Diophantine approximation with improvement of the simultaneous control of the error and of the denominator

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Abstract

In this work we proof the following theorem which is, in addition to some other lemmas, our main result:

theorem. Let $X = \{(x_1, t_1), (x_2, t_2), ..., (x_n, t_n)\}$ be a finite part of $\mathbb{R} \times \mathbb{R}^{*+}$, then there exist a finite part R of $\mathbb{R}^{*+}$ such that for all $\varepsilon > 0$ there exists $r \in R$ such that if $0 < \varepsilon \leq r$ then there exist rational numbers $\left(\frac{p_i}{q}\right)_{i=1,2,...,n}$ such that:

$$\left\{ \begin{array}{c} \left| x_i - \frac{p_i}{q} \right| \leq \varepsilon t_i \\ \varepsilon q \leq t_i \end{array} \right. , \ i = 1, 2, ..., n. \qquad (*)$$

It is clear that the condition $\varepsilon q \leq t_i$ for $i = 1, 2, ..., n$ is equivalent to $\varepsilon q \leq t = \underset{i=1,2,...,n}{Min} (t_i)$. Also, we have (*) for all ε verifying $0 < \varepsilon \leq \varepsilon_0 = \min R$.

The previous theorem is the classical equivalent of the following one which is formulated in the context of the nonstandard analysis ([2], [5], [6], [8]).

theorem. For every positive infinitesimal real ε, there exists an unlimited integer q depending only of ε, such that $\forall^{st} x \in \mathbb{R} \ \exists \ p_x \in \mathbb{Z}$:

$$\left\{ \begin{array}{rcl} x & = & \dfrac{p_x}{q} + \varepsilon\phi \\ \varepsilon q & \cong & 0 \end{array} \right. .$$

For this reason, to prove the nonstandard version of the main result and to get its classical version we place ourselves in the context of the nonstandard analysis.

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1 Introduction, Notations and Rappel

We dispose in the domain of Diophantine approximation of many results (refer for example to [3], [7]). In the following, we give as an example, the two most used theorems:

Theorem (Dirichlet) 1.1. [7]**.** Suppose that x_1, x_2, ... ,x_n *are* n *real numbers and that* $T > 1$ *is an integer. Then there exist integers* q,p_1,p_2,...,p_n *with*

$$\left\{ \begin{array}{c} \left| x_i - \dfrac{p_i}{q} \right| \leq \dfrac{1}{Tq} \qquad (i = 1, 2, ..., n) \\ 1 \leq q < T^n \end{array} \right. . \tag{1.1}$$

Theorem (Kronecker) 1.2. [7]**.** For any reals β_1, β_2, ... ,β_n and any $t > 0$, the system of inequalities

$$\left\{ \begin{array}{l} |q\zeta_1 - p_1 - \beta_1| < t \\ |q\zeta_2 - p_2 - \beta_2| < t \\ \qquad \ldots\ldots\ldots\ldots\ldots \\ |q\zeta_n - p_n - \beta_n| < t \end{array} \right| \tag{1.2}$$

is solvable in integers q, p_1, p_2, ... ,p_n if and only if ζ_1, ζ_2, ... ,ζ_n are not rationally dependent. Note that ζ_1, ζ_2, ... ,ζ_n are said rationally dependent if there exist integers r, r_1, r_2, ... ,r_n not all zero such that

$$r_1\zeta_1 + r_2\zeta_2 + ... + r_n\zeta_n = r.$$

When we take $\beta_1 = \beta_2 = ... = \beta_n = 0$, this theorem is used to approximate the reals ζ_i by using rationals $\frac{p_i}{q}$ to errors smaller than $\frac{t}{q}$.

In general, in these results we observe that the simultaneous control between the error and the common denominator q should be clarified and specified. This, because the approximation to a given error (which is generally small) requires a denominator that is generally too big. Conversely, the approximation with a small denominator might give an error that is not really small. This question has motivated us to give the following theorem which is, in addition to some other lemmas, our main result of this work.

Theorem 1.3. Let $X = \{(x_1, t_1), (x_2, t_2), ..., (x_n, t_n)\}$ be a finite part of $\mathbb{R} \times \mathbb{R}^{*+}$, then there exist a finite part R of $\mathbb{R}^{*+}$ such that for all $\varepsilon > 0$ there exists $r \in R$ such that if $0 < \varepsilon \leq r$ then there exist rational numbers $\left(\frac{p_i}{q}\right)_{i=1,2,...,n}$ such that:

$$\left\{ \begin{array}{c} \left| x_i - \frac{p_i}{q} \right| \leq \varepsilon t_i \\ \varepsilon q \leq t_i \end{array} \right. , \; i = 1, 2, ..., n. \tag{1.3}$$

We note that in (1.3) the condition $\varepsilon q \leq t_i$ for $i = 1, 2, ..., n$ is equivalent to $\varepsilon q \leq t = \underset{i=1,2,...,n}{Min} (t_i)$. Also, under the assumption of theorem 1.3, for all ε verifying $0 < \varepsilon \leq \varepsilon_0 = \min R$ we obtain (1.3).

The theorem 1.3 is the classical equivalent of the following theorem (theorem 1.4.) formulated in the context of the nonstandard analysis.

Theorem 1.4. For every positive infinitesimal real ε, there exists an integer Q depending only of ε, such that $\forall^{st} x \in R \; \exists \; P_x \in \mathbb{Z}$:

$$\left\{ \begin{array}{rcl} x & = & \frac{P_x}{Q} + \varepsilon\phi \\ \varepsilon Q & \cong & 0 \end{array} \right. . \tag{1.4}$$

In the following we make a comparison between our result (theorem 1.3) and the existing results such as Dirichlet's theorem and Kronecker's theorem.

Our main result is used to approximate at a reduced common denominator q since $\varepsilon q \leq t$ (i.e. $q \leq \frac{t}{\varepsilon}$) and at a different errors since $\left| x_i - \frac{p_i}{q} \right| \leq \varepsilon t_i$ for $i = 1, 2, ..., n$. In addition, if we take $t_1 = t_2 = ... = t_n = t > 0$ and $\varepsilon_0 = Min \; R$ then for every $0 < \varepsilon \leq \varepsilon_0$ there exist integers q,p_1,p_2,...,p_n such that

$$\underset{i\in\{1,2,\ldots,n\}}{Max}\left|x_i-\frac{p_i}{q}\right|\leq\varepsilon t \text{ and } q\leq\frac{t}{\varepsilon} \tag{1.5}$$

i.e., a denominator $q\leq\dfrac{t}{\varepsilon}$ enough for an error not exceeding εt.

