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# FOURIER COEFFICIENTS OF SYMMETRIC POWER L-FUNCTIONS

HENGCAI TANG & JIE WU

ABSTRACT. Let  $f$  be a Hecke eigencusp form of even integral weight  $k$  or Maass cusp form for the full modular group  $SL_2(\mathbb{Z})$ . Denote by  $\lambda_{\text{sym}^m f}(n)$  the  $n$ th normalized coefficient of the Dirichlet expansion of the  $m$ th symmetric power  $L$ -function associated to  $f$ . In this paper, we establish some bounds for

$$\sum_{n \leq x} \lambda_{\text{sym}^m f}(n), \quad \sum_{n \leq x} \lambda_f(n^m),$$

which improve the corresponding results of Lau & Lü [9].

## 1. INTRODUCTION

Let  $\mathcal{H}_k^*(\Gamma)$  (resp.  $\mathcal{M}^*(\Gamma)$ ) be the set of normalized Hecke eigencusp forms of even integral weight  $k$  (resp. the set of normalized Maass eigencusp forms) for the full modular group  $\Gamma = SL(2, \mathbb{Z})$ . Each  $f \in \mathcal{H}_k^*(\Gamma) \cup \mathcal{M}^*(\Gamma)$  has the Fourier expansion at the cusp  $\infty$ . Denote by  $\lambda_f(n)$  the  $n$ th normalized Fourier coefficient of  $f$  (also the eigenvalue of  $f$  under the Hecke operator  $T_n$ ). Then from the theory of Hecke operators, it is now well-known that  $\lambda_f(n)$  satisfies the Hecke multiplicity

$$(1.1) \quad \lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right)$$

for all integers  $m \geq 1$  and  $n \geq 1$ . Thus for each prime number  $p$  there are two complex numbers  $\alpha_f(p)$  and  $\beta_f(p)$  such that

$$(1.2) \quad \alpha_f(p)\beta_f(p) = 1$$

and

$$(1.3) \quad \lambda_f(p^\nu) = \alpha_f(p)^\nu + \alpha_f(p)^{\nu-1}\beta_f(p) + \cdots + \beta_f(p)^\nu$$

for all integers  $\nu \geq 1$ . The Ramanujan conjecture states that

$$(1.4) \quad |\alpha_f(p)| = |\beta_f(p)| = 1$$

for all prime numbers  $p$ . For holomorphic  $f$ , this has been proved by Deligne [3] in 1974. Hence for each prime number  $p$  there is a unique  $\theta_f(p) \in [0, \pi]$  such that

$$(1.5) \quad \lambda_f(p) = 2 \cos \theta_f(p).$$

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According to the Sato-Tate conjecture, the sequence  $\{\theta_f(p)\}_p$  is equi-distributed on  $[0, \pi]$  with respect to the measure  $\frac{2}{\pi} \sin^2 \theta d\theta$ , i.e., for any fixed  $[a, b] \subset [0, \pi]$ ,

$$(1.6) \quad |\{p \leq x : \theta_f(p) \in [a, b]\}| \sim \frac{x}{\log x} \cdot \frac{2}{\pi} \int_a^b \sin^2 \theta d\theta \quad (x \rightarrow \infty).$$

This has been proved recently by Barnet-Lamb, Geraghty, Harris et Taylor [1]. For Maass cusp forms, both conjectures are widely open. In place of (1.4), we have only

$$(1.7) \quad p^{-7/64} \leq |\alpha_f(p)| \leq p^{7/64}, \quad p^{-7/64} \leq |\beta_f(p)| \leq p^{7/64}$$

for all primes  $p$ , due to Kim & Sarnak [6].

For each  $f \in \mathcal{H}_k^*(\Gamma) \cup \mathcal{M}^*(\Gamma)$ , its  $m$ th symmetric power  $L$ -function is defined by

$$L(s, \text{sym}^m f) := \prod_p \prod_{0 \leq j \leq m} (1 - \alpha_f(p)^{m-j} \beta_f(p)^j p^{-s})^{-1} \quad (\Re s > 1).$$

Denote by  $\lambda_{\text{sym}^m f}(n)$  the  $n$ th coefficient of the Dirichlet expansion of  $L(s, \text{sym}^m f)$ . Then it is evident that it is multiplicative function and that (1.5) and (1.7) imply

$$(1.8) \quad |\lambda_{\text{sym}^m f}(n)| \leq \begin{cases} d_{m+1}(n) & \text{if } f \in \mathcal{H}_k^*(\Gamma) \\ n^{7/64} d_{m+1}(n) & \text{if } f \in \mathcal{M}^*(\Gamma) \end{cases}$$

for  $n \geq 1$ , where  $d_m(n)$  is the  $n$ th coefficient of the Dirichlet series  $\zeta(s)^m$  (Riemann  $\zeta$ -function). When  $m = 1$ , we write  $L(s, \text{sym}^1 f) = L(s, f)$  and we have  $\lambda_{\text{sym}^1 f}(n) = \lambda_f(n)$  for all integers  $n \geq 1$ .

