Shifted convolution of cusp-forms with θ-series

Guangshi Lu, Jie Wu, Wenguang Zhai

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GUANGSHI LÜ, JIE WU & WENGUANG ZHAI

Abstract. In this paper we apply simple approachs to improve a recent result due to Luo, concerning a shifted convolution sum involving the Fourier coefficients of cusp forms with those of theta series. In order to explore what should be the best possible bound, a mean square result of this shifted convolution sum is also established.

1. Introduction

Suppose $\lambda_1(n)$ and $\lambda_2(n)$ are two arithmetic functions, $b \geqslant 0$ is an integer. It is a classical and important problem in analytic number theory to study the shifted convolution sum

$$\sum_{n\leqslant x} \lambda_1(n)\lambda_2(n+b).$$

There are a large number of papers in this direction. For example, when $\lambda_1(n) = \lambda_2(n) = \tau(n)$, the Dirichlet divisor function, this problem is the so-called additive divisor problem [1, 6, 8, 9, 12, 18, 19]. For other interesting cases, see [14, 15, 7].

Recently Luo [16] investigated a mixed shifted convolution sum

$$\sum_{n\geqslant 1} \lambda_f(n+b)r_\ell(n)\phi(n), \tag{1.1}$$

where $\lambda_f(n)$ is the nth normalized Hecke eigenvalue of a holomorphic cusp form of weight k and level N, $b \geqslant 0$ is a fixed integer,

$$r_\ell(n) := \big|\big\{(n_1,\dots,n_\ell) \in \mathbb{Z}^\ell : n_1^2+\cdots+n_\ell^2 = n\big\}\big|$$

and $\phi(t)$ is a smooth function with support in $[x/2, 5x/2]$, satisfying $\phi^{(j)}(t) \ll_j (x/J)^{-j}$ for all real numbers $t \in \mathbb{R}$, all integers $j \geqslant 0$ and $1 \leqslant J \leqslant x^\beta$ with $\beta < 1$. In order to estimate the sum (1.1), he first established a Voronoi formula for $r_\ell(n)$ and then combined this formula with the upper bound for the Salié sum to derive

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the following result [16, Theorem] : For any $\varepsilon > 0$, the inequality

$$\sum_{n\geqslant 1}\lambda_f(n+b)r_\ell(n)\phi(n) \ll \begin{cases} x^{\ell/2-(\ell-1)/4+\varepsilon}J^{g_\ell} & \text{if } 4\nmid \ell \text{ or } b\neq 0 \\ x^{\ell/2-(\ell-1)/4+\varepsilon}(x^{1/4+\varepsilon}+J^{g_\ell}) & \text{if } 4\mid \ell \text{ and } b=0 \end{cases} \tag{1.2}$$

holds for all $x \geqslant 2$, where g_ℓ is the smallest integer such that $g_\ell \geqslant (\ell+1)/2$ [†] and the implied constant depends on f, b, ℓ and ε. With a suitable choice of $\phi(t)$ and J, the inequality (1.2) implies immediately the following result [16, Corollary] : For $\ell \geqslant 2$, $k \geqslant \ell/2+3$ and $\varepsilon > 0$, the inequality

$$\mathscr{S}_{f,b,\ell}(x) := \sum_{n\leqslant x}\lambda_f(n+b)r_\ell(n) \ll_{f,b,\ell,\varepsilon} x^{\ell/2-\vartheta_\ell+\varepsilon} \tag{1.3}$$

holds for all $x \geqslant 2$, where $\vartheta_\ell := (\ell-1)/(4g_\ell+4)$. In particular

$$\vartheta_2 = \tfrac{1}{12}, \quad \vartheta_3 = \tfrac{1}{6}, \quad \vartheta_4 = \tfrac{3}{16}, \quad \vartheta_5 = \tfrac{1}{4}, \quad \vartheta_6 = \tfrac{1}{4}, \quad \vartheta_\ell \leqslant \tfrac{\ell-1}{2\ell+6} < \tfrac{1}{2} \ \ (\ell \geqslant 7).$$

The first aim of this paper is to propose a better bound for $\mathscr{S}_{f,b,\ell}(x)$.

Theorem 1. *Let f be a cusp form of weight k and level N and let $\ell \geqslant 2$ be an integer. For any $\varepsilon > 0$, we have*

$$\mathscr{S}_{f,b,\ell}(x) := \sum_{n\leqslant x}\lambda_f(n+b)r_\ell(n) \ll_{f,\ell,\varepsilon} x^{\ell/2-\vartheta_\ell+\varepsilon} \tag{1.4}$$

uniformly for $x \geqslant 2$ and $0 \leqslant b \leqslant x$, where

$$\vartheta_3 = \tfrac{1}{4}, \qquad \vartheta_4 = \tfrac{1}{2}, \qquad \vartheta_5 = \tfrac{1}{2}, \qquad \vartheta_\ell = \tfrac{2}{3} \quad (\ell \geqslant 6).$$

In addition, if we assume $N = 1$, then (1.4) holds for $\ell = 2$ with $\vartheta_2 = \frac{1}{6}$.

Remark 1. Theorem 1 improves Luo's (1.3) in three directions; it enlarges the exponent ϑ_ℓ, relaxes the restricted condition $k \geqslant \ell/2+3$ and removes the dependence of b. Our method is completely different from that of [16]. Our new idea is to explore the regularity of $r_\ell(n)$ by the circle method in analytic number theory and Siegel's mass formula. Thanking to these classic tools of analytic number theory, we can show that the influence of $r_\ell(n)$ to the bound [22, (1.10) and Theorem 2]

$$\sum_{n\leqslant x}\lambda_f(n) \ll_f x^{1/3}(\log x)^{2/(\sqrt{\pi}\Gamma(5/2))-1} \tag{1.5}$$

is rather little. More precisely by using the circle method we shall give a very simple proof of $\vartheta_\ell = \frac{1}{2}$ for $\ell \geqslant 3$. Further when $\ell \geqslant 6$, with the help of Siegel's mass formula we shall prove a better exponent $\theta_\ell = \frac{2}{3}$, which means that our bound for $\mathscr{S}_{f,b,\ell}(x)$ is of the same quality as (1.5). Finally the case of $\ell = 2$ requires a more delicate consideration (see the end of Section 3).

[†]It seems that Luo's proof only could show that g_ℓ is the smallest integer such that $g_\ell > (\ell+1)/2$. This means that $\vartheta_3 = \frac{1}{8}$ and $\vartheta_5 = \frac{1}{5}$ instead of $\vartheta_3 = \frac{1}{6}$ and $\vartheta_5 = \frac{1}{4}$.