Look when we use, under the same hypotheses, the Dirichlet's theorem. It may happen that when we take $\dfrac{1}{T}>\varepsilon t$, the common denominator $q\geq 1$ is small enough so that the maximum error is strictly greater than εt i.e. $\varepsilon t<\underset{i\in\{1,2,\ldots,n\}}{Max}\left|x_i-\dfrac{p_i}{q}\right|\leq\dfrac{1}{Tq}\leq\dfrac{1}{T}$. In contrast, when we take T satisfying $\dfrac{1}{T}\leq\varepsilon t$ then we are sure that the maximum error is smaller than or equal to εt i.e. $\underset{i\in\{1,2,\ldots,n\}}{Max}\left|x_i-\dfrac{p_i}{q}\right|\leq\dfrac{1}{Tq}\leq\dfrac{1}{T}\leq\varepsilon t$. But in this case it may happen that the common denominator q, since that $1\leq q<T^n$, is very close to $T^n\geq\dfrac{1}{(\varepsilon t)^n}$ ($q=T^n-1\geq\dfrac{1}{(\varepsilon t)^n}-1$; for instance). Consequently, to be sure of the realization of the approximation asked, it is necessary to choose $\dfrac{1}{T}\leq\varepsilon t$ and q can be too big in this case as we have seen.

On his part the Kronecker's theorem is purely existential and don't say anything on the common denominator.

From the above we can see that the theorem 1.3 ensure the ability to control the size of q and of the maximum error; especially when ε (resp. n) become small (resp. large). For its proof we place ourselves in the framework of the nonstandard analysis and we proceed as follows :

(1) We first show theorem 1.4 (In the sequel noted theorem 2.1.) by using some lemmas.
(2) We translate theorem 1.4 by using the Nelson's algorithm.

1.1 Notations

i) For a number x (integer or non) we have the following usages:
1) Abbreviation, $st(x)$ indicates that x is standard; $\forall^{st}x$ signifies $\forall x\,[st(x)\Longrightarrow ..]$.
2) $x\cong+\infty$ (resp. $x\cong 0$) signifies that x is a positive unlimited (resp. x an infinitesimal). $x\underset{\cong}{>}0$ signifies that x is an infinitesimal real strictly positive.

3) $\pounds$(resp. ϕ) signifies a limited real (resp. an infinitesimal real) on which one doesn't say anything besides.
4) $\|x\|$ is the difference, taken positively, between x and the nearest integer.
5) $E(x)$ (resp. $\{x\}$) is the integral part of x (resp. the fractional part of x; that is $\{x\} = x - E(x)$).
6) Let ε be an infinitesimal real, one designates by $\varepsilon - galaxie(x)$ the set $\{y : y = x + \varepsilon\pounds\}$ and by $\varepsilon - halo(x)$ the set $\{y : y = x + \varepsilon\phi\}$.
7) x^0 signifies, for x limited, the standard part of x.
ii)
8) If E is a given set, E^σ(resp. $|E|$) designates the external set formed, only, by the standard elements of E (resp. the cardinality of E).
9) One notes by $(x_1, x_2, ..., x_n)^T$ the vector column $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$.

1.2 Rappel

1.2.1 Farey series([3])

The Farey series $\mathcal{F}_N$ of order N is the ascending series of irreducible fractions between 0 and 1 whose denominators do not exceed N. Thus $\frac{h}{k}$ belongs to $\mathcal{F}_N$ if

$$0 \leq h \leq k \leq N,\ (h,k) = 1$$

the numbers 0 and 1 are included in the forms $\frac{0}{1}$ and $\frac{1}{1}$. If $\frac{h}{k} < \frac{h'}{k'} < \frac{h''}{k''}$ are three successive elements of $\mathcal{F}_N$ $(N > 1)$, then one has the following properties:

$1^0)$ $kh' - hk' = 1$.
$2^0)$ $\frac{h'}{k'} = \frac{h + h''}{k + k''}$.
$3^0)$ $k + k' > N$ and $\frac{h}{k} < \frac{h + h'}{k + k'} < \frac{h'}{k'}$.
$4^0)$ If $N > 1$, two successive elements of $\mathcal{F}_N$ don't have the same denominator.

5^0) Let $\frac{h_1}{k_1}$, $\frac{h_2}{k_2}$ be two successive elements of $\mathcal{F}_N$ $(N \geq 1)$ with $\frac{h_1}{k_1} < \frac{h_2}{k_2}$, and let the two following sequences:

$$\begin{cases} U_0 = \frac{h_2}{k_2},\ U_1 = \frac{h_2+h_1}{k_2+k_1},\ \ldots\ ,U_i = \frac{h_2+ih_1}{k_2+ik_1},\ \ldots \\ V_0 = \frac{h_1}{k_1},\ V_1 = \frac{h_1+h_2}{k_1+k_2},\ \ldots,\ V_j = \frac{h_1+jh_2}{k_1+jk_2},\ \ldots \end{cases} . \tag{1.6}$$

We prove easily that the sequence $(U_i)_{i\in\mathbb{N}}$ (resp. $(V_j)_{j\in\mathbb{N}}$) is decreasing (resp. increasing); besides we have:

$$\begin{cases} U_i - U_{i+1} = \frac{1}{(k_2+ik_1)(k_2+(i+1)k_1)},\ U_i - \frac{h_1}{k_1} = \frac{1}{k_1(k_2+ik_1)} \\ V_{j+1} - V_j = \frac{1}{(k_1+jk_2)(k_1+(j+1)k_2)},\ \frac{h_2}{k_2} - V_j = \frac{1}{k_2(k_1+jk_2)} \end{cases} . \tag{1.7}$$

1.2.2 Approximation to the infinitesimal sense of reals

Theorem 1.5. [1]. Let ξ be a real number. Then for all positive infinitesimal real ε there exist a rational number $\frac{p}{q}$ and a limited real l such that*:*

$$\begin{cases} \xi &= \frac{p_i}{q} + \varepsilon l \\ \varepsilon q &\cong 0 \end{cases} . \tag{1.8}$$

2 Simultaneous approximation to the infinitesimal sense of standard reals

We prove in this section the following theorem whose translation by the algorithm of Nelson gives the theorem 1.3 .

