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Density regularity for multidimensional jump diffusions with position dependent jump rate

Victor Rabiet
May 10, 2016

Résumé

On considère une diffusion $X = (X_t)_t$, avec des sauts, correspondant au générateur infinitésimal suivant :

$$L\psi(x) = \frac{1}{2} \sum_{1 \leq i, j \leq d} a_{ij}(x) \frac{\partial^2 \psi(x)}{\partial x_i \partial x_j} + g(x) \nabla \psi(x) + \int_{\mathbb{R}^d} (\psi(x + c(z, x)) - \psi(x)) \gamma(z, x) \mu(dz)$$

où $\mu$ est de masse totale infinie. En notant $(X_t(x))$ un tel processus partant du point $x$, et en utilisant une approche basé sur un Calcul de Malliavin fini-dimensionnel, nous étudions la régularité jointe de celui-ci dans le sens suivant : on fixe $b \geq 1$ et $p > 1$, $K$ un ensemble compact de $\mathbb{R}^d$, et nous donnons des conditions suffisantes pour avoir $P(X_t(x) \in dy) = p_t(x, y) dy$ avec $(x, y) \mapsto p_t(x, y)$ appartenant à $W^{bp}(K \times \mathbb{R}^d)$.

Abstract

We consider a jump type diffusion $X = (X_t)_t$ with infinitesimal generator given by

$$L\psi(x) = \frac{1}{2} \sum_{1 \leq i, j \leq d} a_{ij}(x) \frac{\partial^2 \psi(x)}{\partial x_i \partial x_j} + g(x) \nabla \psi(x) + \int_{\mathbb{R}^d} (\psi(x + c(z, x)) - \psi(x)) \gamma(z, x) \mu(dz)$$

where $\mu$ is of infinite total mass. Denoting $(X_t(x))$ such process starting from point $x$, and using an finite-dimensional Malliavin Calculus based approach, we study the joint regularity in the following sense : let $b \geq 1$, $p > 1$ and $K$ be a compact set in $\mathbb{R}^d$, then we give sufficient conditions in order to have $P(X_t(x) \in dy) = p_t(x, y) dy$ with $(x, y) \mapsto p_t(x, y)$ in $W^{bp}(K \times \mathbb{R}^d)$.

Key words : Diffusions with jumps, Malliavin Calculus, Regularity, Density, Finite dimensional Malliavin Calculus.

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1 Differential calculus and Integration by part

1.1 Introduction

The purpose of this work is to study the regularity of the density of the following stochastic equation

\[ X_t = x + \sum_{i=1}^{m} \int_{0}^{t} \sigma_i(x_s) \, dW^i_s + \int_{0}^{t} b(x_s) \, ds \]

\[ + \int_{0}^{t} \int_{E \times \mathbb{R}_+} c(x_{s-}, z) \mathbf{1}_{\{u \leq \gamma(x_{s-}, z)\}} N(ds, dz, du) \]


The way to do so used here is based on a two step strategy. First, we construct an approximation \((F_M)\) of the process \(X_t\) (basically given a non-decreasing sequence of subsets \((B_M)_{M \in \mathbb{N}^*}\), with \(\mu(B_M) < \infty\), recovering \(E\), the approximation \(F_M\) will be constructed (for each \(M\)) from a restriction of the processes \(X_t\) based on the restriction of the random measure \(N\) on the subset \(B_M\) verifying an integration by part formula:

\[ E[\varphi'(F_M)] = E[\varphi(F_M)H_M]. \]

This integration by part is obtained within a general framework developed in [3] by V. Bally and E. Clément, whose main results used in the sequel are presented in this section.

The second step consists in proving the density regularity itself. The idea is to use a certain balance between the error \(E[|F_M - X_t|]\) (which tends to 0) and the weight \(E[|H_M|]\) (which tends to \(\infty\)). This was the strategy used in [3] as well. But here the estimates of \(E[|H_M|]\) will appear to be more delicate than the corresponding one in [3] because of the additional Brownian part \(\sigma \, dW\). Moreover, the balance used in [3] was based on a Fourier transform method while here we use the new method developed by V. Bally and L. Caramellino in [2].

This new method allowed us also to extend the result to the regularity of the density considering additionally the variation of the starting point of the process, which was fixed in [3] ; the part of [2] used for our purpose is presented, in this section, subsection 1.6.

1.2 Notations, tools of differential calculus

1.2.1 Notations and differentials operators

We consider a sequence \((V_i)_{i \in \mathbb{N}^*}\) of random variables on a probability space \((\Omega, \mathcal{F}, P)\), a \(\sigma\)-algebra \(\mathcal{G} \subset \mathcal{F}\) and a \(\mathcal{G}\)-measurable random variable \(J\), with values in \(\mathbb{N}\). We assume that the variables \((V_i)\) and \(J\) satisfy the following integrability condition:

\[ \forall p \geq 1, \quad E[J^p] + E\left[\left(\sum_{i=1}^{J} V_i^2\right)^p\right] < \infty. \]

Following Bally and Clément, we will define a differential calculus based on the variables \((V_i)\), conditionally on \(\mathcal{G}\).

First we will define the following set

**Definition 1.1** Let \(\mathcal{M}\) be the class of functions \(f : \Omega \times \mathbb{R}^{N^*} \rightarrow \mathbb{R}\) such that:

- \(f\) can be written as
  \[ f(\omega, v) = \sum_{j=1}^{\infty} f^j(\omega, v_1, \ldots, v_j) \mathbf{1}_{\{J(\omega) = j\}} \]

  where \(f^j : \Omega \times \mathbb{R}^j \rightarrow \mathbb{R}\) are \(\mathcal{G} \times \mathcal{B}(\mathbb{R}^j)\)-measurable functions;

- there exists a random variable \(C \in \bigcap_{q \geq 1} L^q(\Omega, \mathcal{F}, P)\) and \(p \in \mathbb{N}^*\) such that
  \[ |f(\omega, v)| \leq C(\omega) \left(1 + \left(\sum_{i=1}^{J(\omega)} v_i^2\right)^p\right) \]

  (in other words, conditionally on \(\mathcal{G}\), the functions of \(\mathcal{M}\) have polynomial growth with respect to the variables \((v_i)\)).
We will define also

- \( \mathcal{G}_i \) the \( \sigma \)-algebra generated by \( \mathcal{G} \cup \sigma(V_j, 1 \leq j \leq J, j \neq i) \),
- \( (a_i(\omega)) \) and \( (b_i(\omega)) \) two sequences of \( \mathcal{G} \)-measurable random variables satisfying
  \[-\infty \leq a_i(\omega) < b_i(\omega) \leq +\infty, \quad \forall i \in \mathbb{N}^* \]
- \( \mathcal{O}_i \) the open set of \( \mathbb{R}^{\mathbb{N}^*} \) defined by \( \mathcal{O}_i = P_i^{-1}([a_i, b_i]), \) where \( P_i \) is the coordinate map \( \mathbb{R}^{\mathbb{N}^*} \) (ie. \( P_i(v) = v_i \)).

We localize the differential calculus on the sets \( (\mathcal{O}_i) \) by introducing some weights \( (\pi_i) \), satisfying the following hypothesis.

**Hypothesis 1.1** For all \( n \in \mathbb{N}^* \), \( \pi_i \in \mathcal{M} \) and

\[ \{ \pi_i > 0 \} \subset \mathcal{O}_i. \]

Moreover for all \( j \geq 1 \), \( \pi_i^j \) is infinitely differentiable with bounded derivatives with respect to the variables \((v_1, \ldots, v_j)\).

At last, we associate to these weights \( \pi_i \), the spaces \( C^k_\pi \subset \mathcal{M} \), \( k \in \mathbb{N}^* \), defined recursively as follows.

- For \( k = 1 \), \( C^1_\pi \) denotes the space of functions \( f \in \mathcal{M} \) such that for each \( i \in \mathbb{N}^* \), \( f \) admits a partial derivative with respect to the variable \( v_i \) on the open set \( \mathcal{O}_i \). We then define
  \[ \partial_i^\pi f(\omega, v) \overset{\text{def}}{=} \pi(\omega, v) \frac{\partial}{\partial v_i} f(\omega, v) \]
  and we assume that \( \partial_i^\pi f \in \mathcal{M} \).

- Suppose now that \( C^k_\pi \) is already defined. For a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k \), we define recursively \( \partial_\alpha^\pi = \partial_{\alpha_1}^\pi \cdots \partial_{\alpha_k}^\pi \) and \( C^k_{\pi} \) is the space of functions \( f \in C^k_{\pi} \) such that for every multi-index \( \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k \) we have \( \partial_\alpha^\pi f \in C^k_{\pi} \). Notice that if \( \partial_\alpha^\pi f \in \mathcal{M} \) for each \( \alpha \) with \( |\alpha| \leq k \).

- Finally we define
  \[ C^\infty_\pi \overset{\text{def}}{=} \bigcap_{k \in \mathbb{N}^*} C^k_\pi. \]

**Definition 1.2 (Simple functionals)** A random variable \( F \) is called a simple functional if there exist \( f \in C^\infty_\pi \) such that \( F = f(\omega, V) \), where \( V = (V_i) \). We denote by \( \mathcal{S} \) the space of the simple functionals (it is an algebra) ; moreover, it is worth to notice that, conditionally on \( \mathcal{G} \), \( F = f^J(V_1, \ldots, V_J) \).

**Definition 1.3 (Simple processes.)** A simple process is a sequence of random variables \( U = (U_i)_{i \in \mathbb{N}} \) such that for each \( i \in \mathbb{N}^* \), \( U_i \in \mathcal{S} \). Consequently, conditionally on \( \mathcal{G} \), we have \( U_i = u_i^J(V_1, \ldots, V_J) \). We denote by \( \mathcal{P} \) the space of the simple processes and we define the scalar product

\[ \langle U, V \rangle_J = \sum_{i=1}^J U_i V_i \quad (\in \mathcal{S}). \]

We can now define the derivative operator and state the integration by parts formula.

**Definition 1.4 (The derivative operator.)** We define \( D : \mathcal{S} \to \mathcal{P} \) by

\[ DF \overset{\text{def}}{=} (D_1 F) \in \mathcal{P} \quad \text{where} \quad D_i F \overset{\text{def}}{=} \partial_i^\pi f(\omega, v). \]

Notice that \( D_i F = 0 \), for \( i > J \).

**Definition 1.5 (Malliavin covariance matrix )** For \( F = (F^1, \ldots, F^d) \in \mathcal{S}^d \), the Malliavin covariance matrix is defined by

\[ \sigma^{k,k'}(F) = \langle DF^k, DF^{k'} \rangle_J = \sum_{i=1}^J D_i F^k D_i F^{k'} \]
We denote
\[ \Lambda(F) = \{ \det \sigma(F) \neq 0 \} \quad \text{and} \quad \gamma(F)(\omega) = \sigma^{-1}(F)(\omega), \ \omega \in \Lambda(F) \]

In order to derive an integration by parts formula, we need some additional assumptions on the random variables \((V_i)\). The main hypothesis is that conditionally on \(G\), the law of the vector \((V_1, \ldots, V_J)\), admits a locally smooth density with respect to the Lebesgue measure on \(\mathbb{R}^J\).

**Hypothesis 1.2**  
1. Conditionally on \(G\), the vector \((V_1, \ldots, V_J)\) is absolutely continuous with respect to the Lebesgue measure on \(\mathbb{R}^J\) and we denote by \(p_J\) the conditional density.
2. The set \(\{p_J > 0\}\) is open in \(\mathbb{R}^J\) and on \(\{p_J > 0\}\), \(\ln p_J \in C^\infty_\alpha\).
3. For all \(q > 1\), there exists a constant \(C_q\) such that
\[ (1 + |v|^q)p_J \leq C_q \]
where \(|v|\) stands for the euclidean norm of the vector \((v_1, \ldots, v_J)\).

Assumption 3) implies in particular that conditionally on \(G\), the functions of \(M\) are integrable with respect to \(p_J\) and that for \(f \in M:\)
\[ E_G[f(\omega, V)] = \int_{\mathbb{R}^J} f^j \times p_J(\omega, v_1, \ldots, v_J) \, dv_1, \ldots, dv_J. \]

**Definition 1.6 (The divergence operator)** Let \(U = (U_i)_{i \in \mathbb{N}^*} \in \mathcal{P}\) with \(U \in \mathcal{S}\). We define \(\delta : \mathcal{P} \rightarrow \mathcal{S}\) by
\[ \delta_i(U) \overset{\text{def}}{=} -(\partial_{v_i}(\pi_i U_i) + U_i \mathbf{1}_{\{p_J > 0\}} \partial^T \ln p_J) \]
\[ \delta(U) = \sum_{i=1}^J \delta_i(U) \]

For \(F \in \mathcal{S}\), we then define
\[ L(F) \overset{\text{def}}{=} \delta(D F) \]

### 1.3 Duality and integration by parts formulae

#### 1.4 IPP

The duality between \(\delta\) and \(D\) is given by the following proposition.

**Proposition 1.1** Assuming the two preceding hypothesis, then for all \(F \in \mathcal{S}\) and for all \(U \in \mathcal{P}\) we have
\[ E_G[[DF, U]] = E_G[F \delta(U)]. \]

**Lemma 1.2** Let \(\phi : \mathbb{R}^d \rightarrow \mathbb{R}\) be a smooth function and \(F = (F^1, \ldots, F^d) \in \mathcal{S}^d\). Then \(\phi(F) \in \mathcal{S}\) and
\[ D \phi(F) = \sum_{r=1}^d \partial_r \phi(F) D F^r. \]

If \(F \in \mathcal{S}\) and \(U \in \mathcal{P}\), then
\[ \delta(FU) = F \delta(U) - (DF, U)_J. \]

Moreover, for \(F = (F^1, \ldots, F^d) \in \mathcal{S}^d\), we have
\[ L \phi(F) = \sum_{r=1}^d \partial_r \phi(F) L F^r - \sum_{r,r'=1}^d \partial_{r,r'} \phi(F) (DF^{r'}, DF^r)_J. \]

We can now state the main results of this subsection.

**Theorem 1.3** Assuming the two preceding hypothesis, let \(F = (F^1, \ldots, F^d) \in \mathcal{S}^d, G \in \mathcal{S}\) and \(\phi : \mathbb{R}^d \rightarrow \mathbb{R}\) be a smooth bounded function with bounded derivatives. Let \(\Lambda \in \mathcal{G}, \Lambda \subset \Lambda(F)\) such that
\[ E \left[ \det \gamma(F)^p \mathbf{1}_\Lambda \right] < \infty, \quad \forall p \geq 1. \]

Then,
1. for every $r = 1, \ldots, d$, 
\[
E_G[\partial_r \phi(F)G] \mathbb{1}_\Lambda = E_G[\phi(F)H_r(F,G)] \mathbb{1}_\Lambda
\]
with
\[
H_r(F,G) = \sum_{r' = 1}^d \delta(G \gamma^{r',r}(F) D F^{r'}) = \sum_{r' = 1}^d \left( G \delta(\gamma^{r',r}(F) D F^{r'}) - \gamma^{r',r}(D F^{r'}, D G) \right);
\]  

(1.5)

2. for every multi-index $\beta = (\beta_1, \ldots, \beta_q) \in \{1, \ldots, d\}^q$
\[
E_G[\partial_\beta \phi(F)G] \mathbb{1}_\Lambda = E_G[\phi(F)H_\beta^q(F,G)] \mathbb{1}_\Lambda
\]
where the weights $H^q$ are defined recursively by (1.5) and
\[
H_\beta^q(F,G) = H_{\beta_1} \left( F, H_{(\beta_2, \ldots, \beta_q)}^{q-1}(F,G) \right).
\]  

(1.7)

### 1.5 Estimations of $H^q$

In order to estimate the weights $H^q$ appearing in the integration by parts formulae of the previous subsection, we first need to define iterations of the derivative operator. Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a multi-index, with $\alpha_i \in \{1, \ldots, J\}$, for $i = 1, \ldots, k$ and $|\alpha| = k$.

For $F \in \mathcal{S}$ we define recursively
\[
D_{(\alpha_1, \ldots, \alpha_k)}^k F \overset{\text{def}}{=} D_{\alpha_k} \left( D_{(\alpha_1, \ldots, \alpha_{k-1})}^{k-1} F \right) \quad \text{and} \quad D^k F \overset{\text{def}}{=} \left( D_{(\alpha_1, \ldots, \alpha_k)}^k F \right)_{\alpha_i \in \{1, \ldots, J\}}.
\]

Notice that $D^k F \in \mathbb{R}^{d^k}$, and consequently we define the norm of $D^k F$ as
\[
|D^k F| \overset{\text{def}}{=} \sqrt{\sum_{\alpha_1, \ldots, \alpha_k = 1}^J |D_{(\alpha_1, \ldots, \alpha_k)}^k F|^2}.
\]  

(1.8)

Moreover we introduce the following norms, for $F \in \mathcal{S}$ :
\[
|F|_{1,l} \overset{\text{def}}{=} \sum_{k=1}^l |D^k F| \quad \text{and} \quad |F|_l \overset{\text{def}}{=} |F| + |F|_{1,l} = \sum_{k=0}^l |D^k F|.
\]  

(1.9)

For $F = (F_1, \ldots, F_d) \in \mathcal{S}^d$
\[
|F|_{1,l} \overset{\text{def}}{=} \sum_{r=1}^d |F^r|_{1,l} \quad \text{and} \quad |F|_l \overset{\text{def}}{=} \sum_{r=1}^d |F^r|_l,
\]
and, similarly, for $F = (F^{r,r'})_{r,r'=1,\ldots,d}$
\[
|F|_{1,l} \overset{\text{def}}{=} \sum_{r,r'=1}^d |F^{r,r'}|_{1,l} \quad \text{and} \quad |F|_l \overset{\text{def}}{=} \sum_{r,r'=1}^d |F^{r,r'}|_l.
\]

### Notation 1.4

- **In the sequel, we will generally denote simply $D^k_{\alpha}$ by $D_{\alpha}$ (where $\alpha$ is a multi-index of length $k$).**

- **We will also use the following generalisation for $F \in \mathcal{S}^d$ and $G \in \mathcal{S}^{d \times k}$ : we will simply set**
\[
D_{\alpha} F \overset{\text{def}}{=} \left( D_{\alpha} F_i \right)_{1 \leq i \leq d} \quad \text{and} \quad D^k G \overset{\text{def}}{=} \left( D^k G_{i,j} \right)_{1 \leq i \leq d, 1 \leq j \leq k}.
\]
**1.5.1 Differentiability lemmas**

In this subsection we will use directly the notations from Section 2 defined in 2.9 and 2.10, where we will apply the previous general differential framework.

In order to express the form of the different multi-derivatives we will use in the next section, let us set the following notations:

- If $F \in S^d$, we will denote the $n$-th derivative
  \[ D_{(k_n,r_n)} \left( D_{(k_{n-1},r_{n-1})} \left( \cdots \left( D_{(k_1,r_1)}(F) \right) \right) \right) \]
  by
  \[ D_{\alpha}(F) \]
  with $\alpha = (\alpha_1, \ldots, \alpha_n)$ and, for all $i \in \{1, \ldots, n\}$, $\alpha_i \overset{\text{def}}{=} (k_i, r_i)$;

- For $1 \leq l \leq n$, we denote by
  \[ M_{\alpha}(l) = \left\{ M = (M_1, \ldots, M_l), \bigcup_{i \in [1, l]} M_i = \{1, \ldots, n\} \text{ and } M_i \cap M_j = \emptyset, \text{ for } i \neq j \right\} ; \]
  the set of the partitions of length $l$ of $\{1, \ldots, n\}$.

**Remark 1.5** The multi-derivatives defined above are not commutative: in general

\[ D_{(k,r)} \left( D_{(m,n)}(F) \right) \neq D_{(m,n)} \left( D_{(k,r)}(F) \right). \]

We can now state:

**Lemma 1.6** Let $A, B \in S, \phi : \mathbb{R}^d \to \mathbb{R}$ and $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be smooth functions and $F = (F^1, \ldots, F^d), G = (G^1, \ldots, G^d) \in S^d$. Then

1. for every $(k, r) \in [1, J] \times [1, d]$, \[ D_{k,r}(AB) = D_{k,r}(A)B + A D_{k,r}(B) ; \] and for every $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $\alpha_i \overset{\text{def}}{=} (k_i, r_i) \in [1, J] \times [1, d]$, \[ D_{\alpha}(AB) = \sum_{\alpha_i \overset{\text{def}}{=} \alpha, \alpha_i, i \text{ ordered}} D_{\alpha_i} A D_{\alpha_j} B ; \]
   (by “ordered” we mean that if $\alpha_i = (\alpha_{i_1}, \ldots, \alpha_{i_k})$, then $i_1 < \cdots < i_k$)

2. for every $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $\alpha_i \overset{\text{def}}{=} (k_i, r_i) \in [1, J] \times [1, d]$, \[ D_{\alpha} \phi(F) = \sum_{l=1}^{n} \sum_{\beta_1 = (\beta_{11}, \ldots, \beta_{1l})} \sum_{M \in M_{\alpha}(l)} \partial_{\beta} \phi(F) D_{M_1(\alpha)} F_{\beta_1} \cdots D_{M_l(\alpha)} F_{\beta_l} \]
   \[ = T_{\alpha}(\phi)(F) + \nabla \phi(F) D_{\alpha} F, \]
   where
   - for $M = (M_1, \ldots, M_l) \in M_{\alpha}(l)$, if $M_j = (i_1, \ldots, i_r) \subseteq \{1, \ldots, n\}$,
     \[ M_j(\alpha) \overset{\text{def}}{=} (\alpha_{i_1}, \ldots, \alpha_{i_r}) , \]
   - and
     \[ T_{\alpha}(\phi)(F) \overset{\text{def}}{=} \sum_{l=2}^{n} \sum_{\beta = (\beta_1, \ldots, \beta_l)} \sum_{M \in M_{\alpha}(l)} \partial_{\beta} \phi(F) D_{M_1(\alpha)} F_{\beta_1} \cdots D_{M_l(\alpha)} F_{\beta_l} \]
3. For every $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $\alpha_i \overset{\text{def}}{=} (k_i, r_i) \in [1, J] \times [1, d]$, and using the same notations,

$$D_\alpha c(F, G) = \sum_{l=1}^{n} \sum_{\beta = (\beta_1, \ldots, \beta_l), \beta_i \in [1, d] \cup \{d+1, d+2\}} \partial_\beta c(F, G) D_{M_1(\alpha)} F_{\beta_1} \cdots D_{M_r(\alpha)} F_{\beta_r} \times D_{M_{r+1}(\alpha)} G_{\beta_{r+1}'} \cdots D_{M_1(\alpha)} G_{\beta_1'}$$

$$= U_{\alpha}(c)(F, G) + \nabla f c(F, G) D_\alpha F + \nabla g c(F, G) D_\alpha G.$$

**Remark 1.7** The non-symmetric form (1.13) is used in the sequel in recurrence’s purpose: all the elements $M_1(\alpha)$ from $T_\alpha$ are such that $|M_1(\alpha)| < \alpha$ so the degree of derivation of $D_{M_1(\alpha)}$ is strictly inferior to the one of $D_\alpha$ itself.

With the same notations:

**Lemma 1.8** Let $\phi : \mathbb{R}^d \to \mathbb{R}$ a smooth function and $F = (F^1, \ldots, F^d) \in S^d$, $\alpha$ a multi-index and $n \overset{\text{def}}{=} |\alpha|$. Then there exists $C_{n,p,d} > 0$ such that

$$|D_\alpha \phi(F)|^2 \leq C_{n,p,d} \left( |\phi|_n(F) \right)^2 \sum_{l=0}^{n} \sum_{M \in M_n(l)} \left( |D_{M_1(\alpha)} F|^{2p} + \cdots + |D_{M_d(\alpha)} F|^{2p} \right)$$

(1.15)

with $|\phi|_n(F) \overset{\text{def}}{=} \sup_{|\beta| \leq n} |\partial_\beta \phi(F)|$.

**Proof:**

From Lemma 1.6 we have

$$D_\alpha \phi(F) = \sum_{l=1}^{n} \sum_{\beta = (\beta_1, \ldots, \beta_l), \beta_i \in [1, d]} \partial_\beta \phi(F) D_{M_1(\alpha)} F_{\beta_1} \cdots D_{M_d(\alpha)} F_{\beta_l}.$$

It follows

$$|D_\alpha \phi(F)| \leq C_n |\phi|_n(F) \sum_{l=1}^{n} \sum_{\beta = (\beta_1, \ldots, \beta_l), \beta_i \in [1, d]} \sum_{M \in M_n(l)} |D_{M_1(\alpha)} F_{\beta_1} | \cdots |D_{M_d(\alpha)} F_{\beta_l}|$$

$$|D_\alpha \phi(F)|^{2p} \leq C_{n,p} \left( |\phi|_n(F) \right)^{2p} \sum_{l=1}^{n} \sum_{\beta = (\beta_1, \ldots, \beta_l), \beta_i \in [1, d]} \sum_{M \in M_n(l)} |D_{M_1(\alpha)} F_{\beta_1}|^{2p} \cdots |D_{M_d(\alpha)} F_{\beta_l}|^{2p}$$

Now\(^1\)

$$|D_{M_1(\alpha)} F|^{2p} \cdots |D_{M_d(\alpha)} F|^{2p} \leq \frac{1}{d} \left( |D_{M_1(\alpha)} F|^{2pd} + \cdots + |D_{M_d(\alpha)} F|^{2pd} \right),$$

so

$$|D_\alpha \phi(F)|^{2p} \leq C_{n,p,d} \left( |\phi|_n(F) \right)^{2p} \sum_{l=1}^{n} \sum_{M \in M_n(l)} \left( |D_{M_1(\alpha)} F|^{2pd} + \cdots + |D_{M_d(\alpha)} F|^{2pd} \right).$$

We will also need an extended version of the first item of Lemma 1.6:

\(^1\)Since, if $a_1, \ldots, a_n \in \mathbb{R}_+^*$, it is well-known that $\sqrt[n]{\prod_{i=1}^{n} a_i} \leq \frac{1}{n} \sum_{i=1}^{n} a_i,$

$$\prod_{i=1}^{n} a_i \leq \frac{1}{n} \left( \sum_{i=1}^{n} a_i \right)^n \leq \frac{1}{n} n^{n-1} \sum_{i=1}^{n} a_i^n,$$

so

$$\prod_{i=1}^{n} a_i \leq \frac{1}{n} \sum_{i=1}^{n} a_i^n.$$
Lemma 1.9 Let $A, B \in S^{d \times d}$. Then for every $(k, r) \in [1, J] \times [1, d]$, 
\[ D_{k, r}(AB) = D_{k, r}(A)B + A D_{k, r}(B) ; \]  
and for every $\alpha = (\alpha_1, \ldots, \alpha_n)$, with $\alpha_i \overset{\text{def}}{=} (k_i, r_i) \in [1, J] \times [1, d]$, 
\[ D_{\alpha}(AB) = \sum_{\alpha_i \overset{\text{def}}{=} \alpha} D_{\alpha_i} A D_{\alpha_j} B ; \]  
(1.17)

(by “ordered” we mean that if $\alpha_i = (\alpha_{i_1}, \ldots, \alpha_{i_k})$, then $i_1 < \cdots < i_k$).

Proof: Let $A = (A_{i,j})_{1 \leq i, j \leq d}$ and $B = (B_{i,j})_{1 \leq i, j \leq d}$ with $A_{i,j}, B_{i,j} \in S$. Then, 
\[ D_{k, r}(AB) = D_{k, r} \left( \left( \sum_{m=1}^{d} A_{i,m} B_{m,j} \right)_{1 \leq i, j \leq d} \right) \]
\[ = \left( \left( \sum_{m=1}^{d} D_{k, r} \left( A_{i,m} B_{m,j} \right) \right)_{1 \leq i, j \leq d} \right) \]
\[ = \left( \left( \sum_{m=1}^{d} D_{k, r} \left( A_{i,m} B_{m,j} \right) + A_{i,m} D_{k, r} \left( B_{m,j} \right) \right)_{1 \leq i, j \leq d} \right) \] (using (1.10))
\[ = D_{k, r}(A)B + A D_{k, r}(B). \]

But the proof of (1.17) only requires an induction over the formal relation (1.16) (and does not need any commutativity in the product of $A$ by $B$).

Corollary 1.10 Let $A = (A_{i,j})_{1 \leq i, j \leq d}$, $B = (B_{i,j})_{1 \leq i, j \leq d}$ with $A_{i,j}, B_{i,j} \in S$ and $l \in \mathbb{N}^*$. Then there exists $C_l > 0$ such that 
\[ |AB|_l \leq C_l |A|_l |B|_l . \]  
(1.18)

We also have the following result proven in [3] (Lemma 8):

Lemma 1.11 Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a $C^\infty$ function and $F \in S^d$ then for all $l \geq 1$ we have 
\[ |\phi(F)|_{1,l} \leq |\nabla \phi(F)||F|_{1,l} + C_l \sup_{2 \leq \beta \leq d} |\partial_\beta \phi(F)||F|_{1,l-1}^\beta . \]

Result that we will essentially use in this work through this corollary:

Corollary 1.12 Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a $C^\infty$ bounded function with bounded derivatives of any order and $F \in S^d$ then for all $l \geq 1$ there exists $C_{\phi,l} > 0$ such that 
\[ |\phi(F)|_{l} \leq C_{\phi,l} (1 + |F|_l + |F|_{l-1}^l) . \]  
(1.19)

1.5.2 Some bounds on $H^q$

The further theorem, proven in [3], gives some estimates for the weights $H^q$ in terms of the derivatives of $G, F, LF$ and $\gamma(F)$.

Theorem 1.13 For $F \in S^d$, $G \in S$ and for all $q \in \mathbb{N}^*$ there exists a universal constant $C_{q,d}$ such that for every multi-index $\beta = (\beta_1, \ldots, \beta_q)$ 
\[ \left| H^q_\beta(F, G) \right| \leq \frac{C_{q,d} |G|^q (1 + |F|^q_{q+1})^{(6d+1)q}}{|\det(\sigma(F))|^{q-1}} (1 + |LF|^q_{q-1}) . \]

Remark 1.14 In the sequel, we will simply denote $H^q_\beta(F, 1)$ by $H^q_\beta(F)$.

