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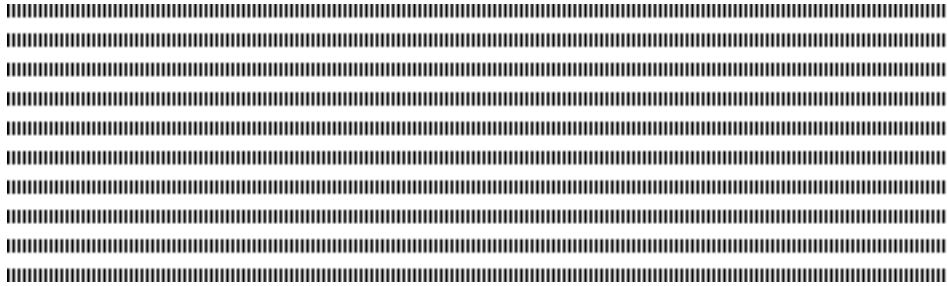
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Inverse heat source problem for a coupled hyperbolic-parabolic system

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RÉSUMÉ. Dans ce papier, on a prouvé une estimation de stabilité de type Höldérienne pour un problème inverse de détermination du terme source de l'équation de la chaleur à l'aide d'une inégalité de Carleman pour un système d'équations hyperbolique-parabolique couplé.

ABSTRACT. In this paper we consider a coupled system of mixed hyperbolic-parabolic type which describes the Biot consolidation model in poro-elasticity. Using a local Carleman estimate for a coupled hyperbolic-parabolic system, we prove the uniqueness and a Hölder stability in determining the heat source by a single measurement of solution over $\omega \times (0, T)$, where $T > 0$ is a sufficiently large time and a suitable subdomain $\omega \subset \Omega$ such that $\partial\omega \supset \partial\Omega$.

MOTS-CLÉS : Problème inverse, estimation de Carleman, système couplé

KEYWORDS : Inverse problem, Carleman estimate, coupled system



1. Introduction

Let $\Omega \subset \mathbb{R}^3$ an open and bounded domain with C^∞ boundary $\Gamma = \partial\Omega$, and let t and $x = (x_1, x_2, x_3) \in \Omega$ denote the time variable and the spatial variable respectively. Given $T > 0$, we consider a coupled hyperbolic-parabolic system

$$\begin{cases} \mathbf{u}_{tt} - \Delta_{\mu,\lambda}\mathbf{u} - \nabla(\lambda^*(x)\operatorname{div}\mathbf{u}_t) + \varrho_1(x)\nabla\theta = 0 & \text{in } Q \equiv \Omega \times (0, T), \\ \theta_t - \Delta\theta + \varrho_2(x)\operatorname{div}\mathbf{u}_t = g & \text{in } Q, \\ \mathbf{u}(x, t) = 0, \quad \theta(x, t) = 0 & \text{on } \Sigma \equiv \Gamma \times (0, T), \end{cases} \quad (1)$$

where the \cdot_t stands for the time derivative, $\nabla = (\partial_1, \partial_2, \partial_3)$, and $\Delta_{\mu,\lambda}$ is the elliptic second order linear differential operator given by

$$\begin{aligned} \Delta_{\mu,\lambda}\mathbf{v}(x) &\equiv \mu\Delta\mathbf{v}(x) + (\mu + \lambda)(\nabla(\operatorname{div}\mathbf{v}(x))) \\ &+ \operatorname{div}\mathbf{v}(x)\nabla\lambda(x) + (\nabla\mathbf{v} + (\nabla\mathbf{v})^T)\nabla\mu(x), \quad x \in \Omega, \end{aligned} \quad (2)$$

for $\mathbf{v} = (v_1, v_2, v_3)^T$, where \cdot^T denotes the transpose of matrices. Throughout this paper, $\mathbf{u} = (u_1, u_2, u_3)^T$ denotes the displacement at the location x and the time t , and $\theta \equiv \theta(x, t)$, the temperature, is a scalar function, $g \in H^1(0, T; L^2(\Omega))$ is a heat source. We will assume that the Lamé parameters $\mu, \lambda \in C^2(\bar{\Omega})$, satisfy

$$\mu(x) \geq \mu_0 > 0, \quad \lambda(x) + 2\mu(x) > 0, \quad \forall x \in \bar{\Omega}.$$

and $\lambda^* \in C^2(\bar{\Omega})$ is the consolidation coefficient which satisfy :

$$\lambda^*(x) \geq k_0 > 0, \quad x \in \bar{\Omega}.$$

Furthermore, the coupling coefficients ϱ_1, ϱ_2 satisfies :

$$(\varrho_1, \varrho_2) \in (C^2(\bar{\Omega}))^2; \quad \varrho_1(x) > \varrho_0 > 0 \quad \text{in for all } x \in \bar{\Omega}.$$

We assume that the heat source is given by

$$g(x, t) = q(x)k(x, t), \quad (3)$$

where $k \in W^{2,\infty}(Q)$ and $q \in H^2(\Omega)$.

We can prove (e.g., [1, 5]) that the system (1) possesses a unique solution

$$(\mathbf{u}, \theta) \equiv \left(\mathbf{u}_{(\lambda^*, \varrho_1, \varrho_2)}, \theta_{(\lambda^*, \varrho_1, \varrho_2)} \right),$$

Let $\omega \subset \Omega$ be a given arbitrarily subdomain such that $\partial\omega \supset \Gamma$, i.e. $\omega = \Omega \cap V$ where V is a neighbourhood of Γ on \mathbb{R}^3 and let k and $t_0 \in (0, T)$ be appropriately given.

Inverse Problem : Determine $q(x)$, $x \in \Omega$, by measurements

$$\mathbf{u}|_{\omega \times (0, T)}, \quad \mathbf{u}(x, t_0), \quad \text{and} \quad \theta(x, t_0) \quad x \in \Omega.$$

The main subject of this paper is the inverse problem of determining of g , in the Biot consolidation model in poro-elasticity, uniquely from observed data of displacement vector \mathbf{u} on a suitable subdomain $\omega \subset \Omega$ and the observation data of \mathbf{u} and θ at given a

suitable time t_0 . Such kinds of observation data are similar to those considered in (e.g. [2], [3], [4]).