Remark 2. In fact our method allows us to consider a more general case. Let $\ell \geqslant 2$, $\mathbf{y} := (y_1, \dots, y_\ell) \in \mathbb{Z}^\ell$ and $\mathbf{A} = (a_{ij})$ be an integral matrix such that $a_{ii} \equiv 0 \,(\mathrm{mod}\, 2)$ for $1 \leqslant i \leqslant \ell$. The positive definite quadratic form $Q(\mathbf{y})$ is defined by $Q(\mathbf{y}) = \frac{1}{2}\mathbf{y}^{\mathrm{t}}\mathbf{A}\mathbf{y}$. For each $n \geqslant 1$, define

$$r(n, Q) := \big|\big\{\mathbf{y} \in \mathbb{Z}^\ell : Q(\mathbf{y}) = n\big\}\big|.$$

Thus $r_\ell(n) = r(n, Q_0)$ with $Q_0 = y_1^2 + \cdots + y_\ell^2$. Similar to $\mathscr{S}_{f,b,\ell}(x)$, we define

$$\mathscr{S}_{f,b,Q}(x) := \sum_{n \leqslant x} \lambda_f(n+b) r(n, Q). \tag{1.6}$$

Define $\mathcal{F}$ to be a class of cusp forms, which consists of holomorphic cusp forms with respect to any finite volume discrete subgroup (such that ∞ is a singular cusp of width 1), any positive real weight and any multiplier systems, as well as Maass cusp forms of any weight and any level [11, 2, 4]. Since the estimate (2.1) below holds for all $f \in \mathcal{F}$ ([10, Theorem 5.3] and [11, Theorem 8.1]), our proof of Theorem 1 implies that

$$\mathscr{S}_{f,b,Q}(x) \ll_{f,Q,\varepsilon} x^{\ell/2-1/2+\varepsilon}$$

holds uniformly for $1 \leqslant b \leqslant x$, provided $\ell \geqslant 5$.

A natural question is what should be the best bound for $\mathscr{S}_{f,b,\ell}(x)$. It is well-known that [3, Theorem 2])

$$\sum_{n \leqslant x} \lambda_f(n) = \Omega_\pm\bigg(x^{1/4} \exp\bigg\{\frac{D(\log\log x)^{1/4}}{(\log\log\log x)^{3/4}}\bigg\}\bigg)$$

where $D > 0$ a constant. A slight weaker Ω-result

$$\sum_{n \leqslant x} \lambda_f(n) = \Omega(x^{1/4})$$

is supported by the asymptotic formula (see [21])

$$\int_1^X \Big|\sum_{n \leqslant x} \lambda_f(n)\Big|^2 \mathrm{d}x = C_f X^{3/2} + O\big(X(\log X)^2\big),$$

for $X \geqslant 2$, where C_f is a positive constant. On the other hand, we have $r_\ell(n) \asymp n^{\ell/2-1}$ ($\ell \geqslant 3$). From the above facts it seems reasonable to propose the following conjecture.

Conjecture. *Let f be a cusp form of weight k and level N. Let $b \geqslant 0$ and $\ell \geqslant 3$ be two integers. For any $\varepsilon > 0$, we have*

$$\mathscr{S}_{f,b,\ell}(x) \ll_{f,b,\ell,\varepsilon} x^{\ell/2-3/4+\varepsilon} \tag{1.7}$$

for $x \to \infty$.

It seems rather difficult to establish (1.7). However we can prove that the bound (1.7) is true on average for $\ell \geqslant 5$ (see (1.8) and (1.9) below). For this, we shall study the square mean of $\mathscr{S}_{f,b,\ell}(x)$. At this time, our method also allows us to deal with the more general case $\mathscr{S}_{f,b,Q}(x)$.

Our second aim is to prove the following asymptotic formula.

Theorem 2. *Let f be a cusp form of weight k and level N. Let $Q(\mathbf{y})$ be a positive definite quadratic form defined as above. If $\ell \geqslant 6$, then we have*

$$\begin{aligned}\int_1^X |\mathscr{S}_{f,b,Q}(x)|^2\,\mathrm{d}x &= C_{f,b,Q}\int_1^X x^{\ell-2}(x+b)^{1/2}\,\mathrm{d}x + O_{f,Q}\big(X^{\ell-7/12}(\log X)^{1/2}\big)\\ &= \frac{C_{f,b,Q}}{\ell-1/2}X^{\ell-1/2} + O_{f,Q}\big(bX^{\ell-3/2} + X^{\ell-7/12}(\log X)^{1/2}\big)\end{aligned} \tag{1.8}$$

uniformly for $X \geqslant 2$ and $0 \leqslant b \leqslant X$, where the constant $C_{f,b,Q}$ is defined by (4.12) *below, and the implied constants depend on f and Q.*

Remark 3. In Section 3, we shall prove also that the inequality

$$\int_1^X |\mathscr{S}_{f,b,5}(x)|^2\,\mathrm{d}x \ll_{f,\varepsilon} X^{\ell-1/2+\varepsilon} \tag{1.9}$$

holds uniformly for $X \geqslant 1$ and $0 \leqslant b \leqslant X$.

2. Preliminary Lemmas

In order to prove our theorems, we need the following lemmas.

Lemma 2.1. *Let f be a cusp form of weight k and level N.*
(a) *The estimate*

$$\sum_{n\leqslant x}\lambda_f(n)\mathrm{e}(\alpha n) \ll_f x^{1/2}\log x \tag{2.1}$$

holds uniformly for $\alpha \in \mathbb{R}$, where $\mathrm{e}(u) := \mathrm{e}^{2\pi \mathrm{i}u}$.
(b) *For any $\varepsilon > 0$, we have*

$$\sum_{\substack{n\leqslant x\\ n\equiv a(\mathrm{mod}\,q)}}\lambda_f(n) \ll_{f,\varepsilon} x^{1/2-1/6+\varepsilon} \tag{2.2}$$

uniformly for $(q, aN) = 1$ and $q \leqslant x^{2/3}$.

Proof. See for example, [10, Theorem 5.3] and [20]. □

Lemma 2.2. *Let f be a cusp form of weight k and level N, $(h, q) = 1$ and*

$$A(x, h/q) := {\sum_{n\leqslant x}}'\lambda_f(n)\mathrm{e}_q(hn), \tag{2.3}$$

where $\mathrm{e}_q(m) := \mathrm{e}(m/q)$ and $\sum'_{n\leqslant x}$ means that if x is an integer, the term $n = x$ should be halved. Then for any $\varepsilon > 0$ we have

$$A(x, h/q) = \frac{q^{1/2}x^{1/4}}{\sqrt{2}\pi}\sum_{n\leqslant M}\frac{\lambda_f(n)}{n^{3/4}}\mathrm{e}_q(-\overline{h}n)\cos\left(\frac{4\pi\sqrt{nx}}{q} - \frac{\pi}{4}\right) + O_{f,\varepsilon}\left(\frac{qx^{1/2+\varepsilon}}{M^{1/2}}\right)$$

uniformly for $1 \leqslant q \leqslant x$ *and* $1 \leqslant M \ll x$, *where* $\overline{h}$ *satisfies* $h\overline{h} \equiv 1 \,(\mathrm{mod}\, q)$. *In particular for any* $\varepsilon > 0$ *we have*

$$A(x, h/q) \ll_{f,\varepsilon} q^{2/3} x^{1/3+\varepsilon} \tag{2.4}$$

uniformly for $x \geqslant 1$ *and* $q \geqslant 1$. *Here the implied constants depend on* f *and* ε *only.*

Proof. The first assertion is [13, Theorem 1.1]. The bound (2.4) is trivial if $q \geqslant x$. When $1 \leqslant q \leqslant x$, it follows from the first assertion with $M = (q^2 x)^{1/3}$. See also, [17, (2.7)]. □

Lemma 2.3. *Let* $A(x, h/q)$ *be defined as in* (2.3).