1.6 Interpolation method : notations and theoretical result

All this subsection is directly taken from the article of Bally and Caramelin [2].
1.6.1 Notations and definitions

Let us define for $\xi$ and $\gamma$ some multi-indexes

$$x^\gamma \overset{\text{def}}{=} \prod_{i=1}^d x_i^{\gamma_i}$$

(1.20)

$$f_{\xi,\gamma}(x) \overset{\text{def}}{=} x^\gamma \partial_\xi f(x)$$

(1.21)

$$\|f\|_{k,t,p} = \sum_{0 \leq |\gamma| \leq l} \sum_{0 \leq |\xi| \leq k} \|f_{\xi,\gamma}\|_p$$

(1.22)

For all $\gamma$ such that $|\gamma| \leq l$,

$$|x^\gamma| \overset{\text{def}}{=} \prod_{i=1}^d |x_i|^{\gamma_i} \leq \prod_{i=1}^d |x|^{\gamma_i} \leq |x|^{\sum \gamma_i} \leq (1 + |x|)^l$$

so

$$\|f\|_{k,t,p} \leq C_d \sum_{0 \leq |\xi| \leq k} \|(1 + |x|)^l \partial_\xi f(x)\|_p.$$  

(1.23)

Since in the sequel we will have to bound the quantity $\|f_M\|_{2m+q,2m,p}$, let us notice that we have directly

$$\|f_M\|_{2m+q,2m,p} \leq C_d \sum_{0 \leq |\xi| \leq 2m+q} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} [(1 + |x,y|^{2m}) \partial_\xi (f_M(x,y))]^p \, dx \, dy \right)^{\frac{1}{p}}.$$  

(1.24)

Moreover, we will define a distance between two measures $\mu$ and $\nu$ in the following way :

$$d_k(\mu,\nu) \overset{\text{def}}{=} \sup \left\{ \left| \int \phi \, d\mu - \int \phi \, d\nu \right| : \phi \in C_c^\infty (\mathbb{R}^d), \sum_{0 \leq |\xi| \leq k} \|\partial_\xi \phi\|_\infty \leq 1 \right\}.$$  

(1.25)

**Remark 1.15** Here, we will only use the case $k = 1$, which is called the bounded variation distance (or, also, the Fortet-Mourier distance).

1.6.2 Main result

**Theorem 1.16** Let $q, k \in \mathbb{N}$, $m \in \mathbb{N}^*$, $p > 1$ and set

$$\eta > \frac{q + k + d/p^*}{2m}.$$  

(1.26)

We consider a non negative finite measure $\mu$ and a family of finite non negative measures

$$\mu_\delta(dx) = f_\delta(x) \, dx, \quad \delta > 0.$$  

We assume that there exist $C$, $r > 0$ such that

$$\lambda_{q,m}(\delta) \overset{\text{def}}{=} \sup_{\delta \leq \delta' \leq 1} \|f_{\delta'}\|_{2m+q,2m,p} \leq C \delta^{-r}$$  

and moreover, with $\eta$ given in (1.26),

$$\lambda_{q,m}(\delta)^\eta d_k(\mu,\mu_\delta) \leq C.$$  

(1.27)

Then $\mu(dx) = f(x) \, dx$ with $f \in W^{q,p}$.  


2 Regularity of the Density

2.1 Introduction

As we briefly mentioned in the introduction of the last section, the main purpose of this second part is to study the regularity of the law of the random variable \( X_t \) solution of the following stochastic equation with jumps:

\[
X_t = x + \int_0^t \sigma(X_s) \, dW_s + \int_0^t \int_{E \times \mathbb{R}_+} c(z, X_{s-}) \mathbb{1}_{\{u \leq \gamma(z, X_{s-})\}} N(ds, dz, du) + \int_0^t g(X_s) \, ds \tag{2.28}
\]

(again, for the existence and the uniqueness of such stochastic equation see [6]).

Our global aim is to give sufficient conditions in order to prove that the law of \( X_t \) is absolutely continuous with respect to the Lebesgue measure and has a smooth density. That was the point of the study made in [3] as well, with an equation of this type but without the Brownian part.

But, here, we will not only consider the existence and the regularity of the density \( y \mapsto p_{X_t}(y) \) (defined by \( \mathbb{P}_{X_t}(dy) = p_{X_t}(y) \, dy \)) with a given starting point \( x \in \mathbb{R}^d \): we will consider instead the behaviour of \( (x, y) \mapsto p_{X_t}(y) \) (with \( \mathbb{P}_{X_t}(dy) = p_{X_t}(y) \, dy \)) where \( X_t \) stands for the solution of (2.28) starting at \( x \).

This joint density regularity property, in addition to being obviously a stronger result, will allow us to obtain an interesting application concerning the Harris-recurrence of the process (section 5).

2.2 Hypothesis and notations

Let us recall that the associated intensity measure of the counting measure \( N \) is given by

\[
\dot{N}(dt, dz, du) = dt \times \mu(dz) \times \mathbb{1}_{(0,\infty)}(u) \, du
\]

where \((z, u) \in X = \mathbb{R}^d \times \mathbb{R}_+ \) and \( \mu(dz) = h(z) \, dz \).

In this subsection we make the following hypothesis on the functions \( \gamma, g, h \) and \( c \).

**Hypothesis 2.1** We assume that \( \gamma, g, h \) and \( c \) are infinitely differentiable functions in both variables \( z \) and \( x \). Moreover we assume that

- \( g \) and its derivatives are bounded;
- \( \ln h \) has bounded derivatives;
- both \( \gamma \) and \( \ln \gamma \) have bounded derivatives.

**Hypothesis 2.2** We assume that there exist two functions \( \overline{\gamma}, \underline{\gamma} : \mathbb{R}^d \to \mathbb{R}_+ \) such that

\[
\overline{\gamma} \geq \overline{\gamma}(z) \geq \gamma(z, x) \geq \underline{\gamma}(z) \geq 0, \quad \forall x \in \mathbb{R}^d
\]

where \( \overline{\gamma} \) is a constant.

**Hypothesis 2.3**

1. We assume that there exists a non negative and bounded function \( \overline{\gamma} : \mathbb{R}^d \to \mathbb{R}_+ \) such that \( \int_{\mathbb{R}^d} \overline{\gamma}(z) \mu(dz) < \infty \) and

\[
|c(z, x)| + |\partial^2_x c(z, x)| \leq \overline{\gamma}(z), \quad \forall z, x \in \mathbb{R}^d.
\]

We need this hypothesis in order to estimate the Sobolev norms.

2. There exists a measurable function \( \hat{c} : \mathbb{R}^d \to \mathbb{R}_+ \) such that \( \int_{\mathbb{R}^d} \hat{c}(z) \mu(dz) < \infty \) and

\[
|\nabla_x c \times (\text{Id} + \nabla_x c)^{-1}(z, x)| \leq \hat{c}(z), \quad \forall z, x \in \mathbb{R}^d.
\]

In order to simplify the notations we assume that \( \hat{c}(z) = \overline{\gamma}(z) \).

3. There exists a non negative function \( \underline{c} : \mathbb{R}^d \to \mathbb{R}_+ \) such that, for all \( z \in \mathbb{R}^d \),

\[
\sum_{r=1}^d (\partial_{x^r} c(z, x), \xi^r)^2 \geq \underline{c}^2(z) |\xi|^2, \quad \forall \xi \in \mathbb{R}^d
\]

and we assume that there exists \( \theta \in \mathbb{R}_+ \) such that

\[
\liminf_{a \to \infty} \frac{1}{\ln a} \int_{\{z^2 \geq \frac{1}{a}\}} \gamma(z) \mu(dz) = \theta. \tag{2.29}
\]
Remark: assumptions 2) and 3) give sufficient conditions to prove the non degeneracy of the Malliavin covariance matrix as defined in the previous section.

2.3 Main result

We are now able to state the density property of $X^x_t$ and the joint regularity (in $x$ and $y$) of it: we fix $q \geq 1$ and $p > 1$, $K$ a compact set of $\mathbb{R}^d$, and we will give sufficient conditions in order to have $P_{X^x_t}(dy) = p_{X^x_t}(x,y)dy$ with $(x,y) \mapsto p_{X^x_t}$ in $W^{q,p}(K \times \mathbb{R}^d)$.

Theorem 2.1 Let $q, p \geq 1$. We assume that hypotheses 2.1, 2.2 and 2.3 hold.
Let $(B_M)_{M \in \mathbb{N}^*}$ such that $\bigcup_{M \in \mathbb{N}^*} B_M = E$ and, for all $i \in \mathbb{N}^*$,
$$B_i \subset B_{i+1} \quad \text{and} \quad \mu(B_i) < +\infty.$$ 

Let $K$ a compact set of $\mathbb{R}^d$ and (with $p^*$ such that $\frac{1}{p} + \frac{1}{p^*} = 1$)
$$\eta > \frac{q + 1 + d/p^*}{2}. \quad (2.30)$$

If there exists $C, r > 0$ such that
$$\mu(B_M)^{6(d+q+3)\eta} \leq CM^r \quad (2.31)$$

and if
$$\limsup_M \left( \mu(B_M)^{6(d+q+3)\eta} \left( \int_{B^c_{M+1}} \tau(z) \gamma(z) d\mu(z) + \sqrt{\int_{B^c_{M+1}} \gamma^2(z) \gamma(z) d\mu(z)} \right) \right) = +\infty, \quad (2.32)$$

then, for $t > \frac{4+d(q'-1)}{q}$, with $q' = d + q + 2$, and for every $x \in K$, the law of $X^x_t$ is absolutely continuous with respect to the Lebesgue measure, i.e. $P_{X^x_t}(dy) = p_{X^x_t}(y)dy$, and the function $(x,y) \mapsto p_{X^x_t}(y)$ belongs to $W^{q,p}(K \times \mathbb{R}^d)$.

Remark 2.2 The quantity $\eta$ and the related condition 2.30, come directly from the main theorem of the interpolation method (Theorem 1.16), in the particular case $k = 1$, as we stated in Remark 1.15, and with $m = 1$ (this last choice is discussed in Remark 2.12).

Remark 2.3 If $\theta = \infty$, then, for all $t > 0$, the law of $X^x_t$ is absolutely continuous with respect to the Lebesgue measure and the density $p_{X^x_t}$ belongs to $W^{q,p}(K \times \mathbb{R}^d)$.

Remark 2.4 Recalling that (cf. Brézis [5], p.168, Corollaire IX.13, for example), with $k \equiv \left[ q - \frac{d}{p} \right]$, we have (with $O \in \mathbb{R}^n$ an open ball), $W^{q,p}(O) \subset C^k(O)$, in the sense that each element of $W^{q,p}(O)$ has a $C^k$ representative, this theorem can also be used to characterize the $C^k$ behaviour of the function $(x,y) \mapsto p_{X^x_t}(y)$ (as we will briefly see in the examples at 2.3.1).

Notation 2.5 In the sequel, since $x$ will belong to a fixed compact set, we will often write simply $Y_t$ instead of $X^x_t$ for any process $Y$ starting at $x$, if this precision is not strictly needed (that is why the starting point will never explicitly appear within the Section 3 but will be always used in Section 4).

Before starting the proof itself, we will first try to give a sketch of the strategy that we will use. The global idea is articulated in two steps:

1. to obtain an integration by part formula on an appropriate approximation of the process $X_t$;
2. to use this last result to prove the regularity of the density.

The terms from the condition (2.32) are a direct consequence of this pattern.

For the first step, and first of all, given a non-decreasing sequence of subsets $(B_M)_{M \in \mathbb{N}^*}$, with $\mu(B_M) < \infty$, recovering $E$, we construct (for each $M$) an approximation $X^x_t$ of the process $X_t$ based on the restriction $N_M$ of the random measure $N$ on the subset $B_M$.

Using a similar result as the Lemma ??, given in the first part of this work, we can then say that the $L^1$-distance between these two processes is bounded as follows:

$$\forall t \leq T, \quad E \left[ \left| X_t - X^x_t \right| \right] \leq C_T \int_{B^c_M} \tau(z) \gamma(z) \mu(dz),$$
which explains the presence of the term \( \int_{B_M^c} \overline{\tau}(z) \overline{\gamma}(z) \, d\mu(z) \) in the condition (2.32).

Since \( \mu(B_M) < +\infty \), the random measure \( N_M \) may be represented as a compound Poisson process (where the jump times will be denoted by \( T_k^M \), \( k \in \mathbb{N} \)) and the Poisson part of process \( X_t^M \) could be expressed as a sum ; nevertheless, because of the indicator function from the original equation, the coefficients of the equation verified by \( X_t^M \) are still (for the Poisson part) discontinuous and therefore, we cannot use directly the differential calculus calculated earlier. Instead we prove that \( X_t^M \) has the same law as the process \( \overline{X}_t^M \) which verifies an equation with smooth coefficients.

At this point, one would like to obtain an integration by part formula for \( \overline{X}_t^M \), but there remains one last difficulty : it is clear that, for \( t < T_1^M \) (the first jump of \( N_M \)), the random measure \( N_M \) produces no noise, and consequently there is no chance to use it for an integration by part (the Malliavin covariance matrix being, of course, degenerated).

That is why one last process will be introduced :

\[
F_M := \overline{X}_t^M + \sqrt{U_M(t)} \times \Delta,
\]

where \( \Delta \) Gaussian and where \( U_M(t) \) is defined by \( U_M(t) = t \int_{B_{M+1}^c} \xi^2(z) \overline{\gamma}(z) \, d\mu(z) \).

The \( L^1 \)-distance between \( F_M \) and \( \overline{X}_t^M \) is then bounded, for \( t \leq T \), by

\[
K_T \sqrt{\int_{B_{M+1}^c} \xi^2(z) \overline{\gamma}(z) \, d\mu(z)},
\]

which gives a natural interpretation for the last term of the condition (2.32).

We are now able to obtain an integration by part formula for the process \( F_M \) :

\[
E [\varphi'(F_M)] = E [\varphi(F_M) H_M]. \tag{I_M}
\]

The second step consists in proving the density regularity. The idea is to use a certain balance between the error \( E [|F_M - X_t|] \) (which tends to 0) and the weight \( E [|H_M|] \) (which tends to \( \infty \)). This was the strategy used in [3] as well. But here the estimates of \( E [|H_M|] \) have been more delicate then the corresponding one in [3] because of the additional brownian part \( \sigma \, dW \). Moreover, the balance used in [3] was based on a Fourier transform method while here we use the new method developed in [2].

This new method allows us also to extend the result to the regularity of the density considering additionally the variation of the starting point of the process, which was fixed in [3] ; finally, we give an application of this improvement since we can then consider a regenerative scheme to obtain an interesting result concerning the Harris-recurrence of the process (section 5).

2.3.1 Examples

In this example we assume that \( h = 1 \) so \( \mu(dz) = dz \) and \( \overline{\gamma}(z) \) is equal to a constant \( \gamma > 0 \). We then have

\[
\mu(B_M) = r_d M^d
\]

where \( r_d \) is the volume of the unit ball in \( \mathbb{R}^d \). We will also assume that \( x \) is in some compact set \( K := B(0, R), R > 0 \).

We will consider two types of behaviour for \( c \).

i) Exponential decay : we assume that \( \overline{\tau}(z) = e^{-a|z|^r} \) for some constants \( 0 < b \leq a \) and \( c > 0 \). We then have

\[
\int_{\{\overline{\tau} > \frac{1}{M}\}} \frac{\overline{\gamma}(z) \, d\mu(dz)}{\left(\frac{2a}{\gamma}\right)^{\frac{r}{2}}} \times (\ln u)^{\frac{d}{2}}.
\]

we then deduce for the constant \( \theta \) (defined in (2.29))

\[
\theta = 0 \quad \text{if} \quad c > d,
\]

\[
\theta = \infty \quad \text{if} \quad 0 < c < d,
\]

\[
\theta = \frac{\gamma r_d}{2a} \quad \text{if} \quad c = d. \tag{2.33}
\]

If \( c > d \), hypothesis 2.3.3 fails, which is coherent with the result of Bichteler, Gravereaux and Jacod in [4]. Now observe that

\[
\int_{B_M^c} \overline{\tau}(z) \overline{\gamma}(z) \, d\mu(z) + \int_{B_{M+1}^c} \xi^2(z) \overline{\gamma}(z) \, d\mu(z) \leq K e^{-\xi M^c},
\]
for some $\xi > 0$, so the condition (2.32) is always well verified.

When $0 < c < d$, since $\theta = \infty$, for every $t > 0$, $(x, y) \mapsto p_{X_t}(y)$ belongs to $W^{\infty,p}(K \times \mathbb{R}^d)$ ($\forall p \geq 1$), which implies, according to the Remark 2.4, that $(x, y) \mapsto p_{X_t}(y)$ can be considered as an element of $C^\infty(K \times \mathbb{R}^d)$.

If $c = d$, then appears a more particular behaviour and it is interesting to compare the result obtained here with the example from [3] (recalling that, in this last case, $X_t$ does not possess a Brownian part, though), assuming here, for that sake, that $x$ is fixed, $k \in \mathbb{N}$ and $p > 1$:

<table>
<thead>
<tr>
<th></th>
<th>[3]</th>
<th>Present work</th>
</tr>
</thead>
<tbody>
<tr>
<td>Domain</td>
<td>$t &gt; \frac{8d^2}{\gamma d^2}(3d + 2)$</td>
<td>$t &gt; \frac{8d^2}{\gamma d^2}(3d + \frac{1}{p} + 3k + 8)$</td>
</tr>
<tr>
<td>Regularity of $p_{X_t}$</td>
<td>$C^k$ with $k \leq \frac{1}{p} \left(1 + \frac{\gamma r_d}{8da}\right) - d$</td>
<td>$C^k$ with $k \leq \frac{1}{p} \left(1 + \frac{\gamma r_d}{8da}\right) - d \left(1 + \frac{1}{p}\right) - 2$</td>
</tr>
</tbody>
</table>

**Remark 2.6** In fact, in this work, $(x, y) \mapsto p_{X_t}(y)$ belongs to $W^{q,p}(K \times \mathbb{R}^d)$ ($\forall p \geq 1$) with

$$q \leq \frac{1}{3} \left(1 + \frac{\gamma r_d}{8da}\right) - d - 2,$$

so, using again the Remark 2.4, $(x, y) \mapsto p_{X_t}(y)$ can be considered as an element of $C^k(K \times \mathbb{R}^d)$, with

$$k \overset{\text{def}}{=} \left[q - \frac{d}{p}\right] \leq q - \frac{d}{p} \leq \frac{1}{3} \left(1 + \frac{\gamma r_d}{8da}\right) - d \left(1 + \frac{1}{p}\right) - 2.$$

In particular it requires, at least, $q \geq \frac{d}{p}$ to obtain some regularity.

**ii) Polynomial decay** : We assume now that $\tau(z) = \frac{b}{1 + |z|^v}$ and $\zeta(z) = \frac{a}{1 + |z|^v}$ for some constants $0 < a \leq b$ and $v > d$. We have here

$$\int_{\{v^2 > \frac{1}{\theta}\}} \gamma(z) \, d\mu(dz) = \gamma r_d \times (a\sqrt{u} - 1)^\theta,$$

so $\theta = \lim_{u \to \infty} \frac{1}{\ln u} \gamma r_d (a\sqrt{u} - 1)^\theta = \infty$ and then, in this case, the regularity result stands for every $t > 0$.

A simple computation gives us the following bounds:

$$\int_{B^c_{\gamma t}} \tau(z) \gamma(z) \, d\mu(z) \leq \frac{C}{M^{v - d}} \quad \text{and} \quad \int_{B^c_{\gamma t + 1}} \zeta^2(z) \gamma(z) \, d\mu(z) \leq \frac{C}{M^{2v - d}}.$$

So with $C$ and $r > 0$ such that $\mu(B_M)^{6(d+q+3)^3} \leq CM^r$ (condition (2.31)), we have

$$\mu(B_M)^{6(d+q+3)^3} \eta \left(\int_{B^c_{\gamma t}} \tau(z) \gamma(z) \, d\mu(z) + \sqrt{\int_{B^c_{\gamma t + 1}} \zeta^2(z) \gamma(z) \, d\mu(z)}\right) \leq C'M^{r\eta} \left(\frac{1}{2M^{v - d}} + \frac{1}{2M^{2v - d}}\right) \leq C'M^{r\eta - v + d}.$$

Hence, the condition (2.32) is true if

$$\eta \leq \frac{v - d}{r};$$

and, since here $\mu(B_M) = r_d M^d$, (2.31) gives $r = 6(d + q + 3)^3$ and, with (2.30), we have the following condition:

$$q + 1 + \frac{d}{p^*} \leq \frac{v - d}{r}.$$

Finally, with $v$ such that

$$v > 6(d + q + 3)^3 \frac{q + 1 + d}{2},$$

$(x, y) \mapsto p_{X_t}(y)$ belongs to $W^{q,p}(K \times \mathbb{R}^d)$, for all $p \geq 1$. 

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2.4 Approximation of \( X_t \)

In order to prove that the process \( X_t \), solution of (2.28), has a smooth density, we will apply the differential calculus and the integration by parts formula from Section 1. But since the random variable \( X_t \) cannot be viewed as a simple functional, the first step consists in approximating it. We describe in this subsection our approximation procedure. We consider a non-negative and smooth function \( \varphi \) such that \( \varphi(z) = 0 \) for \( |z| > 1 \) and \( \int_{\mathbb{R}^d} \varphi(z) \, dz = 1 \). And for \( M \in \mathbb{N} \), we denote

\[
\Phi_M(z) = \varphi * 1_{B_M}
\]

with \( B_M = \{ z \in \mathbb{R}^d : |z| < M \} \). Then \( \Phi_M \in C_b^\infty \) and we have \( 1_{B_{M-1}} < \Phi_M < 1_{B_{M+1}} \). We denote by \( X^M_t \) the solution of the equation

\[
X^M_t = x + \int_0^t \sigma(X^M_s) \, dW_s + \int_0^t \int_E c_M(z, X^M_s) 1_{\{u \leq \gamma(z, X^M_s)\}} N(dz, du) + \int_0^t g(X^M_s) \, ds. \tag{2.35}
\]

where

\[
c_M(z, x) \overset{\text{def}}{=} c(z, x)\Phi_M(z).
\]

If we set

\[
N_M(ds, dz, du) \overset{\text{def}}{=} 1_{B_{M+1}}(z) \times 1_{[0,2C]}(u) N(ds, dz, du),
\]

since \( \{ u < \gamma(z, X^M_s) \} \subset \{ u < 2C \} \) and \( \Phi_M(z) = 0 \) if \( |z| > M + 1 \), we may replace \( N \) by \( N_M \) in the above equation and consequently \( X^M_t \) is solution of the equation

\[
X^M_t = x + \int_0^t \sigma(X^M_s) \, dW_s + \int_0^t \int_E c_M(z, X^M_s) 1_{\{u \leq \gamma(z, X^M_s)\}} dN_M(s, z, u) + \int_0^t g(X^M_s) \, ds.
\]

Since the intensity measure \( N_M \) is finite, we may represent the random measure \( N_M \) by a compound Poisson process. Let

\[
\lambda_M \overset{\text{def}}{=} 2C \times \mu(B_{M+1}) = t^{-1} E[N_M(t, E)]
\]

and let \( J^M_t \) a Poisson process of parameter \( \lambda_M \). We denote by \( T^M_k \), \( k \in \mathbb{N} \), the jump times of \( J^M_t \). We also consider two sequences of independent random variables \( (Z^M_k)_k \in \mathbb{N} \) and \( (U_k)_k \in \mathbb{N} \), respectively in \( \mathbb{R}^d \) and \( \mathbb{R}_+ \), which are independent of \( J^M_t \) and such that

\[
Z^M_k \sim \frac{1}{\mu(B_{M+1})} 1_{B_{M+1}}(z) \mu(dz), \quad \text{and} \quad U_k \sim \frac{1}{2C} 1_{[0,2C]}(u) \, du.
\]

Then, the last equation may be written as

\[
X^M_t = x + \int_0^t \sigma(X^M_s) \, dW_s + \sum_{k=1}^{J^M_t} C_M(Z^M_k, T^M_k) 1_{\{U_k \leq \gamma(z, x)\}}(Z^M_k, T^M_k) + \int_0^t g(X^M_s) \, ds. \tag{2.36}
\]

The random variable \( X^M_t \) solution of (2.36) is a function of \( (Z_1, \ldots, Z_{J^M_t}) \) but it is not a simple functional, as defined in Section 1, because the coefficient \( c_M(z, x) 1_{\{u \leq \gamma(z, x)\}} \) is not differentiable with respect to \( z \). In order to avoid this difficulty we use the following alternative representation. Let \( z^*_M \in \mathbb{R}^d \) such that \( |z^*_M| = M + 3 \). We define

\[
q_M(z, x) \overset{\text{def}}{=} \varphi(z - z^*_M)\theta_M, \gamma(x) + \frac{1}{2C \mu(B_{M+1})} 1_{B_{M+1}}(z) \gamma(z, x) h(z), \tag{2.37}
\]

\[
\theta_M, \gamma(x) \overset{\text{def}}{=} \frac{1}{\mu(B_{M+1})} \int_{\{|z| \leq M+1\}} \left( 1 - \frac{1}{2C} \gamma(z, x) \right) \mu(dz). \tag{2.38}
\]

We recall that \( \varphi \) is the function defined at the beginning of this subsubsection: a non-negative and smooth function with \( \int \varphi = 1 \) and which is null outside the unit ball. Moreover from Hypothesis 2.2, \( 0 \leq \gamma(z, x) \leq C \) and then

\[
1 \geq \theta_M, \gamma(x) \geq \frac{1}{2}. \tag{2.39}
\]

From this last inequality it is easy to deduce the following result:

**Lemma 2.7** Let \( q_M \) defined as in (2.37). Then \( \ln q_M \) has bounded derivatives of any order.
By construction the function $q_M$ satisfies $\int q_M(z,x)\,dz = 1$. Hence we can check that (cf. appendix A.4 for a proof)

$$E\left[f(X_{tk}^M) \mid X_{tk}^M = x\right] = \int_{\mathbb{R}^d} f(x + c_M(z,x))q_M(z,x)\,dz.$$  (2.40)

From the relation (2.40) we construct a process $(\overline{X}_t^M)$ equal in law to $(X_t^M)$ in the following way. Let $0 \leq u \leq v$ and $y \in \mathbb{R}^d$, we denote by $\Psi_{u,v}(y)$ the solution of

$$\Psi_{u,v}(x) = y + \int_u^v \sigma(\Psi_{s,v}(y))\,dW_s + \int_u^v g(\Psi_{s,v}(y))\,ds.$$  

We assume that the times $T_k$, $k \in \mathbb{N}$ are fixed and we consider a sequence $(z_k)_{k \in \mathbb{N}}$ with $z_k \in \mathbb{R}^d$. Then we define $x_t$, $t \geq 0$ by $x_0 = x$ and, if $x_{T_k}$ is given, then

$$x_t = \Psi_{T_k,t}(x_{T_k}), \quad T_k \leq t < T_{k+1},$$

$$x_{T_{k+1}} = x_{T_k} + c_M(z_k, x_{T_k}).$$

We note that for $T_k \leq t < T_{k+1}$, $x_t$ is a function of $x$, $z_1$, ..., $z_k$. Notice also that $x_t$ solves the equation

$$x_t = x + \int_0^t \sigma(x_s)\,dW_s + \int_0^t c_M(z_k, x_{T_k}) + \int_0^t g(x_s)\,ds.$$  

We consider now a sequence of random variables $(Z_k)$, $k \in \mathbb{N}^*$, independent of the Brownian motion $W_t$, and we denote $G_k = \sigma(T_p, p \in \mathbb{N}) \vee \sigma(Z_p, p \leq k)$ and

$$\overline{X}_t^M = x_t(Z_1, \ldots, Z_{J^M}).$$  (2.42)

We assume that the law of $\overline{Z}_{k+1}$ conditionally on $G_k$ is given by

$$P(\overline{Z}_{k+1} \in dz \mid G_k) = q_M(z, x_{T_{k+1}}^{-1}(\overline{Z}_{1}, \ldots, \overline{Z}_{k}))\,dz = q_M(z, \overline{X}_{T_{k+1}}^M)\,dz.$$  (2.43)

Then $\overline{X}_t^M$ satisfies the equation

$$\overline{X}_t^M = x + \int_0^t \sigma(\overline{X}_s^M)\,dW_s + \int_0^t c_M(\overline{Z}_k, \overline{X}_{T_k}^M) + \int_0^t g(\overline{X}_s^M)\,ds$$  (2.44)

and $\overline{X}_t^M$ has the same law as $X_t^M$. Moreover we can prove a bit more.

**Lemma 2.8** For a locally bounded and measurable function $\psi : \mathbb{R}^d \to \mathbb{R}$ let

$$S_k(\psi) = \sum_{k=1}^{J^M}(\Phi_M^k)(Z_k), \quad S_t(\psi) = \sum_{k=1}^{J^M}(\Phi_M^k)(Z_k)\mathbf{1}_{\gamma(z_k, X_{T_k}^M) > U_k},$$

then $(\overline{X}_t^M, S_t(\psi))_{t \geq 0}$ has the same law as $(X_t^M, S_t(\psi))_{t \geq 0}$.

**Proof:** Observing that $(\overline{X}_t^M, S_t(\psi))_{t \geq 0}$ solves a system of equations similar to (2.44), but in dimension $d + 1$, it suffices to prove that $(\overline{X}_t^M)_{t \geq 0}$ has the same law as $(X_t^M)_{t \geq 0}$, which is done in detail in the appendix A.5. 

