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FINITE-TIME STABILIZATION AND H_∞ CONTROL OF NONLINEAR DELAY SYSTEMS VIA OUTPUT FEEDBACK

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ABSTRACT. This paper studies the robust finite-time H_∞ control for a class of nonlinear systems with time-varying delay and disturbances via output feedback. Based on the Lyapunov functional method and a generalized Jensen integral inequality, novel delay-dependent conditions for the existence of output feedback controllers are established in terms of linear matrix inequalities (LMIs). The proposed conditions allow us to design the output feedback controllers which robustly stabilize the closed-loop system in the finite-time sense. An application to H_∞ control of uncertain linear systems with interval time-varying delay is also given. A numerical example is given to illustrate the efficiency of the proposed method.

1. Introduction. The concept of finite-time stability (FTS) (or short-time stability) introduced by Dorato [5] plays an important role in stability theory of differential equations. A system is said to be finite-time stable if its state does not exceed a certain threshold during a specified time interval. Compared with the Lyapunov stability, finite-time stability concerns the boundedness of system during a fixed finite-time interval. It is noted that, a system may be finite-time stable but not Lyapunov asymptotically stable, and vice versa. A lot of interesting results on finite-time stability and stabilization in the context of linear delay systems have been obtained (see, e.g. [2, 8, 11, 14] and the references therein).

On the other hand, one of the most important problems is the H_∞ control of time-delay systems via output feedback controllers. The main principle of the output feedback control is to utilize the measured output to excite the plant. Since the controller can be easily implemented in practice, the output feedback control has attracted a lot of attention over the past few decades and has been applied to many areas for example, in motor engine control, constrained robotics, networked control systems, communication and biological systems, etc. [1, 3, 6, 9, 15, 18, 21, 22]. So far, however, compared with numerous research results on Lyapunov stability with H_∞ control, few results on finite-time H_∞ control have been obtained in the literature.

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The finite-time H_∞ control for switched linear systems with time-varying delay has been studied in [19, 20], but the results were limited either to discrete-time systems or to the systems with constant delays. In [12, 16] some delay-dependent conditions for finite-time H_∞ control are extended to linear systems with time-varying delays, but the delay function is differentiable and the stabilizing control is designed via state feedback.

In this paper, we propose a new design tool to solve the robust finite-time H_∞ control for nonlinear systems with interval time-varying delays via output feedback controls. The novel features here are that the interval time-varying delay is present in the observation output, and the output feedback controllers to be designed must satisfy some robust finite-time stability constraints on the closed-loop poles. Using new generalized Jensen integral inequality we select a new simpler set of Lyapunov-Krasovskii functionals to derive delay-dependent sufficient conditions for solving robust finite-time H_∞ control via output feedback controls. The conditions are obtained in terms of LMIs, which can be determined by utilizing MATLABs LMI Control Toolbox. The approach allows us to apply to H_∞ control of uncertain linear systems with interval non-differentiable time-varying delay.

The paper is organized as follows. The definition of FTS for nonlinear systems with interval time-varying delays and the problem statement are given in Section 2. Sufficient conditions for designing output feedback controllers of finite-time H_∞ control problem, an application to H_∞ control of uncertain linear systems with interval time-varying delay with numerical examples are given in Section 3.

2. Preliminaries. The following notations will be used throughout this paper, \mathbb{R}^+ denotes the set of all nonnegative real numbers; \mathbb{R}^n denotes the n -dimensional space with the scalar product $x^\top y$ and the vector norm $\|\cdot\|$; $\mathbb{R}^{n \times r}$ denotes the space of all matrices of $(n \times r)$ -dimension. A^\top denotes the transpose of A ; a matrix A is symmetric if $A = A^\top$; I denotes the identity matrix; $\lambda(A)$ denotes the set of all eigenvalues of A ; $\lambda_{max}(A) = \max\{Re(\lambda) : \lambda \in \lambda(A)\}$; $\lambda_{min}(A) = \min\{Re(\lambda) : \lambda \in \lambda(A)\}$; $C^1([-\tau, 0], \mathbb{R}^n)$ denotes the set of all \mathbb{R}^n -valued continuously differentiable functions on $[-\tau, 0]$; $L_2([0, T], \mathbb{R}^r)$ stands for the set of all square-integrable \mathbb{R}^r -valued functions on $[0, T]$. The symmetric terms in a matrix are denoted by $*$. Matrix A is positive definite ($A > 0$) if $(Ax, x) > 0$ for all $x \neq 0$. The segment of the trajectory $x(t)$ is denoted by $x_t = \{x(t+s) : s \in [-\tau, 0]\}$ with its norm $\|x_t\| = \sup_{s \in [-\tau, 0]} \|x(t+s)\|$.

Consider the following nonlinear control systems with time-varying delay and disturbances

$$\begin{cases} \dot{x}(t) = A_1 x(t) + A_2 x(t-h(t)) + Bu(t) + Gw(t) \\ \quad + f(t, x(t), x(t-h(t)), u(t), w(t)) \\ z(t) = C_1 x(t) + C_2 x(t-h(t)), \quad t \geq 0, \\ x(t) = \varphi(t), \quad t \in [-h_2, 0], \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $z(t) \in \mathbb{R}^p$ are, respectively, the state, the control, the observation vector, $A_1, A_2 \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $G \in \mathbb{R}^{n \times r}$, $C_1, C_2 \in \mathbb{R}^{p \times n}$ are given constant matrices. The delay function $h(t)$ is continuous and satisfies

$$0 \leq h_1 \leq h(t) \leq h_2, \quad \forall t \geq 0. \quad (2)$$

The initial function $\varphi \in C^1([-h_2, 0], \mathbb{R}^n)$ and the disturbance $w(t)$ is a continuous function satisfying

$$\int_0^T w(t)^\top w(t) dt \leq d. \quad (3)$$

The nonlinear function $f(t, x, y, u, w)$ is globally Lipschitzian in (x, y, u, w) such that

$$\exists a_1, a_2, a_3, a_4 > 0 : \|f\|^2 \leq a_1 \|x\|^2 + a_2 \|y\|^2 + a_3 \|u\|^2 + a_4 \|w\|^2 \quad (4)$$

for all $x, y \in \mathbb{R}^n, u \in \mathbb{R}^m, w \in \mathbb{R}^r$.

Under the above assumptions on $h(\cdot), f(\cdot)$ and the initial function $\varphi(t)$, the system (1) has a unique solution $x(t, \phi)$ on $[0, +\infty)$ (see [10], Theorem 1.2).

To study the robust finite-time H_∞ control of the system (1), the following definitions will be used later.

