Carleman estimates for elliptic operators with complex coefficients Part II: transmission problems

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CARLEMAN ESTIMATES FOR ELLIPTIC OPERATORS WITH COMPLEX COEFFICIENTS PART II: TRANSMISSION PROBLEMS

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ABSTRACT. We consider elliptic transmission problems with complex coefficients across an interface. Under proper transmission conditions, that extend known conditions for well-posedness, and sub-ellipticity we derive microlocal and local Carleman estimates near the interface. Carleman estimates are weighted *a priori* estimates of the solutions of the elliptic transmission problem. The weight is of exponential form, $\exp(\tau\varphi)$ where τ can be taken as large as desired. Such estimates have numerous applications in unique continuation, inverse problems, and control theory. The proof relies on microlocal factorizations of the symbols of the conjugated operators in connection with the sign of the imaginary part of their roots. We further consider weight functions where $\varphi = \exp(\gamma\psi)$, with γ acting as a second large paremeter, and we derive estimates where the dependency upon the two parameters, τ and γ, is made explicit. Applications to unique continuation properties are given.

RÉSUMÉ: Nous considérons des problèmes de transmission elliptiques à coefficients complexes. En étendant des conditions qui rendent ce problème bien posé, et sous condition de sous-ellipticité nous obtenons des inégalités de Carleman microlocales et locales à l'interface qui sont des inégalités a priori à poids pour les solutions du problème. Les fonctions poids sont exponentielles, $\exp(\tau\varphi)$, où le paramètre τ peut être choisi arbitrairement grand. De telles estimations ont de nombreuses applications comme pour les questions de prolongement unique, les problèmes inverses et le contrôle. La démonstration repose sur des factorisations microlocales du symbole des opérateurs conjugués liées aux signes des parties imaginaires de leurs racines. Nous considérons le cas $\varphi = \exp(\gamma\psi)$, où γ peut-être arbitrairement grand et nous obtenons des inégalités de Carleman pour lesquelles la dépendence en les deux grands paramètres, τ et γ, est rendue explicite. Des applications aux questions de prolongement unique sont proposées.

KEYWORDS: Carleman estimate; elliptic operators; transmission problem and condition; unique continuation

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1. Introduction and main result

Let Ω be an open subset of $\mathbb{R}^n$ with a smooth boundary and let Ω_1 be an open subset of Ω such that $\Omega_1 \Subset \Omega$ and such that $S = \partial\Omega_1$ is smooth. We set $\Omega_2 = \Omega \setminus \Omega_1$. We thus have $\partial\Omega_2 = S \cup \partial\Omega$.

Points in $\mathbb{R}^n$ are denoted by $x = (x_1, \dots, x_n)$ and we write $D_j = -i\partial/\partial x_j$ where $i = \sqrt{-1}$. Let us consider two linear partial differential operators P_k, $k = 1, 2$ of respective order $m_k = 2\mu_k$, with $\mu_k \geq 1$,

$$P_k = \sum_{|\alpha| \leq m_k} a_\alpha^k(x) D^\alpha, \quad k = 1, 2, \tag{1.1}$$

where the coefficients $a_\alpha^k(x)$ are bounded measurable complex-valued functions defined in $\overline{\Omega}$. The higher-order coefficients $a_\alpha^k(x)$ with $|\alpha| = m_k$ are required to be $\mathscr{C}^\infty$ in $\overline{\Omega}_k$. In what follows, we assume that boths operators P_k, $k = 1, 2$ are elliptic.

In addition, we consider a system of $m_1 + m_2$ linear transmission operators

$$T_k^j = \sum_{|\alpha| \leq \beta_k^j} t_{k,\alpha}^j(x) D^\alpha, \quad k = 1, 2, \quad j = 1, \dots, m = \mu_1 + \mu_2, \tag{1.2}$$

with $0 \leq \beta_k^j < m_k$, and where the coefficients $t_{k,\alpha}^j(x)$ are $\mathscr{C}^\infty$ complex-valued functions defined in some neighborhood of S. Setting $\beta^j = (\beta_1^j + \beta_2^j)/2$ we assume that

$$m_1 - \beta_1^j = m_2 - \beta_2^j = m - \beta^j, \quad j = 1, \dots, m = \mu_1 + \mu_2. \tag{1.3}$$

We consider a system of $\mu_2 = m_2/2$ linear boundary operators of order less than m_2

$$B^j = \sum_{|\alpha| \leq \beta_\partial^j} b_\alpha^j(x) D^\alpha, \quad j = 1, \dots, \mu_2, \tag{1.4}$$

where the coefficients $b_\alpha^j(x)$ are $\mathscr{C}^\infty$ complex-valued functions defined in some neighborhood of $\partial\Omega$.

We can then consider the following elliptic boundary-value transmission problem

$$\begin{cases} P_k u_k = f_k & \text{in } \Omega_k, \ , k = 1, 2 \\ T_1^j u_1 + T_2^j u_2 = g^j, & \text{in } S, \quad j = 1, \dots, m. \\ B^j u = h^j, & \text{in } \partial\Omega, \quad j = 1, \dots, \mu_2. \end{cases}$$

The aim of the present article is to derive a Carleman estimate for this transmission problem. Carleman estimates are weighted *a priori* inequalities for the solutions of a partial differential equation (PDE), where the weight is of exponential type. For the partial differential operator P *away from the boundary and from the interface*, say for $w \in \mathscr{C}_c^\infty(\Omega_1)$ or $\mathscr{C}_c^\infty(\Omega_2)$, it takes the form:

$$\left\| e^{\tau\varphi} w \right\|_{L^2} \leq C \left\| e^{\tau\varphi} P w \right\|_{L^2}, \qquad \tau \geq \tau_0. \tag{1.5}$$

The exponential weight involves a parameter τ that can be taken as large as desired. The weight function φ needs to be chosen carefully. Additional terms in the l.h.s., involving derivatives of u, can be obtained depending on the order of P and on the *joint* properties of P and φ. For instance for a second-order operator P such an estimate can take the form

$$\tau^3 \left\| e^{\tau\varphi} w \right\|_{L^2}^2 + \tau \left\| e^{\tau\varphi} \nabla_x w \right\|_{L^2}^2 \leq C \left\| e^{\tau\varphi} P w \right\|_{L^2}^2, \qquad \tau \geq \tau_0, \ w \in \mathscr{C}_c^\infty(\Omega_1) \text{ or } \mathscr{C}_c^\infty(\Omega_2). \tag{1.6}$$

This type of estimate was used for the first time by T. Carleman [11] to achieve uniqueness properties for the Cauchy problem of an elliptic operator. Later, A.-P. Calderón and L. Hörmander further developed Carleman's method [10, 18]. To this day, Carleman estimates remain an essential method to prove unique continuation properties; see for instance [50] for manifold results. On such questions more recent advances have been concerned with differential operators with singular potentials, starting with the contribution of D. Jerison and C. Kenig [26]. The reader is also referred to [48, 28, 29]. In more recent years, the field of applications of Carleman estimates has gone beyond the original domain; they are also used in the study of:

- Inverse problems, where Carleman estimates are used to obtain stability estimates for the unknown sought quantity (e.g. coefficient, source term) with respect to norms on measurements performed on the solution of the PDE, see e.g. [6, 24, 30, 23]; Carleman estimates are also fundamental in the construction of complex geometrical optic solutions that lead to the resolution of inverse problems such as the Calderón problem with partial data [27, 13].
- Control theory for PDEs; through unique continuation properties, Carleman estimates are used for the exact controllability of hyperbolic equations [2]. They also yield the null controllability of linear parabolic equations [37] and the null controllability of classes of semi-linear parabolic equations [17, 1, 16].

For general elliptic operators, Carleman estimates away from boundaries and interfaces can be found in [19, Chapter 8]. The essential condition for the derivation of such an estimate is a compatibility property between the elliptic operator P and the weight function φ, the so-called sub-ellipticity condition which is known to be necessary and sufficient for the estimate to hold in the case of an elliptic operator. At the boundary $\partial\Omega$, a Lopatinskii-type compatibility condition involving P, φ, and the operators B^k can be put forward yielding

a Carleman estimate in conjunction with the sub-ellipticity condition [49, 4]. The main goal of the present article is the extension of this analysis to transmission problems.

Note that Carleman estimates of the form given here are local. Yet, they can be patched together to form global estimates. Our goal here is to derive such an estimate in the neighborhood of a point of the interface S. Derivation of Carleman estimates away for the interface can be found in the aforementionned references. Then, the patching procedure allows one to obtain a global estimate in the whole Ω, following for instance [19, Lemma 8.3.1] and [32]. We do not cover this issue here.

Here the weight function φ will be chosen continuous and piecewise smooth, that is, $\varphi_k = \varphi_{|\Omega_k} \in \mathscr{C}^\infty(\overline{\Omega_k})$. The estimate we shall obtain will exhibit additional terms that account for the transmission conditions given by the operators T_k^j, $k = 1, 2$, $j = 1, \dots, \mu_1 + \mu_2$. The key conditions for the derivation of the present Carleman estimate are compatibility properties between the elliptic operator P, the weight function φ, and the transmission operators T_k^j. Those are the sub-ellipticity condition described above that expresses compatibility between P and φ, and in addition a condition that connects them to T_k^j at the interface; we shall refer to this latter condition as to the *transmission condition*. This condition is an extension of the condition presented in [47] in the case of a conjugated operator. There, it was introduced towards to understanding of the well-posedness of the elliptic transmission problem. The condition we use is very close in its formulation to the Lopatinskii type boundary condition used in the first part of this work [4]. In [49, 4] the derivation of Carleman estimates at a boundary is based on the study of interior and boundary differential quadratic forms, an approach that originates in the work of [19] for estimates away from boundaries and in [45, 46, 41] for the treatment of boundaries. This approach is here extended to interface tranmission problems. By proper (tangential) microlocalizations at the interface we show the precise action of our transmission condition. These microlocalizations are important as the transmission condition is function of the sign of the imaginary parts of the roots of[1] $p_{k,\varphi}(x, \xi', \tau, \xi_n) = p_k(x, \xi + i\tau\varphi'(x))$, $k = 1, 2$, viewed as a polynomial in ξ_n. Of course the configuration of the roots changes as the other parameters (x', ξ', τ) are modified. Roots can for instance cross the real axis. Each configuration needs to be addressed separately through a microlocalization procedure. For the Laplace operator at a *boundary* this was exploited to obtain a Carleman estimate in [38] for the purpose of proving a stabilization result for the wave equation. This approach was used for the study of an interface problem in [3, 34, 33] in the case of second-order elliptic operators. The present article provides a generalization of these earlier works, both with respect to the order of the operators and with respect to the generality of the transmission operators used.

The Carleman estimate we prove here is of the form, with $u_k = u_{|\Omega_k}$,

$$\sum_{k=1,2} \|e^{\tau\varphi_k} u_k\|^2 + \sum_{k=1,2} |e^{\tau\varphi} \operatorname{tr}(u_k)|^2 \\ \leq C\Big(\sum_{k=1,2} \|e^{\tau\varphi_k} P(x, D) u_k\|^2 + \sum_{j=1}^{\mu_1+\mu_2} |e^{\tau\varphi_{|S}} (T_1^j(x, D) u_{1|S} + T_2^j(x, D) u_{2|S})|^2 \Big),$$

for u supported near a point at the interface, where $\operatorname{tr}(u_k)$ stands for the trace of $(u_k, D_\nu u_k, \dots, D_\nu^{m-1} u_k)$, the successive normal derivatives of u_k, at the interface S. In this form, the estimate is incorrect as norms needs to be made precise. For a correct statement please refer to Theorem 1.6 below.

For Carleman estimates, one is often inclined to choose a weight function of the form $\varphi = \exp(\gamma\psi)$, with the parameter $\gamma > 0$ chosen large. Several authors have derived Carleman estimates for some operators in which the dependency upon the second parameters γ is kept explicit. See for instance [17]. Such results can be very useful to address systems of PDEs, in particular for the purpose of solving inverse problems. On such questions see for instance [14, 15, 25, 5].

[1] Here to simplify we consider the case $S = \{x_n = 0\}$. Then ξ_n corresponds the (co)normal direction at the interface. In the main text we shall use change of variables to reach this configuration locally.

Compatibility conditions need to be introduced between the operator P and the weight ψ. Those are the so-called strong pseudo-convexity conditions introduced by L. Hörmander [19, 22]. With the weight function φ of the form $\varphi = \exp(\gamma\psi)$, the parameter γ can be viewed as a convexification parameter. As shown in Proposition 28.3.3 in [22] the strong pseudo-convexity of the function ψ with respect to P implies the sub-ellipticity condition for φ mentioned above[2] for γ chosen sufficiently large. Away from any boundary and interface, for a second-order estimate the resulting Carleman estimate can take the form (compare with (1.6)):

(1.7)
$$(\gamma\tau)^3\big\|\varphi^{3/2}e^{\tau\varphi}u\big\|^2_{L^2} + \gamma\tau\big\|\varphi^{1/2}e^{\tau\varphi}\nabla_x u\big\|^2_{L^2} \lesssim \|e^{\tau\varphi}Pu\|^2_{L^2}\,, \qquad \tau \geq \tau_0,\ \gamma \geq \gamma_0,\ u \in \mathscr{C}^\infty_c(\Omega_1) \text{ or } \mathscr{C}^\infty_c(\Omega_2).$$

We aim to extend such estimate in the neighborhood of the interface S. We then assume that the transmission condition holds for the operators P, T^j_k, $k = 1, 2$, $j = 1, \dots, \mu_1 + \mu_2$, and the weight ψ. The work [31] provides a general framework for the analysis and the derivation of Carleman estimates with two large parameters away from boundaries. For that purpose it introduces a pseudo-differential calculus of the Weyl-Hörmander type that resembles the semi-classical calculus and takes into account the two large parameters τ and γ as well as the weight function $\varphi = \exp(\gamma\psi)$. Here, following the first part of this article [4], the analysis of [31] is adapted to the case of an estimate at the interface. Estimates with the two large parameters τ and γ are derived in the case of general elliptic operators.

If we strengthen strong pseudo-convexity condition of ψ and P, assuming the so-called simple characteristic property, sharper estimates can be obtained [31]. We also derive such estimates at the interface.

With the different Carleman estimate that we obtain here she shall be able to achieve unique continuation properties at an interface across some hypersurface for some classes of elliptic operators and some products of such operators.

1.1. **Setting.** Now, we give the precise setting of the problem we consider. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we denote by $\xi = (\xi_1, \dots, \xi_n)$ the corresponding Fourier variables. Moreover, for every $\xi \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$ we define $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$. We denote by

$$p_k(x,\xi) = \sum_{|\alpha|=m_k} a^k_\alpha(x)\xi^\alpha$$

the principal symbol of the operator P_k given in (1.1), $k = 1, 2$. The operators P_k are assumed to be elliptic, viz.,

$$p_k(x,\xi) \neq 0, \quad \forall x \in \overline{\Omega}_k,\ \forall \xi \in \mathbb{R}^n \backslash \{0\}\,.$$

With $m = \mu_1 + \mu_2 = (m_1 + m_2)/2$, we denote by

$$t^j_k(x,\xi) = \sum_{|\alpha|=\beta^j_k} t^j_{k,\alpha}(x)\xi^\alpha, \quad k = 1, 2\ j = 1, \dots, m,$$

the principal symbol of the transmission operator T^j_k defined in (1.2). Each set $\left\{T^j_1\right\}_{1\leq j\leq m}$ and $\left\{T^j_2\right\}_{1\leq j\leq m}$ is assumed normal, that is

$$0 \leq \beta^1_k \leq \beta^2_k \leq \cdots \leq \beta^m_k < m_k,$$

and for all $x \in S$ the conormal vector $\nu(x)$ is non characteristic, i.e., $t^j_k(x, \nu(x)) \neq 0$.

We recall that we assume

$$m_1 - \beta^j_1 = m_2 - \beta^j_2 = m - \beta^j, \quad j = 1, \dots, m = \mu_1 + \mu_2.$$

We now review the definition of two inportant notions that will be used in what follows:

[2]The terminology for the strong pseudo-convexity condition and the sub-ellipticity condition are often confused by authors. Here we make a clear distinction of the two notions.

- the sub-ellipticity condition between the operators P_k and the weight function φ;
- the transmission condition stating the compatibility between the transmission operators T_k^j, the operators P_k, and the weight function φ at a point of the interface.

1.2. **Sub-ellipticity condition.** For any two functions $f(x,\xi)$ and $g(x,\xi)$ in $\mathscr{C}^\infty(\Omega_k\times\mathbb{R}^n)$ we denote their Poisson bracket in phase-space by

$$\{f,g\}=\sum_{j=0}^{n}\big(\frac{\partial f}{\partial\xi_j}\frac{\partial g}{\partial x_j}-\frac{\partial f}{\partial\xi_j}\frac{\partial g}{\partial x_j}\big).$$

It is to be connected with the commutator of two (pseudo-)differential operators. In fact, if f and g are polynomials in ξ, then the principal symbol of the commutator $[f(x,D),g(x,D)]$ is precisely $-i\{f,g\}(x,\xi)$.

The sub-ellipticity condition connecting the symbol p_k and a weight function φ is the following (See [19, Chapter 8] and [22, Sections 28.2–3]).

Definition 1.1. *Let $k\in\{1,2\}$ and let U be an open subset of Ω_k and set $\varphi_k=\varphi_{|\Omega_k}$. The pair $\{P_k,\varphi_k\}$ satisfies the sub-ellipticity condition on $\overline{U}$ if $\varphi_k'(x):=\nabla\varphi(x)\neq 0$ at every point in $\overline{U}$ and if*

$$p_k(x,\xi+i\tau\varphi'(x))=0\quad\Rightarrow\quad\frac{1}{2i}\left\{\overline{p}_k(x,\xi-i\tau\varphi'(x)),p_k(x,\xi+i\tau\varphi'(x))\right\}>0,$$

for all $x\in\overline{U}$ and all non-zero $\xi\in\mathbb{R}^n$, $\tau>0$.

For an elliptic operator the sub-ellipticity condition is *necessary and sufficient* for a Carleman estimate of the form of (1.5) to hold away from the boundary [22, Section 28.2]. For a simple exposition of the derivation of Carleman estimates for second-order elliptic operators under the sub-ellipticity condition we refer to [32].

Note also that the sub-ellipticity condition is invariant under changes of coordinates. This is an important fact here as we shall work in local coordinates in what follows.

Remark 1.2. Note that here, as the operator P_k are elliptic, we have $p_k(x,\xi)\neq 0$ for each $\xi\in\mathbb{R}^n$, $\xi\neq 0$. The sub-ellipticity condition thus holds naturally at $\tau=0$.

Remark 1.3. Setting $p_{k,\varphi}(x,\xi,\tau)=p_k(x,\xi+i\tau\varphi_k')$ and writing $p_{k,\varphi}=a+ib$ with a and b real, we have

$$\frac{1}{2i}\left\{\overline{p_k}(x,\xi-i\tau\varphi_k'),p_k(x,\xi+i\tau\varphi_k')\right\}=\frac{1}{2i}\{\overline{p_{k,\varphi}},p_{k,\varphi}\}(x,\xi,\tau)=\{a,b\}(x,\xi,\tau).$$

Below, we shall use the sub-ellipticity condition in the form

$$p_k(x,\xi+i\tau\varphi_k')=0\quad\Rightarrow\quad\{a,b\}(x,\xi,\tau)>0,$$

for all $x\in\overline{U}$ and all non-zero $\xi\in\mathbb{R}^n$, $\tau>0$.

In connection with the symbol interpretation of the Poisson bracket given above, we see that the sub-ellipticity condition guarantees some positivity for the operator $i[a(x,D,\tau),b(x,D,\tau)]$ on the characteristic set of $p_k(x,D+i\tau\varphi_k')=a(x,D,\tau)+ib(x,D,\tau)$. A proper combination of $a(x,D,\tau)^*a(x,D,\tau)+b(x,D,\tau)^*b(x,D,\tau)$ and $i[a(x,D,\tau),b(x,D,\tau)]$ thus leads to a positive operator. This is the heart of the proof of Carleman estimates.

1.3. **Transmission conditions.** We consider a neighborhood X of a point of the interface S, chosen sufficiently small, so that there exists a smooth function $\theta(x)$ such that $d\theta(x)\neq 0$ in X and

$$\{x\in X;\ \theta(x)<0\}=\Omega_1\cap X,\quad\{x\in X;\ \theta(x)=0\}=S\cap X,\quad\{x\in X;\ \theta(x)>0\}=\Omega_2\cap X. \tag{1.8}$$

For $x\in S$, we denote by $N_x^*(S)$ the conormal space above x given by

$$N_x^*(S)=\left\{\nu\in T_x^*(\Omega);\ \nu(Z)=0,\ \forall Z\in T_x(S)\right\}.$$

The conormal bundle of S is given by

$$N^*(S) = \{(x,\nu) \in T^*(\Omega);\ x \in S,\ \nu \in N_x^*(S)\}\,.$$

In fact, if $x \in X \cap S$ and $(x,\nu) \in N^*(S)$ then $\nu = t\, d\theta(x)$ for some $t \in \mathbb{R}$.

By an interface quadruple $\omega = (x, Y, \nu, \tau)$ we shall mean

$$x \in S \cap X, \quad Y \in T_x^*(S), \quad \nu = td\theta(x) \in N_x^*(S) \text{ with } t > 0, \quad \text{and } \tau \geq 0.$$

In particular ν "points" from Ω_1 into Ω_2. For an interface quadruple ω and $\lambda \in \mathbb{C}$, we set, for $k = 1, 2$,

$$\tilde{p}_{k,\varphi}(\omega,\lambda) := p_k(x, Y + \lambda\nu_k + i\tau d\varphi_k(x)), \quad \text{with } \varphi_k = \varphi_{|\Omega_k}, \text{and } \nu_k = (-1)^k\nu \in N_x^*(S). \tag{1.9}$$

Note that for Ω_k the covector ν_k points inward and we have $\nu_1 = -\nu_2 = -\nu$.

For a fixed interface quadruple $\omega_0 = (x_0, Y_0, \nu_0, \tau_0)$, we denote by σ_k^j, $k = 1, 2$, the roots of $\tilde{p}_{k,\varphi}(\omega_0,\lambda)$ with multiplicity μ_k^j, viewed as a polynomial of degree m in λ, with leading-order coefficient $c_{k,0}$. We can then factorize this polynomial as follows:

$$\tilde{p}_{k,\varphi}(\omega_0,\lambda) = c_{k,0}\tilde{p}^+_{k,\varphi}(\omega_0,\lambda)\tilde{p}^-_{k,\varphi}(\omega_0,\lambda)\tilde{p}^0_{k,\varphi}(\omega_0,\lambda),$$

with

$$\tilde{p}^\pm_{k,\varphi}(\omega_0,\lambda) = \prod_{\pm \operatorname{Im}\sigma_k^j > 0} (\lambda - \sigma_k^j)^{\mu_k^j}, \quad \tilde{p}^0_{k,\varphi}(\omega_0,\lambda) = \prod_{\operatorname{Im}\sigma_k^j = 0} (\lambda - \sigma_k^j)^{\mu_k^j}.$$

We define the polynomial $\kappa_{k,\varphi}(\omega_0,\lambda)$ by

$$\kappa_{k,\varphi}(\omega_0,\lambda) = \tilde{p}^+_{k,\varphi}(\omega_0,\lambda)\tilde{p}^0_{k,\varphi}(\omega_0,\lambda). \tag{1.10}$$

Similarly, for the set of transmission operators, $\{T_k^j\}_{k=1,2;j=1,\dots,m}$, $m = \mu_1 + \mu_2$, and their principal symbols, $t_k^j(x,\xi)$, for an interface quadruple $\omega = (x, Y, \nu, \tau)$ we set

$$\tilde{t}^j_{k,\varphi}(\omega,\lambda) = t_k^j(x, Y + \lambda\nu_k + i\tau d\varphi_k(x)), \tag{1.11}$$

with $\nu_k = (-1)^k\nu$ as above.

Definition 1.4. *We say that $\{P_k, T_k^j, \varphi,\ k = 1, 2; j = 1, \dots, m\}$ satisfies the transmission condition at an interface quadruple $\omega_0 = (x_0, Y_0, \nu_0, \tau_0)$ if for all pairs of polynomials, $q_k(\lambda)$, $k = 1, 2$, there exist U_k, $k = 1, 2$, polynomials and $c_j \in \mathbb{C}$, $j = 1, \dots, m$, such that:*

$$q_1(\lambda) = \sum_{j=1}^m c_j\tilde{t}^j_{1,\varphi}(\omega_0,\lambda) + U_1(\lambda)\kappa_{1,\varphi}(\omega_0,\lambda) \quad \textit{and} \quad q_2(\lambda) = \sum_{j=1}^m c_j\tilde{t}^j_{2,\varphi}(\omega_0,\lambda) + U_2(\lambda)\kappa_{2,\varphi}(\omega_0,\lambda),$$

where the polynomials $\kappa_{k,\varphi}(\omega_0,\lambda)$ are those defined by (1.10).

Additionally, for $x_0 \in S$, we say that $\{P_k, T_k^j, \varphi,\ k = 1, 2; j = 1, \dots, m\}$ satisfies the transmission condition at x_0 if the above property holds for all interace quadruples $\omega = (x_0, Y_0, \nu_0, \tau_0)$ with $Y \in T^*_{x_0}(S)$, $\nu = t\, d\theta(x_0)$ with $t > 0$, and $\tau \geq 0$.

It should be noted that the same coefficients c_j are used in both decompositions.

Remark 1.5. (1) There is a strong similarity in the form between Definition 1.4 and the strong Lopatinskii condition that is used in the derivation of a Carleman estimate at the boundary. The latter condition connects the elliptic operator, the weight functions and the boundary operators given in (1.4) in [49, 4].

(2) Note that we did not choose any particular co-normal vectors ν_k connected with the function θ that locally defines S, apart from their orientation. In fact, for any $t > 0$ replacing ν_k by $t\nu_k$ does not affect the transmission condition of Definition 1.4. We could for instance use normalized conormal vectors, yet keeping the directions of ν_k.

1.3.1. *Alternative formulation.* Setting $n_k = d^\circ \kappa_{k,\varphi}$ we have $n_k = m_k - m_k^-$ with $m_k^- = d^\circ \tilde{p}_{k,\varphi}^-$. We have $n_k \leq m_k$. Hence, it is sufficient to consider the polynomials q_k, $k = 1, 2$, to be of degree less than $m_k - 1$ respectively. Then both polynomials U_k are of degree less than or equal to $m_k - n_k - 1 = m_k^- - 1$, recalling that $\tilde{t}_{k,\varphi}^j$ is of degree $\beta_k^j < m_k$.

We then set

$$\tilde{e}_1^j = \begin{cases} \tilde{t}_{1,\varphi}^j & \text{if } j = 1, \dots, m, \\ \lambda^{j-(m+1)} \kappa_{1,\varphi} & \text{if } j = m+1, \dots, m + m_1^-, \end{cases}$$

$$\tilde{e}_2^j = \begin{cases} \tilde{t}_{2,\varphi}^j & \text{if } j = 1, \dots, m, \\ \lambda^{j-(m+1)} \kappa_{2,\varphi} & \text{if } j = m+1, \dots, m + m_2^-, \end{cases}$$

and the linear map

$$\Phi : \mathbb{C}^m \times \mathbb{C}^{m_1^-} \times \mathbb{C}^{m_2^-} \to \mathbb{C}_{m_1-1}[\lambda] \times \mathbb{C}_{m_2-1}[\lambda], \tag{1.12}$$

$$(c, \gamma_1, \gamma_2) \mapsto \Big(\sum_{j=1}^{m} c_j \tilde{e}_1^j + \sum_{j=1}^{m_1^-} \gamma_{1,j} \tilde{e}_1^{j+m}, \sum_{j=1}^{m} c_j \tilde{e}_2^j + \sum_{j=1}^{m_2^-} \gamma_{2,j} \tilde{e}_2^{j+m} \Big).$$

The transmission condition of Definition 1.4 means precisely that the map Φ is *surjective*. In particular this implies that $m' = m + m_1^- + m_2^- \geq m_1 + m_2 = 2m$.

1.3.2. *Transmission condition and well-posedness.* Transmission conditions across an interface for elliptic problems can be found in [47] to prove well-posedness of the transmission elliptic problem. In [47] it corresponds to the case $\varphi = 0$, as no conjugation with a weight function is performed and, there, the transmission condition reads as follows: let U_k, $k = 1, 2$, polynomials and $c_j \in \mathbb{C}$, $j = 1, \dots, m$, be such that

$$0 = \sum_{j=1}^{m} c_j \tilde{t}_{1,\varphi=0}^j(\omega_0, \lambda) + U_1(\lambda) \kappa_{1,\varphi=0}(\omega_0, \lambda) \quad \text{and} \quad 0 = \sum_{j=1}^{m} c_j \tilde{t}_{2,\varphi=0}^j(\omega_0, \lambda) + U_2(\lambda) \kappa_{2,\varphi=0}(\omega_0, \lambda),$$

then $U_k \equiv 0$, $k = 1, 2$, and $c_j = 0$, $j = 1, \dots, m$. This condition precisely means that the map Φ introduced in (1.12) is *injective* in the case $\varphi = 0$. Above we gave a surjectivity condition that suits the purpose of the present article. We now explain how the two conditions coincide in the cases studied in [47].

In fact if $\varphi = 0$ none of the roots can be real as the operator is elliptic. If $\#\{\operatorname{Im} \sigma_k^j > 0\} = \#\{\operatorname{Im} \sigma_k^j < 0\}$, such an operator is called properly elliptic by Schechter [47] and he only studies this type of elliptic operators. In fact, if the dimension $n \geq 3$ then every elliptic operator is properly elliptic (see e.g. the proof of Proposition 1.1 in [40, Chapter 2, Section 1]). In such a case we have $m_k^- = m_k/2 = \mu_k$. Hence for the map Φ defined in (1.12) we have

$$\dim \mathbb{C}^m \times \mathbb{C}^{m_1^-} \times \mathbb{C}^{m_2^-} = \dim \mathbb{C}_{m_1-1}[\lambda] \times \mathbb{C}_{m_2-1}[\lambda],$$

as $m + m_1^- + m_2^- = 2m = m_1 + m_2$, meaning that in this case the surjectivity of Φ is equivalent to its injectivity. In the case of a properly elliptic operator and $\varphi = 0$ the transmission condition of [47] thus coincides with Definition 1.4 in the case $\varphi = 0$.

Note that the injectivity of the map Φ may be lost for $\varphi \neq 0$ and $\tau > 0$, as the conjugated operator P_φ may not be elliptic, yet the transmission condition we give here precisely states that Φ is surjective.

1.3.3. *Invariance by change of local coordinates.* We finish the presentation of the transmission condition by observing that this definition is of geometrical nature, independent of the choice of coordinates. This fact is important as we shall make use of local coordinates at the interface S in what follows.

In fact, for a point $x \in S$ we consider an open neighborhood $X \subset \Omega$ of x, chosen sufficiently small so that there exists θ as in (1.8).

We consider two coordinate systems $(X^{(i)}, \psi^{(i)})$, $i = 1, 2$, that is $\psi^{(i)} : X \to X^{(i)}$ is a diffeomorphism and $X^{(i)}$ is an open set in $\mathbb{R}^n$. We set $x^{(i)} = \psi^{(i)}(x)$. We moreover set

$$\theta^{(i)} = \theta \circ (\psi^{(i)})^{-1}, \quad X_k^{(i)} = \{x \in X^{(i)};\ (-1)^k \theta^{(i)}(x) > 0\} = \psi^{(i)}(X \cap \Omega_k).$$

We then introduce the diffeomorphism $\kappa : X^{(1)} \to X^{(2)}$ given by $\kappa = \psi^{(2)} \circ (\psi^{(1)})^{-1}$ and we have $\kappa(x^{(1)}) = x^{(2)}$, $\theta^{(1)} = \theta^{(2)} \circ \kappa$, yielding $\kappa(X_k^{(1)}) = X_k^{(2)}$.

We also define $\varphi^{(i)} = \varphi \circ \psi^{(i)}$, $i = 1, 2$, the local versions of the weight function in the coordinate patches and we set $\varphi_k^{(i)} = \varphi^{(i)}{}_{|X_k^{(i)}}$.

Let $Y^{(i)}, \nu_k^{(i)}$, $k = 1, 2$, $i = 1, 2$, be the local versions of Y and ν_k in the two coordinate systems. With standard differential geometry arguments we have the following relations:

$$Y^{(1)} = {}^t\kappa'(x^{(1)}) Y^{(2)}, \quad \nu_k^{(1)} = {}^t\kappa'(x^{(1)}) \nu_k^{(2)}, \quad d\varphi_k^{(1)}(x^{(1)}) = {}^t\kappa'(x^{(1)}) d\varphi_k^{(2)}(x^{(2)}),$$

Similarly let $p_k^{(i)}$ and $t_k^{j(i)}$, $k = 1, 2$, $j = 1, \dots, m$, $i = 1, 2$, be the local versions of the principal symbols of the differential operators P_k and T_k^j. We have

$$p_k^{(1)}(x, \xi) = p_k^{(2)}(\kappa(x), {}^t\kappa'(x)^{-1}\xi), \quad t_k^{j(1)}(x, \xi) = t_k^{j(2)}(\kappa(x), {}^t\kappa'(x)^{-1}\xi).$$

If we set $f_k^{(i)}(\lambda) = p_k^{(i)}(x^{(i)}, Y^{(i)} + \lambda \nu_k^{(i)} + i\tau d\varphi_k^{(i)}(x^{(i)}))$, $i = 1, 2$, we find

$$\begin{aligned} f_k^{(1)}(\lambda) &= p_k^{(1)}(x^{(1)}, Y^{(1)} + \lambda \nu_k^{(1)} + i\tau d\varphi_k^{(1)}(x^{(1)})) \\ &= p_k^{(2)}\big(\kappa(x^{(2)}), {}^t\kappa'(x^{(1)})^{-1}(Y^{(1)} + \lambda \nu_k^{(1)} + i\tau d\varphi_k^{(1)}(x^{(1)}))\big) \\ &= p_k^{(2)}(x^{(2)}, Y^{(2)} + \lambda \nu_k^{(2)} + i\tau d\varphi_k^{(2)}(x^{(2)})) \\ &= f_k^{(2)}(\lambda), \end{aligned}$$

which simply means that the polynomial function $\tilde{p}_{k,\varphi}$ defined in (1.9) does not depend on the coordinate system chosen. The same holds for the polynomial function $\tilde{t}_{k,\varphi}^j$ defined in (1.11), which allows one to conclude that the transmission condition of Definition 1.4 can be stated (and checked) in any coordinate system.

1.4. **Sobolev norms with a parameter.** For non-negative integer m and a real number $\tau \geq 0$, we introduce the Sobolev spaces $H_\tau^m(\Omega_k)$ and $H_\tau^m(S)$ defined by the following norms respectively:

$$\|u\|_{m,\tau}^2 = \sum_{k=0}^m \tau^{2(m-k)} \|u\|_{H^k(\Omega_k)}^2 \quad \text{and} \quad |u|_{m,\tau}^2 = \sum_{k=0}^m \tau^{2(m-k)} |u|_{H^k(S)}^2, \tag{1.13}$$

where we denote the usual Sobolev norms on Ω_k and S by $\|.\|_{H^s(\Omega_k)}$ and $|.|_{H^s(S)}$. The L^2 inner-products on Ω_k and S will be denoted by $(.,.)$ and $(.,.)_\partial$ respectively. Observe that for the norm $\|.\|_{m,\tau}$ on $H_\tau^m(\Omega_k)$ we do not specify explicitly the integer k that refers to which side of the interface we consider, Ω_1 or Ω_2. In the main text there should never be any ambiguity as the norm will be used for functions that are clearly defined on one of the open sets.

For $m \in \mathbb{N}$ and $s \in \mathbb{R}$ we introduce the following interface space

$$H_\tau^{m,s}(S) = \prod_{j=0}^{m} H_\tau^{m-j+s}(S),$$

equipped with the norm

$$|\mathbf{u}|_{m,s,\tau}^2 = \sum_{j=0}^{m} |u_j|_{m-j+s,\tau}^2, \quad \mathbf{u} = (u_0, \dots, u_m). \tag{1.14}$$

If $u \in \mathscr{C}^\infty(\overline{\Omega_k})$ we set $\mathrm{tr}^m(u) = (\mathrm{tr}_0(u), \dots, \mathrm{tr}_m(u))$ where $\mathrm{tr}_j(u) = (\frac{1}{i}\partial_\nu)^j u_{|S}$, with ν conormal to S, is the sectional trace of u of order j and, in accordance with (1.14), we define

$$|\mathrm{tr}^m(u)|_{m,s,\tau}^2 = \sum_{j=0}^{m} |\mathrm{tr}_j(u)|_{m-j+s,\tau}^2 .$$

In what follows we shall write $\mathrm{tr}(u)$ in place of $\mathrm{tr}^m(u)$ for concision. We shall also write norms of the form $|e^{\tau\varphi}\,\mathrm{tr}(u)|_{m,s,\tau}^2$ actually meaning

$$|e^{\tau\varphi}\,\mathrm{tr}^m(u)|_{m,s,\tau}^2 = \sum_{j=0}^{m} |e^{\tau\varphi}\,\mathrm{tr}_j(u)|_{m-j+s,\tau}^2 .$$

1.5. **Statement of the main result.** We can now state the local Carleman estimate that we prove in the neighborhood of a point of the interface, with the sub-ellipticity and transmission conditions.

