of second order terms in the contrast expansion when the considered slices are not too small. This lemma follows from the exact same arguments as Lemma 15 of [Sau12].

Lemma 12 *Let $A_+, A_-, A_l, \beta, C_- > 0$, and assume **(H3)** and (35). If $C_- \geq A_l \frac{D}{n}$ and $A_+ n (\ln n)^{-2} \geq D \geq A_-(\ln n)^2$, then a positive constant $L_{A_-, A_l, \beta}$ exists such that, for all $n \geq n_0(A_\infty, A_{cons}, A_+, A_l)$,*

$$\mathbb{P} \left[\forall C > C_-, \sup_{s \in \mathcal{F}_C} |(P_n - P)((s - s_m)^2)| \leq L_{A_-, A_l, \beta} \sqrt{\frac{CD}{n}} \tilde{R}_{n, D, \alpha} \right] \geq 1 - n^{-\beta}.$$

Having controlled the residual empirical process driven by the remainder terms in the expansion of the contrast, and having proved sharp bounds for the expectation of the increments of the main empirical process on the slices, it remains to combine the above lemmas in order to establish the probability estimates controlling the empirical excess risk on the slices.

The following lemma combines Lemma 9 and Lemma 12 and follows from the same lines as Lemma 16 of [Sau12].

Lemma 13 Let $\beta, A_-, A_+, A_l, C > 0$. Assume that **(H1)**, **(H2)**, **(H3)** and (35) hold. A positive constant A_4 exists, only depending on $A, A_{3,m}, \|\sigma\|_2, \beta$, such that, if

$$A_l \frac{D}{n} \leq C \leq (1 + A_4 \nu_n)^2 \frac{D}{n} \mathcal{C}_m^2 \quad \text{and} \quad A_+ \frac{n}{(\ln n)^2} \geq D \geq A_- (\ln n)^2$$

where $\nu_n = \max \left\{ \sqrt{\frac{\ln n}{D}}, \sqrt{\frac{D \ln n}{n}}, R_{n,D,\alpha} \right\}$ is defined in (39), then for all $n \geq n_0(A_\infty, A_{cons}, A_+, A_l)$,

$$\mathbb{P} \left[\sup_{s \in \mathcal{F}_C} P_n(\gamma(s_m) - \gamma(s)) \geq 2(1 + L_{A_\infty, A, A_{3,m}, \|\sigma\|_2, A_-, A_l, \beta} \times \nu_n) \sqrt{\frac{CD}{n}} \mathcal{C}_m - C \right] \leq 2n^{-\beta}.$$

The following lemma makes use of Lemma 9, Lemma 13 and Lemma 12 and follows from exactly the same arguments as for Lemma 17 in [Sau12].

Lemma 14 Let $\beta, A_-, A_+, C \geq 0$. Assume that **(H1)**, **(H2)**, **(H3)** and (35) hold. A positive constant A_5 , depending on $A, A_{3,m}, A_\infty, \sigma_{\min}, A_-$ and β , exists such that, if it holds

$$C \geq (1 + A_5 \nu_n)^2 \frac{D}{n} \mathcal{C}_m^2 \quad \text{and} \quad A_+ \frac{n}{(\ln n)^2} \geq D \geq A_- (\ln n)^2$$

where $\nu_n = \max \left\{ \sqrt{\frac{\ln n}{D}}, \sqrt{\frac{D \ln n}{n}}, R_{n,D,\alpha} \right\}$ is defined in (39), then for all $n \geq n_0(A_\infty, A_{cons}, A_+)$,

$$\mathbb{P} \left[\sup_{s \in \mathcal{F}_{>C}} P_n(\gamma(s_m) - \gamma(s)) \geq 2(1 + A_5 \nu_n) \sqrt{\frac{CD}{n}} \mathcal{C}_m - C \right] \leq 2n^{-\beta}.$$

Moreover, when we only assume $C \geq 0$, we have for all $n \geq n_0(A_\infty, A_{cons}, A_+)$,

$$\mathbb{P} \left[\sup_{s \in \mathcal{F}_{>C}} P_n(\gamma(s_m) - \gamma(s)) \geq (1 + A_5 \nu_n)^2 \frac{D}{n} \mathcal{C}_m^2 \right] \leq 2n^{-\beta}. \quad (95)$$

The following lemma combines Lemma 10 and Lemma 12 and follows from the exact same lines as Lemma 18 of [Sau12].