In this paper, we are interested in the asymptotical behavior of

$$\sum_{n \leq x} \lambda_{\text{sym}^m f}(n), \quad \sum_{n \leq x} \lambda_f(n^m) \quad \text{as } x \rightarrow \infty.$$

The case of  $m = 1$  has a long and rich history, see [20] for a detailed description. Using the Sato-Tate conjecture (1.6), the best known result is

$$\sum_{n \leq x} \lambda_f(n) \ll_f x^{1/3} (\log x)^{-(1-8/3\pi)}$$

due to Rankin [15]. Without (1.6), this inequality holds with a weaker exponent

$$(1.9) \quad \theta_1 := \frac{33}{35} - \frac{102+7\sqrt{21}}{210} \left(\frac{6-\sqrt{21}}{5}\right)^{1/2} - \frac{102-7\sqrt{21}}{210} \left(\frac{6+\sqrt{21}}{5}\right)^{1/2} = 0.118\dots$$

in place of  $1 - \frac{8}{3\pi} = 0.151\dots$  (see [17] for Ramanujan's  $\tau$ -function and [20] for general case) The higher degree cases have been received recently much attention, see [4, 11, 12, 9, 13]. In particular, Lau & Lü [9] proved the following result : Let  $f \in \mathcal{H}_k^*(\Gamma)$ . Assuming that  $L(s, \text{sym}^m f)$  is automorphic, then we have

$$(1.10) \quad \sum_{n \leq x} \lambda_{\text{sym}^m f}(n) \ll_{f,m} x^{m/(m+2)} \quad (m \geq 2)$$

$$(1.11) \quad \sum_{n \leq x} \lambda_f(n^m) \ll_{f,m} \begin{cases} x^{1/2} \log x & (m = 2) \\ x^{m/(m+2)} & (m \geq 3) \end{cases}$$

for all  $x \geq 2$ . According to [5, 6, 7],  $L(s, \text{sym}^m f)$  is automorphic for  $1 \leq m \leq 4$ . Therefore for these values of  $m$ , bounds (1.10) and (1.11) are true unconditionally. The same bounds also hold for Maass eigenforms if we assume (1.4).

The aim of this paper is to prove better bounds.

**Theorem 1.** *Let  $f \in \mathcal{H}_k^*(\Gamma)$  and  $m \geq 1$  be an integer.*

(i) *Assuming that  $L(s, \text{sym}^m f)$  is automorphic, we have*

$$(1.12) \quad \sum_{n \leq x} \lambda_{\text{sym}^m f}(n) \ll_{f,m} x^{m/(m+2)} (\log x)^{-\delta_m}$$

$$(1.13) \quad \sum_{n \leq x} \lambda_f(n^m) \ll_{f,m} \begin{cases} x^{1/2} (\log x)^{1-\delta_2} & (m = 2) \\ x^{m/(m+2)} (\log x)^{-\delta_m} & (m = 1 \text{ or } m \geq 3) \end{cases}$$

for  $x \geq 3$ , where

$$(1.14) \quad \delta_m := 1 - \frac{4(m+1)}{\pi m(m+2)} \cot\left(\frac{\pi}{2(m+1)}\right).$$

(ii) *Bounds (1.12) and (1.13) hold unconditionally for  $1 \leq m \leq 4$ , and we have*

$$\begin{aligned} \delta_1 &= 1 - \frac{8}{3\pi} = 0.1511\dots, & \delta_2 &= 1 - \frac{3\sqrt{3}}{2\pi} = 0.1730\dots, \\ \delta_3 &= 1 - \frac{16(1+\sqrt{2})}{15\pi} = 0.1803\dots, & \delta_4 &= 1 - \frac{5\sqrt{5+2\sqrt{5}}}{6\pi} = 0.1836\dots \end{aligned}$$

(iii) *The sequence  $\{\delta_m\}_{m \geq 1}$  is strictly increasing and  $\lim_{m \rightarrow \infty} \delta_m = 1 - \frac{8}{\pi^2} = 0.1894\dots$*

*Remark 1.* Recently Lü and Tang [13] obtained the following unconditional bound

$$\sum_{n \leq x} \lambda_f(n^m) \ll_f x e^{-c_m(f)\sqrt{\log x}} \quad (x \geq 3)$$

for  $f \in \mathcal{H}_k^*(\Gamma)$  and  $5 \leq m \leq 8$ , where the  $c_m(f) > 0$  are constants depending on  $f$  and  $m$ . Clearly (1.19) of Theorem 3 below implies trivially that

$$\sum_{n \leq x} \lambda_f(n^m) \ll_{f,m} x (\log x)^{-\delta_m} \quad (x \geq 3, m \geq 9).$$