Definition 2.1 (Robust finite-time stabilization). For given positive numbers $T, c_1, c_2, c_2 > c_1$, and a positive definite matrix R , the system (1) is said to be robustly finite-time stabilizable w.r.t. (c_1, c_2, T, R) if there exists an output feedback controller $u(t) = Fz(t)$ such that the following condition holds for all disturbances satisfying (3) and $\forall t \in [0, T]$

$$\max \left\{ \sup_{-h_2 \leq s \leq 0} \varphi(s)^\top R \varphi(s), \sup_{-h_2 \leq s \leq 0} \dot{\varphi}(s)^\top R \dot{\varphi}(s) \right\} \leq c_1 \Rightarrow x(t)^\top R x(t) \leq c_2.$$

Definition 2.2 (Robust finite-time H_∞ control). Given $\gamma > 0$, the robust finite-time H_∞ control problem for the systems (1) has a solution if

1. The system (1) is robustly finite-time stabilizable w.r.t. (c_1, c_2, T, R) .
2. There is a number $c_0 > 0$ such that

$$\sup \frac{\int_0^T \|z(t)\|^2 dt}{c_0 \|\varphi\|^2 + \int_0^T \|w(t)\|^2 dt} \leq \gamma, \quad (5)$$

where the supremum is taken over all $\varphi \in C^1([-h_2, 0], \mathbb{R}^n)$ and non-zero disturbances $w(\cdot)$ satisfying (3).

We introduce the following technical well-known propositions for the proof of the main result.

Proposition 1 (Schur complement lemma [4]). *Given constant matrices X, Y, Z with appropriate dimensions satisfying $X = X^\top$ and $Y = Y^\top > 0$. Then $X + Z^\top Y^{-1} Z < 0$ if and only if*

$$\begin{pmatrix} X & Z^\top \\ Z & -Y \end{pmatrix} < 0.$$

Proposition 2 (Generalized Jensen integral inequality [17]). *For a given matrix $R > 0$, any differentiable function $\varphi : [a, b] \rightarrow \mathbb{R}^n$, then the following inequality holds*

$$\int_a^b \dot{\varphi}(u) R \dot{\varphi}(u) du \geq \frac{1}{b-a} (\varphi(b) - \varphi(a))^\top R (\varphi(b) - \varphi(a)) + \frac{12}{b-a} \Omega^\top R \Omega,$$

where $\Omega = \frac{\varphi(b) + \varphi(a)}{2} - \frac{1}{b-a} \int_a^b \varphi(u) du$.

3. Output feedback finite-time H_∞ control. Before stating the main result, the following notations of several matrices variables are defined for simplicity.

$$\begin{aligned}\bar{P} &= R^{1/2}PR^{1/2}, \bar{U}_1 = R^{1/2}U_1R^{1/2}, \bar{U}_2 = R^{1/2}U_2R^{1/2}, \bar{X}_1 = R^{1/2}X_1R^{1/2}, \\ \bar{X}_2 &= R^{1/2}X_2R^{1/2}, \bar{S} = R^{1/2}SR^{1/2},\end{aligned}$$

$$\begin{aligned}\alpha_1 &= \lambda_{\min}(P), \quad \alpha_2 = \lambda_{\max}(P) + h_1\lambda_{\max}(U_1) + h_2\lambda_{\max}(U_2) + 0.5h_1^3\lambda_{\max}(X_1) \\ &\quad + 0.5h_2^3\lambda_{\max}(X_2) + 0.5(h_2 - h_1)^2(h_2 + h_1)\lambda_{\max}(S),\end{aligned}$$

$$\begin{aligned}\alpha_3 &= \lambda_{\max}(\bar{P}) + h_1\lambda_{\max}(\bar{U}_1) + h_2\lambda_{\max}(\bar{U}_2) + 0.5h_1^3\lambda_{\max}(\bar{X}_1) + 0.5h_2^3\lambda_{\max}(\bar{X}_2) \\ &\quad + 0.5(h_2 - h_1)^2(h_2 + h_1)\lambda_{\max}(\bar{S}), \quad \Psi^1 = (\Psi_{ij}^1)_{11 \times 11}, \quad \Psi^2 = (\Psi_{ij}^2)_{6 \times 11},\end{aligned}$$

$$\Psi^3 = \text{diag} \left(-0.5N, -0.5N, -\frac{1}{a_3}N + \frac{1}{2a_3}I, -\frac{1}{a_3}N + \frac{1}{2a_3}I, -0.5N, -0.5N \right),$$

$$\begin{aligned}\Psi_{11}^1 &= \bar{P}A_1 + A_1^\top \bar{P} + \bar{U}_1 + \bar{U}_2 + a_1I + \eta C_1^\top C_1 - 4\bar{X}_1 - 4\bar{X}_2 + BKC_1 \\ &\quad + C_1^\top K^\top B^\top - 0.5BNB^\top, \quad \Psi_{22}^1 = -\bar{U}_1 - 4\bar{X}_1 - 4\bar{S},\end{aligned}$$

$$\Psi_{33}^1 = -\bar{U}_2 - 4\bar{X}_2 - 4\bar{S}, \quad \Psi_{44}^1 = -8\bar{S} + a_2I + \eta C_2^\top C_2,$$

$$\Psi_{55}^1 = h_1^2\bar{X}_1 + h_2^2\bar{X}_2 + (h_2 - h_1)^2\bar{S} - 2Q, \quad \Psi_{66}^1 = a_4I - \gamma\eta I, \quad \Psi_{77}^1 = -I,$$

$$\Psi_{88}^1 = -12\bar{X}_1, \quad \Psi_{99}^1 = -12\bar{X}_2, \quad \Psi_{10,10}^1 = \Psi_{11,11}^1 = -12\bar{S}, \quad \Psi_{12}^1 = -2\bar{X}_1,$$

$$\Psi_{13}^1 = -2\bar{X}_2, \quad \Psi_{14}^1 = \bar{P}A_2 + \eta C_1^\top C_2, \quad \Psi_{15}^1 = A_1^\top Q, \quad \Psi_{16}^1 = \bar{P}G, \quad \Psi_{17}^1 = \bar{P}$$

$$\Psi_{18}^1 = \Psi_{28}^1 = 6\bar{X}_1, \quad \Psi_{19}^1 = \Psi_{39}^1 = 6\bar{X}_2, \quad \Psi_{24}^1 = \Psi_{34}^1 = -2\bar{S}, \quad \Psi_{45}^1 = A_2^\top Q, \quad \Psi_{56}^1 = QG,$$

$$\Psi_{2,11}^1 = \Psi_{3,10}^1 = \Psi_{4,10}^1 = \Psi_{4,11}^1 = 6\bar{S}, \quad \Psi_{57}^1 = Q, \quad \text{and } \Psi_{ij}^1 = 0 \text{ for the others,}$$

$$\Psi_{11}^2 = \bar{P}B, \quad \Psi_{12}^2 = C_1^\top K^\top - 0.5BN, \quad \Psi_{13}^2 = C_1^\top K^\top, \quad \Psi_{44}^2 = \Psi_{45}^2 = C_2^\top K^\top,$$

$$\Psi_{56}^2 = QB, \quad \Psi_{ij}^2 = 0 \text{ for the others.}$$

The following theorem gives a sufficient condition for robust finite-time H_∞ control via output feedback of the system (1).