The key ingredient in our argument is an L^2 -weighted inequality of Carleman type for coupled mixed hyperbolic-parabolic system. We prove a Hölder stability estimate in our inverse problem. We note that the uniqueness in the inverse problem follows directly from the Hölder stability.

1.1. Statement of main result

Let $t_0 \in (0, T)$ and $x_0 \in \mathbb{R}^3 \setminus \overline{\Omega}$ such that

$$\frac{1}{\sqrt{r_0}} \max_{x \in \overline{\Omega}} |x - x_0| < \min\{t_0, T - t_0\}, \quad (4)$$

where $r_0 \in (0, \mu_0)$.

We denote (\mathbf{u}, θ) the solution of (1) corresponding to $(\varrho_1, \varrho, \lambda^*)$.

Theorem 1.1 *Let $x_0 \in \mathbb{R}^3 \setminus \overline{\Omega}$ and $t_0 \in (0, T)$ satisfies (4). Let $k \in W^{2,\infty}(Q)$ such that $\|k\|_{W^{2,\infty}} \leq M$ and*

$$k(x, t_0) \neq 0, \quad x \in \overline{\Omega}.$$

We assume that the solution (\mathbf{u}, θ) satisfies the a priori boundedness :

$$\|\mathbf{u}\|_{H^2(0,T;H^2(\Omega))} + \|\theta\|_{H^1(0,T;L^2(\Omega))} \leq M_0, \quad (5)$$

for some given positive constant M_0 .

Then there exist constants $C > 0$ and $\delta \in (0, 1)$ such that the following stability estimate hold

$$\|q(x)\|_{L^2(\Omega)} \leq C(\|\mathbf{u}_t\|_{H^4(\omega \times (0,T))} + \|\theta(\cdot, t_0)\|_{H^2(\Omega)} + \|\mathbf{u}_t(\cdot, t_0)\|_{H^3(\Omega)} + \|\mathbf{u}_{tt}(\cdot, t_0)\|_{H^1(\Omega)})^\delta.$$

By Theorem 1.1, we can readily derive the uniqueness in the inverse problem

Corollary 1.2 *Under the same assumptions as in Theorem 1.1, we have the uniqueness. Let (\mathbf{u}, θ) satisfy the Biot system (1) such that*

$$\mathbf{u}(x, t) = 0, \quad (x, t) \in \omega \times (0, T)$$

and

$$\mathbf{u}(x, t_0) = 0, \quad \theta(x, t_0) = 0, \quad x \in \Omega.$$

Then

$$q(x) = 0 \text{ for all } x \in \Omega.$$

The remainder of the paper is organized as follows. In section 2, we give a Carleman estimate for a coupled hyperbolic-parabolic system. In section 3, we prove Theorem 1.1.

2. Carleman estimate

Here we present a Carleman estimate, which was proved in [3]. For formulating our Carleman estimate, we introduce some notations. Let $\psi : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ by setting

$$\psi(x, t) = \vartheta(x) - \beta((t - t_0)^2 - M) = |x - x_0|^2 - \beta((t - t_0)^2 - M), \quad x \in \overline{\Omega}, \quad (6)$$

for $M > 0$ large, where $x_0 \in \mathbb{R}^3 \setminus \overline{\Omega}$ and $t_0 \in (0, T)$ such that

$$\min\{t_0^2, (T - t_0)^2\} > r_0^{-1} \left(\max_{x \in \overline{\Omega}} \vartheta(x) \right). \quad (7)$$

We fix $\delta > 0$ and $\beta > 0$ satisfying

$$\beta \min\{t_0^2, (T - t_0)^2\} > \max_{x \in \overline{\Omega}} \vartheta(x) + \delta, \quad 0 < \beta < r_0. \quad (8)$$

Therefore, by (6) and (8) the function $\psi(x, t)$ verifies the following properties

$$\psi(x, 0) \leq \beta M - \delta, \quad \psi(x, T) \leq \beta M - \delta, \quad \text{for all } x \in \overline{\Omega}, \quad (9)$$

there exists $\epsilon \in (0, T/4)$ such that

$$\max_{x \in \overline{\Omega}} \psi(x, t) \leq \beta M - \frac{\delta}{2}, \quad \text{for all } t \in (0, 2\epsilon) \cup (T - 2\epsilon, T), \quad (10)$$

and

$$\min_{x \in \overline{\Omega}} \psi(x, t_0) \geq \beta M. \quad (11)$$

We next introduce a function $\varphi : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$\varphi(x, t) = e^{\gamma \psi(x, t)} := \rho(x) \alpha(t), \quad \gamma > 0, \quad (12)$$

where γ is a large parameter, $\rho(x)$ and $\alpha(t)$ are defined by

$$\rho(x) = e^{\gamma(|x - x_0|^2 + \beta M)} \geq e^{\gamma \beta M} \equiv d_1, \quad \forall x \in \Omega \quad \text{and} \quad \alpha(t) = e^{-\beta \gamma (t - t_0)^2} \leq 1, \quad \forall t \in (0, T) \quad (13)$$

and let

$$\sigma \equiv \sigma(x, t) = s \gamma \varphi(x, t). \quad (14)$$

We use usual function spaces, $H^k(Q)$, and

$$H^{2,1}(Q) = H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega)).$$

Let (\mathbf{v}, y) a solution of the linear Biot consolidation system

$$\begin{cases} \mathbf{v}_{tt}(x, t) - \Delta_{\mu, \lambda} \mathbf{v}(x, t) - \nabla(\lambda^* \operatorname{div} \mathbf{v}_t(x, t)) + \varrho_1 \nabla y(x, t) = \mathbf{f}(x, t) & \text{in } Q, \\ y_t(x, t) - \Delta y(x, t) + \varrho_2 \operatorname{div} \mathbf{v}_t(x, t) = h(x, t) & \text{in } Q, \end{cases} \quad (15)$$

such that

$$\begin{aligned} \operatorname{Supp}(\mathbf{v}(\cdot, t)) \subset \Omega, \quad \operatorname{Supp}(y(\cdot, t)) \subset \Omega, & \quad \text{for all } t \in (0, T). \\ \partial_t^j \mathbf{v}(x, 0) = \partial_t^j \mathbf{v}(x, T) = 0, \quad y(x, 0) = y(x, T) = 0 & \quad \text{for all } x \in \Omega, j = 0, 1. \end{aligned} \quad (16)$$

The following theorem is a weighted Carleman estimate with second large parameter for Biot's consolidation system (15) with assumption (16).