**Theorem 2.** *Let  $f \in \mathcal{M}^*(\Gamma)$  and  $m \geq 1$  be an integer. Assuming the Ramanujan conjecture (1.4) and that  $L(s, \text{sym}^j f)$  is automorphic for  $1 \leq j \leq 2m$ , we have*

$$(1.15) \quad \sum_{n \leq x} \lambda_{\text{sym}^m f}(n) \ll_{f,m} x^{m/(m+2)} (\log x)^{-(m\eta_m + 2\theta_1)/(m+2)}$$

$$(1.16) \quad \sum_{n \leq x} \lambda_f(n^m) \ll_{f,m} \begin{cases} x^{1/2} (\log x)^{1-(\eta_2 + \theta_1)/2} & (m = 2) \\ x^{m/(m+2)} (\log x)^{-(m\eta_m + 2\theta_1)/(m+2)} & (m = 1 \text{ or } m \geq 3) \end{cases}$$

for  $x \geq 3$ , where  $\theta_1$  is given as in (1.9) and

$$(1.17) \quad \eta_m := \frac{m(m+2)}{m^2 + 3m + 1} \left( \frac{m+3}{m+2} - \sqrt{\frac{m+2}{m+1}} \right) > 0.$$

*Remark 2.* Assuming the Ramanujan conjecture (1.4) for  $f \in \mathcal{M}^*(\Gamma)$ , estimation (1.22) of Theorem 4 below implies trivially that

$$\sum_{n \leq x} \lambda_f(n^m) \ll_{f,m} x (\log x)^{-\theta_1} \quad (x \geq 3, m \geq 1).$$

We shall follow Rankin's method [15] to prove Theorems 1 and 2. For this, we need to evaluate the quantities

$$\sum_{n \leq x} |\lambda_{\text{sym}^m f}(n)|, \quad \sum_{n \leq x} |\lambda_f(n^m)|.$$

In the holomorphic case, we shall combine Wirsing's well known mean value theorem on multiplicative functions (see Lemma 2.1 below) and Shiu's result on short sums of multiplicative functions [16, Theorem 1] with the Sato-Tate conjecture (1.6) to prove the following result.

**Theorem 3.** *Let  $f \in \mathcal{H}_k^*(\Gamma)$  and  $m \geq 1$  be an integer.*

(i) *We have*

$$(1.18) \quad \sum_{n \leq x} |\lambda_{\text{sym}^m f}(n)| \sim C_m(f) x (\log x)^{-\delta_m},$$

$$(1.19) \quad \sum_{n \leq x} |\lambda_f(n^m)| \sim D_m(f) x (\log x)^{-\delta_m},$$

*unconditionally for  $x \rightarrow \infty$ , where  $\delta_m$  is given as in (1.14) and  $C_m(f), D_m(f)$  are two positive constants depending on  $f$  and  $m$ .*

(ii) *For any  $\varepsilon > 0$ , we have*

$$(1.20) \quad \sum_{x < n \leq x+y} |\lambda_{\text{sym}^m f}(n)|, \quad \sum_{x < n \leq x+y} |\lambda_f(n^m)| \ll_{f,m,\varepsilon} y (\log x)^{-\delta_m},$$

*uniformly for  $x \geq 3$  and  $x^\varepsilon \leq y \leq x$ , where the implied constant depends on  $f, m, \varepsilon$ .*

In the case of Maass cusp forms, we can apply Corollary 3 of Nair & Tenenbaum [14] and Wu & Xu's method [21] to prove the following result.

**Theorem 4.** *Let  $f \in \mathcal{M}^*(\Gamma)$  and  $m \geq 1$  be an integer.*

(i) *Assuming the Ramanujan conjecture (1.4), for any  $\varepsilon > 0$  we have*

$$(1.21) \quad \sum_{x < n \leq x+y} |\lambda_f(n^m)| \ll_{f,m,\varepsilon} y (\log x)^{-\theta_1} \quad (x \geq 3).$$

*uniformly for  $x \geq 3$  and  $x^\varepsilon \leq y \leq x$ , where the implied constants depend on  $f, m$  and  $\varepsilon$ , and  $\theta_1$  is given as in (1.9). In particular we have*

$$(1.22) \quad \sum_{n \leq x} |\lambda_f(n^m)| \ll_{f,m,\varepsilon} x (\log x)^{-\theta_1} \quad (x \geq 3).$$

(ii) *Assuming the Ramanujan conjecture (1.4) and that  $L(s, \text{sym}^j f)$  is automorphic for  $1 \leq j \leq 2m$ , for any  $\varepsilon > 0$  we have*

$$(1.23) \quad \sum_{x < n \leq x+y} |\lambda_{\text{sym}^m f}(n)| \ll_{f,m,\varepsilon} y (\log x)^{-\eta_m}$$

*uniformly for  $x \geq 3$  and  $x^\varepsilon \leq y \leq x$ , where the implied constants depend on  $f, m$  and  $\varepsilon$ , and  $\eta_m$  is given as in (1.17). In particular we have*

$$(1.24) \quad \sum_{n \leq x} |\lambda_{\text{sym}^m f}(n)| \ll_{f,m,\varepsilon} x (\log x)^{-\theta_1}$$

From Theorems 3 and 4, we can deduce Theorems 1 and 2 with the help of Theorem 4.1 of Chandrasekharan and Narasimhan [2] (see Lemma 3.2 below).