(i) *We have*

$$A_1(x, h/q) := \int_1^x A(u, h/q)\, \mathrm{d}u \ll_f q^{3/2} x^{3/4} \tag{2.5}$$

uniformly for $x \geqslant 1$ *and* $1 \leqslant q \leqslant x^{1/2}$.

(ii) *For any* $\varepsilon > 0$ *we have*

$$\int_1^X |A(x, h/q)|^2\, \mathrm{d}x = \frac{1}{(4k+2)\pi^2} \sum_{n=1}^{\infty} \frac{|\lambda_f(n)|^2}{n^{3/2}} q X^{3/2} + O_{f,\varepsilon}\big(q^{3/2} X^{5/4+\varepsilon} + q^2 X^{1+\varepsilon}\big)$$

uniformly for $X \geqslant 1$ *and* $q \geqslant 1$.

Proof. They are (1.6.12) and (1.5.23) of [13], respectively. □

Lemma 2.4. *Let* $\ell \geqslant 2$, $\mathbf{y} := (y_1, \dots, y_\ell) \in \mathbb{Z}^\ell$ *and* $\mathbf{A} = (a_{ij})$ *be an integral matrix such that* $a_{ii} \equiv 0 \,(\mathrm{mod}\, 2)$ *for* $1 \leqslant i \leqslant \ell$. *The positive definite quadratic form* $Q(\mathbf{y})$ *is defined by* $Q(\mathbf{y}) = \frac{1}{2}\mathbf{y}^{\mathrm{t}}\mathbf{A}\mathbf{y}$. *For each* $n \geqslant 1$, *define*

$$r(n, Q) := \big|\big\{\mathbf{y} \in \mathbb{Z}^\ell : Q(\mathbf{y}) = n\big\}\big|.$$

Then for $\ell \geqslant 4$ *we have*

$$r(n, Q) = \sigma_Q n^{\ell/2-1} \sum_{q=1}^{\infty} \sideset{}{^*}\sum_{h=1}^{q} S\Big(\frac{hQ}{q}\Big) \frac{\mathrm{e}_q(-hn)}{q^\ell} + O\big(n^{\ell/4-\delta_\ell+\varepsilon}\big), \tag{2.6}$$

where

$$S(Q) := \sum_{0 \leqslant y_1, \dots, y_\ell \leqslant q-1} \mathrm{e}(Q(\mathbf{y})), \quad \sigma_Q := \frac{(2\pi)^{\ell/2}}{\Gamma(\ell/2)\sqrt{|\mathbf{A}|}}, \quad \delta_\ell := \begin{cases} \frac{1}{4} & \text{if } \ell \text{ is odd,} \\ \frac{1}{2} & \text{if } \ell \text{ is even,} \end{cases}$$

and $\sum^*$ *means the sum is over* $1 \leqslant h \leqslant q$ *with* $(h, q) = 1$. *Furthermore we have*

$$S(hQ/q) \ll q^{\ell/2} \quad ((h, q) = 1), \tag{2.7}$$

Proof. See for example, Theorem 11.2 of [10]. □

3. Proof of Theorem 1

3.1. **Case of $\ell \geqslant 3$.**

Let

$$F(\alpha) := \sum_{n\leqslant x} \lambda_f(n+b)\mathrm{e}(-\alpha n) \qquad \text{and} \qquad S(\alpha) := \sum_{|m|\leqslant x^{1/2}} \mathrm{e}(\alpha m^2).$$

Then

$$\begin{aligned}\int_0^1 F(\alpha)S(\alpha)^\ell \mathrm{d}\alpha &= \sum_{n\leqslant x} \lambda_f(n+b) \sum_{|m_1|\leqslant x^{1/2}} \cdots \sum_{|m_\ell|\leqslant x^{1/2}} \int_0^1 \mathrm{e}\big((m_1^2+\cdots+m_\ell^2-n)\alpha\big)\,\mathrm{d}\alpha \\ &= \sum_{n\leqslant x} \lambda_f(n+b) \sum_{\substack{|m_1|\leqslant x^{1/2} \\ m_1^2+\cdots+m_\ell^2=n}} \cdots \sum_{|m_\ell|\leqslant x^{1/2}} 1 \\ &= \mathscr{S}_{f,b,\ell}(x).\end{aligned}$$

Firstly Lemma 2.1 allows us to deduce

$$F(\alpha) = \mathrm{e}(\alpha b) \sum_{b<n\leqslant x+b} \lambda_f(n)\mathrm{e}(-\alpha n) \ll x^{1/2}\log x$$

uniformly for $x \geqslant 2$ and $0 \leqslant b \leqslant x$. On the other hand, it is easy to see that

$$\begin{aligned}\int_0^1 |S(\alpha)|^2 \mathrm{d}\alpha &= \sum_{\substack{|m|\leqslant x^{1/2} \\ m^2=n^2}} \sum_{|n|\leqslant x^{1/2}} 1 \ll x^{1/2}, \\ \int_0^1 |S(\alpha)|^{2d} \mathrm{d}\alpha &= \sum_{\substack{|m_1|\leqslant x^{1/2},\dots,|m_d|\leqslant x^{1/2} \\ m_1^2+\cdots+m_d^2=n_1^2+\cdots+n_d^2}} \sum_{|n_1|\leqslant x^{1/2},\dots,|n_d|\leqslant x^{1/2}} 1 \quad \leqslant \sum_{n\leqslant dx} r_d(n)^2 \ll x^{d-1}\end{aligned}$$

for $d \geqslant 2$. From the above estimates and the Cauchy inequality we get

$$\mathscr{S}_{f,b,3}(x) \ll x^{1/2}(\log x)\bigg(\int_0^1 |S(\alpha)|^2\mathrm{d}\alpha \int_0^1 |S(\alpha)|^4\mathrm{d}\alpha\bigg)^{1/2} \ll x^{3/2-1/4}\log x$$

and for $\ell \geqslant 4$

$$\begin{aligned}\mathscr{S}_{f,b,\ell}(x) &\ll x^{1/2}(\log x)\bigg(\int_0^1 |S(\alpha)|^4\mathrm{d}\alpha \int_0^1 |S(\alpha)|^{2(\ell-2)}\mathrm{d}\alpha\bigg)^{1/2} \\ &\ll x^{\ell/2-1/2}\log x,\end{aligned}$$

namely we can take $\vartheta_3 = \frac14$ and $\vartheta_\ell = \frac12$ for $\ell \geqslant 4$.