Theorem 1.6. *Let $x_0 \in S$ and let $\varphi \in \mathscr{C}^0(\Omega)$ be such that $\varphi_k = \varphi_{|\Omega_k} \in \mathscr{C}^\infty(\Omega_k)$ for $k = 1, 2$ and such that the pairs $\{P_k, \varphi_k\}$ have the sub-ellipticity property of Definition 1.1 in a neighborhood of x_0 in $\overline{\Omega_k}$. Moreover, assume that $\{P_k, \varphi, T_k^j,\ k = 1, 2,\ j = 1, \dots, m\}$ satisfies the transmission condition at x_0. Then there exist a neighborhood W of x_0 in $\mathbb{R}^n$ and two constants C and $\tau_* > 0$ such that*

$$\begin{aligned} &\sum_{k=1,2} \left(\tau^{-1} \left\|e^{\tau\varphi_k} u_k\right\|_{m_k,\tau}^2 + |e^{\tau\varphi_{|S}}\,\mathrm{tr}(u_k)|_{m_k-1,1/2,\tau}^2\right) \\ &\quad \le C\Big(\sum_{k=1,2} \|e^{\tau\varphi_k} P_k(x,D) u_k\|_{L^2(\Omega_k)}^2 + \sum_{j=1}^{m} \left|e^{\tau\varphi_{|S}} \left(T_1^j(x,D)u_1 + T_2^j(x,D)u_2\right)_{|S}\right|_{m-1/2-\beta^j,\tau}^2\Big), \end{aligned} \tag{1.15}$$

for all $u_k = w_{k|\Omega_k}$ with $w_k \in \mathscr{C}_c^\infty(W)$ and $\tau \ge \tau_*$.

First, this results will be established microlocally: at an interface point x_0 we shall assume that the transmission condition holds for some interface quadruple $(x_0, Y_0, \nu_0, \tau_0)$ and we shall prove that a Carleman estimate of the form above holds in a conic neighborhood of this interface quadruple in phase-space; localization in phase-space will be done by means of cut-off functions and associated pseudo-differential operators. We refer the reader to Section 4.3. Second, we will deduce Theorem 1.6 from such microlocal estimates.

Estimate (1.15) concerns function located near the interface and vanishing near the boundary ∂_Ω. Hence this estimate involves the transmission operators T_k^j and not the boundary operators B^j.

Estimates of the form of (1.15) are local. Yet, such estimates and their counterpart estimates at the boundary proven in [4] can be patched together to form global estimates. We do not cover such details here and we refer to [35] where this is done in the case of a transmission problem.

In Section 6 we shall prove Carleman estimates with a weight function of the form $\varphi(x) = \exp(\gamma\psi(x))$ as is usually done in practice with the parameter γ chosen as large as desired. We shall provide the precise dependency of the Carleman estimate with respect to this second large parameter.

Examples of elliptic transmission problem and weight function for which the above result applies, and other for which it does not, will be given in Section 1.7 below. In fact, as the condition for the Carleman

estimate to hold are geometrical, that is, coordinate invariant, we shall postpone the exposition of example after we introduce local variables that ease the writting of the transmission condition.

1.6. **Local reduction of the problem near the interface.** All the different aspects of the problem we consider –operators, sub-ellipticity condition, transmission condition– are coordinate invariant as we saw above. We shall thus work locally, in the neighborhood of a point of the interface and choose coordinates that allow use to ease the subsequent analysis and derivation of the Carleman estimate.

1.6.1. *Choice of local coordinates.* Let $x_0 \in S$. There exists a neighborhood V of x_0 and a local system of coordinates $x = (x_1, \dots, x_n)$ where $V \cap \Omega_1 \subset \{x_n > 0\}$, $V \cap \Omega_2 \subset \{x_n < 0\}$ and $x' = (x_1, \dots, x_{n-1})$ parametrizes the interface$V \cap S \subset \{x_n = 0\}$. We assume that $V \cap \partial\Omega = \emptyset$, that is, we focus our analysis on the interface and remain away from the boudnary.

We denote by $\mathbb{R}^n_\pm$ the half space $\{\pm x_n > 0\}$ and $V_\pm = V \cap \mathbb{R}^n_\pm$. For our purpose here, without any loss of generality, we may assume that $V_\pm$ is bounded.

In such local coordinates, in $V_\pm$, the differential operator[3] P_k of order m with complex coefficients takes the form

$$P_k = P_k(x, D) = \sum_{j=1}^{m_k} P_{k,j}(x, D')D_n^j, \qquad D_n = \frac{1}{i}\partial_n, \quad k = 1, 2,$$

where $P_{k,j}(x, D')$, $j = 1, \dots, m_k$, $k = 1, 2$, are tangential differential operators with complex coefficients of order $m_k - j$. Similarly the transmission operators take the form

$$T_k^j = T_k^j(x, D) = \sum_{i=0}^{\beta_k^j} T_{k,i}^j(x, D')D_n^i, \quad 1 \le j \le m, \ k = 1, 2$$

where $T_{k,i}^j(x, D')$, $i = 0, \dots, \beta_k^j$, are tangential differential operators of order $\beta_k^j - i$. The local transmission problem we study thus takes the form

$$\begin{cases} P_1 u_1 = f_1 & \text{in } V_1 = \{x_n < 0\}, \\ P_2 u_2 = f_2 & \text{in } V_2 = \{x_n > 0\}, \\ T_1^j u_1 + T_2^j u_2 = g^j, & \text{in } S, \quad j = 1, \dots, m. \end{cases} \tag{1.16}$$

We have $P_{k,m} = P_{k,m}(x) \neq 0$. Upon dividing the functions f_k by $P_{k,m}(x)$ we may assume that $P_{k,m} = 1$.

Calling (ξ', ξ_n) the Fourier variables corresponding to (x', x_n) we have, for the principal symbol of P_k,

$$p_k(x, \xi) = \sum_{j=0}^{m_k} p_{k,j}(x, \xi')\xi_n^j,$$

which is a polynomial homogeneous of degree m_k in the n variables (ξ', ξ_n).

We introduce

$$p_{k,\varphi}(x, \xi, \tau) := p_k(x, \xi + i\tau\varphi_k'(x)),$$

which is the principal symbol of the operator $e^{\tau\varphi_k} P_k e^{-\tau\varphi_k}$ viewed in the class of (pseudo-)differential operators with a large parameter presented in Section 2.

Setting $\varrho' = (x, \xi', \tau)$ and $\varrho = (\varrho', \xi_n)$, for simplicity we shall write $p_{k,\varphi}(\varrho)$ in place of $p_{k,\varphi}(x, \xi, \tau)$ and often $p_{k,\varphi}(\varrho', \xi_n)$ to emphasize that the symbol is polynomial in ξ_n. Similarly we introduce $t_{k,\varphi}^j(x, \xi, \tau) := t_k^j(x, \xi + i\tau\varphi_k'(x)) = t_{k,\varphi}^j(\varrho) = t_{k,\varphi}^j(\varrho', \xi_n)$.

[3]By abuse of notation, in the new local coordinates, we keep the notation P_k and T_k^j, $j = 1, \dots, m$, $k = 1, 2$, for the operators introduced in the beginning of Section 1.

In the present local coordinate, we have $\theta = x_n$ and thus $\nu_k = (-1)^k dx_n$ (see Section 1.3). For a fixed point $\varrho' = (x, \xi', \tau)$ with $x \in S$, setting $Y = \sum_{j=1}^{n-1} \xi_j dx_j$ and $\omega = (x, Y, \tau)$ this gives

$$\begin{aligned} \tilde{p}_{k,\varphi}(\omega, (-1)^k \xi_n) &= p_k(x, Y + (-1)^k \xi_n \nu_k + i\tau d\varphi_k(x)) \\ &= p_k(x, \xi + i\tau d\varphi_k(x)) \\ &= p_{k,\varphi}(x, \xi, \tau), \end{aligned} \tag{1.17}$$

with $\xi = (\xi', \xi_n)$ and $\tilde{p}_{k,\varphi}$ as defined in (1.9). Similarly, we have

$$\tilde{t}^j_{k,\varphi}(\omega, (-1)^k \xi_n) = t^j_{k,\varphi}(x, \xi), \tag{1.18}$$

with $\tilde{t}^j_{k,\varphi}$ as defined in (1.11).

1.6.2. *A system formulation.* To avoid the $(-1)^k$ terms in the two previous equations and to simplify a large part of the analysis that will follow we shall write the local transmission problem as a system of equation in V_+.

Without any loss of generality we can choose the open neighborhood V to be of the form $V' \times (-\varepsilon, \varepsilon)$ with V' an open set of S and $\varepsilon > 0$.

In $V_- = V' \times (\varepsilon, 0)$ we apply the change of variables $(x', x_n) \mapsto \sigma(x', x_n) = (x', -x_n)$. We denote by P_ℓ and T^j_ℓ the operators obtained from P_1 and T^j_1 through this change of variable. For the principal symbols we have

$$p_\ell(x, \xi) = p_1\big(\sigma(x); \sigma(\xi)\big), \;\; t_\ell(x, \xi) = t_1\big(\sigma(x); \sigma(\xi)\big), \quad \text{for } x_n > 0,$$

using that ${}^t\sigma'(x)^{-1}\xi = (\xi', -\xi_n)$. We also define $\varphi_\ell = \varphi_1 \circ \sigma$ for $x_n > 0$. In V_+ we do not apply any change of variable and for the benefit of readibility we set $P_r = P_2$, $p_r = p_2$, and $\varphi_r = \varphi_2$. The subscripts ℓ and r are chosen to keep in mind that part of the system we shall write comes from the left-hand side of the interface and the second part from the right-hand side.

The transmission problems now reads as the following system

$$\begin{cases} P_\ell u_\ell = f_\ell, \;\; P_r u_r = f_r & \text{in } V_+ = \{x_n > 0\}, \\ T^j_\ell u_\ell + T^j_r u_r = g^j, & \text{in } S = \{x_n = 0\}, \quad j = 1, \dots, m, \end{cases} \tag{1.19}$$

where $u_\ell = u_1 \circ \sigma$, $f_\ell = f_1 \circ \sigma$, $u_r = u_2$, $f_r = f_2$.

We set $P_{r/\ell,\varphi} = e^{\tau\varphi_{r/\ell}} P_{r/\ell} e^{-\tau\varphi_{r/\ell}}$ and $T^j_{r/\ell,\varphi} = e^{\tau\varphi_{r/\ell}} T^j_{r/\ell} e^{-\tau\varphi_{r/\ell}}$. They have for respective principal symbols (in the calculus with a large parameter of Section 2)

$$p_{r/\ell,\varphi}(x, \xi, \tau) = p_{r/\ell}(x, \xi + i\tau d\varphi_{r/\ell}(x)), \quad t^j_{r/\ell,\varphi}(x, \xi, \tau) = t^j_{r/\ell}(x, \xi + i\tau d\varphi_{r/\ell}(x)).$$

We have

$$p_{\ell,\varphi}(x, \xi, \tau) = p_1(\sigma(x), \sigma(\xi + i\tau d\varphi_\ell(x))) = p_1\big(\sigma(x), \sigma(\xi) + i\tau d\varphi_1(\sigma(x))\big)\big)$$

Now, as ${}^t\sigma'(x)^{-1}\nu_1 = \nu_2 = \nu$, for a fixed point $\varrho' = (x, \xi', \tau)$ with $x \in S$, i.e., $x_n = 0$ giving $\sigma(x) = x$, if we set $Y = \sum_{j=1}^{n-1} \xi_j dx_j$ and $\omega = (x, Y, \tau)$ we have, for $\tilde{p}_{1,\varphi}$ as defined in (1.9),

$$\begin{aligned} \tilde{p}_{1,\varphi}(\omega, \xi_n) &= p_1(x, Y - \xi_n \nu + i\tau d\varphi_1(x)) \\ &= p_\ell(x, Y + \xi_n \nu + i\tau d\varphi_\ell(x)) \\ &= p_{\ell,\varphi}(x, \xi, \tau), \end{aligned} \tag{1.20}$$

with $\xi = (\xi', \xi_n)$. Similarly we have

$$\tilde{t}^j_{1,\varphi}(\omega, \xi_n) = t^j_{\ell,\varphi}(x, \xi, \tau). \tag{1.21}$$

Naturally by (1.17)–(1.18) we have

$$\tilde{p}_{2,\varphi}(\omega,\xi_n)=p_{r,\varphi}(x,\xi,\tau),\qquad \tilde{t}^{\,j}_{2,\varphi}(\omega,\xi_n)=t^j_{r,\varphi}(x,\xi,\tau).$$

1.6.3. *Symbol factorizations.* For a fixed point $\varrho'_0=(x_0,\xi'_0,\tau_0)\in\mathbb{S}^*_{\mathsf{T},\tau}(V)$ (see the definition below in Section 1.8) with $x_0\in S$ we denote the roots of $p_{r'_\ell,\varphi}(\varrho'_0,\xi_n)$, viewed as a polynomial in ξ_n, by $\alpha_{r'_\ell,1},\dots,\alpha_{r'_\ell,n_{r'_\ell}}$, with respective multiplicities $\mu_{r'_\ell,1},\dots,\mu_{r'_\ell,n_{r'_\ell}}$ satisfying $\mu_{r'_\ell,1}+\cdots+\mu_{r'_\ell,n_{r'_\ell}}=m_{r'_\ell}$, with $m_\ell=m_1$ and $m_r=m_2$. By [4, Lemma A.2], there exists a conic open neighborhood $\mathscr{U}$ of ϱ'_0 such that

$$p_{r'_\ell,\varphi}(\varrho',\xi_n)=p^+_{r'_\ell,\varphi}(\varrho',\xi_n)\,p^-_{r'_\ell,\varphi}(\varrho',\xi_n)\,p^0_{r'_\ell,\varphi}(\varrho',\xi_n),\quad \varrho'\in\mathscr{U},\ \xi_n\in\mathbb{R},\tag{1.22}$$

with $p^\pm_{r'_\ell,\varphi}$ and $p^0_{r'_\ell,\varphi}$, polynomials in ξ_n of constant degrees in $\mathscr{U}$, smooth and homogeneous; in $\mathscr{U}$ the imaginary parts of the roots of $p^+_{r'_\ell,\varphi}(\varrho',\xi_n)$ (resp. $p^-_{r'_\ell,\varphi}(\varrho',\xi_n)$) are all positive (resp. negative) and we have

$$p^\pm_{r'_\ell,\varphi}(\varrho'_0,\xi_n)=\prod_{\pm\operatorname{Im}\alpha_{r'_\ell,j}>0}(\xi_n-\alpha_{r'_\ell,j})^{\mu_{r'_\ell,j}},\quad p^0_{r'_\ell,\varphi}(\varrho'_0,\xi_n)=\prod_{\operatorname{Im}\alpha_{r'_\ell,j}=0}(\xi_n-\alpha_{r'_\ell,j})^{\mu_{r'_\ell,j}}.$$

The polynomials $p_{r'_\ell,\varphi}$ are thus decomposed into three factors in the neighborhood $\mathscr{U}$ of ϱ'_0. For $p^\pm_{r'_\ell,\varphi}$ the sign of the imaginary part of their roots remain constant equal to $\pm$ respectively; for $p^0_{r'_\ell,\varphi}$ this sign may change and the roots are precisely real at $\varrho'=\varrho'_0$.

We then define the polynomial $\kappa_{r'_\ell,\varphi}(\varrho',\xi_n)$ by

$$\kappa_{r'_\ell,\varphi}(\varrho',\xi_n)=p^+_{r'_\ell,\varphi}(\varrho',\xi_n)\,p^0_{r'_\ell,\varphi}(\varrho',\xi_n).\tag{1.23}$$

As above, for the principal symbols of the conjugated transmission operators $T^j_{r'_\ell,\varphi}$, $j=1,\dots,m$, we write $t^j_{r'_\ell,\varphi}(x,\xi,\tau)=t^j_{r'_\ell,\varphi}(\varrho',\xi_n)$ where $\varrho'=(x,\xi',\tau)$ to emphasize that the symbol is polynomial in ξ_n.

Remark 1.7. Observe that the factorizations in (1.22) depends quite significantly on the point ϱ'_0. They may actually be different even for point ϱ' in the neighborhood $\mathscr{U}$ introduced above. We should rather write something like

$$p_{r'_\ell,\varphi}(\varrho',\xi_n)=p^+_{r'_\ell,\varphi,\varrho'_0}(\varrho',\xi_n)\,p^-_{r'_\ell,\varphi,\varrho'_0}(\varrho',\xi_n)\,p^0_{r'_\ell,\varphi,\varrho'_0}(\varrho',\xi_n),\quad \varrho'\in\mathscr{U},\ \xi_n\in\mathbb{R},$$

in place of (1.22) and set

$$\kappa_{r'_\ell,\varphi,\varrho'_0}(\varrho',\xi_n)=p^+_{r'_\ell,\varphi,\varrho'_0}(\varrho',\xi_n)\,p^0_{r'_\ell,\varphi,\varrho'_0}(\varrho',\xi_n).$$

For $\varrho'_1\in\mathscr{U}$ we may very well have

$$p^+_{r'_\ell,\varphi,\varrho'_0}(\varrho',\xi_n)\neq p^+_{r'_\ell,\varphi,\varrho'_1}(\varrho',\xi_n),\ \text{ or }\ p^-_{r'_\ell,\varphi,\varrho'_0}(\varrho',\xi_n)\neq p^-_{r'_\ell,\varphi,\varrho'_1}(\varrho',\xi_n),$$
$$\text{or }\ p^0_{r'_\ell,\varphi,\varrho'_0}(\varrho',\xi_n)\neq p^0_{r'_\ell,\varphi,\varrho'_1}(\varrho',\xi_n).$$

Yet, we shall see below that the notation in (1.22) is sufficiently clear for our purpose.

Still, if we denote by $M^\pm_{r'_\ell}(\varrho')$ the number of roots (counted with their multiplicities) with postive (resp. negative) imaginary parts of $p_{r'_\ell,\varphi}(\varrho',\xi_n)$ for $\varrho'\in\mathscr{U}$ we may have $M^\pm_{r'_\ell}(\varrho'_0)\neq M^\pm_{r'_\ell}(\varrho')$ for some $\varrho'\in\mathscr{U}$. Note that in such case we have $M^\pm_{r'_\ell}(\varrho'_0)\leq M^\pm_{r'_\ell}(\varrho')$ from the construction of the neighborhood $\mathscr{U}$ given in [4, Lemma A.2]. Arguing as in the proof of [4, Lemma A.2], using the continuity of the roots w.r.t. ϱ' we can in fact prove that for $\varrho'_1\in\mathscr{U}$ there exists a conic neighborhood $\mathscr{U}'\subset\mathscr{U}$ of ϱ'_1 such that

$$\kappa_{r'_\ell,\varphi,\varrho'_0}(\varrho',\xi_n)=h_{r'_\ell}(\varrho',\xi_n)\kappa_{r'_\ell,\varphi,\varrho'_1}(\varrho',\xi_n),\qquad \varrho'\in\mathscr{U}',\tag{1.24}$$

where $h_{r'_\ell}(\varrho',\xi_n)$ is polynomial in ξ_n with coefficients that are smooth w.r.t. $\varrho'\in\mathscr{U}'$.

1.6.4. *The transmission conditions in the local coordinates.* In the present coordinate system (x', x_n) in V_+, a conormal vector ν pointing from Ω_1 to Ω_2 is given by $(0, \dots, 0, \nu_n)$ with $\nu_n > 0$. For the statement of the transmission condition we may choose $\nu = (0, \dots, 0, 1)$. A boundary quadruple $\omega = (x, Y, \nu, \tau)$, with $Y = (\xi', 0)$ can thus be identified with $\varrho' = (x, \xi', \tau)$.

The transmission condition of Definition 1.4 being invariant under change of variables as seen in Section 1.3.3, because of (1.20) and (1.21), we may use the polynomials $p_{r/\ell,\varphi}$ and $t^j_{r/\ell,\varphi}$ to state locally this condition for $\{P_{r/\ell}, T^j_{r/\ell}, \varphi_{r/\ell},\ j = 1, \dots, m\}$ at $\varrho'_0 = (x_0, \xi'_0, \tau_0)$. It reads:

For all pairs of polynomials, $q_{r/\ell}(\xi_n)$, there exist $U_{r/\ell}$, polynomials, and $c_j \in \mathbb{C}$, $j = 1, \dots, m$, such that

$$q_\ell(\xi_n) = \sum_{j=1}^m c_j t^j_{\ell,\varphi}(\varrho', \xi_n) + U_\ell(\xi_n)\kappa_{\ell,\varphi}(\varrho', \xi_n), \tag{1.25}$$

and

$$q_r(\xi_n) = \sum_{j=1}^m c_j t^j_{r,\varphi}(\varrho', \xi_n) + U_r(\xi_n)\kappa_{r,\varphi}(\varrho', \xi_n), \tag{1.26}$$

for $\varrho' = \varrho'_0$.

We set $m^-_{r/\ell} = d^\circ\big(p^-_{r/\ell,\varphi}(\varrho', .)\big)$, that is *independent* of $\varrho' \in \mathscr{U}$, with the open conic neighborhood $\mathscr{U}$ as introduced above, and we let $\kappa_{r/\ell,\varphi}(\varrho', \xi_n)$ be the polynomial function given in (1.23). It takes the form

$$\kappa_{r/\ell,\varphi}(\varrho', \xi_n) = \sum_{i=0}^{m_{r/\ell} - m^-_{r/\ell}} \kappa_{r/\ell,i}(\varrho')\xi_n^i, \quad \varrho' \in \mathscr{U},\ \xi_n \in \mathbb{R},$$

where $\kappa_{r/\ell,i}$ is homogeneous of degree $m_{r/\ell} - m^-_{r/\ell} - i$ w.r.t. (ξ', τ). Similarly we write

$$t^j_{r/\ell,\varphi}(\varrho', \xi_n) = \sum_{i=0}^{\beta^j_{r/\ell}} t^j_{r/\ell,i}(\varrho')\xi_n^i, \quad \varrho' \in \mathscr{U},\ \xi_n \in \mathbb{R}, \quad j = 1, 2,$$

with $\beta^j_\ell = \beta^j_1$ and $\beta^j_r = \beta^j_2$, and where $t^j_{r/\ell,i}$ is homogeneous of degree $\beta^j_{r/\ell} - i$ w.r.t. (ξ', τ). We recall that we have (see (1.3))

$$m_\ell - \beta^j_\ell = m_r - \beta^j_r = m - \beta^j, \quad j = 1, \dots, m. \tag{1.27}$$

Now, we introduce two famillies of polynomial functions, $e^j_{r/\ell}(\varrho', .)$, of degree less than or equal to $m_{r/\ell} - 1$, for $j = 1, \dots, m'_{r/\ell}$, with $m'_{r/\ell} = m + m^-_{r/\ell}$. We recall that $m = (m_\ell + m_r)/2$. We set

$$e^j_{r/\ell}(\varrho', \xi_n) = \begin{cases} t^j_{r/\ell,\varphi}(\varrho', \xi_n) & \text{for } 1 \le j \le m, \\ \xi_n^{j-(m+1)}\kappa_{r/\ell,\varphi}(\varrho', \xi_n) & \text{for } m+1 \le j \le m'_{r/\ell}, \quad \text{if } m'_{r/\ell} > m. \end{cases}$$

Observe that $m'_{r/\ell} > m$ if $m^-_{r/\ell} > 0$. If we write

$$e^j_{r/\ell}(\varrho', \xi_n) = \sum_{i=0}^{m_{r/\ell} - 1} e^j_{r/\ell,i}(\varrho')\xi_n^i,$$

we thus obtain

- for $1 \leq j \leq m$, $e^j_{r/\ell,i} = \begin{cases} t^j_{r/\ell,i} & \text{if } 0 \leq i \leq \beta^j_{r/\ell}, \\ 0 & \text{otherwise.} \end{cases}$
- for $m+1 \leq j \leq m'_{r/\ell}$, $e^j_{r/\ell,i} = \begin{cases} \kappa_{r/\ell,i-j+m+1} & \text{if } j-(m+1) \leq i \leq j-(m+1)+m_{r/\ell}-m^-_{r/\ell}, \\ 0 & \text{otherwise,} \end{cases}$

  if $m'_{r/\ell} > m$.

For $1 \leq j \leq m$, $e^j_{r/\ell,i}$ is homogeneous of degree $\beta^j_{r/\ell} - i$ w.r.t. (ξ', τ). If $m'_{r/\ell} > m$, setting

$$\beta^j_{r/\ell} = j + m_{r/\ell} - (m + m^-_{r/\ell} + 1), \quad j = m+1, \dots, m'_{r/\ell}, \tag{1.28}$$

we see that, for $m+1 \leq j \leq m'_{r/\ell}$, the tangential symbol $e^j_{r/\ell,i}$ is homogeneous of degree $\beta^j_{r/\ell} - i$ w.r.t. (ξ', τ) and the symbol $e^j_{r/\ell}$ is homogeneous of degree $\beta^j_{r/\ell}$ w.r.t. (ξ, τ).

Introducing the matrices

$$\mathcal{T}^1_{r/\ell}(\varrho') = \left(e^j_{r/\ell,i-1}\right)_{\substack{1 \leq i \leq m_{r/\ell} \\ 1 \leq j \leq m}}, \qquad \mathcal{T}^2_{r/\ell}(\varrho') = \left(e^{j+m}_{r/\ell,i-1}\right)_{\substack{1 \leq i \leq m_{r/\ell} \\ 1 \leq j \leq m^-_{r/\ell}}},$$

by Section 1.3.1, we see that the transmission condition of Definition 1.4 for $\left\{P_{r/\ell}, T^j_{r/\ell}, \varphi_{r/\ell},\ j = 1, \dots, m\right\}$ at $\varrho'_0 = (x_0, \xi'_0, \tau_0) \in \mathbb{S}^*_{\mathsf{T},\tau}(V)$, with $x_0 \in S$, also stated in (1.25)–(1.26) with the local setting introduced here, reads as follows

$$\operatorname{rank} \mathcal{T}(\varrho') = m_\ell + m_r = 2m, \qquad \text{with } \mathcal{T}(\varrho') = \begin{pmatrix} \mathcal{T}^1_\ell(\varrho') & \mathcal{T}^2_\ell(\varrho') & 0 \\ \mathcal{T}^1_r(\varrho') & 0 & \mathcal{T}^2_r(\varrho') \end{pmatrix}, \tag{1.29}$$

for $\varrho' = \varrho'_0$. Note that $2m$ is the number of rows in $\mathcal{T}(\varrho')$.

We find again that $m' = m + m^-_\ell + m^-_r \geq 2m$. Moreover, there exists a $2m \times 2m$ sub-matrix $\mathcal{T}_0(\varrho'_0)$ such that $\det \mathcal{T}_0(\varrho'_0) \neq 0$. As the coefficients of $\mathcal{T}^1_{r/\ell}(\varrho')$ and $\mathcal{T}^2_{r/\ell}(\varrho')$ are continuous and homogeneous of degree $\beta^j_{r/\ell} - i + 1$ and $\beta^{j+m}_{r/\ell} - i + 1$ w.r.t. (ξ', τ) respectively, where j is the column number and i is the line number, we then have $\det \mathcal{T}_0(\varrho') \neq 0$ homogenous w.r.t. (ξ', τ). It follows that $\det \mathcal{T}_0(\varrho') \neq 0$ for ϱ' in a small conic neighborhood $\mathscr{V} \subset \mathscr{U}$ of ϱ'_0. Note that the homogeneity of the coefficients is important for $\mathscr{V}$ to be chosen conic since $\det \mathcal{T}_0(\varrho')$ is itself homogeneous w.r.t. (ξ', τ). The rank of $\mathcal{T}(\varrho')$ thus remains equal to $2m$ in $\mathscr{V}$, meaning that condition (1.29) is valid for ϱ' in the whole $\mathscr{V}$.

We have thus reached the following result.

Proposition 1.8. *Let the transmission condition $\left\{P_{r/\ell}, T^j_{r/\ell}, \varphi_{r/\ell},\ j = 1, \dots, m\right\}$ hold at $\varrho'_0 = (x_0, \xi'_0, \tau_0)$. Then we have $m^-_\ell + m^-_r \geq m$. Moreover there exists a conic neighborhood $\mathscr{V}$ of ϱ'_0 such that (1.29) is valid for $\varrho' \in \mathscr{V}$.*

Remark 1.9. Observe that the result of Proposition 1.8 implies the condition (1.25)–(1.26) holds for $\varrho' \in \mathscr{V} \subset \mathscr{U}$, yet with $\kappa_{r/\ell,\varphi}$ defined by the symbol factorizations at ϱ'_0, that is $\kappa_{r/\ell,\varphi} = \kappa_{r/\ell,\varphi,\varrho'_0}$ using the notation of Remark 1.7. Now using (1.24) we see that this implies that the transmission condition also holds at ϱ'_1. We thus see that the transmission condition remains valid in a conic neighborhood of ϱ'_0. However, we shall not use this aspect here. The importance aspect we shall use is the local persistence of condition (1.25)–(1.26) for ϱ' in a conic neighborhood of ϱ'_0 as stated in Proposition 1.8 (of course the two are very related). This explains why we do not use the "more precise" notation of Remark 1.7 throughout the article.

With the locally presistant decomposition of the conjugated opeators $p_{r/\ell,\varphi} = p^-_{r/\ell,\varphi}\kappa_{r/\ell,\varphi}$ in the neighborhood $\mathscr{U}$ of ϱ'_0, the following states roughly the proof strategy we shall adopt:

(1) The factors $p^-_{r/\ell,\varphi}$ associated with roots with negative imaginary parts yields two perfect elliptic estimates at the interface and no transmission condition is needed.
(2) Each factors $\kappa_{r/\ell,\varphi}$ yields an estimate at the interface that involves trace terms. These terms will be estimated via the actions of the transmission operators $T^j_{r/\ell,\varphi}$ by means of to the transmission condition.

Note that we have $2m - m^-_\ell - m^-_r \leq m$ by Proposition 1.8. The number of trace relations available at the interface in (1.19) is precisely m. This indicates that we shall have at hand a sufficiently large number of transmission relation to control the terms originating from the estimate with the factors $\kappa_{r/\ell,\varphi}$ that are of degree $m_{r/\ell} - m^-_{r/\ell}$ whose sum is $m_\ell - m^-_\ell + m_r - m^-_r = 2m - m^-_\ell - m^-_r$.

1.7. **Examples.** We provide several examples that illustrate the generality of the elliptic transmission problems that can be addressed through the results of the present article.

1.7.1. *Second-order elliptic operators.* We start by providing fairly classical cases of transmission problems, where the operators are of second order on both sides of the interface. Consider the operators

$$P_k = \sum_{1\leq i,j\leq n} D_i a^{(k)}_{ij}(x) D_j, \quad k = 1, 2,$$

with real coefficients, assumed to be ellitpic, that is, $a^{(k)}_{ij} = a^{(k)}_{ji}$ and $(a^{(k)}_{ij}(x)) \geq C > 0$ uniformly for $x \in \Omega_k$. Let $x_0 \in S$. As above, local coordinates are chosen so that $\Omega_1 = \{x_n < 0\}$ and $\Omega_2 = \{x_n > 0\}$ in a neighborhood of x_0.

A. A natural transmission problem for second-order elliptic operators. A natural and classical transmission problem can be stated with the following interface operators

$$T^1_k = (-1)^k, \quad T^2_k = (-1)^k \sum_{1\leq j\leq n} a^{(k)}_{nj}(x) D_j, \qquad k = 1, 2.$$

The transmission problem thus reads

$$P_1 u_1 = f_1 \text{ in } \Omega_1, \quad P_2 u_2 = f_2 \text{ in } \Omega_2,$$

and

$$u_{1|S} = u_{2|S} \quad \text{and} \quad \sum_{1\leq j\leq n} a^{(1)}_{nj}(x) D_j u_{1|S} = \sum_{1\leq j\leq n} a^{(2)}_{nj}(x) D_j u_{2|S},$$

that is, we impose, at the interface, the continuity of the solution, as well as that of the normal flux (in the sense of the anisotropic diffusion matrices $(a^{(1)}_{ij})$ and $(a^{(2)}_{ij})$) . This is physically very natural, and, mathematically, it implies that piecewise smooth functions satisfying these two conditions are in the domain of the self-adjoint operator $P = \nabla \cdot A\nabla$, where $A = (a_{ij}(x))$ with $a_{ij}(x) = a^{(k)}_{ij}(x)$ if $x \in \Omega_k$, $k = 1, 2$.

This configuration was treated in [33] following some works on some particular "conformal" cases [3, 34]. Since this example has been extensively studied, it is natural to question if the results of the present article generalize those provided in these references.

We choose a weight function $\varphi(x)$ that is smooth on both sides of S and continuous across S. Then for an interface quadruple $\omega = (x_0, Y, \nu, \tau)$, with here $\nu = (0, \dots, 0, 1)$, $Y = (\xi', 0) = (\xi_1, \dots, \xi_{n-1}, 0)$, we have, for $k = 1, 2$,

$$\tilde{t}^1_{k,\varphi}(\omega, \lambda) = (-1)^k,$$

and

$$\tilde{t}^2_{k,\varphi}(\omega,\lambda) = (-1)^k a^{(k)}_{nn}(x_0)\big((-1)^k\lambda + i\tau\partial_{x_n}\varphi_k(x_0)\big) + (-1)^k \sum_{1\le j\le n-1} a^{(k)}_{nj}(x_0)\big(\xi_j + i\tau\partial_{x_j}\varphi_k(x_0)\big).$$

As $a^{(k)}_{nn} \neq 0$ because of the ellipticity of P_k we see that $\tilde{t}^2_{k,\varphi}(\omega,\lambda)$ are exactly of degree 1 in λ.

Observe that we can write the principal symbol of P_k, $k = 1, 2$, as

$$p_k(x,\xi) = a^{(k)}_{nn}\Big(\big(\xi_n + \sum_{j=1}^{n-1} a^{(k)}_{nj}/a^{(k)}_{nn}\xi_j\big)^2 + b_k(x,\xi')\Big),$$

with the quadratic form

$$b_k(x,\xi') = \big(a^{(k)}_{nn}\big)^{-2} \sum_{i,j=1}^{n-1} \big(a^{(k)}_{ij}a^{(k)}_{nn} - a^{(k)}_{ni}a^{(k)}_{nj}\big)\xi_i\xi_j,$$

which is positive definite in ξ'. We thus find

$$\tilde{p}_{k,\varphi}(\omega,\lambda) = a^{(k)}_{nn}(x_0)\Big(\big((-1)^k\lambda + i\tau\partial_{x_n}\varphi_k(x_0) + \sum_{j=1}^{n-1} a^{(k)}_{nj}(x_0)/a^{(k)}_{nn}(x_0)(\xi_j + i\tau\partial_{x_j}\varphi_k(x_0))\big)^2 + b_k(x_0,\xi' + i\tau d_{x'}\varphi_k(x_0))\Big)$$

In fact, we may write $b_k(x,\xi' + i\tau d_{x'}\varphi_k) = (A_k - iB_k)^2$, where A_k and B_k are functions of x, ξ' and τ, homogeneous of degree one in (ξ',τ), with $A_k \geq 0$, and we thus find

$$\tilde{p}_{k,\varphi}(\omega,\lambda) = a^{(k)}_{nn}\prod_{k'=1,2}\Big((-1)^k\lambda + i\tau\partial_{x_n}\varphi_k(x_0) + \sum_{j=1}^{n-1} a^{(k)}_{nj}/a^{(k)}_{nn}(x_0)(\xi_j + i\tau\partial_{x_j}\varphi_k(x_0)) + (-1)^{k'}(B_k + iA_k)\Big).$$

Writing $\tilde{p}_{k,\varphi}(\omega,\lambda) = a^{(k)}_{nn}(\lambda - \sigma^1_k(\omega))(\lambda - \sigma^2_k(\omega))$, several cases can occur depending on the signs of the imaginary part of the roots σ^j_k, $k = 1, 2$, $j = 1, 2$.

Case 1. Either $\tilde{p}_{1,\varphi}(\omega,\lambda)$ or $\tilde{p}_{2,\varphi}(\omega,\lambda)$ has two roots in $\{\mathrm{Im}\, z < 0\}$. Assume that for instance $\tilde{p}_{1,\varphi}(\omega,\lambda)$ has its two roots in $\{\mathrm{Im}\, z < 0\}$. Then $\kappa_{1,\varphi} = 1$ while $\kappa_{2,\varphi}(\omega,\lambda)$ is of degree 0, 1, or 2. Let then $q_1(\lambda), q_2(\lambda)$ be two polynomial functions.

As $\tilde{t}^1_{2,\varphi}(\omega,\lambda) = 1$, and $\tilde{t}^2_{2,\varphi}(\omega,\lambda)$ is exaclty of degree 1, we may write

$$q_2(\lambda) = c_1\tilde{t}^1_{2,\varphi}(\omega,\lambda) + c_2\tilde{t}^2_{2,\varphi}(\omega,\lambda) + U_2(\lambda)\kappa_{2,\varphi}(\omega,\lambda).$$

by means of a Euclidean division by $\kappa_{2,\varphi}(\omega,\lambda)$, writing the remainder polynomial as a linear combination of $\tilde{t}^1_{2,\varphi}(\omega,\lambda)$ and $\tilde{t}^2_{2,\varphi}(\omega,\lambda)$. We then have

$$q_1(\lambda) = c_1\tilde{t}^1_{1,\varphi}(\omega,\lambda) + c_2\tilde{t}^2_{1,\varphi}(\omega,\lambda) + U_1(\lambda)\kappa_{1,\varphi}(\omega,\lambda),$$

by simply choosing $U_1(\lambda) = q_1(\lambda) - c_1\tilde{t}^1_{1,\varphi}(\omega,\lambda) - c_2\tilde{t}^2_{1,\varphi}(\omega,\lambda)$. The transmission condition of Definition 1.4 thus holds in this case.