Theorem 3.1. *For given positive constants T, c_1, c_2, γ and a positive definite matrix R , the robust finite-time H_∞ control of the system (1) has a solution if there exist a positive scalar η , symmetric positive definite matrices $P, U_1, U_2, X_1, X_2, S, N$ and matrices Q, K such that the following conditions hold*

$$\Psi = \begin{bmatrix} \Psi^1 & \Psi^2 \\ * & \Psi^3 \end{bmatrix} < 0, \quad (6)$$

$$\alpha_2 c_1 + \gamma \eta d \leq \alpha_1 c_2 e^{-\eta T}. \quad (7)$$

The output feedback controller is given by $u(t) = N^{-1}Kz(t)$, $t \geq 0$.

Proof. Consider the following Lyapunov-Krasovskii functional associated to the sys-

tem (1): $V(t, x_t) = \sum_{i=1}^4 V_i(t, x_t)$, where

$$V_1(t, x_t) = e^{\eta t} x(t)^\top \bar{P} x(t), \quad V_2(t, x_t) = \sum_{i=1}^2 e^{\eta t} \int_{t-h_i}^t x(s)^\top \bar{U}_i x(s) ds,$$

$$V_3(t, x_t) = \sum_{i=1}^2 h_i e^{\eta t} \int_{-h_i}^0 \int_{t+s}^t \dot{x}(\tau)^\top \bar{X}_i \dot{x}(\tau) d\tau ds,$$

$$V_4(t, x_t) = (h_2 - h_1)e^{\eta t} \int_{-h_2}^{-h_1} \int_{t+s}^t \dot{x}(\tau)^\top \bar{S} \dot{x}(\tau) d\tau ds.$$

It is not difficult to verify that

$$\alpha_1 x(t)^\top R x(t) \leq V(t, x_t), \quad \forall t : 0 \leq t \leq T, \quad (8)$$

$$V(0, x_0) \leq \alpha_2 \sup_{-h_2 \leq s \leq 0} \{x(s)^\top R x(s), \dot{x}(s)^\top R \dot{x}(s)\} \leq \alpha_2 c_1, \quad (9)$$

$$V(0, x_0) \leq \alpha_3 \|\varphi\|^2. \quad (10)$$

Taking the derivative of $V_i(t, x)$, $i = 1, \dots, 4$, along the solution of the system, we get the following estimations

$$\begin{aligned} \dot{V}_1(t, x_t) &\leq \eta V_1 + e^{\eta t} \left(x(t)^\top (\bar{P}A_1 + A_1^\top \bar{P} + 2\bar{P}BN^{-1}B^\top \bar{P} + C_1^\top F^\top NFC_1) x(t) \right. \\ &\quad + 2x(t)^\top \bar{P}Gw(t) + x(t-h(t))^\top C_2^\top F^\top NFC_2 x(t-h(t)) \\ &\quad \left. + 2x(t)^\top \bar{P}A_2 x(t-h(t)) + 2x(t)^\top \bar{P}f(t, x, x_h, u, w) \right), \end{aligned}$$

$$\begin{aligned} \dot{V}_2(t, x_t) &= \eta V_2 + e^{\eta t} \left(x(t)^\top (\bar{U}_1 + \bar{U}_2)x(t) - x(t-h_1)^\top \bar{U}_1 x(t-h_1) \right. \\ &\quad \left. - x(t-h_2)^\top \bar{U}_2 x(t-h_2) \right), \end{aligned}$$

$$\dot{V}_3(t, x_t) = \eta V_3 + e^{\eta t} \left(\dot{x}(t)^\top (h_1^2 \bar{X}_1 + h_2^2 \bar{X}_2) \dot{x}(t) - \sum_{i=1}^2 h_i \int_{t-h_i}^t \dot{x}(s)^\top \bar{X}_i \dot{x}(s) ds \right).$$

Applying Proposition 2, we have

$$\begin{aligned} -h_i \int_{t-h_i}^t \dot{x}(s)^\top \bar{X}_i \dot{x}(s) ds &\leq -4x(t)^\top \bar{X}_i x(t) - 4x(t-h_i)^\top \bar{X}_i x(t-h_i) \\ &\quad - 4x(t)^\top \bar{X}_i x(t-h_i) s + \frac{12}{h_i} x(t-h_i)^\top \bar{X}_i \int_{t-h_i}^t x(s) ds \\ &\quad + \frac{12}{h_i} x(t)^\top \bar{X}_i \int_{t-h_i}^t x(s) ds - \frac{12}{h_i^2} \int_{t-h_i}^t x(s)^\top ds \bar{X}_i \int_{t-h_i}^t x(s) ds. \end{aligned}$$

Hence,

$$\begin{aligned} \dot{V}_3(t, x_t) &\leq \eta V_3 + e^{\eta t} \left(\dot{x}(t)^\top (h_1^2 \bar{X}_1 + h_2^2 \bar{X}_2) \dot{x}(t) + x(t)^\top (-4\bar{X}_1 - 4\bar{X}_2)x(t) \right. \\ &\quad - 4x(t-h_1)^\top \bar{X}_1 x(t-h_1) - 4x(t-h_2)^\top \bar{X}_2 x(t-h_2) - 4x(t)^\top \bar{X}_1 x(t-h_1) \\ &\quad - 4x(t)^\top \bar{X}_2 x(t-h_2) + \sum_{i=1}^2 \frac{12}{h_i} x(t)^\top \bar{X}_i \int_{t-h_i}^t x(s) ds \\ &\quad \left. + \sum_{i=1}^2 \frac{12}{h_i} x(t-h_i)^\top \bar{X}_i \int_{t-h_i}^t x(s) ds - \sum_{i=1}^2 \frac{12}{h_i^2} \int_{t-h_i}^t x(s)^\top ds \bar{X}_i \int_{t-h_i}^t x(s) ds \right). \end{aligned}$$