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## 2. PROOFS OF THEOREM 3

First we cite well know Wirsing's mean value theorem on multiplicative functions [18, 19].

**Lemma 2.1.** *Let  $g(n)$  be a non-negative multiplicative function such that*

$$(2.1) \quad \sum_{p \leq x} g(p) \sim \kappa \frac{x}{\log x} \quad (x \rightarrow \infty),$$

$$(2.2) \quad g(n) \leq Ad(n)^B \quad (n \geq 1),$$

where  $\kappa, A, B$  are positive constants and  $d(n)$  is the divisor function. Then

$$\sum_{n \leq x} g(n) \sim C_g x (\log x)^{\kappa-1} \quad (x \rightarrow \infty),$$

where  $C_g$  is a positive constant depending on  $g$ .

Since  $|\lambda_{\text{sym}^m f}(n)|$  and  $|\lambda_f(n^m)|$  are non-negative multiplicative functions, it is sufficient to show that they verify the conditions (2.1) and (2.2) of Lemma 2.1.

Denoting by  $U_m(u)$  the  $m$ th Chebyshev polynomial of the second kind, then

$$\lambda_{\text{sym}^m f}(p) = \lambda_f(p^m) = U_m(\lambda_f(p)/2) = U_m(\cos \theta_f(p)) = \frac{\sin((m+1)\theta_f(p))}{\sin \theta_f(p)},$$

where  $\theta_f(p)$  is defined as in (1.5). Since  $|U_m(\cos \theta_f(p))| \leq m+1$ , we can write

$$\sum_{p \leq x} |\lambda_{\text{sym}^m f}(p)| = \sum_{p \leq x} \int_0^{|U_m(\cos \theta_f(p))|} 1 dt = \int_0^{m+1} \sum_{\substack{p \leq x \\ t \leq |U_m(\cos \theta_f(p))|}} 1 dt.$$

Clearly

$$t \leq |U_m(\cos \theta_f(p))| \Leftrightarrow \theta_f(p) \in \{\theta \in [0, \pi] : |U_m(\cos \theta)| \geq t\} =: \Theta_m(t).$$

Since  $U_m(u)$  is a polynomial, it is easy to see that  $\Theta_m(t)$  is an union of finite disjoint intervals. With the help of (1.6), we can deduce that

$$(2.3) \quad \sum_{p \leq x} |\lambda_{\text{sym}^m f}(p)| \sim \frac{2}{\pi} \delta_m^* \frac{x}{\log x} \quad (x \rightarrow \infty),$$

where

$$\delta_m^* := \int_0^{m+1} \int_{\Theta_m(t)} \sin^2 \theta d\theta dt.$$

Now we compute  $\delta_m^*$ . Exchanging the order of integrations, we find that

$$\delta_m^* = \int_0^\pi |U_m(\cos \theta)| \sin^2 \theta \, d\theta = \int_0^\pi |\sin((m+1)\theta)| \sin \theta \, d\theta.$$

Setting  $\psi = \psi_m := \pi/(m+1)$ , we can write

$$\begin{aligned} \delta_m^* &= \sum_{k=0}^m (-1)^k \int_{k\psi}^{(k+1)\psi} \sin((m+1)\theta) \sin \theta \, d\theta \\ &= \sum_{k=0}^m \frac{(-1)^k (m+1)}{1 - (m+1)^2} \left[ \sin \theta \cos((m+1)\theta) - \frac{\sin((m+1)\theta) \cos \theta}{m+1} \right]_{k\psi}^{(k+1)\psi} \\ &= \frac{m+1}{(m+1)^2 - 1} \sum_{k=0}^m [\sin((k+1)\psi) + \sin(k\psi)] \\ &= \frac{2(m+1)}{m(m+2)} \cot \left( \frac{\pi}{2(m+1)} \right). \end{aligned}$$

Inserting (2.3), it follows that

$$(2.4) \quad \sum_{p \leq x} |\lambda_{\text{sym}^m f}(p)| \sim (1 - \delta_m) \frac{x}{\log x} \quad (x \rightarrow \infty),$$

where

$$\delta_m = 1 - \frac{2}{\pi} \delta_m^* = 1 - \frac{4(m+1)}{\pi m(m+2)} \cot \left( \frac{\pi}{2(m+1)} \right).$$

This shows that  $|\lambda_{\text{sym}^m f}(n)|$  and  $|\lambda_f(n^m)|$  satisfy the condition (2.1) with  $\kappa = 1 - \delta_m$ .