Case 2. Each symbol $\tilde{p}_{k,\varphi}(\omega,\lambda)$, $k=1,2$ has only one roots in $\{\operatorname{Im} z<0\}$. In this case, both polynomials $\kappa_{1,\varphi}(\omega,\lambda)$ and $\kappa_{2,\varphi}(\omega,\lambda)$ are of degree 1 in λ. As we chose $A_k\geq 0$, the root of $\tilde{p}_{k,\varphi}(\omega,\lambda)$ with non-negative imaginary part is given by $\sigma_k^+=-\zeta_k/a_{nn}^{(k)}(x_0)+B_k+iA_k$ with

$$\zeta_k=(-1)^k\big(\sum_{j=1}^{n-1}a_{nj}^{(k)}(x_0)(\xi_j+i\tau\partial_{x_j}\varphi_j(x_0))+i\tau a_{nn}^{(k)}(x_0)\partial_{x_n}\varphi_k(x_0)\big).$$

To understand whether the transmission condition holds or not, it is handy to use the matrix $\mathcal{T}$ introduced in (1.29). It is a 4×4 matrix here. Using (1.20) and (1.21), we find

$$\mathcal{T}=\begin{pmatrix}-1 & \zeta_1 & -\sigma_1^+ & 0\\ 0 & a_{nn}^{(1)} & 1 & 0\\ 1 & \zeta_2 & 0 & -\sigma_2^+\\ 0 & a_{nn}^{(2)} & 0 & 1\end{pmatrix}.$$

Observe that the rank of $\mathcal{T}$ is 4 if and only if $\zeta_1+\zeta_2\neq -a_{nn}^{(1)}\sigma_1^+-a_{nn}^{(2)}\sigma_2^+$. This yields the condition

$$a_{nn}^{(1)}(B_1+iA_1)+a_{nn}^{(2)}(B_2+iA_2)\neq 0.$$

As we have $A_1\geq 0$ and $A_2\geq 0$ this condition holds if $A_1>0$ or $A_2>0$. In fact if $A_k=0$ this mean that $\operatorname{Im}\sigma_k^1=\operatorname{Im}\sigma_k^2$ which is exluded here. Hence we have $A_1>0$ and $A_2>0$ and the transmission condition holds in this case.

Case 3. One symbol $\tilde{p}_{k,\varphi}(\omega,\lambda)$ has two roots in $\{\operatorname{Im} z\geq 0\}$ and the second one has at most one root in $\{\operatorname{Im} z<0\}$. Assume, for instance, that $\tilde{p}_{1,\varphi}(\omega,\lambda)$ has two roots in $\{\operatorname{Im} z\geq 0\}$. Then, $\kappa_{1,\varphi}$ is of degree 2. With the assumption on $\tilde{p}_{2,\varphi}(\omega,\lambda)$, then, $\kappa_{1,\varphi}$ is at least of degree 1. In this case, we find that the matrix $\mathcal{T}(\varrho')$ has 4 lines and, at most, 3 columns. It cannot be of rank 4. Hence, the transmission condition cannot hold in this case.

From the three exhaustive cases studied above we thus conclude that the derivation of a Carleman estimate in a neighborhood of the point x_0 can be achieved, according to Theorem 1.6, if one chooses the weight function φ so that Case 3 does not occur.

Using the computation made above we write

$$\tilde{p}_{k,\varphi}(\omega,\lambda)=a_{nn}^{(k)}\prod_{k'=1,2}\Big((-1)^k\lambda+\sum_{j=1}^{n-1}a_{nj}^{(k)}/a_{nn}^{(k)}(x_0)\xi_j+i\tau\gamma_k+(-1)^{k'}(B_k+iA_k)\Big).$$

where

$$\gamma_k=\partial_{x_n}\varphi_k(x_0)+\sum_{j=1}^{n-1}a_{nj}^{(k)}/a_{nn}^{(k)}(x_0)\partial_{x_j}\varphi_k(x_0) \tag{1.30}$$

The values of γ_k are fixed by the choice of the weight function φ. The imaginary parts of the roots are given by

$$\operatorname{Im}\alpha_k^{k'}=-(-1)^k\big(\tau\gamma_k+(-1)^{k'}A_k\big). \tag{1.31}$$

If both $b_k(x,\xi'+i\tau d_{x'}\varphi_k)$, $k=1,2$, are nonpositive real numbers, that is $A_k=0$, in particular if $\xi'=0$ and $\tau>0$, then, the imaginary parts of the roots of $\tilde{p}_{k,\varphi}(\omega,\lambda)$ coincide: $\operatorname{Im}\alpha_k^{k'}=-(-1)^k\tau\gamma_k$, $k'=1,2$. In particular, this requires

$$\gamma_2>0 \text{ if } \gamma_1\geq 0, \qquad \text{and} \qquad \gamma_1<0 \text{ if } \gamma_2\leq 0, \tag{1.32}$$

as, otherwise, we face the occurence of Case 3. The case $\gamma_1 \geq 0$ and $\gamma_2 \leq 0$ is thus excluded.

Let us assume that $\gamma_1 \geq 0$ and $\gamma_2 > 0$. On the one hand, as we have assumed $A_k \geq 0$, the root α_1^2 remains in $\{\operatorname{Im} z \geq 0\}$ and the root α_2^2 remains in $\{\operatorname{Im} z < 0\}$. On the other hand, we have

$$\operatorname{Im} \alpha_1^1 = \tau\gamma_1 - A_1, \qquad \operatorname{Im} \alpha_2^1 = -\tau\gamma_2 + A_2.$$

Hence, Case 3 does not occur if and only if $\operatorname{Im} \alpha_2^1 \geq 0 \ \Rightarrow \ \operatorname{Im} \alpha_1^1 < 0$, that is,

$$0 < \tau\gamma_2 \leq A_2 \ \Rightarrow \ 0 \leq \tau\gamma_1 < A_1.$$

A sufficient condition is then

$$\frac{\gamma_1}{\gamma_2} < \frac{A_1}{A_2} \quad \text{if } A_2 \neq 0. \tag{1.33}$$

Observe that in the case where φ_k are only functions of x_n, then A_1 and A_2 are independent of τ and (1.33) becomes a *necessary and sufficient* condition for the transmission condition to hold.

The case $\gamma_1 < 0$ and $\gamma_2 \leq 0$ leads similarly to the sufficient condition

$$\frac{\gamma_2}{\gamma_1} < \frac{A_2}{A_1} \quad \text{if } A_1 \neq 0.$$

Finally, we consider the case $\gamma_1 < 0$ and $\gamma_2 > 0$. Because of (1.31) we then find that Case 3 cannot occur with this choice of γ_1 and γ_2. This particular choice, is however not very interesting, as it somehow corresponds to an observation[4] of the transmission problem both from Ω_2 and Ω_1. The choice $\gamma_1 \geq 0$ and $\gamma_2 > 0$ correspond to an observation of the transmission problem from Ω_2 only. This is relevant for pratical applications, for instance, in unique continuation problems, as one may want to find uniqueness across an interface, having information on the solution on one side of the interface only. Similarly, the choice $\gamma_1 < 0$ and $\gamma_2 \leq 0$ corresponds to an observation from Ω_1 only.

Remark and open question. Note that with (1.33) we recover the condition stated in [33]. There, in the case $\varphi_k = \varphi_k(x_n)$ it is proven to be sharp for a Carleman estimate to hold. This raises the following question: is the transmission condition presented here necessary and sufficient for the Carleman estimate to hold? Sufficiency is the subject of the present article. Necessity is clear in particular cases as shown in [33] but it is not clear in general. Second-order transmission problems, in the case where φ_k depend on x_n and also x' would be a natural field of investigation, but the question extends to higher order transmission problems.

B. Two "non communicating" Dirichlet problems. If we consider

$$T_1^1 = T_2^1 = 1, \qquad T_1^2 = -T_2^2 = 1,$$

observe, then, that the transmission problem

$$P_1 u_1 = f_1 \text{ in } \Omega_1, \quad P_2 u_2 = f_2 \text{ in } \Omega_2,$$

and

$$T_1^1 u_{1|S} + T_2^1 u_{2|S} = g_1, \quad T_1^2 u_{1|S} + T_2^2 u_{2|S} = g_2,$$

corresponds two having the following two problems

$$P_1 u_1 = f_1, \quad u_{1|S} = (g_1 + g_2)/2,$$

and

$$P_2 u_2 = f_2, \quad u_{2|S} = (g_1 - g_2)/2,$$

[4]This interpretation makes sense in the case $\varphi = \varphi(x_n)$. Then $\gamma_k = \partial_{x_n}\varphi_k$. In Carleman estimates, "observation" region are associated with regions where the weight function is the largest.

that is, two well-posed elliptic equations with *independent* Dirichlet boundary conditions.

Using the notation and symbol computations of the previous example, let us assume that one of the symbols $\tilde{p}_{k,\varphi}(\omega,\lambda)$, say for $k=1$, has two roots in $\{\operatorname{Im} z \geq 0\}$. Then, the matrix $\mathcal{T}$ introduced in (1.29) takes the form

$$\mathcal{T} = \left(\begin{array}{cc:c} 1 & 1 & \multirow{3}{*}{0} \\ 0 & 0 & \\ \hdashline 1 & -1 & \multirow{2}{*}{$\otimes$} \\ 0 & 0 & \end{array}\right)$$

and thus cannot be of rank 4. Hence, to derive a Carleman estimate, according to Theorem 1.6, we need to avoid an occurance of two roots associated with the same symbol $\tilde{p}_{k,\varphi}(\omega,\lambda)$ in $\{\operatorname{Im} z \geq 0\}$. With γ_k, as defined in (1.30), and the imaginary parts of the roots given in (1.31), we note that we then need to impose $\gamma_1 < 0$ and $\gamma_2 > 0$. As explained in the previous example, this corresponds to observing the transmission problem from both sides of the interface S. This is totally sensible here, as the example we consider represents two totally decoupled ellitpic problems: observing from one side of the interface cannot yield any information about the system on the other side. As Theorem 1.6 implies unique continuation properties (see Section 7) we see that it is very natural that the Carleman estimate cannot be derived, unless observations are made on both sides.

1.7.2. *Higher-order elliptic operators.* Here, we consider an example that involves both a second- and a fourth-order elliptic operator. In $\mathbb{R}^2$, we consider the operators $P_k(x,D)$, $k=1,2$, such that in the local coordinates as above, the principal symbols are given by

$$p_1(x,\xi_1,\xi_2) = \xi_2^2 + b_1(x,\xi_1), \qquad p_2(x,\xi_1,\xi_2) = \xi_2^4 + b_2^2(x,\xi_1),$$

where $b_k(x,.)$, $k=1,2$, are two positive definite quadratic forms. We assume that the principal symbols of the transmission operators are given by

$$\begin{aligned} t_1^1(x,\xi_1,\xi_2) &= -1, \quad t_2^1(x,\xi_1,\xi_2) = 1, \\ t_1^2(x,\xi_1,\xi_2) &= -\xi_2, \quad t_2^2(x,\xi_1,\xi_2) = \xi_2^3, \\ t_1^3(x,\xi_1,\xi_2) &= 0, \quad t_2^3(x,\xi_1,\xi_2) = \xi_2^2. \end{aligned}$$

We choose a weight function $\varphi(x) = \varphi(x_2)$ that is smooth on both sides of S and continous across S. Then for an interface quadruple $\omega = (x_0, Y, \nu, \tau)$, with $\nu = (0,1)$, $Y = (\xi_1, 0)$, we have for $\mu_k = -i\tau\varphi_k'$,

$$\begin{aligned} \tilde{t}_{1,\varphi}^1(\omega,\lambda) &= -1, \quad \tilde{t}_{2,\varphi}^1(\omega,\lambda) = 1, \\ \tilde{t}_{1,\varphi}^2(\omega,\lambda) &= (\lambda+\mu_1), \quad \tilde{t}_{2,\varphi}^2(\omega,\lambda) = (\lambda-\mu_2)^3, \\ \tilde{t}_{1,\varphi}^3(\omega,\lambda) &= 0, \quad \tilde{t}_{2,\varphi}^3(\omega,\lambda) = (\lambda-\mu_2)^2. \end{aligned}$$

For smooth function φ_k, $k=1,2$, such that φ_k is only a function of x_2, we assume that $\partial_{x_2}\varphi_1(x_0) > 0$ and $\partial_{x_2}\varphi_2(x_0) > 0$. We obtain

$$\tilde{p}_{1,\varphi}(\omega,\lambda) = (\lambda-\alpha_1)(\lambda-\alpha_2),$$

where

$$\alpha_1 = i\tau\partial_{x_2}\varphi_1(x_0) + i\sqrt{b_1(x_0,\xi_1)}, \quad \alpha_2 = i\tau\partial_{x_2}\varphi_1(x_0) - i\sqrt{b_1(x_0,\xi_1)}.$$

We thus have $\operatorname{Im}\alpha_1 > 0$. The sign of $\operatorname{Im}\alpha_2$ may however vary. We also have

$$\tilde{p}_{2,\varphi}(\omega,\lambda) = \prod_{j=1}^{4}(\lambda-\beta_j)$$

where

$$\beta_j = -i\tau\partial_{x_2}\varphi_2(x_0) - e^{i\pi(2j-1)/4}\sqrt{b_2(x_0,\xi_1)}, \quad j = 1,2,3,4.$$

As $\partial_{x_2}\varphi_2(x_0) > 0$, this forbid all the roots to be in the upper complex half plane. Then $\operatorname{Im}\beta_1 < 0$ and $\operatorname{Im}\beta_2 < 0$. Yet, the signs of $\operatorname{Im}\beta_3$ and $\operatorname{Im}\beta_4$ are equal and may vary. We have

$$\operatorname{Im}\beta_3 = \operatorname{Im}\beta_4 = -\tau\partial_{x_2}\varphi_2(x_0) + \sqrt{b_2(x_0,\xi_1)/2}.$$

According to Section 1.6.4, using (1.20) and (1.21), we have,

$$\mathcal{T}_\ell^1 = \begin{pmatrix} -1 & \mu_1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathcal{T}_r^1 = \begin{pmatrix} 1 & -\mu_2^3 & \mu_2^2 \\ 0 & 3\mu_2^2 & -2\mu_2 \\ 0 & -3\mu_2 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Case 1. $\operatorname{Im}\boldsymbol{\alpha_2} < 0$ and $\operatorname{Im}\boldsymbol{\beta_3} = \operatorname{Im}\boldsymbol{\beta_4} < 0$. In this case, we have $\tilde\kappa_{1,\varphi}(\omega,\lambda) = (\lambda-\alpha_1)$ and $\tilde\kappa_{2,\varphi}(\omega,\lambda) = 1$. We then have

$$\mathcal{T}_\ell^2 = \begin{pmatrix} -\alpha_1 \\ 1 \end{pmatrix}$$

and $\mathcal{T}_r^2 = \mathrm{Id}_4$. Recalling the form of $\mathcal{T}$ in (1.29), we have

$$\mathcal{T} = \left(\begin{array}{c:c:c} \mathcal{T}_\ell^1 & \begin{matrix} -\alpha_1 \\ 1 \end{matrix} & 0 \\ \hdashline \mathcal{T}_r^1 & 0 & \mathrm{Id}_4 \end{array}\right)$$

whose rank is 6 as $\operatorname{rank}\mathcal{T}_\ell^1 = 2$. Hence, the transmission condition holds in this case, by its formulation given in (1.29).

Case 2. $\operatorname{Im}\boldsymbol{\alpha_2} \geq 0$ and $\operatorname{Im}\boldsymbol{\beta_3} = \operatorname{Im}\boldsymbol{\beta_4} < 0$. In such case, we have $\tilde\kappa_{1,\varphi}(\omega,\lambda) = (\lambda-\alpha_1)(\lambda-\alpha_2)$ and $\tilde\kappa_{2,\varphi}(\omega,\lambda) = 1$. As $m_\ell^- = 0$ then no matrix $\mathcal{T}_\ell^2$ enters in the composition of $\mathcal{T}$. Still, this matrix has the same form as in Case 1 with the fourth column removed and, in this case, the rank is 6 implying that the transmission condition holds in this case.

Case 3. $\operatorname{Im}\boldsymbol{\alpha_2} < 0$ and $\operatorname{Im}\boldsymbol{\beta_3} = \operatorname{Im}\boldsymbol{\beta_4} \geq 0$. In such case, we have $\tilde\kappa_{1,\varphi}(\omega,\lambda) = (\lambda-\alpha_1)$ and $\tilde\kappa_{2,\varphi}(\omega,\lambda) = (\lambda-\beta_3)(\lambda-\beta_4)$. We thus have

$$\mathcal{T}_\ell^2 = \begin{pmatrix} -\alpha_1 \\ 1 \end{pmatrix}, \quad \mathcal{T}_r^2 = \begin{pmatrix} \beta_3\beta_4 & 0 \\ -(\beta_3+\beta_4) & \beta_3\beta_4 \\ 1 & -(\beta_3+\beta_4) \\ 0 & 1 \end{pmatrix}.$$

Thus, the matrix $\mathcal{T}$ reads

$$\mathcal{T} = \begin{pmatrix} -1 & \mu_1 & 0 & -\alpha_1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -\mu_2^3 & \mu_2^2 & 0 & \beta_3\beta_4 & 0 \\ 0 & 3\mu_2^2 & -2\mu_2 & 0 & -(\beta_3+\beta_4) & \beta_3\beta_4 \\ 0 & -3\mu_2 & 1 & 0 & 1 & -(\beta_3+\beta_4) \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Computing its determinant we find $\det(\mathcal{T}) = b_2(x_0,\xi_1)^2$ that does not vanish as here $\xi_1 \neq 0$. In fact, $\xi_1 = 0$ yields $\operatorname{Im}\alpha_2 = \tau\partial_{x_2}\varphi_1(x_0) > 0$ in contradiction with the assumption $\operatorname{Im}\alpha_2 < 0$ made here.

Case 4. $\operatorname{Im}\boldsymbol{\alpha_2} \geq 0$ and $\operatorname{Im}\boldsymbol{\beta_3} = \operatorname{Im}\boldsymbol{\beta_4} \geq 0$. In this case we have $\tilde{\kappa}_{1,\varphi}(\omega,\lambda) = (\lambda-\alpha_1)(\lambda-\alpha_2)$ and $\tilde{\kappa}_{2,\varphi}(\omega,\lambda) = (\lambda-\alpha_3)(\lambda-\alpha_4)$. In such case the matrix $\mathcal{T}$ is a 6×5 matrix. Its rank cannot be 6. The transmission condition cannot hold in this case.

The four exhaustive cases studied above reveal that the weight function φ needs to be chosen so that Case 4 does not occur. Hence, the following condition needs to be fulfilled:

$$\operatorname{Im}\beta_3 = \operatorname{Im}\beta_4 \geq 0 \quad \Rightarrow \quad \operatorname{Im}\alpha_2 < 0.$$

Recalling the forms of the roots derived above this reads

$$0 < \tau\varphi_1'(x_0) \leq \sqrt{b_2(x_0,\xi_1)/2} \quad \Rightarrow \quad 0 < \tau\varphi_2'(x_0) < \sqrt{b_1(x_0,\xi_1)}.$$

A necessary and sufficient condition is then

$$\frac{\varphi_1'(x_0)}{\varphi_2'(x_0)} < \sqrt{\frac{2b_1(x_0,\xi_1)}{b_2(x_0,\xi_1)}}.$$

Since $b_1(x,\xi_1)/b_2(x,\xi_1)$ is bounded from below, for any ξ_1, locally in x, we see that this yields a precise condition on the weight function φ. The condition prescribes a minimal relative jump of the normal derivative of the weight function across the interface (going from $\{x_2<0\}$ to $\{x_2>0\}$). Note that one can also provide a sufficient condition in the case φ depends also on the x_1 variable, as in Example 1.7.1-A.

1.8. **Notation.** If $V \subset \overline{\mathbb{R}}^n_+$ we denote the semi-classical unit half cosphere bundle over V (in the cotangential direction ξ') by

$$\mathbb{S}^*_{\mathsf{T},\tau}(V) = \{(x,\xi',\tau);\ x\in V,\ \xi'\in\mathbb{R}^{n-1},\ \tau\in\mathbb{R}_+,\ |\xi'|^2+\tau^2=1\}.$$

The canonical inner product in $\mathbb{C}^m$ is denoted by $(\mathbf{z},\mathbf{z}')_{\mathbb{C}^m} = \sum_{j=0}^{m-1} z_j\overline{z'}_j$, for $\mathbf{z} = (z_0,\dots,z_{m-1}), \mathbf{z}' = (z_0',\dots,z_{m-1}') \in \mathbb{C}^m$. The associated norm will be denoted $|\mathbf{z}|^2_{\mathbb{C}^m} = \sum_{j=0}^{m-1}|z_j|^2$.

We shall use some spaces of smooth functions in the closed half space. We set

$$\mathscr{S}(\overline{\mathbb{R}}^n_+) = \{u_{|\overline{\mathbb{R}}^n_+};\ u\in\mathscr{S}(\mathbb{R}^n)\}.$$

For two $u,v\in\mathscr{S}(\overline{\mathbb{R}}^n_+)$ we set

$$(u,v)_+ = (u,v)_{L^2(\mathbb{R}^n_+)} \qquad \left(u_{|x_n=0^+}, v_{|x_n=0^+}\right)_\partial = \left(u_{|x_n=0^+}, v_{|x_n=0^+}\right)_{L^2(\mathbb{R}^{n-1})}.$$

We also set

$$\|u\|_+ = \|u\|_{L^2(\mathbb{R}^n_+)} \qquad \left|u_{|x_n=0^+}\right|_\partial = \left|u_{|x_n=0^+}\right|_{L^2(\mathbb{R}^{n-1})}.$$

In this article, when the constant C is used, it refers to a constant that is independent of the large parameter τ. Its value may however change from one line to another. If we want to keep track of the value of a constant we shall use another letter.

In what follows, for concision, we shall sometimes use the notation $\lesssim$ for $\leq C$, with a constant $C > 0$. We shall write $a \asymp b$ to denote $a \lesssim b \lesssim a$.

2. Pseudo-differential operators with a large parameter

Parameter-dependent pseudo-differential operators have proven to be important tools for the derivation of Carleman estimates. The general aim is to obtain a pseudo-differential calculus with a large parameter, and then to derive estimates with constants that are independent of the parameter. Often such a pseudo-differential calculus is referred to as a semi-classical calculus.

2.1. **Classes of symbols.** We first introduce symbols that depend on a parameter.

Definition 2.1. *Let $a(\varrho) \in \mathscr{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $\varrho = (x, \xi, \tau)$, with τ as a parameter in $[\tau_{\min}, +\infty)$, $\tau_{\min} > 0$, and $m \in \mathbb{R}$, be such that for all multi-indices $\alpha, \beta \in \mathbb{N}^n$ we have*

$$\left|\partial_x^\alpha \partial_\xi^\beta a(\varrho)\right| \leq C_{\alpha,\beta} \lambda^{m-|\beta|}, \quad x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n,\ \tau \in [\tau_{\min}, +\infty), \tag{2.1}$$

where $\lambda = |(\xi, \tau)| = \left(|\xi|^2 + \tau^2\right)^{\frac{1}{2}}$. Thus differentiation with respect to ξ improves the decay in ξ and τ simultaneously. We write $a \in S^m_\tau(\mathbb{R}^n \times \mathbb{R}^n)$ or simply S^m_τ. For $a \in S^m_\tau$ we denote by $\sigma(a)$ its principal part, that is, its equivalence class in S^m_τ / S^{m-1}_τ.

We also introduce tangential symbols. Let $a(\varrho') \in \mathscr{C}^\infty(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1})$, $\varrho' = (x, \xi', \tau)$, with τ as a parameter in $[\tau_{\min}, +\infty)$, $\tau_{\min} > 0$, and $m \in \mathbb{R}$, be such that for all multi-indices $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^{n-1}$ we have

$$\left|\partial_x^\alpha \partial_{\xi'}^\beta a(\varrho')\right| \leq C_{\alpha,\beta} \lambda_{\mathsf{T}}^{m-|\beta|}, \quad x \in \overline{\mathbb{R}}^n_+,\ \xi' \in \mathbb{R}^{n-1},\ \tau \in [\tau_{\min}, +\infty),$$

where $\lambda_{\mathsf{T}} = |(\xi', \tau)| = \left(|\xi'|^2 + \tau^2\right)^{\frac{1}{2}}$. We write $a \in S^m_{\mathsf{T},\tau}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1})$ or simply $S^m_{\mathsf{T},\tau}$. For $a \in S^m_{\mathsf{T},\tau}$ we denote by $\sigma(a)$ its principal part, that is, its equivalence class in $S^m_{\mathsf{T},\tau} / S^{m-1}_{\mathsf{T},\tau}$.

We also introduce symbol classes that behave polynomially in the ξ_n variable. Let $a(\varrho) \in \mathscr{C}^\infty(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)$, with τ as a parameter in $[\tau_{\min}, +\infty)$, $\tau_{\min} > 0$, and $m \in \mathbb{N}$ and $r \in \mathbb{R}$, be such that

$$a(\varrho) = \sum_{j=0}^m a_j(\varrho') \xi_n^j, \quad a_j \in S^{m-j+r}_{\mathsf{T},\tau}, \quad \varrho = (\varrho', \xi_n),\ \varrho' = (x, \xi', \tau),$$

with $x \in \overline{\mathbb{R}}^n_+$, $\xi \in \mathbb{R}^n$, $\tau \geq \tau_{\min}$, and $\xi_n \in \mathbb{R}$. We write $a(\varrho) \in S^{m,r}_\tau(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)$ or simply $S^{m,r}_\tau$.

Note that we have $S^{m,r}_\tau \subset S^{m+m',r-m'}_\tau$, if $m, m' \in \mathbb{N}$ and $r \in \mathbb{R}$. We shall call the principal symbol of a the symbol

$$\sigma(a)(\varrho) = \sum_{j=0}^m \sigma(a_j)(\varrho') \xi_n^j,$$

which is a representative of the class of a in $S^{m,r}_\tau / S^{m,r-1}_\tau$.

Note that $S^{m,r}_\tau \not\subset S^{m+r}_\tau$. For example consider $a(x, \xi, \tau) = |(\xi', \tau)| \xi_n$ for $|(\xi', \tau)| \geq 1$. We have $a \in S^{2,0}_\tau \cap S^{1,1}_\tau$ and yet $a \notin S^2_\tau$. In fact observe that differentiating with respect to ξ' yields

$$|\partial_{\xi'}^\alpha a(x, \xi, \tau)| \leq C_\alpha |(\xi', \tau)|^{1-|\alpha|} |\xi_n|.$$

An estimate of the form of (2.1) is however not achieved for $|\alpha| \geq 2$. A microlocalization is required to repair this flaw and to use the two different symbol classes in a pseudo-differential calculus (See [21, Theorem 18.1.35]).

Finally, we define the corresponding spaces of poly-homogeneous symbols. Such symbols are often referred to as classical symbols; they are characterized by an asymptotic expansion where each term is positively homogeneous with respect to (ξ,τ) (resp. (ξ',τ)):

Definition 2.2. *We shall say $a\in S^m_{\tau,\mathrm{cl}}(\mathbb{R}^n\times\mathbb{R}^n)$ or simply $S^m_{\tau,\mathrm{cl}}$ (resp. $S^m_{\mathsf{T},\tau,\mathrm{cl}}(\overline{\mathbb{R}}^n_+\times\mathbb{R}^{n-1})$ or simply $S^m_{\mathsf{T},\tau,\mathrm{cl}}$) if there exists $a^{(j)}\in S^{m-j}_\tau$ (resp. $S^{m-j}_{\mathsf{T},\tau}$), homogeneous of degree $m-j$ in (ξ,τ) for $|(\xi,\tau)|\geq r_0$, (resp. (ξ',τ) for $|(\xi',\tau)|\geq r_0$), with $r_0\geq 0$, such that*

$$a\sim\sum_{j\geq 0}a^{(j)},\quad \text{in the sense that}\quad a-\sum_{j=0}^{N}a^{(j)}\in S^{m-N-1}_\tau\ (\text{resp. } S^{m-N-1}_{\mathsf{T},\tau}). \tag{2.2}$$

A representative of the principal part is then given by the first term in the expansion.

Finally for $m\in\mathbb{N}$ and $r\in\mathbb{R}$, we shall say that $a(\varrho)\in S^{m,r}_{\tau,\mathrm{cl}}(\overline{\mathbb{R}}^n_+\times\mathbb{R}^n)$ or simply $S^{m,r}_{\tau,\mathrm{cl}}$, if

$$a(\varrho)=\sum_{j=0}^{m}a_j(\varrho')\xi_n^j,\quad \text{with } a_j\in S^{m-j+r}_{\mathsf{T},\tau,\mathrm{cl}},\ \ \varrho=(\varrho',\xi_n).$$

The principal part is given by $\sum_{j=0}^m\sigma(a_j)(\varrho')\xi_n^j$ and is homogeneous of degree m in (ξ,τ).

2.2. **Classes of semi-classical pseudo-differential operators.** For $a\in S^m_\tau(\mathbb{R}^n\times\mathbb{R}^n)$ (resp. $S^m_{\tau,\mathrm{cl}}(\mathbb{R}^n\times\mathbb{R}^n)$) we define the following pseudo-differential operator in $\mathbb{R}^n$:

$$a(x,D,\tau)u(x)=\mathrm{Op}(a)u(x)=(2\pi)^{-n}\int\limits_{\mathbb{R}^n}e^{i(x,\xi)}a(x,\xi,\tau)\hat{u}(\xi)\,d\xi,\qquad u\in\mathscr{S}(\mathbb{R}^n), \tag{2.3}$$

where $\hat{u}$ is the Fourier transform of u. In the sense of oscillatory integrals we have

$$a(x,D,\tau)u(x)=\mathrm{Op}(a)u(x)=(2\pi)^{-n}\iint\limits_{\mathbb{R}^{2n}}e^{i(x-y,\xi)}a(x,\xi,\tau)u(y)\,d\xi\,dy.$$

We write $\mathrm{Op}(a)\in\Psi^m_\tau(\mathbb{R}^n)$ or simply $\Psi^m_{\tau,\mathrm{cl}}$ (resp. $\Psi^m_\tau(\mathbb{R}^n)$ or simply $\Psi^m_{\tau,\mathrm{cl}}$). Here D denotes D_x. The principal symbol of $\mathrm{Op}(a)$ is $\sigma(\mathrm{Op}(a))=\sigma(a)\in S^m_\tau/S^{m-1}_\tau$ (resp. $S^m_{\tau,\mathrm{cl}}/S^{m-1}_{\tau,\mathrm{cl}}$).

Tangential operators are defined similarly. For $a\in S^m_{\mathsf{T},\tau}(\overline{\mathbb{R}}^n_+\times\mathbb{R}^n)$ (resp. $S^m_{\mathsf{T},\tau,\mathrm{cl}}(\overline{\mathbb{R}}^n_+\times\mathbb{R}^n)$) we set

$$a(x,D',\tau)u(x)=\mathrm{Op}(a)u(x)=(2\pi)^{-(n-1)}\iint\limits_{\mathbb{R}^{2n-2}}e^{i(x'-y',\xi')}a(x,\xi',\tau)u(y',x_n)\,d\xi'\,dy', \tag{2.4}$$

for $u\in\mathscr{S}(\overline{\mathbb{R}}^n_+)$, where $x\in\overline{\mathbb{R}}^n_+$. Here D' denotes $D_{x'}$. We write $A=\mathrm{Op}(a)\in\Psi^m_{\mathsf{T},\tau}(\overline{\mathbb{R}}^n_+)$ or simply $\Psi^m_{\mathsf{T},\tau}$ (resp. $\Psi^m_{\mathsf{T},\tau,\mathrm{cl}}(\overline{\mathbb{R}}^n_+)$ or simply $\Psi^m_{\mathsf{T},\tau,\mathrm{cl}}$). The principal symbol of $A=\mathrm{Op}(a)$ is $\sigma(A)=\sigma(a)\in S^m_{\mathsf{T},\tau}/S^{m-1}_{\mathsf{T},\tau}$ (resp. $S^m_{\mathsf{T},\tau,\mathrm{cl}}/S^{m-1}_{\mathsf{T},\tau,\mathrm{cl}}$).

Finally for $m\in\mathbb{N}$, $r\in\mathbb{R}$, and $a\in S^{m,r}_\tau$ (resp. $S^{m,r}_{\tau,\mathrm{cl}}$) with

$$a(\varrho)=\sum_{j=0}^{m}a_j(\varrho')\xi_n^j,\quad a_j\in S^{m-j+r}_{\mathsf{T},\tau}(\text{resp. } S^{m-j+r}_{\mathsf{T},\tau,\mathrm{cl}}),\ \ \varrho=(\varrho',\xi_n),$$

we set

$$a(x,D,\tau)=\mathrm{Op}(a)=\sum_{j=0}^{m}a_j(x,D',\tau)D_n^j,$$

and we write $A=\mathrm{Op}(a)\in\Psi^{m,r}_\tau(\mathbb{R}^n_+)$ or simply $\Psi^{m,r}_\tau$ (resp. $\Psi^{m,r}_{\tau,\mathrm{cl}}(\mathbb{R}^n_+)$ or simply $\Psi^{m,r}_{\tau,\mathrm{cl}}$). The principal symbol of A is $\sigma(A)(\varrho)=\sigma(a)(\varrho)=\sum_{j=0}^m\sigma(a_j)(\varrho')\xi_n^j$ in $S^{m,r}_\tau/S^{m,r-1}_\tau$ (resp. $S^{m,r}_{\tau,\mathrm{cl}}/S^{m,r-1}_{\tau,\mathrm{cl}}$).

We provide some basic calculus rules in the case of tangential operators.

Proposition 2.3 (composition)**.** *Let $a \in S^m_{\mathsf{T},\tau}$ (resp. $S^m_{\mathsf{T},\tau,\mathrm{cl}}$) and $b \in S^{m'}_{\mathsf{T},\tau}$(resp. $S^{m'}_{\mathsf{T},\tau,\mathrm{cl}}$) be two tangential symbols. Then $\mathrm{Op}(a)\,\mathrm{Op}(b) = \mathrm{Op}(c) \in \Psi^{m+m'}_{\mathsf{T},\tau}$ (resp. $\Psi^{m+m'}_{\mathsf{T},\tau,\mathrm{cl}}$) with $c \in S^{m+m'}_{\mathsf{T},\tau}$ (resp. $S^{m+m'}_{\mathsf{T},\tau,\mathrm{cl}}$) defined by the (oscillatory) integral:*

$$\begin{aligned} c(\varrho') = (a\,\#\,b)(\varrho') &= (2\pi)^{-(n-1)} \iint e^{-i(y',\eta')} a(x,\xi'+\eta',\tau)\, b(x'+y',x_n,\xi',\tau)\, dy'\, d\eta' \\ &= \sum_{|\alpha|<N} \frac{(-i)^{|\alpha|}}{\alpha!} \partial^\alpha_{\xi'} a(\varrho')\, \partial^\alpha_{x'} b(\varrho') + r_N, \end{aligned}$$

where $r_N \in S^{m+m'-N}_{\mathsf{T},\tau}$ (resp. $S^{m+m'-N}_{\mathsf{T},\tau,\mathrm{cl}}$) is given by

$$r_N = \frac{(-i)^N}{(2\pi)^{(n-1)}} \sum_{|\alpha|=N} \int_0^1 \frac{N(1-s)^{N-1}}{\alpha!} \iint e^{-i(y',\eta')} \partial^\alpha_{\xi'} a(x,\xi'+\eta',\tau) \partial^\alpha_{x'} b(x'+sy',x_n,\xi',\tau)\, dy' d\eta' ds.$$

Proposition 2.4 (formal adjoint)**.** *Let $a \in S^m_{\mathsf{T},\tau}$ (resp. $S^m_{\mathsf{T},\tau,\mathrm{cl}}$). There exists $a^* \in S^m_{\mathsf{T},\tau}$ (resp. $S^m_{\mathsf{T},\tau,\mathrm{cl}}$) such that*

$$(\mathrm{Op}(a)u, v)_+ = (u, \mathrm{Op}(a^*)v)_+\,, \qquad u, v \in \mathscr{S}(\overline{\mathbb{R}}^n_+).$$

and a^* is given be the following asymptotic expansion

$$\begin{aligned} a^*(\varrho') &= (2\pi)^{-(n-1)} \iint e^{-i(y',\eta')} \overline{a}(x'+y',x_n,\xi'+\eta',\tau)\, dy'\, d\eta' \\ &= \sum_{|\alpha|<N} \frac{(-i)^{|\alpha|}}{\alpha!} \partial^\alpha_{\xi'} \partial^\alpha_{x'} \overline{a}(\varrho') + r_N, \quad r_N \in S^{m-N}_{\mathsf{T},\tau} (\textit{resp. } S^{m-N}_{\mathsf{T},\tau,\mathrm{cl}}), \end{aligned}$$

where

$$r_N = \frac{(-i)^N}{(2\pi)^{(n-1)}} \sum_{|\alpha|=N} \int_0^1 \frac{N(1-s)^{N-1}}{\alpha!} \iint e^{-i(y',\eta')} \partial^\alpha_{\xi'} \partial^\alpha_{x'} \overline{a}(x'+sy',x_n,\xi'+\eta',\tau)\, dy' d\eta' ds.$$

We denote $\mathrm{Op}(a)^* = \mathrm{Op}(a^*)$. We refer to $\mathrm{Op}(a)^*$ as to the formal adjoint of $\mathrm{Op}(a)$.

A consequence of the previous calculus results is the following proposition.