3.2. **Case of $\ell \geqslant 6$.**

When $\ell \geqslant 6$, we can apply Lemmas 2.2 and 2.4 to improve the exponent $\vartheta_\ell = \frac12$. From Lemma 2.4 with $Q_0(\mathbf{y}) = y_1^2+\cdots+y_\ell^2$ we get

$$r_\ell(n) = \sigma_{Q_0} n^{\ell/2-1} \sum_{q=1}^{\infty} \sideset{}{^*}\sum_{h=1}^{q} S\bigg(\frac{hQ_0}{q}\bigg)\frac{\mathrm{e}_q(-hn)}{q^\ell} + O_\varepsilon\big(n^{\ell/4-\delta_\ell+\varepsilon}\big). \tag{3.1}$$

In view of (2.4) in Lemma 2.2 and the bound (2.7) we easily deduce, via a simple integration by parts, that

$$\begin{aligned}\mathscr{S}_{f,b,\ell}(x) &\ll \sum_{q\geqslant 1}\frac{1}{q^{\ell/2}}\sum_{\substack{1\leqslant h\leqslant q\\(h,q)=1}}\Big|\sum_{1+b\leqslant n\leqslant x+b} n^{\ell/2-1}\lambda_f(n)\mathrm{e}_q(-hn)\Big| + x^{\ell/4-\delta_\ell+1+\varepsilon}\\ &\ll_{f,\varepsilon} x^{\ell/2-2/3+\varepsilon}\sum_{q\geqslant 1}\frac{1}{q^{\ell/2-5/3}} + x^{\ell/4-\delta_\ell+1+\varepsilon}\\ &\ll_{f,\varepsilon} x^{\ell/2-2/3+\varepsilon}\quad (\text{recall } \ell\geqslant 6).\end{aligned}$$

3.3. **Case of $\ell = 2$.**

We need the following Lemmas.

Lemma 3.1. *Let $m\geqslant 2$ be a positive integer. There is an arithmetic function $h_m(n)$ such that*

$$h_m(n) = 0 \ \text{ if } \exists\, p \text{ such that } p\mid n \text{ and } p\nmid m, \tag{3.2}$$

$$|h_m(n)|\leqslant \tau(m)\tau_4(n) \ \text{ if } n\mid m^\infty, \tag{3.3}$$

$$\lambda_f(mn) = \sum_{d\mid n} h_m(d)\lambda_f(n/d), \tag{3.4}$$

where $\tau_k(n)$ denotes the number of solutions of $n = n_1\cdots n_k$ with positive numbers $n_1,\dots,n_k$, and $\tau(n) := \tau_2(n)$.

Proof. As usual, we denote by $v_p(n)$ the p-adic valuation of n. By using the formula

$$\lambda_f(mn) = \prod_{p\mid d}\lambda_f(p^{v_p(n)+v_p(m)})\prod_{p\nmid d}\lambda_f(p^{v_p(n)}),$$

we can write, for $\Re e\, s > 1$,

$$\begin{aligned}\sum_{n=1}^\infty\frac{\lambda_f(mn)}{n^s} &= \prod_{p\mid m}\sum_{\nu=0}^\infty\frac{\lambda_f(p^{\nu+v_p(m)})}{p^{\nu s}}\prod_{p\nmid m}\sum_{\nu=0}^\infty\frac{\lambda_f(p^\nu)}{p^{\nu s}}\\ &= L(s,f)\prod_{p\mid m}\Big(\sum_{\nu=0}^\infty\frac{\lambda_f(p^{\nu+v_p(m)})}{p^{\nu s}}\Big)\Big(\sum_{\nu=0}^\infty\frac{\lambda_f(p^\nu)}{p^{\nu s}}\Big)^{-1}.\end{aligned}$$

This implies that there is an arithmetic function $h_m(n)$ such that (3.2) and (3.4) hold.

Next we prove (3.3). If $p\mid m$, it is easy to see that

$$h_m(p^\nu) = \begin{cases}\lambda_f(p^{v_p(m)}) & \text{if } \nu = 0,\\ \lambda_f(p^{v_p(m)+1}) - \lambda_f(p^{v_p(m)})\lambda_f(p) & \text{if } \nu = 1,\\ \lambda_f(p^{v_p(m)+\nu}) - \lambda_f(p^{v_p(m)+\nu-1})\lambda_f(p) + \lambda_f(p^{v_p(m)+\nu-2}) & \text{if } \nu\geqslant 2.\end{cases}$$

In view of Degline's bound $|\lambda_f(n)| \leqslant \tau(n)$, for any $n \mid m^\infty$ we have

$$\begin{aligned}|h_m(n)| = \prod_{p^\nu \| n} |h_m(p^\nu)| &\leqslant \prod_{p^\nu \| n} 4(v_p(m)+\nu) \\ &\leqslant \prod_{p^\nu \| n} (v_p(m)+1)\binom{\nu+3}{3} \leqslant \tau(m)\tau_4(n).\end{aligned}$$

This completes the proof. □

Lemma 3.2. *Let f be a cusp form of weight k and level 1. Then the estimate*

$$S_f(x;a,q) := \sum_{\substack{n\leqslant x\\ n\equiv a(\mathrm{mod}\, q)}} \lambda_f(n) \ll_f x^{1/3+\varepsilon} \tag{3.5}$$

holds uniformly for $x \geqslant 1$ and $q \geqslant a \geqslant 1$.

Proof. Note that (2.2) and (3.5) is trivial for $q > x^{2/3}$. Now suppose $q \leqslant x^{2/3}$. Writting $m := (a,q)$, $a = ma_1$ and $q = mq_1$ such that $(a_1, q_1) = 1$ and using (3.4) of Lemma 3.1, we have

$$\begin{aligned}S_f(x;a,q) &= \sum_{\substack{n\leqslant x/m\\ n\equiv a_1(\mathrm{mod}\, q_1)}} \lambda_f(mn) \\ &= \sum_{\substack{n_1n_2\leqslant x/m\\ n_1n_2\equiv a_1(\mathrm{mod}\, q_1)}} h_m(n_1)\lambda_f(n_2) \\ &= \sum_{n_1\leqslant x/m} h_m(n_1) \sum_{\substack{n_2\leqslant x/mn_1\\ n_2\equiv a_1\overline{n_1}(\mathrm{mod}\, q_1)}} \lambda_f(n_2),\end{aligned}$$

where $n_1\overline{n_1} \equiv 1 \,(\mathrm{mod}\, q_1)$. Noting that $(a_1\overline{n_1}, q_1) = 1$, we can apply (2.2) with $N = 1$ and (3.2)-(3.3) of Lemma 3.1 to write

$$\begin{aligned}S_f(x;a,q) &\ll \tau(m)\left(\frac{x}{m}\right)^{1/3+\varepsilon} \sum_{n_1\mid m^\infty} \frac{\tau_4(n_1)}{n_1^{1/3+\varepsilon}} \\ &\ll \tau(m)\left(\frac{x}{m}\right)^{1/3+\varepsilon} \prod_{p\mid m}\left(1-\frac{1}{p^{1/3}}\right)^{-4} \\ &\ll x^{1/3+\varepsilon}.\end{aligned} \tag{3.6}$$