Using the same calculation as in $\dot{V}_3(t, x_t)$, we get

$$\begin{aligned} \dot{V}_4(t, x_t) &= \eta V_4 + e^{\eta t} \left((h_2 - h_1)^2 \dot{x}(t)^\top \bar{S} \dot{x}(t) - (h_2 - h_1) \int_{t-h_2}^{t-h(t)} \dot{x}(s)^\top \bar{S} \dot{x}(s) ds \right. \\ &\quad \left. - (h_2 - h_1) \int_{t-h(t)}^{t-h_1} \dot{x}(s)^\top \bar{S} \dot{x}(s) ds \right) \\ &\leq \eta V_4 + e^{\eta t} \left((h_2 - h_1)^2 \dot{x}(t)^\top \bar{S} \dot{x}(t) - 8x(t-h(t))^\top \bar{S} x(t-h(t)) \right) \end{aligned}$$

$$\begin{aligned}
& -4x(t-h_2)^\top \bar{S}x(t-h_2) - 4x(t-h_1)^\top \bar{S}x(t-h_1) - 4x(t-h(t))^\top \bar{S}x(t-h_2) \\
& -4x(t-h(t))^\top \bar{S}x(t-h_1) - \frac{12}{(h_2-h(t))^2} \int_{t-h_2}^{t-h(t)} x(s)^\top ds \bar{S} \int_{t-h_2}^{t-h(t)} x(s) ds \\
& - \frac{12}{(h(t)-h_1)^2} \int_{t-h(t)}^{t-h_1} x(s)^\top ds \bar{S} \int_{t-h(t)}^{t-h_1} x(s) ds \\
& + \frac{12}{h_2-h(t)} x(t-h(t))^\top \bar{S} \int_{t-h_2}^{t-h(t)} x(s) ds + \frac{12}{h_2-h(t)} x(t-h_2)^\top \bar{S} \int_{t-h_2}^{t-h(t)} x(s) ds \\
& + \frac{12}{h(t)-h_1} x(t-h_1)^\top \bar{S} \int_{t-h(t)}^{t-h_1} x(s) ds + \frac{12}{h(t)-h_1} x(t-h(t))^\top \bar{S} \int_{t-h(t)}^{t-h_1} x(s) ds.
\end{aligned}$$

Multiplying both sides of (1) by $e^{\eta t} 2\dot{x}(t)^\top Q$ from the right, we obtain

$$\begin{aligned}
e^{\eta t} \left(-2\dot{x}(t)^\top Q\dot{x}(t) + 2\dot{x}(t)^\top Q(A_1 + BFC_1)x(t) + 2\dot{x}(t)^\top Q(A_2 + BFC_2)x(t-h(t)) \right. \\
\left. + 2\dot{x}(t)^\top QGw(t) + 2\dot{x}(t)^\top Qf(t, x, x_h, u, w) \right) = 0.
\end{aligned}$$

Consequently,

$$\begin{aligned}
0 \leq e^{\eta t} \left(\dot{x}(t)^\top (-2Q + 2QBN^{-1}B^\top Q)\dot{x}(t) + x(t)^\top C_1^\top F^\top NFC_1x(t) \right. \\
+ 2\dot{x}(t)^\top QA_1x(t) + x(t-h(t))^\top C_2^\top F^\top NFC_2x(t-h(t)) \\
\left. + 2\dot{x}(t)^\top QA_2x(t-h(t)) + 2\dot{x}(t)^\top QGx(t) + 2\dot{x}(t)^\top Qf(t, x, x_h, u, w) \right). \quad (11)
\end{aligned}$$

Adding the inequality (11) and the zero term

$$\begin{aligned}
e^{\eta t} \left(f(t, x, x_h, u, w)^\top f(t, x, x_h, u, w) - f(t, x, x_h, u, w)^\top f(t, x, x_h, u, w) \right. \\
\left. + \gamma\eta w(t)^\top w(t) - \gamma\eta w(t)^\top w(t) + \eta z(t)^\top z(t) - \eta z(t)^\top z(t) \right) = 0
\end{aligned}$$

to $\dot{V}(t, x_t)$, and using the estimation (4) for $f(t, x, x_h, u, w)$, and noting that

$$\begin{aligned}
f(t, x, x_h, u, w)^\top f(t, x, x_h, u, w) & \leq x(t)^\top (a_1I + 2a_3C_1^\top F^\top FC_1)x(t) + a_3w(t)^\top w(t) \\
& \quad + x(t-h(t))^\top (a_2I + 2a_3C_2^\top F^\top FC_2)x(t-h(t)), \\
\eta z(t)^\top z(t) & = \eta x(t)^\top C_1^\top C_1x(t) + 2\eta x(t)^\top C_1^\top C_2x(t-h(t)) \\
& \quad + \eta x(t-h(t))^\top C_2^\top C_2x(t-h(t)),
\end{aligned}$$

we have

$$\begin{aligned}
\dot{V}(t, x_t) & \leq \eta V(t, x_t) + e^{\eta t} \left(x(t)^\top (\bar{P}A_1 + A_1^\top \bar{P} + \bar{U}_1 + \bar{U}_2 + a_1I + \eta C_1^\top C_1 \right. \\
& - 4\bar{X}_1 - 4\bar{X}_2 + 2\bar{P}BN^{-1}B^\top \bar{P} + 2C_1^\top F^\top NFC_1 + 2a_3C_1^\top F^\top FC_1B) x(t) \\
& + w(t)^\top (a_4I - \gamma\eta I)w(t) + x(t-h_1)^\top (-\bar{U}_1 - 4\bar{X}_1 - 4\bar{S})x(t-h_1) \\
& + x(t-h_2)^\top (-\bar{U}_2 - 4\bar{X}_2 - 4\bar{S})x(t-h_2) + x(t-h(t))^\top \left(-8\bar{S} + a_2I + \eta C_2^\top C_2 \right. \\
& \left. + 2C_2^\top F^\top NFC_2 + 2a_3C_2^\top F^\top FC_2 \right) x(t-h(t)) \\
& + \dot{x}(t)^\top (h_1^2\bar{X}_1 + h_2^2\bar{X}_2 + (h_2-h_1)^2\bar{S} - 2Q + 2QBN^{-1}B^\top Q)\dot{x}(t) \\
& \left. - f(\cdot)^\top f(\cdot) + 2x(t)^\top (\bar{P}A_2 + \eta C_1^\top C_2)x(t-h(t)) + 2x(t)^\top \bar{P}Gw(t) + 2x(t)^\top \bar{P}f(\cdot) \right)
\end{aligned}$$

$$\begin{aligned}
& -4x(t)^\top \overline{X}_1 x(t-h_1) - 4x(t)^\top \overline{X}_2 x(t-h_2) - 4x(t-h(t))^\top \overline{S} x(t-h_2) \\
& + 2\dot{x}(t)^\top QGw(t) + 2\dot{x}(t)^\top Qf(\cdot) - \sum_{i=1}^2 \frac{12}{h_i^2} \int_{t-h_i}^t x(s)^\top ds \overline{X}_i \int_{t-h_i}^t x(s) ds \\
& + \sum_{i=1}^2 \frac{12}{h_i} x(t)^\top \overline{X}_i \int_{t-h_i}^t x(s) ds + \sum_{i=1}^2 \frac{12}{h_i} x(t-h_i)^\top \overline{X}_i \int_{t-h_i}^t x(s) ds \\
& - \frac{12}{(h_2-h(t))^2} \int_{t-h_2}^{t-h(t)} x(s)^\top ds \overline{S} \int_{t-h_2}^{t-h(t)} x(s) ds - 4x(t-h(t))^\top \overline{S} x(t-h_1) \\
& - \frac{12}{(h(t)-h_1)^2} \int_{t-h(t)}^{t-h_1} x(s)^\top ds \overline{S} \int_{t-h(t)}^{t-h_1} x(s) ds + 2\dot{x}(t)^\top QA_1 x(t) \\
& + \frac{12}{h_2-h(t)} x(t-h(t))^\top \overline{S} \int_{t-h_2}^{t-h(t)} x(s) ds + 2\dot{x}(t)^\top QA_2 x(t-h(t)) \\
& + \frac{12}{h_2-h(t)} x(t-h_2)^\top \overline{S} \int_{t-h_2}^{t-h(t)} x(s) ds + \frac{12}{h(t)-h_1} x(t-h_1)^\top \overline{S} \int_{t-h(t)}^{t-h_1} x(s) ds \\
& + \frac{12}{h(t)-h_1} x(t-h(t))^\top \overline{S} \int_{t-h(t)}^{t-h_1} x(s) ds \Big) + e^{nt} \gamma \eta w(t)^\top w(t) - e^{nt} \eta z(t)^\top z(t).
\end{aligned}$$