**2.5 The integration by part formula**

The random variable $\overline{X}_t^M$ constructed previously is a simple functional but unfortunately its Malliavin covariance matrix is degenerated. To avoid this problem we use a classical regularization procedure. Instead of the variable $\overline{X}_t^M$, we consider the regularized one $F_M$ defined by

$$F_M \overset{\text{def}}{=} \overline{X}_t^M + \sqrt{U_M(t)} \times \Delta,$$  (2.45)

where $\Delta$ is a $d$-dimensional standard Gaussian variable independent of the variables $(Z_k)_{k \geq 1}$ and $(T_k)_{k \geq 1}$ and $U_M(t)$ is defined by

$$U_M(t) = t \int_{B_{M+1}} \xi^2(z)\gamma^{2}d\mu(z).$$  (2.46)
Notation 2.9 We observe that $F_M \in S^d$ where $S$ is the space of simple functionals for the differential calculus based on the variables $(Z_k)_{k \geq 1}$ with $Z_0 = (\Delta^i)_{1 \leq i \leq d}$ and $Z_k = (Z_k^j)_{1 \leq j \leq d}$ and we are now in the framework of the previous section (subsection 1.2) by taking $G = \sigma(T_k, k \in \mathbb{N})$ and defining the weights $(\pi_k)$ by setting $\pi_k = 1$ and

$$\pi^r_k \overset{def}{=} \phi_M(\tilde{Z}_k)$$

(2.47)

for $1 \leq r \leq d$.

Conditionally on $G$, the density of the law of $(Z_1, \ldots, Z_{J_M})$ is given by

$$p_M(\omega, z_1, \ldots, z_{J_M}) = \prod_{j=1}^{J_M} q_M(z_j, \Psi_{T_{j-1}, T_j}(X_{T_{j-1}}^M))$$

(2.48)

where $X_{T_{j-1}}^M$ is a function of $z_j$, $1 \leq i \leq j - 1$ (moreover, we can notice that $\Psi_{T_{j-1}, T_j}(X_{T_{j-1}}^M) = X_{T_{j-1}}^M$); we can check that $p_M$ satisfies the Hypothesis 1.2 of Section 1.

Notation 2.10 To clarify the notation, the derivative operator can be written in this framework for $F \in S$ by $DF = (D_{k,r} F)$ where $D_{k,r} = \pi^r_k \partial_{Z_k}$ for $k \geq 0$ and $1 \leq r \leq d$. Consequently we deduce that

$$D_{k,r} F_M = D_{k,r} X_{T_{j-1}}^M,$$

for $k \geq 1$ and $D_{0,r} F_M = \sqrt{U_M(t)} \delta_{r,r'}$ with $\delta_r = 0$ if $r \neq r'$, $\delta_{r,r'} = 1$ otherwise.

The Malliavin covariance matrix of $X_t^M$ is equal to

$$\sigma(X_t^M)_{i,j} = \sum_{k=1}^{J_M} \sum_{r=1}^{d} D_{k,r} D_{k,r}^\prime X_{t}^{M,i} X_{t}^{M,j}$$

for $1 \leq i, j \leq d$ and finally the Malliavin covariance matrix of $F_M$ is given by

$$\sigma(F_M) = \sigma(X_t^M) + U_M(t) \times \text{Id}.$$
\[ F_M \defeq X_i^M + \sqrt{U_M(t)} \times \Delta, \] where \( \Delta \) is Gaussian and where \( U_M(t) \) is defined by \( U_M(t) = t \int_{B_{M+1}} c^2(z) \gamma(z) \, d\mu(z) \).

\[ f_M \defeq \Psi_K(x)p_{P,M}(x,y). \]

We will then use the Theorem 1.16, with \( \delta \defeq M^{-1} \) (to be rigorous with the notations, we should define and work with \( f_M \), but we will simply use \( f_M \)).

On the one hand, using Lemma 2.11 with \( m = 1 \), we find that
\[ \|f_M\|_{2+q,2,p} \leq C \mu(B_M)^{6(d+q+3)} \] (2.52)
where \( C \) does not depend on \( M \).

On the other hand, using the definition (1.25) of the distance \( d_k \) in the case \( k = 1 \):
\[
d_1(\mu_X, f_M) \defeq \sup_{g \in C^\infty([R^d \times R^d], \|g\|_{\infty}, \|\nabla g\|_{\infty} \leq 1}) \left| \iint_{R^d \times R^d} g(x, y) \mu_X(dx, dy) \right| - \left| \iint_{R^d \times R^d} g(x, y) f_M(x, y) \, dx \, dy \right|
\]
\[
= \int_{R^d} \Psi_K(x) \left( E\left[ g(x, X_t(x)) \right] - E\left[ g(x, F_t^M(x)) \right] \right) \, dx
\]
\[
\leq \int_{R^d} \Psi_K(x) E\left[ |X_t(x) - F_t^M(x)| \right] \, dx
\]
so
\[
d_1(\mu_X, f_M) \leq C_K \left( \int_{B_{M'}} \sigma(z) \gamma(z) \, d\mu(z) + \sqrt{\int_{B_{M'+1}} c^2(z) \gamma(z) \, d\mu(z)} \right). \] (2.53)

It follows that, with the conditions (2.31) and (2.32), the conditions of the Theorem 1.16 are well verified and we can directly conclude.

\[ \text{Remark 2.2.} \text{ Let us give a quick explanation why we only considered the case where the parameter } m \text{ is equal to 1. This parameter was made to loosen up the lower bound condition for } \eta \text{ in Theorem 1.16 (roughly speaking this lower bound is a } O\left( \frac{1}{m^2} \right) \text{) which could help to obtain a better condition from (1.27). But, the upper bound obtained for } \|f_M\|_{2m+q,2m,p} \text{ with respect to } m, \text{ is a } O(m^3), \text{ so we lose here completely the possible advantage of taking } m > 1. \]

3 \ Bounding of the weights \( H_M^\beta(F_M) \)

3.1 Introduction

In this section we consider that the starting point \( x \) is fixed.

The final result of that part is to bound the quantity \( H_M^\beta(F_M) \); to do so, we will use the bounding given by the Theorem 1.13, which implies:
\[
\left| H_M^\beta(F_M) \right| \leq C_{q,d} \frac{1}{|\det \sigma(F_M)|^{\frac{1}{q-1}}} \left( 1 + |F_M|_{q+1} \right)^{(6d+1)\beta} \left( 1 + |LF_M|_{q-1} \right),
\]
and so will be brought to bound, in particular, on one hand \( \|\frac{1}{|\det \sigma(F_M)|}\|_p \) which will be done at Lemma 3.14 and, on the other hand \( \|F_M\|_p \) and \( \|LF_M\|_p \) (where \( \|\|_p \) is the \( L^p \)-norm). To do this last thing, because of the similar structure of the linear equations verified by the different processes involved here, we will develop in the first place a way to bound this type of processes, in a recursive way (which is natural, since we want, in particular, to bound successive derivatives of our process). Moreover, this theoretical result will be helpful in the Section 4, when we will study further the density continuity of the process \( X_t \).

The upper bound of this quantity allows us to prove, under some similar conditions to (2.32), the existence of a regular density for \( X_t \) : (with \( q \geq 1 \) and \( p > 1 \) fixed) we have \( P_X(dy) = p_X \, dy \) with \( p_X \in W^{q,p}(R^d) \) (using a Fourier transform method as in [3], or some weaker version of the interpolation method (cf. [2]) quoted here). In this sense, this section is "self-contained"; that is one of the reasons why we give the Lemma 3.15. The other reason (and it is globally true for the whole section) is to show a pattern of the proof, in a simpler case, which will be used again in the more general Lemma 2.11 (proved in the subsection 4.3).

Even though, to conclude in the general case (joint regularity), we need some further results, made in the next section, the main part of the needed techniques is presented in this one, with less heavier notations, since the starting point \( x \) is momentarily put aside.
Notations

- In all the sequel we will denote by $E_W$ the expectation with respect to the Brownian motion; i.e. conditionally with respect to the Poisson measure.
- As we have already pointed out in 2.5, in all this section, since $x$ will belong to a fixed compact set, we will always write $Y_t$ instead of $Y_t^x$ for any process $Y$ starting at $x$.

3.2 An upper bound lemma for a family of linear SDE’s

In this subsection we give $L^p$ bounds for the solution of a family of linear equations which represent the general framework in which the Malliavin derivatives fit.

Hypothesis

We fix a finite set $I$ and we consider the multi-indexes of the type $\alpha \stackrel{\text{def}}{=} (\alpha_1, \ldots, \alpha_n)$, with $\alpha_i \in I$. We define and denote the length of $\alpha$ by $|\alpha| \stackrel{\text{def}}{=} n$. We also consider the void multi-index $\alpha = \emptyset$ and in this case we put $|\alpha| \stackrel{\text{def}}{=} 0$.

Then we denote

$$A_n \stackrel{\text{def}}{=} \{ \alpha \stackrel{\text{def}}{=} (\alpha_1, \ldots, \alpha_n) : \alpha_i \in I \}$$

the set of multi-indexes of length $n$ and define then

$$A \stackrel{\text{def}}{=} \bigcup_{n \in \mathbb{N}} A_n$$

and

$$n_k \stackrel{\text{def}}{=} \#\{ \alpha \in A, |\alpha| \leq k \}$$

(since $I$ is finite, $n_k$ is a defined finite number).

We define a family of process $(\overline{V}_t^\alpha)_{t \in \mathbb{R}}$, $\alpha \in A$ in the following way.

- If $|\alpha| = 0$ we put $\overline{V}_t^\emptyset \stackrel{\text{def}}{=} \overline{X}_t^M$ with $\overline{X}_t^M$ solution of the equation (2.44):

  $$\overline{X}_t^M = x + \int_0^t \sigma(\overline{X}_s^M) \, dW_s + \sum_{k=1}^{J} c(M) x_{k} M + \int_0^t g(\overline{X}_s^M) \, ds.$$

- Suppose now that we have already defined $\overline{V}^\alpha$ for $|\alpha| < n - 1$. We denote by

  $$\overline{V}_{(k-1)}(t) \stackrel{\text{def}}{=} (\overline{V}_t^\alpha)_{|\alpha| \leq k-1}$$

  (so $\overline{V}(0)(t) = (\overline{V}_t^\emptyset) = (\overline{X}_t^M)$, a family of $d$-dimensional one element). Then let $\overline{V}^\alpha$ be, for $|\alpha| = n$, the solution of

  $$\overline{V}_t^\alpha = \overline{V}_0^\alpha + \int_0^t G^\alpha(\overline{V}_{(k-1)}(s)) \, dW_s + \sum_{j=1}^{J} d_j^\alpha(z_j, \overline{V}_{(k-1)}(s_j^-)) + \int_0^t g^\alpha(\overline{V}_{(k-1)}(s)) \, ds$$

  with the functions $G^\alpha : \mathbb{R}^{d \times n_{k-1}} \rightarrow \mathbb{R}^{d \times m}, d_j^\alpha : \mathbb{R}^d \times \mathbb{R}^{d \times n_{k-1}} \rightarrow \mathbb{R}^d, g^\alpha : \mathbb{R}^{d \times n_{k-1}} \rightarrow \mathbb{R}^d, \rho^\alpha : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}, \beta^\alpha : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}, b^\alpha : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ and the following Hypothesis:

**Hypothesis 3.1**

1. There exist $w \in \mathbb{N}$ and $K \in \mathbb{R}_+$ such that, for all $v \in \mathbb{R}^{n_{k-1}}$ and $z \in \mathbb{R}^d$,

   $$|G^\alpha(v)| \leq K(1 + |v|)^w,$$

   $$|g^\alpha(v)| \leq K(1 + |v|)^w,$$  \hspace{1cm} (3.55)

   and

   $$|\beta^\alpha(z, v)| \leq \overline{J}(z),$$

   $$|d_j^\alpha(z, v)| \leq K \overline{c}(z)(1 + |v|)^w \quad (\forall j \in \mathbb{N}),$$  \hspace{1cm} (3.56)

   with, for all $n \in \mathbb{N}$,
\[ f(x) = \int_{x \in T} f(x) dx \]

- \[ \int_{x \in T} f(x) dx < \infty \]
- \[ \int_{x \in T} g(x) dx < \infty \]

2. Bounding conditions:

\[ P \equiv \sup_{s \in \mathbb{R}^+} \sup_{\alpha \in A} \sup_{1 \leq l \leq m} |\rho^0_l(s)| < \infty \]
\[ \delta \equiv \sup_{s \in \mathbb{R}^+} \sup_{\alpha \in A} |b^\alpha(s)| < \infty \]

Lemma 3.1 Let \( p \in \mathbb{N}^+ \). We assume that the Hypothesis 3.1 holds and we set

\[ \Theta_{p,k}(t) = \sup_{\alpha \in A} \mathbb{E}_t \left[ |V_t^\alpha|^{2p} \right] \]

For all \( T > 0 \), there exists a constant \( C_{T,p,k} \) which does not depend on \( M \) nor on the set \( A \), and, in particular, does not depend on the size of \( I \) such that

\[ \mathbb{E} \left[ \Theta_{p,k}(T) \right] \leq C_{T,p,k} \quad (3.57) \]

Proof: In order to use stochastic calculus, we will come back to the process \( X_t^M \), so, with the same notations as before, and

\[ V_t^{(k)}(t) \equiv \left( V_t^{(k)} \right)_{|\alpha| \leq k} \]

with the convention \( V_t^{(0)} \equiv X_t^M \) (so \( V(0)(t) = X_t^M \)), \( V_t^{(k)} \) is then defined as a solution of the following SDE (where \( k = |\alpha|) :

\[ V_t^{(k)} = V_0^{(k)} + \int_0^t G^{(k)}(V_{k-1}(s)) dW_s + \sum_{j=0}^{J^M} d_{T_j}^k \left( Z_j, V_{k-1}(T_j^-) \right) \mathbb{1}_{\{U_j \leq \gamma(Z_j, X_{T_j^-}^M)\}} + \int_0^t g^\alpha(V_{k-1}(s)) ds \]

\[ + \sum_{l=1}^{M} \int_0^t \rho^\alpha_l(V_s^0) V_s^{\alpha_l} dW_s + \sum_{j=0}^{J^M} b^\alpha(Z_j, V_{T_j}^-) \mathbb{1}_{\{U_j \leq \gamma(Z_j, X_{T_j}^-)\}} V_T^{(k)} + \int_0^t b^\alpha(V_s^0) V_s^{(k)} ds \]

In order to express the first compound Poisson process with an integral with respect to the Poisson measure, we put (with the convention \( T_0 = 0) \)

\[ e_s^\alpha(z,v) \equiv \sum_{j=1}^{\infty} \mathbb{1}_{\{T_{j-1}, T_j\}}(s) d_{T_j}^k (z,v) \]

from (3.56) it is clear that,

\[ |e_s^\alpha(z,v)| \leq K\gamma(z)(1 + |v|)^w. \quad (3.59) \]

Therefore

\[ V_t^{(k)} = V_0^{(k)} + \int_0^t G^{(k)}(V_{k-1}(s)) dW_s + \sum_{j=0}^{J^M} e_{T_j}^\alpha \left( Z_j, V_{k-1}(T_j^-) \right) \mathbb{1}_{\{U_j \leq \gamma(Z_j, X_{T_j}^-)\}} + \int_0^t g^\alpha(V_{k-1}(s)) ds \]

\[ + \sum_{l=1}^{M} \int_0^t \rho^\alpha_l(V_s^0) V_s^{\alpha_l} dW_s + \sum_{j=0}^{J^M} b^\alpha(Z_j, V_{T_j}^-) \mathbb{1}_{\{U_j \leq \gamma(Z_j, X_{T_j}^-)\}} V_T^{(k)} + \int_0^t b^\alpha(V_s^0) V_s^{(k)} ds \]

\[ = V_0^{(k)} + \int_0^t G^{(k)}(V_{k-1}(s)) dW_s + \int_0^t \int_{E} e_s^\alpha(z, V_{k-1}(s^-)) \mathbb{1}_{\{U \leq \gamma(z, X_{s^-}^M)\}} N(ds, dz, du) + \int_0^t g^\alpha(V_{k-1}(s)) ds \]

\[ + \sum_{l=1}^{M} \int_0^t \rho^\alpha_l(V_s^0) V_s^{\alpha_l} dW_s + \int_0^t \beta^\alpha(z, V_{s^-}) \mathbb{1}_{\{U \leq \gamma(z, X_{s^-}^M)\}} V_{s^-} N(ds, dz, du) + \int_0^t b^\alpha(V_s^0) V_s^{(k)} ds \]

From Lemma 2.8, \( V_t^{(k)} \) and \( V_t^{(0)} \) are sharing the same law, so we will prove that, for all \( k \geq 0 \),

\[ \mathbb{E} \left[ \sup_{\alpha \in A} \mathbb{E}_t \left[ |V_t^{(k)}|^{2p} \right] \right] \leq C_{T,p} \]

which will be shown by recurrence on \( k \).
For \( k = 0 \), \( V_0^\alpha = X^M_t \), it is the Proposition B.1.

For \( k \geq 1 \), we first simplify the notations by writing:

\[
G^\alpha(s) \overset{\text{def}}{=} G^\alpha(V_{(k-1)}(s)), \quad g^\alpha(s) \overset{\text{def}}{=} g^\alpha(V_{(k-1)}(s)), \quad \rho^\alpha(s) \overset{\text{def}}{=} \rho^\alpha(V^0_s), \quad b^\alpha(s) \overset{\text{def}}{=} b^\alpha(V^0_s), \\
h^\alpha(s^-, z, u) \overset{\text{def}}{=} e^\alpha_s(z, V_{(k-1)}(s^-)) \mathbb{I}_{(s^-(z, X^M_s))}, \quad \beta^\alpha(s^-, z, u) \overset{\text{def}}{=} \beta^\alpha(z, V^0_s) \mathbb{I}_{(s^-(z, X^M_s))},
\]

which gives

\[
V^\alpha_t = V^0_0 + \int_0^t G^\alpha(s) \, dW_s + \int_0^t \int E h^\alpha(s^-, z, u) N(ds, dz, du) + \int_0^t g^\alpha(s) \, ds + \sum_{l=1}^m \int_0^t \rho^\alpha_l(s) V^\alpha_s \, dW^l_s + \int_0^t \int E \beta^\alpha(s^-, z, u) V^\alpha_s^- N(ds, dz, du) + \int_0^t b^\alpha(s) V^\alpha_s \, ds.
\]

In order to use the recurrence hypothesis, we bound the coefficients of this last equation in the following way: with

\[
\overline{\ell}_k(s) \overset{\text{def}}{=} K(1 + |V_{(k-1)}(s)|)^w
\]

and

\[
\overline{\xi}(z, u) \overset{\text{def}}{=} (\overline{c}(z) + \overline{\beta}(z)) \mathbb{I}_{(z \leq \overline{\tau}(z))}
\]

we have (according to the hypothesis 3.1)

\[
\sup_{\alpha \in A} G^\alpha(s) \leq \overline{\ell}_k(s) \quad \text{and} \quad \overline{\ell}_k(s) \overset{\text{def}}{=} \sup_{\alpha \in A} g^\alpha(s) \leq \overline{\ell}_k(s)
\]

and (using (3.59) and (3.63))

\[
\forall s, z, u, \ |h^\alpha(s^-, z, u)| \leq \overline{h}_k(s^-) \overline{\xi}(z, u) \quad \text{and} \quad |\beta^\alpha(s^-, z, u)| \leq \overline{\xi}(z, u). \tag{3.61}
\]

At last, we have to notice, using the recurrence hypothesis for \( k - 1 \), that, for all \( n \in \mathbb{N}^* \),

\[
\sup_{0 \leq s \leq T} \mathbb{E} \left[ |\overline{\ell}_k(s)|^n \right] < \infty; \tag{3.62}
\]

this last result comes directly from the following bounding:

\[
\mathbb{E} \left[ |V_{(k-1)}(s)|^{2m} \right] = \mathbb{E} \left[ E_W \left[ |V_{(k-1)}(s)|^{2m} \right] \right] \leq K \mathbb{E} \left[ \sum_{\beta \in A_{k-1}} E_W \left[ |V^\beta|^{2m} \right] \right] \leq K' \sqrt{\mathbb{E} \left[ \left( \sup_{\beta \in A_{k-1}} E_W \left[ |V^\beta|^{2m} \right] \right)^2 \right]} \leq K' \sqrt{C_{T, m, k-1}} < \infty.
\]

**Step 1**

In order to use the Itô’s formula, we will first have to localize our problem by using the sequence \((\tau^M_K)_K \in \mathbb{N}^*\) of stopping times defined by

\[
\tau^M_K(k) \overset{\text{def}}{=} \inf\{t > 0 : \sup_{s \leq t} \sum_{|\alpha| \leq k} |V^\alpha_s| \geq K \}. \tag{3.63}
\]

Let us prove that a.s. \( \lim_{K \to \infty} \tau^M_K = \infty \).

From the hypothesis made on the coefficients of \( V_0^\alpha \), it is clear that, for all \( t \geq 0 \),

\[
\sum_{|\alpha| \leq k} \mathbb{E} \left[ \sup_{s \leq t} |V^\alpha_s| \right] \leq \infty. \tag{3.64}
\]
We have, for $t \geq 0$

$$
\lim_{K \to \infty} \Pr(\tau^M_K < t) = \lim_{K \to \infty} \Pr(\sup_{s \leq t} \sum_{|\alpha| \leq k} |V_{s\alpha}^\alpha| > K)
\leq \lim_{K \to \infty} \frac{1}{K} E\left[ \sup_{s \leq t} \sum_{|\alpha| \leq k} |V_{s\alpha}^\alpha| \right] = 0.
$$

$(\tau^M_K)_{K \in \mathbb{N}^*}$ tends to $\infty$ in probability and so, there exists a subsequence (that we will continue to denote by $(\tau^M_K)_{K \in \mathbb{N}^*}$) which tends to $\infty$ a.s.

In this case we have

$$
|V_{t\alpha}^\alpha|^2 \mathbb{1}_{\tau^M_K > t} \uparrow |V_{t\alpha}^\alpha|^2 \text{ a.s.}
$$

so

$$
E_W \left[ |V_{t\alpha}^\alpha|^2 \mathbb{1}_{\tau^M_K > t} \right] \uparrow E_W \left[ |V_{t\alpha}^\alpha|^2 \right] \text{ a.s.}
$$

and

$$
\sup_{\alpha \in A} E_W \left[ |V_{t\alpha}^\alpha|^2 \mathbb{1}_{\tau^M_K > t} \right] \uparrow \sup_{\alpha \in A} E_W \left[ |V_{t\alpha}^\alpha|^2 \right] \text{ a.s.}
$$

If we admit for the moment that there exists a constant $C_{p,T,k}$ which does not depend on $K$ and $M$ and such that, for all $0 \leq t \leq T$,

$$
E \left[ \sup_{\alpha \in A} E_W \left[ |V_{t\wedge \tau^M_K}^\alpha|^2 \right] \right] \leq C_{T,p,k}
$$

(3.65)

The monotone convergence theorem implies then

$$
E \left[ \sup_{\alpha \in A} E_W \left[ |V_{t\alpha}^\alpha|^2 \right] \right] = \sup_K E \left[ \sup_{\alpha \in A} E_W \left[ |V_{t\wedge \tau^M_K}^\alpha|^2 \right] \right] = \sup_K E \left[ \sup_{\alpha \in A} E_W \left[ |V_{t\wedge \tau^M_K}^\alpha|^2 \right] \right] \leq C_{T,p,k}.
$$

Step 2

We have to establish now (3.65).

For a single component we have (omitting for a moment the parameter $\alpha$ in order to simplify the notations)

$$
V^i_t = V^i_0 + \sum_{l=1}^{d} \int_0^t G^i_l(s) \, dW^l_s + \int_0^t \int_E h^i(s^-, z, u) N(ds, dz, du) + \int_0^t g^i(s) \, ds
$$

$$
+ \sum_{l=1}^{d} \int_0^t \int_E \rho^i_{ih}(s) V^h_s \, dW^l_s + \sum_{h=1}^{d} \int_0^t \int_E \beta^i_{ih}(s^-, z, u) V^h_s \, N(ds, dz, du) + \sum_{h=1}^{d} \int_0^t b^i_{ih}(s) V^h_s \, ds
$$

Then, applying Itô’s formula with $f(x) = x^{2p}$

$$
(V_{t\wedge \tau^M_K}^i)^{2p} = (V^i_0)^{2p} + \sum_{l=1}^{d} \int_0^{t\wedge \tau^M_K} 2p (V^i_s)^{2p-1} \left( G^i_l(s) + \sum_{h=1}^{d} \rho^i_{ih}(s) V^h_s \right) \, dW^l_s
$$

$$
+ 2p \int_0^{t\wedge \tau^M_K} (V^i_s)^{2p-1} \left( g^i(s) + \sum_{h=1}^{d} b^i_{ih}(s) V^h_s \right) \, ds
$$

$$
+ p(2p-1) \sum_{l=1}^{d} \int_0^{t\wedge \tau^M_K} (V^i_s)^{2p-2} \left( G^i_l(s) + \sum_{h=1}^{d} \rho^i_{ih}(s) V^h_s \right)^2 \, ds
$$

$$
+ \int_0^{t\wedge \tau^M_K} \int_E \left( V^i_{s^+} + h^i(s^-, z, u) + \sum_{h=1}^{d} \beta^i_{ih}(s^-, z, u) V^h_{s^-} \right)^{2p} \, N(ds, dz, du)
$$

- (V^i_{s^-})^{2p} N(ds, dz, du)
We take now the expectation with respect to the Brownian motion (i.e. conditionally with respect to all the others random quantities):

\[
E_W \left[ (V^i_{t \wedge \tau^M_K})^{2p} \right] = E_W \left[ (V^i_0)^{2p} \right] + 2p \int_0^{t \wedge \tau^M_K} E_W \left[ (V^i_s)^{2p-1} \left( g_i(s) + \sum_{h=1}^{d} b_{ih}(s) V^h_s \right) \right] \, ds \\
+ p(2p-1) \sum_{l=1}^{m} \int_0^{t \wedge \tau^M_K} E_W \left[ (V^i_s)^{2p-2} \left( G_{id}(s) + \sum_{h=1}^{d} \rho_{ih}(s) V^h_s \right)^2 \right] \, ds \\
+ \int_0^{t \wedge \tau^M_K} \int_E E_W \left[ \left( (V^i_{s^-} + h_i(s^-, z, u) + \sum_{h=1}^{d} \beta_{ih}(s^-, z, u) V^h_{s^-})^{2p} - (V^i_{s^-})^{2p} \right) \right] N(ds, dz, du).
\]

Since \( s < t \wedge \tau^M_K \), we have \( X_s = X_{s \wedge \tau^M_K} \), and obviously \( t < t \wedge \tau^M_K \), so we have

\[
E_W \left[ (V^i_{t \wedge \tau^M_K})^{2p} \right] = E_W \left[ (V^i_0)^{2p} \right] + 2p \int_0^{t} E_W \left[ (V^i_{s \wedge \tau^M_K})^{2p-1} \left( g_i(s) + \sum_{h=1}^{d} b_{ih}(s) V^h_{s \wedge \tau^M_K} \right) \right] \, ds \\
+ p(2p-1) \sum_{l=1}^{m} \int_0^{t} E_W \left[ (V^i_{s \wedge \tau^M_K})^{2p-2} \left( G_{id}(s) + \sum_{h=1}^{d} \rho_{ih}(s) V^h_{s \wedge \tau^M_K} \right)^2 \right] \, ds \\
+ \int_0^{t \wedge \tau^M_K} \int_E E_W \left[ \left( (V^i_{s^-} + h_i(s^-, z, u) + \sum_{h=1}^{d} \beta_{ih}(s^-, z, u) V^h_{s^-})^{2p} - (V^i_{s^-})^{2p} \right) \right] N(ds, dz, du).
\]

Since

\[
\left| \left( V^i_{s^-} + h_i(s^-, z, u) + \sum_{h=1}^{d} \beta_{ih}(s^-, z, u) V^h_{s^-} \right)^{2p} - (V^i_{s^-})^{2p} \right| \\
= \left| \left( \sum_{k=1}^{2p} \binom{2p}{k} \left( h_i(s^-, z, u) + \sum_{h=1}^{d} \beta_{ih}(s^-, z, u) V^h_{s^-} \right)^k V^i_{s^-}^{2p-k} \right) \right| \\
\leq \left| \sum_{k=1}^{2p} \binom{2p}{k} \left( \left| h_i(s^-, z, u) + \beta(s^-, z, u) \right| V^i|s^-|^{2p-k} \right) \right| \\
= \left( \left| V^i|s^-| + h_i(s^-, z, u) + \beta(s^-, z, u) \right| V^i|s^-|^{2p} - (V^i|s^-|) \right) \\
\]

it follows (with \( \rho \overset{\text{def}}{=} \sup_t |\rho_t| \))

\[
E_W \left[ (V^i_{t \wedge \tau^M_K})^{2p} \right] \leq E_W \left[ (V^i_0)^{2p} \right] + 2p \int_0^{t \wedge \tau^M_K} E_W \left[ (V^i_s)^{2p-1} \left( g_i(s) + |b|(s)|V^i_s| \right) \right] \, ds \\
+ p(2p-1) \int_0^{t \wedge \tau^M_K} E_W \left[ (V^i_s)^{2p-2} \left( G(s) + \rho(s)|V^i_s| \right) \right] \, ds \\
+ \int_0^{t \wedge \tau^M_K} \int_E E_W \left[ \left( |V^i|s^-| + h_i(s^-, z, u) + \beta(s^-, z, u) \right| V^i|s^-|^{2p} - (V^i|s^-|) \right] N(ds, dz, du).
\]

Then (writing again from now on the parameter \( \alpha \), in order to see clearly which components depend of it or not), we have (using, among others things, the inequality (3.61))

\[
E_W \left[ (V^\alpha_{t \wedge \tau^M_K})^{2p} \right] \leq E_W \left[ (V^\alpha_0)^{2p} \right] + 2p \int_0^{t \wedge \tau^M_K} E_W \left[ (V^\alpha_s)^{2p-1} (\bar{h}_k(s) + \bar{b}|V^\alpha_s|) \right] \, ds \\
+ p(2p-1) \int_0^{t \wedge \tau^M_K} E_W \left[ (V^\alpha_s)^{2p-2} (\bar{h}_k(s) + \bar{b}|V^\alpha_s|) \right] \, ds \\
+ \int_0^{t \wedge \tau^M_K} \int_E E_W \left[ \left( |V^\alpha|s^-| + \bar{h}_k(s^-) \bar{z}(z, u) + \bar{z}(z, u) |V^\alpha|s^-| \right|^{2p} - |V^\alpha|s^-|^{2p} \right] N(ds, dz, du).
\]

(Notice that, since \( V^\alpha \) is an adapted process, \( |V^\alpha_0| \) is a constant, so \( E_W \left[ (V^\alpha_0)^{2p} \right] = |V^\alpha_0|^{2p} \).)