On the other hand, with the help of (1.8) and (1.5), it is easy to see that the condition (2.2) also is verified. Thus the required asymptotical formulas (1.18) and (1.19) follow from Lemma 2.1.

It is easy to verify that functions  $|\lambda_{\text{sym}^m f}(n)|$  and  $|\lambda_f(n^m)|$  satisfy the conditions of Theorem 1 of Shiu [16] on short sums of multiplicative functions. Thus we have

$$(2.5) \quad \sum_{x < n \leq x+y} |\lambda_{\text{sym}^m f}(n)|, \quad \sum_{x < n \leq x+y} |\lambda_f(n^m)| \ll \frac{y}{\log x} \exp \left( \sum_{p \leq x} \frac{|\lambda_{\text{sym}^m f}(p)|}{p} \right)$$

uniformly for  $x \geq 3$  and  $x^\varepsilon \leq y \leq x$ . Now (1.20) is an immediate consequence of this and (2.3). This completes the proof of Theorem 3.  $\square$

### 3. PROOF OF THEOREM 4

#### 3.1. Proof of (1.21) and (1.22).

Firstly, we introduce Corollary 3 of Nair & Tenenbaum [14] which gave a sharp bound for short sums of nonnegative arithmetic functions. Their result is very general. We only need a very special case, i.e., [14, estimation (2)], for our purpose. As usual denote by  $\Omega(m)$  the total number of prime factors of  $m$  counted with multiplicity. For  $A > 0$ ,  $B > 0$  and  $\varepsilon > 0$ , we define  $\mathcal{M}(A, B, \varepsilon)$  the class of non-negative arithmetic functions  $F(n)$  such that

$$(3.1) \quad F(mn) \leq \min \{A^{\Omega(m)}, m^\varepsilon\} F(n) \quad \text{for } (m, n) = 1.$$

Let  $Q \in \mathbb{Z}[X]$  be an irreducible polynomial of degree  $g$ , having no fixed prime divisor. Denote by  $\rho_Q(n)$  the number of roots of  $Q$  in  $\mathbb{Z}/m\mathbb{Z}$  and by  $\|Q\|$  the maximal value of absolute values of its coefficients.

Taking  $k = 1$  in Corollary 3 of [14], we obtain the following result.

**Lemma 3.1.** *Under the previous notation, for any  $A \geq 1$ ,  $B \geq 1$ ,  $0 < \varepsilon < (8g^2)^{-1}$ ,  $0 < \delta \leq 1$  and  $F \in \mathcal{M}(A, B, \varepsilon\delta/3)$ , we have*

$$\sum_{x < n \leq x+y} F(Q(n)) \ll y \prod_{p \leq x} \left(1 - \frac{\rho_Q(p)}{p}\right) \sum_{n \leq x} \frac{F(n)\rho_Q(n)}{n}$$

uniformly for  $x \geq c_0 \|Q\|^\delta$  and  $x^{4g^2\varepsilon} \leq y \leq x$ , where the implied constant and the constant  $c_0$  depend at most  $A, B, \delta, \varepsilon$  and  $g$ .

Now we apply this result to

$$F(n) = |\lambda_f(n)|, \quad Q(x) = x^m, \quad \delta = 1.$$

Clearly  $Q(n)$  has no fixed prime divisor and we have  $g = m$ ,  $\|Q\| = 1$  and

$$\rho_Q(p) = 1 \text{ (for all primes } p), \quad \rho_Q(n) \leq m \text{ (for all integers } n \geq 1).$$

Since we assume the Ramanujan conjecture (1.4), then we have  $|\lambda_f(n)| \leq d(n)$  for all  $n \geq 1$ . Thus the function  $|\lambda_f(n)| \in \mathcal{M}(2, B_\varepsilon, \varepsilon)$  for any  $\varepsilon \in (0, (8m^2)^{-1})$  and some suitable constant  $B_\varepsilon$  depending on  $\varepsilon$ . Hence, we have

$$(3.2) \quad \sum_{x < n \leq x+y} |\lambda_f(n^m)| \ll_{f,m,\varepsilon} y \prod_{p \leq x} \left(1 - \frac{1}{p}\right) \sum_{n \leq x} \frac{|\lambda_f(n)|}{n}$$

uniformly for  $x \geq 3$  and  $x^\varepsilon \leq y \leq x$ .