We obtain

$$\dot{V}(t, x_t) - \eta V(t, x_t) \leq e^{nt} \xi(t)^\top W \xi(t) + e^{nt} \gamma \eta w(t)^\top w(t) - e^{nt} \eta z(t)^\top z(t), \quad (12)$$

where

$$\begin{aligned}
\xi(t)^\top = & \left[x(t)^\top, x(t-h_1)^\top, x(t-h_2)^\top, x(t-h(t))^\top, \dot{x}(t)^\top, w(t)^\top, f(\cdot)^\top, \right. \\
& \frac{1}{h_1} \int_{t-h_1}^t x(s)^\top ds, \frac{1}{h_2} \int_{t-h_2}^t x(s)^\top ds, \frac{1}{h_2-h(t)} \int_{t-h_2}^{t-h(t)} x(s)^\top ds, \\
& \left. \frac{1}{h(t)-h_1} \int_{t-h(t)}^{t-h_1} x(s)^\top ds \right],
\end{aligned}$$

$$W = (W_{ij})_{11 \times 11},$$

$$\begin{aligned}
W_{11} = & \overline{P}A_1 + A_1^\top \overline{P} + \overline{U}_1 + \overline{U}_2 + a_1 I + \eta C_1^\top C_1 - 4\overline{X}_1 - 4\overline{X}_2 \\
& + 2\overline{P}BN^{-1}B^\top \overline{P} + 2C_1^\top F^\top NFC_1 + 2a_3 C_1^\top F^\top FC_1,
\end{aligned}$$

$$W_{22} = -\overline{U}_1 - 4\overline{X}_1 - 4\overline{S}, W_{33} = -\overline{U}_2 - 4\overline{X}_2 - 4\overline{S},$$

$$W_{44} = -8\overline{S} + a_2 I + \eta C_2^\top C_2 + 2C_2^\top F^\top NFC_2 + 2a_3 C_2^\top F^\top FC_2,$$

$$W_{55} = h_1^2 \overline{X}_1 + h_2^2 \overline{X}_2 + (h_2-h_1)^2 \overline{S} - 2Q + 2QBN^{-1}B^\top Q, W_{66} = a_4 I - \gamma \eta I,$$

$$W_{77} = -I, W_{88} = -12\overline{X}_1, W_{99} = -12\overline{X}_2, W_{10,10} = W_{11,11} = -12\overline{S},$$

$$W_{12} = -2\overline{X}_1, W_{13} = -2\overline{X}_2, W_{14} = \overline{P}A_2 + \eta C_1^\top C_2, W_{15} = A_1^\top Q, W_{16} = \overline{P}G,$$

$$W_{17} = \overline{P}, W_{18} = W_{28} = 6\overline{X}_1, W_{19} = W_{39} = 6\overline{X}_2, W_{24} = W_{34} = -2\overline{S},$$

$$W_{45} = A_2^\top Q, W_{56} = QG, W_{2,11} = W_{3,10} = W_{4,10} = W_{4,11} = 6\overline{S},$$

$$W_{57} = Q, \text{ and } W_{ij} = 0 \text{ for the others.}$$

Therefore, from (12) it follows that

$$\frac{d}{dt} (e^{-nt} V(t, x_t)) \leq \xi(t)^\top W \xi(t) + \gamma \eta w(t)^\top w(t) - \eta z(t)^\top z(t). \quad (13)$$

We prove that the matrix inequality $W < 0$ holds if the LMI (6) holds. By using Proposition 1 for each nonlinear items W_{11}, W_{44}, W_{55} , then the condition $W < 0$ holds if and only if

$$\Omega = \begin{bmatrix} \Omega^1 & \Omega^2 \\ * & \Omega^3 \end{bmatrix} < 0,$$

where

$$\begin{aligned} \Omega^1 &= (\Omega_{ij}^1)_{11 \times 11}, \quad \Omega^2 = (\Omega_{ij}^2)_{6 \times 11}, \\ \Omega^3 &= \text{diag} \left(-0.5N, -0.5N^{-1}, -\frac{1}{2a_3}I, -\frac{1}{2a_3}I, -0.5N^{-1}, -0.5N \right), \\ \Omega_{11}^1 &= \bar{P}A_1 + A_1^\top \bar{P} + \bar{U}_1 + \bar{U}_2 + a_1I + \eta C_1^\top C_1 - 4\bar{X}_1 - 4\bar{X}_2, \\ \Omega_{44}^1 &= -8\bar{S} + a_2I + \eta C_2^\top C_2, \Omega_{55}^1 = h_1^2 \bar{X}_1 + h_2^2 \bar{X}_2 + (h_2 - h_1)^2 \bar{S} - 2Q, \\ &\text{and } \Omega_{ij}^1 = W_{ij} \text{ for the others, } \Omega_{11}^2 = \bar{P}B, \Omega_{12}^2 = \Omega_{13}^2 = C_1^\top F^\top, \\ \Omega_{44}^2 &= \Omega_{45}^2 = C_2^\top F^\top, \Omega_{56}^2 = QB, \Omega_{ij}^2 = 0 \text{ for the others.} \end{aligned}$$

Define the matrix Δ as follows

$$\Delta = \begin{bmatrix} I_n & 0 & BN & 0 & 0 & 0 & 0 \\ 0 & I_{9n+m+r} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & N & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & N & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & N & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_m \end{bmatrix}.$$