To bound the first integral, using the elementary inequality

\[
\forall x, y \geq 0, \forall u, v > 0 \quad x^u y^v \leq x^{u+v} + y^{u+v},
\]

(3.66)
we notice that:
\[
||V_{s}^{α}|^2|^{2p-1} \lambda_k(s) + \overline{b}|V_{s}^{α}|^{2p} \leq \lambda_k(s)^{2p} + |V_{s}^{α}|^{2p} + \overline{b}|V_{s}^{α}|^{2p} \\
= \lambda_k(s)^{2p} + (1 + \overline{b})|V_{s}^{α}|^{2p},
\]
and, similarly, for the second one,
\[
|V_{s}^{α}|^{2p-2}(\lambda_k(s) + \overline{p}|V_{s}^{α}|)^{2} \leq 2\lambda_k(s)^{2p} + 2|V_{s}^{α}|^{2p} + 2\overline{p}^2|V_{s}^{α}|^{2p} \\
= 2\lambda_k(s)^{2p} + 2(1 + \overline{p}^2)|V_{s}^{α}|^{2p}.
\]

It follows that,
\[
2p \int_{0}^{t \wedge \tau_k^M} E_W[|V_{s}^{α}|^{2p-1}(\lambda_k(s) + \overline{b}|V_{s}^{α}|)] \, ds + p(2p - 1) \int_{0}^{t \wedge \tau_k^M} E_W[|V_{s}^{α}|^{2p-2}(\lambda_k(s) + \overline{p}|V_{s}^{α}|)] \, ds \\
\leq C_p \int_{0}^{t \wedge \tau_k^M} E_W[\lambda_k(s)^{2p}] \, ds + C'_p \int_{0}^{t \wedge \tau_k^M} E_W[|V_{s}^{α}|^{2p}] \, ds.
\]

For the third integral, we will see that
\[
(|V_{s}^{α}| + \lambda_k(s^-)\xi(z, u) + \xi(z, u)|V_{s}^{α}|)^{2p} - |V_{s}^{α}|^{2p} \leq \xi(z, u)P(\xi(z, u))(|V_{s}^{α}|^{2p} + (\lambda_k(s^-))^2)
\]
where \(P\) is a polynomial function.

Let us prove now (3.67) : if \(u, v \geq 0\),
\[
|u^{2p} - v^{2p}| \leq |u - v|(u + v)^{2p-1},
\]
so, it follows that, for \(a, b, c \geq 0\):
\[
(a + c(b + a))^{2p} - a^{2p} \leq c(b + a)(a(2 + c) + cb)^{2p-1} \\
\leq 2^{2p-1}c(b + a)(a^{2p-1}(2 + c)^{2p-1} + (cb)^{2p-1}) \\
\leq 2^{2p-1}c(a^{2p}(2 + c)^{2p-1} + a(cb)^{2p-1} + ba^{2p-1}(2 + c)^{2p-1} + c^{2p-1}b^{2p})
\]
using (3.66), we have
\[
a(cb)^{2p-1} \leq a^{2p} + (cb)^{2p} \quad \text{and} \quad ba^{2p-1}(2 + c)^{2p-1} \leq a^{2p} + b^{2p}(2 + c)^{2p-1}
\]
which brings to
\[
(a + c(b + a))^{2p} - a^{2p} \leq 2^{2p-1}c[a^{2p}(2 + (2 + c)^{2p-1}) + b^{2p}(c^{2p-1} + c^{2p} + (2 + c)^{2p-1})]
\]
or, more generally, to
\[
(a + c(b + a))^{2p} - a^{2p} \leq cP(c)[a^{2p} + b^{2p}],
\]
where \(P \in \mathbb{R}[X]\), which proves (3.67).

Gathering all those results,
\[
E_W[|V_{t \wedge \tau_k^M}^{α}|^{2p}] \leq E_W[|V_{0}^{α}|^{2p}] + C_p \int_{0}^{t \wedge \tau_k^M} E_W[\lambda_k(s)^{2p}] \, ds + C'_p \int_{0}^{t \wedge \tau_k^M} E_W[|V_{s}^{α}|^{2p}] \, ds \\
+ \int_{0}^{t \wedge \tau_k^M} \int_{E} P(\xi) E_W[|\lambda_k(s^-)|^{2p}]N(ds, dz, du) + \int_{0}^{t \wedge \tau_k^M} \int_{E} P(\xi) E_W[|V_{s}^{α}|^{2p}]N(ds, dz, du).
\]

We then have, directly, with \(\Theta_{p,k}(t \wedge \tau_k^M) \overset{\text{def}}{=} \sup_{|α| \leq k} E_W[|V_{t \wedge \tau_k^M}^{α}|^{2p}]\),
\[
\Theta_{p,k}(t \wedge \tau_k^M) \leq \Theta_{p,k}(0) + C_p \int_{0}^{t \wedge \tau_k^M} E_W[\lambda_k(s)^{2p}] \, ds + C'_p \int_{0}^{t \wedge \tau_k^M} \Theta_{p,k}(s) \, ds \\
+ \int_{0}^{t \wedge \tau_k^M} \int_{E} P(\xi) E_W[|\lambda_k(s^-)|^{2p}]N(ds, dz, du) + \int_{0}^{t \wedge \tau_k^M} \int_{E} P(\xi) \Theta_{p,k}(s-)N(ds, dz, du),
\]
(3.70)
Since $s \leq t \wedge \tau_K^M$, we have, for any process $Y$, $Y_s = Y_{s \wedge \tau_K^M}$, and obviously $t \leq t \wedge \tau_K^M$, so we have

$$\Theta_{p,k}(t \wedge \tau_K^M) \leq \Theta_{p,k}(0) + C_p \int_0^t E[W[h_k(s \wedge \tau_K^M)^{2p}]] ds + C'_p \int_0^t \Theta_{p,k}(s \wedge \tau_K^M) ds$$

$$+ \int_0^t \int_B \xi P(\xi) E[h_k^{2p}(s \wedge \tau_K^M)] N(ds, dz, du) + \int_0^t \int_B \xi P(\xi) \Theta_{p,k}(s \wedge \tau_K^M) N(ds, dz, du)$$

(3.71)

**Step 2**

The last step is to bound

$$E[\Theta_{p,k}(t)].$$

By setting

$$R_1 \overset{\text{def}}{=} \int E[\xi P(\xi) du \mu(dz)]$$

we have (using the isometry in $F^I$, cf. (??))

$$E[\Theta_{p,k}(t \wedge \tau_K^M)]$$

$$\leq E[\Theta_{p,k}(0)] + C_p E\left[ \int_0^t E[W[h_k(s \wedge \tau_K^M)^{2p}]] ds \right] + C'_p E\left[ \int_0^t \Theta_{p,k}(s \wedge \tau_K^M) ds \right]$$

$$+ E\left[ \int_0^t \int_B \xi P(\xi) E[h_k^{2p}(s \wedge \tau_K^M)] N(ds, dz, du) \right] + E\left[ \int_0^t \int_B \xi P(\xi) \Theta_{p,k}(s \wedge \tau_K^M) N(ds, dz, du) \right]$$

$$= E[\Theta_{p,k}(0)] + C_p \int_0^t E[h_k(s \wedge \tau_K^M)^{2p}] ds + C'_p \int_0^t E[\Theta_{p,k}(s \wedge \tau_K^M)] ds$$

$$+ R_1 \int_0^t E[h_k^{2p}(s \wedge \tau_K^M)] ds + \int_0^t E[\Theta_{p,k}(s \wedge \tau_K^M)] ds$$

$$= E[\Theta_{p,k}(0)] + (C_p + R_1) \int_0^t E[h_k(s \wedge \tau_K^M)^{2p}] ds + (C'_p + R_1) \int_0^t E[\Theta_{p,k}(s \wedge \tau_K^M)] ds$$

With $A_p(T) \overset{\text{def}}{=} E[\Theta_{p,k}(0)] + (C_p + R_1) \int_0^T E[h_k(s)^{2p}] ds$, which, by virtue of (3.62), is a finite quantity, the Gronwall's lemma gives here:

$$E[\Theta_{p,k}(t \wedge \tau_K^M)] \leq A_p(T) \exp((C_p + R_1) t)$$

which proves the assertion (3.65).

To bound (in $L^p$, $p \geq 1$) the Sobolev norm $|X_t^M|$, we will proceed by recurrence on $l \in \mathbb{N}^*$, and we will show in detail the case corresponding to the first order norm, since in this particular case, the structure of the general method already appears with lesser notations than used in the general case.

In all the following we will set

$$A_k(L) \overset{\text{def}}{=} ([1, L] \times [1, d])^k$$

and

$$A(L) \overset{\text{def}}{=} \bigcup_{k \in \mathbb{N}^*} A_k(L).$$

We will also need the following lemma’s:

**Lemma 3.2** Let $n, p \geq 1$ and $F \in \mathcal{S}^d$. Then

$$E[|D^n F|^{2p}] \leq d^{np} \sqrt{E[(J_t^M)^{2np}]} \sup_L \sqrt{E\left[ \max_{\alpha \in A_n(L)} E_W \left[ |D_\alpha F|^{4p} \right] \right]}.$$

**Proof:** By definition

$$E[|D^n F|^{2p}] = E\left[ \sum_{\alpha \in ([1, J_t^M] \times [1, d])^n} |D_\alpha F|^2 \right]^{p}.$$
on the other hand
\[
E \left[ |D^n F|^{2p} \right] = \sup_L E \left[ |D^n F|^{2p} \mathbb{1}_{J^M \leq L} \right]
\]
and
\[
E \left[ |D^n F|^{2p} \mathbb{1}_{J^M \leq L} \right] = E \left[ \left( \sum_{\alpha \in ([1,J^M] \times [1,d]^n)} |D_\alpha F|^2 \right)^p \mathbb{1}_{J^M \leq L} \right]
\[
\leq E \left[ (dJ)^{(p-1)} \sum_{\alpha \in A_\alpha(L)} |D_\alpha F|^{2p} \prod_{i=1}^n \mathbb{1}_{k_i \leq J^M} \right]
\[
\leq E \left[ (dJ)^{(p-1)} \sum_{\alpha \in A_\alpha(L)} E_W \left[ |D_\alpha F|^{2p} \right] \right]
\[
\leq E \left[ (dJ)^{2p} \max_{\alpha \in A_\alpha(L)} E_W \left[ |D_\alpha F|^{2p} \right] \right]
\[
\leq d^{np} \sqrt{E \left[ (J^M)^{2np} \right]} \sqrt{E \left[ \max_{\alpha \in A_\alpha(L)} E_W \left[ |D_\alpha F|^{4p} \right] \right]}.
\]

Lemma 3.3 Let \( j \geq 1 \). Then there exists \( C_{|\alpha|} > 0 \) such that
\[
|D_\alpha \tilde{Z}_j| \leq C_{|\alpha|}.
\]

Proof: For \( |\alpha| = 1 \), \( \alpha = (k, r) \) and (recalling that \( \pi_k^r = \phi_M(\tilde{Z}_k) \))
\[
D_{k,r} \tilde{Z}_j = \begin{pmatrix}
\pi_k^r \partial_{\tilde{Z}_k} Z_j^1 \\
\vdots \\
\pi_k^r \partial_{\tilde{Z}_k} Z_j^r \\
\end{pmatrix} = \begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
\]
(3.72)

so \( |D_{k,r} \tilde{Z}_j| \leq \|\phi\|_\infty = 1 \).

Since \( \phi \) has bounded derivatives of any order, the recursive differentiation of (3.72) gives the general bounding property; although this recursive differentiation is rather clear, we show the case \( |\alpha| = 2 \), to highlight the mechanism of it:

let \( \alpha = ((m,n)(k,r)) \), then (using (3.72)),
\[
D_\alpha \tilde{Z}_j = D_{m,n} D_{k,r} \tilde{Z}_j = \begin{pmatrix}
\sum_{\beta} c \prod_{\beta \in (\tilde{Z}_i)} \partial_{\tilde{Z}_j} \phi_M(\tilde{Z}_k) \delta_{k,j}
\end{pmatrix}
\]
with
\[
\pi_m^r \partial_{\tilde{Z}_m} \phi_M(\tilde{Z}_k) \delta_{k,j} = \phi_M(\tilde{Z}_m) \delta_{m,k} \delta_{m,k} \delta_{k,j},
\]
and a derivative of higher order will be of the following form:
\[
D_\alpha \tilde{Z}_j = \begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
\]
so (since the sum and the product are finite),
\[
|D_\alpha \tilde{Z}_j| \leq \sum c \prod \|\partial_{\tilde{Z}_j} \phi\|_\infty \overset{\text{def}}{=} C_{|\alpha|}.
\]

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Recalling the following notation (cf. subsection 1.5.1) : for $1 \leq l \leq n$,
\[ \mathcal{M}_n(l) \overset{\text{def}}{=} \left\{ M = (M_1, \ldots, M_l), \bigcup_{i \in [1,l]} M_i = \{1, \ldots, n\} \text{ and } M_i \cap M_j = \emptyset, \text{ for } i \neq j \right\}, \]
we have, in fact, a more precise result :

**Lemma 3.4** Let $k \geq 1$ and $\alpha = ((k_n, r_n), \ldots, (k_1, r_1))$.
\[ D_{\alpha}^\varnothing \bar{Z}_k = \delta_{k_n,k} \cdots \delta_{k_1,k} \delta_{r_1,r_1} \varnothing f_{\alpha}(\bar{Z}_k) \tag{3.73} \]
where (with $r \overset{\text{def}}{=} (r_n, \ldots, r_2)$)
\[ f_{\alpha}(\bar{Z}_k) \overset{\text{def}}{=} \sum_{\beta = (\beta_1, \ldots, \beta_n) \in \mathcal{M}_{n-1}(n)} c_{\beta} \prod_{i=1}^{n} \partial_{\beta_i(r_i)} \theta_M(\bar{Z}_k) \tag{3.74} \]
with
\begin{itemize}
  \item $c_{\beta} \in \mathbb{N}$ ;
  \item we denote again $\mathcal{M}_{n-1}(n)$, but, here, we allow $\beta_i$ to be empty.
\end{itemize}

**Proof** : By induction over the length $k$ of $\alpha$.

3.3 First order norm

**Proposition 3.5** If $p \geq 1$, for all $T > 0$, exists a constant $C_{T,p} > 0$ such that
\[ \forall t \in [0,T], \quad \|D^1\bar{X}_t\|_{2p} \leq C_{T,p} \sqrt{\|J_{T}^M\|_{2p}}. \tag{3.75} \]

**Proof** :
First, from Lemma 3.2, we have
\[ E \left[ \|D^1\bar{X}_t^M\|_{2p}^2 \right] \leq d^p \sqrt{E \left[ (\|J_{T}^M\|_{2p})^3 \right]} \sup_L \sqrt{E \left[ \max_{\alpha \in A(L)} E_W [\|D_{\alpha} \bar{X}_t^M\|_{4p}] \right]} \]
Hence, to conclude, it remains to bound, independently of $L$, the quantity $E \left[ \max_{\alpha \in A_L(L)} E_W [\|D_{\alpha} \bar{X}_t^M\|_{4p}] \right]$. Recalling that $\bar{X}_t^M$ solves the following diffusion equation :
\[ \bar{X}_t^M = x + \int_0^t \sigma(\bar{X}_s^M) \, dW_s + \sum_{j=1}^{J_{M}} c_M(\bar{Z}_j, \bar{X}_{T_j^-}^M) + \int_0^t g(\bar{X}_s^M) \, ds, \]
we have
\[ D_{k,r} \bar{X}_t = \nabla_x c_M(\bar{Z}_k, \bar{X}_{T_k^-}^M) \ D_{k,r} \bar{Z}_k + \sum_{l=1}^{J_{M}} \int_{T_k}^{t} \nabla \sigma_l(\bar{X}_s^M) \, D_{k,r} \bar{X}_s^M \, dW_s^l \]
\[ + \sum_{j=k+1}^{m} \nabla_x c_M(\bar{Z}_j, \bar{X}_{T_j^-}^M) \ D_{k,r} \bar{X}_{T_j^-}^M \]
\[ + \int_{T_k}^{t} \nabla g(\bar{X}_s^M) \, D_{k,r} \bar{X}_s^M \, ds. \]
We can then apply the bounding Lemma 3.1 with $\alpha = (k, r)$ and $\bar{V}_t^\alpha \overset{\text{def}}{=} D_{k,r} \bar{X}_t$, since the Hypothesis 3.1 are well verified, for we have ($\alpha = (k, r)$)
\[ G^\alpha = 0, \quad d^\varnothing_j (\bar{Z}_j, \nabla_{(0)}(T_j^-)) = \partial_x c_M(\bar{Z}_j, \bar{X}_{T_j^-}^M) \Phi_M(\bar{Z}_j) \delta_{k,j}, \quad g^\alpha = 0 \]
\[ \rho_{x}^\alpha (\bar{V}_s^\alpha) = \nabla \sigma_l(\bar{X}_s^M), \quad \beta^\alpha(\bar{Z}_j, \bar{V}_{T_j}^\alpha) = \nabla_x c_M(\bar{Z}_j, \bar{X}_{T_j^-}^M) \ D_{k,r} \bar{X}_{T_j^-}^M, \]
\[ \beta^\varnothing = 0, \quad \rho_{x}^\varnothing(\bar{V}_s^\varnothing) = \nabla \sigma_l(\bar{X}_s^M), \quad \gamma^\varnothing = 0 \]

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If Proposition 3.6 which completes the proof.

So, if we admit for a moment that, for
and, consequently, there exists a constant
order to have
Following the very same path as before, we find, recalling that
3.4 Norm of higher order
which proves the proposition.

Proof : We have
Lemma 3.7
Let
which will be done by recurrence on
Proof :

\[ E \left[ |\bar{X}_t^M|^{2p} \right] \leq C_p \sum_{k=0}^{l} E \left[ |D^k \bar{X}_t^M|^{2p} \right] \]

and, using Lemma 3.2,

\[ C_p \sum_{k=0}^{l} E \left[ |D^k \bar{X}_t^M|^{2p} \right] \leq C_p \sum_{k=0}^{l} d^{kp} \sqrt{E \left[ (I_M^{2kp}) \sup_L E \left[ \max_{\alpha \in A_k(L)} E_W \left[ |D_\alpha \bar{X}_t^M|^{4p} \right] \right) \right]}. \]

So, if we admit for a moment that, for \( k \in [0, l] \), \( \sup_L E \left[ \max_{\alpha \in A_k(L)} E_W \left[ |D_\alpha \bar{X}_t^M|^{4p} \right] \right] \leq C_{k,p} \), then

\[ E \left[ |\bar{X}_t^M|^{2p} \right] \leq C_p \sum_{k=0}^{l} d^{kp} \sqrt{E \left[ (I_M^{2kp}) C_{k,p} \right] \leq C_p (1 + \sqrt{E \left[ (I_M^{2lp}) \right]}) \]  \]

which proves the proposition.

Hence, to conclude, it remains to bound, independently of \( L \), the quantity \( E \left[ \max_{\alpha \in A_k(L)} E_W \left[ |D_\alpha \bar{X}_t^M|^{4p} \right] \right] \), which will be done by recurrence on \( |\alpha| \) in the next lemma.

\[ \sup_L E \left[ \max_{\alpha \in A(L)} \left| D_\alpha \bar{X}_t^M \right|^{2p} \right] \leq C_{n,p}. \]

Proof :

The case \( |\alpha| = 1 \) is corresponding to the first order norm case.
Else, starting again from

\[ \bar{X}_t^M = x + \int_0^t \sigma(X_s^M) \, dW_s + \sum_{j=1}^{J_t^M} c_M(\bar{Z}_j, \bar{X}_{t_j^-}^M) + \int_0^t g(\bar{X}_s^M) \, ds, \]

\[ 2 \text{We note, indeed, that } J_t^M \sim P(t \lambda_M) \text{ which implies } E \left[ J_t^{M,n} \right] = P(t \lambda_M), \text{ where } P \text{ is polynomial of degree } n : \text{ when } M \text{ is growing, (for } t \leq T), \text{ we have } E \left[ J_t^{M,n} \right] = O(\lambda_M^n). \]
we have (using Lemma 1.6, with

$$
\sum_{l=2}^{k} \sum_{\beta_{1}=\langle \beta_{1}, \ldots, \beta_{l} \rangle} \sum_{M \in M_{k}(l)} \text{ and } \sum_{l=1}^{k} \sum_{\beta_{1}=\langle \beta_{1}, \ldots, \beta_{l}, \beta'_{1}, \ldots, \beta'_{l} \rangle} \sum_{M \in M_{k}(l)} \text{,}
$$

where \( k \overset{\text{def}}{=} |\alpha| \), in order to shorten the equation)

$$
D_{\alpha} \hat{X}_{t}^{M} = \sum_{l=1}^{m} \int_{0}^{t} D_{\alpha}(\sigma_{l}(\hat{X}_{s}^{M})) dW_{s}^{l} + \sum_{j=1}^{l} D_{\alpha}(c_{M}(\hat{Z}_{j}, \hat{X}_{T_{j-}^{M}})) + \int_{0}^{t} D_{\alpha}(g(\hat{X}_{s}^{M})) ds
$$

$$
= \sum_{l=1}^{m} \int_{0}^{t} \sum_{(1)} \partial_{\beta} \sigma_{l}(\hat{X}_{s}^{M}) D_{M_{1}(\alpha)}(\hat{X}_{s}^{M})_{\beta_{1}} \cdots D_{M_{l}(\alpha)}(\hat{X}_{s}^{M})_{\beta_{l}} + \nabla \sigma_{l}(\hat{X}_{s}^{M}) D_{\alpha}(\hat{X}_{s}^{M}) dW_{s}^{l}
$$

$$
+ \sum_{j=1}^{l} \sum_{(2)} \partial_{\beta} c_{M}(\hat{Z}_{j}, \hat{X}_{T_{j-}^{M}}) D_{M_{1}(\alpha)}(\hat{Z}_{j}^{\beta_{1}}) \cdots D_{M_{l}(\alpha)}(\hat{Z}_{j}^{\beta_{l}}) \times D_{M_{r+1}(\alpha)}(\hat{X}_{T_{j-}^{M}})_{d-\beta'_{r+1}} \cdots D_{M_{l}(\alpha)}(\hat{X}_{T_{j-}^{M}})_{d-\beta'_{l}}
$$

$$
+ \nabla_{x} c_{M}(\hat{Z}_{j}, \hat{X}_{T_{j-}^{M}}) D_{\alpha}(\hat{Z}_{j}) + \nabla_{x} c_{M}(\hat{Z}_{j}, \hat{X}_{T_{j-}^{M}}) D_{\alpha}(\hat{X}_{T_{j-}^{M}})
$$

$$
+ \int_{0}^{t} \sum_{(1)} \partial_{\beta}(g(\hat{X}_{s}^{M})) D_{M_{1}(\alpha)}(\hat{X}_{s}^{M})_{\beta_{1}} \cdots D_{M_{l}(\alpha)}(\hat{X}_{s}^{M})_{\beta_{l}} + \nabla g(\hat{X}_{s}^{M}) D_{\alpha}(\hat{X}_{s}^{M}) ds
$$

(3.80)

Then we apply the upper bound Lemma 3.1 with \( \nabla_{x}^{\alpha} \overset{\text{def}}{=} D_{\alpha} \hat{X}_{t} \) (and consequently \( \nabla_{(k-1)}(t) = \left( D_{\beta} \hat{X}_{t} \right)_{|\beta|<|\alpha|} \)). Using Lemma 3.3, it follows that the Hypothesis 3.1 are well verified; for example :

$$
G_{t}^{\alpha}(\nabla_{(k-1)}(s)) = G_{t}^{\alpha}(\left( D_{\beta} \hat{X}_{t} \right)_{|\beta|<|\alpha|})
$$

$$
\overset{\text{def}}{=} \sum_{(1)} \partial_{\beta}(\sigma_{l}(\hat{X}_{s}^{M})) D_{M_{1}(\alpha)}(\hat{X}_{s}^{M})_{\beta_{1}} \cdots D_{M_{l}(\alpha)}(\hat{X}_{s}^{M})_{\beta_{l}}
$$

so there exists \( w \in \mathbb{N} \) such that

$$
|G^{\alpha}(v)| \leq K(1 + |v|)^{w};
$$

and

$$
d_{T}^{\alpha}(\hat{Z}_{j}, \nabla_{(k-1)}(T_{j}^{-}))
$$

$$
= d_{T}^{\alpha}(\hat{Z}_{j}, \left( D_{\beta} \hat{X}_{T_{j-}^{M}} \right)_{|\beta|<|\alpha|})
$$

$$
\overset{\text{def}}{=} \sum_{(2)} \partial_{\beta} c_{M}(\hat{Z}_{j}, \bar{X}_{T_{j-}^{M}}) D_{M_{1}(\alpha)}(\bar{Z}_{j}^{\beta_{1}}) \cdots D_{M_{l}(\alpha)}(\bar{Z}_{j}^{\beta_{l}}) \times D_{M_{r+1}(\alpha)}(\bar{X}_{T_{j-}^{M}})_{d-\beta'_{r+1}} \cdots D_{M_{l}(\alpha)}(\bar{X}_{T_{j-}^{M}})_{d-\beta'_{l}}.
$$

Notice that it is legitimate to consider each \( D_{M_{l}(\alpha)}(\bar{Z}_{j}^{\beta_{l}}) \) as a function of \( j \) and \( \bar{Z}_{j} \) since, using directly the Lemma 3.4

$$
D_{M_{l}(\alpha)}(\bar{Z}_{j}^{\beta_{l}}) = \delta_{(\alpha_{l}(\alpha))} \cdots \delta_{(\alpha_{l}(\alpha))} \delta_{(\alpha_{l}(\alpha))} f_{\alpha}(\bar{Z}_{j})
$$

where \( f_{\alpha} \) is defined in (3.74).

Since, from the Lemma 3.3, every \( |D_{M_{l}(\alpha)}(\bar{Z}_{j}^{\beta_{l}})| \) is bounded, there comes an inequality of the form

$$
|d_{T}^{\alpha}(\hat{Z}_{j}, \nabla_{(k-1)}(T_{j}^{-}))| \leq K'\nu(\hat{Z}_{j})(1 + |\nabla_{(k-1)}(T_{j}^{-})|)^{w},
$$

and likewise to the other quantities.

\[ \bullet \]

### 3.5 Operator L

**Proposition 3.8** For \( p \leq 1 \), and all \( l \in \mathbb{N}^{*} \), it exists \( N_{T, l, p} > 0 \) such that

$$
\| L \hat{X}_{t}^{M} \|_{2p} \leq N_{T, l, p}(\lambda M)^{(l+2)^{2}},
$$

(3.81)

with \( \lambda M = \mu(E_{M}) \).
Proof:
In the following $C_p, C_l, C_{p,l}$ are “flying constants” which may change during the calculation; we have
\[
L(F) = -\sum_{k=1}^{J} \sum_{t=1}^{d} \partial_{k,r}(\pi_{k,r}) D_{k,r} F + D_{k,r}(D_{k,r} F) + D_{k,r} \ln p_{J} D_{k,r} F, \tag{3.82}
\]
hence, using the fact that $||_t$ is a norm (and with $|AB|_t \leq C_t |A|_t |B|_t$; cf. 1.10)
\[
|L(F)|_t \leq \sum_{k=1}^{J} \sum_{t=1}^{d} |\partial_{k,r}(\pi_{k,r}) D_{k,r} F|_t + |D_{k,r}(D_{k,r} F)|_t + \sum_{k=1}^{J} \sum_{t=1}^{d} |D_{k,r} \ln p_{J} D_{k,r} F|_t
\leq C_t \sum_{k=1}^{J} \sum_{t=1}^{d} |\partial_{k,r}(\pi_{k,r})||D_{k,r} F|_t + \sum_{k=1}^{J} \sum_{t=1}^{d} |D_{k,r}(D_{k,r} F)|_t + C_t \sum_{k=1}^{J} \sum_{t=1}^{d} |D_{k,r} \ln p_{J}||D_{k,r} F|_t
\leq C_t \sum_{k=1}^{J} \sum_{t=1}^{d} D_{k,r}(D_{k,r} F)|_t + C_t (|\pi_{k,r}|_{l+1}^2 + |\ln p_{J}|_{l+1}^2 + 2|F|_{l+1}^2)
\]
(the last inequality follows from the fact that $\sum_{k=1}^{J} \sum_{r=1}^{d} |D_{k,r} G|^2 = \sum_{k=1}^{J} \sum_{r=1}^{d} \sum_{|\alpha| \leq t} |D_{\alpha} D_{k,r} G|^2 \leq \sum_{|\alpha| \leq t+1} |D_{\alpha} G|^2 = |G|_{t+1}^2$, which implies:
\[
|L(F)|_{l}^{2p} \leq C_p, l \left( \left( \sum_{k=1}^{J} \sum_{t=1}^{d} |D_{k,r}(D_{k,r} F)|_t \right)^{2p} + |\pi_{k,r}|_{l+1}^{4p} + |F|_{l+1}^{4p} + |\ln p_{J}|_{l+1}^{4p} \right)
\leq C_p, l \left( \left( \sum_{k=1}^{J} \sum_{t=1}^{d} |D_{k,r}(D_{k,r} F)|_t^2 \right)^2 + |\pi_{k,r}|_{l+1}^{4p} + |F|_{l+1}^{4p} + |\ln p_{J}|_{l+1}^{4p} \right)
\leq C_p, l \left( \sum_{k=1}^{J} \sum_{t=1}^{d} |F|_{l+1}^{2p} + |\pi_{k,r}|_{l+1}^{4p} + |F|_{l+1}^{4p} + |\ln p_{J}|_{l+1}^{4p} \right).
\]
And, finally (noticing that, as a consequence of Lemma 3.3, $|\pi_{k,r}|_{l+1}^{4p} \leq C_{l,p}$),
\[
E \left[ |L(F)|_{l}^{2p} \right] \leq C_p, l \left( 1 + \sqrt{E \left[ (J_t^M)^{2p} \right]} \sqrt{E \left[ |F|_{l+1}^{4p} \right]} + E \left[ |F|_{l+1}^{4p} \right] + E \left[ |\ln p_{J}|_{l+1}^{4p} \right] \right) \tag{3.83}
\]
We put $F = \bar{X}_t^M$. Let us recall that, from (3.78)
\[
E \left[ \bar{X}_t^M |_{l+1}^{4n} \right] \leq C_{p,l,T} (1 + \sqrt{E \left[ (J_t^M)^{4n(l+1)} \right]}) \leq C_{p,l,T} \sqrt{\lambda_M}^{4n(l+1)}.
\]
Then, if we admit for the moment Lemma 3.9, we obtain the following upper bound:
\[
E \left[ |L(\bar{X}_t^M)|_{l}^{2p} \right] \leq N_{T,l,p} (\lambda_M)^{2p(l+2)}
\]
which ends the proof.