Lemma 15 Let $r > 1$ and $C, \beta > 0$. Assume that **(H1)**, **(H2)**, **(H4)** and (35) hold and let $\varphi = (\varphi_k)_{k=1}^D$ be an orthonormal basis of $(m, \|\cdot\|_2)$ satisfying **(H4)**. If positive constants A_-, A_+, A_l, A_u exist such that

$$A_+ \frac{n}{(\ln n)^2} \geq D \geq A_- (\ln n)^2 \quad \text{and} \quad A_l \frac{D}{n} \leq rC \leq A_u \frac{D}{n},$$

and if the constant A_∞ defined in (38) satisfies

$$A_\infty \geq 32B_2 A \sqrt{2A_u} \|\sigma\|_2^{-1} u_m(\varphi),$$

then a positive constant $L_{A_-, A_+, A_u, A_l, A, A_\infty, \|\sigma\|_2, u_m(\varphi), \beta}$ exists such that, for all $n \geq n_0(A_-, A_+, A_u, A_l, A, A_\infty, A_{cons}, B_2, u_m(\varphi), \|\sigma\|_2)$,

$$\mathbb{P} \left(\sup_{s \in \mathcal{F}_{(C, rC]}} P_n(\gamma(s_m) - \gamma(s)) \leq 2 \left(1 - L_{A_-, A_+, A_u, A_l, A, A_\infty, \|\sigma\|_2, u_m(\varphi), \beta} \times \nu_n \right) \sqrt{\frac{rCD}{n}} \mathcal{C}_m - rC \right) \leq 2n^{-\beta},$$

where $\nu_n = \max \left\{ \sqrt{\frac{\ln n}{D}}, \sqrt{\frac{D \ln n}{n}}, R_{n, D, \alpha} \right\}$ is defined in (39).

5.2.2 Probabilistic Tools

We recall here the main probabilistic results that are instrumental in our proofs. Let us begin with the L_p -version of Hoffmann-Jørgensen's inequality, that can be found for example in [LT91], Proposition 6.10, p.157.

Theorem 16 *For any independent mean zero random variables Y_j , $j = 1, \dots, n$ taking values in a Banach space $(\mathcal{B}, \|\cdot\|)$ and satisfying $\mathbb{E}[\|Y_j\|^p] < +\infty$ for some $p \geq 1$, we have*

$$\mathbb{E}^{1/p} \left\| \sum_{j=1}^n Y_j \right\|^p \leq B_p \left(\mathbb{E} \left\| \sum_{j=1}^n Y_j \right\| + \mathbb{E}^{1/p} \left(\max_{1 \leq j \leq n} \|Y_j\| \right)^p \right)$$

where B_p is a universal constant depending only on p .

We will use this theorem for $p = 2$ in order to control suprema of empirical processes. In order to be more specific, let \mathcal{F} be a class of measurable functions from a measurable space \mathcal{Z} to \mathbb{R} and (X_1, \dots, X_n) be independent variables of common law P taking values in \mathcal{Z} . We then denote by $\mathcal{B} = l^\infty(\mathcal{F})$ the space of uniformly bounded functions on \mathcal{F} and, for any $b \in \mathcal{B}$, we set $\|b\| = \sup_{f \in \mathcal{F}} |b(f)|$. Thus $(\mathcal{B}, \|\cdot\|)$ is a Banach space. Indeed we shall apply Theorem 16 to the independent random variables, with mean zero and taking values in \mathcal{B} , defined by

$$Y_j = \{f(X_j) - Pf, f \in \mathcal{F}\}.$$

More precisely, we will use the following result, which is a straightforward application of Theorem 16. Denote by

$$P_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$$

the empirical measure associated to the sample (X_1, \dots, X_n) and by

$$\|P_n - P\|_{\mathcal{F}} = \sup_{f \in \mathcal{F}} |(P_n - P)(f)|$$

the supremum of the empirical process over \mathcal{F} .

Corollary 17 *If \mathcal{F} is a class of measurable functions from a measurable space \mathcal{Z} to \mathbb{R} satisfying*

$$\sup_{z \in \mathcal{Z}} \sup_{f \in \mathcal{F}} |f(z) - Pf| = \sup_{f \in \mathcal{F}} \|f - Pf\|_\infty < +\infty$$

and (X_1, \dots, X_n) are n i.i.d. random variables taking values in \mathcal{Z} , then an absolute constant B_2 exists such that,

$$\mathbb{E}^{1/2} [\|P_n - P\|_{\mathcal{F}}^2] \leq B_2 \left(\mathbb{E} [\|P_n - P\|_{\mathcal{F}}] + \frac{\sup_{f \in \mathcal{F}} \|f - Pf\|_\infty}{n} \right). \quad (96)$$

Another tool we need is a comparison theorem for Rademacher processes, see Theorem 4.12 of [LT91]. A function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is called a contraction if $|\varphi(u) - \varphi(v)| \leq |u - v|$ for all $u, v \in \mathbb{R}$. Moreover, for a subset $T \subset \mathbb{R}^n$ we set

$$\|h(t)\|_T = \|h\|_T = \sup_{t \in T} |h(t)|.$$

Theorem 18 *Let $(\varepsilon_1, \dots, \varepsilon_n)$ be n i.i.d. Rademacher variables and $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a convex and increasing function. Furthermore, let $\varphi_i : \mathbb{R} \rightarrow \mathbb{R}$, $i \leq n$, be contractions such that $\varphi_i(0) = 0$. Then, for any bounded subset $T \subset \mathbb{R}^n$,*

$$\mathbb{E} F \left(\left\| \sum_i \varepsilon_i \varphi_i(t_i) \right\|_T \right) \leq 2 \mathbb{E} F \left(\left\| \sum_i \varepsilon_i t_i \right\|_T \right).$$

The next tool is the well known Bernstein's inequality, that can be found for example in [Mas07], Proposition 2.9.