Theorem 2.1 (Carleman estimate) *There exist two constants $\gamma_* > 0$ and $C > 0$ such that for any $\gamma > \gamma_*$, there exists $s_* = s_*(\gamma) > 0$ such that the following estimate holds*

$$\begin{aligned}
& C \int_Q \left(\sigma |\nabla_{x,t} \mathbf{v}|^2 + \sigma^3 |\mathbf{v}|^2 + \sigma^4 |\operatorname{div} \mathbf{v}|^2 + \sigma^3 |\operatorname{div} \mathbf{v}_t|^2 + \sigma^2 |\nabla \operatorname{div} \mathbf{v}|^2 + \sigma |\nabla \operatorname{div} \mathbf{v}_t|^2 \right. \\
& \quad \left. + |\Delta y|^2 + \sigma^2 |\nabla y|^2 + \sigma^4 |y|^2 \right) e^{2s\varphi} dx dt \\
& \leq \int_Q (|\mathbf{f}|^2 + |\nabla \mathbf{f}|^2 + \gamma^{-1} \sigma |h|^2) e^{2s\varphi} dx dt \\
& \quad + \int_Q \left(\gamma^{-1} |\Delta \operatorname{div} \mathbf{v}(x, t_0)|^2 + \gamma^{-1} |\operatorname{div} \mathbf{v}_t(x, t_0)|^2 \right. \\
& \quad \left. + \sigma^4 |\operatorname{div} \mathbf{v}(x, t_0)|^2 + \sigma^2 |\nabla \operatorname{div} \mathbf{v}(x, t_0)|^2 \right) e^{2s\varphi} dx dt, \tag{17}
\end{aligned}$$

for any solution $(\mathbf{v}, y) \in H^2(Q) \times H^{2,1}(Q)$ to problem (15) which satisfy (16) and any $s \geq s_*$.

The proof is given by Bellassoued and Riahi [3].

3. Proof of the main result

This section is devoted to the proof of the Theorem 1.1. The idea of the proof is based on the Carleman estimate method in [3]. A usual methodology by [2, 3].

3.1. Preliminaries estimate

Lemma 3.1 *Let ω be an open subdomain of Ω with regular boundary $\partial\omega \supset \Gamma$. There exists constants γ_* , s_* and $C > 0$ such that for any $s \geq s_*$ and any $\gamma \geq \gamma_*$ the following estimate holds :*

$$\int_{\omega \times (0, T)} \sigma^4 |v|^2 e^{2s\varphi} dx dt \leq C \int_{\omega \times (0, T)} \sigma^2 |\nabla v|^2 e^{2s\varphi} dx dt \tag{18}$$

for any $v \in H^1(\omega \times (0, T))$ such that $v(x, t) = 0$ on $\partial\omega \times (0, T)$.

Proof We multiply ∇v by $(\nabla \varphi) v e^{2s\varphi}$ and using the divergence theorem, we obtain

$$\begin{aligned}
\int_{\omega} \nabla v \cdot (\nabla \varphi) v e^{2s\varphi} dx &= - \int_{\omega} v \operatorname{div}((\nabla \varphi) v e^{2s\varphi}) dx \\
&= - \int_{\omega} |v|^2 \Delta \varphi e^{2s\varphi} dx - 2s \int_{\omega} |v|^2 |\nabla \varphi|^2 e^{2s\varphi} dx \\
&\quad - \int_{\omega} \nabla v \cdot (\nabla \varphi) v e^{2s\varphi} dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
2 \int_{\omega} \sigma \nabla v \cdot (\nabla \varphi) v e^{2s\varphi} dx &= -2 \int_{\omega} \sigma^2 |v|^2 |\nabla \varphi|^2 e^{2s\varphi} dx \\
&\quad - \int_{\omega} \sigma |v|^2 \Delta \varphi e^{2s\varphi} dx - \gamma \int_{\omega} \sigma |v|^2 |\nabla \varphi|^2 e^{2s\varphi} dx.
\end{aligned}$$

Taking $\gamma \geq \gamma_*$ and $s \geq s_*$ sufficiently large, we obtain for any $\varepsilon > 0$

$$C \int_{\omega} \sigma^2 |v|^2 e^{2s\varphi} dx \leq C_{\varepsilon} \int_{\omega} |\nabla v|^2 e^{2s\varphi} dx + \varepsilon \int_{\omega} \sigma^2 |v|^2 e^{2s\varphi} dx. \tag{19}$$

Integrating in $(0, T)$ and taking ε small we obtain

$$C \int_{\omega \times (0, T)} \sigma^2 |v|^2 e^{2s\varphi} dx \leq C \int_{\omega \times (0, T)} |\nabla v|^2 e^{2s\varphi} dx. \quad (20)$$

Applying the last inequality to $\sigma^2 v$ we obtain

$$\int_{\omega \times (0, T)} \sigma^4 |v|^2 e^{2s\varphi} dx \leq C \int_{\omega \times (0, T)} \sigma^2 |\nabla v|^2 e^{2s\varphi} dx, \quad (21)$$

for any $\gamma \geq \gamma_*$ and $s \geq s_*$. This completes the proof. Hence, by Lemma 3.1, we obtain the following Lemma.

Lemma 3.2 *Let $(\mathbf{v}, y) \in H^2(Q) \times H^{2,1}(Q)$, satisfying*

$$\begin{cases} \mathbf{v}_{tt} - \Delta_{\mu, \lambda} \mathbf{v} - \nabla(\lambda^*(x) \operatorname{div} \mathbf{v}_t) + \varrho_1(x) \nabla y = \mathbf{f} & (x, t) \in Q, \\ y_t - \Delta y + \varrho_2(x) \operatorname{div} \mathbf{v}_t = h & (x, t) \in Q, \\ \mathbf{v} = 0, \quad y = 0 & (x, t) \in \Sigma, \end{cases} \quad (22)$$

and

$$\partial_t^j \mathbf{v}(x, 0) = \partial_t^j \mathbf{v}(x, T) = 0, \quad y(x, 0) = y(x, T) = 0 \quad \text{for all } x \in \Omega, j = 0, 1. \quad (23)$$