Theorem 2.1. For every positive infinitesimal real ε, there exists *an integer* Q *depending only of* ε, such that $\forall^{st} x \in \mathbb{R}\ \exists\ P_x \in \mathbb{Z}$:

$$\begin{cases} x &= \frac{P_x}{Q} + \varepsilon\phi \\ \varepsilon Q &\cong 0 \end{cases} . \tag{2.1}$$

Let ε be a positive infinitesimal real. We need to the following lemmas

Lemma 2.2. Let $(\xi_1, \xi_2, ..., \xi_N)$ a system of real numbers with $N \geq 1$ limited. Then for all positive infinitesimal real θ there are rational numbers $\left(\frac{p_i}{q}\right)_{i=1,2,...,N}$ and limited reals $(l_i)_{i=1,2,...,N}$ such that for $i = 1, 2, ..., N$:

$$\begin{cases} \xi_i &= \frac{p_i}{q} + \theta l_i \\ \theta q &\cong 0 \end{cases}. \tag{2.2}$$

Proof. Consider, for every $n \in \mathbb{N}^*$, the formula:

$$B(n) = \begin{array}{c} \text{"}\forall\ (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n \text{ with } n \geq 1 \text{ and } \forall \theta \underset{\cong}{\geq} 0\ \exists \left(\frac{P_i}{Q}\right)_{i=1,2,...,n} \\ \text{such that for every } i \in \{1, 2, ..., n\} : \begin{cases} x_i - \frac{P_i}{Q} = \theta \pounds \\ \theta Q \cong 0 \end{cases} \text{"} \end{array}.$$

By theorem 1.5, we have $B(1)$. Suppose, for $1 \leq n$ a standard integer, $B(n)$ and prove $B(n+1)$. Let $(\xi_1, \xi_2, ..., \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+1}$ and let $\theta \underset{\cong}{\geq} 0$, then by $B(n)$ there are rational numbers $\left(\frac{p_i}{q}\right)_{i=1,2,...,n}$ such that

$$\begin{cases} \xi_1 &= \frac{p_1}{q} + \theta \pounds \\ \xi_2 &= \frac{p_2}{q} + \theta \pounds \\ \vdots &= \vdots \\ \xi_n &= \frac{p_n}{q} + \theta \pounds \end{cases} \tag{2.3}$$

where $\theta q \cong 0$. Now, since $\theta q \cong 0$, the application of theorem 1.5 implies $q\xi_{n+1} = \frac{p_{n+1}}{q_{n+1}} + (\theta q)\, \pounds$, $(\theta q)\, q_{n+1} \cong 0$. Hence

$$\xi_{n+1} = \frac{p_{n+1}}{q q_{n+1}} + \theta \pounds,\ \theta q q_{n+1} \cong 0. \tag{2.4}$$

We deduct from (2.3) and (2.4) that:

$$\left\{\begin{array}{lllll} \xi_1 & = & \dfrac{p_1 q_{n+1}}{q q_{n+1}} + \theta\pounds & = & \dfrac{P_1}{Q} + \theta\pounds \\ \xi_2 & = & \dfrac{p_2 q_{n+1}}{q q_{n+1}} + \theta\pounds & = & \dfrac{P_2}{Q} + \theta\pounds \\ \vdots & = & \vdots & = & \vdots \\ \xi_n & = & \dfrac{p_n q_{n+1}}{q q_{n+1}} + \theta\pounds & = & \dfrac{P_n}{Q} + \theta\pounds \\ \xi_{n+1} & = & \dfrac{p_{n+1}}{q q_{n+1}} + \theta\pounds & = & \dfrac{P_{n+1}}{Q} + \theta\pounds \end{array}\right.$$

where, from (2.4), $\theta Q = \theta q q_{n+1} \cong 0$. Consequently $B(n+1)$. Therefore, by the external recurrence principle, we have $\forall^{st} n \geq 1\ B(n)$.

Lemma 2.3. Let E be a given set. For all integer $\omega \cong +\infty$, there is a finite subset $F \subset E$ containing all standard elements of E (i.e. $E^{\sigma} \subset F$) and whose cardinal is strictly inferior to ω ($|F| < \omega$).

Proof. Let $\omega \cong +\infty$. Let $B(F, z)$ be the internal formula: "$F \subset E$, $|F| < \omega$, $z \in F$ ". Let $Z \subset E$ be a standard finite part. Then there exists a finite part $F \subset E$ with $|F| < \omega$ such that every element z of Z belongs to F, i.e. we have $B(F, z)$. Indeed it suffices to take $F = Z$. Therefore, the principle of idealization (I) asserts the existence of a finite part $F \subset E$ with $|F| < \omega$ such that any standard element of L belongs to F.

Lemma 2.4. Let $\lambda \cong +\infty$ be a real number such that $\sqrt{\varepsilon}\lambda \cong 0$. Let F_M be the Farey sequence of order $M = E\left(\dfrac{\lambda}{\sqrt{\varepsilon}}\right)$. If $\dfrac{p_1}{q_1}, \dfrac{p_2}{q_2}$ are two elements of F_M such that $q_1 \simeq +\infty$, $q_2 \simeq +\infty$ and $\left[\dfrac{p_1}{q_1}\ ,\ \dfrac{p_2}{q_2}\right]$ doesn't contain any standard rational number (in this case $\dfrac{p_1}{q_1} \cong \dfrac{p_2}{q_2}$). Then there exist a finite sequence of irreducible rational numbers $\left(\dfrac{l_i}{m_i}\right)_{i=1,2,...,g}$ such that:

$$\frac{p_1}{q_1} = \frac{l_1}{m_1} < \frac{l_2}{m_2} < ... < \frac{l_g}{m_g} = \frac{p_2}{q_2}$$

where $\dfrac{l_{i+1}}{m_{i+1}} - \dfrac{l_i}{m_i} = \varepsilon\phi$ for $i = 1, 2, ..., g-1$. Besides for $i = 1, 2, ..., g$ we

have $\varepsilon m_i \cong 0$ and $m_i \cong +\infty$.

Proof. Let us consider the case where $\dfrac{p_2}{q_2} - \dfrac{p_1}{q_1}$ is not of $\varepsilon\phi$ form; otherwise the lemma is proved. Let $\left(\dfrac{t_i}{\gamma_i}\right)_{i=1,2,...,r}$ be the elements of $\mathcal{F}_M$ such that

$$\frac{p_1}{q_1} = \frac{t_1}{\gamma_1} < \frac{t_2}{\gamma_2} < ... < \frac{t_r}{\gamma_r} = \frac{p_2}{q_2}.$$

Let $i_0 \in \{1, 2, ..., r-1\}$ such that $\dfrac{t_{i_0+1}}{\gamma_{i_0+1}} - \dfrac{t_{i_0}}{\gamma_{i_0}}$ is not of $\varepsilon\phi$ form, because if a such i_0 does not exist the lemma is proved. From the properties of $\mathcal{F}_M$ (**1.2.1**), γ_{i_0+1} and γ_{i_0} cannot be equal. Then there are two cases:
A) $\gamma_{i_0+1} > \gamma_{i_0}$: Let us take, in this case, $g_0 \cong +\infty$ an integer such that $\dfrac{g_0}{\gamma_{i_0}} \cong 0$ (the existence of g_0 is assured by Robinson's lemma). Let $X = E\left(\dfrac{g_0}{\varepsilon\gamma_{i_0}}\right)$ and

$$H = \left\{\frac{t_{i_0}}{\gamma_{i_0}}, U_p, U_{p-1}, ..., U_0\right\}$$

where $p = E\left(\dfrac{X - \gamma_{i_0+1}}{\gamma_{i_0}}\right)$ and $U_i = \dfrac{t_{i_0+1} + i.t_{i_0}}{\gamma_{i_0+1} + i.\gamma_{i_0}}$ $(i = 0, 1, ..., p-1, p)$. Now we prove that : p is an unlimited integer, the product of the denominator of every element of H by ε is an infinitesimal and the distance between two successive elements of H is of the $\varepsilon\phi$ form.