Hence, all that remains is to prove the following result:

Lemma 3.9 Let $l, n \in \mathbb{N}$.
\[
E \left[ |\ln p_{J}|_{l+1}^{2n} \right] \leq C_{q_{n,l,n},T} (\mu(E_M))^{n(l+2)}.
\]

Proof:
We recall that
\[
\ln p_{J} = \sum_{j=1}^{J} \ln q_{M}(Z_j, \bar{X}_{j-1}^M),
\]
and recalling that in \( q_M \) has bounded derivatives of any order, using Lemma 2.7, which implies, by corollary 1.12, the existence of \( C_{q_M,l} > 0 \) such that

\[
|q_M(F)|_{l+1} \leq C_{q_M,l}(1 + |F|_{l+1} + |F|_{l+1}^{l+1})
\]

we have,

\[
|\ln p_j|_{l+1} \leq \sum_{j=1}^{J_F} |\ln q_M(\tilde{Z}_j, \tilde{X}_{T_j-})|_{l+1}
\]

\[
\leq C_{q_M,l} \sum_{j=1}^{J_F} 1 + |(\tilde{Z}_j, \tilde{X}_{T_j-})|_{l+1} + |(\tilde{Z}_j, \tilde{X}_{T_j-})|_{l+1}^{l+1}
\]

\[
\leq C_{q_M,l} \left( J_F^M + \sum_{j=1}^{J_F} |\tilde{Z}_j|_{l+1} + |\tilde{X}_{T_j-}|_{l+1} + 2(|\tilde{Z}_j|_{l+1} + |\tilde{X}_{T_j-}|_{l+1}^{l+1}) \right)
\]

\[
\leq C_{q_M,l} \left( J_F^M + \sum_{j=1}^{J_F} |\tilde{X}_{T_j-}|_{l+1} + |\tilde{X}_{T_j-}|_{l+1}^{l+1} \right)
\]

it directly follows

\[
E \left[ |\ln p_j|_{2n} \right] \leq C_{q_M,l,n} \left( E \left[ (J_F^M)^{2n} \right] + E \left[ \left( J_F^M \sum_{j=1}^{J_F} |\tilde{X}_{T_j-}|_{l+1}^2 + (|\tilde{X}_{T_j-}|_{l+1}^{l+1})^n \right) \right] \right)
\]

\[
\leq C_{q_M,l,n} \left( E \left[ (J_F^M)^{2n} \right] + E \left[ (J_F^M)^n \left( \sum_{j=1}^{J_F} |\tilde{X}_{T_j-}|_{l+1}^2 + |\tilde{X}_{T_j-}|_{l+1}^{2(l+1)} \right)^n \right) \right)
\]

\[
\leq C_{q_M,l,n} \left( E \left[ (J_F^M)^{2n} \right] + \sqrt{E \left[ (J_F^M)^{2n} \right]} \sqrt{E \left[ \left( \sum_{j=1}^{J_F} |\tilde{X}_{T_j-}|_{l+1}^2 + |\tilde{X}_{T_j-}|_{l+1}^{2(l+1)} \right)^{2n} \right]} \right)
\]

Our aim is then to bound the second term of the rhs of this last inequality.

We have

\[
|\tilde{X}_{T_j-}|_{l+1}^2 = \sum_{|\alpha| \leq l+1} |D_\alpha \tilde{X}_{T_j-}|^2
\]

and from (3.80) we know that we can put \( \nabla_t^\alpha \overset{\text{def}}{=} D_\alpha \tilde{X}_t \) with \( \nabla_t^\alpha \) defined in (3.54) ; then, there exists a process \( V_t^\alpha \) with the same law and verifying (3.58). So :

\[
E \left[ \left( \sum_{j=1}^{J_F} |\tilde{X}_{T_j-}|_{l+1}^2 \right)^{2n} \right] = E \left[ \left( \sum_{j=1}^{J_F} \sum_{|\alpha| \leq l+1} |D_\alpha \tilde{X}_{T_j-}|^2 \right)^{2n} \right]
\]

\[
= E \left[ \left( \sum_{j=1}^{J_F} \sum_{|\alpha| \leq l+1} |D_\alpha \tilde{X}_{T_j-}|^2 \right)^{2n} \right]
\]

\[
= E \left[ \left( \sum_{j=1}^{J_F} \sum_{|\alpha| \leq l+1} |\nabla_t^\alpha|^2 \right)^{2n} \right]
\]

\[
= E \left[ \left( \sum_{j=1}^{J_F} \sum_{|\alpha| \leq l+1} |V_t^\alpha|^2 \right)^{2n} \right]
\]

\[
\leq C_n \sqrt{E \left[ (J_F^M)^{2n} \right]} \sqrt{E \left[ \sum_{j=1}^{J_F} \left( \sum_{|\alpha| \leq l+1} |V_t^\alpha|^2 \right)^{2n} \right]}.
\]
In the same way,
\[
E \left[ \left( \sum_{j=1}^{J_T^M} |X_{T_j-}^{M,2(l+1)}|^2 \right)^{2n} \right] \leq C_n \sqrt{E \left[ (J_T^M)^{4n-2} \right]} \sqrt{E \left[ \sum_{j=1}^{J_T^M} \left( \sum_{|\alpha| \leq l} |V_{T_j}^\alpha|^2 \right)^{2n(l+1)} \right]}. 
\]

But, using the \( F_p \) isometry, with \( v, w \in \mathbb{N}^* \),
\[
E \left[ \sum_{j=1}^{J_T^M} \left( \sum_{|\alpha| \leq v} |V_{T_j}^\alpha|^2 \right)^{w} \right] = E \left[ \int_0^t \int_{E_M \times [0,2\zeta]} \left( \sum_{|\alpha| \leq v} |V_{s-}^\alpha|^2 \right)^w N(ds, dz, du) \right] 
= 2\tilde{C} \mu(E_M) E \left[ \int_0^t \left( \sum_{|\alpha| \leq v} |V_{s-}^\alpha|^2 \right)^w ds \right] 
= 2\tilde{C} \mu(E_M) E \left[ \int_0^t |X_{s-}^{M,2w} ds \right] 
= 2\tilde{C} \mu(E_M) \int_0^t E \left[ |X_{s}^{M,2w} \right] ds.
\]

Now from (3.78) we have
\[
E \left[ |X_{T}^{M,v}| \right] \leq C_{p,l,T} (1 + \sqrt{E \left[ (J_T^M)^{2v} \right]}) 
\]
Gathering these results, we obtain :
\[
E \left[ \ln p_{J_{l+1}^{2n}} \right] \leq C_{q,M,a,n,T} \left( E \left[ (J_T^M)^{2n} \right] + \sqrt{E \left[ (J_T^M)^{2n} \right]} \left( \mu(E_M) E \left[ (J_T^M)^{4n-2} \right] \sqrt{E \left[ (J_T^M)^{4n(l+1)} \right]} \right)^{1/2} \right)
\]
Since (for \( t \leq T \)),
\[
E \left[ J_T^{Mn} \right] = O_{M \to + \infty} (\lambda_M^n),
\]
we have
\[
E \left[ \ln p_{J_{l+1}^{2n}} \right] \leq C_{q,M,a,n,T} \left( \mu(E_M) \right)^{1/2} \left( \mu(E_M) \right)^{2n-1+\frac{n(l+1)}{2}} \leq C_{q,M,a,n,T} \left( \mu(E_M) \right)^{n(l+2)^2}
\]

3.6 The covariance matrix

3.6.1 Preliminaries

We consider a Poisson point measure \( N(ds, dz, du) \) on \( \mathbb{R}^d \times \mathbb{R} \); with compensator \( \mu(dz) \times \mathbb{I}_{(0,\infty)}(u) du \) and two non-negative measurable functions \( f, g : \mathbb{R}^d \to \mathbb{R}^+ \). For a measurable set \( B \subset \mathbb{R}^d \) we denote \( B_g = \{ (z,u) : z \in B, u < g(z) \} \subset \mathbb{R}^d \times \mathbb{R}^+ \), and we consider the process
\[
N_t(\mathbb{I}_B f) \overset{\text{def}}{=} \int_0^t \int_{B_g} f(z) N(ds, dz, du).
\]
Moreover we note \( \nu_g(dz) = g(z) \mu(dz) \) and
\[
\alpha_{\eta, f}(s) = \int_{\mathbb{R}^d} (1 - e^{-sf(z)}) d\nu_g(dz), \quad \beta_{B,g, f}(s) = \int_{B^c} (1 - e^{-sf(z)}) d\nu_g(dz)
\]
We have the following result.

**Lemma 3.10** Let \( \phi(s) = E \left[ e^{-SN_t(1_{B_g}f)} \right] \) the Laplace transform of the random variable \( N_t(1_{B_g} f) \) then we have
\[
\phi(s) = e^{-t(\alpha_{\eta, f}(s) - \beta_{B,g, f}(s))}.
\]
Proof : From Itô’s formula we have
\[
\exp(N_t(\mathbb{1}_{B_s}f)) = 1 - \int_0^t \int_{\mathbb{R}^4 \times \mathbb{R}^+} \exp(-s(N_r-\mathbb{1}_{B_s}f))(1 - \exp(-sf(z)\mathbb{1}_{B_s}(z,u))) \, dN(r,z,u)
\]
and consequently
\[
E[\exp(N_t(\mathbb{1}_{B_s}f))] = 1 - \int_0^t E[\exp(-s(N_r-\mathbb{1}_{B_s}f))] \, dr \int_{\mathbb{R}^4 \times \mathbb{R}^+} (1 - \exp(-sf(z)\mathbb{1}_{B_s}(z,u))) \, d\mu(dz) \, du.
\]
But
\[
\int_{\mathbb{R}^4 \times \mathbb{R}^+} (1 - \exp(-sf(z)\mathbb{1}_{B_s}(z,u))) \, d\mu(dz) \, du = \int_{\mathbb{R}^4 \times \mathbb{R}^+} \mathbb{1}_{B_s}(z,u)(1 - \exp(-sf(z))) \, d\mu(dz) \, du
\]
\[= \int_{\mathbb{R}^4} \mathbb{1}_{B}(z)(1 - \exp(-sf(z))) \int_{\mathbb{R}^+} \mathbb{1}_{\{u < g(z)\}} \, d\mu(dz) \, du
\]
\[= \int_{\mathbb{R}^4} (1 - \exp(-sf(z))) g(z) \, d\mu(dz) = \alpha_{g,s}(t) - \beta_{B,g,s}(t),
\]
It follows that
\[
E[\exp(N_t(\mathbb{1}_{B_s}f))] = \exp(-t(\alpha_{g,s}(t) - \beta_{B,g,s}(t))). \quad \Box
\]

We consider an abstract measurable space $E$, a measure $\nu$ on this space and a non-negative measurable function $f : E \to \mathbb{R}^+$. Suppose that for $f \, d\nu < \infty$.

For $t > 0$ and $p \geq 1$ we notice
\[
\alpha_f(t) = \int_E (1 - e^{-tf(a)}) \, d\nu(a), \quad \text{and} \quad P_t^p(f) = \int_0^{+\infty} s^{p-1} e^{-t\alpha_f(s)} \, ds.
\]

Lemma 3.11 1. Suppose that for $p \geq 1$ and $t > 0$
\[
\liminf_{u \to \infty} \frac{1}{\ln u} \alpha_f(u) > \frac{p}{t} \quad (3.84)
\]
then
\[
P_t^p(f) < \infty.
\]
2. A sufficient condition for (3.84) is
\[
\liminf_{u \to \infty} \frac{1}{\ln u} \nu\left(f \geq \frac{1}{u}\right) > \frac{p}{t}. \quad (3.85)
\]
In particular, if $\liminf_{u \to \infty} \frac{1}{\ln u} \nu\left(f \geq \frac{1}{u}\right) = \infty$, then \(\forall p \geq 1\) and \(\forall t > 0\),
\[
P_t^p(f) < \infty.
\]
We notice that if $\nu$ is finite then (3.85) cannot be satisfied.

Proof : 1) From (3.84) one can find $\varepsilon > 0$ such that as $s$ goes to infinity $s^{p-1}e^{-t\alpha_f(s)} \leq \frac{1}{s^{\varepsilon}}$ and consequently $P_t^p(f) < \infty$.

2) With the notation $n(dz) = \nu \circ f^{-1}(dz)$, we have
\[
\alpha_f(u) = \int_0^{+\infty} (1 - e^{-yu}) \, dn(z) = \int_0^{+\infty} e^{-yu} n\left(\frac{y}{u}, \infty\right) \, dy.
\]
Using Fatou’s lemma and (3.85), we obtain
\[
\liminf_{u \to \infty} \frac{1}{\ln u} \int_0^{+\infty} e^{-yu} n\left(\frac{y}{u}, \infty\right) \, dy \geq \int_0^{+\infty} e^{-y} \liminf_{u \to \infty} \frac{1}{\ln u} n\left(\frac{y}{u}, \infty\right) \, dy > \frac{p}{t}.
\]
\[\Box\]
We consider the Poisson point measure $N(ds, dz, du)$ on $\mathbb{R}^d \times \mathbb{R}_+$ with compensator $\mu(dz) \times \mathbb{1}_{(0, \infty)}(u) du$. We recall that

$$N_t(\mathbb{1}_{B_g} f) \overset{def}{=} \int_0^t \int_{B_g} f(z) N(ds, dz, du),$$

for $f, g : \mathbb{R}^d \to \mathbb{R}_+$ and $B_g = \{(z, u) : z \in B, u < g(z)\} \subset \mathbb{R}^d \times \mathbb{R}_+$ and that (with $\nu_g(dz) \overset{def}{=} g(z) \mu(dz)$)

$$\alpha_{g,f}(s) = \int_{\mathbb{R}^d} (1 - e^{-s f(z)}) d\nu_g(dz), \quad \beta_{B,g,f}(s) = \int_{B^c} (1 - e^{-s f(z)}) d\nu_g(dz).$$

We have the following result (with $\Gamma(p) = \int_0^{+\infty} s^{p-1} e^{-s} ds$).

**Lemma 3.12** Let $U_t = t \int_{B^c} f(z) d\nu_g(z)$, then, for all $p \geq 1$,

$$E \left[ \frac{1}{(N_t(\mathbb{1}_{B_g} f) + U_t)^p} \right] \leq \frac{1}{\Gamma(p)} \int_0^{+\infty} s^{p-1} \exp(-t \alpha_{g,f}(s)) \, ds. \quad (3.86)$$

If it is supposed that, for some $0 < \theta \leq \infty$,

$$\liminf_{a \to \infty} \frac{1}{\ln a} \nu_g(f \geq \frac{1}{a}) = \theta, \quad (3.87)$$

then for every $t > 0$ and $p \geq 1$ such that $\frac{2}{t} < \theta$

$$E \left[ \frac{1}{(N_t(\mathbb{1}_{B_g} f) + U_t)^p} \right] < \infty.$$

**Proof**: By a change of variables we obtain for every $\lambda > 0$,

$$\lambda^{-p} \Gamma(p) = \int_0^{+\infty} s^{p-1} e^{-\lambda s} \, ds.$$

Taking the expectation in the previous equality with $\lambda = N_t(\mathbb{1}_{B_g} f) + U_t$ we obtain

$$E \left[ \frac{1}{(N_t(\mathbb{1}_{B_g} f) + U_t)^p} \right] = \frac{1}{\Gamma(p)} \int_0^{+\infty} s^{p-1} E \left[ \exp(-s(N_t(\mathbb{1}_{B_g} f) + U_t)) \right] \, ds.$$

Now from Lemma 3.10 we have

$$E \left[ \exp(-s N_t(\mathbb{1}_{B_g} f)) \right] = \exp(-t(\alpha_{g,f}(s) - \beta_{B,g,f}(s))).$$

Moreover, from the definition of $U_t$ one can easily verify that $\exp(-s U_t) \leq \exp(-t \beta_{B,g,f}(s))$ and then

$$E \left[ \exp(-s(N_t(\mathbb{1}_{B_g} f) + U_t)) \right] \leq \exp(-t \alpha_{g,f}(s))$$

this completes the proof of (3.86). The second part of the lemma follows directly from Lemma 3.11.

3.6.2 The Malliavin covariance matrix

In this subsection, we prove, under some additional assumptions on $p$ and $t$, that $E \left[ \frac{1}{|det \sigma(F_M)|^p} \right]$ is bounded (uniformly on $M$), for the Malliavin matrix $\sigma(F_M)$ defined at the definition 1.5.

From the diffusion equation (2.44)

$$X^M_t = x + \int_0^t \sigma(X^M_s) \, dW_s + \sum_{j=1}^{J^M} c_M(\bar{Z}_j, \bar{X}^M_{T_j-}) + \int_0^t g(\bar{X}^M_s) \, ds,$$

let us consider the tangent flow

$$Y^M_t = \text{Id} + \sum_{l=1}^{M} \int_0^t \nabla \sigma_l(\bar{X}^M_s) Y^M_s \, dW^l_s + \sum_{j=1}^{J^M} \nabla c_M(\bar{Z}_j, \bar{X}^M_{T_j-}) Y^M_{T_j} + \int_0^t \nabla g(\bar{X}^M_s) Y^M_s \, ds.$$
We then define the following process (with $\nabla_x c_j = \nabla_x c_M(\bar{Z}_j, \bar{X}_M^{T_{j-1}})$):

$$\hat{Y}_t^M \overset{def}{=} \text{Id} - \sum_{l=1}^{m} \int_0^t \hat{Y}_s^M \nabla \sigma_l(\bar{X}_s^M) \, dW_s^l - \sum_{j=1}^{J_t^M} \nabla_x c_j(\text{Id} + \nabla_x c_j)^{-1} + \int_0^t \hat{Y}_s^M \left( \frac{1}{2} \sum_{l=1}^{m} \nabla \sigma_l(\bar{X}_s^M)^2 - \nabla x g(\bar{X}_s^M) \right) \, ds.$$

**Lemma 3.13** We have, for all $t \geq 0$,

$$Y_t^M \hat{Y}_t^M = \text{Id}. \quad (3.88)$$

**Proof:** The proof is postponed in the Appendix C.

**Lemma 3.14** Assuming hypothesis 2.1, 2.2, 2.3 we have, for $p \geq 1$, $T > t > 0$ such that $\frac{2dp}{T} \theta < \theta$

$$\mathbb{E} \left[ \frac{1}{| \det \sigma(F_M) |^p} \right] \leq C_p, \quad (3.89)$$

where the constant $C_p$ does not depend on $M$.

**Proof:**

Since

$$D_{k,r} \bar{X}_t = \nabla_x c_M(\bar{Z}_k, \bar{X}_M^{T_{k-1}}) D_{k,r} \bar{Z}_k + \sum_{l=1}^{m} \int_0^t \nabla \sigma_l(\bar{X}_s^M) D_{k,r} \bar{X}_s^M \, dW_s^l + \sum_{j=k+1}^{J_t^M} \nabla_x c_M(\bar{Z}_j, \bar{X}_M^{T_{j-1}}) D_{k,r} \bar{X}_M^{T_{j-1}} + \int_0^t \nabla x g(\bar{X}_s^M) D_{k,r} \bar{X}_s^M \, ds.$$

We have

$$Y_t^M = Y_t^{M T_k} + \sum_{l=1}^{m} \int_{T_k}^t \nabla \sigma_l(\bar{X}_s^M) Y_s^M \, dW_s^l + \sum_{j=k+1}^{J_t^M} \nabla_x c_M(\bar{Z}_j, \bar{X}_M^{T_{j-1}}) Y_t^M \, dW_s^l$$

and with $A_k = \nabla_x c_M(\bar{Z}_k, \bar{X}_M^{T_{k-1}}) D_{k,r} \bar{Z}_k$,

$$Y_t^M \hat{Y}_t^M A_k = Y_t^{M T_k} \hat{Y}_t^{M T_k} A_k + \sum_{l=1}^{m} \int_{T_k}^t \nabla \sigma_l(\bar{X}_s^M) Y_s^M \hat{Y}_s^M A_k \, dW_s^l + \sum_{j=k+1}^{J_t^M} \nabla_x c_M(\bar{Z}_j, \bar{X}_M^{T_{j-1}}) Y_t^M \hat{Y}_t^M A_k + \int_{T_k}^t \nabla x g(\bar{X}_s^M) Y_s^M \hat{Y}_s^M A_k \, ds.$$

Then

$$D_{k,r} \bar{X}_t = Y_t^M \hat{Y}_t^M A_k = Y_t^M \hat{Y}_t^M \nabla_x c_M(\bar{Z}_k, \bar{X}_M^{T_{k-1}}) D_{k,r} \bar{Z}_k.$$

Therefore

$$\sum_{r=1}^{d} (D_{k,r} \bar{X}_t, \xi)^2 = \sum_{r=1}^{d} (\hat{Y}_t^M \hat{Y}_t^M \nabla_x c_M(\bar{Z}_k, \bar{X}_M^{T_{k-1}}) D_{k,r} \bar{Z}_k, \xi)^2$$

$$= \sum_{k=1}^{d} \sigma_k^2 \langle \partial_{x'} c_M(\bar{Z}_k, \bar{X}_M^{T_{k-1}}), (Y_t^M \hat{Y}_t^M)^* \xi \rangle^2$$

$$\geq \sum_{k=1}^{d} I_{B_{M-1}}(\bar{Z}_k) \langle \partial_{x'} c(\bar{Z}_k, \bar{X}_M^{T_{k-1}}), (Y_t^M \hat{Y}_t^M)^* \xi \rangle^2$$

since $\sigma_k \geq I_{B_{M-1}}(\bar{Z}_k)$ and $c_M = c$ on $B_{M-1}$ ; using Hypothesis 2.3 item 3., it follows that

$$\rho_t \geq \inf_{|\xi|=1} \sum_{k=1}^{J_t^M} I_{B_{M-1}}(\bar{Z}_k) \xi^2(\bar{Z}_k) \langle (Y_t^M \hat{Y}_t^M)^* \xi \rangle^2 \geq \langle (Y_t^M \hat{Y}_t^M) \xi \rangle^2 - \sum_{k=1}^{d} I_{B_{M-1}}(\bar{Z}_k) \xi^2(\bar{Z}_k).$$
With \( \sigma(F_M) = \sigma(\bar{X}_M) + U_M(t) \), we have
\[
E \left[ \frac{1}{\det \sigma(F_M)} \right]^p \leq E \left[ \frac{1}{\rho_t + U_M(t)} \right]^{dp} \leq E \left[ \frac{1 + (\|Y_t^M \bar{Y}_t^M\|)^2}{\sum_{r=1}^{\lambda_M} (\bar{Z}_k)^2 (\bar{Z}_k) + U_M(t)} \right]^{dp}
\]

Now observe that the denominator of the last fraction is equal in law to
\[
\sum_{r=1}^{\lambda_M} (B_{M-1}(Z_k))^2 (Z_k) \mathbf{1}_{\nu_k < \gamma(z, X_{M-1}^\gamma)} + U_M(t) \geq N_t (B_{M-1})^2 + U_M(t),
\]
with \( B_M = \{(z, u) : z \in B_{M-1}, u < \gamma(z)\} \). Assuming Hypothesis 2.3 item 3., we can apply Lemma 3.12 with \( f = c^2 \) and \( \nu(z) = \gamma(z) \mu(dz) \). This gives \( p' \geq 1 \) such that \( \frac{p'}{t} < \theta \)
\[
E \left[ \left( \frac{1}{B_{M-1}^2 + U_M(t)} \right)^{p'} \right] \leq C_p'.
\]

Finally, since the moments of \( \|\bar{Y}_t^M\| \) are bounded uniformly on \( M \), the result follows from Cauchy-Schwarz inequality:
\[
E \left[ \frac{1}{\det \sigma(F_M)} \right]^p \leq C_p.
\]

### 3.7 Bounding the weights

**Lemma 3.15** Let \( q, p \in \mathbb{N}^* \) and \( T > 0 \). For \( T > t > 0 \) with \( \frac{4d(3q-1)}{t} < \theta \), there exists a constant \( C_{p,q,T} \) such that

\[
\|H^q_{\beta}(F_M)\|_p \leq C_{p,q,T} \lambda_M^{q(4q+6d+9)}. \tag{3.90}
\]

**Proof:** Let \( q \in \mathbb{N}^* \) and \( \beta = (\beta_1, \ldots, \beta_q) \) a multi-index. We have to bound \( H^q_{\beta}(F_M) \equiv H^q_{\beta}(F_M, 1) \) from Theorem 1.13 there exists a universal constant \( C_{q,d} \) such that (recalling that \( F_M \in \mathbb{R}^d \))
\[
\|H^q_{\beta}(F_M)\| \leq C_{q,d} \left( \frac{1}{\det \sigma(F_M)^{3q-1}} \right) \left( 1 + |F_M|_{q+1}^{(6d+1)q} \right) \left( 1 + |L_F|_{q-1}^4 \right).
\]

So (for \( T > t > 0 \) such that \( \frac{2d(6q-2)}{t} < \theta \); \( C_q \) will be, in the following lines, a “flying constant”)
\[
E \left[ H^q_{\beta}(F_M) \right] \leq C_{q,d} \sqrt{E \left[ \frac{1}{\det \sigma(F_M)^{3q-1}} \right]} \sqrt{E \left[ \left( 1 + |F_M|_{q+1}^{(6d+1)q} \right)^2 \left( 1 + |L_F|_{q-1}^4 \right)^2 \right]}
\]
\[
\leq C_q \left( E \left[ \left( 1 + |F_M|_{q+1}^{(6d+1)q} \right)^4 \right] E \left[ \left( 1 + |L_F|_{q-1}^4 \right)^4 \right] \right)^{1/2}
\]
since we know that, from Lemma 3.14, for \( p \geq 1 \) and \( T > t > 0 \) such that \( \frac{dp}{T} < \theta \),
\[
E \left[ \frac{1}{\det \sigma(F_M)} \right]^p \leq C_p.
\]

But, from (3.76) and (3.81), there exists \( C_{q,T} > 0 \) such that
\[
E \left[ (|F_M|_{q+1})^{4q(6d+1)} \right] \leq C_{q,T} \lambda_M^{2q(q+1)(6d+1)} \quad \text{and} \quad E \left[ |L_F|_{q-1}^{4q} \right] \leq C_{q,T} \lambda_M^{q(q+1)2},
\]
so (since \( q \geq 1, q^2 \geq \frac{q(q+1)}{2} \))
\[
E \left[ H^q_{\beta}(F_M) \right] \leq C_{p,q,T} \lambda_M^{\frac{q(q+1)}{2}(6d+1)} \lambda_M^{q(q+1)2}.
\]

\footnote{If \( a, b, c \) and \( d \) are non-negative real numbers,
\[
\frac{1}{a} + \frac{1}{d} \leq \frac{1+c}{a+d}.
\]}
4 Joint density regularity

4.1 Introduction

We recall that we made an approximation in law $F^x_M$ of our process $X^x_t$. It is clear, from its definition (cf. (2.45)), that the law $P_{F_M}$ of $F^x_M$ possesses a smooth density: $P_{F_M}(dy) = p_{F_M}(y)dy$. Then, we have defined

$$f_M(x,y) \overset{\text{def}}{=} \Psi_K(x)p_{F_M}(y)$$

where $\Psi_K$ is a smooth version with bounded derivatives of any order of the indicator function $1_K$.

In this section we will highlight the behaviour of $f_M(x,y)$ with respect to the norm defined by (1.22), which will prove the Lemma 2.11 and, consequently, will end the proof of our main result.

4.2 Bounds for the Sobolev norms of the tangent flow and its derivatives

A simple generalisation (with a little bit heavier with respect to the notations, but using the very same ideas and methods) of Proposition 3.6 gives straightforwardly the following result:

**Proposition 4.1** Let $l,q \in \mathbb{N}^*$, $p \geq 1$ and $t < T$. For every multi-index $\beta \in \{1, \ldots, d\}^t$, there exists $C_{l,p,q,T} > 0$ such that

$$\|\partial_\beta X^M_t\|_p \leq C_{l,p,q,T} \sqrt{(\lambda_M)^t}.$$  \hspace{1cm} (4.91)

4.3 Proof of 2.11 : an upper bound for $\|f_M\|_{2m+q,2m,p}$

We already have all the tools to prove the Proposition 2.11, which was the key for proving our main joint density result 2.1. To bound the quantity $\|f_M\|_{2m+q,2m,p}$, it is sufficient, by (1.24), to bound the quantities

$$\partial_\xi (f_M(x,y)), \quad \xi \leq 2m + q,$$

which is the exact point of the Proposition 4.4 that we will prove now ; we will deal first with the case $\xi = 0$ (which is the point of the next proposition), to show more conveniently the method that we used.