There exist positive constants γ_* and $C > 0$ such that, for any $\gamma \geq \gamma_*$ we can find s_* and D , the following inequality holds

$$\begin{aligned} & \int_Q e^{2s\varphi} \left(\sigma |\nabla_{x,t} \mathbf{v}|^2 + \sigma^3 |\mathbf{v}|^2 + \sigma^4 |\operatorname{div} \mathbf{v}|^2 + \sigma^3 |\operatorname{div} \mathbf{v}_t|^2 + \sigma^2 |\nabla \operatorname{div} \mathbf{v}|^2 \right. \\ & \left. + \sigma |\nabla \operatorname{div} \mathbf{v}_t|^2 + |\Delta y|^2 + \sigma^2 |\nabla y|^2 + \sigma^4 |y|^2 \right) dx dt \leq C \int_Q \left(\gamma^{-1} \sigma |h|^2 \right. \\ & \left. + \sigma^2 |\mathbf{f}|^2 + |\nabla \mathbf{f}|^2 \right) dx dt + C e^{Ds} \left(\|\mathbf{v}\|_{H^4(\omega \times (0, T))}^2 \right. \\ & \left. + \|\mathbf{v}(\cdot, t_0)\|_{H^3(\Omega)}^2 + \|\mathbf{v}_t(\cdot, t_0)\|_{H^1(\Omega)}^2 \right), \end{aligned} \quad (24)$$

for any $s \geq s_*$.

Proof Let $\omega' \subset \omega$ such that $\partial \omega' \supset \Gamma$. In order to apply Carleman estimate, we introduce a cut-off function χ satisfying $0 \leq \chi \leq 1$, $\chi \in \mathcal{C}^\infty(\mathbb{R}^3)$, $\chi = 1$ in $\overline{\Omega} \setminus \omega'$ and $\operatorname{supp} \chi \subset \Omega$. Put

$$\widehat{\mathbf{v}}(x, t) = \chi(x) \mathbf{v}(x, t), \quad \widehat{y}(x, t) = \chi(x) y(x, t).$$

Noting that $(\widehat{\mathbf{v}}, \widehat{y}) \in H^2(Q) \times H^{2,1}(Q)$, and satisfying

$$\begin{cases} \widehat{\mathbf{v}}_{tt} - \Delta_{\mu, \lambda} \widehat{\mathbf{v}} - \nabla(\lambda^* \operatorname{div} \widehat{\mathbf{v}}_t) + \varrho_1 \nabla \widehat{y} = \widehat{\mathbf{f}} & (x, t) \in Q, \\ \widehat{y}_t - \Delta \widehat{y} + \varrho_2 \operatorname{div} \widehat{\mathbf{v}}_t = \widehat{g} & (x, t) \in Q, \end{cases} \quad (25)$$

with

$$\operatorname{Supp}(\widehat{\mathbf{v}}(\cdot, t)) \subset \Omega \quad \operatorname{Supp}(\widehat{y}(\cdot, t)) \subset \Omega, \quad \forall t \in (0, T).$$

Here

$$\begin{aligned} \widehat{\mathbf{f}} &= \chi(x) \mathbf{f}(x, t) - [\Delta_{\mu, \lambda}, \chi] \mathbf{v} - \nabla(\lambda^* \nabla \chi \cdot \mathbf{v}_t) - \lambda^* (\nabla \chi) \operatorname{div} \mathbf{v}_t + \varrho_1 y \nabla \chi, \\ &\equiv \chi(x) \mathbf{f}(x, t) + P_1 \mathbf{v} + Q_1 \mathbf{v}_t + A_0 y \end{aligned}$$

$$\begin{aligned}\widehat{g} &= \chi(x)h(x, t) - 2\nabla\chi \cdot \nabla y - y\Delta\chi + \varrho_2\nabla\chi \cdot \mathbf{v}_t \\ &\equiv \chi(x)h(x, t) + A_1y + Q_0\mathbf{v}_t\end{aligned}$$

and P_1, Q_1, A_1 are a first order partial differential operators with the coefficients are supported in ω' and A_0, Q_0 are zeroth order partial differential operators supported also in ω' . Noting that $(\widehat{\mathbf{v}}, \widehat{y})$ satisfies (25), then we can apply the Carleman estimate for Biot's system (17) to $(\widehat{\mathbf{v}}, \widehat{y})$, we obtain

$$\begin{aligned}& C \int_{(\Omega \setminus \omega') \times (0, T)} e^{2s\varphi} \left(\sigma |\nabla_{x,t}\mathbf{v}|^2 + \sigma^3 |\mathbf{v}|^2 + \sigma^4 |\operatorname{div}\mathbf{v}|^2 + \sigma^3 |\operatorname{div}\mathbf{v}_t|^2 \right. \\ & \left. + \sigma^2 |\nabla\operatorname{div}\mathbf{v}|^2 + \sigma |\nabla\operatorname{div}\mathbf{v}_t|^2 + |\Delta y|^2 + \sigma^2 |\nabla y|^2 + \sigma^4 |y|^2 \right) dxdt \\ & \leq \int_Q e^{2s\varphi} \left(\gamma^{-1}\sigma |\widehat{g}|^2 + |\widehat{\mathbf{f}}|^2 + |\nabla\widehat{\mathbf{f}}|^2 \right) dxdt + \int_Q \left(\gamma^{-1} |\Delta\operatorname{div}\widehat{\mathbf{v}}(x, t_0)|^2 \right. \\ & \left. + \gamma^{-1} |\operatorname{div}\widehat{\mathbf{v}}_t(x, t_0)|^2 + \sigma^4 |\operatorname{div}\widehat{\mathbf{v}}(x, t_0)|^2 + \sigma^2 |\nabla\operatorname{div}\widehat{\mathbf{v}}(x, t_0)|^2 \right) e^{2s\varphi} dxdt. \quad (26)\end{aligned}$$

Using the first equation of system (22) after taking divergence, we obtain the following estimate

$$\begin{aligned}\int_{\omega' \times (0, T)} |\Delta y|^2 e^{2s\varphi} dxdt &\leq C e^{Ds} \|\mathbf{v}\|_{H^4(\omega \times (0, T))}^2 \\ &\quad + \int_{\omega' \times (0, T)} |\nabla y|^2 e^{2s\varphi} dxdt + \int_Q e^{2s\varphi} |\nabla\mathbf{f}|^2 dxdt.\end{aligned}$$