Under the Ramanujan conjecture (1.4) for  $f \in \mathcal{M}^*(\Gamma)$ , Theorem 1 of Wu [20] gives

$$(3.3) \quad \sum_{n \leq x} |\lambda_f(n)| \ll x(\log x)^{-\theta_1}$$

for all  $x \geq 3$ , where  $\theta_1$  is given as in (1.9). (Wu & Xu [21, Theorem 1] proved that this inequality also holds unconditionally with a weaker exponent  $\frac{3}{2} - \sqrt{2} = 0.0857\dots$ )

Now the required result (1.21) follows immediately from (3.2) and (3.3).

For any  $x \geq 10$ , inequality (1.21) with  $x = y = x/2^k$  gives us

$$(3.4) \quad \begin{aligned} \sum_{n \leq x} |\lambda_f(n^m)| &= \sum_{1 \leq k \leq (\log x)/\log 2} \sum_{x/2^k < n \leq x/2^{k-1}} |\lambda_f(n^m)| \\ &\ll_{f,m} \sum_{1 \leq k \leq (\log x)/\log 2} \frac{x}{2^k} \log^{-\theta_1} \left(\frac{x}{2^k}\right) \\ &\ll_{f,m} x(\log x)^{-\theta_1}. \end{aligned}$$

This proves (1.22).



### 3.2. Proof of (1.23) and (1.24).

Firstly we prove two preliminary lemmas.

**Lemma 3.2.** *Let  $f \in \mathcal{M}^*(\Gamma)$  and  $m \geq 1$  be an integer. Assuming the Ramanujan conjecture (1.4) and that  $L(s, \text{sym}^j f)$  is automorphic for  $1 \leq j \leq 2m$ , then we have*

$$(3.5) \quad \sum_{p \leq x} \frac{\lambda_f(p^m)^2}{p} = \log_2 x + O(1).$$

$$(3.6) \quad \sum_{p \leq x} \frac{\lambda_f(p^m)^4}{p} = (m+1) \log_2 x + O(1).$$

*Proof.* As usual denote by  $U_n(x)$  the  $n$ th Chebyshev polynomial of the second kind. Put  $T_n(x) := U_n(x/2)$ . In view of the identity (see [10, page 198])

$$T_m(x)^2 = \sum_{n=0}^m T_{2n}(x),$$

we have

$$T_m(x)^4 = \sum_{n=0}^m \sum_{n'=0}^m T_{2n}(x) T_{2n'}(x).$$

According to the Rankin-Selberg theory, it is known that

$$\sum_{p \leq x} \frac{\lambda_{\text{sym}^n f \times \text{sym}^{n'} f}(p)}{p} = \begin{cases} \log_2 x + O(1) & \text{if } n = n', \\ O(1) & \text{if } n \neq n'. \end{cases}$$

In view of  $\lambda_{\text{sym}^n f \times \text{sym}^{n'} f}(p) = \lambda_{\text{sym}^n f}(p) \lambda_{\text{sym}^{n'} f}(p) = \lambda_f(p^n) \lambda_f(p^{n'})$ , the required asymptotic formulas follow.  $\square$

The next lemma is a generalization of Wu-Xu's inequality [21, page 452, line 18].

**Lemma 3.3.** *Let  $m \geq 1$  be an integer,  $\kappa_m := m^2 + 3m + 2 = (m+2)(m+1)$  and*

$$\begin{aligned} a_0(m) &:= \frac{(\kappa_m - 3)\sqrt{\kappa_m} + 2}{2(\kappa_m - 1)^2} > 0, \\ a_1(m) &:= \frac{(\kappa_m^2 + 3)\sqrt{\kappa_m} - 4\kappa_m}{2(\kappa_m - 1)^2} > 0, \\ a_2(m) &:= -\frac{(\kappa_m^2 + \kappa_m)\sqrt{\kappa_m} - 2\kappa_m^2}{2(\kappa_m - 1)^2} < 0. \end{aligned}$$

Then we have

$$(3.7) \quad t^{1/2} \leq a_0(m) + a_1(m)t + a_2(m)t^2 \quad (0 \leq t \leq 1).$$

*Proof.* Consider the function

$$h(t) := t^{1/2} - a_1(m)t - a_2(m)t^2 \quad (0 \leq t \leq 1).$$

It is easy to check that  $h(\kappa_m^{-1}) = h(1)$ ,  $h'(\kappa_m^{-1}) = 0$ ,  $h''(\kappa_m^{-1}) = -\frac{(1+3\kappa_m)(1-\kappa_m)}{4\kappa_m^3(\kappa_m+1)^2} < 0$  and  $\lim_{t \rightarrow \infty} h(t) = -\infty$ . Thus  $h'(t) = \frac{1}{2}t^{-1/2}(1 - 2a_1(m)t^{1/2} - 4a_2(m)t^{3/2})$  has three zeros  $\kappa_m^{-1}, t_m, t'_m$  on  $(0, \infty)$  such that  $0 < \kappa_m < t_m < 1 < t'_m < \infty$ . So  $h(t)$  takes the maximum at  $\kappa_m^{-1}$  and 1 on  $[0, 1]$ . The result follows.  $\square$

Now we are ready to prove (1.23) and (1.24).