This completes the proof. □

Now we prove Theorem 1 for $\ell = 2$. By the classical expression

$$r_2(n) = 4\sum_{d\mid n}\chi(d)$$

($\chi(n)$ is the non trivial Dirichlet character modulo 4), we can write

$$\mathscr{S}_{f,b,2}(x) = 4S_1 + 4S_2 - 4S_3, \tag{3.7}$$

where

$$S_1 := \sum_{d\leqslant\sqrt{x}}\sum_{dm\leqslant x}\chi(d)\lambda_f(dm+b),$$
$$S_2 := \sum_{m\leqslant\sqrt{x}}\sum_{dm\leqslant x}\chi(d)\lambda_f(dm+b),$$
$$S_3 := \sum_{d\leqslant\sqrt{x}}\sum_{m\leqslant\sqrt{x}}\chi(d)\lambda_f(dm+b).$$

By Lemma 3.2, we have

$$\begin{aligned} S_1 &= \sum_{d\leqslant\sqrt{x}}\chi(d)\sum_{\substack{n\leqslant x+b\\ n\equiv b(\mathrm{mod}\, d)}}\lambda_f(n)\\ &\ll \sum_{d\leqslant\sqrt{x}}(x+b)^{1/3+\varepsilon}\ll x^{5/6+\varepsilon}. \end{aligned} \tag{3.8}$$

Note that $\chi(d)=1$ if $d\equiv 1\,(\mathrm{mod}\,4)$, $\chi(d)=-1$ if $d\equiv 3\,(\mathrm{mod}\,4)$ and $\chi(d)=0$ if $2\mid d$, we have by Lemma 3.2 that

$$\begin{aligned} S_2 &= \sum_{m\leqslant\sqrt{x}}\Big(\sum_{(4d+1)m\leqslant x}\lambda_f((4d+1)m+b)-\sum_{(4d+3)m\leqslant x}\lambda_f((4d+3)m+b)\Big)\\ &= \sum_{m\leqslant\sqrt{x}}\Big(\sum_{\substack{n\leqslant x+b\\ n\equiv m+b(\mathrm{mod}\,4m)}}\lambda_f(n)-\sum_{\substack{n\leqslant x+b\\ n\equiv 3m+b(\mathrm{mod}\,4m)}}\lambda_f(n)\Big)\\ &\ll \sum_{m\leqslant\sqrt{x}}(x+b)^{1/3+\varepsilon}\ll x^{5/6+\varepsilon}. \end{aligned} \tag{3.9}$$

For S_3 we have by Lemma 3.2 again that

$$\begin{aligned} S_3 &= \sum_{d\leqslant\sqrt{x}}\chi(d)\sum_{\substack{n\leqslant d\sqrt{x}+b\\ n\equiv b(\mathrm{mod}\, d)}}\lambda_f(n)\\ &\ll \sum_{d\leqslant\sqrt{x}}(d\sqrt{x}+b)^{1/3+\varepsilon}\ll x^{5/6+\varepsilon}. \end{aligned} \tag{3.10}$$

Now Theorem 1 for the case $\ell=2$ follows from (3.7)-(3.10).

4. Proof of Theorem 2 and (1.9)

Suppose $Q(y)$ is the quadratic form defined in Lemma 2.4 for $\ell\geqslant 6$, and $Q(\mathbf{y})=y_1^2+\cdots+y_5^2$ for $\ell=5$. By Deligne's inequality $|\lambda_f(n)|\leqslant\tau(n)$ and the bound $r(n,Q)\ll n^{\ell/2-1}$, we have, with the notation $\mathcal{L}:=\log(2X)$,

$$\int_1^{\sqrt{X}}|\mathscr{S}_{f,b,Q}(x)|^2\,\mathrm{d}x\ll\int_1^{\sqrt{X}}x^{\ell}(\log x)^2\,\mathrm{d}x\ll X^{(\ell+1)/2}\mathcal{L}^2\ll X^{\ell-2}. \tag{4.1}$$

It suffices for us to evaluate $\int_{\sqrt{X}}^{X}|\mathscr{S}_{f,b,Q}(x)|^2\,\mathrm{d}x$. For any $x\in[X^{1/2},X]\smallsetminus\mathbb{Z}$, Lemma 2.4 and [5, p. 256, (2.25), p. 270] allow us to write

$$\mathscr{S}_{f,b,Q}(x)=\sigma_Q\sum_{q=1}^{\infty}\sideset{}{^*}\sum_{h=1}^{q}S\Big(\frac{hQ}{q}\Big)\frac{\mathrm{e}_q(bh)}{q^\ell}A_{\ell,b}(x,-h/q)+O\big(\delta_Q x^{\ell/4-\delta_\ell+1+\varepsilon}\big),\tag{4.2}$$

where

$$A_{\ell,b}(x,-h/q):=\sum_{1+b\leqslant n\leqslant x+b}(n-b)^{\ell/2-1}\lambda_f(n)\mathrm{e}_q(-hn),$$

$$\delta_Q:=\begin{cases}1 & \text{if } \ell\geqslant 6,\\ 0 & \text{if } \ell=5 \text{ and } Q(\mathbf{y})=y_1^2+\cdots+y_5^2.\end{cases}$$

By partial summation, we can deduce, with the notation (2.3),

$$A_{\ell,b}(x,-h/q)=x^{\ell/2-1}A(x+b,-h/q)-(\ell/2-1)I\tag{4.3}$$

with

$$I:=\int_b^{x+b}(u-b)^{\ell/2-2}A(u,-h/q)\,\mathrm{d}u.\tag{4.4}$$

It is easy to see that Lemma 2.1 implies that

$$A_{\ell,b}(x,-h/q)\ll X^{\ell/2-1/2}\mathcal{L}$$

uniformly for $1\leqslant x\leqslant X$, $0\leqslant b\leqslant X$ and $(h,n)=1$. By this and (2.7), we have

$$\sum_{q>X^{1/2}}\frac{1}{q^\ell}\sideset{}{^*}\sum_{h=1}^{q}\bigg|S\Big(\frac{hQ}{q}\Big)A_{\ell,b}(x,-h/q)\bigg|\ll X^{\ell/4+1/2}\mathcal{L}.$$