By the last inequality and (26), we deduce

$$\begin{aligned}& C \int_{\Omega \times (0, T)} e^{2s\varphi} \left(\sigma |\nabla_{x,t}\mathbf{v}|^2 + \sigma^3 |\mathbf{v}|^2 + \sigma^4 |\operatorname{div}\mathbf{v}|^2 + \sigma^3 |\operatorname{div}\mathbf{v}_t|^2 + \sigma^2 |\nabla\operatorname{div}\mathbf{v}|^2 \right. \\ & \left. + \sigma |\nabla\operatorname{div}\mathbf{v}_t|^2 + |\Delta y|^2 + \sigma^2 |\nabla y|^2 + \sigma^4 |y|^2 \right) dxdt \\ & \leq C \int_Q \left(\gamma^{-1}\sigma |h|^2 + |\mathbf{f}|^2 + |\nabla\mathbf{f}|^2 \right) dxdt \\ & \quad + C \left(\int_{\omega' \times (0, T)} \sigma^4 |y|^2 e^{2s\varphi} dxdt + \int_{\omega' \times (0, T)} \sigma^2 |\nabla y|^2 e^{2s\varphi} dxdt \right) \\ & \quad + e^{Ds} \left(\|\mathbf{v}\|_{H^4(\omega \times (0, T))}^2 + \|\mathbf{v}(\cdot, t_0)\|_{H^3(\Omega)}^2 + \|\mathbf{v}_t(\cdot, t_0)\|_{H^1(\Omega)}^2 \right). \quad (27)\end{aligned}$$

Let χ_1 be a cut-off function satisfying $0 \leq \chi_1 \leq 1$, $\chi_1 \in \mathcal{C}^\infty(\mathbb{R}^3)$, $\chi_1 = 1$ in $\overline{\omega'}$ and $\operatorname{Supp}(\chi_1) \subset \omega$. Let us consider $z(x, t) = \chi_1(x)y(x, t) \in H^1(\omega \times (0, T))$ and $z(x, t) = 0$ for all $(x, t) \in \partial\omega \times (0, T)$, so that by Lemma 3.1, we have

$$\begin{aligned}\int_{\omega' \times (0, T)} \sigma^4 |y|^2 \sigma e^{2s\varphi} dxdt &\leq \int_{\omega \times (0, T)} \sigma^4 |z|^2 e^{2s\varphi} dxdt \\ &\leq C \int_{\omega \times (0, T)} \sigma^2 |\nabla y|^2 e^{2s\varphi} dxdt + C \int_Q \sigma^2 |y|^2 e^{2s\varphi} dxdt. \quad (28)\end{aligned}$$

Furthermore by the first equation of (22), we have

$$\int_{\omega' \times (0, T)} \sigma^2 |\nabla y|^2 e^{2s\varphi} dxdt \leq C e^{Ds} \|\mathbf{v}\|_{H^3(\omega \times (0, T))}^2$$

$$+ \int_{\Omega \times (0, T)} \sigma^2 |\mathbf{f}|^2 e^{2s\varphi} dxdt. \quad (29)$$

Inserting (28) and (29) in (27), we obtain (24). This completes the proof of the Lemma.

Let $(\mathbf{v}, y) \in H^2(Q) \times H^{2,1}(Q)$ satisfying

$$\begin{aligned} \mathbf{v}_{tt} - \Delta_{\mu, \lambda} \mathbf{v} - \nabla(\lambda^* \operatorname{div} \mathbf{v}_t) + \varrho_1 \nabla y &= \mathbf{f} & (x, t) \in Q, \\ y_t - \Delta y + \varrho_2 \operatorname{div} \mathbf{v}_t &= h & (x, t) \in Q, \\ \mathbf{v} = 0, y = 0 & & (x, t) \in \Sigma. \end{aligned}$$

Put

$$\tilde{\mathbf{v}}(x, t) = \eta(t) \mathbf{v}(x, t), \quad \tilde{y}(x, t) = \eta(t) y(x, t).$$

Noting that $(\tilde{\mathbf{v}}, \tilde{y}) \in H^2(Q) \times H^{2,1}(Q)$ satisfies

$$\begin{aligned} \tilde{\mathbf{v}}_{tt} - \Delta_{\mu, \lambda} \tilde{\mathbf{v}} - \nabla(\lambda^* \operatorname{div} \tilde{\mathbf{v}}_t) + \varrho_1 \nabla \tilde{y} &= \eta \mathbf{f} + \eta_{tt} \mathbf{v} + 2\eta_t \mathbf{v}_t - \eta_t \nabla(\lambda^* \operatorname{div} \mathbf{v}) & (x, t) \in Q, \\ \tilde{y}_t - \Delta \tilde{y} + \varrho_2 \operatorname{div} \tilde{\mathbf{v}}_t &= \eta h + \eta_t (y - \varrho_2 \operatorname{div} \mathbf{v}) & (x, t) \in Q, \\ \tilde{\mathbf{v}} = 0, \tilde{y} = 0 & & (x, t) \in \Sigma, \end{aligned} \quad (30)$$

Henceforth we fix $\gamma > 0$ sufficiently large. By $N_{s, \varphi}$ we denote the quantity

$$\begin{aligned} N_{s, \varphi}(\mathbf{v}, y) &= \int_Q e^{2s\varphi} \left(s |\nabla_{x, t} \mathbf{v}|^2 + s^3 |\mathbf{v}|^2 + s^4 |\operatorname{div} \mathbf{v}|^2 + s^3 |\operatorname{div} \mathbf{v}_t|^2 \right. \\ &\quad \left. + s^2 |\nabla \operatorname{div} \mathbf{v}|^2 + s |\nabla \operatorname{div} \mathbf{v}_t|^2 + |\Delta y|^2 + s^2 |\nabla y|^2 + s^4 |y|^2 \right) dxdt. \end{aligned} \quad (31)$$

We introduce a cut-off function η satisfying $0 \leq \eta \leq 1$, $\eta \in C^\infty(\mathbb{R})$, $\eta = 1$ in $(2\varepsilon, T - 2\varepsilon)$ and