Proposition 2.5. *Let $a(\varrho') \in S^m_{\mathsf{T},\tau,\mathrm{cl}}$ (resp. $S^m_{\mathsf{T},\tau,\mathrm{cl}}$) and $b(\varrho') \in S^{m'}_{\mathsf{T},\tau}$ (resp. $S^{m'}_{\mathsf{T},\tau,\mathrm{cl}}$), with $m, m' \in \mathbb{R}$. Define $h(\varrho') = D_{x'}(b\partial_{\xi'}\overline{a})(\varrho') \in S^{m+m'-1}_\tau$. Then we have*

$$\mathrm{Op}(a)^*\,\mathrm{Op}(b) - \mathrm{Op}(\overline{a}b + h) \in \Psi^{m+m'-2}_{\mathsf{T},\tau}\ (\textit{resp. } \Psi^{m+m'-2}_{\mathsf{T},\tau,\mathrm{cl}}),$$

or equivalently $a^*\#b - \overline{a}b - h \in S^{m+m'-2}_{\mathsf{T},\tau}$ (resp. $S^{m+m'-2}_{\mathsf{T},\tau,\mathrm{cl}}$).

For semi-classical operators in the half space with symbols that are polynomial in ξ_n we also provide a notion of formal adjoint.

Definition 2.6. *Let $b \in S^{m,r}_\tau$ (resp. $S^{m,r}_{\tau,\mathrm{cl}}$), with*

$$b(x,D,\tau) = \sum_{j=0}^m b_j(x,D',\tau) D_n^j, \qquad b_j \in S^{m+r-j}_{\mathsf{T},\tau}\ (\textit{resp. } S^{m+r-j}_{\mathsf{T},\tau,\mathrm{cl}}).$$

We set

$$b(x,D,\tau)^* = \sum_{j=0}^m D_n^j b_j(x,D',\tau)^*.$$

In other words, in this definition we ignore the possible occurrence of boundary terms when performing the operator transposition.

Note that for $a \in S^m_{\mathsf{T},\tau,\mathrm{cl}}$ we have $[D_n, \mathrm{Op}(a)] = \mathrm{Op}(D_n a) \in \Psi^m_{\mathsf{T},\tau,\mathrm{cl}}$ and more generally, for $j \geq 1$, we have

$$\big[D_n^j, \mathrm{Op}(a)\big] = \sum_{k=0}^{j-1} \mathrm{Op}(\alpha_k) D_n^k, \qquad \alpha_k \in S^m_{\mathsf{T},\tau,\mathrm{cl}},$$

where the symbols α_k involve various derivatives of a in the x_n-direction. As an application we see that if we consider $a_j \in S^{m-j+r}_{\mathsf{T},\tau,\mathrm{cl}}$ then we have

$$\sum_{j=0}^{m} D_n^j a_j(x, D', \tau) = \sum_{j=0}^{m} \tilde{a}_j(x, D', \tau) D_n^j,$$

where $\tilde{a}_j \in S^{m-j+r}_{\mathsf{T},\tau,\mathrm{cl}}$ and its principal part satisfies $\sigma(\tilde{a}_j) \equiv a_j$ in $S^{m-j+r}_{\mathsf{T},\tau}/S^{m-j+r-1}_{\mathsf{T},\tau}$. Hence

$$\sigma\Big(\sum_{j=0}^{m} D_n^j a_j(x, D', \tau)\Big) = \sum_{j=0}^{m} a_j(x, \xi', \tau)\xi_n^j \mod S^{m,r-1}_{\tau}.$$

From the calculus rules given above for the tangential operators and the above observation we have the following results on the principal symbols.

Proposition 2.7. *Let $a \in S^{m,r}_{\tau}$ (resp. $S^{m,r}_{\tau,\mathrm{cl}}$) and $b \in S^{m',r'}_{\tau}$ (resp. $S^{m',r'}_{\tau,\mathrm{cl}}$) with*

$$a(\varrho) = \sum_{j=0}^{m} a_j(\varrho')\xi_n^j, \quad b(\varrho) = \sum_{j=0}^{m'} b_j(\varrho')\xi_n^j, \quad \varrho = (\varrho', \xi_n),\ \varrho' = (x, \xi', \tau).$$

(1) *We have $a(x, D, \tau)^* \in \Psi^{m,r}_{\tau}$ (resp. $\Psi^{m,r}_{\tau,\mathrm{cl}}$) and*

$$\sigma\big(a(x, D, \tau)^*\big) \equiv \sum_{j=0}^{m} \overline{a}_j(\varrho')\xi_n^j \in S^{m,r}_{\tau}/S^{m,r-1}_{\tau} \text{ (resp. } S^{m,r}_{\tau,\mathrm{cl}}/S^{m,r-1}_{\tau,\mathrm{cl}}).$$

Moreover we have $\mathrm{Op}(a)^* - \mathrm{Op}(\overline{a}) \in \Psi^{m,r-1}_{\tau}$ (resp. $\Psi^{m,r-1}_{\tau,\mathrm{cl}}$).

(2) *$a(x, D, \tau)b(x, D, \tau) \in \Psi^{m+m',r+r'}_{\tau}$ (resp. $\Psi^{m+m',r+r'}_{\tau,\mathrm{cl}}$) and*

$$\sigma\big(a(x, D, \tau)b(x, D, \tau)\big) \equiv \sum_{\substack{0\leq j\leq m\\ 0\leq k\leq m'}} a_j(\varrho')b_k(\varrho')\xi_n^{j+k} \in S^{m+m',r+r'}_{\tau}/S^{m+m',r+r'-1}_{\tau}$$

$$\text{(resp. } S^{m+m',r+r'}_{\tau,\mathrm{cl}}/S^{m+m',r+r'-1}_{\tau,\mathrm{cl}}).$$

We have $\mathrm{Op}(a)\,\mathrm{Op}(b)u - \mathrm{Op}(ab)u \in \Psi^{m+m',r+r'-1}_{\tau}$ (resp. $\Psi^{m+m',r+r'-1}_{\tau,\mathrm{cl}}$).

2.3. **Sobolev continuity results.** Here we state continuity results for the operators defined above using the Sobolev norms with parameters introduced in Section 1.4. Such results can be obtained from their standard counterparts.

Let $\lambda_{\mathsf{T}}(\xi', \tau) = \Big(\tau^2 + |\xi'|^2\Big)^{1/2}$ and $\Lambda_{\mathsf{T}} := \mathrm{Op}(\lambda_{\mathsf{T}})$. For a given real number s, the boundary norm given by (1.14) is equivalent to the following norms (see (1.13) for the definition of $|.|_{p,\tau}$):

$$|\mathbf{u}|^2_{m,s,\tau} = \sum_{k=0}^{m} |\Lambda^s_{\mathsf{T}} u_k|^2_{m-k,\tau}, \quad \mathbf{u} = (u_0, \dots, u_m) \in \big(\mathscr{S}(\mathbb{R}^{n-1})\big)^{m+1}. \tag{2.5}$$

Moreover, we define the following semi-classical interior norm

$$\|u\|^2_{m,s,\tau} = \|\Lambda^s_{\mathsf{T}} u\|^2_{m,\tau}, \quad u \in \mathscr{S}(\overline{\mathbb{R}}^n_+). \tag{2.6}$$

Proposition 2.8. *If $a(\varrho) \in S^{m,r}_{\tau}$, with $m \in \mathbb{N}$ and $r \in \mathbb{R}$, then for $m' \in \mathbb{N}$ and $r' \in \mathbb{R}$ there exists $C > 0$ such that*

$$\|\mathrm{Op}(a)u\|_{m',r',\tau} \leq C \|u\|_{m+m',r+r',\tau}, \qquad u \in \mathscr{S}(\overline{\mathbb{R}}^n_+).$$

A consequence of this results and Proposition 2.7 is the following property.

Corollary 2.9. *Let $a \in S^{m,r}_{\tau}$ and $m' \in \mathbb{N}$ and $s \in \mathbb{R}$. We have*

$$\|a(x,D,\tau)^* u - \overline{a}(x,D,\tau)u\|_{m',s,\tau} \leq C \|u\|_{m+m',r+s-1,\tau}, \qquad u \in \mathscr{S}(\overline{\mathbb{R}}^n_+).$$

The following simple inequality will be used implicitly at many places in what follows when we invoke the parameter τ to be chosen sufficiently large. This will then allow us to absorb semi-classical norms of lower order.

Corollary 2.10. *Let $m \in \mathbb{N}$ and $s \in \mathbb{R}$ and $\ell \geq 0$. For some $C > 0$, we have*

$$\|u\|_{m,s,\tau} \leq C\tau^{-\ell} \|u\|_{m,s+\ell,\tau}, \qquad u \in \mathscr{S}(\overline{\mathbb{R}}^n_+).$$

This implies that $\|u\|_{m,s,\tau} \ll \|u\|_{m,s+\ell,\tau}$ for τ sufficiently large.

3. Interface quadratic forms

For $a(\varrho) \in S^{p,\sigma}_{\tau,\mathrm{cl}}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)$, we have

$$a(\varrho) = \sum_{j=0}^{p} a_j(\varrho')\xi_n^j, \quad \text{with } a_j \in S^{p-j+\sigma}_{\mathsf{T},\tau,\mathrm{cl}}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1}),$$

and for $\mathbf{z} = (z_0, \ldots, z_p) \in \mathbb{C}^{p+1}$ we set

$$\Sigma_a(\varrho', \mathbf{z}) = \sum_{j=0}^{p} a_j(\varrho')z_j. \tag{3.1}$$

We let m_ℓ and m_r be two integers. For applications of the results of this section we shall use the values of $m_{r/\ell}$ that come with the elliptic transmission problem we consider in the present article.

Definition 3.1 (interface quadratic forms)**.** *Let $w = (w_\ell, w_r) \in \big(\mathscr{S}(\overline{\mathbb{R}}^n_+)\big)^2$. We say that*

$$\mathscr{G}(w) = \sum_{s=1}^{N} \big(A^s_\ell w_{\ell|x_n=0^+} + A^s_r w_{r|x_n=0^+}, B^s_\ell w_{\ell|x_n=0^+} + B^s_r w_{r|x_n=0^+}\big)_\partial,$$

with $A^s_{r/\ell} = a^s_{r/\ell}(x,D,\tau)$ and $B^s_{r/\ell} = b^s_{r/\ell}(x,D,\tau)$, is an interface quadratic form of type $(m_\ell - 1, m_r - 1, \sigma)$ with $\mathscr{C}^\infty$ coefficients, if for each $s = 1, \ldots, N$, we have $a^s_{r/\ell}(\varrho), b^s_{r/\ell}(\varrho) \in S^{m_{r/\ell}-1,\sigma_{r/\ell}}_{\tau,\mathrm{cl}}(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)$, with $\sigma_\ell + \sigma_r = 2\sigma$, $\varrho = (\varrho', \xi_n)$ with $\varrho' = (x, \xi', \tau)$.

For $\mathbf{w} = (\mathbf{z}^\ell, \mathbf{z}^r), \tilde{\mathbf{w}} = (\tilde{\mathbf{z}}^\ell, \tilde{\mathbf{z}}^r) \in \mathbb{C}^{m_\ell} \times \mathbb{C}^{m_r}$, $\mathbf{z}^{r/\ell} = (z_0^{r/\ell}, \ldots, z_{m_{r/\ell}-1}^{r/\ell}), \tilde{\mathbf{z}}^{r/\ell} = (\tilde{z}_0^{r/\ell}, \ldots, \tilde{z}_{m_{r/\ell}-1}^{r/\ell}) \in \mathbb{C}^{m_{r/\ell}}$ with the interface quadratic form $\mathscr{G}$ we associate the following bilinear symbol

$$\Sigma_{\mathscr{G}}(\varrho', \mathbf{w}, \tilde{\mathbf{w}}) = \sum_{s=1}^{N} \big(\Sigma_{a^s_\ell}(\varrho', \mathbf{z}^\ell) + \Sigma_{a^s_r}(\varrho', \mathbf{z}^r)\big)\overline{\big(\Sigma_{b^s_\ell}(\varrho', \tilde{\mathbf{z}}^\ell) + \Sigma_{b^s_r}(\varrho', \tilde{\mathbf{z}}^r)\big)}.$$

with $\Sigma_{a^s_{r/\ell}}$ and $\Sigma_{b^s_{r/\ell}}$ defined as in (3.1).

Definition 3.2. *Let $\mathscr{W}$ be an open conic set in $\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}\times\mathbb{R}_+$ and let $\mathscr{G}$ be an interface quadratic form of type $(m_\ell-1,m_r-1,\sigma)$ associated with the bilinear symbol $\Sigma_{\mathscr{G}}(\varrho',\mathbf{w},\tilde{\mathbf{w}})$. We say that $\mathscr{G}$ is positive definite in $\mathscr{W}$ if there exists $C>0$ and $R>0$ such that*

$$\operatorname{Re}\Sigma_{\mathscr{G}}(\varrho'',x_n=0^+,\mathbf{w},\mathbf{w})\geq C\Big(\sum_{j=0}^{m_\ell-1}\lambda_{\mathsf{T}}^{2(m_\ell-1-j+\sigma_\ell)}\big|z_j^\ell\big|^2+\sum_{j=0}^{m_r-1}\lambda_{\mathsf{T}}^{2(m_r-1-j+\sigma_r)}\big|z_j^r\big|^2\Big),$$

for any $\mathbf{w}=(\mathbf{z}^\ell,\mathbf{z}^r)$, $\mathbf{z}^{\gamma_\ell}=(z_0^{\gamma_\ell},\dots,z_{m_{\gamma_\ell}-1}^{\gamma_\ell})\in\mathbb{C}^{m_{\gamma_\ell}}$, and $\varrho''=(x',\xi',\tau)\in\mathscr{W}$, $\tau\geq 0$, such that $\lambda_{\mathsf{T}}=|(\xi',\tau)|\geq R$.

Then we have the following Lemma

Lemma 3.3. *Let $\mathscr{W}$ be an open conic set in $\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}\times\mathbb{R}_+$ and let $\mathscr{G}$ be an interface quadratic form of type $(m_\ell-1,m_r-1,\sigma)$ that is positive definite in $\mathscr{W}$. Let $\chi\in S^0_{\mathsf{T},\tau}$ be homogeneous of degree 0, with $\operatorname{supp}(\chi_{|x_n=0^+})\subset\mathscr{W}$ and let $N\in\mathbb{N}$. Then there exist $\tau_*\geq 1$, $C>0$, $C_N>0$ such that*

$$\begin{aligned}\operatorname{Re}\mathscr{G}(\operatorname{Op}(\chi)u)\geq{}& C\big(\,|\operatorname{tr}(\operatorname{Op}(\chi)u_\ell)|^2_{m_\ell-1,\sigma_\ell,\tau}+|\operatorname{tr}(\operatorname{Op}(\chi)u_r)|^2_{m_r-1,\sigma_r,\tau}\,\big)\\&-C_N\big(\,|\operatorname{tr}(u_\ell)|^2_{m_\ell-1,\sigma_\ell-N,\tau}+|\operatorname{tr}(u_r)|^2_{m_r-1,\sigma_r-N,\tau}\,\big)\end{aligned}$$

for $u=(u_\ell,u_r)\in\big(\mathscr{S}(\overline{\mathbb{R}}^n_+)\big)^2$ and $\tau\geq\tau_*$.

Proof. The interface quadratic form can be written as

$$\begin{aligned}\mathscr{G}(u)=&\sum_{j,k=0}^{m_\ell-1}\big(G^{\ell\ell}_{kj}\Lambda_{\mathsf{T}}^{m_\ell-1-j+\sigma_\ell}D_n^ju_{\ell|x_n=0^+},\Lambda_{\mathsf{T}}^{m_\ell-1-k+\sigma_\ell}D_n^ku_{\ell|x_n=0^+}\big)_\partial\\&+\sum_{j,k=0}^{m_r-1}\big(G^{rr}_{kj}\Lambda_{\mathsf{T}}^{m_r-1-j+\sigma_r}D_n^ju_{r|x_n=0^+},\Lambda_{\mathsf{T}}^{m_r-1-k+\sigma_r}D_n^ku_{r|x_n=0^+}\big)_\partial\\&+\sum_{j=0}^{m_\ell-1}\sum_{k=0}^{m_r-1}\big(G^{r\ell}_{kj}\Lambda_{\mathsf{T}}^{m_\ell-1-j+\sigma_\ell}D_n^ju_{\ell|x_n=0^+},\Lambda_{\mathsf{T}}^{m_r-1-k+\sigma_r}D_n^ku_{r|x_n=0^+}\big)_\partial\\&+\sum_{j=0}^{m_r-1}\sum_{k=0}^{m_\ell-1}\big(G^{\ell r}_{kj}\Lambda_{\mathsf{T}}^{m_r-1-j+\sigma_r}D_n^ju_{r|x_n=0^+},\Lambda_{\mathsf{T}}^{m_\ell-1-k+\sigma_\ell}D_n^ku_{\ell|x_n=0^+}\big)_\partial,\end{aligned}$$

where $G^{ii'}_{jk}=\operatorname{Op}(g^{ii'}_{jk})\in\Psi^0_{\mathsf{T},\tau,\mathrm{cl}}$, with $i,i'=\gamma_\ell$.

We set the $2m\times 2m$-matrix tangential symbol

$$\mathbf{g}(\varrho')=\begin{pmatrix}g^{\ell\ell}&g^{\ell r}\\g^{r\ell}&g^{rr}\end{pmatrix}(\varrho'),\qquad g^{ii'}(\varrho')=\big(g^{ii'}_{jk}(\varrho')\big)_{\substack{0\leq j\leq m_i-1\\0\leq k\leq m_{i'}-1}},\quad i,i'=\gamma_\ell.$$

We introduce $\tilde\chi\in S^0_{\mathsf{T},\tau}$ that has the same properties as χ with moreover $0\leq\tilde\chi\leq 1$ and $\tilde\chi=1$ in a neighborhood of $\operatorname{supp}\chi$. We then set

$$\tilde{\mathbf{g}}=\tilde\chi\mathbf{g}+(1-\tilde\chi)I_{2m},$$

where I_{2m} is the $2m\times 2m$ identity matrix.

As $\mathscr{G}$ is positive definite in $\mathscr{W}$ we have, for some $C>0$,

$$\operatorname{Re}\big(\mathbf{g}(\varrho'',x_n=0^+)\mathbf{w},\mathbf{w}\big)\geq C\,|\mathbf{w}|^2_{\mathbb{C}^{2m}},\quad \varrho''\in\mathscr{W},\ \mathbf{w}\in\mathbb{C}^{2m}.$$

Therefore we have, for some $C'>0$,

$$\operatorname{Re}\big(\tilde{\mathbf{g}}(\varrho'',x_n=0^+)\mathbf{w},\mathbf{w}\big)\geq C'\,|\mathbf{w}|^2_{\mathbb{C}^{2m}},\quad \varrho''\in\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}\times\mathbb{R}_+,\ \mathbf{w}\in\mathbb{C}^{2m}.\tag{3.2}$$

For a function v we define the $m_{r/\ell}$-tuple functions $V_{r/\ell} = (v_{r/\ell,0}, \dots, v_{r/\ell,m_{r/\ell}-1})$ by

$$v_{r/\ell,k} = \Lambda_{\mathsf{T}}^{m_{r/\ell}+\sigma_{r/\ell}-1-k} D_n^k v_{r/\ell|x_n=0^+}, \quad k = 0, \dots, m_{r/\ell} - 1.$$

We then have, for $N \in \mathbb{Z}$,

$$\begin{aligned} \big|V_{r/\ell}\big|_{N,\tau}^2 &= \sum_{k=0}^{m_{r/\ell}-1} \big|v_{r/\ell,k}\big|_{N,\tau}^2 = \sum_{k=0}^{m_{r/\ell}-1} \big|\Lambda_{\mathsf{T}}^{m_{r/\ell}+\sigma_{r/\ell}-1-k} D_n^k v_{r/\ell|x_n=0^+}\big|_{N,\tau}^2 \\ &\asymp \sum_{k=0}^{m_{r/\ell}-1} \big|\Lambda_{\mathsf{T}}^{\sigma_{r/\ell}+N} D_n^k v_{r/\ell}\big|_{m_{r/\ell}-1-k,\tau}^2 = \big|\mathrm{tr}(v_{r/\ell})\big|_{m_{r/\ell}-1,\sigma_{r/\ell}+N,\tau}^2 . \end{aligned} \tag{3.3}$$

We set $\underline{u}_{r/\ell} = \mathrm{Op}(\chi) u_{r/\ell}$ and introduce $U_{r/\ell} = (u_{r/\ell,0}, \dots, u_{r/\ell,m_{r/\ell}-1})$ and $\underline{U}_{r/\ell} = (\underline{u}_{r/\ell,0}, \dots, \underline{u}_{r/\ell,m-1})$ as above:

$$u_{r/\ell,k} = \Lambda_{\mathsf{T}}^{m_{r/\ell}+\sigma_{r/\ell}-1-k} D_n^k u_{r/\ell|x_n=0^+}, \quad \underline{u}_{r/\ell,k} = \Lambda_{\mathsf{T}}^{m_{r/\ell}+\sigma_{r/\ell}-1-k} D_n^k \underline{u}_{r/\ell|x_n=0^+}, \quad k = 0, \dots, m_{r/\ell} - 1.$$

Setting ${}^t\underline{U} = (\underline{U}_\ell, \underline{U}_r)$ we obtain

$$\mathscr{G}(\underline{u}) = \big(\mathrm{Op}(\mathbf{g}_{|x_n=0^+}) \underline{U}, \underline{U} \big)_\partial .$$

Writing $\mathbf{g} = \tilde{\mathbf{g}} + \mathbf{r}$ with $\mathbf{r} = (\mathbf{g} - I_{2m})(1 - \tilde{\chi})$ we find

$$\mathscr{G}(\underline{u}) = \big(\mathrm{Op}(\tilde{\mathbf{g}}_{|x_n=0^+}) \underline{U}, \underline{U} \big)_\partial + \big(\mathrm{Op}(\mathbf{r}_{|x_n=0^+}) \underline{U}, \underline{U} \big)_\partial$$

As the supports of $1 - \tilde{\chi}$ and χ are disjoint, with the pseudo-differential calculus, for any $N \in \mathbb{N}$ we have for some $C_N > 0$

$$\big| \big(\mathrm{Op}(\mathbf{r}_{|x_n=0^+}) \underline{U}, \underline{U} \big)_\partial \big| \le C_N \, |U|_{-N,\tau}^2 . \tag{3.4}$$

Next, from (3.2) with the Gårding inequality in the tangential direction we deduce that for some $C > 0$ we have

$$\mathrm{Re} \, \big(\mathrm{Op}(\tilde{\mathbf{g}}_{|x_n=0^+}) \underline{U}, \underline{U} \big)_\partial \ge C \, |\underline{U}|_\partial^2 , \tag{3.5}$$

for τ sufficiently large. Combining (3.4)–(3.5) with (3.3) yields the conclusion. □

Proposition 3.4. *Assume that the transmission condition of Definition 1.4 holds at $\varrho_0' = (x_0, \xi_0', \tau_0) \in \mathbb{S}_{\mathsf{T},\tau}^*(V)$ with $x_0 \in S$ (see also (1.25)–(1.26) and (1.29) for a formulation in the local setting). Then there exists $\mathscr{U}_1$ a conic open neighborhood of ϱ_0' in $\overline{V_+} \times \mathbb{R}^{n-1} \times \mathbb{R}_+$ such that*

$$\begin{aligned} &\sum_{j=1}^{m} \lambda_{\mathsf{T}}^{2(m-1/2-\beta^j)} \big| \Sigma_{t_{\ell,\varphi}^j}(\varrho', \mathbf{z}^\ell) + \Sigma_{t_{r,\varphi}^j}(\varrho', \mathbf{z}^r) \big|^2 \\ &\qquad + \sum_{j=m+1}^{m_\ell'} \lambda_{\mathsf{T}}^{2(m_\ell-1/2-\beta_\ell^j)} \big| \Sigma_{e_{\ell,\varphi}^j}(\varrho', \mathbf{z}^\ell) \big|^2 + \sum_{j=m+1}^{m_r'} \lambda_{\mathsf{T}}^{2(m_r-1/2-\beta_r^j)} \big| \Sigma_{e_{r,\varphi}^j}(\varrho', \mathbf{z}^r) \big|^2 \\ &\qquad\qquad \ge C \Big(\sum_{j=0}^{m_\ell-1} \lambda_{\mathsf{T}}^{2(m_\ell-1/2-j)} |z_j^\ell|^2 + \sum_{j=0}^{m_r-1} \lambda_{\mathsf{T}}^{2(m_r-1/2-j)} |z_j^r|^2 \Big), \end{aligned}$$

for $\varrho \in \mathscr{U}_1$ and $\mathbf{z}^{r/\ell} = (z_0^{r/\ell}, \dots, z_{m_{r/\ell}-1}^{r/\ell}) \in \mathbb{C}^{m_{r/\ell}}$.

We recall that $\beta^j = (\beta_\ell^j + \beta_r^j)/2$ for $j = 1, \dots, m$, and that we have (1.27).

Proof. With the transmission condition holding at ϱ_0', by Proposition 1.8 there exists a conic open set $\mathscr{U}_1$ neighborhood of ϱ_0' where condition (1.29) is valid. Observe that $\mathcal{K} = \overline{\mathscr{U}_1} \cap \overline{\mathbb{S}^*_{\mathsf{T},\tau}(V)}$ is compact, recalling that V_+ is bounded.

Let $\varrho_1' \in \mathcal{K}$ and $\mathcal{T}(\varrho_1')$ be as in (1.29). We have

$$\operatorname{rank} \mathcal{T}(\varrho_1') = \operatorname{rank} \overline{\mathcal{T}(\varrho_1')}\, {}^t\mathcal{T}(\varrho_1') = 2m,$$

For $\mathbf{w} = (\mathbf{z}^\ell, \mathbf{z}^r) \in \mathbb{C}^{2m}$, $\mathbf{z}^{r/\ell} = (z_0^{r/\ell}, \dots, z_{m_{r/\ell}-1}^{r/\ell}) \in \mathbb{C}^{m_{r/\ell}}$, recalling that $2m = m_\ell + m_r$, we thus have $\big(\overline{\mathcal{T}(\varrho_1')}\, {}^t\mathcal{T}(\varrho_1')\mathbf{w}, \mathbf{w}\big) \geq C|\mathbf{w}|^2_{\mathbb{C}^{2m}}$, for some $C > 0$.

Observe that we have

$$\overline{\mathcal{T}(\varrho_1')}\, {}^t\mathcal{T}(\varrho_1') = \begin{pmatrix} \overline{\mathcal{T}_\ell^1(\varrho_1')}\, {}^t\mathcal{T}_\ell^1(\varrho_1') & \overline{\mathcal{T}_\ell^1(\varrho_1')}\, {}^t\mathcal{T}_r^1(\varrho_1') \\ \overline{\mathcal{T}_r^1(\varrho_1')}\, {}^t\mathcal{T}_\ell^1(\varrho_1') & \overline{\mathcal{T}_r^1(\varrho_1')}\, {}^t\mathcal{T}_r^1(\varrho_1') \end{pmatrix} + \begin{pmatrix} \overline{\mathcal{T}_\ell^2(\varrho_1')}\, {}^t\mathcal{T}_\ell^2(\varrho_1') & 0 \\ 0 & \overline{\mathcal{T}_r^2(\varrho_1')}\, {}^t\mathcal{T}_r^2(\varrho_1') \end{pmatrix}.$$

For $\mathbf{w} = (\mathbf{z}^\ell, \mathbf{z}^r)$, $\mathbf{z}^{r/\ell} = (z_0^{r/\ell}, \dots, z_{m_{r/\ell}-1}^{r/\ell}) \in \mathbb{C}^{m_{r/\ell}}$, we have

$$\begin{aligned}
\big(\overline{\mathcal{T}(\varrho_1')}\, {}^t\mathcal{T}(\varrho_1')\mathbf{w}, \mathbf{w}\big) &= \big|{}^t\mathcal{T}_\ell^1(\varrho_1')\mathbf{z}^\ell + {}^t\mathcal{T}_r^1(\varrho_1')\mathbf{z}^r\big|^2_{\mathbb{C}^m} + \big|{}^t\mathcal{T}_\ell^2(\varrho_1')\mathbf{z}^\ell\big|^2_{\mathbb{C}^{m_\ell^-}} + \big|{}^t\mathcal{T}_r^2(\varrho_1')\mathbf{z}^r\big|^2_{\mathbb{C}^{m_r^-}} \\
&= \sum_{j=1}^{m} \Big| \sum_{i=0}^{m_\ell - 1} t^j_{\ell,i}(\varrho_1')z_i^\ell + \sum_{i=0}^{m_r-1} t^j_{r,i}(\varrho_1')z_i^r \Big|^2 \\
&\quad + \sum_{j=1}^{m_\ell^-} \Big| \sum_{i=0}^{m_\ell - 1} e^{j+m}_{\ell,i}(\varrho_1')z_i^\ell \Big|^2 + \sum_{j=1}^{m_r^-} \Big| \sum_{i=0}^{m_r-1} e^{j+m}_{r,i}(\varrho_1')z_i^r \Big|^2 \\
&= \sum_{j=1}^{m} \Big| \Sigma_{t^j_{\ell,\varphi}}(\varrho_1', \mathbf{z}^\ell) + \Sigma_{t^j_{r,\varphi}}(\varrho_1', \mathbf{z}^r) \Big|^2 + \sum_{j=1}^{m_\ell^-} \big| \Sigma_{e^{j+m}_{\ell,\varphi}}(\varrho_1', \mathbf{z}^\ell) \big|^2 + \sum_{j=1}^{m_r'^-} \big| \Sigma_{e^{j+m}_{r,\varphi}}(\varrho_1', \mathbf{z}^r) \big|^2.
\end{aligned}$$

We thus obtain

$$\sum_{j=1}^{m} \Big| \Sigma_{t^j_{\ell,\varphi}}(\varrho_1', \mathbf{z}^\ell) + \Sigma_{t^j_{r,\varphi}}(\varrho_1', \mathbf{z}^r) \Big|^2 + \sum_{j=1}^{m_\ell^-} \big| \Sigma_{e^{j+m}_{\ell,\varphi}}(\varrho_1', \mathbf{z}^\ell) \big|^2 + \sum_{j=1}^{m_r^-} \big| \Sigma_{e^{j+m}_{r,\varphi}}(\varrho_1', \mathbf{z}^r) \big|^2 \gtrsim |(\mathbf{z}^\ell, \mathbf{z}^r)|^2.$$

By continuity this inequality remains true in a small neighborhood of ϱ_1' in $\mathcal{K}$. Using the compactness of $\mathcal{K}$ we thus find

$$\sum_{j=1}^{m} \Big| \Sigma_{t^j_{\ell,\varphi}}(\varrho', \mathbf{z}^\ell) + \Sigma_{t^j_{r,\varphi}}(\varrho', \mathbf{z}^r) \Big|^2 + \sum_{j=1}^{m_\ell^-} \big| \Sigma_{e^{j+m}_{\ell,\varphi}}(\varrho', \mathbf{z}^\ell) \big|^2 + \sum_{j=1}^{m_r^-} \big| \Sigma_{e^{j+m}_{r,\varphi}}(\varrho', \mathbf{z}^r) \big|^2 \gtrsim |(\mathbf{z}^\ell, \mathbf{z}^r)|^2,$$

for $\varrho' \in \mathcal{K}$ and $\mathbf{z}^{r/\ell} = (z_0^{r/\ell}, \dots, z_{m_{r/\ell}-1}^{r/\ell}) \in \mathbb{C}^{m_{r/\ell}}$. Introducing the map

$$M_t\varrho' = (x, t\eta), \qquad \varrho' = (x, \eta) \in \overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1} \times \mathbb{R}_+, \quad t > 0,$$

as we have $\overline{\mathscr{U}_1} = \{M_t\varrho';\ t > 0,\ \varrho' \in \mathcal{K}\}$, we find

$$\sum_{j=1}^{m} \Big| \Sigma_{t^j_{\ell,\varphi}}(M_t\varrho', \tilde{\mathbf{z}}^\ell) + \Sigma_{t^j_{r,\varphi}}(M_t\varrho', \tilde{\mathbf{z}}^r) \Big|^2 + \sum_{j=1}^{m_\ell^-} \big| \Sigma_{e^{j+m}_{\ell,\varphi}}(M_t\varrho', \tilde{\mathbf{z}}^\ell) \big|^2 + \sum_{j=1}^{m_r^-} \big| \Sigma_{e^{j+m}_{r,\varphi}}(M_t\varrho', \tilde{\mathbf{z}}^r) \big|^2 \gtrsim |(\tilde{\mathbf{z}}^\ell, \tilde{\mathbf{z}}^r)|^2,$$

where $t=\lambda_{\mathsf{T}}^{-1}=|(\xi',\tau)|^{-1}$ and $\tilde{\mathbf{z}}^{r/\ell}=(\tilde{z}_0^{r/\ell},\dots,\tilde{z}_{m_{r/\ell}-1}^{r/\ell})\in\mathbb{C}^{m_{r/\ell}}$, with $\tilde{z}_j^{r/\ell}=t^{-m_{r/\ell}+1/2+j}z_j^{r/\ell}$, yielding

$$\begin{aligned}\sum_{j=1}^{m}\big|\lambda_{\mathsf{T}}^{-(\beta_\ell^j-m_\ell+1/2)}\Sigma_{t_{\ell,\varphi}^j}(\varrho',\mathbf{z}^\ell)+\lambda_{\mathsf{T}}^{-(\beta_r^j-m_r+1/2)}\Sigma_{t_{r,\varphi}^j}(\varrho',\mathbf{z}^r)\big|^2\\ +\sum_{j=1}^{m_\ell^-}\big|\lambda_{\mathsf{T}}^{-(\beta_\ell^j-m_\ell+1/2)}\Sigma_{e_{\ell,\varphi}^{j+m}}(\varrho',\mathbf{z}^\ell)\big|^2+\sum_{j=1}^{m_r^-}\big|\lambda_{\mathsf{T}}^{-(\beta_r^j-m_r+1/2)}\Sigma_{e_{r,\varphi}^{j+m}}(\varrho',\mathbf{z}^r)\big|^2\\ \gtrsim\sum_{j=0}^{m_\ell-1}|\lambda_{\mathsf{T}}^{m_\ell-1/2-j}z_j^\ell|^2+\sum_{j=0}^{m_r-1}|\lambda_{\mathsf{T}}^{m_r-1/2-j}z_j^r|^2.\end{aligned}$$

With (1.27) we obtain the sought result. □

4. Proof of the Carleman estimate

As is usual in the proof of Carleman estimates we consider the following conjugated operators

$$P_{r/\ell,\varphi}=e^{\tau\varphi_{r/\ell}}P_{r/\ell}e^{-\tau\varphi_{r/\ell}}.$$

As $e^{\tau\varphi_{r/\ell}}D_je^{-\tau\varphi_{r/\ell}}=D_j+i\tau\partial_j\varphi_{r/\ell}\in\Psi^{1,0}_{\tau,\mathrm{cl}}$, we see that $P_{r/\ell,\varphi}\in\Psi^{2m,0}_{\tau,\mathrm{cl}}$. Their principal symbols are given by $p_{r/\ell,\varphi}(\varrho)=p_{r/\ell}(x,\xi+i\tau\varphi'_{r/\ell}(x))\in S^{2m,0}_{\tau,\mathrm{cl}}$.

Similarly we recall that we set

$$T^j_{r/\ell,\varphi}=e^{\tau\varphi_{r/\ell}}T^j_{r/\ell}e^{-\tau\varphi_{r/\ell}}\in\Psi^{\beta^j_{r/\ell},0}_{\tau,\mathrm{cl}},\quad j=1,\dots,m,$$

with principal symbols $t_{r/\ell,\varphi}(\varrho)=t^j_{r/\ell}(x,\xi+i\tau\varphi'_{r/\ell}(x))\in S^{\beta^j_{r/\ell},0}_{\tau,\mathrm{cl}}$.

We start the proof of the main theorem with a microlocal estimate that exploits the transmission condition.

4.1. Estimate with the transmission condition.

Let us first consider a polynomial function with roots with negative imaginary parts in a microlocal region. Then, we have the following perfect microlocal elliptic estimate. We refer to Lemma 4.1 in Part I [4] for a proof.

Lemma 4.1. *Let $h(\varrho',\xi_n)\in S^{k,0}_\tau$, $\varrho'=(x,\xi',\tau)$, with $k\geq 1$, be polynomial in ξ_n with homogeneous coefficients in (ξ',τ) and $H=h(x,D,\tau)$. When viewed as a polynomial in ξ_n the leading coefficient is 1. Let $\mathscr{U}$ be a conic open subset of $\overline{V_+}\times\mathbb{R}^{n-1}\times\mathbb{R}_+$. We assume that all the roots of $h(\varrho',\xi_n)=0$ have negative imaginary part for $\varrho'=(x,\xi',\tau)\in\mathscr{U}$. Letting $\chi(\varrho')\in S^0_{\mathsf{T},\tau}$ be homogeneous of degree 0 and such that $\operatorname{supp}(\chi)\subset\mathscr{U}$, and $N\in\mathbb{N}$, there exist $C>0$, $C_N>0$, and $\tau_*>0$ such that*

$$\|\operatorname{Op}(\chi)w\|^2_{k,\tau}+|\operatorname{tr}(\operatorname{Op}(\chi)w)|^2_{k-1,1/2,\tau}\leq C\,\|H\operatorname{Op}(\chi)w\|^2_++C_N\big(\,\|w\|^2_{k,-N,\tau}+|\operatorname{tr}(w)|^2_{k-1,-N,\tau}\big),$$

for $w\in\mathscr{S}(\overline{\mathbb{R}}^n_+)$ and $\tau\geq\tau_*$.

Now, we consider a point in the cotangent bundle, at the interface where the transmission condition holds. We then obtain an estimate of an interface norm.