**Notation 4.2** In all the sequel, $\varphi_\varepsilon$ will represent a mollifier converging weakly, as $\varepsilon$ tends to 0, to the Dirac distribution. We will also define

$$\Phi_\varepsilon(x_1, \ldots, x_d) \overset{\text{def}}{=} \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_d} \varphi_\varepsilon(t_1, \ldots, t_d) dt_1 \cdots dt_d. \hspace{1cm} (4.92)$$

**Proposition 4.3** Let $T > 0$. For every $t \in [\frac{4d(3d-1)}{d}, T]$

$$f_M(x,y) \leq C_{d,T} \Psi_K(x) \left( 1 + \frac{1}{(y-3)d+T} \right) \lambda_M^{5(d+1)^3} \hspace{1cm} (4.93)$$

**Proof :** Let us note that, formally, $p_{F^x_M}(x,y) = \mathbb{E}\left[\delta_0(F^x_M(x) - y)\right]$, where $\delta_0$ is the Dirac distribution.

In order to work in the direction of this last representation, we will therefore consider the following approximation of $f_M$:

$$f_{M,\varepsilon} \overset{\text{def}}{=} \Psi_K(x) \mathbb{E}\left[\varphi_\varepsilon(F^x_M(x) - y)\Psi_2(F^x_M(x) - y)\right], \hspace{1cm} (4.94)$$

where $\varphi_\varepsilon$ is, as we said before, a mollifier and where $\Psi_2$ is a smooth version (with bounded derivatives of any order) of the indicator function with respect to the ball centred at 0 with radius 2.

We will consequently look, in the first place, for an upper-bound of $f_{M,\varepsilon} = \Psi_K(x) \mathbb{E}\left[\varphi_\varepsilon(F^x_M(x) - y)\Psi_2(F^x_M(x) - y)\right]$; using Theorem 1.3,

$$\mathbb{E}\left[\varphi_\varepsilon(F^x_M - y)\Psi_2(F^x_M(x) - y)\right] = \mathbb{E}\left[\Phi_\varepsilon(F^x_M - y)H^d_M(F_M, \Psi_2(F^x_M(x) - y))\right]$$

It directly follows, since $\|\Phi_\varepsilon\|_\infty \leq 1$ (which is clear from the definition (4.92)) and the weight $H_M$ does not depend on $\varepsilon$, the pointwise convergence of $f_{M,\varepsilon}(x,y)$ when $\varepsilon$ tends to 0.

Using Theorem 1.13 and denoting temporarily $G_M \overset{\text{def}}{=} \Psi_2(F^x_M(x) - y)$,

$$\left|H^d_M(F_M, G_M)\right| \leq C_d |G_M|_d \frac{(1 + |F_M|_{d+1})^{(6d+1)d}}{|\det \sigma(F_M)|_{3d-1}} \left( 1 + |LF_M|_{d-1} \right).$$
Following the same pattern as we did in the proof of Lemma 3.15, for $T > t > 0$ such that $\frac{2d(6q-2)}{t} < \theta$ (here $q = d$), we have

$$
E \left[ |H_M^d(F_M, G_M)| \right] \leq C_d \left( E \left[ \frac{1}{|\det \sigma(F_M)|^{6d-2}} \right] E \left[ (1 + |F_M|^d)^{4|6d|^d} \right] \right)^\frac{1}{4} \left| |G_M|_d \right|
$$

$$\leq C'_d \left( E \left[ F_M^{4|6d|^d} \right] E \left[ |L_f|^{4|d-1|} \right] \right)^\frac{1}{4} \left| |G_M|_d \right|
$$

and, for $M$ big enough (provided that $\lambda_M \to +\infty$, when $M \to +\infty$) we found that

$$
E \left[ |H_M^d(F_M, G_M)| \right] \leq C_{d,T} \lambda_M^3 \left| |G_M|_d \right|
$$

The Lemma 1.11, with $\phi(\cdot) \overset{\text{def}}{=} \Psi_2(\cdot - y)$ (and with $|\phi|_n(F) \overset{\text{def}}{=} \sup_{|\beta| \leq n} |\partial \phi(F)|$ ; it is clear then that $|\phi_n(F^M(x))| = |\Psi_2_n(F^M(x) - y)|$ implies:

$$
|\phi(F^M_t)|_d \leq C_d |\phi|_d(F^M_t)(1 + |F^M_t|_1, d + |F^M_t|_{d-1})
$$

$$\leq C_d |\phi|_d(F^M_t)(1 + |X^M_t|_d + |X^M_t|_{d-1})
$$

So, noting that $|\Psi_2_n(u) \leq C_n T (|u| \leq 3)$,

$$
\left| |G_M|_d \right| \leq \left| |\Psi_2(F^M_t(x) - y)| \right|_d
\leq K_d \left( E \left[ \mathbb{1}_{|\{F^M_t(x) - y| \leq 3\}} \right]\right) \left( E \left[ (1 + |\Psi_2|_d(1 + |\Psi_2|_{d-1})^{R_t}) \right] \right)^\frac{1}{8}
$$

$$= C_{d,T} \left( E \left[ |F^M_t(x) - y| \leq 3 \right] \right)^\frac{1}{8} \left( \lambda_M \right)^\frac{d}{2}
$$

and, since

$$
P \left[ |F^M_t(x) - y| \leq 3 \right] \leq P \left[ |F^M_t(x)| \geq |y| - 3 \right]
$$

$$\leq \frac{E \left[ |F^M_t(x)|^{8(d+1)} \right]}{(|y| - 3)^{8(d+1)}}
$$

$$\leq \frac{K'_d}{(|y| - 3)^{8(d+1)}}
$$

(we used the fact that $E \left[ |F^M_t(x)|^{8(d+1)} \right] = \left| |F^M_t(x)|_d \right|^{8(d+1)}$ which is bounded from Proposition 3.6) so

$$
\left| |G_M|_d \right| \leq \frac{C'_{d,T} \left( \lambda_M \right)^\frac{d}{2}}{(|y| - 3)^{d+1}}
$$

So, since by definition $\left| |\Phi_\xi \right| \leq 1$,

$$
E \left[ \phi\xi(F^M_t - y)\Psi_2(F^M_t(x) - y) \right] \leq K_{d,T} \lambda_M^3 \left( \lambda_M \right)^\frac{d}{2}
$$

$$\leq \frac{K_d \lambda_M^3}{(|y| - 3)^{d+1}}
$$

And

$$
f_{M,\xi}(x, y) \leq C_{d,T} \Psi_K(x) \left( 1 + \frac{1}{(|y| - 3)^{d+1}} \right) \lambda_M^3
$$

(4.95)

We are now ready to deal with the general case $|\xi| > 1$. We have to be aware that the density, conditionally on $G = \sigma(T_k, k \in \mathbb{N})$, of the law of $(Z_1, \ldots, Z_{j+M})$, density given by

$$
p_{j+M, x}(\omega, z_1, \ldots, z_{j+M}) = \prod_{j=1}^{j+M} q_M(z_j, \Psi_{T_j - T_{j-1}}(X_{T_{j-1}}))
$$

depends on $x$, which makes the differentiation more complicated.

---

4 using the non-optimal inequality:

$$d((d-1)^2 + 2) + \frac{d}{2}(6d+1)(d+1) < 4(d+1)^3$$

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Proposition 4.4 Let $T > 0$, $m$, $q \in \mathbb{N}$. For every $t \in \left[ \frac{1}{q}T, T \right]$, with $q' = d + 2m + q$, and every multi-index $\xi$ such that $|\xi| \leq 2m + q$, 

$$|\partial_{\xi} f_M(x, y)| \leq C_{d, m, q, T} K + \left( 1 + \frac{1}{(1 - 3)^{2m+d+1}} \right) \lambda M^{6(d+2m+q+1)}, \quad (4.96)$$

Proof:Because of what we said concerning the density, we will separate the differentiation with respect to $x$ and to $y$, and hence define two multi-indexes $\alpha$ and $\beta$ such that

$$\partial_\alpha \partial_\beta \equiv \partial_\xi.$$

Then, we will start to work on the quantity $\partial_{\xi} f_M(x, y) = \partial_\alpha \partial_\beta (f_{M, \varepsilon}(x, y)).$

It is clear that $\partial_\beta (\varphi_\varepsilon(F^M - y)) = (-1)^\beta \partial_\beta \varphi_\varepsilon(F^M - y)$, so

$$\partial_\alpha \partial_\beta (f_{M, \varepsilon}(x, y)) = \partial_\alpha \partial_\beta \left( \Psi_K(x) E \left[ \varphi_\varepsilon(X_1^M(x) - y) \varphi_2(X_2^M(x) - y) \right] \right)$$

$$= \partial_\alpha \left( \Psi_K(x) E \left[ \sum_{\beta' \beta'' = \beta} \partial_\beta' \varphi_\varepsilon(X_1^M(x) - y) \partial_\beta'' \varphi_2(X_2^M(x) - y) \right] \right)$$

$$= (-1)^\beta \sum_{\beta' \beta'' = \beta} \partial_\alpha \partial_\beta \left( \Psi_K(x) E \left[ \partial_\beta' \varphi_\varepsilon(X_1^M(x) - y) \partial_\beta'' \varphi_2(X_2^M(x) - y) \right] \right)$$

Lemma 4.8 implies then that $\partial_\alpha \partial_\beta (f_{M, \varepsilon}(x, y))$ converges (when $\varepsilon$ tends to 0) and the existence of $C_{d, m, q, T} > 0$ such that

$$\partial_{\xi} f_{M, \varepsilon}(x, y) \leq C_{d, m, q, T} K + \left( 1 + \frac{1}{(1 - 3)^{2m+d+1}} \right) \lambda M^{6(d+2m+q+1)}, \quad (4.97)$$

which allows us to state that $\lim_{\varepsilon \to 0} \partial_{\xi} f_{M, \varepsilon}(x, y) = \partial_{\xi} f_M(x, y)$, and letting $\varepsilon \to 0$ in (4.97), we obtain (4.96).

In this last proof, we used the Lemma 4.8 ; to prove it, we will first need two preliminary lemmas :

Lemma 4.5 Let $f : \mathbb{R}^d \to \mathbb{R}_+$ and $\beta$ a multi-index, then (with the notations used in Lemma 1.6)

$$\partial_\beta f = f(x) \sum_{\beta} \sum_{M \in M_{|\beta|}} c_M \prod_{i=1}^{l} \partial_\beta M_i \ln f(x) \quad \text{(4.98)}$$

with $c_M \in \mathbb{N}$.

Proof: By induction on $|\beta|$.

We will just show the mechanism, which will then be rather clear for a higher range, for $|\beta| = 1$, 2.

For $|\beta| = 1$, let $\beta = (x_i)$ ; then it is clear that

$$\partial_{x_i} f(x) = f(x) \partial_{x_i} \ln f(x)$$

(so $c_M = 1$ for $M = \{x_i\}$).

For $|\beta| = 2$, let $\beta = (x_j, x_i)$ then

$$\partial_{x_j} \partial_{x_i} f(x) = \partial_{x_j} \left( f(x) \partial_{x_i} \ln f(x) \right)$$

$$= \partial_{x_j} f(x) \partial_{x_i} \ln f(x) + f(x) \partial_{x_j} \partial_{x_i} \ln f(x)$$

$$= (f(x) \partial_{x_j} \ln f(x)) \partial_{x_i} f(x) + f(x) \partial_{x_j} \partial_{x_i} \ln f(x)$$

$$= f(x) (\partial_{x_j} \ln f(x) \partial_{x_i} f(x) + \partial_{x_j} \partial_{x_i} \ln f(x))$$

(so $c_M = 1$ for $M = \{x_j, x_i\}$ or $M = \{x_j\}$). 

\[ \blacksquare \]
Lemma 4.6 Let $l, q \in \mathbb{N}$, $p \geq 1$ and $t < T$. For every multi-index $\beta \in \{1, \ldots, d\}^q$, there exists $C_{l, p, q, T} > 0$ such that

$$E \left[ \left( \sum_{j=1}^{J_T^M} |\partial_x^\beta \bar{X}^M_{T_j} |_v^{2w} \right)^w \right] \leq C_{u, v, w, |\beta|, T}(\lambda_M)^{uw+1}.$$  \hspace{1cm} (4.99)

Proof: We have

$$|\partial_x^\beta \bar{X}^M_{T_j} |_v^2 = \sum_{|\alpha| \leq v} |D_\alpha \partial_x^\beta \bar{X}^M_{T_j} |^2.$$

In the very same way as we did in (3.80) (with an obvious generalisation of Lemma 1.6) we can set

$$V_t^{\alpha, \beta} \overset{\text{def}}{=} D_\alpha \partial_x^\beta \bar{X}_t$$

with $V_t^{\alpha, \beta}$ defined as in (3.54) ; then, there exists a process $V_t^{\alpha, \beta}$ with the same law and verifying (3.58), so we have:

$$E \left[ \left( \sum_{j=1}^{J_T^M} |\partial_x^\beta \bar{X}^M_{T_j} |_v^{2w} \right)^w \right] = E \left[ \left( \sum_{j=1}^{J_T^M} \left( \sum_{|\alpha| \leq v} |D_\alpha \partial_x^\beta \bar{X}^M_{T_j} |^2 \right)^w \right)^w \right]$$

$$= E \left[ \left( \sum_{j=1}^{J_T^M} \left( \sum_{|\alpha| \leq v} |V_{T_j}^{\alpha, \beta} |^2 \right)^w \right)^w \right]$$

$$= E \left[ \left( \sum_{j=1}^{J_T^M} \left( \sum_{|\alpha| \leq v} |V_{T_j}^{\alpha, \beta} |^2 \right)^w \right)^w \right]$$

$$\leq \sqrt{E \left[ (J_T^M)^{2w-2} \right]} \sqrt{E \left( \left( \sum_{j=1}^{J_T^M} \left( \sum_{|\alpha| \leq v} |V_{T_j}^{\alpha, \beta} |^2 \right)^{wu} \right)^2 \right]}.$$  

---

5 With the convention $\{1, \ldots, d\}^0 = \{0\}$ and $\partial_x^\beta F = F$.

6 Indeed, in this lemma, the derivatives were purely formal, therefore, we can use it directly with an operator $D_{\alpha'}$ where $\alpha' = (\alpha, \beta)$ and $D_{\alpha'} \overset{\text{def}}{=} D_\alpha \partial_x^\beta$. 


But, using the $F^2_p$ isometry, with $w' \in \mathbb{N}^*$, and setting $f(y) \overset{\text{def}}{=} \left( \sum_{|\alpha| \leq v} |V_{y}^{\alpha,\beta}|^2 \right)^{w'}$, we have:

\[
\mathbb{E} \left[ \left( \sum_{j=1}^{J^M_t} \left( \sum_{|\alpha| \leq v} |V_{T_j}^{\alpha,\beta}|^2 \right)^{w'} \right)^2 \right] \\
= \mathbb{E} \left[ \left( \int_0^t \int_{E_M \times [0,2C]} f(s^-) \ d\mathcal{N}(s,z,u) + \int_0^t \int_{E_M \times [0,2C]} f(s^-) \ d\mathcal{N}(s,z,u) \right)^2 \right] \\
\leq \mathbb{E} \left[ \left( \int_0^t \int_{E_M \times [0,2C]} f(s^-)^2 \ d\mathcal{N}(s,z,u) \right) + \mathbb{E} \left[ \left( \int_0^t \int_{E_M \times [0,2C]} f(s^-) \ d\mathcal{N}(s,z,u) \right)^2 \right] \right] \\
\leq 2C\mu(E_M) \mathbb{E} \left[ \int_0^t f(s^-)^2 \ ds \right] + (2C\mu(E_M))^2 \mathbb{E} \left[ \int_0^t f(s^-)^2 \ ds \right] \\
= (2C\mu(E_M) + (2C\mu(E_M))^2) \mathbb{E} \left[ \int_0^t f(s^-)^2 \ ds \right] \\
= (2C\mu(E_M) + (2C\mu(E_M))^2) \mathbb{E} \left[ \int_0^t \left( \sum_{|\alpha| \leq v} |V_{s}^{\alpha,\beta}|^2 \right)^{2w'} \ ds \right] \\
= (2C\mu(E_M) + (2C\mu(E_M))^2) \mathbb{E} \left[ \int_0^t \left( \sum_{|\alpha| \leq v} |V_{s}^{\alpha,\beta}|^2 \right)^{2w'} \ ds \right] \\
= (2C\mu(E_M) + (2C\mu(E_M))^2) \mathbb{E} \int_0^t \left[ |\partial_x^{\alpha} \mathcal{X}^M_{T_j} \right] \ ds \\
= (2C\mu(E_M) + (2C\mu(E_M))^2) \int_0^t \mathbb{E} \left[ |\partial_x^{\alpha} \mathcal{X}^M_{T_j} \right] \ ds.
\]

Now from (4.1) we have (with $\lambda_M = \mu(E_M)$),

\[
\mathbb{E} \left[ \left( \sum_{j=1}^{J^M_t} |\partial_x^{\alpha} \mathcal{X}^M_{T_j} \right) |_{v} \right] \leq C_{v,w',|\beta|,T}(\lambda_M)^{2w'}
\]

Gathering these results, we obtain:

\[
\mathbb{E} \left[ \left( \sum_{j=1}^{J^M_t} |\partial_x^{\alpha} \mathcal{X}^M_{T_j} \right) |_{v} \right] \leq C_{u,v,w',|\beta|,T}(\lambda_M)^{u-1}(\lambda_M)^{uw+1}.
\]

\[\bullet\]

**Lemma 4.7** Let $l,q \in \mathbb{N}^*$, $p \geq 1$ and $t < T$. For every multi-index $\alpha \in \{1, \ldots , d\}^q$, there exists $C_{t,p,q,T} > 0$ such that

\[
||| \partial_x^\alpha \ln p_{J^M_t, x}(Z_{j_1}, \ldots , Z_{j^M_t}) |||_{2p} \leq C_{t,p,q,T}(\lambda_M)^{(q+l)+1}.
\] (4.100)

**Proof:** In this proof we will denote $J^M_t$ simply by $J$ (and sometimes $\mathcal{X}^M_{T_j}(x)$ simply by $\mathcal{X}^M_{T_j}$).

Using a similar formula as (1.12) (the Malliavin derivatives were used in a formal way, so it is in particular true with the usual differential operator), we have (with $\alpha \in \{1, \ldots , d\}^q$),

\[
\partial_x^\alpha \phi(F) = \sum_{l=1}^{n} \sum_{\beta=(\beta_1, \ldots , \beta_l)} \sum_{M \in M_n(l)} \partial_\beta \phi(F) \partial_x^{M_l(\alpha)} F_{\beta_1} \cdots \partial_x^{M_l(\alpha)} F_{\beta_l},
\]

so, with $\alpha \overset{\text{def}}{=} \sum_{l=1}^{n} \sum_{\beta=(\beta_1, \ldots , \beta_l)} \sum_{M \in M_n(l)}^{1,d}$,

\[
\partial_x^\alpha (\ln q_M(\mathcal{X}^M_{T_j}(x), Z_j)) = \sum_{(\alpha)} \partial_\beta (\ln q_M(\mathcal{X}^M_{T_j}(x), Z_j)) \partial_x^{M_l(\alpha)} (\mathcal{X}^M_{T_j}(x))_{\beta_1} \cdots \partial_x^{M_l(\alpha)} (\mathcal{X}^M_{T_j}(x))_{\beta_l}.
\]

By corollary 1.12, we have the existence of $C'_{q_M,l} > 0$ such that

\[
|\ln q_M(F)|_t \leq C'_{q_M,l}(1 + |F|_t + |F|_{t-1})
\]
which leads to (recalling \( q = |\alpha| \))

\[
|\partial_x^q (\ln q_M(\bar{X}^{M}_{T_j-}(x),\bar{Z}_j))| \leq C_{q,M,l} \left( 1 + |\bar{X}^{M}_{T_j-}|^2 + |\bar{X}^{M}_{T_j-}|^2 \sum_{\beta \subseteq \alpha} |\partial_x^\beta (\bar{X}^{M}_{T_j-}(x))|^{2q} \right).
\]

Then

\[
|\partial_x^q \ln p_j| \leq \sum_{j=1}^{J^M_{T_j}} |\partial_x^q (\ln q_M(\bar{X}^{M}_{T_j-}(x),\bar{Z}_j))| \leq C_{q,M,l} \left( \sum_{j=1}^{J^M_{T_j}} 1 + |\bar{X}^{M}_{T_j-}|^2 + |\bar{X}^{M}_{T_j-}|^2 \sum_{\beta \subseteq \alpha} |\partial_x^\beta (\bar{X}^{M}_{T_j-}(x))|^{2q} \right)
\]

\[
\leq C_{q,M,l} \left( J^M_{T_j} + \sum_{j=1}^{J^M_{T_j}} |\bar{X}^{M}_{T_j-}|^2 + |\bar{X}^{M}_{T_j-}|^2 \sum_{\beta \subseteq \alpha} j=1 \sum_{\beta \subseteq \alpha} |\partial_x^\beta (\bar{X}^{M}_{T_j-}(x))|^{2q} \right)
\]

it directly follows, using the Lemma 4.6,

\[
E \left[ |\partial_x^q \ln p_j|^{2n} \right] \leq C_{q,M,l,n,|\alpha|} \left( E \left[ (J^M_{T_j})^{2n} \right] + E \left[ \left( \sum_{j=1}^{J^M_{T_j}} |\bar{X}^{M}_{T_j-}|^2 \right)^{2n} \right] \right)
\]

\[
+ E \left[ \left( \sum_{j=1}^{J^M_{T_j}} \sum_{\beta \subseteq \alpha} |\partial_x^\beta (\bar{X}^{M}_{T_j-}(x))|^{2q} \right) \sum_{\beta \subseteq \alpha} \right] \leq C_{q,M,l,n,|\alpha|} \left( (\lambda_M)^{2n} + C_{n,l,T}(\lambda_M)^{2n|l+1|} + C_{n,l,T}(\lambda_M)^{2n|l+1|} + \sum_{\beta \subseteq \alpha} C_{n,l,q,|\beta|,T}(\lambda_M)^{2n|q+1|} \right)
\]

\[
\leq C_{n,l,|\alpha|} (\lambda_M)^{2n|l+1|}.
\]

Now we can prove the wanted result:

**Lemma 4.8** Let \( T > 0 \). Let \( \alpha, \beta \) and \( \gamma \) multi-indexes such that \( |\alpha| + |\beta| + |\gamma| \leq 2m + q \). For every \( t \in [\frac{4d(3q+1)}{\theta} - 1, T] \), with \( q' = d + |\beta| \), the quantity \( \partial_x^\alpha \partial_y^\beta \partial_z^\gamma \varphi(z,F^M - y) \partial_z^\gamma \Psi_2(F^M_t(x) - y) \) converges (when \( \varepsilon \) tends to 0) and

\[
\partial_x^\alpha \partial_y^\beta \partial_z^\gamma \left( \varphi(z,F^M - y) \partial_z^\gamma \Psi_2(F^M_t(x) - y) \right) \leq K_{d,m,q,T} \left( 1 + \frac{1}{(|y| - 3)^{2m+q+1}} \right) \lambda_M^{(d+2m+q+1)^3}.
\]

**Proof:** (in this proof we will denote simply \( J^M_{T_j} \) by \( J \); let us set \( \delta_M \equiv \sqrt{U_M(t)} \) and temporarily \( \Psi \equiv \partial_y \Psi_2 \) and \( f(u,z) \equiv \delta_M u + x_i(x,z_1,\ldots,z_j) \))

\[
\partial_x^\alpha \partial_y^\beta \partial_z^\gamma \left( \varphi(z,F^M - y) \Psi(F^M_t(x) - y) \right)
\]

\[
= E \left[ \partial_x^\alpha \int_{R^d} \nu(du) \int_{R^d} \partial_y^\beta \partial_z^\gamma (\varphi(u,z) \Psi(\delta_M u + x_i(x,z_1,\ldots,z_j) P_{J,x}(z_1,\ldots,z_j)) dz_1 \cdots dz_j \right]
\]

\[
+ \sum_{\alpha_1 \oplus \alpha_2 \oplus \alpha_3 \neq \beta} E \left[ \int_{R^d} \nu(du) \int_{R^d} \partial_x^{\alpha_1} \left( \partial_y^\beta \partial_z^\gamma (f(u,z)) \right) \partial_x^{\alpha_2} \left( \Psi(f(u,z)) \right) P_{J,x}(z_1,\ldots,z_j) dz_1 \cdots dz_j \right]
\]

\[
= (1) + (2).
\]
For the part (1), since
\[
E \left[ \int_{\mathbb{R}^d} \nu(du) \int_{\mathbb{R}^d} \left( \partial^2 \varphi_x \left( f(u, z) \right) \right) \partial^2 \varphi_x \left( \Psi(f(u, z)) \right) P_{J,x}(z_1, \ldots, z_J) \, dz_1 \cdots dz_J \right]
\]
\[
= E \left[ \partial^2 \varphi_x \left( F^M - y \right) \partial^2 \varphi_x \left( \Psi(F^M - y) \right) \right]
\]
\[
= E \left[ \left( \sum_{(\alpha_1)} \partial \varphi_{\alpha_1} (F^M_t(x) - y) \partial N_{(\alpha_1)}^{\epsilon} \right)^{\beta_1} \cdots \partial N_{(\alpha_1)}^{\epsilon} \left( \sum_{(\alpha_2)} \partial \varphi_{\alpha_2} (F^M_t(x) - y) \partial N_{(\alpha_2)}^{\epsilon} \right)^{\beta_2} \right],
\]
we are brought, on one hand, to prove the convergence (when \( \varepsilon \) tends to 0) and afterwards to bound the quantity
\[
E \left[ \partial \varphi_{\epsilon} (F^M_t(x) - y) \partial \varphi_{\epsilon} \Psi_2 (F^M_t(x) - y) Y_{N,N'} \right]
\]
(4.102)
with \( Y_{N,N'} \) def \( \partial N_{(\alpha_1)}^{\epsilon} (x_{\beta_1} \cdots \partial N_{(\alpha_1)}^{\epsilon} (x_{\beta_1}) \right)^{\beta_1} \cdots \partial N_{(\alpha_2)}^{\epsilon} (x_{\beta_2} \cdots \partial N_{(\alpha_2)}^{\epsilon} (x_{\beta_2}) \right)^{\beta_2} \).

On the other hand, for the part (2), Lemma 4.5 leads to
\[
\partial^2 \varphi_x P_{J,x} = P_{J,x} \sum_{l=1}^{[\alpha_3]} \sum_{M \in M_{(\alpha_3)}} c_M \prod_{i=1}^{l} \partial^M_{(\alpha_3)} \ln P_{J,x}
\]
(4.103)
so, letting \( P_{J,x}^{[\alpha_3]} = \sum_{l=1}^{[\alpha_3]} \sum_{M \in M_{(\alpha_3)}} c_M \prod_{i=1}^{l} \partial^M_{(\alpha_3)} \ln P_{J,x} \), we have :
\[
E \left[ \int_{\mathbb{R}^d} \nu(du) \int_{\mathbb{R}^d} \partial^2 \varphi_x \left( f(u, z) \right) \partial^2 \varphi_x \left( \Psi(f(u, z)) \right) P_{J,x}(z_1, \ldots, z_J) \, dz_1 \cdots dz_J \right]
\]
\[
= E \left[ \partial^2 \varphi_x \left( F^M - y \right) \partial^2 \varphi_x \left( \Psi(F^M - y) \right) \right]
\]
\[
= E \left[ \left( \sum_{(\alpha_1)} \partial \varphi_{\alpha_1} (F^M_t(x) - y) \partial N_{(\alpha_1)}^{\epsilon} \right)^{\beta_1} \cdots \partial N_{(\alpha_1)}^{\epsilon} \left( \sum_{(\alpha_2)} \partial \varphi_{\alpha_2} (F^M_t(x) - y) \partial N_{(\alpha_2)}^{\epsilon} \right)^{\beta_2} \right],
\]
so we are brought, again, to prove the convergence (when \( \varepsilon \) tends to 0) and afterwards to bound the quantity
\[
E \left[ \partial \varphi_{\epsilon} (F^M_t(x) - y) \partial \varphi_{\epsilon} \Psi_2 (F^M_t(x) - y) Y_{N,N'} \right]
\]
(4.104)
with \( Y_{N,N'} \) def \( \partial N_{(\alpha_1)}^{\epsilon} (x_{\beta_1} \cdots \partial N_{(\alpha_1)}^{\epsilon} (x_{\beta_1}) \right)^{\beta_1} \cdots \partial N_{(\alpha_2)}^{\epsilon} (x_{\beta_2} \cdots \partial N_{(\alpha_2)}^{\epsilon} (x_{\beta_2}) \right)^{\beta_2} \).

We can see a similar structure between (4.102) and (4.104) ; for the moment we will treat them at the same time ; we will temporarily denote by \( Y \) either \( Y_{N,N'} \) or \( \tilde{Y}_{N,N'} \), and keep working on
\[
E \left[ \partial \varphi_{\epsilon} (F^M_t(x) - y) \partial \varphi_{\epsilon} \Psi_2 (F^M_t(x) - y) Y \right].
\]
(4.105)

Letting \( G_M \) def \( \partial \varphi_{\epsilon} \Psi_2 (F^M_t(x) - y) \), using Theorem 1.3, we have
\[
E \left[ \partial \varphi_{\epsilon} (F^M - y) \partial \varphi_{\epsilon} \Psi_2 (F^M_t(x) - y) \right] = E \left[ \Phi_{\epsilon} (F^M - y) H_{M}^{d+|\beta|} (F_M, G_M) \right].
\]
(4.106)
It directly follows, since \( \|\Phi_{\epsilon}\|_{\infty} \leq 1 \) and the weight \( H_{M} \) does not depend on \( \varepsilon \), the pointwise convergence of \( \partial_{\alpha}(f_M, \varphi(x, y)) \) when \( \varepsilon \) tends to 0.