Indeed, we have $X = E\left(\dfrac{g_0}{\varepsilon\gamma_{i_0}}\right) = \dfrac{g_0}{\varepsilon\gamma_{i_0}} - \rho_X$ where $\rho_X \in [0, 1[$.

$$\begin{aligned}\frac{X - \gamma_{i_0+1}}{\gamma_{i_0}} &= \frac{g_0}{\varepsilon\gamma_{i_0}\gamma_{i_0}} - \frac{\rho_X}{\gamma_{i_0}} - \frac{\gamma_{i_0+1}}{\gamma_{i_0}} \\ &= \frac{g_0 - \varepsilon\gamma_{i_0}\rho_X - \varepsilon\gamma_{i_0}\gamma_{i_0+1}}{\varepsilon\gamma_{i_0}\gamma_{i_0}}.\end{aligned}$$

Since $\varepsilon\gamma_{i_0}\rho_X \cong 0$, $\varepsilon\gamma_{i_0}\gamma_{i_0+1}$ is a limited real number otherwise $\dfrac{t_{i_0+1}}{\gamma_{i_0+1}} - \dfrac{t_{i_0}}{\gamma_{i_0}} = \dfrac{1}{\gamma_{i_0}\gamma_{i_0+1}} = \varepsilon\phi$ what contradicts the supposition. Then $g_0 - \varepsilon\gamma_{i_0}\rho_X - \varepsilon\gamma_{i_0}\gamma_{i_0+1}$ is a positive unlimited real. On the other hand $\varepsilon\gamma_{i_0}\gamma_{i_0}$ is limited; then

$\dfrac{X-\gamma_{i_0+1}}{\gamma_{i_0}}$ is a positive unlimited real, therefore p is also. The greatest denominator in H is $\gamma_{i_0+1}+p\gamma_{i_0}$ where $p=\dfrac{g_0}{\varepsilon\gamma_{i_0}\gamma_{i_0}}-\dfrac{\rho_X}{\gamma_{i_0}}-\dfrac{\gamma_{i_0+1}}{\gamma_{i_0}}-\rho$ with $\rho\in[0,1[$.

$$\begin{aligned}\varepsilon\left(\gamma_{i_0+1}+p\gamma_{i_0}\right) &= \varepsilon\left(\gamma_{i_0+1}+\left(\frac{g_0}{\varepsilon\gamma_{i_0}\gamma_{i_0}}-\frac{\rho_X}{\gamma_{i_0}}-\frac{\gamma_{i_0+1}}{\gamma_{i_0}}-\rho\right)\gamma_{i_0}\right)\\ &= \varepsilon\gamma_{i_0+1}+\frac{g_0}{\gamma_{i_0}}-\varepsilon\rho_X-\varepsilon\gamma_{i_0+1}-\varepsilon\rho\gamma_{i_0}\cong 0\ .\end{aligned}$$

Hence the product of the denominator of every element of H by ε is an infinitesimal. It remains to prove that the distance between two elements of H is of the $\varepsilon\phi$ form; Indeed: Let $i\in\{0,1,...,p-1\}$, from (1.7) we have

$$U_i-U_{i+1}=\frac{1}{\left(\gamma_{i_0+1}+i.\gamma_{i_0}\right)\left(\gamma_{i_0+1}+(i+1).\gamma_{i_0}\right)}\ .$$

By hypothesis we have $\gamma_{i_0+1}>\gamma_{i_0}$, then of properties of Farey's series **(1.2.1)**) $2\gamma_{i_0+1}>\gamma_{i_0+1}+\gamma_{i_0}>M$, then $\gamma_{i_0+1}>\dfrac{M}{2}$.
Let $d_i=\varepsilon\left(\gamma_{i_0+1}+i.\gamma_{i_0}\right)\left(\gamma_{i_0+1}+(i+1).\gamma_{i_0}\right)$.
Seen that $\left(\gamma_{i_0+1}\right)^2>\left(\dfrac{M}{2}\right)^2$, d_i is unlimited, therefore $U_i-U_{i+1}=\varepsilon\phi$. To finish the proof, we have of (1.7):

$$U_p-\frac{t_{i_0}}{\gamma_{i_0}}=\frac{1}{\left(\gamma_{i_0+1}+p\gamma_{i_0}\right)\gamma_{i_0}}.$$

Let $d_p=\varepsilon\left(\gamma_{i_0+1}+p.\gamma_{i_0}\right)\gamma_{i_0}$, after the replacement by the value of p, we obtain

$$d_p=\varepsilon\gamma_{i_0+1}\gamma_{i_0}+g_0-\varepsilon\rho_X\gamma_{i_0}-\varepsilon\gamma_{i_0+1}\gamma_{i_0}-\varepsilon\rho\gamma_{i_0}\gamma_{i_0}.$$

Since $\varepsilon\gamma_{i_0}\cong 0$, $\varepsilon\gamma_{i_0}\gamma_{i_0}$ is limited, then d_p is unlimited; hence

$$U_p-\frac{t_{i_0}}{\gamma_{i_0}}=\varepsilon\phi.$$

Thus, we end what we perceived.

B) $\gamma_{i_0} > \gamma_{i_0+1}$: Let us take, in this case, $g_1 \cong +\infty$ an integer such that $\dfrac{g_1}{\gamma_{i_0+1}} \cong 0$ (the existence of g_1 is assured by Robinson's lemma). Let $\widetilde{X} = E\left(\dfrac{g_1}{\varepsilon\gamma_{i_0+1}}\right)$ and

$$\widetilde{H} = \left\{V_0, V_1, ..., V_{p'-1}, V_{p'}, \frac{t_{i_0+1}}{\gamma_{i_0+1}}\right\}$$

where $p' = E\left(\dfrac{\widetilde{X} - \gamma_{i_0}}{\gamma_{i_0+1}}\right)$ and $V_j = \dfrac{t_{i_0} + j.t_{i_0+1}}{\gamma_{i_0} + j.\gamma_{i_0+1}}$ ($j = 0, 1, ..., p' - 1, p'$). Since the symmetry of this case with the case **A**) we prove, as in the case of H, that p' is an unlimited integer, the product of the denominator of every element of $\widetilde{H}$ by ε is an infinitesimal and the distance between two successive elements of $\widetilde{H}$ is of the $\varepsilon\phi$ form.