$\operatorname{Supp}(\eta) \subset (\varepsilon, T - \varepsilon)$. Finally we denote

$$\tilde{\mathbf{v}} = \eta \mathbf{v}, \quad \tilde{y} = \eta y.$$

Setting $d_0 = e^{(\beta M - \delta/2)\gamma}$, by (9), we have

$$\max_{x \in \Omega} \varphi(x, t) \leq d_0, \quad t \in (0, 2\varepsilon) \cup (T - 2\varepsilon, T). \quad (32)$$

We have the following lemma :

Lemma 3.3 *There exist three positive constants s_* , $C > 0$ and D such that the following inequality holds :*

$$\begin{aligned} CN_{s, \varphi}(\tilde{\mathbf{v}}, \tilde{y}) &\leq \int_Q \left(s |h|^2 + s^2 |\mathbf{f}|^2 + |\nabla \mathbf{f}|^2 \right) e^{2s\varphi} dxdt \\ &\quad + e^{Ds} \left(\|\mathbf{v}\|_{H^4(\omega \times (0, T))}^2 + \|\mathbf{v}(\cdot, t_0)\|_{H^3(\Omega)}^2 + \|\mathbf{v}_t(\cdot, t_0)\|_{H^1(\Omega)}^2 \right) \\ &\quad + Cs^2 e^{2d_0 s} \left(\|\mathbf{v}\|_{H^1(0, T; H^2(\Omega))}^2 + \|y\|_{L^2(Q)}^2 \right), \end{aligned}$$

for any $s \geq s_*$ and any $(\tilde{\mathbf{v}}, \tilde{y}) \in H^2(Q) \times H^{2,1}(Q)$ satisfying (30).

Proof Applying Carleman estimate (24) to $(\tilde{\mathbf{v}}, \tilde{y})$, we obtain

$$\begin{aligned}
CN_{s,\varphi}(\tilde{\mathbf{v}}, \tilde{y}) &\leq \int_Q \left(s |h|^2 + s^2 |\mathbf{f}|^2 + |\nabla \mathbf{f}|^2 \right) e^{2s\varphi} dxdt \\
&\quad + e^{Ds} \left(\|\mathbf{v}\|_{H^4(\omega \times (0,T))}^2 + \|\mathbf{v}(\cdot, t_0)\|_{H^3(\Omega)}^2 + \|\mathbf{v}_t(\cdot, t_0)\|_{H^1(\Omega)}^2 \right) \\
&+ \int_Q (s^2 (|\eta_{tt}|^2 + |\eta_t|^2) (|y|^2 + |\mathbf{v}|^2 + |\mathbf{v}_t|^2 + |\nabla \mathbf{v}|^2 + |\nabla \mathbf{v}_t|^2 + |\nabla(\operatorname{div} \mathbf{v})|^2)) e^{2s\varphi} dxdt,
\end{aligned}$$

for any $\gamma \geq \gamma_*$ and $s \geq s_*$. Since $\operatorname{Supp}(\eta_{tt}), \operatorname{Supp}(\eta_t) \subset (0, 2\epsilon) \cup (T - 2\epsilon, T)$, we obtain from (32)

$$\begin{aligned}
&\int_Q (s^2 (|\eta_{tt}|^2 + |\eta_t|^2) (|y|^2 + |\mathbf{v}|^2 + |\mathbf{v}_t|^2 + |\nabla \mathbf{v}|^2 + |\nabla \mathbf{v}_t|^2 + |\nabla(\operatorname{div} \mathbf{v})|^2)) e^{2s\varphi} dxdt \\
&\leq Cs^2 e^{2d_0s} \left(\|\mathbf{v}\|_{H^1(0,T;H^2(\Omega))}^2 + \|y\|_{L^2(Q)}^2 \right).
\end{aligned}$$

This completes the proof of the lemma.

3.2. Completion of the proof of the main result

Consider now the following system

$$\begin{aligned}
\mathbf{u}_{tt} - \Delta_{\mu,\lambda} \mathbf{u} - \nabla(\lambda^* \operatorname{div} \mathbf{u}_t) + \varrho_1 \nabla \theta &= 0 & (x, t) \in Q, \\
\theta_t - \Delta \theta + \varrho_2 \operatorname{div} \mathbf{u}_t &= g & (x, t) \in Q, \\
\mathbf{u} = 0, \theta = 0 & & (x, t) \in \Sigma,
\end{aligned} \tag{33}$$

where the heat source term g is given by

$$g(x, t) = q(x)k(x, t).$$

Let $\mathbf{v} = \mathbf{u}_t$ and $y = \theta_t$. Then, we have

$$\begin{aligned}
\mathbf{v}_{tt} - \Delta_{\mu,\lambda} \mathbf{v} - \nabla(\lambda^* \operatorname{div} \mathbf{v}_t) + \varrho_1 \nabla y &= 0 & (x, t) \in Q, \\
y_t - \Delta y + \varrho_2 \operatorname{div} \mathbf{v}_t &= g_t & (x, t) \in Q, \\
\mathbf{v} = 0, y = 0 & & (x, t) \in \Sigma.
\end{aligned} \tag{34}$$

We apply Lemma 3.3 to $(\tilde{\mathbf{v}}, \tilde{y})$ solution of the following system

$$\begin{aligned}
\tilde{\mathbf{v}}_{tt} - \Delta_{\mu,\lambda} \tilde{\mathbf{v}} - \nabla(\lambda^* \operatorname{div} \tilde{\mathbf{v}}_t) + \varrho_1 \nabla \tilde{y} &= \eta_{tt} \mathbf{v} + 2\eta_t \mathbf{v}_t - \eta_t \nabla(\lambda^* \operatorname{div} \mathbf{v}) & (x, t) \in Q, \\
\tilde{y}_t - \Delta \tilde{y} + \varrho_2 \operatorname{div} \tilde{\mathbf{v}}_t &= \eta g_t + \eta_t (y - \varrho_2 \operatorname{div} \mathbf{v}) & (x, t) \in Q, \\
\tilde{\mathbf{v}} = 0, \tilde{y} = 0 & & (x, t) \in \Sigma,
\end{aligned} \tag{35}$$

we obtain the following estimate :

$$\begin{aligned}
CN_{s,\varphi}(\tilde{\mathbf{v}}, \tilde{y}) &\leq \int_Q s |g_t|^2 e^{2s\varphi} dxdt \\
&+ e^{Ds} \left(\|\mathbf{v}\|_{H^4(\omega \times (0,T))}^2 + \|\mathbf{v}(\cdot, t_0)\|_{H^3(\Omega)}^2 + \|\mathbf{v}_t(\cdot, t_0)\|_{H^1(\Omega)}^2 \right) \\
&+ Cs^2 e^{2d_0s} \left(\|\mathbf{v}\|_{H^1(0,T;H^2(\Omega))}^2 + \|y\|_{L^2(Q)}^2 \right),
\end{aligned} \tag{36}$$

for sufficiently large $s > 0$.