Now, with Theorem 1.13, and following the same pattern as we did in the proof of Lemma 3.15, for \( T > t > 0 \) such that \( \frac{d(3^q - 1)}{t} < \theta \) (recalling \( q' = d + |\beta| \)),
\[
E \left[ H_{M}^{d+|\beta|} (F_M, G_M) \right] \leq C_d \left( E \left[ (1 + |F_M|_{q'-1}^q)^4 \right] \right)^{\frac{4}{q'}} \left( E \left[ (1 + |F_M|_{q'+1}^q)^4 \right] \right)^{\frac{4}{q'}} \|G_M\|_{4}. 
\]
So, for $M$ big enough (provided that $\lambda_M \to +\infty$, when $M \to +\infty$) we find that

\[ \mathbb{E}\left[ H_M^{d+\beta}(F_M, G_M) \right] \leq C_{d,q,T} \lambda_M^{(q'-1)^2+2}(q' + \frac{d(6d+1)(d+1)}{2}) ||G_M||_{q'} \]

\[ \leq C_{d,q,T} \lambda_M^{(q'+1)^2+3(d+1)^3} ||G_M||_{q'} \]

Moreover (with Cauchy-Swartz and 1.10)

\[ ||G_M||_{q'} \leq C_{q'} ||\partial_{\beta'} \psi_2(F_M(x) - y)||_{q'} ||Y||_{q'} \]

\[ \leq C_{q'} ||\partial_{\beta'} \psi_2(F_M(x) - y)||_{q'} ||Y||_{q'} \]

With the fact that, for all multi-index $\tau$

\[ \partial_{\beta'} \psi_2(u) \leq C_{1} |u| \leq 3 \] \hspace{1cm} (4.107)

using 4.7, 4.1 (and also 1.10), it appears that

\[ ||Y_N,N'||_{q'} \leq C_{l,p,T} \lambda_M^{\frac{q'^2}{2}} ||Y||_{q'} \leq C_{l,p,T} \lambda_M^{\frac{q'^2}{2}} \]

and

\[ ||\bar{Y}_{N,N'}||_{q'} \leq C_{l,p,q,T} \lambda_M^{\frac{q'^2}{2}} ||\partial_{\beta'} \psi_2(F_M(x) - y)||_{q'} \leq C_{l,p,q,T} \lambda_M^{\frac{q'^2}{2}} \]

Since $\lambda_M \to +\infty$, we have finally

\[ \partial_{\alpha} \mathbb{E}\left[ \partial_{\beta'} \varphi(F_M - y) \partial_{\gamma} \psi_2(F_M(x) - y) \right] \leq K_{d,m,q,T} \lambda_M^{\frac{q'^2}{2}} \lambda_M^{A(d,m,q,q')} \]

with

\[ A(d, m, q, q') \overset{\text{def}}{=} (q' + 1)^3 + 3(d + 1)^3 + \frac{q'^2}{2} + (2m + q)(q'^2 + (2m + q + 1)q' + 1). \]

Now, setting $\bar{q} \overset{\text{def}}{=} d + 2m + q \geq q'$,

\[ A(d, m, q, q') \leq 4(\bar{q} + 1)^3 + 2\bar{q}^3 + \frac{3}{2}\bar{q}^2 + \bar{q} \leq 6(\bar{q} + 1)^3. \]
5 Regenerative scheme and Harris-recurrence

We assume that the process \( X_t \), solution of (??), admits a transition density for any time \( t > 0 \), denoted \( p_t(x, y) \), which is strictly positive and continuous in \( x \) and \( y \); a criteria for such a situation is given by Theorem 2.1.

As a consequence, for any \( t > 0 \) and any compact set \( C \), there exists a probability measure \( \nu \) and a constant \( \alpha > 0 \) such that the local Doeblin condition is verified:

\[
P_t(x, dy) \geq \alpha \mathbb{1}_C(x) \nu(dy).
\]

(5.109)

In order to obtain some ergodic result over the stochastic process \( X_t \), the main heuristic idea is to approximate, when \( t \to +\infty \), the quantity \( \frac{1}{n} \sum_{i=1}^{n} f(X_s) ds \) by \( \frac{1}{n} \sum_{i=1}^{n} f_{R_i} f(X_s) ds \), where the r.v. \( R_i \) are to be defined and where the r.v. \( f_{R_i} f(X_s) ds \) would be i.i.d which will allow to conclude by applying the strong law of large numbers.

To do so in a rigorous way, we will follow the path developed by Eva Löcherbach in [9], Ergodicity and speed of convergence to equilibrium for diffusion processes, 2013 (cf. also Ikeda, Nagasawa and Watanabe (1966) [8]). First, given a càdlàg Markov process \( Y_t \), we will define the notion of regeneration times:

**Definition 5.1** A sequence \( (R_n)_{n \geq 1} \) is called generalized sequence of regeneration times, if

1. \( R_n \uparrow \infty \) as \( n \to +\infty \).
2. \( R_{n+1} = R_n + R_1 \circ \vartheta_{R_n} (\vartheta \text{ is the shift operator defined by } \vartheta f = f(t + .), \text{ where } f : \mathbb{R}_+ \to E \text{ is càdlàg}). \)
3. \( Y_{R_{n+1}} \) is independent of \( \mathcal{F}_{S_{R_n}} \).
4. At regeneration times, the process starts afresh from \( Y_{R_n} \sim \nu(dy) \).
5. The trajectories \( (Y_{R_{n+1}}, 0 \leq s \leq R_{n+1} - R_n) \) are 2-independent, i.e. \( (Y_{R_{n+1}}, 0 \leq s \leq R_{n+1} - R_n) \) and \( (Y_{R_{m+1}}, 0 \leq s \leq R_{m+1} - R_m) \) are independent if and only if \( \mid m - n \mid \geq 2 \).

These regeneration times do not exist for the original solution \( X_t \), but they exist for a version of the process on an extended probability space, rich enough to support the driving Brownian motion, the Poisson measure and an i.i.d sequence of uniform random variable \((U_n)_{n \geq 1}\).

We will construct, then, a stochastic process \((Y_t)_{t \geq 0}\) on this richer probability space, equal in law to \((X_t)_{t \geq 0}\).

First we will fix a compact \( C \) and a time parameter \( t_* > 0 \) such that (5.109) is true:

\[
P_{t_*}(x, dy) \geq \alpha \mathbb{1}_C(x) \nu(dy).
\]

Then we set \( Y_t = X_t \) for all \( 0 \leq t \leq \tilde{S}_1 \), where

\[
\tilde{S}_1 \overset{\text{def}}{=} \inf \{ t \geq t_* : X_t \in C \} \quad \text{and} \quad \tilde{R}_1 \overset{\text{def}}{=} \tilde{S}_1 + t_*.
\]

At time \( \tilde{S}_1 \), we choose \( U_1 \), the first of the uniform random variables. If \( U_1 \leq \alpha \), we choose

\[
Y_{\tilde{R}_1} \sim \nu(dy).
\]

(5.110)

Else, if \( U_1 > \alpha \), given \( Y_{\tilde{S}_1} = x \), we choose

\[
Y_{\tilde{R}_1} \sim \frac{P_{t_*}(x, dy) - \alpha \nu(dy)}{1 - \alpha}.
\]

(5.111)

Finally, given \( Y_{\tilde{R}_1} = y \), we fill in the missing trajectory \((Y_t)_{t \in [\tilde{S}_1, \tilde{R}_1]}\) between time \( \tilde{S}_1 \) and time \( \tilde{R}_1 \) according to the diffusion bridge law

\[
\frac{p_{t - \tilde{S}_1}(x, z) p_{\tilde{R}_1 - t}(z, y)}{p_{t_*}(x, y)} \, dz.
\]

(5.112)

Notice that by construction, if we do not care about the exact choice of the auxiliary random variable \( U_1 \), then we have that \((Y_t)_{t \leq \tilde{R}_1} \overset{\text{L}}{=} (X_t)_{t \leq \tilde{R}_1}\).
We continue this construction after time \( \tilde{R}_1 \) : choose \( Y_t \) equal to \( X_t \) for all \( t \in [\tilde{R}_1, \tilde{S}_2] \) where

\[
\tilde{S}_2 \overset{\text{def}}{=} \inf\{ t > \tilde{R}_1 : X_t \in C \} \quad \text{and} \quad \tilde{R}_2 \overset{\text{def}}{=} \tilde{S}_2 + t_*.
\]

At time \( \tilde{S}_2 \), we choose \( U_2 \) in order to realize the choice of \( Y_{\tilde{R}_2} \) according to the splitting of the transition kernel \( P_{\tilde{R}_2} \), as in (5.110) and (5.111). More generally, the construction is therefore achieved along the sequence of stopping times

\[
\tilde{S}_{n+1} \overset{\text{def}}{=} \inf\{ t > \tilde{R}_n : X_t \in C \} \quad \text{and} \quad \tilde{R}_{n+1} \overset{\text{def}}{=} \tilde{S}_{n+1} + t_* , \quad n \geq 1,
\]

where during each \( [\tilde{R}_n, \tilde{S}_{n+1}] \), \( Y \) follows the original solution of the SDE, whereas the intervals \([\tilde{S}_{n+1}, \tilde{R}_{n+1}]\) are used to construct the splitting. In particular, every time that we may choose a transition according to (5.110), we introduce a regeneration event for the process \( Y \), and therefore the following two sequences of generalized stopping times will play a role. Firstly,

\[
S_1 = \inf\{ \tilde{S}_n : U_n \leq \alpha \} , \quad \ldots , \quad S_n = \inf\{ \tilde{S}_m > S_{n-1} : U_m \leq \alpha \} , \quad n \geq 2,
\]

and secondly,

\[
R_n \overset{\text{def}}{=} S_n + t_* , \quad n \geq 1 .
\]

The above construction of the process \( X \), since at each time \( \tilde{S}_n \), a projection into the future is made.

Let \( \tilde{N}_t \overset{\text{def}}{=} \sup\{ n : U_n \leq t \} \) and

\[
\mathcal{F}_t^Y = \sigma\{ Y_s, s \leq t, U_n, Y_{\tilde{R}_n}, n \leq \tilde{N}_t \} , \quad t \geq 0 ,
\]

be the canonical filtration of the process \( Y \). The sequence of \( (\mathcal{F}_t^Y)_{t \geq 0} \)-stopping times \( (R_n)_{n \geq 1} \) is a generalized sequence of regeneration times as it was defined in Definition 5.1.

**Remark 5.1** The trajectories of \( Y \) are not the same as those of the original solution \( X \) of the SDE. However, by definition, the Harris-recurrence is only a property in law. As a consequence, if, for a given set \( A \), we succeed to show that almost surely, \( Y \) visits it infinitely often, the same is automatically true for \( X \) as well.

We can now state the theorem we were looking for (the demonstration of it is directly taken from Eva Löcherbach’s lecture, *Ergodicity and speed of convergence to equilibrium for diffusion processes*, and we give it here only for the convenience of the reader):

**Theorem 5.2** If for all \( x \in \mathbb{R}^d \) we have \( P_x[R_1 < \infty] = 1 \), then the process \( X \) is recurrent in the sense of Harris.

**Proof :**

Define a measure \( \pi \) on \( (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \) by

\[
\pi(A) \overset{\text{def}}{=} \mathbb{E} \left[ \int_{\tilde{R}_1}^{\tilde{R}_2} 1(Y_s) \, ds \right] , \quad A \in \mathcal{B}(\mathbb{R}^d).
\]

For any \( n \geq 2 \), put \( \xi_n \overset{\text{def}}{=} \int_{\tilde{R}_{n-1}}^{\tilde{R}_n} 1(Y_s) \, ds \). By construction, the random variables \( \xi_{2n}, n \geq 1 \), are i.i.d. and so are, on the other hand, as well, the random variables \( \xi_{2n+1} \). Put

\[
N_t = \sup\{ n : R_n \leq t \}
\]

and observe that \( N_t \to \infty \) as \( t \to \infty \). Hence, applying the strong law of large numbers separately to the sequence \( (\xi_{2n})_{n \geq 1} \) and the sequence \( (\xi_{2n+1})_{n \geq 1} \), we have that

\[
\lim_{t \to \infty} \frac{\int_{0}^{t} 1(Y_s) \, ds}{N_t} = \pi(A)
\]

P\(_x\)-almost surely, for any \( x \in \mathbb{R}^d \). This implies that any set \( A \) such that \( \pi(A) > 0 \) is visited infinitely often by the process \( Y \) almost surely. Thus, we have the recurrence property also for the process \( X \), for any set \( A \) such that \( \pi(A) > 0 \). Then, by a deep theorem of Azéma, Duflo and Revuz (1969) [1], see also Theorem 1.2. of Höpfner and Löcherbach (2003) [7], the process is indeed Harris.

•
A Miscellaneous

A.1 Regularisation

Let us define on $\mathbb{R}$ the following function:

$$\varphi(x) \overset{\text{def}}{=} \begin{cases} \alpha \exp\left(-\frac{1}{1-x^2}\right) & \text{si } |x| < 1 \\ 0 & \text{si } |x| \geq 1 \end{cases}$$

where $\alpha$ is chosen in order to have $\int_{\mathbb{R}} \varphi(x) \, dx = 1$. Numerically, $\alpha \approx (0, 44399)^{-1}$.

**Proposition A.1** $\varphi : \mathbb{R} \to \mathbb{R}$ is $C^\infty$ compact support function.

We then define $\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}$ by:

$$\varphi_\varepsilon(x) \overset{\text{def}}{=} \frac{1}{\varepsilon} \varphi\left(\frac{x}{\varepsilon}\right), \quad \forall x \in \mathbb{R}.$$ 

Thus defined, function $\varphi_\varepsilon$ converges weakly, as $\varepsilon$ tends to 0, to the Dirac distribution; it is a mollifier: for every continuous function $f$,

1. $\lim_{\varepsilon \to 0} f \ast \varphi_\varepsilon(x) = f(x)$
2. $f \ast \varphi_\varepsilon$ is $C^\infty$

(with $f \ast \varphi_\varepsilon(x) \overset{\text{def}}{=} \int_{\mathbb{R}} f(y) \varphi_\varepsilon(x-y) \, dy$).

**Proposition A.2** $\varphi_\varepsilon : \mathbb{R} \to \mathbb{R}$ remains a $C^\infty$ compact support function and there exists $M > 0$ such that:

$$\forall x \in \mathbb{R}, \quad |\varphi'_\varepsilon(x)| \leq \frac{M}{\varepsilon^2} \quad \text{and} \quad |\varphi''_\varepsilon(x)| \leq \frac{M}{\varepsilon^3}.$$ 

**Proof:** If $|x| \geq \varepsilon$,

$$\varphi'_\varepsilon(x) = \varphi''_\varepsilon(x) = 0.$$

Elsewhere, $\varphi_\varepsilon(x) = \frac{\alpha}{\varepsilon} \exp\left(-\frac{1}{1-\left(\frac{x}{\varepsilon}\right)^2}\right)$ and

$$\varphi'_\varepsilon(x) = \frac{\alpha}{\varepsilon^2} \exp\left(-\frac{1}{1-\left(\frac{x}{\varepsilon}\right)^2}\right) \times \frac{-2x}{\varepsilon^2} \times \frac{1}{1-\left(\frac{x}{\varepsilon}\right)^2} = \frac{-2\alpha}{\varepsilon^2} \exp\left(-\frac{1}{1-\left(\frac{x}{\varepsilon}\right)^2}\right) \times \frac{\frac{x}{\varepsilon}}{1-\left(\frac{x}{\varepsilon}\right)^2}.$$ 

The function $y \mapsto \exp\left(-\frac{1}{1-y^2}\right) \frac{y}{(1-y^2)^2}$ is defined and continuous (considering a continuous extension for $y = \pm 1$) on $\mathbb{R}$ and bounded, as we can easily prove by considering its limits when $y$ tends respectively to $\pm \infty$, $-1$ and 1.

Denoting now by $M_1$ an upperbound and letting $C_1 = 2\alpha M_1$, we obtain the first assumption.

Now,

$$\varphi''_\varepsilon(x) = \frac{\alpha}{\varepsilon^3} \exp\left(-\frac{1}{1-\left(\frac{x}{\varepsilon}\right)^2}\right) \times \frac{-2x}{\varepsilon^2} \times \frac{1}{1-\left(\frac{x}{\varepsilon}\right)^2} \times \frac{-2x}{\varepsilon^2} \left(1-\left(\frac{x}{\varepsilon}\right)^2\right)^2 + \frac{\partial}{\partial x} \left(\frac{-2x}{\varepsilon^2} \left(1-\left(\frac{x}{\varepsilon}\right)^2\right)^2\right)$$

$$= \frac{\alpha}{\varepsilon^3} \exp\left(-\frac{1}{1-\left(\frac{x}{\varepsilon}\right)^2}\right) \left(\frac{4\left(\frac{x}{\varepsilon}\right)^2}{(1-\left(\frac{x}{\varepsilon}\right)^2)^4} - \frac{2}{(1-\left(\frac{x}{\varepsilon}\right)^2)^2} + \frac{4x}{\varepsilon^2} \times \frac{-2x}{(1-\left(\frac{x}{\varepsilon}\right)^2)^3}\right).$$

Similarly, the function $y \mapsto \exp\left(-\frac{1}{1-y^2}\right) \left(\frac{4y^2}{(1-y^2)^2} - \frac{2}{(1-y^2)^2} - \frac{8y^2}{(1-y^2)^3}\right)$ is bounded on $\mathbb{R}$ by a constant $M_2$. With $C_2 = \alpha M_2$ and $M = \max(C_1, C_2)$ we then obtain the last property.

We then define $h_\varepsilon : \mathbb{R} \to \mathbb{R}$ by:

$$h_\varepsilon(x) \overset{\text{def}}{=} |x| \vee 2\varepsilon$$

and $\phi_\varepsilon : \mathbb{R} \to \mathbb{R}$ by:

$$\phi_\varepsilon(x) = h_\varepsilon \ast \varphi_\varepsilon(x).$$
Proposition A.3  

1. $\phi_\varepsilon$ converges pointwise to the absolute value function $x \mapsto |x|$ and

$$
\phi_\varepsilon(x) = \begin{cases} 
2\varepsilon & \text{if } |x| \leq \varepsilon \\
|x| & \text{if } |x| > 3\varepsilon 
\end{cases}
$$

and

$$
0 \leq \phi_\varepsilon(x) \leq 4\varepsilon \quad \text{if } |x| \in [\varepsilon, 3\varepsilon].
$$

2. There exists $C > 0$ such that

$$
\forall x \in \mathbb{R}, \quad |\phi'_\varepsilon(x)| \leq C \quad \text{and} \quad |\phi''_\varepsilon(x)| \leq \frac{C}{\varepsilon} \mathbb{1}_{|x| \leq 3\varepsilon}.
$$

**Proof:** First, let us remark that, since $\phi_\varepsilon$ is defined by a convolution, it is a smooth function and, for all $n \in \mathbb{N}^*$, $\phi_\varepsilon^{(n)}(x) = h_\varepsilon \ast \varphi_\varepsilon^{(n)}(x)$.

We will now prove the two items by dividing the problem into the following three cases.

- **Case $|x| \leq \varepsilon$:**
  
  We have

  $$
  \phi_\varepsilon(x) = h_\varepsilon \ast \varphi_\varepsilon(x) = \int_{-2\varepsilon}^{2\varepsilon} \varphi_\varepsilon(x - y) h_\varepsilon(y) \, dy \\
  = 2\varepsilon \int_{-2\varepsilon}^{2\varepsilon} \varphi_\varepsilon(x - y) \, dy \\
  = 2\varepsilon \int_{\mathbb{R}} \varphi_\varepsilon(z) \, dz = 2\varepsilon.
  $$

  Hence, for all $|x| < \varepsilon$,

  $$
  \phi'_\varepsilon(x) = \phi''_\varepsilon(x) = 0,
  $$

  (which remains true when $|x| = \varepsilon$, since $\phi_\varepsilon$ is a smooth function).

- **Case $|x| > 3\varepsilon$:**
  
  noticing that $z \mapsto z \varphi_\varepsilon(z)$ is an odd function (with compact support),

  $$
  \phi_\varepsilon(x) = \int_{\mathbb{R}} h_\varepsilon(x - z) \varphi_\varepsilon(z) \, dz = \int_{|z| \leq \varepsilon} h_\varepsilon(x - z) \varphi_\varepsilon(z) \, dz \\
  = \int_{|z| \leq \varepsilon} |x - z| \varphi_\varepsilon(z) \, dz \\
  = \int_{|z| \leq \varepsilon} \text{sign}(x - z) \varphi_\varepsilon(z) \, dz \\
  = \text{sign}(x) \times \left( x - \int_{\mathbb{R}} \varphi_\varepsilon(z) z \, dz \right) = |x|,
  $$

  Hence, for all $|x| > 3\varepsilon$,

  $$
  \phi'_\varepsilon(x) = \pm 1 \quad \text{and} \quad \phi''_\varepsilon(x) = 0,
  $$

  (and obviously again, it remains true when $|x| = 3\varepsilon$).

- **Case $|x| \in [\varepsilon, 3\varepsilon]$:**
  
  Let $g$ be a continuous function (with compact support), then

  $$
  h_\varepsilon \ast g(x) = \int_{\mathbb{R}} g(x - y) h_\varepsilon(y) \, dy = \int_{-4\varepsilon}^{4\varepsilon} g(x - y) h_\varepsilon(y) \, dy \\
  \leq 4\varepsilon \int_{\mathbb{R}} |g(z)| \, dz.
  $$
With \( g \overset{\text{def}}{=} \phi_\varepsilon \), \( \int_{\mathbb{R}} |g(z)| \, dz = 1 \) and it follows that
\[
\phi_\varepsilon(x) \leq 4\varepsilon ;
\]
with respectively \( g \overset{\text{def}}{=} \phi'_\varepsilon \) and \( g \overset{\text{def}}{=} \phi''_\varepsilon \), and using the Proposition A.2, it follows that
\[
\phi'_\varepsilon(x) = h_\varepsilon \ast \phi'_\varepsilon \leq 4\varepsilon \int_{-\varepsilon}^{\varepsilon} M \frac{dy}{\varepsilon^2} = 8M ;
\]
and
\[
\phi''_\varepsilon(x) = h_\varepsilon \ast \phi''_\varepsilon \leq 4\varepsilon \int_{-\varepsilon}^{\varepsilon} M \frac{dy}{\varepsilon} = \frac{8M}{\varepsilon}.
\]
Gathering the three cases, we may conclude.

### A.2 Gronwall’s lemma

In all this work we used the following version of the Gronwall’s lemma:

**Proposition A.4** If a measurable function \( g : [0, T] \rightarrow \mathbb{R}^+ \) is such that

1. \( G = \sup_{t \in [0, T]} g(t) < +\infty \);
2. for all \( t \in [0, T] \),
\[
g(t) \leq A + B \int_0^t g(s) \, ds
\]
then, for all \( t \in [0, T] \),
\[
g(t) \leq A \exp(Bt).
\]

**Proof:** It is easy to obtain by induction that, for every \( n \in \mathbb{N}^* \),
\[
g(t) \leq A \left( 1 + \sum_{k=1}^{n-1} \frac{(Bt)^k}{k!} + B^n \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} g(t_n) \, dt_n \cdots dt_1 \right),
\]
which implies
\[
g(t) \leq A \left( 1 + \sum_{k=1}^{n-1} \frac{(Bt)^k}{k!} + G\left(\frac{(Bt)^n}{n!}\right)\right).
\]

Since \( \lim_{n \rightarrow +\infty} G\left(\frac{(Bt)^n}{n!}\right) = 0 \), the assertion follows.

### A.3 Moment inequalities

In this subsection of the appendix we prove some moment inequalities which we used in Subsection ??, so the notations introduced in that subsection prevail.

We assume in this subsection that \( \mu(E) < \infty \). This is just to simplify the notations — in concrete applications we will replace \( \mu \) by \( 1_{G} \mu \). Then we consider an index set \( \Lambda \) and we denote by \( \alpha \) the elements of \( \Lambda \). Moreover we consider a family of processes \( V^\alpha_t \in \mathbb{R}^d, \alpha \in \Lambda \) which verify the following equation
\[
V^\alpha_t = V^\alpha_0 + \sum_{l=1}^m \int_0^t (H^\alpha_l(s) + \langle \nabla \sigma_l(X_s), V^\alpha_s \rangle) \, dW^l_s + \int_0^t \left( h^\alpha(s) + \langle \nabla b(X_s), V^\alpha_s \rangle \right) \, ds + \int_0^t \int_{E \times (0,1)} (Q^\alpha(s-\varepsilon, z) + \langle \nabla_c c(X_{s-\varepsilon}, \cdot), V^\alpha_{s-\varepsilon} \rangle) \, 1_{\{u \leq \gamma(X_{s-\varepsilon}, \cdot)\}} \, dN(s, u, z).
\]

Here \( H^\alpha_l, h^\alpha \) and \( Q^\alpha \) are adapted càdlàg processes which verify
\[
\int_0^T \left( |H^\alpha_l(s)|^2 + |h^\alpha(s)| + \int_E |Q^\alpha(s, z)| \, d\mu(z) \right) \, ds < \infty.
\]
(Where the functions \( \sigma, b, c \) and \( \gamma \) are the ones introduced in Subsection ??.) So the corresponding stochastic integrals in (A.113) make sense.
Proposition A.5 We suppose that
\[ |Q^\alpha(s, z)| \leq \bar{q}(z) |R^\alpha_s| \]  
for some adapted càdlàg process \( R^\alpha \) and some measurable function \( \bar{q} : E \to \mathbb{R}_+ \) and we denote
\begin{align*}
\bar{c}_1(p) &= \int_E (\bar{q}(z) + \bar{c}(1)(z))(1 + \bar{q}(z))^{2p} \tau(z) d\mu(z), \\
\bar{c}_2(p) &= \int_E (\bar{q}(z) + \bar{c}(1)(z))(1 + \bar{c}(1)(z))^{2p} \tau(z) d\mu(z).
\end{align*}

For every \( p \in \mathbb{N} \) there exists a universal constant \( C_p \) such that
\[ 0 \leq t \leq T \]
\[ E \left[ |V^\alpha_t|^2p \right] \leq \exp \left( C_p (1 + \|\nabla \sigma\|^2p_\infty + \|\nabla b\|^2p_\infty + \bar{c}_2(p)) \right) 
\times \left( |V^\alpha_0|^2p + C_p \int_0^t E \left[ \sum_{l=1}^m |H^\alpha_l(s)|^2p + |h^\alpha(s)|^2p + \bar{c}_1(p) |R^\alpha_{s-}|^2p \right] ds \right)
\]

Proof: Using Itô’s formula for \( f(x) = x^{2p} \), we obtain\(^7\)
\[ (V^\alpha_t)^{2p} = (V^\alpha_0)^{2p} + M^\alpha_t + I^\alpha_t + J^\alpha_t \]
with
\[ M^\alpha_t = \sum_{l=1}^m \int_0^t 2p(V^\alpha_{s-})^{2p-1} (H^\alpha_l(s) + <\nabla \sigma_l(X_s), V^\alpha_s>) dW^s, \]
\[ I^\alpha_t = \sum_{l=1}^m \int_0^t p(2p - 1)(V^\alpha_{s-})^{2p-2} \sum_{l=1}^m (H^\alpha_l(s) + <\nabla \sigma_l(X_s), V^\alpha_s>)^2 ds \]
\[ + 2p \int_0^t (V^\alpha_{s-})^{2p-1} (h^\alpha(s) + <\nabla b(X_s), V^\alpha_s>) ds \]
and
\[ J^\alpha_t = \int_0^t \int_{E \times (0,1)} \left( (V^\alpha_{s-} + Q^\alpha(s-, z) + <\nabla xe(X_{s-}, z), V^\alpha_{s-}>)^{2p} - (V^\alpha_{s-})^{2p} \right) 1_{\{u \leq \gamma(X_{s-}, z)\}} dN(s, u, z). \]

Using the trivial inequality \( a^u b^v \leq a^{u+v} + b^{u+v} \) we obtain
\[ E \left[ |H^\alpha_t|^2 \right] \leq C_p \int_0^t E \left[ \sum_{l=1}^m |H^\alpha_l(s)|^2p + |h^\alpha(s)|^2p \right] ds \]
\[ + C_p (1 + \|\nabla \sigma\|^2p_\infty + \|\nabla b\|^2p_\infty) \int_0^t E \left[ |V^\alpha_s|^2p \right] ds. \]

We estimate now \( J^\alpha_t \). Using the elementary inequality
\[ (a + b)^{2p} - a^{2p} \leq C_p |b| (|a|^{2p-1} + |b|^{2p-1}) \]
we obtain
\[ |V^\alpha_{s-} + Q^\alpha(s-, z) + <\nabla xe(X_{s-}, z), V^\alpha_{s-}>|^{2p} - |V^\alpha_{s-}|^{2p} \]
\[ \leq C_p (|Q^\alpha(s-, z)| + \bar{c}(1)(z) |V^\alpha_{s-}|)(|V^\alpha_{s-}|^{2p-1} (1 + \bar{c}(1)(z)) + |Q^\alpha(s-, z)|^{2p-1}). \]
Recall that \( |Q^\alpha(s-, z)| \leq \bar{q}(z) |R^\alpha_{s-}| \) so the above term is upper bounded by
\[ C_p (\bar{q}(z) |R^\alpha_{s-}| + \bar{c}(1)(z) |V^\alpha_{s-}|)(|V^\alpha_{s-}|^{2p-1} (1 + \bar{c}(1)(z))^{2p-1} + \bar{q}(z) R^\alpha_{s-}^{2p-1}) \]
\[ \leq C_p (\bar{q}(z) + \bar{c}(1)(z))(|R^\alpha_{s-}| + |V^\alpha_{s-}|)((1 + \bar{c}(1)(z))^{2p-1} |V^\alpha_{s-}|^{2p-1} + \bar{q}(z) R^\alpha_{s-}^{2p-1}). \]

We use once again the inequality \( a^u b^v \leq a^{u+v} + b^{u+v} \) and we upper bound the above term by
\[ C_p (\bar{q}(z) + \bar{c}(1)(z))(|R^\alpha_{s-}|^{2p}(1 + \bar{q}(z))^{2p} + (1 + \bar{c}(1)(z))^{2p} |V^\alpha_{s-}|^{2p}). \]

\(^7\)For \( x = (x_i)_{1 \leq i \leq d} \in \mathbb{R}^d \) we simply denote \((x_i^{2p})_{1 \leq i \leq d}\) by \( x^{2p}\).
It follows that
\[ E \left[ \left| J_1^\alpha \right| \right] \leq C_p \int_{t}^{t_f} \left( \eta(z) + \bar{c}_1(\alpha)(1 + \eta(z)) \right)^{2p} \left( \tau(z) \right) d\mu(z) + \int_{0}^{t} E \left[ \left| R_{s-}^\alpha \right|^{2p} \right] ds \]
\[ + C_p \int_{t}^{t_f} \left( \eta(z) + \bar{c}_1(\alpha)(1 + \eta(z)) \right)^{2p} \left( \tau(z) \right) d\mu(z) + \int_{0}^{t} E \left[ \left| V_{s}^\alpha \right|^{2p} \right] ds. \]