Proposition 4.2. *Assume that the transmission condition of Definition 1.4 is satisfied at $\varrho'_0=(x_0,\xi'_0,\tau_0)\in\mathbb{S}^*_{\mathsf{T},\tau}(V)$ with $x_0\in S\cap V$. Then there exists $\mathscr{U}$, a conic open neighborhood of ϱ'_0 in $\overline{V_+}\times\mathbb{R}^{n-1}\times\mathbb{R}_+$, such*

that for $\chi \in S^0_{\mathsf{T},\tau}$, homogeneous of degree 0, with $\mathrm{supp}(\chi) \subset \mathscr{U}$*, there exist $C > 0$ and $\tau_* > 0$ such that*

$$\begin{aligned} C\big(|\mathrm{tr}(\mathrm{Op}(\chi)v_\ell)|^2_{m_\ell-1,1/2,\tau} + |\mathrm{tr}(\mathrm{Op}(\chi)v_r)|^2_{m_r-1,1/2,\tau} \big) \\ \leq \sum_{j=1}^{m} \big| T^j_{\ell,\varphi} v_{\ell|x_n=0^+} + T^j_{r,\varphi} v_{r|x_n=0^+} \big|^2_{m-1/2-\beta^j,\tau} + \|P_{\ell,\varphi} v_\ell\|^2_+ + \|P_{r,\varphi} v_r\|^2_+ \\ + \|v_\ell\|^2_{m_\ell,-1,\tau} + \|v_r\|^2_{m_r,-1,\tau} + |\mathrm{tr}(v_\ell)|^2_{m_\ell-1,-1/2,\tau} + |\mathrm{tr}(v_r)|^2_{m_r-1,-1/2,\tau}\,, \end{aligned}$$

for $\tau \geq \tau_*$, $v_\ell, v_r \in \mathscr{S}(\overline{\mathbb{R}^n_+})$.

Proof. As the transmission condition holds at ϱ'_0 the local smooth symbol factorizations of Section 1.6.3,

$$p_{r/\ell,\varphi}(\varrho', \xi_n) = p^-_{r/\ell,\varphi}(\varrho', \xi_n)\, \kappa_{r/\ell,\varphi}(\varrho', \xi_n),$$

is such that condition (1.29) is valid for $\varrho' \in \mathscr{U}_0$ with U_0 a conic neighborhood of ϱ'_0 in $\overline{V_+} \times \mathbb{R}^{n-1} \times \mathbb{R}_+$. Moreover, by Proposition 3.4 there exists $\mathscr{U}_1 \subset \mathscr{U}_0$, a conic open neighborhood of ϱ'_0, such that

$$\begin{aligned} &\sum_{j=m+1}^{m'_\ell} \lambda_{\mathsf{T}}^{2(m_\ell-1/2-\beta^j_\ell)} \big|\Sigma_{e^j_{\ell,\varphi}}(\varrho', \mathbf{z}^\ell)\big|^2 + \sum_{j=m+1}^{m'_r} \lambda_{\mathsf{T}}^{2(m_r-1/2-\beta^j_r)} \big|\Sigma_{e^j_{r,\varphi}}(\varrho', \mathbf{z}^r)\big|^2 \\ &+ \sum_{j=1}^{m} \lambda_{\mathsf{T}}^{2(m-1/2-\beta^j)} \big|\Sigma_{t^j_{\ell,\varphi}}(\varrho', \mathbf{z}^\ell) + \Sigma_{t^j_{r,\varphi}}(\varrho', \mathbf{z}^r)\big|^2 \gtrsim \sum_{j=0}^{m_\ell-1} \lambda_{\mathsf{T}}^{2(m_\ell-1/2-j)} |z^\ell_j|^2 + \sum_{j=0}^{m_r-1} \lambda_{\mathsf{T}}^{2(m_r-1/2-j)} |z^r_j|^2, \end{aligned}$$

for $\varrho' \in \mathscr{U}_1$ and $\mathbf{z}^{r/\ell} \in \mathbb{C}^{m_{r/\ell}}$. We now choose $\mathscr{U}$ a conic open subset, neighborhood of ϱ'_0, such that $\overline{\mathscr{U}} \subset \mathscr{U}_1$. We let χ be as in the statement and we also choose $\tilde\chi \in S^0_{\mathsf{T},\tau}$ homogeneous of degree 0 with $\mathrm{supp}(\tilde\chi) \subset \mathscr{U}_1$ and $\tilde\chi = 1$ in a neighborhood of $\overline{\mathscr{U}}$. Then,

$$\begin{aligned} &\sum_{j=m+1}^{m'_\ell} \lambda_{\mathsf{T}}^{2(m_\ell-1/2-\beta^j_\ell)} \big|\tilde\chi(\varrho')\Sigma_{e^j_{\ell,\varphi}}(\varrho', \mathbf{z}^\ell)\big|^2 + \sum_{j=m+1}^{m'_r} \lambda_{\mathsf{T}}^{2(m_r-1/2-\beta^j_r)} \big|\tilde\chi(\varrho')\Sigma_{e^j_{r,\varphi}}(\varrho', \mathbf{z}^r)\big|^2 \\ &+ \sum_{j=1}^{m} \lambda_{\mathsf{T}}^{2(m-1/2-\beta^j)} \big|\Sigma_{t^j_{\ell,\varphi}}(\varrho', \mathbf{z}^\ell) + \Sigma_{t^j_{r,\varphi}}(\varrho', \mathbf{z}^r)\big|^2 \gtrsim \sum_{j=0}^{m_\ell-1} \lambda_{\mathsf{T}}^{2(m_\ell-1/2-j)} |z^\ell_j|^2 + \sum_{j=0}^{m_r-1} \lambda_{\mathsf{T}}^{2(m_r-1/2-j)} |z^r_j|^2, \end{aligned} \tag{4.1}$$

for $\varrho' \in \overline{\mathscr{U}}$ and $\mathbf{z}^{r/\ell} \in \mathbb{C}^{m_{r/\ell}}$.

We set[5] $E_{r/\ell,\varphi} = \mathrm{Op}(\tilde\chi e_{r/\ell,\varphi})$ and we define the following interface quadratic form (see Definition 3.1):

$$\begin{aligned} \mathscr{G}_S(u) = &\sum_{j=1}^{m} \big| T^j_{\ell,\varphi} u_{\ell|x_n=0^+} + T^j_{r,\varphi} u_{r|x_n=0^+} \big|^2_{m-1/2-\beta^j,\tau} \\ &+ \sum_{j=m+1}^{m'_\ell} \big| E^j_{\ell,\varphi} u_{\ell|x_n=0^+} \big|^2_{m+m^-_\ell+1/2-j,\tau} + \sum_{j=m+1}^{m'_r} \big| E^j_{r,\varphi} u_{\ell|x_n=0^+} \big|^2_{m+m^-_r+1/2-j,\tau}. \end{aligned}$$

[5]The introduction of $\tilde\chi$ is made so that $\tilde\chi e^k_\varphi$ is defined on the whole tangential phase-space.

Observe that with (1.28) we have $m+m^-_{r/\ell}+1/2-j=m_{r/\ell}-1/2-\beta^j_{r/\ell}$ for $j=m+1,\dots,m'_{r/\ell}$. We thus find that $\mathscr{G}_S$ is of type $(m^\ell-1,m^r-1,\frac{1}{2})$ and its bilinear symbol is given by

$$\begin{aligned}\Sigma_{\mathscr{G}_S}(\varrho',\mathbf{w},\tilde{\mathbf{w}})=\sum_{j=1}^{m}\lambda_{\mathsf{T}}^{2(m-1/2-\beta^j)}\big(\Sigma_{t^j_{\ell,\varphi}}(\varrho',\mathbf{z}^\ell)+\Sigma_{t^j_{r,\varphi}}(\varrho',\mathbf{z}^r)\big)\big(\overline{\Sigma_{t^j_{\ell,\varphi}}}(\varrho',\overline{\tilde{\mathbf{z}}^\ell})+\overline{\Sigma_{t^j_{r,\varphi}}}(\varrho',\overline{\tilde{\mathbf{z}}^r})\big)\\ +|\tilde\chi(\varrho')|^2\sum_{j=m+1}^{m'_\ell}\lambda_{\mathsf{T}}^{2(m_\ell-1/2-\beta^j_\ell)}\Sigma_{e^j_{\ell,\varphi}}(\varrho',\mathbf{z}^\ell)\overline{\Sigma_{e^j_{\ell,\varphi}}}(\varrho',\overline{\tilde{\mathbf{z}}^\ell})\\ +|\tilde\chi(\varrho')|^2\sum_{j=m+1}^{m'_r}\lambda_{\mathsf{T}}^{2(m_r-1/2-\beta^j_r)}\Sigma_{e^j_{r,\varphi}}(\varrho',\tilde{\mathbf{z}}^r)\overline{\Sigma_{e^j_{r,\varphi}}}(\varrho',\overline{\tilde{\mathbf{z}}^r}),\end{aligned}$$

with $\mathbf{w}=(\mathbf{z}^\ell,\mathbf{z}^r),\tilde{\mathbf{w}}=(\tilde{\mathbf{z}}^\ell,\tilde{\mathbf{z}}^r)\in\mathbb{C}^{m_\ell}\times\mathbb{C}^{m_r}$. Hence (4.1) gives

$$\Sigma_{\mathscr{G}_S}(\varrho',\mathbf{w},\tilde{\mathbf{w}})\gtrsim\sum_{j=0}^{m_\ell-1}\lambda_{\mathsf{T}}^{2(m_\ell-1/2-j)}|z^\ell_j|^2+\sum_{j=0}^{m_r-1}\lambda_{\mathsf{T}}^{2(m_r-1/2-j)}|z^r_j|^2,$$

for $\varrho'\in\overline{\mathscr{U}}$. For any $N\in\mathbb{N}$, by Lemma 3.3 there exists $\tau_*\geq 1$, $C>0$, $C_N>0$ such that

$$\begin{aligned}\mathscr{G}_S(\underline{v})=\operatorname{Re}\mathscr{G}_S(\underline{v})\geq C\big(\,|\operatorname{tr}(\underline{v}_\ell)|^2_{m_\ell-1,1/2,\tau}+|\operatorname{tr}(\underline{v}_r)|^2_{m_r-1,1/2,\tau}\,\big)\\ -C_N\big(\,|\operatorname{tr}(v_\ell)|^2_{m_\ell-1,1/2-N,\tau}+|\operatorname{tr}(v_r)|^2_{m_r-1,1/2-N,\tau}\,\big),\end{aligned}\tag{4.2}$$

with $\underline{v}=(\underline{v}_\ell,\underline{v}_r)$ and $\underline{v}_{r/\ell}=\operatorname{Op}(\chi)v_{r/\ell}$, for $\mathbf{v}=(v_\ell,v_r)\in\big(\mathscr{S}(\overline{\mathbb{R}}^n_+)\big)^2$ and $\tau\geq\tau_*$.

The functions $p^-_{r/\ell,\varphi}(\varrho',\xi_n)$ and $\kappa_{r/\ell,\varphi}(\varrho',\xi_n)$ in the symbol factorization recalled at the begining of the proof are polynomial in ξ_n with homogeneous coefficients in $\varrho'\in\mathscr{U}_0$ and the leading coefficient of $p^-_{r/\ell,\varphi}(\varrho',\xi_n)$ is equal to 1. For $\varrho'\in\mathscr{U}_0$, their degrees are constant and equal to $m^-_{r/\ell}$ and $m_{r/\ell}-m^-_{r/\ell}$ respectively. We smoothly extend $p^-_{r/\ell,\varphi}(\varrho',\xi_n)$ for ϱ' outside of $\mathscr{U}_0$ keeping the leading coefficient equal to 1 and we denote this extension by $\tilde p^-_{r/\ell,\varphi}$. In fact we have $\chi p_{r/\ell,\varphi}=\chi\kappa_{r/\ell,\varphi}p^-_{r/\ell,\varphi}=\chi\tilde\chi\kappa_{r/\ell,\varphi}\tilde p^-_{r/\ell,\varphi}$. We thus obtain $\operatorname{Op}(\chi)P_{r/\ell,\varphi}=\operatorname{Op}(\tilde p^-_{r/\ell,\varphi})\operatorname{Op}(\chi)\operatorname{Op}(\tilde\chi\kappa_{r/\ell,\varphi})+R_{r/\ell}$ with $R_{r/\ell}$ in $\Psi^{m,-1}_\tau$ by the last point of Proposition 2.7. Observe that $\tilde\chi\kappa_{r/\ell,\varphi}$ is a well defined symbol.

Applying Lemma 4.1 to $\operatorname{Op}(\tilde p^-_{r/\ell,\varphi})$ and $w_{r/\ell}=\operatorname{Op}(\tilde\chi\kappa_{r/\ell,\varphi})v_{r/\ell}$ we obtain

$$\begin{aligned}&\|\operatorname{Op}(\chi)w_\ell\|^2_{m^-_\ell,\tau}+\|\operatorname{Op}(\chi)w_r\|^2_{m^-_r,\tau}+|\operatorname{tr}(\operatorname{Op}(\chi)w_\ell)|^2_{m^-_\ell-1,1/2,\tau}+|\operatorname{tr}(\operatorname{Op}(\chi)w_r)|^2_{m^-_r-1,1/2,\tau}\\ &\quad\lesssim\big\|\operatorname{Op}(\tilde p^-_{\ell,\varphi})\operatorname{Op}(\chi)w_\ell\big\|^2_+\big\|\operatorname{Op}(\tilde p^-_{r,\varphi})\operatorname{Op}(\chi)w_r\big\|^2_+\|w_\ell\|^2_{m^-_\ell,-N,\tau}+\|w_r\|^2_{m^-_r,-N,\tau}\\ &\qquad+|\operatorname{tr}(w_\ell)|^2_{m^-_\ell-1,-N,\tau}+|\operatorname{tr}(w_r)|^2_{m^-_r-1,-N,\tau}\\ &\quad\lesssim\big\|\operatorname{Op}(\chi)P_{\ell,\varphi}v_\ell\big\|^2_+\big\|\operatorname{Op}(\chi)P_{r,\varphi}v_r\big\|^2_+\|v_\ell\|^2_{m_\ell,-1,\tau}+\|v_r\|^2_{m_r,-1,\tau}\\ &\qquad+\|v_\ell\|^2_{m_\ell,-N,\tau}+\|v_r\|^2_{m_r,-N,\tau}+|\operatorname{tr}(v_\ell)|^2_{m_\ell-1,-N,\tau}+|\operatorname{tr}(v_r)|^2_{m_r-1,-N,\tau}\\ &\quad\lesssim\|P_{\ell,\varphi}v_\ell\|^2_+\|P_{r,\varphi}v_r\|^2_+\|v_\ell\|^2_{m_\ell,-1,\tau}+\|v_r\|^2_{m_r,-1,\tau}\\ &\qquad+|\operatorname{tr}(v_\ell)|^2_{m_\ell-1,-N,\tau}+|\operatorname{tr}(v_r)|^2_{m_r-1,-N,\tau},\end{aligned}$$

yielding

$$\sum_{j=0}^{m_\ell^- -1} \left| D_n^j \operatorname{Op}(\chi) w_{\ell|x_n=0^+} \right|^2_{m_\ell^- -1/2-j,\tau} + \sum_{j=0}^{m_r^- -1} \left| D_n^j \operatorname{Op}(\chi) w_{r|x_n=0^+} \right|^2_{m_r^- -1/2-j,\tau}$$
$$\lesssim \|P_{\ell,\varphi} v_\ell\|_+^2 + \|P_{r,\varphi} v_r\|_+^2 + \|v_\ell\|^2_{m_\ell,-1,\tau} + \|v_r\|^2_{m_r,-1,\tau} + |\operatorname{tr}(v_\ell)|^2_{m_\ell-1,-N,\tau} + |\operatorname{tr}(v_r)|^2_{m_r-1,-N,\tau},$$

Recalling that $e^{j+m+1}_{r/\ell,\varphi} = \kappa_{r/\ell,\varphi}\xi_n^j$, $j = 0,\dots,m^-_{r/\ell} - 1$ in $\mathscr{U}_1$ we have $D_n^j \operatorname{Op}(\chi) \operatorname{Op}(\tilde\chi \kappa_{r/\ell,\varphi}) v_{r/\ell} = E^{j+m+1}_{r/\ell,\varphi} \underline{v}_{r/\ell} + R_{r/\ell,j} v_{r/\ell}$ with $R_{r/\ell,j} \in \Psi_\tau^{m_{r/\ell} - m^-_{r/\ell} + j, -1}$ by the last point of Proposition 2.7. We then obtain, for τ chosen sufficiently large

$$\sum_{j=0}^{m_\ell^- -1} \left| E^{j+m+1}_{\ell,\varphi} \underline{v}_{\ell|x_n=0^+} \right|^2_{m_\ell^- -1/2-j,\tau} + \sum_{j=0}^{m_r^- -1} \left| E^{j+m+1}_{r,\varphi} \underline{v}_{r|x_n=0^+} \right|^2_{m_r^- -1/2-j,\tau}$$
$$\lesssim \|P_{\ell,\varphi} v_\ell\|_+^2 + \|P_{r,\varphi} v_r\|_+^2 + \|v_\ell\|^2_{m_\ell,-1,\tau} + \|v_r\|^2_{m_r,-1,\tau}$$
$$+ |\operatorname{tr}(v_\ell)|^2_{m_\ell-1,-1/2,\tau} + |\operatorname{tr}(v_r)|^2_{m_r-1,-1/2,\tau},$$

which we write, by a shift of indices,

$$\sum_{j=m+1}^{m'_\ell} \left| E^j_{\ell,\varphi} \underline{v}_{\ell|x_n=0^+} \right|^2_{m+m_\ell^- +1/2-j,\tau} + \sum_{j=m+1}^{m'_r} \left| E^j_{r,\varphi} \underline{v}_{r|x_n=0^+} \right|^2_{m+m_r^- +1/2-j,\tau} \tag{4.3}$$
$$\lesssim \|P_{\ell,\varphi} v_\ell\|_+^2 + \|P_{r,\varphi} v_r\|_+^2 + \|v_\ell\|^2_{m_\ell,-1,\tau} + \|v_r\|^2_{m_r,-1,\tau}$$
$$+ |\operatorname{tr}(v_\ell)|^2_{m_\ell-1,-1/2,\tau} + |\operatorname{tr}(v_r)|^2_{m_r-1,-1/2,\tau},$$

Collecting estimates (4.2) and (4.3) we thus obtain

$$|\operatorname{tr}(\underline{v}_\ell)|^2_{m_\ell-1,1/2,\tau} + |\operatorname{tr}(\underline{v}_r)|^2_{m_r-1,1/2,\tau} \lesssim \sum_{j=1}^{m} \left| T^j_{\ell,\varphi} \underline{v}_{\ell|x_n=0^+} + T^j_{r,\varphi} \underline{v}_{r|x_n=0^+} \right|^2_{m-1/2-\beta^j,\tau}$$
$$+ \|P_{\ell,\varphi} v_\ell\|_+^2 + \|P_{r,\varphi} v_r\|_+^2 + \|v_\ell\|^2_{m_\ell,-1,\tau} + \|v_r\|^2_{m_r,-1,\tau}$$
$$+ |\operatorname{tr}(v_\ell)|^2_{m_\ell-1,-1/2,\tau} + |\operatorname{tr}(v_r)|^2_{m_r-1,-1/2,\tau}.$$

Writing $T^j_{\ell,\varphi} \operatorname{Op}(\chi) = \operatorname{Op}(\chi) T^j_{\ell,\varphi} + [T^j_{\ell,\varphi}, \operatorname{Op}(\chi)]$ we observe that (using that $m_\ell - \beta^j_\ell = m_r - \beta^j_r = m - \beta^j$)

$$\left| T^j_{\ell,\varphi} \underline{v}_{\ell|x_n=0^+} + T^j_{r,\varphi} \underline{v}_{r|x_n=0^+} \right|_{m-1/2-\beta^j,\tau} \lesssim \left| T^j_{\ell,\varphi} v_{\ell|x_n=0^+} + T^j_{r,\varphi} v_{r|x_n=0^+} \right|_{m-1/2-\beta^j,\tau}$$
$$+ |\operatorname{tr}(v_\ell)|_{\beta^j_\ell, m_\ell-1/2-\beta^j_\ell-1,\tau} + |\operatorname{tr}(v_r)|_{\beta^j_r, m_r-1/2-\beta^j_r-1,\tau}$$
$$\lesssim \left| T^j_{\ell,\varphi} v_{\ell|x_n=0^+} + T^j_{r,\varphi} v_{r|x_n=0^+} \right|_{m-1/2-\beta^j,\tau}$$
$$+ |\operatorname{tr}(v_\ell)|_{m_\ell-1,-1/2,\tau} + |\operatorname{tr}(v_r)|_{m_r-1,-1/2,\tau}.$$

This concludes the proof. □

4.2. **Estimate with a positive Poisson bracket on the characteristic set.** If we consider the case of two symbols a, b such that the Poisson bracket $\{a, b\}$ is positive on the characterisitic set $\{a = b = 0\}$, an estimate with the control of a volume norm can be achieved.

Lemma 4.3. *Let U be an open set of $\overline{V_+}$. Let $a \in S^{m,0}_\tau$ and $b \in S^{m-1,1}_\tau$ be real symbols homogeneous of degree m in (τ,ξ), and set*

$$Q_{a,b}(v) = 2\operatorname{Re}(Av, iBv)_+, \quad A = a(x,D,\tau),\ B = b(x,D,\tau).$$

We assume that

$$a(\varrho)=b(\varrho)=0 \;\;\Rightarrow\;\; \{a,b\}>0, \quad \varrho=(x,\xi,\tau),$$

for $x\in\overline{U}$, $(\xi,\tau)\neq(0,0)$. *Then there exist* $C>0$, $C'>0$, *and* $\tau_*>0$ *such that*

$$C\,\|v\|^2_{m,\tau}\le C'\big(\,\|Av\|^2_+ + \|Bv\|^2_+ + |\mathrm{tr}(v)|^2_{m-1,1/2,\tau}\,\big)+\tau\big(Q_{a,b}(v)-\mathrm{Re}\,\mathscr{B}_{a,b}(v)\big),$$

for $\tau>\tau_*$ *and for* $v\in\mathscr{C}^\infty(\overline{\mathbb{R}}^n_+)$ *with* $\mathrm{supp}(v)\subset U$, *with* $\mathscr{B}_{a,b}$ *satisfying*

$$\big|\mathscr{B}_{a,b}(v)\big|\le C'\big|\,\mathrm{tr}(v)\big|^2_{m-1,1/2,\tau}.$$

We refer to [4] for a proof.

4.3. **A microlocal Carleman estimate.** With the results of Sections 4.1 and 4.2, if the transmission condition holds at one point of the cotangent bundle at the interface and if the sub-ellipticity property also holds we can then derive a Carleman estimate that holds microlocally, that is, with a cut-off in phase-space applied through a tangential pseudo-differential operator.

Theorem 4.4. *Let* $x_0\in S\cap V$. *Assume that* $\{P_{r/\ell},\varphi_{r/\ell}\}$ *satisfies the sub-ellipticity condition on a neighborhood of* x_0 *in* $\overline{V_+}$. *Assume moreover that* $\{P_{r/\ell},T^j_{r/\ell},\varphi_{r/\ell},\ j=1,\dots,m\}$ *satisfies the transmission condition at* $\varrho'_0=(x_0,\xi'_0,\tau_0)\in\mathbb{S}^*_{\mathsf{T},\tau}(\overline{V_+})$. *Then there exists* $\mathscr{U}$ *a conic open neighborhood of* ϱ'_0 *in* $\overline{V_+}\times\mathbb{R}^{n-1}\times\mathbb{R}_+$ *such that for* $\chi\in S^0_{\mathsf{T},\tau}$, *homogeneous of degree 0, with* $\mathrm{supp}(\chi)\subset\mathscr{U}$, *there exist* $C>0$ *and* $\tau_*>0$ *such that*

(4.4)

$$\begin{aligned}\|P_{\ell,\varphi}v_\ell\|^2_+ + \|P_{r,\varphi}v_r\|^2_+ + \sum_{j=1}^m \big|T^j_{\ell,\varphi}v_{\ell|x_n=0^+}+T^j_{r,\varphi}v_{r|x_n=0^+}\big|^2_{m-1/2-\beta^j,\tau} + \|v_\ell\|^2_{m_\ell,-1,\tau}+\|v_r\|^2_{m_r,-1,\tau}\\ + |\mathrm{tr}(v_\ell)|^2_{m_\ell-1,-1/2,\tau}+|\mathrm{tr}(v_r)|^2_{m_r-1,-1/2,\tau}\ge C\big(\tau^{-1}\,\|\mathrm{Op}(\chi)v_\ell\|^2_{m_\ell,\tau}+\tau^{-1}\,\|\mathrm{Op}(\chi)v_r\|^2_{m_r,\tau}\\ + |\mathrm{tr}(\mathrm{Op}(\chi)v_\ell)|^2_{m_\ell-1,1/2,\tau}+|\mathrm{tr}(\mathrm{Op}(\chi)v_r)|^2_{m_r-1,1/2,\tau}\big),\end{aligned}$$

for $\tau\ge\tau_*$, $v_\ell,v_r\in\mathscr{S}(\overline{\mathbb{R}}^n_+)$.

Note that there are remainder terms, viz.

$$\big\|v_{r/\ell}\big\|^2_{m_{r/\ell},-1,\tau},\quad \big|\mathrm{tr}(v_{r/\ell})\big|^2_{m_{r/\ell}-1,-1/2,\tau}$$

that concern the unknown functions $v_{r/\ell}$ everywhere and not only in the microlocal region $\mathscr{U}$ we consider here. The norms of these remainder terms are weaker that those in the r.h.s. of the estimates. When patching microlocal estimates of the form of (4.4) together these remainder terms can be dealt with; see Section 4.4 below.

Proof. Let U_0 be a open neighborhood of x_0 in $\overline{V_+}$ with the sub-ellipticity condition holding in $\overline{U_0}$.

In the local coordinates we have chosen we have

$$P_{r/\ell}=P_{r/\ell}(x,D)=\sum_{j=1}^m P_{r/\ell,j}(x,D')D^j_n,$$

with $P_{r/\ell,m}=1$ (see Section 1.6). We decompose the conjugated operator $P_{r/\ell,\varphi}=e^{\tau\varphi_{r/\ell}}P_{r/\ell}e^{-\tau\varphi_{r/\ell}}$ as

$$P_{r/\ell,\varphi}=P_{r/\ell,2}+iP_{r/\ell,1},\quad P_{r/\ell,2}=\frac{1}{2}(P_{r/\ell,\varphi}+P^*_{r/\ell,\varphi}),\ \ P_{r/\ell,1}=\frac{1}{2i}(P_{r/\ell,\varphi}-P^*_{r/\ell,\varphi}).$$

The operators $P_{r/\ell,2}$ and $P_{r/\ell,1}$ are thus formally self-adjoint. Their respective principal symbols $a_{r/\ell}(x,\xi,\tau) \in S_\tau^{m_{r/\ell},0}$ and $b_{r/\ell}(x,\xi,\tau) \in S_\tau^{m_{r/\ell}-1,1}$ are both real and homogeneous. We set $p_{r/\ell,\varphi} = a_{r/\ell} + ib_{r/\ell}$. We then set

$$Q_{a,b}^{r/\ell}(v) = 2\,\mathrm{Re}(A_{r/\ell}v_{r/\ell}, iB_{r/\ell}v_{r/\ell})_+, \qquad A_{r/\ell} = \mathrm{Op}(a_{r/\ell}),\ B_{r/\ell} = \mathrm{Op}(b_{r/\ell}).$$

Note that we have

$$P_{r/\ell,\varphi} = A_{r/\ell} + iB_{r/\ell} + R_{r/\ell}, \qquad R_{r/\ell} \in \Psi_\tau^{m_{r/\ell},-1}. \tag{4.5}$$

The sub-ellipticity condition of Definition 1.1 reads

$$p_{r/\ell,\varphi}(x,\xi,\tau) = 0 \ \Rightarrow\ \{a_{r/\ell}, b_{r/\ell}\}(x,\xi,\tau) > 0,$$

for $x \in U_0$ and $(\xi,\tau) \neq (0,0)$. Note that the case $\tau = 0$ is achieved because of the ellipticity of P (see Definition 1.1 and Remark 1.2).

Let now $\mathscr{U}$ be as given by Proposition 4.2, possibly reduced so that $\mathscr{U} \subset U_0 \times \mathbb{R}^{n-1} \times \mathbb{R}_+$, and let χ be as in the statement of the theorem. By Lemma 4.3 we then have, for $\underline{v}_{r/\ell} = \mathrm{Op}(\chi)v_{r/\ell}$,

$$\begin{aligned} Q_{a,b}^{r/\ell}(\underline{v}_{r/\ell}) - \mathrm{Re}\,\mathscr{B}_{a_{r/\ell},b_{r/\ell}}(\underline{v}_{r/\ell}) &\geq C\tau^{-1}\big\|\underline{v}_{r/\ell}\big\|^2_{m_{r/\ell},\tau} \\ &\quad - C'\tau^{-1}\big(\big\|A_{r/\ell}\underline{v}_{r/\ell}\big\|_+^2 + \big\|B_{r/\ell}\underline{v}_{r/\ell}\big\|_+^2 + \big|\,\mathrm{tr}(\underline{v}_{r/\ell})\big|^2_{m_{r/\ell}-1,1/2,\tau}\big), \end{aligned} \tag{4.6}$$

with $\mathscr{B}_{a_{r/\ell},b_{r/\ell}}$ satisfying

$$\big|\mathscr{B}_{a_{r/\ell},b_{r/\ell}}(\underline{v}_{r/\ell})\big| \lesssim \big|\,\mathrm{tr}(\underline{v}_{r/\ell})\big|^2_{m_{r/\ell}-1,1/2,\tau}.$$

With Proposition 4.2, making use of the transmission condition, we obtain for M chosen sufficiently large

$$\begin{aligned} \mathrm{Re}\,\mathscr{B}_{a_\ell,b_\ell}(\underline{v}_\ell) + \mathrm{Re}\,\mathscr{B}_{a_r,b_r}(\underline{v}_r) + M\sum_{j=1}^m \big|T^j_{\ell,\varphi}v_{\ell|x_n=0^+} + T^j_{r,\varphi}v_{r|x_n=0^+}\big|^2_{m-1/2-\beta^j,\tau} \\ \geq C\,|\mathrm{tr}(\underline{v}_\ell)|^2_{m_\ell-1,1/2,\tau} + |\mathrm{tr}(\underline{v}_r)|^2_{m_r-1,1/2,\tau} - C'\big(\,\|v_\ell\|^2_{m_\ell,-1,\tau} + \|v_r\|^2_{m_r,-1,\tau} \\ + |\mathrm{tr}(v_\ell)|^2_{m_\ell-1,-1/2,\tau} + |\mathrm{tr}(v_r)|^2_{m_r-1,-1/2,\tau} + \|P_{\ell,\varphi}v_\ell\|_+^2 + \|P_{r,\varphi}v_r\|_+^2\,\big). \end{aligned} \tag{4.7}$$

Summing $(4.6)_\ell$, $(4.6)_r$, and (4.7) we find, by taking τ sufficiently large,

$$\begin{aligned} \|P_{\ell,\varphi}v_\ell\|_+^2 + \|P_{r,\varphi}v_r\|_+^2 + \sum_{j=1}^m \big|T^j_{\ell,\varphi}v_{\ell|x_n=0^+} + T^j_{r,\varphi}v_{r|x_n=0^+}\big|^2_{m-1/2-\beta^j,\tau} \\ + Q^\ell_{a,b}(\underline{v}_\ell) + Q^r_{a,b}(\underline{v}_r) + \tau^{-1}(\|A_\ell\underline{v}\|_+^2 + \|B_\ell\underline{v}_\ell\|_+^2 + \|A_r\underline{v}_r\|_+^2 + \|B_r\underline{v}_r\|_+^2) \\ + \|v_\ell\|^2_{m_\ell,-1,\tau} + \|v_r\|^2_{m_r,-1,\tau} + |\mathrm{tr}(v_\ell)|^2_{m_\ell-1,-1/2,\tau} + |\mathrm{tr}(v_r)|^2_{m_r-1,-1/2,\tau} \\ \gtrsim \tau^{-1}\big(\,\|\underline{v}_\ell\|^2_{m_\ell,\tau} + \|\underline{v}_r\|^2_{m_r,\tau}\,\big) + |\mathrm{tr}(\underline{v}_\ell)|^2_{m_\ell-1,1/2,\tau} + |\mathrm{tr}(\underline{v}_r)|^2_{m_r-1,1/2,\tau}\,. \end{aligned}$$

Finally, noting that

$$\begin{aligned} \tau^{-1}\big(\big\|A_{r/\ell}\underline{v}_{r/\ell}\big\|_+^2 + \big\|B_{r/\ell}\underline{v}_{r/\ell}\big\|_+^2\big) + Q^{r/\ell}_{a,b}(\underline{v}_{r/\ell}) &\leq \big\|(A_{r/\ell} + iB_{r/\ell})\underline{v}_{r/\ell}\big\|_+^2 \\ &\lesssim \big\|P_{r/\ell,\varphi}\underline{v}_{r/\ell}\big\|_+^2 + \big\|\underline{v}_{r/\ell}\big\|^2_{m_{r/\ell},-1,\tau} \\ &\lesssim \big\|P_{r/\ell,\varphi}v_{r/\ell}\big\|_+^2 + \big\|v_{r/\ell}\big\|^2_{m_{r/\ell},-1,\tau}\,, \end{aligned}$$

by (4.5) and pseudo-differential calculus (last point of Proposition 2.7), we obtain the sought microlocal estimate. □

4.4. **Proof of Theorem 1.6.** We shall patch together estimates of the form given in Theorem 4.4.

With x_0 as in the statement of Theorem 1.6 the transmission condition holds for all boundary quadruples $\omega = (x_0, Y, \nu_0, \tau)$ with $Y \in T^*_{x_0}(\partial\Omega)$, $\nu_0 \in N^*_{x_0}(S)$, and $\tau \geq 0$. In the local coordinates that we use here this means that this property is satisfied for $\nu = d_{x_n}$ equal to the (oriented) unit conormal to $\{x_n = 0\}$ and all $\varrho' = (x_0, \xi', \tau)$ with $\xi' \in \mathbb{R}^{n-1}$ and $\tau \geq 0$. (See Section 1.6.4.) It is fact sufficient to consider $(\xi', \tau) \in \mathbb{S}^{n-1}_+ = \{(\xi', \tau) \in \mathbb{R}^n,\ \tau \geq 0,\ |(\xi', \tau)| = 1\}$.

By Theorem 4.4 for all $(\xi'_0, \tau_0) \in \mathbb{S}^{n-1}_+$ there exists a conic openneighborhood $\mathscr{U}_{\varrho'_0}$ of $\varrho'_0 = (x_0, \xi'_0, \tau_0)$ in $\overline{V_+} \times \mathbb{R}^{n-1} \times \mathbb{R}_+$ such that the estimate (4.4) holds. In fact by reducing $\mathscr{U}_{\varrho'_0}$ we can choose $\mathscr{U}_{\varrho'_0} = \mathcal{O}_{\varrho'_0} \times \Gamma_{\varrho'_0}$ where $\mathcal{O}_{\varrho'_0}$ is an open set in $\overline{V_+}$ and $\Gamma_{\varrho'_0}$ is a conic open set in $\mathbb{R}^{n-1} \times \mathbb{R}_+$. With the compactness of $\mathbb{S}^{n-1}_+$ we can thus find finitely many such open sets $\mathscr{U}_j = \mathcal{O}_j \times \Gamma_j$, $j \in J$, such that $\mathbb{S}^{n-1}_+ \subset \cup_{j\in J}\Gamma_j$. We then set $\mathcal{O} = \cap_{j\in J}\mathcal{O}_j$ that is an open neighborhood of x_0 in $\overline{V_+}$ and we set $\mathscr{V}_j = \mathcal{O} \times \Gamma_j \subset \mathscr{U}_j$. We also choose an open neighborhood W of x_0 in $\mathbb{R}^n$ such that $W^+ = W \cap \overline{V_+} \Subset \mathcal{O}$.

We then choose a partition of unity, $\chi_j \in S^0_{\mathsf{T},\tau}$, $j \in J$, on $\overline{W_+} \times \mathbb{R}^{n-1} \times R_+$ subordinated by the covering by the open sets $\mathscr{V}_j$:

$$\sum_{j\in J} \chi_j(\varrho') = 1, \text{ for } \varrho' = (x, \xi', \tau) \in \overline{W_+} \times \mathbb{R}^{n-1} \times R_+ \text{ and } |(\xi', \tau)| \geq r_0 > 0, \qquad \operatorname{supp}(\chi_j) \subset \mathscr{V}_j.$$

The symbols χ_j are chosen homogeneous of degree 0 for $|(\xi', \tau)| \geq r_0 > 0$. We set $\underline{\chi} = 1 - \sum_{j\in J}\chi_j$ and have $\underline{\chi} \in \cap_{N\in\mathbb{N}} S^{-N}_{\mathsf{T},\tau}$.