We have the following Lemma

Lemma 3.4 *There exists a positive constant $C > 0$ such that the following estimate*

$$\int_{\Omega} |z(x, t_0)|^2 dx \leq C \int_Q (s |z(x, t)|^2 + s^{-1} |z_t(x, t)|^2) dx dt,$$

for any $z \in H^1(0, T; L^2(\Omega))$.

Proof By direct computations, we have

$$\begin{aligned} \int_{\Omega} \eta^2(t_0) |z(x, t_0)|^2 dx &= \int_0^{t_0} \frac{d}{dt} \left(\int_{\Omega} \eta^2(t) |z(x, t)|^2 dx \right) dt \\ &= 2 \int_0^{t_0} \int_{\Omega} \eta^2(t) z(x, t) z_t(x, t) dx dt + 2 \int_0^{t_0} \int_{\Omega} \eta_t(t) \eta(t) |z(x, t)|^2 dx dt. \end{aligned}$$

Then, we have

$$\int_{\Omega} |z(x, t_0)|^2 dx \leq C \int_Q (s |z(x, t)|^2 + s^{-1} |z_t(x, t)|^2) dx dt.$$

This complete the proof of the lemma.

Second, we apply lemma 3.4 to $y_1(x, t) = \eta(t)e^{2s\varphi(x,t)}y(x, t) = e^{2s\varphi(x,t)}\tilde{y}(x, t)$ and by the second equation of (35), we obtain

$$\begin{aligned} \int_{\Omega} e^{2s\rho(x)} |y(x, t_0)|^2 dx &\leq \int_Q \sigma e^{2s\varphi} |\tilde{y}(x, t)|^2 dx dt + \int_Q \sigma^{-1} e^{2s\varphi} |\tilde{y}_t(x, t)|^2 dx dt \\ &\leq \int_Q \sigma e^{2s\varphi} |\tilde{y}(x, t)|^2 dx dt + \int_Q \sigma^{-1} e^{2s\varphi} |g_t(x, t)|^2 dx dt \\ &+ \int_Q \sigma^{-1} e^{2s\varphi} |\Delta \tilde{y}(x, t)|^2 dx dt + \int_Q \sigma^{-1} e^{2s\varphi} |\operatorname{div} \tilde{\mathbf{v}}_t|^2 dx dt \\ &+ \int_Q \sigma^{-1} e^{2s\varphi} |\eta_t|^2 (|y|^2 + |\operatorname{div} \mathbf{v}|^2) dx dt \\ &\leq \int_Q \sigma e^{2s\varphi} |\tilde{y}(x, t)|^2 dx dt + \int_Q \sigma^{-1} e^{2s\varphi} |g_t(x, t)|^2 dx dt \\ &+ \int_Q \sigma^{-1} e^{2s\varphi} |\Delta \tilde{y}(x, t)|^2 dx dt + \int_Q \sigma^{-1} e^{2s\varphi} |\operatorname{div} \tilde{\mathbf{v}}_t|^2 dx dt \\ &+ e^{2sd_0} \left(\|\mathbf{v}\|_{H^1(Q)}^2 + \|y\|_{L^2(Q)}^2 \right). \end{aligned} \quad (37)$$

Then,

$$\begin{aligned} Cs \int_{\Omega} e^{s\rho(x)} |y(x, t_0)|^2 dx &\leq \int_Q e^{2s\varphi} |g_t(x, t)|^2 dx dt \\ &+ \int_Q e^{2s\varphi} \left(|\Delta \tilde{y}|^2 + \sigma^2 |\tilde{y}|^2 + |\operatorname{div} \tilde{\mathbf{v}}_t|^2 \right) dx dt + s e^{2sd_0} \left(\|\mathbf{v}\|_{H^1(Q)}^2 + \|y\|_{L^2(Q)}^2 \right) \\ &\leq \int_Q e^{2s\varphi} |g_t(x, t)|^2 dx dt + C \mathcal{N}_{s, \varphi}(\tilde{\mathbf{v}}, \tilde{y}) + s e^{2sd_0} \left(\|\mathbf{v}\|_{H^1(Q)}^2 + \|y\|_{L^2(Q)}^2 \right). \end{aligned} \quad (38)$$

Moreover, using (36), we get

$$s \int_{\Omega} e^{2s\rho(x)} |y(x, t_0)|^2 dx \leq Cs \int_Q e^{2s\varphi} |g_t(x, t)|^2 dx dt$$

$$\begin{aligned}
& +e^{Ds} \left(\|\mathbf{v}\|_{H^4(\omega \times (0,T))}^2 + \|\mathbf{v}(\cdot, t_0)\|_{H^3(\Omega)}^2 + \|\mathbf{v}_t(\cdot, t_0)\|_{H^1(\Omega)}^2 \right) \\
& +Cs^2e^{2sd_0} \left(\|\mathbf{v}\|_{H^1(0,T;H^2(\Omega))}^2 + \|y\|_{L^2(Q)}^2 \right). \tag{39}
\end{aligned}$$

On the one hand, by the second equation in (33), we have

$$y(x, t_0) = \Delta\theta(x, t_0) - \rho_2 \operatorname{div}\mathbf{v}(x, t_0) + q(x)k(x, t_0). \tag{40}$$

Moreover, we have

$$|q(x)| \leq |q(x)| |k(x, t_0)| \leq |y(x, t_0)| + |\Delta\theta(x, t_0)| + |\operatorname{div}\mathbf{v}(x, t_0)|. \tag{41}$$

Then,

$$\begin{aligned}
& \int_{\Omega} e^{2s\rho(x)} |q(x)|^2 dx \leq \int_{\Omega} e^{2s\rho(x)} |y(x, t_0)|^2 dx \\
& + \int_{\Omega} e^{2s\rho(x)} |\operatorname{div}\mathbf{v}(x, t_0)|^2 dx + \int_{\Omega} e^{2s\rho(x)} |\Delta\theta(x, t_0)|^2 dx. \tag{42}
\end{aligned}$$