Theorem 19 *(Bernstein's inequality) Let (X_1, \dots, X_n) be independent real valued random variables and define*

$$S = \frac{1}{n} \sum_{i=1}^n (X_i - \mathbb{E}[X_i]).$$

Assuming that

$$v = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[X_i^2] < \infty$$

and

$$|X_i| \leq b \text{ a.s.}$$

we have, for every $x > 0$,

$$\mathbb{P} \left[|S| \geq \sqrt{2v \frac{x}{n}} + \frac{bx}{3n} \right] \leq 2 \exp(-x). \quad (97)$$

We turn now to concentration inequalities for the empirical process around its mean. Bousquet's inequality [Bou02] provides optimal constants for the deviations at the right. Klein-Rio's inequality [KR05] gives sharp constants for the deviations at the left, that slightly improves Klein's inequality [Kle02].

Theorem 20 Let (ξ_1, \dots, ξ_n) be n i.i.d. random variables having common law P and taking values in a measurable space \mathcal{Z} . If \mathcal{F} is a class of measurable functions from \mathcal{Z} to \mathbb{R} satisfying

$$|f(\xi_i) - Pf| \leq b \quad \text{a.s., for all } f \in \mathcal{F}, i \leq n,$$

then, by setting

$$\sigma_{\mathcal{F}}^2 = \sup_{f \in \mathcal{F}} \{P(f^2) - (Pf)^2\},$$

we have, for all $x \geq 0$,

Bousquet's inequality :

$$\mathbb{P} \left[\|P_n - P\|_{\mathcal{F}} - \mathbb{E} [\|P_n - P\|_{\mathcal{F}}] \geq \sqrt{2(\sigma_{\mathcal{F}}^2 + 2b\mathbb{E} [\|P_n - P\|_{\mathcal{F}}]) \frac{x}{n}} + \frac{bx}{3n} \right] \leq \exp(-x) \quad (98)$$

and we can deduce that, for all $\varepsilon, x > 0$, it holds

$$\mathbb{P} \left[\|P_n - P\|_{\mathcal{F}} - \mathbb{E} [\|P_n - P\|_{\mathcal{F}}] \geq \sqrt{2\sigma_{\mathcal{F}}^2 \frac{x}{n}} + \varepsilon \mathbb{E} [\|P_n - P\|_{\mathcal{F}}] + \left(\frac{1}{\varepsilon} + \frac{1}{3}\right) \frac{bx}{n} \right] \leq \exp(-x). \quad (99)$$

Klein-Rio's inequality :

$$\mathbb{P} \left[\mathbb{E} [\|P_n - P\|_{\mathcal{F}}] - \|P_n - P\|_{\mathcal{F}} \geq \sqrt{2(\sigma_{\mathcal{F}}^2 + 2b\mathbb{E} [\|P_n - P\|_{\mathcal{F}}]) \frac{x}{n}} + \frac{bx}{n} \right] \leq \exp(-x) \quad (100)$$

and again, we can deduce that, for all $\varepsilon, x > 0$, it holds

$$\mathbb{P} \left[\mathbb{E} [\|P_n - P\|_{\mathcal{F}}] - \|P_n - P\|_{\mathcal{F}} \geq \sqrt{2\sigma_{\mathcal{F}}^2 \frac{x}{n}} + \varepsilon \mathbb{E} [\|P_n - P\|_{\mathcal{F}}] + \left(\frac{1}{\varepsilon} + 1\right) \frac{bx}{n} \right] \leq \exp(-x). \quad (101)$$

The following proposition is proved in [Sau12], Corollary 25.

Proposition 21 Under notations of Theorem 20, if some $\varkappa_n \in (0, 1)$ exists such that

$$\varkappa_n^2 \mathbb{E} [\|P_n - P\|_{\mathcal{F}}^2] \geq \frac{\sigma^2}{n}$$

and

$$\varkappa_n^2 \sqrt{\mathbb{E} [\|P_n - P\|_{\mathcal{F}}^2]} \geq \frac{b}{n}$$

then we have, for a numerical constant $A_{1,-}$,

$$(1 - \varkappa_n A_{1,-}) \sqrt{\mathbb{E} [\|P_n - P\|_{\mathcal{F}}^2]} \leq \mathbb{E} [\|P_n - P\|_{\mathcal{F}}].$$

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