Under the Ramanujan conjecture (1.4), we can apply inequality (3.7) with  $t = (|\lambda_f(p^m)|/(m+1))^2$  to write

$$|\lambda_f(p^m)| \leq a_0(m)(m+1) + \frac{a_1(m)}{m+1} |\lambda_f(p^m)|^2 + \frac{a_2(m)}{(m+1)^3} |\lambda_f(p^m)|^4$$

for all primes  $p$ . By using this and Lemma 3.2, it follows that

$$(3.8) \quad \sum_{p \leq x} \frac{|\lambda_f(p^m)|}{p} \leq \left( a_0(m)(m+1) + \frac{a_1(m)}{m+1} + \frac{a_2(m)}{(m+1)^2} \right) \log_2 x + O(1) \\ = (1 - \eta_m) \log_2 x + O(1).$$

A simple calculation shows that

$$2(m+1)(\kappa_m - 1)^2 \eta_m = 2(m+1)(\kappa_m - 1)^2 - 2(m+1)^2 + 4\kappa_m - 2(m+2)\kappa_m \\ + [(m+2)(\kappa_m + 1) - (m+1)^2(\kappa_m - 3) - \kappa_m^2 - 3] \sqrt{\kappa_m} \\ = 2(\kappa_m - 1)[m(m+1)(m+3) - m(m+2)\sqrt{\kappa_m}],$$

which implies the expression (1.17) of  $\eta_m$ .

Under the Ramanujan conjecture (1.4), the function  $|\lambda_{\text{sym}^m f}(n)|$  satisfies the conditions of Theorem 1 of Shiu [16]. Thus (1.23) follows from (2.5) and (3.8).

In view of (1.1) and the Ramanujan conjecture (1.4), it is easy to show that

$$(3.9) \quad L(s, \text{sym}^m f) = G_m(s) \sum_{n \geq 1} \lambda_f(n^m) n^{-s}$$

for  $\Re s > 1$ , where  $G_m(s)$  is a Dirichlet series absolutely convergent for  $\Re s > \frac{1}{2}$  and  $G_2(s) = \zeta(2s)$ . Thus with the the help of (1.22), a simple convolution argument allows us to deduce (1.24).

## 4. PROOF OF THEOREMS 1 AND 2

### 4.1. A theorem of Chandrasekharan and Narasimhan.

First we prove a lemma, which is a variant of Theorem 4.1 of [2] in the case of the symmetric power  $L$ -functions.

**Lemma 4.1.** *Let  $f \in \mathcal{H}_k^*(\Gamma) \cup \mathcal{M}^*(\Gamma)$  and  $m \geq 1$ . Suppose that  $L(s, \text{sym}^m f)$  are automorphic. Then for any  $1 \leq y \leq x^{m/(m+1)}$ , we have*

$$(4.1) \quad \sum_{n \leq x} \lambda_{\text{sym}^m f}(n) \ll_{f,m} x^{m/2(m+1)} \sum_{n \leq x^m/y^{m+1}} |\lambda_{\text{sym}^m f}(n)| n^{-1/2-1/2(m+1)} \\ + x^{m+m/2(m+1)} y^{-(m+1)} \sum_{n > x^m/y^{m+1}} |\lambda_{\text{sym}^m f}(n)| n^{-3/2-1/2(m+1)} \\ + \sum_{x < n \leq x+(m+1)y} |\lambda_{\text{sym}^m f}(n)| + O(1).$$

*Proof.* Under our hypothesis,  $L(s, \text{sym}^m f)$  satisfies all conditions of Theorem 4.1 of [2] with the parameters  $\mu_n = \lambda_n = n$ ,  $a_n = b_n = \lambda_{\text{sym}^m f}(n)$  and

$$A = \frac{1}{2}(m+1), \quad \rho = m+1, \quad \delta = 1, \quad q = -\frac{1}{2} + \varepsilon.$$

Thus (4.14) of [2] becomes

$$(4.2) \quad \sum_{n \leq x} \lambda_{\text{sym}^m f}(n) = y^{-(m+1)} \Delta_y^{m+1} W_{m+1}(x) + O\left( \sum_{x < n \leq x + (m+1)y} |\lambda_{\text{sym}^m f}(n)| \right),$$

where we have removed the term  $Q_0(x) \equiv 0$  (since  $L(s, \text{sym}^m f)$  is an entire function under our hypothesis) and the term  $O(yx^{q-1}(\log x)^{r-1}) = O(1)$ .