Since the matrix Δ is regular (full column), we have $\Lambda = \Delta \Omega \Delta^\top < 0$ where

$$\Lambda = \begin{bmatrix} \Lambda^1 & \Lambda^2 \\ * & \Lambda^3 \end{bmatrix} < 0,$$

where

$$\begin{aligned} \Lambda^1 &= (\Lambda_{ij}^1)_{11 \times 11}, \quad \Lambda^2 = (\Lambda_{ij}^2)_{6 \times 11}, \\ \Lambda^3 &= \text{diag} \left(-0.5N, -0.5N, -\frac{1}{2a_3}N^2, -\frac{1}{2a_3}N^2, -0.5N, -0.5N \right), \\ \Lambda_{11}^1 &= \bar{P}A_1 + A_1^\top \bar{P} + \bar{U}_1 + \bar{U}_2 + a_1I + \eta C_1^\top C_1 - 4\bar{X}_1 - 4\bar{X}_2 \\ &\quad + BNFC_1 + C_1^\top F^\top NB^\top - 0.5BNB^\top, \quad \text{and } \Lambda_{ij}^1 = \Omega_{ij}^1 \text{ for the others,} \\ \Lambda_{11}^2 &= \bar{P}B, \Lambda_{12}^2 = C_1^\top F^\top N - 0.5BN, \Lambda_{13}^2 = C_1^\top F^\top N, \Lambda_{44}^2 = \Lambda_{45}^2 = C_2^\top F^\top N, \\ \Lambda_{56}^2 &= QB, \Lambda_{ij}^2 = 0 \text{ for the others.} \end{aligned}$$

Let $F^\top N = K^\top$, then $F = N^{-1}K$. Since $-(N-I)^2 \leq 0$, we have $-N^2 \leq -2N + I$. Then

$$-\frac{1}{2a_3}N^2 \leq -\frac{1}{a_3}N + \frac{1}{2a_3}I.$$

Then the condition $\Lambda < 0$ holds if $\Psi < 0$, therefore $W < 0$ and hence from the inequality (13) we obtain

$$\frac{d}{dt} (e^{-\eta t} V(t, x_t)) < \gamma \eta w(t)^\top w(t). \quad (14)$$

Integrating (14) from 0 to t , with $t \in [0, T]$, we get

$$\begin{aligned} V(t, x_t) &\leq e^{\eta t} \left(V(0, x_0) + \gamma\eta \int_0^t w(s)^\top w(s) ds \right) \\ &\leq e^{\eta T} (\alpha_2 c_1 + \gamma\eta d). \end{aligned}$$

Therefore,

$$\alpha_1 x(t)^\top R x(t) \leq V(t, x_t) \leq e^{\eta T} (\alpha_2 c_1 + \gamma\eta d),$$

or equivalently,

$$x(t)^\top R x(t) \leq \frac{e^{\eta T} (\alpha_2 c_1 + \gamma\eta d)}{\alpha_1} \leq c_2, \quad \forall t \in [0, T],$$

which implies that the system of (1) is robustly finite-time stabilizable w.r.t. (c_1, c_2, T, R) . To complete the proof of the theorem, it remains to show the γ -optimal level condition (5). For this, we consider the following relation

$$\begin{aligned} \int_0^T [\eta \|z(t)\|^2 - \gamma\eta \|w(t)\|^2] dt &= \int_0^T \left[\eta \|z(t)\|^2 - \gamma\eta \|w(t)\|^2 + \frac{d}{dt} (e^{-\eta t} V(t, x_t)) \right] dt \\ &\quad - \int_0^T \frac{d}{dt} (e^{-\eta t} V(t, x_t)) dt. \end{aligned}$$

Since $V(t, x_t) \geq 0$, we have

$$- \int_0^T \frac{d}{dt} (e^{-\eta t} V(t, x_t)) dt = -e^{-\eta T} V(T, x_T) + V(0, x_0) \leq \alpha_3 \|\varphi\|^2.$$

On the other hand, from (13) we have

$$\eta \|z(t)\|^2 - \gamma\eta \|w(t)\|^2 + \frac{d}{dt} (e^{-\eta t} V(t, x_t)) < 0,$$

therefore

$$\int_0^T [\eta \|z(t)\|^2 - \gamma\eta \|w(t)\|^2] dt \leq \alpha_3 \|\varphi\|^2.$$

Setting $c_0 = \frac{\alpha_3}{\gamma\eta} > 0$, the above inequality yields

$$\sup \frac{\int_0^T \|z(t)\|^2 dt}{c_0 \|\varphi\|^2 + \int_0^T \|w(t)\|^2 dt} \leq \gamma.$$

This estimation holds for all non-zero $w \in L_2([0, T], \mathbb{R}^r)$, $\varphi \in C^1([-h_2, 0], \mathbb{R}^n)$, and hence the condition (5) is derived. This completes the proof of the theorem. \square

Remark 1. We note that the condition (7) is not an LMI with respect to η , since η appears in a nonlinear item. However, the condition (6) is an LMI, so we can find the scalar η from the condition (6), and check the condition (7). If the problem is feasible, the output feedback controller $F = N^{-1}K$ solves the robust finite-time H_∞ control problem.

In the sequel, we apply the result of Theorem 3.1 to study the robust finite-time H_∞ control problem for uncertain linear systems with interval time-varying delay

[13, 19, 20, 21]. Consider the following uncertain linear system with time-varying delay:

$$\begin{cases} \dot{x}(t) = [A_1 + \Delta A_1(t)]x(t) + [A_2 + \Delta A_2(t)]x(t - h(t)) \\ \quad + [B + \Delta B(t)]u(t) + [G + \Delta G(t)]w(t), \quad t \geq 0, \\ z(t) = C_1x(t) + C_2x(t - h(t)), \\ x(t) = \varphi(t), \quad t \in [-h_2, 0], \end{cases} \quad (15)$$

where the delay function $h(t)$ satisfies the condition (2), the uncertainties $\Delta A_1(t)$, $\Delta A_2(t)$, $\Delta B(t)$, $\Delta G(t)$ are given as

$$[\Delta A_1(t) \quad \Delta A_2(t) \quad \Delta B(t) \quad \Delta G(t)] = DE(t)[M_{a_1} \quad M_{a_2} \quad M_b \quad M_g],$$

where $D, M_{a_1}, M_{a_2}, M_b, M_g$ are known real constant matrices of appropriate dimensions and $E(t)$ is an unknown uncertain matrix satisfying

$$E(t)^\top E(t) \leq I, \quad \forall t \geq 0.$$

To apply Theorem 3.1, we denote

$$\begin{aligned} f(t, x, x_h, u, \omega) &= \Delta A_1(t)x(t) + \Delta A_2(t)x(t - h(t)) + \Delta B(t)u(t) + \Delta G(t)\omega(t), \\ \lambda_d &= \lambda_{\max}(D^\top D), \quad \lambda_{m_1} = \lambda_{\max}(M_{a_1}^\top M_{a_1}), \quad \lambda_{m_2} = \lambda_{\max}(M_{a_2}^\top M_{a_2}), \\ \lambda_{m_b} &= \lambda_{\max}(M_b^\top M_b), \quad \lambda_{m_g} = \lambda_{\max}(M_g^\top M_g). \end{aligned}$$