Thus we can write (4.2) as, for $x\in[X^{1/2},X]\smallsetminus\mathbb{Z}$,

$$\begin{aligned}\mathscr{S}_{f,b,Q}(x)=\sigma_Q\sum_{q\leqslant X^{1/2}}\sideset{}{^*}\sum_{h=1}^{q}S\Big(\frac{hQ}{q}\Big)\frac{\mathrm{e}_q(bh)}{q^\ell}A_{\ell,b}(x,-h/q)\\ +O\big(X^{\ell/4+1/2}\mathcal{L}+\delta_Q X^{\ell/4-\delta_\ell+1+\varepsilon}\big).\end{aligned}\tag{4.5}$$

Suppose now $1\leqslant q\leqslant X^{1/2}$ and we estimate the integral I defined as in (4.4). If $q^2\leqslant b$, by partial integration and Lemma 2.3 we have, with the notation (2.5),

$$\begin{aligned}I&=x^{\ell/2-2}A_1(x+b,-h/q)-\frac{\ell-4}{2}\int_b^{x+b}(u-b)^{\ell/2-3}A_1(u,-h/q)\,\mathrm{d}u\\ &\ll q^{3/2}X^{\ell/2-5/4}.\end{aligned}\tag{4.6}$$

If $q^2>b$, we can write

$$\begin{aligned}I&=\int_b^{q^2}A(u,-h/q)(u-b)^{\ell/2-2}\,\mathrm{d}u+\int_{q^2}^{x+b}A(u,-h/q)(u-b)^{\ell/2-2}\,\mathrm{d}u\\ &\ll q^{\ell-1}\log(2q)+q^{3/2}X^{\ell/2-5/4}\ll q^{3/2}X^{\ell/2-5/4}\mathcal{L},\end{aligned}\tag{4.7}$$

where we estimated the first integral by Lemma 2.1 and the second as in (4.6). In view of (4.6), (4.7) and (2.7), the contribution of I to the last sum in (4.5) is

$$\begin{aligned}&\ll X^{\ell/2-5/4}\mathcal{L}\sum_{q\leqslant X^{1/2}} q^{-(\ell-5)/2}\\ &\ll R_\ell(X):=\begin{cases}X^{\ell/2-1+\varepsilon} & \text{if } \ell\geqslant 6,\\ X^{\ell/2-3/4+\varepsilon} & \text{if } \ell=5.\end{cases}\end{aligned}$$

Since $\ell/4-\delta_\ell+1\leqslant \ell/2-1$ ($\ell\geqslant 6$), thus we can write (4.5) as, for $x\in[X^{1/2},X]\smallsetminus\mathbb{Z}$,

$$\mathscr{S}_{f,b,Q}(x)=\sigma_Q x^{\ell/2-1}\sum_{q\leqslant X^{1/2}}\sideset{}{^*}\sum_{h=1}^{q} S\Big(\frac{hQ}{q}\Big)\frac{\mathrm{e}_q(bh)}{q^\ell}A(x+b,-h/q)+O(R_\ell(X)). \tag{4.8}$$

Now we split the sum over $q\leqslant X^{1/2}$ into two parts according to $q\leqslant X^{1/6}$ or $X^{1/6}<q\leqslant X^{1/2}$. In view of $X^{1/2}\leqslant x$, we can apply Lemma 2.2 to the sum $\sum_{q\leqslant X^{1/6}}$ by taking $M=x+b$. It is easy to see that the contribution of the error term to $\mathscr{S}_{f,b,Q}(x)$ is $O(X^{\ell/2-1+\varepsilon})$ if $\ell\geqslant 6$ and $O(X^{19/12})$ if $\ell=5$. Thus the formula (4.8) becomes

$$\mathscr{S}_{f,b,Q}(x)=S_1(x)+S_2(x)+O(R_\ell(X)), \tag{4.9}$$

where

$$\begin{aligned}S_1(x)&:=\frac{\sigma_Q}{\sqrt{2}\pi}x^{\ell/2-1}(x+b)^{1/4}\sum_{q\leqslant X^{1/6}}\sideset{}{^*}\sum_{h=1}^{q}\sum_{n\leqslant x+b}\psi(h,n,q)\cos(g_{n,q}(x)),\\ S_2(x)&:=\sigma_Q x^{\ell/2-1}\sum_{X^{1/6}<q\leqslant X^{1/2}}\sideset{}{^*}\sum_{h=1}^{q} S\Big(\frac{hQ}{q}\Big)\frac{\mathrm{e}_q(bh)}{q^\ell}A(x+b,-h/q),\end{aligned}$$

and

$$\psi(h,n,q):=\frac{\lambda_f(n)\mathrm{e}_q(bh-\overline{h}n)}{n^{3/4}q^{\ell-1/2}}S\Big(\frac{hQ}{q}\Big),\qquad g_{n,q}(x):=\frac{4\pi\sqrt{n(x+b)}}{q}-\frac{\pi}{4}.$$

We first evaluate the integral $\int_{\sqrt{X}}^{X}|S_1(x)|^2\,\mathrm{d}x$. By the elementary formula

$$\cos u\cos v=\frac{\cos(u-v)+\cos(u+v)}{2},$$

we can write

$$|S_1(x)|^2=S_{11}(x)+S_{12}(x)+S_{13}(x), \tag{4.10}$$

where $h_j\overline{h_j} \equiv 1 \,(\mathrm{mod}\, q_j)$ for $j = 1, 2$, and

$$\begin{aligned}
S_{11}(x) &:= \Big(\frac{\sigma_Q}{2\pi}\Big)^2 x^{\ell-2}(x+b)^{1/2} \sum_{q_1\leqslant X^{1/6}} \sideset{}{^*}\sum_{h_1=1}^{q_1} \sum_{\substack{n_1\leqslant x+b\\ \sqrt{n_1}/q_1=\sqrt{n_2}/q_2}} \sum_{q_2\leqslant X^{1/6}} \sideset{}{^*}\sum_{h_2=1}^{q_2} \sum_{n_2\leqslant x+b} \\
&\quad \times \psi(h_1, n_1, q_1)\overline{\psi(h_2, n_2, q_2)}, \\
S_{12}(x) &:= \Big(\frac{\sigma_Q}{2\pi}\Big)^2 x^{\ell-2}(x+b)^{1/2} \sum_{q_1\leqslant X^{1/6}} \sideset{}{^*}\sum_{h_1=1}^{q_1} \sum_{\substack{n_1\leqslant x+b\\ \sqrt{n_1}/q_1\neq\sqrt{n_2}/q_2}} \sum_{q_2\leqslant X^{1/6}} \sideset{}{^*}\sum_{h_2=1}^{q_2} \sum_{n_2\leqslant x+b} \\
&\quad \times \psi(h_1, n_1, q_1)\overline{\psi(h_2, n_2, q_2)}\cos(g_{n_1,q_1}(x) - g_{n_2,q_2}(x)), \\
S_{13}(x) &:= \Big(\frac{\sigma_Q}{2\pi}\Big)^2 x^{\ell-2}(x+b)^{1/2} \sum_{q_1\leqslant X^{1/6}} \sideset{}{^*}\sum_{h_1=1}^{q_1} \sum_{n_1\leqslant x+b} \sum_{q_2\leqslant X^{1/6}} \sideset{}{^*}\sum_{h_2=1}^{q_2} \sum_{n_2\leqslant x+b} \\
&\quad \times \psi(h_1, n_1, q_1)\overline{\psi(h_2, n_2, q_2)}\cos(g_{n_1,q_1}(x) + g_{n_2,q_2}(x)).
\end{aligned}$$