Thus the elements of H (or of $\widetilde{H}$) form a subdivision of the interval $\left[\dfrac{t_{i_0}}{\gamma_{i_0}}, \dfrac{t_{i_0+1}}{\gamma_{i_0+1}}\right]$. For the other intervals $\left[\dfrac{t_i}{\gamma_i}, \dfrac{t_{i+1}}{\gamma_{i+1}}\right]_{i\in\{1,2,...,r-1\}-\{i_0\}}$ which don't have a length of $\varepsilon\phi$ form we do the same construction as we did with $\left[\dfrac{t_{i_0}}{\gamma_{i_0}}, \dfrac{t_{i_0+1}}{\gamma_{i_0+1}}\right]$.

By regrouping rational numbers which subdivide intervals $\left[\dfrac{t_i}{\gamma_i}, \dfrac{t_{i+1}}{\gamma_{i+1}}\right]$ ($i \in \{1, 2, ..., r-1\}$) not having a length of the $\varepsilon\phi$ form and the rationals which are borders of intervals having a length of the $\varepsilon\phi$ form, we obtain the finite sequence $\left(\dfrac{l_i}{m_i}\right)_{i=1,2,...,g}$. The irreducibility of the elements of the sequence $\left(\dfrac{l_i}{m_i}\right)_{i=1,2,...,g}$ results from properties of Farey's series.□

Lemma 2.5. Let $\xi \in [0\ ,\ 1]$ be a real, if ξ is not in the ε-galaxie of a standard rational number then there exists two irreducible rational numbers $\dfrac{h_1}{k_1}, \dfrac{h_2}{k_2}$ of the interval $[0, 1]$ such that

$$\xi \in \left[\frac{h_1}{k_1}, \frac{h_2}{k_2}\right],\ k_1 \cong +\infty,\ k_2 \cong +\infty,\ \varepsilon k_1 \cong \varepsilon k_2 \cong 0 \text{ and } \frac{h_2}{k_2} - \frac{h_1}{k_1} = \varepsilon\phi.$$

Proof. Let us take, as in the lemma 2.4, a positive unlimited real number λ

such that $\sqrt{\varepsilon}\lambda \cong 0$ and let F_M *be* the Farey sequence of order $M = E\left(\frac{\lambda}{\sqrt{\varepsilon}}\right)$. Let $\frac{p_1}{q_1}$, $\frac{p_2}{q_2}$ be two successive elements of F_M such that $\xi \in \left[\frac{p_1}{q_1}, \frac{p_2}{q_2}\right]$. Two cases are distinguished:

A) Nor $\frac{p_1}{q_1}$ nor $\frac{p_2}{q_2}$ is a standard rational : In this case by applying the lemma 2.4, we obtain two irreducible rationals $\frac{l_{i_0}}{m_{i_0}}$ and $\frac{l_{i_0+1}}{m_{i_0+1}}$ such that $\xi \in \left[\frac{l_{i_0}}{m_{i_0}}, \frac{l_{i_0+1}}{m_{i_0+1}}\right]$, $m_{i_0} \cong +\infty$, $m_{i_0+1} \cong +\infty$, $\varepsilon m_{i_0} \cong \varepsilon m_{i_0+1} \cong 0$, $\frac{l_{i_0+1}}{m_{i_0+1}} - \frac{l_{i_0}}{m_{i_0}} = \varepsilon\phi$. Hence the lemma is proved by taking $\frac{l_{i_0}}{m_{i_0}}$ for $\frac{h_1}{k_1}$ and $\frac{l_{i_0+1}}{m_{i_0+1}}$ for $\frac{h_2}{k_2}$.

B) $\frac{p_1}{q_1}$ or $\frac{p_2}{q_2}$ is standard (cannot be both at the same time standard). Let us suppose that $\frac{p_1}{q_1}$ is standard (the other case, seen the symmetry, can be treated by the same way.). Then $\xi - \frac{p_1}{q_1} = \varepsilon w$ where $w \cong +\infty$. Let us put $L = E\left(2/\left(\xi - \frac{p_1}{q_1}\right)\right)$ then $\varepsilon L \cong 0$ and $\frac{p_1}{q_1} + \frac{1}{L} < \xi$. Let $\frac{l}{m}$ be the reduced form of $\frac{p_1}{q_1} + \frac{1}{L}$, then $\varepsilon m \cong 0$ because $m \leq Lq_1$ and q_1 is a standard. $m > M$ because $\frac{l}{m}$ is not an element of $\mathcal{F}_M$. Therefore εm^2 is an unlimited because $\varepsilon m^2 > \varepsilon M^2$ and εM^2 is an unlimited. This means that m is of the $E\left(\frac{\lambda'}{\sqrt{\varepsilon}}\right)$ form where λ' is a positive unlimited real verifying $\sqrt{\varepsilon}\lambda' \cong 0$. Now if we consider $\mathcal{F}_m$, then $\xi \in \left[\frac{p_1'}{q_1'}, \frac{p_2'}{q_2'}\right]$ where $\frac{p_1'}{q_1'}$ and $\frac{p_2'}{q_2'}$ are two successive non standard elements of $\mathcal{F}_m$. Thus the case **B)** comes back itself to the case **A)**, therefore the proposition is also proved for this case. □

Remark. We easily see that this proof is also a proof for the theorem 1.5.

Let γ be a positive unlimited real such that $\varepsilon.\gamma \simeq 0$, then

Lemma 2.6. There exists a finite set

$$S = \{l_1, l_2, ..., l_n\} \subset [0,1] \tag{2.5}$$

containing all standard elements of $[0,1]$ such that $|l_{i+1} - l_i| \geq \varepsilon\gamma$ for $i \in \{1, 2, ..., n-1\}$.

Proof. Let $B(S,z)$ be the internal formula: "$S \subset [0,1]$ is finite, $z \in S$ & $\forall (x_1\ ,\ x_2) \in S \times S\ (|x_1 - x_2| \geq \varepsilon\gamma)$". Let $Z \subset [0,1]$ be a standard finite part. Then there exists a finite part $S \subset [0,1]$ such that every element z of Z belongs to S and $\forall (x_1\ ,\ x_2) \in S \times S\ (|x_1 - x_2| \geq \varepsilon\gamma)$, i.e. we have $B(S,z)$. Indeed it suffices to take $S = Z$. Therefore, the principle of idealization (I) asserts the existence of a finite part $S \subset [0,1]$ such that any standard element of $[0,1]$ belongs to S and $\forall (x_1\ ,\ x_2) \in S \times S\ (|x_1 - x_2| \geq \varepsilon\gamma)$. Put $S = \{l_1, l_2, ..., l_n\}$, where $|l_{i+1} - l_i| \geq \varepsilon\gamma$ for $i \in \{1, 2, ..., n-1\}$ and any standard element of $[0,1]$ belongs to S.

Corollary 2.7. For every element l_i of S (S is the set that has been constructed in the lemma 2.6) we have only one of the two cases:
1) l_i is a standard rational number.
2) l_i is outside of $\varepsilon-$galaxies of all standard rational number.