We have

$$\begin{aligned} \|f\|^2 &\leq 4\|\Delta A_1x\|^2 + 4\|\Delta A_2x_h\|^2 + 4\|\Delta Bu\|^2 + 4\|\Delta G\omega\|^2 \\ &\leq 4\lambda_d\lambda_{m_1}\|x\|^2 + 4\lambda_d\lambda_{m_2}\|x_h\|^2 + 4\lambda_d\lambda_{m_b}\|u\|^2 + 4\lambda_d\lambda_{m_g}\|\omega\|^2 \end{aligned}$$

By the same notations used in Theorem 3.1

$$a_1 = 4\lambda_d\lambda_{m_1}, \quad a_2 = 4\lambda_d\lambda_{m_2}, \quad a_3 = 4\lambda_d\lambda_{m_b}, \quad a_4 = 4\lambda_d\lambda_{m_g},$$

we have

Corollary 1. *The robust finite-time H_∞ control of the system (15) has a solution if there exist a positive scalar η , symmetric positive definite matrices $P, U_1, U_2, X_1, X_2, S, N$ and matrices Q, K such that the following conditions hold*

$$\Psi = \begin{bmatrix} \Psi^1 & \Psi^2 \\ * & \Psi^3 \end{bmatrix} < 0, \quad (16)$$

$$\alpha_2 c_1 + \gamma \eta d \leq \alpha_1 c_2 e^{-\eta T}. \quad (17)$$

The output feedback controller is given by $u(t) = N^{-1}Kz(t)$, $\forall t \geq 0$.

Remark 2. The proposed output feedback controller can ensure robustly finite-time stability of the closed-loop system while also guaranteeing an adequate level of system performance which is expressed in terms of LMIs. The result in this paper advances recent findings H_∞ controller reported in [6, 12, 16, 19, 20], where the time delays considered are interval time-varying as opposed to constant delays. Moreover, we construct Lyapunov-like functionals different from the ones in [12, 16, 19, 20] and estimate the derivative of $V(\cdot)$ by the generalized integral inequality, which leads to a less conservative LMI condition and reduced numerical complexity, and also as shown in the numerical example below, the proposed LMI condition in this paper can be solved with less free weighting matrix unknowns comparatively.

Example 1. Consider the system (1) where

$$A_1 = \begin{bmatrix} 0.8 & 0.05 \\ 0 & -1.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ 0.02 & 0.9 \end{bmatrix}, \quad B = \begin{bmatrix} -6 & 1 \\ 2 & 4 \end{bmatrix},$$

$$G = \begin{bmatrix} 0.01 & 0 \\ 0.5 & 0.2 \end{bmatrix}, \quad C_1 = [0.01 \quad -0.2], \quad C_2 = [-0.02 \quad 0.1],$$

$$f(\cdot) = 0.1 \begin{bmatrix} \sqrt{\sin(t)x_1^2(t) + x_1^2(t-h(t)) + \cos(t)u_2^2(t) + \omega_2^2(t)} \\ \sqrt{\sin(t)x_2^2(t) + x_2^2(t-h(t)) + \cos(t)u_1^2(t) + \omega_1^2(t)} \end{bmatrix}$$

and $a_1 = a_2 = a_3 = a_4 = 0.01$,

$$h(t) = \begin{cases} 0.1 + 0.3 \cos(t), & t \in I = \left(0, \frac{\pi}{2}\right) \bigcup_{k \in \mathbb{N}^+} \left(-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi\right) \\ 0.1, & t \in \mathbb{R}^+ \setminus I. \end{cases}$$

$$\varphi(t) = [2, 2.4], \quad t \in [-0.4, 0].$$

Note that the functions $h(t)$ are non-differentiable, therefore, the methods proposed in [12, 16, 19, 20] are not applicable to this system. For given $h_1 = 0.1, h_2 = 0.4, T = 5, d = 1, \gamma = 4, c_1 = 1, c_2 = 37, R = 0.1I$, by using the LMI Toolbox in Matlab (see [7]), the LMI in Theorem 3.1 is satisfied with $\eta = 0.4138$ and

$$P = \begin{bmatrix} 1.5584 & -0.2630 \\ -0.2630 & 1.6955 \end{bmatrix}, U_1 = \begin{bmatrix} 0.9896 & -0.9271 \\ -0.9271 & 1.6102 \end{bmatrix}, U_2 = \begin{bmatrix} 0.5054 & -0.6833 \\ -0.6833 & 1.2911 \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 194.7508 & 24.4056 \\ 24.4056 & 135.9976 \end{bmatrix}, X_2 = \begin{bmatrix} 8.7212 & 1.9574 \\ 1.9574 & 5.4938 \end{bmatrix}, S = \begin{bmatrix} 49.2089 & -6.5256 \\ -6.5256 & 52.7801 \end{bmatrix},$$

$$N = 10^4 \begin{bmatrix} 1.3599 & 0.3567 \\ 0.3567 & 0.9026 \end{bmatrix}, Q = \begin{bmatrix} 0.8179 & 0.0048 \\ 0.0048 & 0.7321 \end{bmatrix}, K = \begin{bmatrix} 0.0105 \\ -0.0335 \end{bmatrix}.$$

By Theorem 3.1, the robust finite-time H_∞ control problem for the systems (1) has a solution, and the output feedback control $u(t) = N^{-1}Kz(t)$ is defined as

$$u(t) = 10^{-6} \begin{bmatrix} 0.0195 & -0.3901 \\ -0.0448 & 0.8962 \end{bmatrix} x(t) + 10^{-6} \begin{bmatrix} -0.0390 & 0.1950 \\ 0.0896 & -0.4481 \end{bmatrix} x(t-h(t)).$$

Moreover, the solution $x(t, \varphi)$ satisfies

$$x(t)^\top R x(t) \leq 37, \quad \forall t \in [0, 5].$$

Example 2. Consider the uncertain linear systems (15) where

$$A_1 = \begin{bmatrix} 1.3 & 0.01 \\ 0.2 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1.5 & 0 \\ 0.02 & 1.8 \end{bmatrix}, \quad B = \begin{bmatrix} -12 & 5 \\ 2 & 8 \end{bmatrix},$$

$$G = \begin{bmatrix} 0.01 & 0 \\ 0.5 & 0.02 \end{bmatrix}, \quad C_1 = [0.001 \quad -0.05], \quad C_2 = [-0.04 \quad 0.01],$$

$$D = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.025 \end{bmatrix}, \quad M_{a_1} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}, \quad M_{a_2} = M_b = M_g = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$\varphi(t) = [3, 3.8]$ and with the delay $h(t)$ is given as in Example 1. We also note that the function $h(t)$ is non-differentiable, therefore, the methods proposed in [13, 19, 20] are not applicable to this system. By using the LMI Toolbox in Matlab, the LMI in Corollary 1 is satisfied with $T = 8, d = 2, \gamma = 1, c_1 = 1, c_2 = 74, R = 0.04I, \eta = 0.4$ and