By (4.6) and the formula below (4.18) of [2], we have

$$\begin{aligned} \Delta_y^{m+1} W_{m+1}(x) &\ll y^{m+1} x^{m/2(m+1)} \sum_{n \leq x^m/y^{m+1}} |\lambda_{\text{sym}^m f}(n)| n^{-1/2-1/2(m+1)} \\ &\quad + x^{m+m/2(m+1)} \sum_{n > x^m/y^{m+1}} |\lambda_{\text{sym}^m f}(n)| n^{-3/2-1/2(m+1)}. \end{aligned}$$

Inserting it into (4.2), we obtain (4.1). □

#### 4.2. Proof of Theorem 1.

Now we are ready to complete the proof of Theorem 1.

With the help of (1.18), a simple partial integration gives

$$\begin{aligned} \sum_{n \leq x^m/y^{m+1}} |\lambda_{\text{sym}^m f}(n)| n^{-1/2-1/2(m+1)} &\ll (x^m/y^{m+1})^{1/2-1/2(m+1)} (\log x)^{-\delta_m}, \\ \sum_{n > x^m/y^{m+1}} |\lambda_{\text{sym}^m f}(n)| n^{-3/2-1/2(m+1)} &\ll (x^m/y^{m+1})^{-1/2-1/2(m+1)} (\log x)^{-\delta_m}. \end{aligned}$$

Inserting these and (1.20) into (4.1) of Lemmas 4.1, it follows that

$$\sum_{n \leq x} \lambda_{\text{sym}^m f}(n) \ll x^{m/2} y^{-m/2} (\log x)^{-\delta_m} + y (\log x)^{-\delta_m}.$$

for all  $x \geq 3$  and  $x^\varepsilon \leq y \leq x^{m/(m+1)}$ . Thus (1.12) follows from the choice of  $y = x^{m/(m+2)}$ .

In view of (3.9), bound (1.13) is an immediate consequence of (1.12) by a simple convolution argument. This proves the assertion (i).

Next we treat the assertion (iii). Consider the function

$$g(t) := \frac{t}{1-t^2} \cot\left(\frac{\pi}{2}t\right) \quad (t \in (0, \frac{1}{2})).$$

We have

$$\begin{aligned}
g'(t) &= \frac{\pi t(1+t^2)}{2(1-t^2)^2} \left( \frac{\sin(\pi t)}{\pi t} - \frac{1-t^2}{1+t^2} \right) \\
&= \frac{\pi t(1+t^2)}{2(1-t^2)^2} \sum_{k \geq 1} (-1)^{k-1} \left( 2 - \frac{\pi^{2k}}{(2k+1)!} \right) t^{2k} \\
&= \frac{\pi t(1+t^2)}{2(1-t^2)^2} \sum_{n \geq 1} \left\{ \left( 2 - \frac{\pi^{4n-2}}{(4n-1)!} \right) - \left( 2 - \frac{\pi^{4n}}{(4n+1)!} \right) t^2 \right\} t^{4n-2} \\
&\geq \frac{\pi t(1+t^2)}{2(1-t^2)^2} \sum_{n \geq 1} \left\{ \left( 2 - \frac{\pi^{4n-2}}{(4n-1)!} \right) - \frac{1}{4} \left( 2 - \frac{\pi^{4n}}{(4n+1)!} \right) \right\} t^{4n-2} \\
&> 0 \quad \left( t \in \left( 0, \frac{1}{2} \right] \right),
\end{aligned}$$

since each term of the last series is positive. Thus  $g(t)$  is increasing on  $(0, \frac{1}{2}]$ . Noticing that  $\delta_m = 1 - \frac{4}{\pi} g\left(\frac{1}{m+1}\right)$ , the sequence  $\{\delta_m\}_{m \geq 1}$  is increasing.

### 4.3. Proof of Theorem 2.

With the help of (1.24), a simple partial integration gives

$$\begin{aligned}
\sum_{n \leq x^m/y^{m+1}} |\lambda_{\text{sym}^m f}(n)| n^{-1/2-1/2(m+1)} &\ll (x^m/y^{m+1})^{1/2-1/2(m+1)} (\log x)^{-\theta_1}, \\
\sum_{n > x^m/y^{m+1}} |\lambda_{\text{sym}^m f}(n)| n^{-3/2-1/2(m+1)} &\ll (x^m/y^{m+1})^{-1/2-1/2(m+1)} (\log x)^{-\theta_1}.
\end{aligned}$$

Inserting these and (1.23) into (4.1) of Lemmas 4.1, it follows that

$$\sum_{n \leq x} \lambda_{\text{sym}^m f}(n) \ll x^{m/2} y^{-m/2} (\log x)^{-\theta_1} + y (\log x)^{-\eta_m}$$

for all  $x \geq 3$  and  $x^\varepsilon \leq y \leq x^{m/(m+1)}$ . Thus (1.15) follows from the choice of  $y = x^{m/(m+2)} (\log x)^{-2(\theta_1 - \eta_m)/(m+2)}$ .

Bound (1.16) is an immediate consequence of (1.15) by a simple convolution argument as before. This finishes the proof of Theorem 2.

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