Proof. Let $l_i \in S$, then
1) l_i can be a standard rational because S contains all standard elements of $[0,1]$.
2) l_i is not a standard rational then l_i is not in the $\varepsilon-$galaxy of any standard rational. Indeed, suppose that $l_i = \dfrac{p}{q} + \varepsilon\pounds\ (\pounds \neq 0)$, where $\dfrac{p}{q}$ is standard. Then l_i and $\dfrac{p}{q}$ are elements of S with $\left| l_i - \dfrac{p}{q} \right| = |\varepsilon\pounds| < \varepsilon\gamma$ which contradicts lemma 2.6 .

Lemma 2.8. For every standard integer $n \geq 1$. The real numbers x_i of all system $\{x_1, x_2, ..., x_n\} \subset S$ (S is the set that has been constructed in the lemma 2.6.) are approximated by rational numbers $\left(\dfrac{P_i}{Q}\right)_{i=1,2,...,n}$ to $\varepsilon\phi$ near with $\varepsilon Q \cong 0$. that is to say:

$$\left\{\begin{array}{rcl} x_i & = & \dfrac{P_i}{Q}+\varepsilon\phi \\ \varepsilon Q & \cong & 0 \end{array}\right. ;i=1,2,...,n. \tag{2.6}$$

Proof. Consider the formula:

$$A(n)\equiv \text{" } \forall\{x_1,x_2,...,x_n\}\subset S\ \exists \left(\frac{P_i}{Q}\right)_{i=1,2,...,n} \text{ such}$$
$$\text{that:}\left\{\begin{array}{rcl} x_i & = & \dfrac{P_i}{Q}+\varepsilon\phi \\ \varepsilon Q & \cong & 0 \end{array}\right. ;\ i=1,2,...,n \text{ ".}$$

According to the corollary 2.7, a real x of S is a standard rational or is outside of $\varepsilon-$galaxies of standard rationals. In addition, according to lemma 2.5, if x is not in the $\varepsilon-$galaxy of a rational standard, x is written in the form $\left\{\begin{array}{rcl} x & = & \dfrac{P}{Q}+\varepsilon\phi \\ \varepsilon Q & \cong & 0 \end{array}\right.$. Then in all cases x is written in the form $\left\{\begin{array}{rcl} x & = & \dfrac{P}{Q}+\varepsilon\phi \\ \varepsilon Q & \cong & 0 \end{array}\right.$. Consequently we have $A\,(1)$.
Suppose $A\,(n)$, for a standard integer n, and prove $A\,(n+1)$.
Let $(x_1,x_2,...,x_n,x_{n+1})\subset S$. Since A is verified for n we have

$$\left\{\begin{array}{l} x_i=\dfrac{p_i}{q}+\varepsilon\phi \quad ; \quad i=1,2,...,n \\ \varepsilon q\cong 0 \end{array}\right. . \tag{2.7}$$

If $x_{n+1}=\dfrac{h_1}{k_1}$ is standard, then because k_1 is standard and of (2.7) we have

$$\left\{\begin{array}{l} x_i=\dfrac{p_ik_1}{qk_1}+\varepsilon\phi=\dfrac{P_i}{Q}+\varepsilon\phi \qquad ; \quad i=1,2,...,n \\ x_{n+1}=\dfrac{h_1q}{k_1q}+\varepsilon.0=\dfrac{P_{n+1}}{Q}+\varepsilon\phi \\ \varepsilon Q=\varepsilon qk_1\cong 0 \end{array}\right. . \tag{2.8}$$

Let us look at the case where x_{n+1} is not a rational standard. In this case the application of the theorem 1.5 to the real qx_{n+1} with the infinitesimal εq implies:

$$\left\{\begin{array}{rcl} qx_{n+1} & = & \dfrac{M}{N}+(\varepsilon q)\,a \\ (\varepsilon q)\,N & \cong & 0 \end{array}\right.$$

where a is limited. If $a \cong 0$, then from this and (2.7) :

$$\left\{\begin{array}{l} x_i = \dfrac{p_i N}{qN}+\varepsilon\phi = \dfrac{P_i}{Q}+\varepsilon\phi \quad ; \quad i=1,2,...,n \\ x_{n+1} = \dfrac{M}{qN}+\varepsilon a = \dfrac{P_{n+1}}{Q}+\varepsilon\phi \\ \varepsilon Q = \varepsilon qN \cong 0 \end{array}\right. . \tag{2.9}$$

Let us look at the case where a is appreciable. Suppose $a > 0$, then

$$\left\{\begin{array}{l} x_i = \dfrac{Np_i}{Nq}+\varepsilon\phi \quad ; \quad i=1,2,...,n \\ x_{n+1} = \dfrac{M}{Nq}+\varepsilon a \\ \varepsilon Nq \cong 0 \end{array}\right. . \tag{2.10}$$

The reduced form of $\dfrac{M}{Nq}$ cannot be a rational standard. Otherwise, x_{n+1} and $\dfrac{M}{Nq}$ become two elements of S such that the separating distance between them, is of the εa form. What, according to lemma 2.6, is not true for two elements of S; for the same reason x_{n+1} cannot be in the $\varepsilon-$galaxy of a standard rational. According to the lemma 2.5:

$$\left\{\begin{array}{rcl} x_{n+1} = \dfrac{h_1}{k_1}+\varepsilon\phi_1 & = & \dfrac{h_2}{k_2}-\varepsilon\phi_2 \\ \varepsilon k_1 \cong \varepsilon k_2 \cong 0 & ; & k_1 \cong k_2 \cong +\infty \end{array}\right. \tag{2.11}$$

Where $\phi_1 \geq 0$ and $\phi_2 \geq 0$ are two infinitesimal reals and $\dfrac{h_1}{k_1}$, $\dfrac{h_2}{k_2}$ are irreducibles. Let ξ the element of S succeeding immediately x_{n+1} in S ($x_{n+1} < \xi$). Then by lemma 2.6 :

$$\xi - x_{n+1} = \varepsilon\omega \cong 0,\ \omega \geq \gamma.$$

The real number $\dfrac{x_{n+1}+\xi}{2}$ is not in the $\varepsilon-$galaxy of a rational standard, otherwise, x_{n+1} and ξ does not become two successive elements of S. Hence,

according to the lemma 2.5

$$\left\{ \begin{array}{l} \dfrac{x_{n+1}+\xi}{2} = \dfrac{s}{l} - \varepsilon\phi_4 \\ \varepsilon l \cong 0 \ , \ l \cong +\infty \ , \ \phi_4 \geq 0 \text{ and } \phi_4 \cong 0 \end{array} \right. . \tag{2.12}$$