$$P = \begin{bmatrix} 3.3747 & -0.7143 \\ -0.7143 & 1.3600 \end{bmatrix}, \quad U_1 = \begin{bmatrix} 2.3774 & -0.1270 \\ -0.1270 & 2.1229 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0.7881 & 0.2115 \\ 0.2115 & 1.2668 \end{bmatrix},$$

$$\begin{aligned}
X_1 &= \begin{bmatrix} 323.1831 & -12.8844 \\ -12.8844 & 121.4385 \end{bmatrix}, & X_2 &= \begin{bmatrix} 13.8824 & -0.6318 \\ -0.6318 & 5.1041 \end{bmatrix}, \\
S &= \begin{bmatrix} 84.9282 & -1.2178 \\ -1.2178 & 34.5612 \end{bmatrix}, & N &= 10^3 \begin{bmatrix} 6.3315 & 0.4824 \\ 0.4824 & 4.6013 \end{bmatrix}, \\
Q &= \begin{bmatrix} 0.5451 & -0.0093 \\ -0.0093 & 0.2147 \end{bmatrix}, & K &= \begin{bmatrix} 0.0591 \\ 0.0496 \end{bmatrix}.
\end{aligned}$$

The robust finite-time H_∞ control problem for the systems (15) has a solution, and the output feedback control $u(t) = N^{-1}Kz(t)$ is defined by

$$u(t) = 10^{-6} \begin{bmatrix} 0.0086 & -0.4288 \\ 0.0099 & -0.4942 \end{bmatrix} x(t) + 10^{-6} \begin{bmatrix} -0.3430 & 0.0858 \\ -0.3953 & 0.0988 \end{bmatrix} x(t - h(t)).$$

Moreover, the solution $x(t, \varphi)$ satisfies

$$x(t)^\top R x(t) \leq 74, \quad \forall t \in [0, 8].$$

4. Conclusions. This paper has investigated the robust finite-time H_∞ control problem via the output feedback controls for nonlinear systems with the interval and non-differentiable time-varying delays. Based on constructing the improved Lyapunov functionals and by utilizing a new generalized integral inequality, new LMI-based sufficient conditions for designing output feedback controller are derived for the considered system. An application to H_∞ control of uncertain linear systems with the interval time-varying delays is given. Numerical examples have been given to illustrate the effectiveness of the proposed results. The foregoing results have the potential to be useful for the study of finite-time H_∞ control via output feedback for nonlinear non-autonomous systems with time-varying delay and disturbances.

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REFERENCES

- [1] F. Amato, M. Ariola and C. Cosentino, Finite-time stabilization via dynamic output feedback, *Automatica*, **42** (2006), 337–342.
- [2] F. Amato, G. De Tommasi and A. Pironti, Necessary and sufficient conditions for finite-time stability of impulsive dynamical linear systems, *Automatica*, **49** (2013), 2546–2550.
- [3] E. K. Boukas, Static output feedback control for stochastic hybrid systems: LMI approach, *Automatica*, **42** (2006), 183–188.
- [4] S. Boyd, L. El. Ghaoui and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM, Philadelphia, PA, 1994.
- [5] P. Dorato, Short time stability in linear time-varying systems, *In Proc IRE Int Convention Record*, **4** (1961), 83–87.
- [6] E. Fridman and U. Shaked, Delay-dependent stability and H_∞ control: constant and time-varying delays, *International Journal of Control*, **76** (2003), 48–60.
- [7] P. Gahinet, A. Nemirovskii, A. J. Laub and M. Chilali, *LMI Control Toolbox For use with MATLAB*, The MathWorks, Inc, 1995.
- [8] G. Garcia, S. Tarbouriech and J. Bernussou, Finite-time stabilization of linear time-varying continuous systems, *IEEE Transactions on Automatic Control*, **54** (2009), 364–369.
- [9] L. Gollmann and H. Maurer, Theory and applications of optimal control problems with multiple time-delays, *Journal of Industrial and Management Optimization*, **10** (2014), 413–441.

- [10] V. Kharitonov, *Time-Delay Systems: Lyapunov Functionals and Matrices*, Control Engineering. Birkhäuser/Springer, New York, 2013.
- [11] O. M. Kwon, J. H. Park and S. M. Lee, Exponential stability for uncertain dynamic systems with time-varying delays: LMI optimization approach, *Journal of Optimization Theory and Applications*, **137** (2008), 521–532.
- [12] H. Liu, Y. Shen and X. Zhao, Delay-dependent observer-based H_∞ finite-time control for switched systems with time-varying delay, *Nonlinear Analysis: Hybrid Systems*, **6** (2012), 885–898.
- [13] Q. Y. Meng and Y. J. Shen, Finite-time H_∞ control for linear continuous system with norm-bounded disturbance, *Communications in Nonlinear Science and Numerical Simulation*, **14** (2009), 1043–1049.
- [14] E. Moulay, M. Dambrine, N. Yeganefar and W. Perruquetti, Finite-time stability and stabilization of time-delay systems, *Systems and Control Letters*, **57** (2008), 561–566.
- [15] T. Senthilkumar and P. Balasubramaniam, Delay-dependent robust stabilization and H_∞ control for nonlinear stochastic systems with Markovian jump parameters and interval time-varying delays, *Journal of Optimization Theory and Applications*, **151** (2011), 100–120.
- [16] T. Senthilkumar and P. Balasubramaniam, Delay-dependent robust stabilization and H_∞ control for nonlinear stochastic systems with Markovian jump parameters and interval time-varying delays, *Journal of Optimization Theory and Applications*, **151** (2011), 100–120.
- [17] A. Seuret and F. Gouaisbaut, Wirtinger-based integral inequality: Application to time-delay systems, *Automatica*, **49** (2013), 2860–2866.
- [18] L. Wu, J. Lam and C. Wang, Robust H_∞ dynamic output feedback control for 2D linear parameter-varying systems, *IMA journal of mathematical control and information*, **26** (2009), 23–44.
- [19] Z. Xiang, Y. N. Sun and M. S. Mahmoud, Robust finite-time H_∞ control for a class of uncertain switched neutral systems, *Communications in Nonlinear Science Numerical Simulations*, **17** (2012), 1766–1778.
- [20] W. Xiang and J. Xiao, H_∞ finite-time control for nonlinear switched discrete-time systems with norm-bounded disturbance, *Journal of the Franklin Institute*, **348** (2011), 331–352.
- [21] H. Xu and K. L. Teo, H_∞ optimal stabilization of a class of uncertain impulsive systems: An LMI approach, *Journal of Industrial and Management Optimization*, **5** (2009), 153–159.
- [22] Y. Zhang, C. Liu and X. Mu, Robust finite-time $H - \infty$ control of singular stochastic systems via static output feedback, *Applied Mathematics and Computation*, **218** (2012), 5629–5640.

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