As $\operatorname{supp}(\chi_j) \subset \mathscr{U}_j$, we can apply the microlocal estimate of Theorem 4.4:

(4.8)

$$\begin{aligned}\|P_{\ell,\varphi}v_\ell\|^2_+ + \|P_{r,\varphi}v_r\|^2_+ + \sum_{j=1}^m \big|T^j_{\ell,\varphi}v_{\ell|x_n=0^+} + T^j_{r,\varphi}v_{r|x_n=0^+}\big|^2_{m-1/2-\beta^j,\tau} + \|v_\ell\|^2_{m_\ell,-1,\tau} + \|v_r\|^2_{m_r,-1,\tau}\\ + |\mathrm{tr}(v_\ell)|^2_{m_\ell-1,-1/2,\tau} + |\mathrm{tr}(v_r)|^2_{m_r-1,-1/2,\tau} \gtrsim \tau^{-1}\big(\|v_\ell\|^2_{m_\ell,\tau} + \|v_r\|^2_{m_r,\tau}\big)\\ + |\mathrm{tr}(\mathrm{Op}(\chi_j)v_\ell)|^2_{m_\ell-1,1/2,\tau} + |\mathrm{tr}(\mathrm{Op}(\chi_j))v_r)|^2_{m_r-1,1/2,\tau}\,,\end{aligned}$$

for τ chosen sufficiently large and for $v_{r/\ell} = e^{\tau\varphi_{r/\ell}}u_{r/\ell}$ with $u_{r/\ell} = w_{r/\ell|\overline{\mathbb{R}^n_+}}$ with $w_{r/\ell} \in \mathscr{C}^\infty_c(W)$. (see the statement of Theorem 1.6).

Observe then that, for any $N \in \mathbb{N}$,

$$\big\|v_{r/\ell}\big\|_{m,\tau} \leq \sum_{j\in J}\big\|\mathrm{Op}(\chi_j)v_{r/\ell}\big\|_{m,\tau} + \big\|\mathrm{Op}(\underline{\chi})v_{r/\ell}\big\|_{m,\tau} \lesssim \sum_{j\in J}\big\|\mathrm{Op}(\chi_j)v_{r/\ell}\big\|_{m,\tau} + \big\|v_{r/\ell}\big\|_{m,-N,\tau}\,,$$

and

$$\begin{aligned}\big|\mathrm{tr}(v_{r/\ell})\big|_{m-1,1/2,\tau} &\leq \sum_{j\in J}\big|\mathrm{tr}(\mathrm{Op}(\chi_j)v_{r/\ell})\big|_{m-1,1/2,\tau} + \big|\mathrm{tr}(\mathrm{Op}(\underline{\chi})v_{r/\ell})\big|_{m-1,1/2,\tau}\\ &\lesssim \sum_{j\in J}\big|\mathrm{tr}(\mathrm{Op}(\chi_j)v_{r/\ell})\big|_{m-1,1/2,\tau} + \big|\mathrm{tr}(v_{r/\ell})\big|_{m-1,-N,\tau}\,.\end{aligned}$$

Summing estimates (4.8) for each χ_j we thus obtain

$$\begin{aligned}\|P_{\ell,\varphi}v_\ell\|^2_+ + \|P_{r,\varphi}v_r\|^2_+ + \sum_{j=1}^m \big|T^j_{\ell,\varphi}v_{\ell|x_n=0^+} + T^j_{r,\varphi}v_{r|x_n=0^+}\big|^2_{m-1/2-\beta^j,\tau} + \|v_\ell\|^2_{m_\ell,-1,\tau} + \|v_r\|^2_{m_r,-1,\tau}\\ + |\mathrm{tr}(v_\ell)|^2_{m_\ell-1,-1/2,\tau} + |\mathrm{tr}(v_r)|^2_{m_r-1,-1/2,\tau} \gtrsim \tau^{-1}\big(\|v_\ell\|^2_{m_\ell,\tau} + \|v_r\|^2_{m_r,\tau}\big)\\ + |\mathrm{tr}(v_\ell)|^2_{m_\ell-1,1/2,\tau} + |\mathrm{tr}(v_r)|^2_{m_r-1,1/2,\tau}\,,\end{aligned}$$

Choosing now τ sufficiently large we obtain

$$\|P_{\ell,\varphi}v_\ell\|_+^2+\|P_{r,\varphi}v_r\|_+^2+\sum_{j=1}^m\big|T^j_{\ell,\varphi}v_{\ell|x_n=0^+}+T^j_{r,\varphi}v_{r|x_n=0^+}\big|^2_{m-1/2-\beta^j,\tau}$$
$$\gtrsim\tau^{-1}\big(\|v_\ell\|^2_{m_\ell,\tau}+\|v_r\|^2_{m_r,\tau}\big)+|\mathrm{tr}(v_\ell)|^2_{m_\ell-1,1/2,\tau}+|\mathrm{tr}(v_r)|^2_{m_r-1,1/2,\tau}\,,$$

Setting $v_{r/\ell}=e^{\tau\varphi_{r/\ell}}u_{r/\ell}$ the conclusion of the proof of Theorem 1.6 is then classical. □

4.5. **Shifted estimates.** As in [4] it may be interesting to consider shifted estimates in the Sobolev scales. Namely we may wish to have an estimate of the following form.

Corollary 4.5. *Let $x_0\in S$ and let $\varphi\in\mathscr{C}^0(\Omega)$ be such that $\varphi_k=\varphi_{|\Omega_k}\in\mathscr{C}^\infty(\Omega_k)$ for $k=1,2$ and such that the pairs $\{P_k,\varphi_k\}$ have the sub-ellipticity property of Definition 1.1 in a neighborhood of x_0 in $\overline{\Omega_k}$. Moreover, assume that $\big\{P_k,\varphi,T^j_k,\ k=1,2,\ j=1,\dots,m\big\}$ satisfies the transmission condition at x_0. Let $\ell\in\mathbb{N}$. Then there exist a neighborhood W of x_0 in $\mathbb{R}^n$ and two constants C and $\tau_*>0$ such that*

$$\sum_{k=1,2}\big(\tau^{-1}\|e^{\tau\varphi_k}u_k\|^2_{\ell+m_k,\tau}+|e^{\tau\varphi_{|S}}\,\mathrm{tr}(u_k)|^2_{\ell+m_k-1,1/2,\tau}\big)$$
$$\leq C\Big(\sum_{k=1,2}\|e^{\tau\varphi_k}P_k(x,D)u_k\|^2_{\ell,\tau}+\sum_{j=1}^m\big|e^{\tau\varphi_{|S}}\big(T^j_1(x,D)u_1+T^j_2(x,D)u_2\big)_{|S}\big|^2_{\ell,m-1/2-\beta^j,\tau}\Big),\tag{4.9}$$

for all $u_k=w_{k|\Omega_k}$ with $w_k\in\mathscr{C}^\infty_c(W)$ and $\tau\geq\tau_*$.

The proof of this corollary can be adapted from that of its counterpart at a boundary, namely Corollary 4.5 in [4].

4.6. **Interior-eigenvalue transmission problems.** Interior-eigenvalue transmission problems are very related to the transmission problem we have considered. In fact, for Ω, a bounded open set in $\mathbb{R}^n$, we consider two elliptic operators P_1 and P_2 of respective orders m_1 and m_2, as in Section 1, yet *both* defined on Ω.

In addition, we consider $2m=m_1+m_2$ boundary operators operators

$$T^j_k=\sum_{|\alpha|\leq\beta^j_k}t^j_{k,\alpha}(x)D^\alpha,\quad k=1,2,\quad j=1,\dots,m,\tag{4.10}$$

with $0\leq\beta^j_k<m_k$, and where the coefficients $t^j_{k,\alpha}(x)$ are $\mathscr{C}^\infty$ complex-valued functions defined in some neighborhood of $\partial\Omega$. Setting $\beta^j=(\beta^j_1+\beta^j_2)/2$ we assume that

$$m_1-\beta^j_1=m_2-\beta^j_2=m-\beta^j,\quad j=1,\dots,m.\tag{4.11}$$

The interior-eigenvalue transmission problem consist in resolving a system of the form

$$\begin{cases}(P_k-\tau^{m_k})u_k=f_k & \text{in }\Omega,\ ,k=1,2\\ T^j_1u_1+T^j_2u_2=g^j, & \text{in }\Gamma,\quad j=1,\dots,m.\end{cases}$$

We refer to [12, 7, 42, 8, 9, 44, 43] and the reference therein for more details on this very active field of research.

In the analysis of such problems, resolvant estimates are central. In the proof of such resolvent estimates, a Carleman inequality at the boundary can be a very efficient tool. Here, we provide such an estimate in a neighborhood of a point of $\partial\Omega$, as the proof is in fact given by the analysis of the previous section, in particular, as we used the system formulation of Section 1.6.2, which yield a formulation close to that of the interior-eigenvalue transmission problem.

Let $x_0\in\partial\Omega$ and V be a neighborhood of x_0 where $\Omega=\{x_n>0\}$. We consider two smooth weight functions φ_1 and φ_2 in V such that $\varphi_{1|x_n=0^+}=\varphi_{2|x_n=0^+}$.

With the notation of Sections 1.6.2–1.6.3, with the letter ℓ replaced by 1 and the letter r replaced by 2, and moreover P_ℓ (resp. P_r) replaced by $P_1 - \tau^{m_1}$ (resp. $P_2 - \tau^{m_2}$), we say that the transmission condition holds at x_0 for $\{P_k - \tau^{m_k}, T_k^j, \varphi_k,\ k = 1, 2,\ j = 1, \dots, m\}$ if for all $\varrho' = (x_0, \xi', \tau)$ and for all pairs of polynomials, $q_1(\xi_n), q_2(\xi_n)$, there exist U_1, U_2, polynomials, and $c_j \in \mathbb{C}$, $j = 1, \dots, m$, such that

$$q_1(\xi_n) = \sum_{j=1}^{m} c_j t^j_{\ell,\varphi}(\varrho', \xi_n) + U_1(\xi_n)\kappa_1(\varrho', \xi_n), \tag{4.12}$$

and

$$q_2(\xi_n) = \sum_{j=1}^{m} c_j t^j_{r,\varphi}(\varrho', \xi_n) + U_2(\xi_n)\kappa_2(\varrho', \xi_n). \tag{4.13}$$

Then the proof of the following local Carleman estimate is the same as that of Theorem 1.6.

Theorem 4.6. *Let $x_0 \in \partial\Omega$ and let $\varphi_k \in \mathscr{C}^\infty(\Omega)$, $k = 1, 2$, as above, such that the pairs $\{P_k - \tau^{m_k}, \varphi_k\}$ satisfy the sub-ellipticity property of Definition 1.1 in a neighborhood of x_0 in $\overline{\Omega}$. Moreover, assume that $\{P_k - \tau^{m_k}, T_k^j, \varphi_k,\ k = 1, 2,\ j = 1, \dots, m\}$ satisfies the transmission condition at x_0. Then there exist a neighborhood W of x_0 in $\mathbb{R}^n$ and two constants C and $\tau_* > 0$ such that*

$$\sum_{k=1,2} \left(\tau^{-1} \left\|e^{\tau\varphi_k} u_k\right\|^2_{m_k,\tau} + \left|e^{\tau\varphi_k}\gamma(u_k)\right|^2_{m_k-1,1/2,\tau}\right) \tag{4.14}$$

$$\leq C\Big(\sum_{k=1,2} \left\|e^{\tau\varphi_k}(P_k(x,D) - \tau^{m_k})u_k\right\|^2_{L^2} + \sum_{j=1}^{m} \left|e^{\tau\varphi_{|\partial\Omega}}(T_1^j(x,D)u_1 + T_2^j(x,D)u_2)_{|\partial\Omega}\right|^2_{m-1/2-\beta^j,\tau}\Big),$$

for all $u_k = w_{k|\Omega}$ with $w_k \in \mathscr{C}^\infty_c(W)$ and $\tau \geq \tau_*$.

5. A pseudo-differential calculus with two large parameters

The weight function we shall consider below is of the form $\varphi(x) = \exp(\gamma\psi(x))$. The function ψ is assumed to be $\mathscr{C}^0$, piecewise smooth, and to satisfy

$$0 < C \leq \psi \quad \text{and } \left\|\psi^{(k)}\right\|_{L^\infty} < \infty,\ k \in \mathbb{N}.$$

We take $\gamma \geq 1$. The goal of what follows is to achieve estimates as in Theorem 1.6 with the explicit dependency upon the additional parameter γ. This can be done by the introduction of an appropriate pseudo-differential calculus. Assumption of the function ψ will be made in Section 6.1, namely, the strong pseudo-convexity conditions, to obtain a Carleman estimate.

5.1. Metric, symbols and Sobolev norms.

Here, by ϱ and ϱ' we shall denote $\varrho = (x, \xi, \tau, \gamma) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+$ and $\varrho' = (x, \xi', \tau, \gamma) \in \mathbb{R}^n \times \mathbb{R}^{n-1} \times \mathbb{R}_+ \times \mathbb{R}_+$.

We set $\tilde{\tau}(x) = \tau\gamma\varphi(x)$. Following [31] we consider the metrics on phase-space

$$g = \gamma^2|dx|^2 + \frac{|d\xi|^2}{\mu^2}, \qquad \text{with } \mu^2 = \mu^2(\varrho) = |(\tilde{\tau}(x), \xi)|^2 = \tilde{\tau}(x)^2 + |\xi|^2,$$

and on tangent phase space

$$g_{\mathsf{T}} = \gamma^2|dx|^2 + \frac{|d\xi'|^2}{\mu_{\mathsf{T}}^2}, \qquad \text{with } \mu_{\mathsf{T}}^2 = \mu_{\mathsf{T}}^2(\varrho') = |(\tilde{\tau}(x), \xi')|^2 = \tilde{\tau}(x)^2 + |\xi'|^2,$$

for $\tau \geq 1$ and $\gamma \geq 1$. Below, the explicit dependencies of μ and μ_{T} upon ϱ and ϱ' are dropped to ease notation.

The metric g (resp. g_{T}) along with the order function μ (resp. μ_{T}) generates a (resp. tangential) Weyl-Hörmander pseudo-differential calculus as proven in [31, Proposition 2.2]. Note that this uses the conditions $0 < C \leq \psi$ and $\left\|\psi'\right\| < \infty$.

For a presentation of the Weyl-Hörmander calculus we refer to [39], [21, Sections 18.4–6] and [20].

Let $a(x,\xi,\tau,\gamma) \in \mathscr{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, with τ,γ as parameter in $[\tau_{\min},+\infty)$ and $[\gamma_{\min},+\infty)$, $\tau_{\min} > 0$, $\gamma_{\min} > 0$, and $m \in \mathbb{R}$, be such that for all multi-indices $\gamma,\beta \in \mathbb{N}^n$ we have

$$\left|\partial_x^\gamma \partial_\xi^\beta a(\varrho)\right| \leq C_{\gamma,\beta}\gamma^\gamma \mu^{m-|\beta|}, \quad \varrho \in \mathbb{R}^n \times \mathbb{R}^n \times [\tau_{\min},+\infty) \times [\gamma_{\min},+\infty). \tag{5.1}$$

With the notation of [21, Sections 18.4-18.6] we then have $a(\varrho) \in S(\mu^m, g)(\mathbb{R}^n \times \mathbb{R}^n)$, which we write $S^m(g)$ for simplicity[6]. The associated class of pseudo-differential operators is denoted by $\Psi^m(g) = \Psi(\mu^m,g)(\mathbb{R}^n \times \mathbb{R}^n)$.

Similarly we define tangential symbols and operators. Let $a(x,\xi',\tau,\gamma) \in \mathscr{C}^\infty(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1})$ and $m \in \mathbb{R}$, be such that for all multi-indices $\gamma,\beta \in \mathbb{N}^n$ we have

$$\left|\partial_x^\gamma \partial_{\xi'}^\beta a(\varrho')\right| \leq C_{\gamma,\beta}\gamma^\gamma \mu_{\mathsf{T}}^{m-|\beta|}, \quad \varrho' \in \overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1} \times [\tau_{\min},+\infty) \times [\gamma_{\min},+\infty). \tag{5.2}$$

We then have $a(\varrho') \in S^m(g_{\mathsf{T}}) = S(\mu^m_{\mathsf{T}}, g_{\mathsf{T}})(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1})$. The associated class of tangential pseudo-differential operators is denoted by $\Psi^m(g_{\mathsf{T}}) = \Psi(\mu^m, g_{\mathsf{T}})(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1})$.

Note that the condition $\left\|\psi^{(k)}\right\| < \infty$, $k \in \mathbb{N}$, is used to prove[7] that $\tilde\tau \in S(\tilde\tau, g) \cap S(\tilde\tau, g_{\mathsf{T}})$.

With $\varrho = (x,\xi,\tau,\gamma) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+$ (resp. $\varrho' = (x,\xi',\tau,\gamma) \in \overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1} \times \mathbb{R}_+ \times \mathbb{R}_+$) we shall associate $\tilde\varrho = (x,\xi,\tilde\tau(x)) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$ (resp. $\tilde\varrho' = (x,\xi',\tilde\tau(x)) \in \overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1} \times \mathbb{R}_+$).

Note that if $\hat a(x,\xi,\hat\tau) \in S^m_\tau$, with the notation of Section 2.1, satisfying moreover, for all multi-indices $\gamma,\beta',\beta'' \in \mathbb{N}^n$, with $\beta = \beta' + \beta''$,

$$\left|\partial_x^\gamma \partial_\xi^{\beta'} \partial_{\hat\tau}^{\beta''} \hat a(x,\xi,\hat\tau)\right| \leq C_{\gamma,\beta',\beta''} \left|(\xi,\hat\tau)\right|^{m-|\beta|}, \quad x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n,\ \hat\tau \in [\tau_{\min},+\infty), \tag{5.3}$$

i.e., differentiation w.r.t. $\hat\tau$ yields the same additional decay as a differentiation w.r.t. ξ, then

$$a(x,\xi,\tau,\gamma) = \hat a(x,\xi,\tilde\tau(x)) \in S^m(g),$$

which we shall write $a(\varrho) = \hat a(\tilde\varrho)$. Similarly if $\hat a(x,\xi',\hat\tau) \in S^m_{\mathsf{T},\tau}$ with the same additional property regarding differentiation w.r.t. $\hat\tau$ we have $a(\varrho') = \hat a(\tilde\varrho') \in S^m(g_{\mathsf{T}})$. In what follows we shall assume that symbols in S^m_τ and $S^m_{\mathsf{T},\tau}$ have this additional regularity property. We then say that $a \in S^m(g)$ (resp. $S^m(g_{\mathsf{T}})$) is homogeneous of degree m with respect to $(\xi,\tilde\tau)$ (resp. $(\xi',\tilde\tau)$) if we have $a(\varrho) = \hat a(\tilde\varrho)$) (resp. $a(\varrho') = \hat a(\tilde\varrho')$) with $\hat a(x,\xi,\hat\tau) \in S^m_\tau$ (resp. $\hat a(x,\xi',\hat\tau) \in S^m_{\mathsf{T},\tau}$) homogeneous of degree m in $(\xi,\hat\tau)$ (resp. $(\xi',\hat\tau)$).

We shall also use the following classes of symbols $S(\tilde\tau^r \mu^m_{\mathsf{T}}, g_{\mathsf{T}}) = \tilde\tau^r S^m(g_{\mathsf{T}})$ on $\overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1}$, for $r,m \in \mathbb{R}$. The associated class of tangential pseudo-differential operators is denoted by $\tilde\tau^r \Psi^m(g_{\mathsf{T}}) = \tilde\tau^r \Psi(\mu^m_{\mathsf{T}}, g_{\mathsf{T}})(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1})$. We have the following lemma whose proof is similar to that of Lemma 2.7 in [31].

Lemma 5.1. *Let $r,m \in \mathbb{R}$ and $a \in \tilde\tau^r S^m(g_{\mathsf{T}})$. There exists $C > 0$ such that for τ sufficiently large*

$$\left|\left(\mathrm{Op}(a)u, v\right)_\partial\right| \leq C \left\|\mathrm{Op}(\tilde\tau^{r'}\mu^{m'}_{\mathsf{T}})u\right\|_+ \left\|\mathrm{Op}(\tilde\tau^{r''}\mu^{m''}_{\mathsf{T}})v\right\|_+, \qquad u \in \mathscr{S}(\overline{\mathbb{R}}^n_+).$$

for $r = r' + r''$, $m = m' + m''$.

[6] The dependence upon the metric g is kept explicit here as we shall actually have to face two calculi simultaneously, associated with the weight functions on both sides of the interface. Interactions between the two calculi will only occur at the interface where they coincide. See Section 5.2.

[7] This condition was not written in [31] and [4]. This is however made precise in [36], including the proof of $\tilde\tau \in S(\tilde\tau, g) \cap S(\tilde\tau, g_{\mathsf{T}})$.

This contains the estimate

$$\big\| \operatorname{Op}(\tilde\tau^s \mu_{\mathsf T}^p) \operatorname{Op}(a) u\big\|_+ \le C \big\| \operatorname{Op}(\tilde\tau^{s+r} \mu_{\mathsf T}^{p+m}) u\big\|_+, \qquad u \in \mathscr{S}(\overline{\mathbb{R}}^n_+),$$

for $s, p \in \mathbb{R}$. Note also that we have

$$\big\| \operatorname{Op}(\tilde\tau^r \mu_{\mathsf T}^m) u\big\|_+ \asymp \big\| \operatorname{Op}(\mu_{\mathsf T}^m) \tilde\tau^r u\big\|_+, \tag{5.4}$$

for τ chosen sufficiently large.

Next we say that $a(x, \xi', \tau, \gamma) \in \tilde\tau^r S^m_{\mathrm{cl}}(g_{\mathsf T})$ on $\overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1}$ if there exists a sequence $a^{(j)} \in \gamma^j \tilde\tau^r S^{m-j}(g_{\mathsf T})$, with $\gamma^{-j} a^{(j)}$ homogeneous of degree $m + r - j$ in $(\xi', \tilde\tau)$ for $|(\xi', \tilde\tau)| \ge r_0$, with $r_0 \ge 0$, such that

$$a \sim \sum_{j \ge 0} a^{(j)}, \quad \text{in the sense that} \quad a - \sum_{j=0}^N a^{(j)} \in \gamma^{N+1} \tilde\tau^r S^{m-N-1}(g_{\mathsf T}). \tag{5.5}$$

A representative of the principal part, denoted by $\sigma(a)$, is then given by the first term in the expansion. Then we shall say that $a(\varrho) \in \tilde\tau^r S^{m,\sigma}_{\mathrm{cl}}(g)$ on $\overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1}$ if

$$a(\varrho) = \sum_{j=0}^m a_j(\varrho') \xi_n^j, \quad \text{with } a_j \in \tilde\tau^r S^{m-j+\sigma}_{\mathrm{cl}}(g_{\mathsf T}).$$

The principal part is given by $\sum_{j=0}^m \sigma(a_j)(\varrho') \xi_n^j$. With these symbol classes we associate classes of pseudo-differential operators, $\tilde\tau^r \Psi^m_{\mathrm{cl}}(g_{\mathsf T}) = \tilde\tau^r \Psi^m_{\mathrm{cl}}(g_{\mathsf T})(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1})$ and $\tilde\tau^r \Psi^{m,\sigma}_{\mathrm{cl}}(g) = \tilde\tau^r \Psi^{m,\sigma}_{\mathrm{cl}}(g)(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1})$, as is done in Section 2.2.

We define the following semi-classical interior norm, for $m \in \mathbb{N}$,

$$\|u\|^2_{m,\tilde\tau} = \sum_{j=0}^m \big\| \operatorname{Op}(\mu_{\mathsf T}^{m-j}) D_n^j u\big\|^2_+, \qquad u \in \mathscr{S}(\overline{\mathbb{R}}^n_+). \tag{5.6}$$

We also set, for $m \in \mathbb{N}$ and $\sigma \in \mathbb{R}$,

$$\|u\|^2_{m,\sigma,\tilde\tau} = \big\| \operatorname{Op}(\mu_{\mathsf T}^\sigma) u\big\|^2_{m,\tilde\tau} \sim \sum_{j=0}^m \big\| \operatorname{Op}(\mu_{\mathsf T}^{m-j+\sigma}) D_n^j u\big\|^2_+, \qquad u \in \mathscr{S}(\overline{\mathbb{R}}^n_+). \tag{5.7}$$

At the interface $\{x_n = 0^+\}$ we define the following norms, for $m \in \mathbb{N}$ and $\sigma \in \mathbb{R}$,

$$|\operatorname{tr}(u)|^2_{m,\sigma,\tilde\tau} = \sum_{j=0}^m \big| \operatorname{Op}(\mu_{\mathsf T}^{m-j+\sigma}) \operatorname{tr}_j(u)\big|^2_\partial, \qquad u \in \mathscr{S}(\overline{\mathbb{R}}^n_+). \tag{5.8}$$

5.2. **Transmission problem with two calculi.** In the present setting, using the system formulation of Section 1.6.2 we shall in fact work in $\{x_n \ge 0\}$ with two weight functions, namely $\varphi_{r/\ell} = e^{\gamma \psi_{r/\ell}}$. With each weight function we shall associate a pseudo-differential calculus, classes of symbols and pseudo-differential operators, and Sobolev norms, as introduced in the previous section.

We shall thus define $\tilde\tau_{r/\ell}(x) = \tau \gamma \varphi_{r/\ell}(x)$,

$$\mu^2_{r/\ell} = \tilde\tau^2_{r/\ell} + |\xi|^2, \qquad \mu^2_{\mathsf T, r/\ell} = \tilde\tau^2_{r/\ell} + |\xi|^2,$$

the associated metrics

$$g_{r/\ell} = \gamma^2 |dx|^2 + \frac{|d\xi|^2}{\mu^2_{r/\ell}}, \qquad g_{\mathsf T, r/\ell} = \gamma^2 |dx|^2 + \frac{|d\xi'|^2}{\mu^2_{\mathsf T, r/\ell}},$$

and the symbol classes $S^m(g_{r/\ell})$, $S^m(g_{\mathsf T, r/\ell})$, $S^m_{\mathrm{cl}}(g_{\mathsf T, r/\ell})$, $S^{m,\sigma}_{\mathrm{cl}}(g_{\mathsf T, r/\ell})$ and the associated operator classes $\Psi^m(g_{r/\ell})$, $\Psi^m(g_{\mathsf T, r/\ell})$, $\Psi^m_{\mathrm{cl}}(g_{\mathsf T, r/\ell})$, $\Psi^{m,\sigma}_{\mathrm{cl}}(g_{\mathsf T, r/\ell})$.

Accordingly for a function defined in $\{x_n \geq 0\}$ we denote by $\|u\|_{m,\tilde{\tau}_{r/\ell}}$ and $\|u\|_{m,\sigma,\tilde{\tau}_{r/\ell}}$ the associated norms as in (5.6)–(5.7).

Observe that the two calculi coincide at $x_n = 0$, that is, on the interface, since $\psi_{\ell|x_n=0^+} = \psi_{r|x_n=0^+}$, implying $\varphi_{\ell|x_n=0^+} = \varphi_{r|x_n=0^+}$ and $\mu_{\ell|x_n=0^+} = \mu_{r|x_n=0^+}$. In particular we shall keep the notation

$$|\mathrm{tr}(u)|_{m,\sigma,\tilde{\tau}} = |\mathrm{tr}(u)|_{m,\sigma,\tilde{\tau}_\ell} = |\mathrm{tr}(u)|_{m,\sigma,\tilde{\tau}_r}$$

as in (5.8) for interface norms.

5.3. **Interface quadratic forms.**

Definition 5.2. *Let $w = (w_\ell, w_r) \in \big(\mathscr{S}(\overline{\mathbb{R}}^n_+)\big)^2$. We say that*

$$\mathscr{G}(w) = \sum_{s=1}^{N} \big(A^s_\ell w_{\ell|x_n=0^+} + A^s_r w_{r|x_n=0^+}, B^s_\ell w_{\ell|x_n=0^+} + B^s_r w_{r|x_n=0^+}\big)_\partial\,,$$

with $A^s_{r/\ell} = a^s_{r/\ell}(x, D, \tau, \gamma)$ and $B^s_{r/\ell} = b^s_{r/\ell}(x, D, \tau, \gamma)$, is an interface quadratic form of type $(m_\ell - 1, m_r - 1, \sigma)$ with $\mathscr{C}^\infty$ coefficients, if for each $s = 1, \dots, N$, we have $a^s_{r/\ell}(\varrho), b^s_{r/\ell}(\varrho) \in S^{m_{r/\ell}-1,\sigma_{r/\ell}}_{\mathrm{cl}}(g_{\mathsf{T},r/\ell})(\overline{\mathbb{R}}^n_+ \times \mathbb{R}^n)$, with $\sigma_\ell + \sigma_r = 2\sigma$, $\varrho = (\varrho', \xi_n)$ with $\varrho' = (x, \xi', \tau, \gamma)$.

As in Section 3 we associate to $\mathscr{G}$ a bilinear symbol $\Sigma_{\mathscr{G}}(\varrho', \mathbf{w}, \tilde{\mathbf{w}})$.

We let $\mathscr{W}$ be an open conic set in $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}_+$.

Definition 5.3. *Let $\mathscr{G}$ be an interface quadratic form of type $(m_\ell - 1, m_r - 1, \sigma)$ associated with the bilinear symbol $\Sigma_{\mathscr{G}}(\varrho', \mathbf{w}, \tilde{\mathbf{w}})$. We say that $\mathscr{G}$ is positive definite in $\mathscr{W}$ if there exists $C > 0$ and $R > 0$ such that*

$$\mathrm{Re}\,\Sigma_{\mathscr{G}}(\varrho'', x_n = 0^+, \mathbf{w}, \mathbf{w}) \geq C\Big(\sum_{j=0}^{m_\ell-1} \mu_{\mathsf{T},\ell|x_n=0^+}^{2(m_\ell-1-j+\sigma_\ell)} \big|z^\ell_j\big|^2 + \sum_{j=0}^{m_r-1} \mu_{\mathsf{T},r|x_n=0^+}^{2(m_r-1-j+\sigma_r)} \big|z^r_j\big|^2\Big),$$

for any $\mathbf{w} = (\mathbf{z}^\ell, \mathbf{z}^r)$, $\mathbf{z}^{r/\ell} = (z^{r/\ell}_0, \dots, z^{r/\ell}_{m_{r/\ell}-1}) \in \mathbb{C}^{m_{r/\ell}}$, and $\tilde{\varrho}'' \in \mathscr{W}$, such that $\mu_{\mathsf{T},\ell|x_n=0^+} = \mu_{\mathsf{T},r|x_n=0^+} \geq R$, with $\varrho'' = (x', \xi', \tau, \gamma)$ and $\tilde{\varrho}'' = (x', \xi', \tilde{\tau}(x', x_n = 0^+))$.

We have the following Gårding estimate.

Lemma 5.4. *Let $\mathscr{W}$ be an open conic set in $\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}_+$ and let $\mathscr{G}$ be an interface quadratic form of type $(0, m_\ell - 1, m_r - 1, \sigma)$ that is positive definite in $\mathscr{W}$. Let $\hat{\chi} \in S^0_{\mathsf{T},\tau}$ be homogeneous of degree 0, with $\mathrm{supp}(\hat{\chi}_{|x_n=0^+}) \subset \mathscr{W}$ and let $N \in \mathbb{N}$. Then there exist $\tau_* \geq 1$, $\gamma_* \geq 1$, $C > 0$, $C_N > 0$ such that*

$$\begin{aligned}\mathrm{Re}\,\mathscr{G}(\mathrm{Op}(\chi)u) \geq C\big(|\mathrm{tr}(\mathrm{Op}(\chi_\ell)u_\ell)|^2_{m_\ell-1,\sigma_\ell,\tilde{\tau}} + |\mathrm{tr}(\mathrm{Op}(\chi_r)u_r)|^2_{m_r-1,\sigma_r,\tilde{\tau}}\big)\\ - C_N\big(|\mathrm{tr}(u_\ell)|^2_{m_\ell-1,\sigma_\ell-N,\tilde{\tau}} + |\mathrm{tr}(u_r)|^2_{m_r-1,\sigma_r-N,\tilde{\tau}}\big)\end{aligned}$$

for $u = (u_\ell, u_r) \in \big(\mathscr{S}(\overline{\mathbb{R}}^n_+)\big)^2$, $\tau \geq \tau_*$, $\gamma \geq \gamma_*$, and $\chi_{r/\ell}(\varrho') = \hat{\chi}(\tilde{\varrho}'_{r/\ell}) \in S^0_{\mathsf{T},\tilde{\tau}}$, with $\varrho' = (x, \xi', \tau, \gamma)$ and $\tilde{\varrho}'_{r/\ell} = (x, \xi', \tilde{\tau}_{r/\ell}(x))$.

The proof is similar to that of Lemma 3.3 using that the two calculi, associated with ψ_ℓ and ψ_r respectively, coincide on the interface $S = \{x_n = 0\}$. In particular note that $\chi_{\ell|x_n=0^+} = \chi_{r|x_n=0^+}$ as $\tilde{\tau}_\ell$ coincides with $\tilde{\tau}_r$ at the interface.

6. Carleman estimate with two large parameters

With a weight function of the form $\varphi(x) = \exp(\gamma\psi(x))$, some condition on ψ can yield φ to fulfill the sub-ellipticity condition of Definition 1.1. Those are the strong pseudo-convexity conditions introduced by L. Hörmander (see [18], [19, Section 8.6] and [22, Section 28.3]). We shall see that along with the transmission condition they are sufficient to derive Carleman estimates with an explicit dependency upon the additional parameter γ. In fact the strong pseudo-convexity condition is also necessary if one considers a weight function of this form; for such question we refer to [31].

6.1. Strong pseudo-convexity.

We recall the notion of strong pseudo-convexity and then adapt it to the geometry we consider.

As we restrict ourselves to elliptic operators in the present article, the classical notion of strong pseudo-convexity then reduces to the following one (the reader can compare with Section 28.3 in [22]).

Definition 6.1 (strong pseudo-convexity up to a boundary)**.** *Let $\mathcal{O}$ be an open set. We say that a smooth function ψ is strongly pseudo-convex at $x \in \overline{\mathcal{O}}$ w.r.t. p if $\psi'(x) \neq 0$ and if for all $\xi \in \mathbb{R}^n$ and $\hat\tau > 0$,*

$$\text{(s-}\Psi\text{c)}\quad p(x,\xi + i\hat\tau\psi'(x)) = 0 \text{ and } \big\{p,\psi\big\}(x,\xi+i\hat\tau\psi'(x)) = 0 \\ \Rightarrow \quad \frac{1}{2i}\big\{\overline{p}(x,\xi - i\hat\tau\psi'(x)), p(x,\xi+i\hat\tau\psi'(x))\big\} > 0.$$

Let U be an open subset of $\mathcal{O}$. The function ψ is said to be strongly pseudo-convex w.r.t. p in U up to the boundary *if* (s-Ψc) *is valid for all $x \in \overline{U}$.*

Definition 6.2 (strong pseudo-convexity at an interface)**.** *Let Ω, Ω_1, Ω_2, and S be as in Section 1. Let ψ be a continuous function such that $\psi_k = \psi_{|\Omega_k}$ are smooth for $k = 1, 2$. Let U be an open subset of Ω that meets S. The function ψ is said to be strongly pseudo-convex w.r.t. P_1 and P_2 in U up to the interface if both ψ_k, $k = 1, 2$, are strongly pseudo-convex w.r.t. P_k in $U_k = U \cap \Omega_k$ up to the boundary.*

Note in particular that for $x \in S \cap U$ (s-Ψc) *is required to hold for both $k = 1$ and $k = 2$.*

6.2. Conjugated operators and transmission condition.

Here we use directly the notation introduced in Section 1.6.2 with the weight functions of the form $\varphi_{r/\ell} = \exp(\gamma\psi_{r/\ell})$, which is sensible as the transmission is a coordinate invariant property.

The principal symbol of $P_{r/\ell,\varphi} = e^{\tau\varphi_{r/\ell}} P_{r/\ell} e^{-\tau\varphi_{r/\ell}} \in \Psi^{m,0}_{\mathrm{cl}}(g_{r/\ell})$ in the present calculus is

$$p_{r/\ell,\varphi}(x,\xi,\tau) = p_{r/\ell}(x,\xi + i\tau\varphi'_{r/\ell}(x)) = p_{r/\ell}(x,\xi+i\tilde\tau_{r/\ell}(x)\psi'_{r/\ell}(x)) = p_{r/\ell,\psi}(x,\xi,\tilde\tau_{r/\ell}(x)) \in S^{m,0}_{\mathrm{cl}}(g_{r/\ell}),$$

Similarly, the principal symbol of $T^j_{r/\ell,\varphi} = e^{\tau\varphi_{r/\ell}} T^j_{r/\ell} e^{-\tau\varphi_{r/\ell}} \in \Psi^{\beta_k,0}_{\mathrm{cl}}(g_{r/\ell})$, $j = 1,\dots,m$, is

$$t^j_{r/\ell,\varphi}(x,\xi,\tau) = t^j_{r/\ell}(x,\xi+i\tau\varphi'(x)) = t^j_{r/\ell}(x,\xi+i\tilde\tau_{r/\ell}(x)\psi'(x)) = t^j_{r/\ell,\psi}(x,\xi,\tilde\tau_{r/\ell}(x)) \in S^{\beta_k,0}_{\mathrm{cl}}(g_{r/\ell}).$$

The dependency upon γ is hidden either in φ or in $\tilde\tau$.

Setting $\kappa_{r/\ell,\varphi} = p^+_{r/\ell,\varphi} p^0_{r/\ell,\varphi}$ and $\kappa_{r/\ell,\psi} = p^+_{r/\ell,\psi} p^0_{r/\ell,\psi}$, we then find

$$\kappa_{r/\ell,\varphi}(x,\xi,\tau) = \kappa_{r/\ell,\psi}(x,\xi,\tilde\tau_{r/\ell}(x)).$$

From these simple observations we thus conclude that $\{P_{r/\ell}, T^j_{r/\ell}, \varphi,\ j = 1,\dots,m\}$ satisfies the transmission condition at (x_0,ξ'_0,τ_0), with $x_0 \in S$, *if and only if* $\{P_{r/\ell}, T^j_{r/\ell}, \psi,\ j = 1,\dots,m\}$ satisfies the transmission condition at $(x_0,\xi'_0,\tilde\tau_0)$ with $\tilde\tau_0 = \tilde\tau_\ell(x_0) = \tilde\tau_r(x_0)$.