where $\dfrac{s}{l}$ is irreducible. Let $\overline{\gamma}$ be an unlimited natural number such that $\sqrt{\epsilon}.\overline{\gamma} \cong 0$ and $\overline{N} = E\left(\dfrac{\overline{\gamma}}{\sqrt{\epsilon}}\right)$. Let us take $\underline{N} = \max\left(\overline{N}, k_2, l\right)$. Then $\underline{N} \cong +\infty$ and is of the $E\left(\dfrac{\lambda}{\sqrt{\epsilon}}\right)$ form with λ is a positive unlimited real verifying $\sqrt{\epsilon}.\lambda \cong 0$. In the other hand $\dfrac{h_2}{k_2}$ and $\dfrac{s}{l}$ are two elements of $\mathcal{F}_{\underline{N}}$ such that $\left[\dfrac{h_2}{k_2} \ , \ \dfrac{s}{l}\right]$ doesn't contain any rational standard and $k_2 \cong +\infty$ and $l \cong +\infty$. In this situation the lemma 2.4 is applicable and consequently there is a finite sequence of irreducible rational numbers $\left(\dfrac{s_i}{l_i}\right)_{1\leq i\leq e}$ such that

$$\frac{h_2}{k_2} = \frac{s_1}{l_1} < \frac{s_2}{l_2} < ... < \frac{s_e}{l_e} = \frac{s}{l}$$

where $e \cong +\infty$ and for $i = 1, 2, ..., e-1$ we have :

$$\frac{s_{i+1}}{l_{i+1}} - \frac{s_i}{l_i} = \varepsilon\phi.$$

Besides we have $\varepsilon l_i \cong 0$, $l_i \cong +\infty$ for $i = 1, 2, ..., e$; $\dfrac{s_e}{l_e} - \dfrac{s_1}{l_1} = \varepsilon\left(\dfrac{\omega}{2} + \phi_4 - \phi_2\right)$.

In this paragraph we will associate to each $i \in \{1, 2, ..., e\}$ a vector V_i in $\mathbb{Q}^{n+1}$ such that the n first components of V_i are in the ε-galaxie of the n first components of $(x_1, x_2, ..., x_n, x_{n+1})$, respectively. Whereas the $(n+1)-$th component of V_i is equal to $\dfrac{s_i}{l_i}$. Indeed, for $i = 1$ apply lemma 2.2 to the system $(l_1x_1, l_1x_2, ..., l_1x_n)$ with the infinitesimal εl_1:

$$\left\{ \begin{array}{rcl} l_1x_i & = & \dfrac{T_{i,1}}{t_1} + (\varepsilon l_1)\,\pounds \quad ; \quad i = 1, 2, ..., n \\ & & \epsilon l_1 t_1 \cong 0 \end{array} \right. .$$

Hence $\left\{ \begin{array}{rcl} x_i & = & \dfrac{T_{i,1}}{l_1 t_1} + \varepsilon\pounds \quad ; \quad i = 1, 2, ..., n \\ & & \varepsilon l_1 t_1 \cong 0 \end{array} \right.$. Then

$$\begin{cases} x_i &= \dfrac{T_{i,1}}{l_1 t_1} + \varepsilon \pounds \quad ; \quad i = 1, 2, ..., n \\ x_{n+1} &= \dfrac{T_{n+1,1}}{l_1 t_1} - \lambda_1 \end{cases} \tag{2.13}$$

where$T_{n+1,1} = s_1 t_1$, $\varepsilon l_1 t_1 \cong 0$ and $\lambda_1 = \varepsilon \phi_2$. Then we obtain the vector $V_1 = \left(\dfrac{T_{1,1}}{l_1 t_1} \ , \ \dfrac{T_{2,1}}{l_1 t_1} \ , \ ..., \dfrac{T_{n+1,1}}{l_1 t_1} \right)^T$, where $x_{n+1} = \dfrac{T_{n+1,1}}{l_1 t_1} = \dfrac{s_1}{l_1}$.
Again the application of the lemma 2.2 to the system $(l_2 x_1, l_2 x_2, ..., l_2 x_n)$ with the infinitesimal εl_2, gives:

$$\begin{cases} l_2 x_i &= \dfrac{T_{i,2}}{t_2} + (\varepsilon l_2) \pounds \quad ; \quad i = 1, 2, ..., n \\ & \epsilon l_2 t_2 \cong 0 \end{cases} .$$

Hence $\begin{cases} x_i &= \dfrac{T_{i,2}}{l_2 t_2} + \varepsilon \pounds \quad ; \quad i = 1, 2, ..., n \\ & \varepsilon l_2 t_2 \cong 0 \end{cases}$. Then

$$\begin{cases} x_i &= \dfrac{T_{i,2}}{l_2 t_2} + \varepsilon \pounds \quad ; \quad i = 1, 2, ..., n \\ x_{n+1} &= \dfrac{T_{n+1,2}}{l_2 t_2} - \lambda_2 \end{cases} \tag{2.14}$$

where $T_{n+1,2} = s_2 t_2$, $\varepsilon l_2 t_2 \cong 0$ and $\lambda_2 = \varepsilon \phi_2 + \varepsilon \phi$ with $0 < \lambda_1 < \lambda_2$. Then we obtain the vector $V_2 = \left(\dfrac{T_{1,2}}{l_2 t_2} \ , \ \dfrac{T_{2,2}}{l_2 t_2} \ , \ ..., \dfrac{T_{n+1,2}}{l_2 t_2} \right)^T$, where $x_{n+1} = \dfrac{T_{n+1,2}}{l_2 t_2} = \dfrac{s_2}{l_2}$.
Thus we construct the following vectors:

$$V_i = \left(\frac{T_{1,i}}{l_i t_i} \ , \ \frac{T_{2,i}}{l_i t_i} \ , \ ..., \frac{T_{n+1,i}}{l_i t_i} \right)^T ; \quad i = 1, 2, ..., e \tag{2.15}$$

where for $i = 1, 2, ..., e$: $x_{n+1} = \dfrac{T_{n+1,i}}{l_i t_i} - \lambda_i = \dfrac{s_i}{l_i} - \lambda_i$ with $\varepsilon l_i t_i \cong 0$.
Besides $0 < \varepsilon \phi_2 = \lambda_1 < \lambda_2 < ... < \lambda_e = \dfrac{\varepsilon \omega}{2} + \varepsilon \phi_4$ and for $i = 1, 2, ..., e-1$:

$$\lambda_{i+1} - \lambda_i = \varepsilon \phi.$$

Let h be the smallest integer such that $hNq \geq \max_i (l_i t_i)$, then $\varepsilon hNq \cong 0$. On the other hand and according to Robinson's lemma it exists an integer

$W \cong +\infty$ such that:

$$\varepsilon W h N q \cong 0.$$

Put $K = hNq$. From (2.10):

$$\left\{ \begin{array}{rcccccccl} x_i & = & \dfrac{hNp_i}{hNq} & + & \varepsilon\phi & = & \dfrac{H_i}{K} & + & \varepsilon\phi \quad ; \quad i = 1, 2, ..., n \\ x_{n+1} & = & \dfrac{hM}{hNq} & + & \varepsilon a & = & \dfrac{H_{n+1}}{K} & + & \varepsilon a \end{array} \right. \tag{2.16}$$

where $\varepsilon K \cong 0$, $K \geq \max\limits_i (l_i t_i)$.