The relation $\sqrt{n_1}/q_1 = \sqrt{n_2}/q_2$ implies that n_1 and n_2 have the same square-free factor. We write $n_j = d_j^2 m$ with $\mu(m) \neq 0$ for $j = 1, 2$. Recalling (2.7), Deligne's well-known bound $|\lambda_f(n)| \leqslant \tau(n)$ and the trivial inequality $\tau(uv) \leqslant \tau(u)\tau(v)$, we have

$$\begin{aligned}
\psi(h_1, n_1, q_1)\overline{\psi(h_2, n_2, q_2)} &\ll \frac{\tau(d_1^2 m)\tau(d_2^2 m)}{(d_1 d_2 m)^{3/2}(q_1 q_2)^{\ell/2-1/2}} \\
&\ll \frac{\tau(d_1)^2\tau(d_2)^2\tau(m)^2}{(d_1 d_2 m)^{3/2}(q_1 q_2)^{\ell/2-1/2}}.
\end{aligned} \tag{4.11}$$

The above estimate (4.11) implies that the infinite series

$$C_{f,b,Q} := \Big(\frac{\sigma_Q}{2\pi}\Big)^2 \sum_{q_1\geqslant 1} \sum_{\substack{1\leqslant h_1\leqslant q_1\\ \sqrt{n_1}/q_1=\sqrt{n_2}/q_2}}^{*} \sum_{n_1\geqslant 1} \sum_{q_2\geqslant 1} \sum_{1\leqslant h_2\leqslant q_2}^{*} \sum_{n_2\geqslant 1} \psi(h_1, n_1, q_1)\overline{\psi(h_2, n_2, q_2)} \tag{4.12}$$

converges absolutely provided $\ell \geqslant 4$ and

$$C_{f,b,Q} \ll_{f,Q} 1 \quad (\forall\, b). \tag{4.13}$$

If a quadruple (q_1, q_2, n_1, n_2) satisfies $\sqrt{n_1}/q_1 = \sqrt{n_2}/q_2$ but is not included in $S_{11}(x)$, then we must have that $q_1 > X^{1/6}$, or $q_2 > X^{1/6}$, or $n_1 > x + b \geqslant X^{1/2}$, or $n_2 > x + b \geqslant X^{1/2}$. Thus $n_1 q_2^2 = n_2 q_1^2 > X^{1/3}$. In view of the following elementary estimates

$$t_\ell(m) := \sum_{m=nq^2} \tau(n) q^{-\ell/2+3} \leqslant \tilde{t}_\ell(m) := \begin{cases} \tau_3(m) & \text{if } \ell \geqslant 6, \\ m^{1/4}\tau_3(m) & \text{if } \ell = 5, \end{cases}$$

$$\sum_{n\leqslant u} \tau_3(n)^2 \ll u(\log u)^8 \quad (u \geqslant 2), \tag{4.14}$$

we can derive, for all $x \in [X^{1/2}, X] \smallsetminus \mathbb{Z}$,

$$\begin{aligned}\left|\frac{S_{11}(x)}{x^{\ell-2}(x+b)^{1/2}} - C_{f,b,Q}\right| &\ll \sum_{\substack{n_1,n_2,q_1,q_2\\ n_2q_1^2=n_1q_2^2>X^{1/3}}} \frac{\tau(n_1)\tau(n_2)}{(n_1q_2^2n_2q_1^2)^{3/4}(q_1q_2)^{\ell/2-3}}\\ &\ll \sum_{m>X^{1/3}} \frac{t_\ell(m)^2}{m^{3/2}} \leqslant \sum_{m>X^{1/3}} \frac{\tilde{t}_\ell(m)^2}{m^{3/2}} \\ &\ll \begin{cases} X^{-1/6}\mathcal{L}^8 & \text{if } \ell \geqslant 6,\\ X^{-1/12}\mathcal{L}^8 & \text{if } \ell = 5.\end{cases}\end{aligned} \tag{4.15}$$

On the other hand, we have, in view of (4.13),

$$C_{f,b,Q}\int_{\sqrt{X}}^{X} x^{\ell-2}(x+b)^{1/2}\,\mathrm{d}x = C_{f,b,Q}\int_{1}^{X} x^{\ell-2}(x+b)^{1/2}\,\mathrm{d}x + O(X^{\ell/2})$$

uniformly for $X \geqslant 1$ and $0 \leqslant b \leqslant X$. And hence we obtain

$$\int_{\sqrt{X}}^{X} S_{11}(x)\,\mathrm{d}x = C_{f,b,Q}\int_{1}^{X} x^{\ell-2}(x+b)^{1/2}\,\mathrm{d}x + \begin{cases} O(X^{\ell-2/3}\mathcal{L}^8) & \text{if } \ell \geqslant 6,\\ O(X^{\ell-7/12}\mathcal{L}^8) & \text{if } \ell = 5.\end{cases} \tag{4.16}$$

By the first derivative test, (4.11) and (2.7), we have (note $\ell \geqslant 5$)

$$\begin{aligned}\int_{X^{1/2}}^{X} |S_{13}(x)|\,\mathrm{d}x &\ll X^{\ell-1} \sum_{\substack{n_1,n_2\leqslant x+b\\ q_1,q_2\leqslant X^{1/6}}} \frac{\tau(n_1)\tau(n_2)}{(n_1n_2)^{3/4}(q_1q_2)^{\ell/2-3/2}} \frac{1}{\sqrt{n_1}/q_1+\sqrt{n_2}/q_2}\\ &\ll X^{\ell-1} \sum_{\substack{n_1,n_2\leqslant x+b\\ q_1,q_2\leqslant X^{1/6}}} \frac{\tau(n_1)\tau(n_2)}{n_1n_2(q_1q_2)^{\ell/2-2}}\\ &\ll X^{\ell-5/6}\mathcal{L}^6,\end{aligned} \tag{4.17}$$

where we used the inequality $u+v \geqslant 2\sqrt{uv}$ for $u, v > 0$.