6.3. **Statement of the Carleman estimate with two large parameters.** We shall prove the following theorem, counterpart of Theorem 1.6 in the case of a weight function of the form $\varphi = \exp(\gamma\psi)$, with an explicit dependency with respect to the second large parameter γ.

Theorem 6.3. *Let $x_0 \in S$ and let $\psi \in \mathscr{C}^0(\Omega)$ be such that $\psi_k = \psi_{|\Omega_k} \in \mathscr{C}^\infty(\Omega_k)$ for $k = 1, 2$ and such that ψ has the strong pseudo-convexity property of Definition 6.2 with respect to P_1 and P_2 in a neighborhood of x_0 in Ω. Moreover, assume that $\{P_k, T_k^j, \psi, k = 1, 2,\ j = 1, \dots, m\}$ satisfies the transmission condition at x_0. Then there exist a neighborhood W of x_0 in $\mathbb{R}^n$ and three constants C, $\tau_* > 0$, and $\gamma_* > 0$ such that for $\varphi_k = \exp(\gamma\psi_k)$ and $\tilde\tau_k = \tau\gamma\varphi_k$:*

$$\sum_{k=1,2} \big(\big\| \tilde\tau_k^{-1/2} e^{\tau\varphi_k} u_k \big\|^2_{m_k,\tilde\tau_k} + |e^{\tau\varphi_{|S}} \operatorname{tr}(u_k)|^2_{m_k-1,1/2,\tilde\tau} \big) \tag{6.1}$$
$$\leq C \Big(\sum_{k=1,2} \|e^{\tau\varphi_k} P_k(x,D) u_k\|^2_{L^2(\Omega_k)} + \sum_{j=1}^m |e^{\tau\varphi_{|S}} (T_1^j(x,D)u_1 + T_2^j(x,D)u_2)_{|S}|^2_{m-1/2-\beta^j,\tilde\tau} \Big),$$

for all $u_k = w_{k|\Omega_k}$ with $w_k \in \mathscr{C}^\infty_c(W)$, $\tau \geq \tau_*$, and $\gamma \geq \gamma_*$.

Here norms of defined on Ω_k and S. They are locally equivalent to their counterpart defined on $\{x_n > 0\}$ and $\{x_n = 0\}$ above.

6.4. **Preliminary estimates.** The following lemma is the counterpart of Lemma 4.1, that is, an elliptic estimate. It will be applied on both the ℓ and r "sides". Hence, we formulate it for a weight funtion φ, $\tilde\tau$ and phase-space metric g in place of $\varphi_{r/\ell}$, $\tilde\tau_{r/\ell}$, and $g_{r/\ell}$.

With $\varrho' = (x, \xi', \tau, \gamma) \in \overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1} \times \mathbb{R}_+ \times \mathbb{R}_+$ we shall associate $\tilde\varrho' = (x, \xi', \tilde\tau(x)) \in \overline{\mathbb{R}}^n_+ \times \mathbb{R}^{n-1} \times \mathbb{R}_+$, with $\tilde\tau(x) = \tau\gamma\varphi(x)$.

Lemma 6.4. *Let $h(\varrho) \in S^{k,0}_{\mathrm{cl}}(g)$, with $\varrho = (x, \xi, \tau, \gamma)$ and $k \geq 1$, be polynomial in ξ_n with homogeneous coefficients in $(\xi', \tilde\tau)$ and $H = h(x, D, \tau, \gamma)$. When viewed as a polynomial in ξ_n the leading coefficient is 1. Let $\mathscr{U}$ be a conic open subset of $\overline{V_+} \times \mathbb{R}^{n-1} \times \mathbb{R}_+$. We assume that all roots of $h(\varrho', \xi_n) = 0$ have negative imaginary part for $\tilde\varrho' \in \mathscr{U}$. Letting $\hat\chi(\hat\varrho') \in S^0_{\mathsf{T},\tau}$, $\hat\varrho' = (x, \xi', \hat\tau)$, be homogeneous of degree 0 and such that $\operatorname{supp}(\hat\chi) \subset \mathscr{U}$, and $N \in \mathbb{N}$, there exist $C > 0$, $C_N > 0$, $\tau_* > 0$ and γ_*, such that*

$$\|\operatorname{Op}(\chi) w\|^2_{k,\tilde\tau} + |\operatorname{tr}(\operatorname{Op}(\chi) w)|^2_{k-1,1/2,\tilde\tau} \leq C \|H \operatorname{Op}(\chi) w\|^2_+ + C_N \big(\|w\|^2_{k,-N,\tilde\tau} + |\operatorname{tr}(w)|^2_{k-1,-N,\tilde\tau} \big),$$

for $w \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$ and $\tau \geq \tau_*$, $\gamma \geq \gamma_*$ and $\chi(\varrho') = \hat\chi(\tilde\varrho') \in S^0(g_{\mathsf{T}})$.

We refer to [4] for a proof.

The following proposition is the counterpart of Proposition 4.2, that is, an estimate exploiting the transmission condition, yielding an estimate of an interface norm.

Proposition 6.5. *Assume that the transmission condition for $\{P_{r/\ell}, T^j_{r/\ell}, \psi_{r/\ell},\ j = 1, \dots, m\}$ is satisfied at $(x_0, \xi_0', \hat\tau_0) \in \mathbb{S}^*_{\mathsf{T},\tau}(V)$ with $x_0 \in S \cap V$. Then there exists $\mathscr{U}$, a conic open neighborhood of $(x_0, \xi_0', \hat\tau_0)$ in $\overline{V_+} \times \mathbb{R}^{n-1} \times \mathbb{R}_+$, such that for $\hat\chi \in S^0_{\mathsf{T},\tau}$, homogeneous of degree 0, with $\operatorname{supp}(\hat\chi) \subset \mathscr{U}$, and there exist $C > 0$, $\tau_* > 0$, and $\gamma_* > 0$ such that*

$$C \big(|\operatorname{tr}(\operatorname{Op}(\chi_\ell) v_\ell)|^2_{m_\ell-1,1/2,\tilde\tau} + |\operatorname{tr}(\operatorname{Op}(\chi_r) v_r)|^2_{m_r-1,1/2,\tilde\tau} \big)$$
$$\leq \sum_{j=1}^m \big| T^j_{\ell,\varphi} v_{\ell|x_n=0^+} + T^j_{r,\varphi} v_{r|x_n=0^+} \big|^2_{m-1/2-\beta^j,\tilde\tau} + \|P_{\ell,\varphi} v_\ell\|^2_+ + \|P_{r,\varphi} v_r\|^2_+$$
$$+ \gamma^2 \big(\|v_\ell\|^2_{m_\ell,-1,\tilde\tau_\ell} + \|v_r\|^2_{m_r,-1,\tilde\tau_r} + |\operatorname{tr}(v_\ell)|^2_{m_\ell-1,-1/2,\tilde\tau} + |\operatorname{tr}(v_r)|^2_{m_r-1,-1/2,\tilde\tau} \big),$$

for $\tau \geq \tau_*$, $\gamma \geq \gamma_*$, $v_\ell, v_r \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$ *and* $\chi_{r\!/\!_\ell}(\varrho') = \hat\chi(\tilde\varrho'_{r\!/\!_\ell}) \in S^0(g_{\mathsf{T},r\!/\!_\ell})$, *with* $\varrho' = (x,\xi',\tau,\gamma)$ *and* $\tilde\varrho'_{r\!/\!_\ell} = (x,\xi',\tilde\tau_{r\!/\!_\ell}(x))$.

Proof. The beginning of the proof is nearly identical to that of Proposition 4.2. In particular the neighborhood $\mathscr{U}$ is chosen similarly. Inequality (4.1) becomes

$$\sum_{j=m+1}^{m'_\ell} \lambda_{\mathsf{T}}^{2(m_\ell-1/2-\beta^j_\ell)} \big|\hat{\tilde\chi}(\hat\varrho')\Sigma_{e^j_{\ell,\psi}}(\hat\varrho',\mathbf{z}^\ell)\big|^2 + \sum_{j=m+1}^{m'_r} \lambda_{\mathsf{T}}^{2(m_r-1/2-\beta^j_r)} \big|\hat{\tilde\chi}(\hat\varrho')\Sigma_{e^j_{r,\psi}}(\hat\varrho',\mathbf{z}^r)\big|^2$$
$$+\sum_{j=1}^{m} \lambda_{\mathsf{T}}^{2(m-1/2-\beta^j)} \big|\Sigma_{t^j_{\ell,\psi}}(\hat\varrho',\mathbf{z}^\ell)+\Sigma_{t^j_{r,\psi}}(\hat\varrho',\mathbf{z}^r)\big|^2 \geq C\Big(\sum_{j=0}^{m_\ell-1} \lambda_{\mathsf{T}}^{2(m_\ell-1/2-j)} \big|z^\ell_j\big|^2 + \sum_{j=0}^{m_r-1} \lambda_{\mathsf{T}}^{2(m_r-1/2-j)} |z^r_j|^2\Big),$$

for $\mathbf{z}^{r\!/\!_\ell} \in \mathbb{C}^{m_{r\!/\!_\ell}}$ and $\hat\varrho' = (x', x_n = 0^+, \xi', \hat\tau) \in \overline{\mathscr{U}} \cap \{x_n = 0\}$ with $\lambda_{\mathsf{T}}^2 = |\xi|'^2 + \hat\tau^2$. Here, $\hat{\tilde\chi} \in S^0_{\mathsf{T},\tau}$ is homogeneous of degree 0 and such that $\hat{\tilde\chi} = 1$ in a neighborhood of $\overline{\mathscr{U}}$. We set $\tilde\chi_{r\!/\!_\ell}(x,\xi',\tau,\gamma) = \hat{\tilde\chi}(x,\xi',\tilde\tau_{r\!/\!_\ell}) \in S^0(g_{\mathsf{T},r\!/\!_\ell})$.

We then obtain, taking $\hat\varrho' = \tilde\varrho' = (x', x_n = 0^+, \xi', \tilde\tau(x))$,

$$\sum_{j=m+1}^{m'_\ell} \mu_{\mathsf{T}|x_n=0^+}^{2(m_\ell-1/2-\beta^j_\ell)} \big|\tilde\chi(\varrho'')\Sigma_{e^j_{\ell,\varphi}}(\varrho',\mathbf{z}^\ell)\big|^2 + \sum_{j=m+1}^{m'_r} \mu_{\mathsf{T}|x_n=0^+}^{2(m_r-1/2-\beta^j_r)} \big|\tilde\chi(\varrho'')\Sigma_{e^j_{r,\varphi}}(\varrho',\mathbf{z}^r)\big|^2$$
$$+\sum_{j=1}^{m} \mu_{\mathsf{T}|x_n=0^+}^{2(m-1/2-\beta^j)} \big|\Sigma_{t^j_{\ell,\varphi}}(\varrho',\mathbf{z}^\ell)+\Sigma_{t^j_{r,\varphi}}(\varrho',\mathbf{z}^r)\big|^2 \geq C\Big(\sum_{j=0}^{m_\ell-1} \mu_{\mathsf{T}|x_n=0^+}^{2(m_\ell-1/2-j)} |z^\ell_j|^2 + \sum_{j=0}^{m_r-1} \mu_{\mathsf{T}|x_n=0^+}^{2(m_r-1/2-j)} |z^r_j|^2\Big),$$

for all $\mathbf{z}^{r\!/\!_\ell} \in \mathbb{C}^{m_{r\!/\!_\ell}}$ and $\varrho' = (x', x_n = 0+, \xi', \tau, \gamma)$ and $\varrho'' = (x', \xi', \tau, \gamma)$ such that $\tilde\varrho' \in \overline{\mathscr{U}} \cap \{x_n = 0\}$. Here $\tilde\chi(\varrho'') = \hat{\tilde\chi}(\tilde\varrho'', x_n = 0^+)$ with $\tilde\varrho'' = (x', \xi', \tilde\tau(x', x_n = 0^+))$. We have $\tilde\chi(\varrho'') = \tilde\chi_\ell(\varrho')_{|x_n=0^+} = \tilde\chi_r(\varrho')_{|x_n=0^+}$. We set

$$\mathscr{G}_S(u) = \sum_{j=1}^{m} \big|T^j_{\ell,\varphi} u_{\ell|x_n=0^+} + T^j_{r,\varphi} u_{r|x_n=0^+}\big|^2_{m-1/2-\beta^j,\tau}$$
$$+ \sum_{j=m+1}^{m'_\ell} \big|E^j_{\ell,\varphi} u_{\ell|x_n=0^+}\big|^2_{m+m^-_\ell+1/2-j,\tau} + \sum_{j=m+1}^{m'_r} \big|E^j_{r,\varphi} u_{\ell|x_n=0^+}\big|^2_{m+m^-_r+1/2-j,\tau}.$$

with $E_{r\!/\!_\ell,\varphi} = \mathrm{Op}(\tilde\chi_{r\!/\!_\ell} e_{r\!/\!_\ell,\varphi})$. Then, according to the Gårding inequality of Lemma 5.4 for interface quadratic forms of type $(m_\ell - 1, m_r - 1, 1/2)$, there exists $\tau_* > 0$, $\gamma_* > 0$, $C > 0$, and $C_N > 0$ such that

$$\mathscr{G}_S(\underline v) = \mathrm{Re}\,\mathscr{G}_S(\underline v) \geq C\big(\,|\mathrm{tr}(\underline v_\ell)|^2_{m_\ell-1,1/2,\tilde\tau} + |\mathrm{tr}(\underline v_r)|^2_{m_r-1,1/2,\tilde\tau}\,\big) - C_N\big(\,|\mathrm{tr}(v_\ell)|^2_{m_\ell-1,1/2-N,\tilde\tau} + |\mathrm{tr}(v_r)|^2_{m_r-1,1/2-N,\tilde\tau}\,\big), \tag{6.2}$$

with $\underline v = (\underline v_\ell, \underline v_r)$ and $\underline v_{r\!/\!_\ell} = \mathrm{Op}(\chi_{r\!/\!_\ell}) v_{r\!/\!_\ell}$, for $\mathbf{v} = (v_\ell, v_r) \in \big(\mathscr{S}(\overline{\mathbb{R}}^n_+)\big)^2$, $\tau \geq \tau_*$, and $\gamma \geq \gamma_*$.

Now, arguing as in the proof of Proposition 4.2 we write $\chi_{r\!/\!_\ell} p_{r\!/\!_\ell,\varphi} = \chi_{r\!/\!_\ell}\kappa_{r\!/\!_\ell,\varphi} p^-_{r\!/\!_\ell,\varphi} = \chi_{r\!/\!_\ell}\tilde\chi_{r\!/\!_\ell}\kappa_{r\!/\!_\ell,\varphi}\tilde p^-_{r\!/\!_\ell,\varphi}$, where $\tilde p^-_{r\!/\!_\ell,\varphi}$ denotes an extension of $p^-_{r\!/\!_\ell,\varphi}$ to the whole phase space. Then

$$\mathrm{Op}(\chi_{r\!/\!_\ell}) P_{r\!/\!_\ell,\varphi} = \mathrm{Op}(\tilde p^-_{r\!/\!_\ell,\varphi})\,\mathrm{Op}(\chi_{r\!/\!_\ell})\,\mathrm{Op}(\tilde\chi_{r\!/\!_\ell}\kappa_{r\!/\!_\ell,\varphi}) + R_{r\!/\!_\ell},$$

with $R_{r/\ell}$ in $\gamma\Psi_{\mathrm{cl}}^{m,-1}(g_{r/\ell})$. Applying Lemma 6.4 to $\mathrm{Op}(\tilde{p}^-_{r/\ell,\varphi})$ and $w_{r/\ell}=\mathrm{Op}(\tilde{\chi}_{r/\ell}\kappa_{r/\ell,\varphi})v_{r/\ell}$ we find

$$\begin{aligned}&\|\mathrm{Op}(\chi_\ell)w_\ell\|^2_{m^-_\ell,\tilde{\tau}_\ell}+\|\mathrm{Op}(\chi_r)w_r\|^2_{m^-_r,\tilde{\tau}_r}+|\mathrm{tr}(\mathrm{Op}(\chi_\ell)w_\ell)|^2_{m^-_\ell-1,1/2,\tilde{\tau}}+|\mathrm{tr}(\mathrm{Op}(\chi_r)w_r)|^2_{m^-_r-1,1/2,\tilde{\tau}}\\&\lesssim\|P_{\ell,\varphi}v_\ell\|^2_+\!+\|P_{r,\varphi}v_r\|^2_+\!+\gamma^2\big(\|v_\ell\|^2_{m_\ell,-1,\tilde{\tau}_\ell}+\|v_r\|^2_{m_r,-1,\tilde{\tau}_r}\big)\\&\quad+|\mathrm{tr}(v_\ell)|^2_{m_\ell-1,-N,\tilde{\tau}}+|\mathrm{tr}(v_r)|^2_{m_r-1,-N,\tilde{\tau}}\,,\end{aligned}$$

yielding

$$\begin{aligned}&\sum_{j=0}^{m^-_\ell-1}\big|D_n^j\,\mathrm{Op}(\chi_\ell)w_{\ell|x_n=0^+}\big|^2_{m^-_\ell-1/2-j,\tilde{\tau}}+\sum_{j=0}^{m^-_r-1}\big|D_n^j\,\mathrm{Op}(\chi_r)w_{r|x_n=0^+}\big|^2_{m^-_r-1/2-j,\tilde{\tau}}\\&\quad\lesssim\|P_{\ell,\varphi}v_\ell\|^2_+\!+\|P_{r,\varphi}v_r\|^2_+\!+\gamma^2\big(\|v_\ell\|^2_{m_\ell,-1,\tilde{\tau}_\ell}+\|v_r\|^2_{m_r,-1,\tilde{\tau}_r}\big)\\&\qquad+|\mathrm{tr}(v_\ell)|^2_{m_\ell-1,-N,\tilde{\tau}}+|\mathrm{tr}(v_r)|^2_{m_r-1,-N,\tilde{\tau}}\,,\end{aligned}$$

Recalling that $e^{j+m+1}_{r/\ell,\varphi}=\kappa_{r/\ell,\varphi}\xi_n^j$, $j=0,\dots,m^-_{r/\ell}-1$ in a neighborhood of $\mathscr{U}$ we have

$$D_n^j\,\mathrm{Op}(\chi_{r/\ell})\,\mathrm{Op}(\tilde{\chi}_{r/\ell}\kappa_{r/\ell,\varphi})v_{r/\ell}=E^{j+m+1}_{r/\ell,\varphi}\underline{v}_{r/\ell}+R_{r/\ell,j}v_{r/\ell},$$

with $R_{r/\ell,j}\in\gamma\Psi_{\mathrm{cl}}^{m_{r/\ell}-m^-_{r/\ell}+j,-1}(g_{r/\ell})$. We thus obtain

$$\begin{aligned}&\sum_{j=0}^{m^-_\ell-1}\big|E^{j+m+1}_{\ell,\varphi}\underline{v}_{\ell|x_n=0^+}\big|^2_{m^-_\ell-1/2-j,\tilde{\tau}}+\sum_{j=0}^{m^-_r-1}\big|E^{j+m+1}_{r,\varphi}\underline{v}_{r|x_n=0^+}\big|^2_{m^-_r-1/2-j,\tilde{\tau}}\\&\quad\lesssim\|P_{\ell,\varphi}v_\ell\|^2_+\!+\|P_{r,\varphi}v_r\|^2_+\!+\gamma^2\Big(\|v_\ell\|^2_{m_\ell,-1,\tilde{\tau}_\ell}+\|v_r\|^2_{m_r,-1,\tilde{\tau}_r}\big)\\&\qquad+|\mathrm{tr}(v_\ell)|^2_{m_\ell-1,-1/2,\tilde{\tau}}+|\mathrm{tr}(v_r)|^2_{m_r-1,-1/2,\tilde{\tau}}\Big).\end{aligned}\tag{6.3}$$

Collecting (6.2) and (6.3) we obtain the result of Proposition 6.5, for τ and γ chosen sufficiently large, with an additional commutator argument as in the end of the proof of Proposition 4.2. □

6.5. **Proof of the Carleman estimate with two-large parameters.** We prove a microlocal result, counterpart of that of Theorem 4.4. Patching microlocal estimates of this type, arguing as in Section 4.4 we can then obtain the local Carleman estimate of Theorem 6.3. The proof is left to the reader.

Theorem 6.6. *Let $x_0\in S\cap V$ and let $\psi\in\mathscr{C}^0(V)$ be such that $\psi_{r/\ell}\in\mathscr{C}^\infty(\overline{V^+})$ has the strong pseudo-convexity property of Definition 6.1 with respect to $P_{r/\ell}$ in a neighborhood of x_0 in $\overline{V_+}$. Moreover, assume that $\big\{P_{r/\ell},\psi_{r/\ell},T^j_{r/\ell},\ j=1,\dots,m\big\}$ satisfies the transmission condition at $(x_0,\xi_0',\hat{\tau}_0)\in\mathbb{S}^*_{\mathsf{T},\tau}(\overline{V_+})$. Then there exists $\mathscr{U}$ a conic open neighborhood of $(x_0,\xi_0',\hat{\tau}_0)$ in $\overline{V_+}\times\mathbb{R}^{n-1}\times\mathbb{R}_+$ such that for $\hat{\chi}\in S^0_{\mathsf{T},\tau}$, homogeneous of degree 0, with $\mathrm{supp}(\hat{\chi})\subset\mathscr{U}$, there exist $C>0$, $\tau_*>0$, and $\gamma_*>0$ such that, for $\varphi_{r/\ell}=\exp(\gamma\psi_{r/\ell})$ and $\tilde{\tau}_{r/\ell}=\tau\gamma\varphi_{r/\ell}$,*

$$\begin{aligned}&\|P_{\ell,\varphi}v_\ell\|^2_+\!+\|P_{r,\varphi}v_r\|^2_+\!+\sum_{j=1}^m\big|T^j_{\ell,\varphi}v_{\ell|x_n=0^+}+T^j_{r,\varphi}v_{r|x_n=0^+}\big|^2_{m-\beta^j-1/2,\tilde{\tau}}\\&\quad+\gamma^2\big(\|v_\ell\|^2_{m_\ell,-1,\tilde{\tau}_\ell}+\|v_r\|^2_{m_r,-1,\tilde{\tau}_r}+|\mathrm{tr}(v_\ell)|^2_{m_\ell-1,-1/2,\tilde{\tau}}+|\mathrm{tr}(v_r)|^2_{m_r-1,-1/2,\tilde{\tau}}\big)\\&\qquad\geq C\big(\big\|\tilde{\tau}_\ell^{-\frac{1}{2}}\,\mathrm{Op}(\chi_\ell)v_\ell\big\|^2_{m_\ell,\tilde{\tau}_\ell}+\big\|\tilde{\tau}_r^{-\frac{1}{2}}\,\mathrm{Op}(\chi_r)v_r\big\|^2_{m_r,\tilde{\tau}_r}\\&\qquad\qquad+|\mathrm{tr}(\mathrm{Op}(\chi_\ell)v_\ell)|^2_{m_\ell-1,1/2,\tilde{\tau}}+|\mathrm{tr}(\mathrm{Op}(\chi_r)v_r)|^2_{m_r-1,1/2,\tilde{\tau}}\big),\end{aligned}\tag{6.4}$$

for all $v_\ell, v_r \in \mathscr{S}(\overline{\mathbb{R}^n_+})$, $\tau \geq \tau_*$, $\gamma \geq \gamma_*$, *and* $\chi_{r/\ell}(\varrho') = \hat{\chi}(\tilde{\varrho}'_{r/\ell}) \in S^0(g_{\mathsf{T},r/\ell})$, *with* $\tilde{\varrho}'_{r/\ell} = (x, \xi', \tilde{\tau}_{r/\ell}(x))$ *for* $\varrho' = (x, \xi', \tau, \gamma)$.

Proof. Applying Lemma 6.8 in [4] we obtain that there exists $\mathscr{U}$ a conic open neighborhood of $(x_0, \xi'_0, \hat{\tau}_0)$ in $\overline{V_+} \times \mathbb{R}^{n-1} \times \mathbb{R}_+$ such that for $\hat{\chi} \in S^0_{\mathsf{T},\tau}$, homogeneous of degree 0, with $\operatorname{supp}(\hat{\chi}) \subset \mathscr{U}$, there exist $C > 0$, $\tau_0 > 0$, and $\gamma_0 > 0$ such that

$$
\begin{aligned}
(6.5)\quad & C\big\|P_{r/\ell,\varphi} v_{r/\ell}\big\|^2_+ - \operatorname{Re} \mathscr{B}_{a_{r/\ell}, b_{r/\ell}}(\operatorname{Op}(\chi_{r/\ell}) v_{r/\ell}) \geq C' \big\|\tilde{\tau}_{r/\ell}^{-\frac{1}{2}} \operatorname{Op}(\chi_{r/\ell}) v_{r/\ell}\big\|^2_{m_{r/\ell}, \tilde{\tau}_{r/\ell}} \\
& - C''\big(\gamma^2 \big\|v_{r/\ell}\big\|^2_{m_{r/\ell}, -1, \tilde{\tau}_{r/\ell}} + \big|\tilde{\tau}_{r/\ell}^{-\frac{1}{2}} \operatorname{tr}(\operatorname{Op}(\chi_{r/\ell}) v_{r/\ell})\big|^2_{m_{r/\ell}-1, 1/2, \tilde{\tau}_{r/\ell}} + \gamma \big|\operatorname{tr}(\operatorname{Op}(\chi_{r/\ell}) v_{r/\ell})\big|^2_{m_{r/\ell}-1, 0, \tilde{\tau}_{r/\ell}}\big),
\end{aligned}
$$

for $\tau \geq \tau_0$, $\gamma \geq \gamma_0$, and $\chi_{r/\ell}(\varrho') = \hat{\chi}(\tilde{\varrho}'_{r/\ell}) \in S^0(g_{\mathsf{T},r/\ell})$, where $\mathscr{B}_{a_{r/\ell}, b_{r/\ell}}$ satisfies

$$
\Big|\mathscr{B}_{a_{r/\ell}, b_{r/\ell}}(\operatorname{Op}(\chi_{r/\ell}) v_{r/\ell})\Big| \lesssim \big|\operatorname{tr}(\operatorname{Op}(\chi_{r/\ell}) v_{r/\ell})\big|^2_{m_{r/\ell}-1, 1/2, \tilde{\tau}}\,.
$$

With Proposition 6.5, making use of the transmission condition, we obtain for M chosen sufficiently large, in a possibly reduced neighborhood $\mathscr{U}$,

$$
\begin{aligned}
(6.6)\quad & \operatorname{Re} \mathscr{B}_{a_\ell, b_\ell}(\operatorname{Op}(\chi_\ell) v_\ell) + \operatorname{Re} \mathscr{B}_{a_r, b_r}(\operatorname{Op}(\chi_r) v_r) + M \sum_{j=1}^m \big|T^j_{\ell,\varphi} v_{\ell|x_n=0^+} + T^j_{r,\varphi} v_{r|x_n=0^+}\big|^2_{m-1/2-\beta^j, \tilde{\tau}} \\
& \geq C\big(\,|\operatorname{tr}(\operatorname{Op}(\chi_\ell) v_\ell)|^2_{m_\ell-1,1/2,\tilde{\tau}} + |\operatorname{tr}(\operatorname{Op}(\chi_r) v_r)|^2_{m_r-1,1/2,\tilde{\tau}}\big) - C'\big(\,\|P_{\ell,\varphi} v_\ell\|^2_+ + \|P_{r,\varphi} v_r\|^2_+ \\
& \quad + \gamma^2\big(\,\|v_\ell\|^2_{m_\ell,-1,\tilde{\tau}_\ell} + \|v_r\|^2_{m_r,-1,\tilde{\tau}_r} + |\operatorname{tr}(v_\ell)|^2_{m_\ell-1,-1/2,\tilde{\tau}} + |\operatorname{tr}(v_r)|^2_{m_r-1,-1/2,\tilde{\tau}}\big),
\end{aligned}
$$

Summing $(6.5)_\ell$, $(6.5)_r$, and (6.6) we obtain the result of Theorem 6.6, by taking τ and γ sufficiently large. □

6.6. **Estimate with the simple characterisitic property.** As in [31] and [4] a stronger estimate with two parameters can be achieved if one assumes that the operator P and the weight function ψ fulfills the so-called simple characterisitic property.

We introduce the map

$$
\begin{aligned}
\rho_{x,\xi} : \mathbb{R}^+ &\to \mathbb{C}, \\
\hat{\tau} &\mapsto p(x, \xi + i\hat{\tau}\psi'(x)),
\end{aligned}
\tag{6.7}
$$

where $x \in \overline{\Omega}$ and $\xi \in \mathbb{R}^n$.

Definition 6.7. *Let U be an open subset of Ω. Given a weight function ψ and an operator P we say that the simple-characteristic property is satisfied in $\overline{U}$ if, for all $x \in \overline{U}$, we have $\xi = 0$ and $\hat{\tau} = 0$ when the map $\rho_{x,\xi}$ has a double root.*

Remark 6.8. In fact the simple-characteristic property implies the property of strong pseudo-convexity. We refer the reader to [31] and [4].

We have the following result.

Theorem 6.9. *Let $x_0 \in S$ and let $\psi \in \mathscr{C}^0(\Omega)$ be such that $\psi_k = \psi_{|\Omega_k} \in \mathscr{C}^\infty(\Omega_k)$ for $k = 1, 2$ and such that ψ_k and P_k have the simple characteristic property of Definition 6.7 in a neighborhood of x_0 in $\overline{\Omega_k}$. Moreover, assume that $\big\{P_k, T^j_k, \psi,\ k = 1, 2,\ j = 1, \dots, m\big\}$ satisfies the transmission condition at x_0.*

Then there exist a neighborhood W of x_0 in $\mathbb{R}^n$ and three constants C, $\tau_* > 0$, and $\gamma_* > 0$ such that, for $\varphi_k = \exp(\gamma\psi_k)$ and $\tilde\tau_k = \tau\gamma\varphi_k$,

$$\begin{aligned}(6.8)\quad &\sum_{k=1,2}\big(\gamma\big\|\tilde\tau_k^{-\frac12}e^{\tau\varphi_k}u_k\big\|^2_{m_k,\tilde\tau_k}+|e^{\tau\varphi_{|S}}\operatorname{tr}(u_k)|^2_{m_k-1,1/2,\tilde\tau}\big)\\ &\qquad\le C\Big(\sum_{k=1,2}\|e^{\tau\varphi_k}P_k(x,D)u_k\|^2_{L^2(\Omega_k)}+\sum_{j=1}^m\big|e^{\tau\varphi_{|S}}(T_1^j(x,D)u_1+T_2^j(x,D)u_2)_{|S}\big|^2_{m-1/2-\beta^j,\tilde\tau}\Big),\end{aligned}$$

for all $u_k = w_{k|\Omega_k}$ with $w_k \in \mathscr{C}^\infty_c(W)$, $\tau \ge \tau_*$ and $\gamma \ge \gamma_*$.

We prove the following microlocal result and the result of Theorem 6.9 can be deduced, arguing as in Section 4.4.

Theorem 6.10. *Let $x_0 \in S \cap V$ and let $\psi \in \mathscr{C}^0(V)$ be such that $\psi_{r/\ell} \in \mathscr{C}^\infty(\overline{V+})$ and such that $\psi_{r/\ell}$ and $P_{r/\ell}$ have the simple characteristic property of Definition 6.7 in a neighborhood of x_0 in $\overline{V+}$. Moreover, assume that $\{P_{r/\ell}, T^j_{r/\ell}, \psi_{r/\ell},\ j=1,\dots,m\}$ satisfies the transmission condition at $(x_0,\xi_0',\hat\tau_0) \in \mathbb{S}^*_{\mathsf{T},\tau}(\overline{V_+})$. Then there exists $\mathscr{U}$ a conic open neighborhood of $(x_0,\xi_0',\hat\tau_0)$ in $\overline{V_+}\times\mathbb{R}^{n-1}\times\mathbb{R}_+$ such that for $\hat\chi \in S^0_{\mathsf{T},\tau}$, homogeneous of degree 0, with $\operatorname{supp}(\hat\chi)\subset\mathscr{U}$, there exist $C>0$, $\tau_*>0$, and $\gamma_*>0$ such that, for $\varphi_{r/\ell} = \exp(\gamma\psi_{r/\ell})$ and $\tilde\tau_{r/\ell} = \tau\gamma\varphi_{r/\ell}$,*

$$\begin{aligned}(6.9)\quad &\|P_{\ell,\varphi}v_\ell\|^2_+ + \|P_{r,\varphi}v_r\|^2_+ + \sum_{j=1}^m\big|T^j_{\ell,\varphi}v_{\ell|x_n=0^+}+T^j_{r,\varphi}v_{r|x_n=0^+}\big|^2_{m-\beta^j-1/2,\tilde\tau}\\ &\quad+\gamma^2\big(\|v_\ell\|^2_{m_\ell,-1,\tilde\tau_\ell}+\|v_r\|^2_{m_r,-1,\tilde\tau_r}+|\operatorname{tr}(v_\ell)|^2_{m_\ell-1,-1/2,\tilde\tau}+|\operatorname{tr}(v_r)|^2_{m_r-1,-1/2,\tilde\tau}\big)\\ &\qquad\ge C\big(\gamma\big\|\tilde\tau_\ell^{-\frac12}\operatorname{Op}(\chi_\ell)v_\ell\big\|^2_{m_\ell,\tilde\tau_\ell}+\gamma\big\|\tilde\tau_r^{-\frac12}\operatorname{Op}(\chi_r)v_r\big\|^2_{m_r,\tilde\tau_r}\\ &\qquad\qquad+|\operatorname{tr}(\operatorname{Op}(\chi_\ell)v_\ell)|^2_{m_\ell-1,1/2,\tilde\tau}+|\operatorname{tr}(\operatorname{Op}(\chi_r)v_r)|^2_{m_r-1,1/2,\tilde\tau}\big),\end{aligned}$$

for all $v_\ell, v_r \in \mathscr{S}(\overline{\mathbb{R}}^n_+)$, $\tau\ge\tau_*$, $\gamma\ge\gamma_*$, and $\chi_{r/\ell}(\varrho') = \hat\chi(\tilde\varrho'_{r/\ell}) \in S^0(g_{\mathsf{T},r/\ell})$, with $\tilde\varrho'_{r/\ell} = (x,\xi',\tilde\tau_{r/\ell}(x))$ for $\varrho' = (x,\xi',\tau,\gamma)$.

Proof. Applying Lemma 6.13 in [4] we obtain that there exists $\mathscr{U}$ a conic open neighborhood of $(x_0,\xi_0',\hat\tau_0)$ in $\overline{V_+}\times\mathbb{R}^{n-1}\times\mathbb{R}_+$ such that for $\hat\chi\in S^0_{\mathsf{T},\tau}$, homogeneous of degree 0, with $\operatorname{supp}(\hat\chi)\subset\mathscr{U}$, there exist $C>0$, $\tau_0>0$, and $\gamma_0>0$ such that

$$\begin{aligned}(6.10)\quad &C\big\|P_{r/\ell,\varphi}v_{r/\ell}\big\|^2_+ - \operatorname{Re}\mathscr{B}_{a_{r/\ell},b_{r/\ell}}(\operatorname{Op}(\chi_{r/\ell})v_{r/\ell}) \ge C'\gamma\big\|\tilde\tau_{r/\ell}^{-\frac12}\operatorname{Op}(\chi_{r/\ell})v_{r/\ell}\big\|^2_{m_{r/\ell},\tilde\tau_{r/\ell}}\\ &\quad-C''\big(\gamma^2\big\|v_{r/\ell}\big\|^2_{m_{r/\ell},-1,\tilde\tau_{r/\ell}}+\big|\tilde\tau_{r/\ell}^{-\frac12}\operatorname{tr}(\operatorname{Op}(\chi_{r/\ell})v_{r/\ell})\big|^2_{m_{r/\ell}-1,1/2,\tilde\tau}+\gamma\big|\operatorname{tr}(\operatorname{Op}(\chi_{r/\ell})v_{r/\ell})\big|^2_{m_{r/\ell}-1,0,\tilde\tau}\big),\end{aligned}$$

for $\tau\ge\tau_0$, $\gamma\ge\gamma_0$, and $\chi_{r/\ell}(\varrho') = \hat\chi(\tilde\varrho'_{r/\ell}) \in S^0(g_{\mathsf{T},r/\ell})$, where $\mathscr{B}_{a_{r/\ell},b_{r/\ell}}$ satisfies

$$\big|\mathscr{B}_{a_{r/\ell},b_{r/\ell}}(\operatorname{Op}(\chi_{r/\ell})v_{r/\ell})\big| \lesssim \big|\operatorname{tr}(\operatorname{Op}(\chi_{r/\ell})v_{r/\ell})\big|^2_{m_{r/\ell}-1,1/2,\tilde\tau}.$$

Summing $(6.10)_\ell$, $(6.10)_r$, and (6.6) we obtain the result of Theorem 6.10 by taking τ and γ sufficiently large. □

6.7. **Shifted estimates.** As in [4] it may be interesting to consider shifted estimates in the Sobolev scales. Namely we may wish to have an estimate of the following form.