Since \( M_t^\alpha \) is a martingale we obtain
\[ E \left[ (V_t^\alpha)^{2p} \right] = E \left[ (V_0^\alpha)^{2p} \right] + E \left[ I_t^\alpha \right] + E \left[ J_t^\alpha \right] \]
and (we recall the notation in (A.115))
\[ E \left[ |V_t^\alpha|^{2p} \right] \leq |V_0^\alpha|^{2p} + E \left[ |I_t^\alpha| \right] + E \left[ |J_t^\alpha| \right] \]
\[ \leq |V_0^\alpha|^{2p} + C_p \int_{0}^{t} E \left[ \sum_{i=1}^{m} |H_i^\alpha(s)|^{2p} + |h^\alpha(s)|^{2p} + \bar{c}_1(p) \right] \left| R_{s-}^\alpha \right|^{2p} \right] ds \]
\[ + C_p(1 + \|\nabla a\|_{\infty}^{2p} + \|\nabla b\|_{\infty}^{2p} + \bar{c}_2(p)) \int_{0}^{t} E \left[ |V_s^\alpha|^{2p} \right] ds. \]
The Gronwall’s lemma then gives
\[ E \left[ |V_t^\alpha|^{2p} \right] \leq \exp \left( C_p t(1 + \|\nabla a\|_{\infty}^{2p} + \|\nabla b\|_{\infty}^{2p} + \bar{c}_2(p)) \right) \]
\[ \times \left( |V_0^\alpha|^{2p} + C_p \int_{0}^{t} E \left[ \sum_{i=1}^{m} |H_i^\alpha(s)|^{2p} + |h^\alpha(s)|^{2p} + \bar{c}_1(p) \right] \left| R_{s-}^\alpha \right|^{2p} \right] ds. \]

\[ \bullet \]

### A.4 Proof of (2.40)

\[ I = E \left[ f(X_{T_k}^M) \mathbb{1}_{\{U_k \geq \gamma(Z_k, X_{T_k}^M)\}} \left| X_{T_k}^M = x \right. \right] \]
\[ = \int_{\{X_{T_k}^M = x\}} f(X_{T_k}^M) \mathbb{1}_{\{U_k \geq \gamma(Z_k, X_{T_k}^M)\}} \frac{dP}{\mathbb{P}(X_{T_k}^M = x)} \]
\[ = \int_{\{X_{T_k}^M = x\} \cap \{U_k \geq \gamma(Z_k, X_{T_k}^M)\}} f(X_{T_k}^M) \frac{dP}{\mathbb{P}(X_{T_k}^M = x)} \]

On the event \( \{X_{T_k}^M = x\} \cap \{U_k \geq \gamma(Z_k, X_{T_k}^M)\} \) we have \( X_{T_k}^M = x \), so \( I \) becomes
\[ I = f(x) \frac{P((X_{T_k}^M = x) \cap \{U_k \geq \gamma(Z_k, X_{T_k}^M)\})}{P(X_{T_k}^M = x)} \]
\[ = f(x) \frac{P(U_k \geq \gamma(Z_k, x))}{P(X_{T_k}^M = x)} \]
\[ = f(x) \frac{1}{\mu(B_{M+1})} \int_{B_{M+1}}^{2\pi} \mathbb{1}_{\{u \geq \gamma(z, x)\}} d\mu(z) \]
\[ = f(x) \frac{1}{\mu(B_{M+1})} \int_{B_{M+1}} \left( 1 - \frac{\gamma(z, x)}{2\pi} \right) d\mu(z) \]
\[ = f(x) \theta_{M, \gamma}(x) = f(x) \theta_{M, \gamma}(x) \int_{\mathbb{R}^d} \phi(z - z_{M}^\ast) dz \]
\[ = \int_{|z| > M+1} f(x + c_M(z, x)) \phi(z - z_{M}^\ast) \theta_{M, \gamma}(x) dz \]
\[ = \int_{|z| > M+1} f(x + c_M(z, x)) q_M(z, x) dz. \]
On the other hand:

\[ J = \mathbb{E} \left[ f(X_{T_k}^M) \mathbb{1}_{\{U_k < \gamma(Z_k, X_{T_k}^M)\}} | X_{T_k}^M = x \right] \]

\[ = \int_{\{X_{T_k}^M = x\}} f(X_{T_k}^M) \mathbb{1}_{\{U_k < \gamma(Z_k, X_{T_k}^M)\}} \frac{dP}{P(X_{T_k}^M = x)} \]

\[ = \int_{\{X_{T_k}^M = x\} \cap \{U_k \geq \gamma(Z_k, X_{T_k}^M)\}} f(x + c_M(Z_k, x)) \frac{dP}{P(X_{T_k}^M = x)} \]

\[ = \int_{\Omega} f(x + c_M(Z_k, x)) \mathbb{1}_{\{U_k < \gamma(z, x)\}} \ dP \ 	ext{ (since } U_k, Z_k \text{ and } X_{T_k}^M \text{ are independent)} \]

\[ = \frac{1}{\mu(B_{M+1})} \int_{B_{M+1}} \int_0^2 \int f(x + c_M(z, x)) \frac{\gamma(z, x)}{2c} \ d\mu(z) \ 	ext{ (with } d\mu(z) = h(z) \ dz) \]

We finally have

\[ \mathbb{E} \left[ f(X_{T_k}^M) | X_{T_k}^M = x \right] = \int_{\mathbb{R}^d} f(x + c_M(z, x)) q_M(z, x) \ dz. \]

Now \( \sigma(X_{T_k+1}^M) \subset \mathcal{G}_k \), so

\[ \mathbb{E} \left[ f(X_{T_k+1}^M) | \overline{X}_{T_k+1}^M = x \right] \]

\[ = \mathbb{E} \left[ f(X_{T_k+1}^M) + c_M(Z_{k+1}, \overline{X}_{T_k+1}^M) | \overline{X}_{T_k+1}^M = x \right] \]

\[ = \mathbb{E} \left[ \mathbb{E} \left[ f(X_{T_k+1}^M) + c_M(Z_{k+1}, \overline{X}_{T_k+1}^M) | \mathcal{G}_k \right] | \overline{X}_{T_k+1}^M = x \right] \]

\[ = \mathbb{E} \left[ \int_{\mathbb{R}^d} f(X_{T_k+1}^M) + c_M(z, \overline{X}_{T_k+1}^M) q_M(\overline{X}_{T_k+1}, z) \frac{dP}{P(\overline{X}_{T_k+1}^M = x)} \ dz \right] \]

\[ = \int_{\mathbb{R}^d} f(x + c_M(z, x)) q_M(z, x) \ dz \]

So we have:

\[ \mathbb{E} \left[ f(X_{T_k}^M) | X_{T_k}^M = x \right] = \mathbb{E} \left[ \mathbb{E} \left[ f(X_{T_k}^M) | \overline{X}_{T_k}^M = x \right] \right] \]

(A.117)

We can prove now that the processes \( X_t^M \) and \( \overline{X}_t^M \) are sharing the same law.

A.5 \( X_t^M \) and \( \overline{X}_t^M \) share the same law

- First, if \( 0 \leq t < T_1 \), \( x_t = \Psi_t(x) \) and then \( \overline{X}_t^M = \Psi_t(x) \sim X_t^M \).

- Moreover, if \( \overline{X}_{T_k}^M \sim X_{T_k}^M \), we have \( \overline{X}_{T_k}^M \sim X_{T_k}^M \); recalling that, if \( \phi(x) := \mathbb{E} [Y | X = x] \), we have \( \int_A \phi(x) \ dP_X(x) = \int_{\{x \in A\}} Y \ dP \), since \( P_{X_{T_k}^M} = P_{X_{T_k}^M} \) the relation (A.117) leads to

\[ \mathbb{E} \left[ f(X_{T_k}^M) \right] = \mathbb{E} \left[ \mathbb{E} \left[ f(X_{T_k}^M) \right] \right] \]

- Finally, if \( T_k \leq t < T_{k+1} \), \( \overline{X}_t^M = \Psi_{t-T_k}(\overline{X}_{T_k}^M) \sim \Psi_{t-T_k}(X_{T_k}^M) = X_t^M \).

B Sobolev norms of \( X_t^M \) and its derivatives.

We have used, within the proof of the Lemma 3.1 that \( X_t^M \) has moments of any order. The proof of that result follows the same pattern of this same lemma, although a bit simpler. The following result is the equation (3.60) in the special case \( k = 0 \).
Proposition B.1 Let $M \in \mathbb{N}_*$. For all $T > 0$ and $p \geq 1$, there exists a constant $C_{T,p} > 0$ (which does not depend on $M$) such that
\[
E \left[ |X_t^M |^{2p} \right] \leq C_{T,p}.
\] (B.118)

Proof:

We localize our problem by using the sequence $(\tau_K^M)_{K \in \mathbb{N}_*}$ of stopping times defined by
\[
\tau_K^M \overset{\text{def}}{=} \inf \{ t > 0 : |X_t^M | \geq K \}.
\] (B.119)

We can prove that a.s. $\lim_{K \to \infty} \tau_K^M = \infty$:
From the hypothesis made on the coefficients of $X_t^M$, it is clear that, for all $t \geq 0$,
\[
E \left[ \sup_{s \leq t} |X_s^M | \right] < \infty.
\] (B.120)

We have, for $t \geq 0$
\[
\lim_{K \to \infty} P(\tau_K^M < t) = \lim_{K \to \infty} P(\sup_{s \leq t} |X_s^M | > K) \leq \lim_{K \to \infty} \frac{1}{K} E \left[ \sup_{s \leq t} |X_s^M | \right] = 0.
\]

$(\tau_K^M)_{K \in \mathbb{N}_*}$ tends to $\infty$ in probability and so, there exists a subsequence (that we will continue to denote by $(\tau_K^M)_{K \in \mathbb{N}_*}$) which tends to $\infty$ a.s.

If we admit for the moment the Lemma B.2, we know that there exists a constant $C_{p,T}$ which does not depend on $K$ and $M$ and such that, for all $0 \leq t \leq T$,
\[
E \left[ |X_{t\wedge \tau_K^M}^{M} |^{2p} \right] \leq C_{T,p}.
\]

The monotone convergence theorem implies then
\[
E \left[ |X_t^M |^{2p} \right] = \sup_K E \left[ |X_t^M |^{2p} \mathbf{1}_{\tau_K^M > t} \right]
\]
and
\[
\sup_K E \left[ |X_t^M |^{2p} \mathbf{1}_{\tau_K^M > t} \right] = \sup_K E \left[ |X_{t\wedge \tau_K^M}^{M} |^{2p} \right] \leq C_{T,p}.
\]

Lemma B.2 Let $M \in \mathbb{N}_*$ and a sequence $(\tau_K^M)_{K \in \mathbb{N}_*}$ of stopping times defined by (B.119). There exists a constant $C_{p,T}$, which does not depend on $K$ and $M$, and such that, for all $0 \leq t \leq T$,
\[
E \left[ |X_{t\wedge \tau_K^M}^{M} |^{2p} \right] \leq C_{T,p}.
\]

Proof:

Recalling the definition (2.35) of $X_t^M$:
\[
X_t^M = x + \int_0^t \sigma(X_s^M) \, dW_s + \int_0^t \int_\mathcal{E} c_M(z, X_{s-}^M) \mathbf{1}_{\{ u \geq \gamma(z, X_{s-}^M) \}} N(\, ds, dz, du \) + \int_0^t g(X_s^M) \, ds,
\]
we have, for a single component (omitting for a moment the parameter $M$ in order to simplify the notations), applying Itô’s formula with $f(x) = x^{2p}$ with respect to every component of the process $X_{t\wedge \tau_K^M}^M$,
\[
(X_{t\wedge \tau_K^M}^i)_{2p} = (X_{0}^i)_{2p} + \sum_{l=1}^{m} \int_0^{t\wedge \tau_K^M} 2p(X_s^i)^{2p-1} \sigma_i(X_s) \, dW_s^l
\]
\[
+ 2p \int_0^{t\wedge \tau_K^M} (X_s^i)^{2p-1} g_i(X_s) \, ds
\]
\[
+ p(2p-1) \sum_{l=1}^{m} \int_0^{t\wedge \tau_K^M} (X_s^i)^{2p-2} (\sigma_i(X_s))^2 \, ds
\]
\[
+ \int_0^{t\wedge \tau_K^M} \int_\mathcal{E} (X_{s-}^i + c_M(z, X_{s-}^M) \mathbf{1}_{\{ u \geq \gamma(z, X_{s-}^M) \}})^{2p} - (X_{s-}^i)^{2p} N(\, ds, dz, du \)
\]
\]
\]
We now take the expectation with respect to the Brownian motion (i.e. conditionally with respect to all the other random quantities):

\[
EW \left[ (X_{t \wedge \tau_K}^i)^{2p} \right] = EW \left[ (X_0^i)^{2p} \right] + 2p \int_0^{t \wedge \tau_K} EW \left[ (X_s^i)^{2p-1} g_i(X_s) \right] ds \\
+ p(2p-1) \sum_{i=1}^{m} \int_0^{t \wedge \tau_K} EW \left[ (X_s^i)^{2p-2} (\sigma_{ii}(X_s))^2 \right] ds \\
+ \int_0^{t \wedge \tau_K} \int_E EW \left[ (X_s^i + c_M(z, X_{s-}^M)1_{\{u \leq \gamma(z, X_{s-}^M)\}})^{2p} - (X_s^i)^{2p} \right] N(ds, dz, du)
\]

Since \( s \leq t \wedge \tau_K \), we have \( X_s = X_{s,m} \), and obviously \( t \geq t \wedge \tau_K \), so we have

\[
EW \left[ |X_{t \wedge \tau_K}^i|^{2p} \right] \leq EW \left[ |X_0^i|^{2p} \right] + 2p \int_0^t EW \left[ |X_{s \wedge \tau_K}^i|^{2p-1} |g_i(X_s)\right] ds \\
+ p(2p-1) \sum_{i=1}^{m} \int_0^{t \wedge \tau_K} EW \left[ |X_{s \wedge \tau_K}^i|^{2p-2} |\sigma_{ii}(X_s)|^2 \right] ds \\
+ \int_0^{t \wedge \tau_K} \int_E EW \left[ \left( |X_s^i + c_M(z, X_{s-}^M)1_{\{u \leq \gamma(z, X_{s-}^M)\}}\right)^{2p} - (X_s^i)^{2p} \right] N(ds, dz, du)
\]

It follows (since \((a+b)^{2n} - a^{2n} \leq (|a| + |b|)^{2n} - a^{2n}\), for all \(a, b \in \mathbb{R}\) and using the inequality \(x^a y^b \leq x^{a+b} + y^{a+b}\))

\[
EW \left[ |X_{t \wedge \tau_K}^i|^{2p} \right] \leq EW \left[ |X_0^i|^{2p} \right] + 2p \int_0^t EW \left[ |X_{s \wedge \tau_K}^i|^{2p} \right] ds + 2pT\|g^{2p}\|_{\infty} \\
+ p(2p-1)m \int_0^t EW \left[ |X_{s \wedge \tau_K}^i|^{2p} \right] ds + p(2p-1)mT\|\sigma^{2p}\|_{\infty} \\
+ \int_0^{t \wedge \tau_K} \int_E EW \left[ \left( |X_s^i + c(z)1_{\{u \leq \gamma(z)\}}\right)^{2p} - |X_s^i|^{2p} \right] N(ds, dz, du);
\]

using now \(|a^{2p} - b^{2p}| \leq |a - b|(a+b)^{2p-1}\), we have \(|X_s^i + c(z)1_{\{u \leq \gamma(z)\}}\)^{2p} - |X_s^i|^{2p} \leq 2^{2p} c(z)1_{\{u \leq \gamma(z)\}}|X_s^i|^{2p} - |X_s^i|^{2p},

so,

\[
EW \left[ |X_{t \wedge \tau_K}^i|^{2p} \right] \leq C_T + A_p \int_0^t EW \left[ |X_{s \wedge \tau_K}^i|^{2p} \right] ds + B_p \int_0^t \int_E \tau(z)1_{\{u \leq \gamma(z)\}} \theta^K_{s-} N(ds, dz, du) \\
\leq C_T + A_p \int_0^t EW \left[ |X_{s \wedge \tau_K}^i|^{2p} \right] ds + B_p \int_0^t \int_E \tau(z)1_{\{u \leq \gamma(z)\}} \theta^K_{s-} N(ds, dz, du).
\]

With

\[
\theta^K_t \overset{\text{def}}{=} \sup_{0 \leq u \leq t} EW \left[ |X_{u \wedge \tau_K}^M|^{2p} \right],
\]

we then have

\[
\theta^K_t \leq C_T + A_p \int_0^t \theta^K_s ds + B_p \int_0^t \int_E \tau(z)1_{\{u \leq \gamma(z)\}} \theta^K_{s-} N(ds, dz, du).
\]

With

\[
R_1 \overset{\text{def}}{=} \int_0^t \int_E \tau(z)1_{\{u \leq \gamma(z)\}} d\mu(z) du
\]

we have

\[
E[\theta^K_t] \leq C_T + A_p E \left[ \int_0^t \theta^K_s ds \right] + B_p \left[ \int_0^t \int_E \tau(z)1_{\{u \leq \gamma(z)\}} \theta^K_{s-} N(s, z, u) \right] \\
\leq C_T + A_p \int_0^t E[\theta^K_s] ds + B_p \int_0^t \int_E \tau(z)1_{\{u \leq \gamma(z)\}} E[\theta^K_{s-}] ds d\mu(dz) du \\
\leq C_T + (A_p + R_1) \int_0^t E[\theta^K_s] ds.
\]

The Grönwall’s lemma ends then the proof.
C  Tangent flow

Let us recall the expression of the tangent flow \( Y^M_t \):

\[
Y^M_t = \text{Id} + \sum_{i=1}^m \int_0^t \nabla \sigma_t(\tilde{X}_s^M) Y_s^M \, dW_s^i + \sum_{j=1}^{J^M_t} \nabla \sigma_M(Z_{j,-}) Y_{T_{j,-}}^M + \int_0^t \nabla g(\tilde{X}_s^M) Y_s^M \, ds.
\]

We have then defined the following process (with \( \nabla x c_j = \nabla x c_M(Z_j, X_{T_{j,-}}^M) \)):

\[
\hat{Y}_t^M \overset{\text{def}}{=} \text{Id} - \sum_{i=1}^m \int_0^t \hat{Y}_s^M \nabla \sigma_t(\tilde{X}_s^M) \, dW_s^i - \sum_{j=1}^{J^M_t} \hat{Y}_{T_{j,-}} \nabla \sigma_M(\text{Id} + \nabla x c_j)^{-1} + \int_0^t \hat{Y}_s^M \left( \frac{1}{2} \sum_{i=1}^m \nabla \sigma_t(\tilde{X}_s^M)^2 - \nabla g(\tilde{X}_s^M) \right) \, ds,
\]

and stated that:

**Lemma C.1** For all \( t \geq 0 \),

\[
Y^M_t \hat{Y}^M_t = \text{Id}.
\]  

**Proof**: 

**Step 1**

Let us consider (for the moment \( m = 1 \)) the stochastic process sharing the same law as \( Y_t \) (that we will continue to denote \( Y_t \), in order to simplify the notations) defined by

\[
Y_t = \text{Id} + \int_0^t \Sigma Y_s \, dW_s + \sum_{j=1}^{J^M_t} C_j Y_{T_{j,-}}^M + \int_0^t GY_s \, ds
\]

with

\[
C_j = \nabla x c_M(Z_j, X_{T_{j,-}}^M) I_{\{U_j \leq \gamma(Z_j, X_{T_{j,-}}^M)\}}
\]

and let us set

\[
\tilde{Y}_t = \text{Id} + \int_0^t \tilde{Y}_s A \, dW_s + \sum_{j=1}^{J^M_t} \tilde{Y}_{T_{j,-}}^M - H_j + \int_0^t \tilde{Y}_s B \, ds
\]

that is

\[
Y^{i,j}_t = \delta_{i,j} + \sum_{h=1}^d \int_0^t \Sigma_{i,h} Y^{h,j}_s \, dW_s + \sum_{k'=1}^{J^M_t} \sum_{h=1}^d C^{i,h}_{k'} Y^{h,j}_{T_{k',-}} + \sum_{h=1}^{J^M_t} \int_0^t G_{i,h} Y^{h,j}_s \, ds
\]

and

\[
\hat{Y}^{i,j}_t = \delta_{i,j} + \sum_{h=1}^d \int_0^t \tilde{Y}^{i,h}_s A_{h,j} \, dW_s + \sum_{k'=1}^{J^M_t} \sum_{h=1}^d \hat{Y}^{i,h}_{T_{k',-}} - H^{i,j}_{k'} + \sum_{h=1}^{J^M_t} \int_0^t \hat{Y}^{i,h}_s B_{h,j} \, ds
\]

We have \((\tilde{Y}_t Y_t - \text{Id})_{p,q} = \sum_{n=1}^d \tilde{Y}_t^{p,n} Y_t^{n,q} - \delta_{p,q} \). Using Itô’s formula it follows:

\[
\tilde{Y}_t^{p,n} Y_t^{n,q} - \delta_{p,q,n}
\]

\[
= \int_0^t \Sigma^{p,n} \sum_{h=1}^d \tilde{Y}^{p,h}_s A_{h,n} \, dW_s + \int_0^t \tilde{Y}_s^{p,n} \sum_{h=1}^d \Sigma_{n,h} Y^{h,q}_s \, dW_s
\]

\[
+ \frac{1}{2} \int_0^t \left( \sum_{g=1}^d \tilde{Y}_s^{p,g} A_{g,n} \right) \left( \sum_{h=1}^d \Sigma_{n,h} Y^{h,q}_s \right) \, ds
\]

\[
+ \int_0^t \tilde{Y}_s^{p,n} \sum_{h=1}^d \Sigma_{n,h} Y^{h,q}_s \, ds + \int_0^t \tilde{Y}_s^{p,n} \sum_{h=1}^d G_{n,h} Y^{h,q}_s \, ds
\]

\[
+ \int_0^t \int_E \left( \sum_{h=1}^d C^{n,h}_{s,-}(z, u) Y^{h,q}_s \right) \left( \sum_{h=1}^d \tilde{Y}_s^{p,h} H^{h,n}_{s,-}(z, u) \right) \, ds + \int_0^t \int_E \left( \sum_{h=1}^d \tilde{Y}_s^{p,h} H^{h,n}_{s,-}(z, u) \right) \, N(ds, dz, du)
\]
The integrated term with respect to $dW_s$ of $(\hat{Y}_t Y_t - \text{Id})_{p,q}$ has the following form:

$$
\sum_{n=1}^{d} \sum_{h=1}^{d} \hat{Y}_{s}^{p,q} \hat{Y}_{s}^{p,h} A_{h,n} + \sum_{n=1}^{d} \sum_{h=1}^{d} \hat{Y}_{s}^{p,q} \hat{Y}_{s}^{p,h} Y_{s}^{h,q} + \sum_{n=1}^{d} \sum_{h=1}^{d} \hat{Y}_{s}^{p,q} \hat{Y}_{s}^{p,h} Y_{s}^{h,q} = \sum_{n=1}^{d} \sum_{h=1}^{d} \hat{Y}_{s}^{p,q} \hat{Y}_{s}^{p,h} (A_{n,h} + \Sigma_{n,h}).
$$

This term is null for

$$A = -\Sigma$$

which we will suppose in the following of this proof.

For the third term

$$
\sum_{n=1}^{d} \sum_{g=1}^{d} \left( \sum_{h=1}^{d} \left( \sum_{n,h}^{d} A_{g,n} \Sigma_{n,h} Y_{s}^{h,q} \right) \right) = \sum_{n=1}^{d} \sum_{g=1}^{d} \left( \sum_{h=1}^{d} A_{g,n} \Sigma_{n,h} Y_{s}^{h,q} \right) = \sum_{n=1}^{d} \sum_{g=1}^{d} \left( A_{n,h} \Sigma_{g,h} \right) Y_{s}^{p,n} Y_{s}^{h,q} = \sum_{n=1}^{d} \sum_{h=1}^{d} (A \Sigma)_{n,h} Y_{s}^{p,n} Y_{s}^{h,q}
$$

The integrated term with respect to $dW_s$ of $(\hat{Y}_t Y_t - \text{Id})_{p,q}$ has the following form:

$$
\frac{1}{2} \sum_{n=1}^{d} \sum_{h=1}^{d} (A \Sigma)_{n,h} Y_{s}^{p,n} Y_{s}^{h,q} + \sum_{n=1}^{d} \sum_{h=1}^{d} Y_{s}^{p,n} Y_{s}^{h,q} + \sum_{n=1}^{d} \sum_{h=1}^{d} \hat{Y}_{s}^{p,n} B_{h,n} + \sum_{n=1}^{d} \sum_{h=1}^{d} \hat{Y}_{s}^{p,n} G_{n,h} Y_{s}^{h,q}
$$

$$
= \frac{1}{2} \sum_{n=1}^{d} \sum_{h=1}^{d} (A \Sigma)_{n,h} Y_{s}^{p,n} Y_{s}^{h,q} + \sum_{n=1}^{d} \sum_{h=1}^{d} Y_{s}^{p,n} Y_{s}^{h,q} + \sum_{n=1}^{d} \sum_{h=1}^{d} \hat{Y}_{s}^{p,n} B_{h,n} + \sum_{n=1}^{d} \sum_{h=1}^{d} \hat{Y}_{s}^{p,n} G_{n,h} Y_{s}^{h,q}
$$

$$
= \sum_{n=1}^{d} \sum_{h=1}^{d} \left( \frac{1}{2} A \Sigma + B + G \right) Y_{s}^{p,n} Y_{s}^{h,q}.
$$

This term is null if (with $A = -\Sigma$)

$$B = \frac{1}{2} \Sigma^2 - G.
$$

**Step 2**

Multidimensional Brownian case:

$$Y_t = \text{Id} + \int_0^t \sum_{i=1}^d \Sigma_i W_s^i \, ds + \int_0^t \sum_{i=1}^d \Delta_i Y_t^i \, ds + \int_0^t \sum_{i=1}^d \sum_{j=1}^d C_i^{j} Y_t^j \, ds + \int_0^t G Y_t \, ds$$

and let us set

$$\hat{Y}_t = \text{Id} + \int_0^t \hat{Y}_s A_1 \, dW_s^1 + \cdots + \int_0^t \hat{Y}_s A_m \, dW_s^m + \int_0^t \hat{Y}_s M \, ds + \int_0^t \hat{Y}_s B \, ds$$

that is

$$Y_t^{i,j} = \delta_{i,j} + \sum_{l=1}^d \sum_{h=1}^d \int_0^t \Sigma_{i,h} Y_s^{h,j} \, dW_s^l + \sum_{k=1}^d \sum_{h=1}^d \int_0^t C_{i,k}^{h,j} Y_s^{h,j} \, dt + \sum_{h=1}^d \int_0^t G_{i,h} Y_s^{h,j} \, ds$$

$$= \delta_{i,j} + \sum_{l=1}^d \sum_{h=1}^d \int_0^t \Sigma_{i,h} Y_s^{h,j} \, dW_s^l + \sum_{k=1}^d \sum_{h=1}^d \int_0^t C_{i,k}^{h,j} Y_s^{h,j} \, dt + \sum_{h=1}^d \int_0^t G_{i,h} Y_s^{h,j} \, ds$$

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and
\[
\hat{Y}_{t}^{i,j} = \delta_{i,j} + \sum_{l=1}^{m} \sum_{h=1}^{d} \int_{0}^{t} \hat{Y}_{s}^{i,h} A_{h,j}^{l} dW_{s}^{l} + \sum_{l=1}^{m} \sum_{h=1}^{d} \hat{Y}_{s}^{i,h} T_{s,h}^{l} - H_{s,h}^{l} + \sum_{h=1}^{d} \int_{0}^{t} \hat{Y}_{s}^{i,h} B_{h,j} ds
\]
and
\[
\hat{Y}_{t}^{p,n} Y_{t}^{q,n} - \delta_{p,q,n} = \int_{0}^{t} Y_{s}^{n,q} \sum_{l=1}^{m} \sum_{h=1}^{d} \hat{Y}_{s}^{p,h} A_{h,n}^{l} dW_{s}^{l} + \int_{0}^{t} \hat{Y}_{s}^{p,n} \sum_{l=1}^{m} \sum_{h=1}^{d} \Sigma_{n,h} Y_{s}^{l,h,q} dW_{s}^{l} + \int_{0}^{t} \hat{Y}_{s}^{p,n} \sum_{h=1}^{d} \Sigma_{n,h} Y_{s}^{l,h,q} ds
\]
\[
+ \int_{0}^{t} \int_{0}^{t} \left( \sum_{g=1}^{d} \hat{Y}_{s}^{p,q} A_{s,g}^{l} \right) \left( \sum_{h=1}^{d} \Sigma_{n,h} Y_{s}^{l,h,q} \right) + \int_{0}^{t} \hat{Y}_{s}^{p,n} \sum_{h=1}^{d} G_{n,h} Y_{s}^{l,h,q} ds
\]
\[
+ \int_{0}^{t} \int_{0}^{t} E \left( \sum_{h=1}^{d} C_{s-h}^{n,h} (z,u) Y_{s-h}^{l,h,q} \right) \left( \sum_{h=1}^{d} \hat{Y}_{s}^{p,n} H_{s-h}^{l,h,n} (z,u) - Y_{s-h}^{n,q} \hat{Y}_{s-h}^{p,n} N (ds, dz, du) \right)
\]

The same computations, by a straight superposition, give us then, for all \( l \in [0, m] \),
\[
A_{l} = -\Sigma_{l}
\]
and
\[
B = \frac{1}{2} \left( \sum_{i=1}^{d} \Sigma_{i}^{2} \right) - G.
\]

•
References