We deduce that

$$\begin{aligned}
& s \int_{\Omega} e^{2s\rho(x)} |q(x)|^2 dx \leq s \int_{\Omega} e^{2s\rho(x)} |y(x, t_0)|^2 dx \\
& + s \int_{\Omega} e^{2s\rho(x)} |\operatorname{div}\mathbf{v}(x, t_0)|^2 dx + Ce^{sD} \|\Delta\theta(x, t_0)\|_{L^2(\Omega)}^2. \tag{43}
\end{aligned}$$

Inserting (39) into (43), we get

$$\begin{aligned}
& s \int_{\Omega} e^{2s\rho(x)} |q(x)|^2 dx \leq Cs \int_Q e^{2s\varphi} |g_t(x, t)|^2 dxdt + Cs^2e^{2sd_0} M_0^2 \\
& + Ce^{Ds} \left(\|\mathbf{v}\|_{H^4(\omega \times (0,T))}^2 + \|\mathbf{v}(\cdot, t_0)\|_{H^3(\Omega)}^2 \right. \\
& \left. + \|\mathbf{v}_t(\cdot, t_0)\|_{H^1(\Omega)}^2 + \|\Delta\theta(x, t_0)\|_{L^2(\Omega)}^2 \right). \tag{44}
\end{aligned}$$

Selecting $\kappa \in (d_0, d_1)$ such that $Cs^2e^{2sd_0} \leq e^{2\kappa s}$ for any s large, we get

$$\begin{aligned}
& s \int_{\Omega} e^{2s\rho(x)} |q(x)|^2 dx \leq Cs \int_Q e^{2s\varphi} |g_t(x, t)|^2 dxdt + e^{2\kappa s} M_0^2 \\
& + Ce^{Ds} \left(\|\mathbf{v}\|_{H^4(\omega \times (0,T))}^2 + \|\mathbf{v}(\cdot, t_0)\|_{H^3(\Omega)}^2 \right. \\
& \left. + \|\mathbf{v}_t(\cdot, t_0)\|_{H^1(\Omega)}^2 + \|\theta(\cdot, t_0)\|_{H^2(\Omega)}^2 \right). \tag{45}
\end{aligned}$$

Now we return to the first integral term in Q on the right-hand side term of (45).

$$\int_Q e^{2s\varphi} |q(x)k_t|^2 dxdt \leq \int_{\Omega} e^{2s\rho(x)} |q(x)|^2 \left(\int_0^T e^{-2s(\rho-\varphi)} \|k_t(\cdot, t)\|_{L^\infty(\Omega)}^2 dt \right) dx. \tag{46}$$

By the Lebesgue theorem, we obtain

$$\int_0^T e^{-2s(\rho-\varphi)} \|k_t(\cdot, t)\|_{L^\infty(\Omega)}^2 dt = \int_0^T e^{-2s\rho(x)(1-\alpha(t))} \|k_t(\cdot, t)\|_{L^\infty(\Omega)}^2 dt$$

$$\leq \int_0^T e^{-2s(1-\alpha(t))} \|k_t(\cdot, t)\|_{L^\infty}^2 dt = o(1), \quad (47)$$

as $s \rightarrow \infty$. By (45), we have

$$\begin{aligned} s \|e^{s\rho} q\|_{L^2(\Omega)}^2 &\leq o(1) \int_Q s e^{2s\rho} |q(x)|^2 dx dt + e^{2\kappa s} M_0^2 \\ &+ C e^{Ds} \left(\|\mathbf{v}\|_{H^4(\omega \times (0, T))}^2 + \|\mathbf{v}(\cdot, t_0)\|_{H^3(\Omega)}^2 \right. \\ &\left. + \|\mathbf{v}_t(\cdot, t_0)\|_{H^1(\Omega)}^2 + \|\theta(\cdot, t_0)\|_{H^2(\Omega)}^2 \right). \end{aligned} \quad (48)$$

for all $s \geq s_*$.

On the other hand, using the fact that $\rho(x) \geq d_1$ for all $x \in \Omega$. Then for sufficiently large $s > 0$, we have

$$\begin{aligned} \|q\|_{L^2(\Omega)}^2 &\leq C e^{\gamma_1 s} (\|\mathbf{v}\|_{H^4(\omega \times (0, T))}^2 + \|\mathbf{v}(\cdot, t_0)\|_{H^3(\Omega)}^2 \\ &+ \|\mathbf{v}_t(\cdot, t_0)\|_{H^1(\Omega)}^2 + \|\theta(\cdot, t_0)\|_{H^2(\Omega)}^2) + C e^{-\gamma_2 s} M_0^2. \end{aligned} \quad (49)$$

Finally, minimizing the right hand side with respect to s , we obtain

$$\|q\|_{L^2(\Omega)} \leq C (\|\mathbf{v}\|_{H^4(\omega \times (0, T))} + \|\theta(\cdot, t_0)\|_{H^2(Q)} + \|\mathbf{v}(\cdot, t_0)\|_{H^3(\Omega)} + \|\mathbf{v}_t(\cdot, t_0)\|_{H^1(\Omega)})^\delta.$$

Thus proof of Theorem 1.1 is completed.

4. Bibliographie

- [1] H. BARUCQ, M.MADAUNE-TORT AND P. SAINT-MACARY, « Theoretical aspects of wave propagation for Biot's consolidation problem », *Monografías del Seminario Matemático García de Galdeano*, vol. 31, 2004.
- [2] M.BELLASSOUED, M. YAMAMOTO, « Carleman estimates and inverse heat source problem for thermoelasticity system », *Inverse problems*, vol. 27, 2011.
- [3] M.BELLASSOUED, B. RIAHI « Carleman estimate for Biot consolidation system in poro-elasticity and application to inverse problem », *Mathematical Method in the Applied Sciences*, 2016.
- [4] B. WU, J. LIU, « Conditional stability and uniqueness for determining two coefficients in a hyperbolic-parabolic system », *Journal*, vol. 27, 2011.
- [5] J-L.LIONS, E.MAGENES, « Non-homogenous boundary value problems and applications », *Springer-Verlag*, vol. I and II, 1972.