Corollary 6.11. *Let $x_0 \in S$ and let $\psi \in \mathscr{C}^0(\Omega)$ be such that $\psi_k = \psi_{|\Omega_k} \in \mathscr{C}^\infty(\Omega_k)$ for $k = 1, 2$ and such that ψ has the strong pseudo-convexity property of Definition 6.2 with respect to P_1 and P_2 in a neighborhood of x_0 in Ω. Moreover, assume that $\{P_k, T_k^j, \psi, k = 1, 2,\ j = 1, \dots, m\}$ satisfies the transmission condition at x_0. Let $\ell \in \mathbb{N}$ and $s \in \mathbb{R}$. Then there exist a neighborhood W of x_0 in $\mathbb{R}^n$ and three constants C, $\tau_* > 0$, and $\gamma_* > 0$ such that for $\varphi_k = \exp(\gamma\psi_k)$ and $\tilde\tau_k = \tau\gamma\varphi_k$:*

$$\sum_{k=1,2} \big(\big\| \tilde\tau_k^{s-1/2} e^{\tau\varphi_k} u_k \big\|^2_{\ell+m_k,\tilde\tau_k} + |\tilde\tau_k^s e^{\tau\varphi_{|S}} \operatorname{tr}(u_k)|^2_{\ell+m_k-1,1/2,\tilde\tau} \big) \tag{6.11}$$
$$\le C \Big(\sum_{k=1,2} \|\tilde\tau_k^s e^{\tau\varphi_k} P_k(x,D) u_k\|^2_{\ell,\tilde\tau_k} + \sum_{j=1}^m |\tilde\tau_k^s e^{\tau\varphi_{|S}} (T_1^j(x,D)u_1 + T_2^j(x,D)u_2)_{|S}|^2_{\ell,m-1/2-\beta^j,\tilde\tau} \Big),$$

for all $u_k = w_{k|\Omega_k}$ with $w_k \in \mathscr{C}_c^\infty(W)$, $\tau \ge \tau_*$, and $\gamma \ge \gamma_*$.

Corollary 6.12. *Let $x_0 \in S$ and let $\psi \in \mathscr{C}^0(\Omega)$ be such that $\psi_k = \psi_{|\Omega_k} \in \mathscr{C}^\infty(\Omega_k)$ for $k = 1, 2$ and such that ψ_k and P_k have the simple characteristic property of Definition 6.7 in a neighborhood of x_0 in $\overline{\Omega_k}$. Moreover, assume that $\{P_k, T_k^j, \psi,\ k = 1, 2,\ j = 1, \dots, m\}$ satisfies the transmission condition at x_0. Let $\ell \in \mathbb{N}$ and $s \in \mathbb{R}$. Then there exist a neighborhood W of x_0 in $\mathbb{R}^n$ and three constants C, $\tau_* > 0$, and $\gamma_* > 0$ such that, for $\varphi_k = \exp(\gamma\psi_k)$ and $\tilde\tau_k = \tau\gamma\varphi_k$,*

$$\sum_{k=1,2} \big(\gamma \big\| \tilde\tau_k^{s-\frac12} e^{\tau\varphi_k} u_k \big\|^2_{\ell+m_k,\tilde\tau_k} + |\tilde\tau_k^s e^{\tau\varphi_{|S}} \operatorname{tr}(u_k)|^2_{\ell+m_k-1,1/2,\tilde\tau} \big) \tag{6.12}$$
$$\le C \Big(\sum_{k=1,2} \|\tilde\tau_k^s e^{\tau\varphi_k} P_k(x,D) u_k\|^2_{\ell,\tilde\tau_k} + \sum_{j=1}^m \big|\tilde\tau_k^s e^{\tau\varphi_{|S}} (T_1^j(x,D)u_1 + T_2^j(x,D)u_2)_{|S}\big|^2_{\ell,m-1/2-\beta^j,\tilde\tau} \Big),$$

for all $u_k = w_{k|\Omega_k}$ with $w_k \in \mathscr{C}_c^\infty(W)$, $\tau \ge \tau_*$ and $\gamma \ge \gamma_*$.

The proofs of these two corollaries can be adapted from the proofs of their counterpart at a boundary, namely Corollaries 6.14 and 6.15 in [4].

7. Application to unique continuation

With the Carleman estimates we have derived here we can obtain unique continuation results near an interface for high-order elliptic operators with the transmission condition, if we make a geometrical assumption, namely the strong pseudo-convexity condition. Result for a product of two operators can be obtained if additionaly the simple characteristic propery holds for one of them.

7.1. Uniqueness under strong pseudo-convexity and transmission condition.

Theorem 7.1. *Let P_k and T_k^j, $j = 1, \dots, m$ be given as in Section 1. Let $x_0 \in S$, $f \in \mathscr{C}^0(\Omega)$, and V be a neighborhood of x_0, be such that f has the strong pseudo-convexity property of Definition 6.2 with respect to P_1 and P_2 in V . Moreover, assume that $\{P_k, f, T_k^j,\ k = 1, 2,\ j = 1, \dots, m\}$ satisfies the transmission condition at x_0. Assume that u is such that $u_k = u_{|\Omega_k} \in H^{m_k}(\Omega_k)$ and satisfies*

- $$|P_k u_k(x)| \le C \sum_{|\alpha| \le m-1} |D^\alpha u_k(x)|, \quad \text{a.e. in } V_k = V \cap \Omega_k,\ k = 1, 2; \tag{7.13}$$
- *for $j = 1, \dots, m$ and $|\alpha| \le m - \beta^j$, with $\alpha \in \mathbb{N}^{n-1}$,*
$$|D_\tau^\alpha \big(T_1^j u_1(x) + T_2^j u_2(x)\big)| \le C \sum_{k=1,2} \sum_{\substack{|\alpha'| \le |\alpha| \\ +\beta_k^j - 1}} |D^{\alpha'} u_k(x)|, \quad \text{a.e. in } V \cap S; \tag{7.14}$$
- *and u vanishes in $\{x \in V;\ f(x) \ge f(x_0)\}$.*

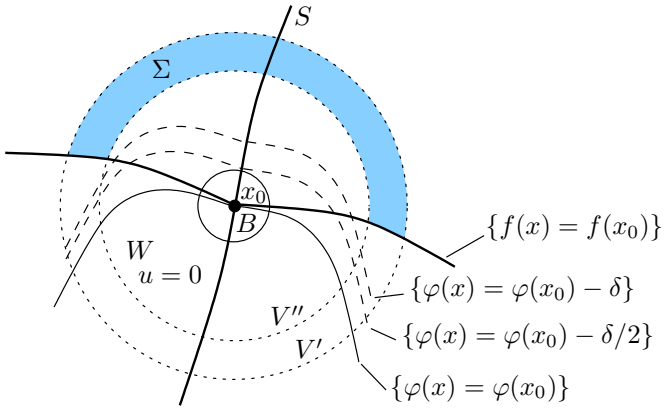

FIGURE 1. Local geometry for the unique continuation problem. The shaded region Σ contains the supports of $[P, \chi]u$ and $[\chi, D^j_{x_k}]u$. In the figure the funtions f and φ are continuous and yet only piecewise smooth. Here unique continuation is performed across a surface that is not related to S.

Then u vanishes in a neighborhood of x_0.

Here $D^\alpha_\intercal$ denotes a familly of differential operators that act tangentially to the interface S and, in local coordinates near x_0, where $S = \{x_n = 0\}$, their principal symbol is ξ'^α.

Proof. Strong pseudo-convexity is a stable notion in $\mathscr{C}^2$ (see Proposition 28.3.2 in [22]). Here the function f is continuous and piecewise smooth. The argument of [22] applies on both sides of the interface. For ε chosen sufficiently small, there exists a neighborhood V' of x_0 such that the function $\psi(x) = f(x) - \varepsilon|x - x_0|^2$ has the strong pseudo-convexity property of Definition 6.2 with respect to P_1 and P_2 in V'. Similary we saw in Section 1.6.4 for the proof of Proposition 1.8, that the transmission condition (or rather property (1.29)) is robust upon perturbation of the weight function. Hence if ε is chosen sufficiently small $\{P, \psi, B^k,\ k = 1, \dots, \mu\}$ will also satisfy this condition.

We set $\varphi = \exp(\gamma\psi)$. As shown in Proposition 28.3.3 in [22] the strong pseudo-convexity of the function ψ with respect to P_1 and P_2 implies the sub-ellipticity condition for $\{P_k, \varphi_k\}$ for γ chosen sufficiently large for both $k = 1, 2$ with $\varphi_k = \varphi_{|\Omega_k}$. Moreover, as seen in Section 6.2 $\left\{P_k, T^j_k, \varphi;\ k = 1, 2,\ j = 1, \dots, \mu\right\}$ also satisfies the transmission condition at x_0.

The geometrical situation we describre is illustrated in Figures 1 and 2. The two figures show different unique continuation configuration: across an hypersuface that is not related to the interface S, or across the interface S. We call W the region $\{x \in V;\ f(x) \geq f(x_0)\}$ (region beneath $\{f(x) = f(x_0)\}$ in Figure 1) where u vanishes by assumption. We choose V'' a neighborhood of x_0 such that $V'' \Subset V'$.

We pick a function $\chi \in \mathscr{C}^\infty_c(\mathbb{R}^n)$ such that $\chi = 1$ in V'' and $\mathrm{supp}(\chi) \cap V \subset V'$. We observe that the Carleman estimate of Theorem 1.6 applies to χu by density (possibly by reducing the neighborhoods V and V' of x_0):

$$\sum_{k=1,2} \left(\tau^{-1/2} \left\|e^{\tau\varphi_k}\chi u_k\right\|_{m_k,\tau} + \left|e^{\tau\varphi_{|S}} \operatorname{tr}(\chi u_k)\right|_{m_k-1,1/2,\tau}\right) \tag{7.15}$$

$$\lesssim \sum_{k=1,2} \left\|e^{\tau\varphi_k} P_k(\chi u_k)\right\|_{L^2(\Omega_k)} + \sum_{j=1}^m \left|e^{\tau\varphi_{|S}} \left(T^j_1(\chi)u_1 + T^j_2(\chi u_2)\right)_{|S}\right|_{m-1/2-\beta^j,\tau},$$

for $\tau \geq \tau_0$.

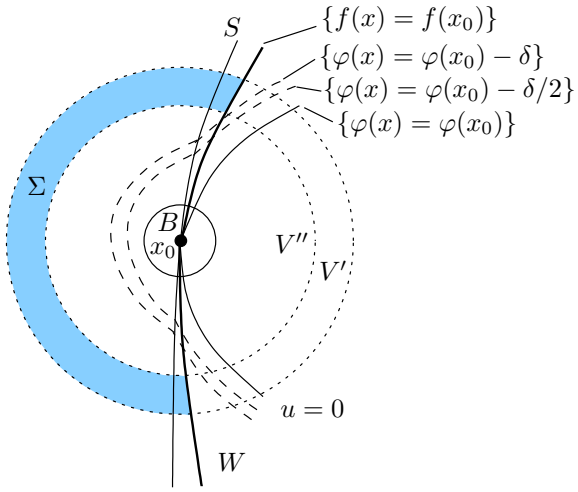

FIGURE 2. Local geometry for the unique continuation problem. Here, unique continuation is performed across the interface S.

We have $P_k(\chi u_k) = \chi P_k u_k + [P_k, \chi]u_k$, where the commutator is a differential operator of order $m-1$. With (7.13) we have

$$\begin{aligned}\|e^{\tau\varphi_k} P_k(\chi u_k)\|_{L^2(\Omega_k)} &\lesssim \sum_{|\alpha|\leq m-1} \|e^{\tau\varphi_k}\chi D^\alpha u_k\|_{L^2(\Omega_k)} + \|e^{\tau\varphi_k}[P_k,\chi]u_k\|_{L^2(\Omega_k)} \\ &\lesssim \sum_{|\alpha|\leq m-1} \|e^{\tau\varphi_k} D^\alpha(\chi u_k)\|_{L^2(\Omega_k)} + \sum_{i\in I_k} \|e^{\tau\varphi_k} M_{k,i} u_k\|_{L^2(\Omega_k)}\,,\end{aligned}$$

where I_k is finite and the operators $M_{k,i}$ are commutators fo χ and differential operators. They are of order $m_k - 1$ at most.

We also write

$$\begin{aligned}&\big|e^{\tau\varphi|_S}\big(T_1^j(\chi u_1) + T_2^j(\chi u_2)\big)_{|S}\big|_{m-1/2-\beta^j,\tau} \\ &\quad\leq \big|e^{\tau\varphi|_S}\big(T_1^j(\chi u_1) + T_2^j(\chi u_2)\big)_{|S}\big|_{m-\beta^j,\tau} \\ &\quad= \sum_{\substack{r+|\alpha|\\ \leq m-\beta^j}} \big|\tau^r e^{\tau\varphi|_S} D_{\mathsf{T}}^\alpha\big(T_1^j(\chi u_1) + T_2^j(\chi u_2)\big)_{|S}\big|_{L^2(S)}\end{aligned}$$

We write $D_{\mathsf{T}}^{\alpha}T_k^j(\chi u_k)=\chi D_{\mathsf{T}}^{\alpha}T_k^j u_k+[D_{\mathsf{T}}^{\alpha}T_k^j,\chi]u_k$ and have

$$\begin{aligned}
&\big|e^{\tau\varphi_{|S}}\big(T_1^j(\chi u_1)+T_2^j(\chi u_2)\big)_{|S}\big|_{m-1/2-\beta^j,\tau}\\
&\quad\lesssim \sum_{\substack{r+|\alpha|\\ \leq m-\beta^j}}\Big(\big|\tau^r e^{\tau\varphi_{|S}}\chi\big(D_{\mathsf{T}}^{\alpha}T_1^j u_1+D_{\mathsf{T}}^{\alpha}T_2^j u_2\big)_{|S}\big|_{L^2(S)}\\
&\qquad+\big|\tau^r e^{\tau\varphi_{|S}}\big([D_{\mathsf{T}}^{\alpha}T_1^j,\chi]u_1+[D_{\mathsf{T}}^{\alpha}T_2^j,\chi]u_2\big)_{|S}\big|_{L^2(S)}\Big)\\
&\quad\lesssim \sum_{k=1,2}\sum_{\substack{r+|\alpha|\\ \leq m-\beta^j}}\sum_{\substack{|\alpha'|\leq|\alpha|\\ +\beta_k^j-1}}\big|\tau^r e^{\tau\varphi_{|S}}\chi(D^{\alpha'}u_k)_{|S}\big|_{L^2(S)}\\
&\qquad+\sum_{k=1,2}\sum_{\substack{r+|\alpha|\\ \leq m-\beta^j}}\big|\tau^r e^{\tau\varphi_{|S}}\big([D_{\mathsf{T}}^{\alpha}T_k^j,\chi]u_k\big)_{|S}\big|_{L^2(S)},
\end{aligned}$$

Using commutators once more we write

$$\big|\tau^r e^{\tau\varphi_{|S}}\chi(D^{\alpha'}u_k)_{|S}\big|_{L^2(S)}\leq\big|\tau^r D^{\alpha'}(e^{\tau\varphi_{|S}}\chi u_k)_{|S}\big|_{L^2(S)}+\big|\tau^r([e^{\tau\varphi_{|S}}\chi,D^{\alpha'}]u_k)_{|S}\big|_{L^2(S)}.$$

We thus find

$$\begin{aligned}
&\sum_{j=1}^m\big|e^{\tau\varphi_{|S}}\big(T_1^j(\chi u_1)+T_2^j(\chi u_2)\big)_{|S}\big|_{m-1/2-\beta^j,\tau}\\
&\quad\lesssim\sum_{j=1}^m\sum_{k=1,2}\sum_{\substack{r+|\alpha|\\ \leq m-\beta^j}}\sum_{\substack{|\alpha'|\leq|\alpha|\\ +\beta_k^j-1}}\big|\tau^r\big(D^{\alpha'}(e^{\tau\varphi_{|S}}\chi u_k)\big)_{|S}\big|_{L^2(S)}+\sum_{k=1,2}\sum_{i\in J_k}\big|(\tilde{M}_{k,i}u_k)_{|S}\big|_{L^2(S)}\\
&\quad\lesssim\sum_{k=1,2}\big|\operatorname{tr}(e^{\tau\varphi}\chi u_k)\big|_{m_k-1,0,\tau}+\sum_{k=1,2}\sum_{i\in J_k}\big|\tilde{M}_{k,i}u_k\big|_{L^2(S)},
\end{aligned}$$

where J_k is finite and $\tilde{M}_{k,i}$ is the commutator of χ with a differential operator. The operator $\tilde{M}_{k,i}$ is of order at most m_k-1 (w.r.t. to τ and ξ).

As

$$|e^{\tau\varphi_{|S}}\operatorname{tr}(\chi u)|_{m_k-1,1/2,\tau}\geq\tau^{\frac{1}{2}}|e^{\tau\varphi_{|S}}\operatorname{tr}(\chi u_k)|_{m_k-1,0,\tau}\geq\tau^{\frac{1}{2}}|\operatorname{tr}(e^{\tau\varphi_k}\chi u_k)|_{m_k-1,0,\tau},$$

for τ chosen sufficiently large, from (7.15) we thus obtain

$$\begin{aligned}
&\sum_{k=1,2}\big(\tau^{-1/2}\|e^{\tau\varphi_k}\chi u_k\|_{m_k,\tau}+|e^{\tau\varphi_k}\operatorname{tr}(\chi u_k)|_{m_k-1,1/2,\tau}\big)\\
&\qquad\lesssim\sum_{k=1,2}\Big(\sum_{i\in I_k}\big\|e^{\tau\varphi_k}M_{k,i}u_k\big\|_{L^2(\Omega_k)}+\sum_{i\in J_k}\big|(\tilde{M}_{k,i}u_k)_{|S}\big|_{L^2(S)}\Big),
\end{aligned}$$

We set $\Sigma:=V'\setminus(V''\cup W)$ (see the shaded region in Figure 1). We have

$$\operatorname{supp}(M_{k,i}u_k)\subset\Sigma,\ \ i\in I_k\quad\text{and}\quad\operatorname{supp}(\tilde{M}_{k,j}u_k)\subset\Sigma,\ \ i\in J_k,$$

as they are confined in the region where χ varies and u does not vanish.

We thus obtain

$$\begin{aligned}
&\sum_{k=1,2}\big(\tau^{-1/2}\|e^{\tau\varphi_k}\chi u_k\|_{m_k,\tau}+|e^{\tau\varphi_k}\operatorname{tr}(\chi u_k)|_{m_k-1,1/2,\tau}\big)\\
&\qquad\lesssim\sum_{k=1,2}\Big(\sum_{|\alpha|\leq m_k-1}\big\|e^{\tau\varphi_k}D^{\alpha}u_k\big\|_{L^2(\Sigma)}+\sum_{r+|\alpha|\leq m_k-1}\big|\tau^r(D^{\alpha}u_k)_{|S}\big|_{L^2(\Sigma\cap S)}\Big).
\end{aligned}$$

For all $\delta > 0$, we set $V_\delta = \{x \in V;\ \varphi(x) \leq \varphi(x_0) - \delta\}$. There exists $\delta > 0$ such that $\Sigma \Subset V_\delta$. We then choose B a neighborhood of x_0 such that $\overline{B} \subset V'' \setminus V_{\delta/2}$ and obtain, as $\chi \equiv 1$ on B,

$$e^{\tau \inf_B \varphi} \sum_{k=1,2} \|u_k\|_{H_k^m(B)} \lesssim e^{\tau(\sup_\Sigma \varphi + \delta/2)} \sum_{k=1,2} \Big(\|u_k\|_{H_k^m(\Sigma)} + \sum_{|\alpha| \leq m_k - 1} \big|D^\alpha u_{|S}\big|_{L^2(\Sigma \cap S} \Big), \quad \tau \geq \tau_1$$

for some $\tau_1 > 0$. Since $\inf_B \varphi > \sup_\Sigma \varphi + \delta/2$, letting τ go to ∞, we obtain $u = 0$ in B. □

7.2. **Uniqueness for products of operators.** We now consider two sets of elliptic operators: P_1 and Q_1 defined on Ω_1 and P_2 and Q_2 defined on Ω_2. We denote their respective orders by m_1^p, m_1^q, m_2^p, and m_2^q. We assume that $m_1^p = m_2^p = m^p$.

We also consider interface operators $T_k^{p,j}$, $k = 1, 2$, $j = 1, \dots, m^p$ of order $\beta_k^{p,j}$ and $T_k^{q,j}$, $k = 1, 2$, $j = 1, \dots, m^q$ of order $\beta_k^{q,j}$ with $m^q = (m_1^q + m_2^q)/2$. We assume that $m^p - \beta_1^{p,j} = m^p - \beta_2^{p,j} = m^p - \beta^{p,j}$ and $m_1^q - \beta_1^{q,j} = m_2^q - \beta_2^{q,j} = m^q - \beta^{q,j}$. Then the operators P_1, P_2, $T_k^{p,j}$, $k = 1, 2$, $j = 1, \dots, m^p$, allow one to define an elliptic transmission problem as presented in Section 1. The same is valid for Q_1, Q_2, $T_k^{q,j}$, $k = 1, 2$, $j = 1, \dots, m^q$.

We observe that P_1Q_1, P_2Q_2, and the intreface operators $T_k^{p,j}Q_k$, $k = 1, 2$, $j = 1, \dots, m^p$, and $T_k^{q,j}$, $k = 1, 2$, $j = 1, \dots, m^q$ also allow to define an elliptic transmission problem. In fact, the operators P_1Q_1 and P_2Q_2 are of respective order $m_1 = m^p + m_1^q$ and $m_2 = m^p + m_2^q$. We set $m = m^p + m^q = (m_1 + m_2)/2$. The interface operator $T_k^{q,j}$ is of order $\beta_k^{q,j}$ and we have $m_1 - \beta_1^{q,j} = m_2 - \beta_2^{q,j} = m - \beta^{q,j}$. The interface operator $T_k^{p,j}Q_k$ is of order $\beta_k^{p,j} + m_k^q$ and we have $m_1 - (\beta_1^{p,j} + m_1^q) = m_2 - (\beta_2^{p,j} + m_2^q)$.

One may possibly wonder about unique continuation for this product transmission problem, in particuler in the case no Carleman estimate of the type derived here can be achieved. Let us however assume that for a function ψ and the weigth function $\varphi = \exp(\gamma\psi)$ we can derive Carleman estimates for the transmission problems associated with P_1, P_2 and Q_1, Q_2. More precisely we assume that the first problem satisfies the simple characteristic property while the second one only satisfies the strong pseudo-convexity condition.

Theorem 7.2. *Let P_k, $T_k^{p,j}$, $k = 1, 2$, $j = 1, \dots, m^p$, and Q_k, $T_k^{q,j}$, $k = 1, 2$, $j = 1, \dots, m^q$, be given as above. Let $x_0 \in S$, $f \in \mathscr{C}^0(\Omega)$, with $f_k = f_{|\Omega_k}$ and V a neighborhood of x_0, be such that*

1. *f_k and P_k have the simple characteristic property of Definition 6.7 in $V \cap \overline{\Omega_k}$;*
2. *f has the strong pseudo-convexity property of Definition 6.2 with respect to Q_1 and Q_2 in V;*
3. *$\big\{P_k, f, T_k^{p,j},\ k = 1, 2,\ j = 1, \dots, m^p\big\}$ satisfies the transmission condition at x_0;*
4. *$\big\{Q_k, f, T_k^{q,j},\ k = 1, 2,\ j = 1, \dots, m^q\big\}$ satisfies the transmission condition at x_0.*

Assume that u is such that $u_k = u_{|\Omega_k} \in H^{m^p + m_k^q}(\Omega_k)$ and satisfies

- $$|P_kQ_ku_k(x)| \leq C \sum_{|\alpha| \leq m^p + m_k^q - 1} |D^\alpha u_k(x)|, \quad \text{a.e. in } V_k = V \cap \Omega_k,\ k = 1, 2; \tag{7.16}$$

- *for $j = 1, \dots, m^p$, $|\alpha| \leq m^p - \beta^{p,j}$, with $\alpha \in \mathbb{N}^{n-1}$,*

$$|D_\mathsf{T}^\alpha \big(T_1^{p,j} Q_1 u_1(x) + T_2^{p,j} Q_2 u_2(x)\big)| \leq C \sum_{k=1,2} \sum_{\substack{|\alpha'| \leq |\alpha| + m_k^q \\ + \beta_k^{p,j} - 1}} |D^{\alpha'} u_k(x)|, \quad \text{a.e. in } V \cap S; \tag{7.17}$$

- *for $j = 1, \dots, m^q$, $|\alpha_1| \leq m^p$, $|\alpha_2| \leq m^q - \beta^{q,j}$, with $\alpha_1 \in \mathbb{N}^n$ and $\alpha_2 \in \mathbb{N}^{n-1}$,*

$$|D^{\alpha_1} D_\mathsf{T}^{\alpha_2} \big(T_1^{q,j} u_1(x) + T_2^{q,j} u_2(x)\big)| \leq C \sum_{k=1,2} \sum_{\substack{|\alpha'| \leq |\alpha_1| + |\alpha_2| \\ + \beta_k^{q,j} - 1}} |D^{\alpha'} u_k(x)|, \quad \text{a.e. in } V \cap S; \tag{7.18}$$

- *and u vanishes in $\{x \in V;\ f(x) \geq f(x_0)\}$.*

Then u vanishes in a neighborhood of x_0.

Here D_{T}^{α} denotes a familly of differential operators that act tangentially to the interface S and, in local coordinates near x_0, where $S = \{x_n = 0\}$, their principal symbol is ξ'^{α}.

Remark 7.3. Here we have assume that $m_1^p = m_2^p$. It would be interesting to know if such an assumption can be removed. This assumption is connected to the shifted estimate of Corollary 6.11, where the shift is the same on both sides of the interface. Having different Sobolev-scale shifts from one side to the other leads to technical difficulties with the transmission terms on the interface. It is not clear whether such estimates can be achieved without modifying the properties of the transmission operators.

Proof. The proof follows that of Theorem 7.1. We set $\psi(x) = f(x) - \varepsilon|x - x_0|^2$ and conditions (1)-(4) in the statement of the theorem are also satisfied by ψ for ε chosen sufficiently small in a neighborhood $V' \subset V$ of x_0. We then set $\varphi = \exp(\gamma\psi)$.

We derive an estimate for P_kQ_k. We first write an estimate for P_k. By Theorem 6.9, there exists V_1 neighborhood of x_0 in $\mathbb{R}^n$ such that $V_1 \subset V'$ and

$$\sum_{k=1,2} \big(\gamma^{1/2}\big\|\tilde\tau_k^{-\frac12} e^{\tau\varphi_k} v_k\big\|_{m^p,\tilde\tau_k} + |e^{\tau\varphi_{|S}}\,\mathrm{tr}(v_k)|_{m^p-1,1/2,\tilde\tau}\big)$$
$$\lesssim \sum_{k=1,2} \|e^{\tau\varphi_k} P_k v_k\|_{L^2(\Omega_k)} + \sum_{j=1}^{m^p} \big|e^{\tau\varphi_{|S}}(T_1^{p,j}v_1 + T_2^{p,j}v_2)_{|S}\big|_{m^p-1/2-\beta^{p,j},\tilde\tau},$$

for all $v_k = w_{k|\Omega_k}$ with $w_k \in \mathscr{C}_c^\infty(V_1)$, $\tau \geq \tau_1$ and $\gamma \geq \gamma_1$, for τ_1 and γ_1 chosen sufficiently large.

For Q_k, by Corollary 6.11, there exists V_2 neighborhood of x_0 in $\mathbb{R}^n$ such that $V_2 \subset V'$ and

$$\sum_{k=1,2} \big(\big\|\tilde\tau_k^{-1} e^{\tau\varphi_k} v_k\big\|_{m^p+m_k^q,\tilde\tau_k} + \big|\tilde\tau^{-1/2} e^{\tau\varphi_{|S}}\,\mathrm{tr}(v_k)\big|_{m^p+m_k^q-1,1/2,\tilde\tau}\big)$$
$$\lesssim \sum_{k=1,2} \big\|\tilde\tau_k^{-1/2} e^{\tau\varphi_k} Q_k v_k\big\|_{m^p,\tilde\tau_k} + \sum_{j=1}^{m^q} \big|\tilde\tau^{-1/2} e^{\tau\varphi_{|S}}(T_1^{q,j}v_1 + T_2^{q,j}v_2)_{|S}\big|_{m^p,m^q-1/2-\beta^{q,j},\tilde\tau},$$

for all $v_k = w_{k|\Omega_k}$ with $w_k \in \mathscr{C}_c^\infty(V_2)$, $\tau \geq \tau_2$ and $\gamma \geq \gamma_2$, for τ_2 and γ_2 chosen sufficiently large.

Letting $V_3 = V_1 \cap V_2$ with the previous two estimates we obtain

$$\begin{aligned}(7.19)\quad \gamma^{1/2} \sum_{k=1,2} \big(\big\|\tilde\tau_k^{-1} e^{\tau\varphi_k} v_k\big\|_{m^p+m_k^q,\tilde\tau_k} + \big|\tilde\tau^{-1/2} e^{\tau\varphi_{|S}}\,\mathrm{tr}(v_k)\big|_{m^p+m_k^q-1,1/2,\tilde\tau}\big)&\\ \lesssim \sum_{k=1,2} \|e^{\tau\varphi_k} P_k Q_k v_k\|_{L^2(\Omega_k)} + \sum_{j=1}^{m^p} \big|e^{\tau\varphi_{|S}}(T_1^{p,j}Q_1v_1 + T_2^{p,j}Q_2v_2)_{|S}\big|_{m^p-1/2-\beta^{p,j},\tilde\tau}&\\ + \gamma^{1/2} \sum_{j=1}^{m^q} \big|\tilde\tau^{-1/2} e^{\tau\varphi_{|S}}(T_1^{q,j}v_1 + T_2^{q,j}v_2)_{|S}\big|_{m^p,m^q-1/2-\beta^{q,j},\tilde\tau},&\end{aligned}$$

for all $v_k = w_{k|\Omega_k}$ with $w_k \in \mathscr{C}_c^\infty(V_3)$, $\tau \geq \tau_3$ and $\gamma \geq \gamma_3$, for τ_3 and γ_3 chosen sufficiently large.

We choose χ as in the proof of Theorem 7.1 and we apply estimate (7.19) to $v_k = \chi u_k$ as can be done by a density argument. We now sketch how the remainder of the proof can be carried out.

We have $P_kQ_k(\chi u)=\chi P_kQ_ku+[P_kQ_k,\chi]u$. The term $[P_kQ_k,\chi]u$ is supported in the set Σ introduced in the proof of Theorem 7.1 and can be handle as it is done there. For the first term, with (7.16) we have

$$\begin{aligned}\|e^{\tau\varphi}\chi P_kQ_ku\|_{L^2(\Omega_k)}&\lesssim\sum_{|\alpha|\leq m^p+m_k^q-1}\|e^{\tau\varphi}\chi D^\alpha u\|_{L^2(\Omega_k)}\\&\lesssim\sum_{|\alpha|\leq m^p+m_k^q-1}\|e^{\tau\varphi}D^\alpha(\chi u)\|_{L^2(\Omega_k)}+\sum_{|\alpha|\leq m^p+m_k^q-1}\|e^{\tau\varphi}[\chi,D^\alpha]u\|_{L^2(\Omega_k)}.\end{aligned}$$

The second term in the r.h.s. concerns functions with support located in Σ and their treatment is done as in the proof of Theorem 7.1. For the first term we have

$$\sum_{|\alpha|\leq m^p+m_k^q-1}\|e^{\tau\varphi}D^\alpha(\chi u)\|_{L^2(\Omega_k)}\lesssim\|e^{\tau\varphi}\chi u\|_{m^p+m_k^q-1,\tilde\tau_k}\lesssim\big\|\tilde\tau_k^{-1}e^{\tau\varphi}\chi u\big\|_{m^p+m_k^q,\tilde\tau_k},$$

which can be absorbed by the first term in (7.19) by chosing γ sufficiently large.

Next we have

$$\begin{aligned}\big|e^{\tau\varphi_{|S}}(T_1^{p,j}Q_1v_1+T_2^{p,j}Q_2v_2)_{|S}\big|_{m^p-1/2-\beta^{p,j},\tilde\tau}&\leq\big|e^{\tau\varphi_{|S}}(T_1^{p,j}Q_1v_1+T_2^{p,j}Q_2v_2)_{|S}\big|_{m^p-\beta^{p,j},\tilde\tau}\\&=\sum_{\substack{r+|\alpha|\\\leq m^p-\beta^{p,j}}}\big|\tilde\tau^rD_{\mathsf T}^\alpha e^{\tau\varphi_{|S}}(T_1^{p,j}Q_1v_1+T_2^{p,j}Q_2v_2)_{|S}\big|_{L^2(S)}\\&\lesssim\sum_{\substack{r+|\alpha|\\\leq m^p-\beta^{p,j}}}\big|\tilde\tau^re^{\tau\varphi_{|S}}D_{\mathsf T}^\alpha(T_1^{p,j}Q_1v_1+T_2^{p,j}Q_2v_2)_{|S}\big|_{L^2(S)}.\end{aligned}$$

Writing $D_{\mathsf T}^\alpha T_k^{p,j}Q_kv_k=\chi D_{\mathsf T}^\alpha T_k^{p,j}Q_ku_k+[D_{\mathsf T}^\alpha T_k^{p,j}Q_k,\chi]u$ we have

$$\begin{aligned}&\big|e^{\tau\varphi_{|S}}(T_1^{p,j}Q_1v_1+T_2^{p,j}Q_2v_2)_{|S}\big|_{m^p-1/2-\beta^{p,j},\tilde\tau}\\&\quad\lesssim\sum_{\substack{r+|\alpha|\\\leq m^p-\beta^{p,j}}}\big|\tilde\tau^re^{\tau\varphi_{|S}}\chi D_{\mathsf T}^\alpha(T_1^{p,j}Q_1u_1+T_2^{p,j}Q_2u_2)_{|S}\big|_{L^2(S)}\\&\qquad+\sum_{\substack{r+|\alpha|\\\leq m^p-\beta^{p,j}}}\big|\tilde\tau^re^{\tau\varphi_{|S}}([D_{\mathsf T}^\alpha T_1^{p,j}Q_1,\chi]u_k+[D_{\mathsf T}^\alpha T_2^{p,j}Q_2,\chi]v_2)_{|S}\big|_{L^2(S)}.\end{aligned}$$

The terms $[D_{\mathsf T}^\alpha T_k^{p,j}Q_k,\chi]u$ are supported in the set Σ and can be treated as in the proof of Theorem 7.1. For the first term, with (7.17) we have

$$\begin{aligned}&\big|\tilde\tau^re^{\tau\varphi_{|S}}\chi D_{\mathsf T}^\alpha(T_1^{p,j}Q_1u_1+T_2^{p,j}Q_2u_2)_{|S}\big|_{L^2(S)}\\&\quad\lesssim\sum_{k=1,2}\sum_{\substack{|\alpha'|\leq|\alpha|+m_k^q\\+\beta_k^{p,j}-1}}\big|\tilde\tau^re^{\tau\varphi_{|S}}(\chi D^{\alpha'}u_k)_{|S}\big|_{L^2(S)}\\&\quad\lesssim\sum_{k=1,2}\sum_{\substack{|\alpha'|\leq|\alpha|+m_k^q\\+\beta_k^{p,j}-1}}\big|\tilde\tau^rD^{\alpha'}(e^{\tau\varphi_k}\chi u_k)_{|S}\big|_{L^2(S)}+\sum_{k=1,2}\sum_{\substack{|\alpha'|\leq|\alpha|+m_k^q\\+\beta_k^{p,j}-1}}\big|\tilde\tau^r([e^{\tau\varphi_{|S}}\chi,D^{\alpha'}]u_k)_{|S}\big|_{L^2(S)}\end{aligned}$$

The second term in the r.h.s. concerns functions with support located in Σ and their treatment is done as in the proof of Theorem 7.1. For the first term we have $r+|\alpha|\leq m^p-\beta^{p,j}$ and $|\alpha'|\leq|\alpha|+m_k^q+\beta_k^{p,j}-1$ and thus we write

$$\big|\tilde\tau^rD^{\alpha'}(e^{\tau\varphi_k}\chi u_k)_{|S}\big|_{L^2(S)}\lesssim\big|\operatorname{tr}(e^{\tau\varphi_k}\chi u_k)\big|_{m^p+m_k^q-1,0,\tilde\tau}.$$

As

$$\begin{aligned}\gamma^{1/2}\big|\tilde\tau^{-1/2}e^{\tau\varphi_{|S}}\operatorname{tr}(\chi u_k)\big|_{m^p+m_k^q-1,1/2,\tilde\tau}&\geq\gamma^{1/2}\big|e^{\tau\varphi_{|S}}\operatorname{tr}(\chi u_k)\big|_{m^p+m_k^q-1,0,\tilde\tau}\\&\gtrsim\gamma^{1/2}\big|\operatorname{tr}(e^{\tau\varphi_k}\chi u_k)\big|_{m^p+m_k^q-1,0,\tilde\tau}\end{aligned}$$

we see that the above terms can be absorbed by the second term in (7.19) by chosing γ sufficiently large.

We finally write

$$\begin{aligned}&\sum_{j=1}^{m^q}\big|\tilde\tau^{-1/2}e^{\tau\varphi_{|S}}(T_1^{q,j}v_1+T_2^{q,j}v_2)_{|S}\big|_{m^p,m^q-1/2-\beta^{q,j},\tilde\tau}\\&\quad\leq\sum_{j=1}^{m^q}\big|\tilde\tau^{-1/2}e^{\tau\varphi_{|S}}(T_1^{q,j}v_1+T_2^{q,j}v_2)_{|S}\big|_{m^p,m^q-\beta^{q,j},\tilde\tau}\\&\quad\lesssim\sum_{\substack{|\alpha_1|\leq m^p\\|\alpha_2|\leq m^q-\beta^{q,j}\\r+|\alpha_1|+|\alpha_2|\leq m^p+m^q-\beta^{q,j}}}\big|\tilde\tau^{r-1/2}D^{\alpha_1}D_{\mathsf T}^{\alpha_2}e^{\tau\varphi_{|S}}(T_1^{q,j}v_1+T_2^{q,j}v_2)_{|S}\big|,\end{aligned}$$

which can be treated as above by using (7.18). We then conclude the proof as in that of Theorem 7.1. □

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