Similarly by the first derivative test again

$$\int_{X^{1/2}}^{X} |S_{12}(x)|\,\mathrm{d}x \ll X^{\ell-1}\big(\Sigma_{11}+\Sigma_{12}\big), \tag{4.18}$$

where

$$\Sigma_{11} := \sum_{\substack{n_1,n_2\leqslant x+b,\ q_1,q_2\leqslant X^{1/6}\\ |\sqrt{n_1}/q_1-\sqrt{n_2}/q_2|\geqslant\frac{1}{10}(\sqrt{n_1n_2}/q_1q_2)^{1/2}}} \frac{\tau(n_1)\tau(n_2)}{(n_1n_2)^{3/4}(q_1q_2)^{\ell/2-3/2}} \frac{1}{|\sqrt{n_1}/q_1-\sqrt{n_2}/q_2|},$$

$$\Sigma_{12} := \sum_{\substack{n_1,n_2\leqslant x+b,\ q_1,q_2\leqslant X^{1/6}\\ 0<|\sqrt{n_1}/q_1-\sqrt{n_2}/q_2|<\frac{1}{10}(\sqrt{n_1n_2}/q_1q_2)^{1/2}}} \frac{\tau(n_1)\tau(n_2)}{(n_1n_2)^{3/4}(q_1q_2)^{\ell/2-3/2}} \frac{1}{|\sqrt{n_1}/q_1-\sqrt{n_2}/q_2|}.$$

As in the argument of (4.17) we get easily that (recall $\ell \geqslant 5$)

$$\Sigma_{11} \ll \sum_{n_1,n_2\leqslant x+b,\ q_1,q_2\leqslant X^{1/6}} \frac{\tau(n_1)\tau(n_2)}{n_1n_2(q_1q_2)^{\ell/2-2}} \ll X^{1/6}\mathcal{L}^6. \tag{4.19}$$

If $|\sqrt{n_1}/q_1 - \sqrt{n_2}/q_2| < \frac{1}{10}(\sqrt{n_1n_2}/q_1q_2)^{1/2}$ holds, then it follows that

$$\sqrt{n_1}q_2 \asymp \sqrt{n_2}q_1 \qquad \text{and} \qquad \sqrt{n_1}q_2 + \sqrt{n_2}q_1 \asymp (n_1n_2)^{1/4}(q_1q_2)^{1/2}.$$

So in view of these facts, by a similar argument as in (4.15) we have, for $\ell \geqslant 6$,

$$\begin{aligned}\Sigma_{12} &\ll \sum_{\substack{q_1,q_2\leqslant X^{1/6};\ n_1,n_2\leqslant x+b\\ n_1q_2^2-n_2q_1^2\neq 0}} \frac{d(n_1)d(n_2)}{(n_1n_2q_1^2q_2^2)^{1/2}(q_1q_2)^{\ell/2-4}|n_1q_2^2-n_2q_1^2|}\\ &\ll X^{1/3} \sum_{\substack{m_1,m_2\leqslant X^{1/6}(X+b)\\ m_1\neq m_2}} \frac{d_3(m_1)d_3(m_2)}{(m_1m_2)^{1/2}|m_1-m_2|}\\ &\ll X^{1/3} \sum_{1\leqslant h\leqslant X^{1/6}(X+b)} \frac{1}{h} \sum_{m\leqslant X^{1/6}(X+b)} \frac{d_3(m)d_3(m+h)}{(m(m+h))^{1/2}}.\end{aligned}$$

By applying the Cauchy inequality to the sum over m and (4.14), we can deduce

$$\Sigma_{12} \ll X^{1/3}\mathcal{L}^9 \qquad (\ell \geqslant 6). \tag{4.20}$$

When $\ell = 5$, the similar estimate with $X^{1/2}$ in place of $X^{1/3}$ holds. Inserting (4.19) and (4.20) into (4.18), we find that

$$\int_{\sqrt{X}}^{X} |S_{12}(x)|\,\mathrm{d}x \ll R_\ell^*(X) := \begin{cases} X^{\ell-2/3}\mathcal{L}^9 & \text{if } \ell \geqslant 6,\\ X^{\ell-1/2}\mathcal{L}^9 & \text{if } \ell = 5.\end{cases} \tag{4.21}$$

Combining (4.16), (4.21) and (4.17) with (4.10) yields

$$\int_{\sqrt{X}}^{X} |S_1(x)|^2\,\mathrm{d}x = C_{f,b,Q}\int_1^X x^{\ell-2}(x+b)^{1/2}\,\mathrm{d}x + O\big(R_\ell^*(X)\big). \tag{4.22}$$

Finally we estimate $\int_{\sqrt{X}}^{X} |S_2(x)|^2\,\mathrm{d}x$. By (2.7) and Cauchy's inequality, it follows

$$\begin{aligned}|S_2(x)|^2 &\ll x^{\ell-2}\Big(\sum_{X^{1/6}<q\leqslant X^{1/2}} q^{-\ell/2+1}|A(x+b,-h/q)|\Big)^2\\ &\ll X^{\ell-2}\mathcal{L}\sum_{X^{1/6}<q\leqslant X^{1/2}} q^{-\ell+3}|A(x+b,-h/q)|^2.\end{aligned}$$

From this and Lemma 2.3(ii), we deduce (recall $\ell \geqslant 5$ and $0 \leqslant b \leqslant X$)

$$\begin{aligned}\int_{\sqrt{X}}^{X} |S_2(x)|^2\,\mathrm{d}x &\ll X^{\ell-2}\mathcal{L}\sum_{X^{1/6}<q\leqslant X^{1/2}} q^{-\ell+3}\big(qX^{3/2} + q^{3/2}X^{5/4+\varepsilon} + q^2X^{1+\varepsilon}\big)\\ &\ll \begin{cases} X^{\ell-2/3}\mathcal{L} & \text{if } \ell \geqslant 6,\\ X^{\ell-1/2+\varepsilon} & \text{if } \ell = 5.\end{cases}\end{aligned} \tag{4.23}$$

On the other hand, (4.22) implies immediately

$$\int_{\sqrt{X}}^{X} |S_1(x)|^2 \,\mathrm{d}x \ll X^{\ell-1/2}.$$

In view of this and (4.23), a simple application of Cauchy's inequality yields

$$\int_{\sqrt{X}}^{X} |S_1(x)S_2(x)| \,\mathrm{d}x \ll \begin{cases} X^{\ell-7/12}\mathcal{L}^{1/2} & \text{if } \ell \geqslant 6, \\ X^{\ell-1/2+\varepsilon} & \text{if } \ell = 5. \end{cases} \tag{4.24}$$

Now the first assertion of Theorem 2 and (1.9) follow from (4.1), (4.9), (4.22), (4.23) and (4.24). The second is an immediate consequence by noticing that

$$(x+b)^{1/2} = x^{1/2} + O(bx^{-1/2})$$

valid uniformly for $x \geqslant 1$ and $b \geqslant 1$.

$$\begin{aligned} \sum_{p_1+p_2+p_3=N} \tau(p_1) &= \sum_{p_1+p_2+p_3=N} \tau(N-p_2-p_3) \\ &\ll N^{15/2} \exp\big\{-C\sqrt{\log N}\big\} \end{aligned}$$

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