Let $\overline{W} = \min\left(W, \dfrac{\omega}{2} + \phi_4 - \phi_2\right)$ and $\dfrac{T_{n+1,i_0}}{l_{i_0} t_{i_0}}$ be the element of the sequence $\left(\dfrac{T_{n+1,i}}{l_i t_i}\right)_{i=1,2,...,e}$ which is the farthest from $\dfrac{T_{n+1,1}}{l_1 t_1}$ verifying

$$\frac{T_{n+1,i_0}}{l_{i_0} t_{i_0}} - \frac{T_{n+1,1}}{l_1 t_1} = \varepsilon\overline{\overline{W}}$$

with $\overline{\overline{W}} \leq \overline{W}$. One notices that $\overline{\overline{W}} \cong +\infty$ because by construction $\overline{W} - \overline{\overline{W}} = \phi$.

Let $R \geq 1$ be the integer such that $R l_{i_0} t_{i_0} \leq K < (R+1) l_{i_0} t_{i_0}$. In this case $R l_{i_0} t_{i_0}$ and K are of the same order of magnitude i.e. : $\dfrac{K}{R l_{i_0} t_{i_0}} = \delta$ where δ is a positive appreciable. Consider, the rationals of the following vector:

$$\left(\frac{RT_{1,i_0}}{R l_{i_0} t_{i_0}}, \frac{RT_{2,i_0}}{R l_{i_0} t_{i_0}}, ..., \frac{RT_{n,i_0}}{R l_{i_0} t_{i_0}}, \frac{RT_{n+1,i_0}}{R l_{i_0} t_{i_0}}\right)^T. \tag{2.17}$$

Where the n first components of $(x_1, x_2, ..., x_n, x_{n+1})$ are in the $\varepsilon-$galaxies of the n first components of the (2.17), respectively. Whereas x_{n+1} is far from the last component of (2.17) by $\varepsilon\overline{\overline{W}} + \varepsilon\phi_2$. We will search a positive integer j_0 for which the rational $\dfrac{RT_{n+1,i_0} + j_0 H_{n+1}}{R l_{i_0} t_{i_0} + j_0 K}$ becomes equal to $\dfrac{H_{n+1}}{K} + \varepsilon a + \varepsilon\phi$ i.e. equal to $x_{n+1} + \varepsilon\phi$. Indeed, put

$$\Delta_j = \frac{RT_{n+1,i_0} + jH_{n+1}}{R l_{i_0} t_{i_0} + jK} - \frac{H_{n+1}}{K}. \tag{2.18}$$

Then $\Delta_j = \dfrac{\Delta}{1 + j\delta}$ where Δ is the distance between $\dfrac{RT_{n+1,i_0}}{R l_{i_0} t_{i_0}}$ and $\dfrac{H_{n+1}}{K}$ which is equal to $\varepsilon\overline{\overline{W}} + \varepsilon\phi_2 + \varepsilon a$.

Put $\dfrac{\Delta}{1+j\delta} = \varepsilon a$. For this $1 + j\delta = \dfrac{\Delta}{\varepsilon a}$. Hence

$$\begin{aligned} j &= \frac{1}{\delta}\left(\frac{\Delta - \varepsilon a}{\varepsilon a}\right) = \frac{1}{\delta}\left(\frac{\varepsilon\overline{\overline{W}} + \varepsilon\phi_2}{\varepsilon a}\right) \\ &= \frac{\overline{\overline{W}} + \phi_2}{\delta a} \cong +\infty \end{aligned} .$$

Let us take $j_0 = E\left(\dfrac{\overline{\overline{W}} + \phi_2}{\delta a}\right)$, hence $j_0 = \dfrac{\overline{\overline{W}} + \phi_2}{\delta a} - \rho$ with $\rho \in [0, 1[$. Then $\Delta_{j_0} = \dfrac{\Delta}{1 + j_0\delta}$. After the substitution by the value of Δ and of j_0:

$$\begin{aligned} \Delta_{j_0} &= a.\frac{\varepsilon\overline{\overline{W}} + \varepsilon\phi_2 + \varepsilon a}{a + \overline{\overline{W}} + \phi_2 - \rho a\delta} \\ &= \varepsilon a\left(\frac{\overline{\overline{W}} + \phi_2 + a}{\overline{\overline{W}} + \phi_2 + a - \rho a\delta}\right) \end{aligned} .$$

Hence

$$\begin{aligned} \Delta_{j_0} &= \varepsilon a.\frac{\left(\overline{\overline{W}} + \phi_2 + a\right)}{\left(\overline{\overline{W}} + \phi_2 + a\right)\left(1 - \dfrac{\rho a\delta}{\overline{\overline{W}} + \phi_2 + a}\right)} \\ &= \varepsilon a\frac{1}{1-\phi} \quad . \end{aligned}$$

Since $\dfrac{1}{1-\phi} = 1 + \phi$, then :

$$\Delta_{j_0} = \varepsilon a + \varepsilon\phi. \tag{2.19}$$

On the other hand j_0 and $\overline{\overline{W}}$ are of the same order of magnitude; indeed:

$$\begin{aligned} \frac{j_0}{\left(\overline{\overline{W}}\right)} &= \frac{1}{\left(\overline{\overline{W}}\right)}\left(\frac{\overline{\overline{W}} + \phi_2 - \rho a\delta}{a\delta}\right) \\ &= \frac{1+\phi}{a\delta}. \end{aligned}$$

Therefore $\dfrac{j_0}{\left(\overline{\overline{W}}\right)} = A$ with A is appreciable, hence $j_0 = A\overline{\overline{W}}$. Since $\overline{\overline{W}} \leq \overline{W}$ one has: $j_0 = A\overline{\overline{W}} \leq A\overline{W} \leq AW$.

Lemma 2.9. The denominator of $\dfrac{RT_{i,i_0} + j_0 H_i}{Rl_{i_0}t_{i_0} + j_0 K}$ ($i = 1, 2, ..., n+1$) verifies $\varepsilon\left(Rl_{i_0}t_{i_0} + j_0 K\right) \cong 0$ and for $i = 1, 2, ..., n, n+1$ we have:

$$x_i = \frac{RT_{i,i_0} + j_0 H_i}{Rl_{i_0}t_{i_0} + j_0 K} + \varepsilon\phi \tag{2.20}$$

Proof.

$$\begin{aligned} Rl_{i_0}t_{i_0} + j_0 K &= \frac{K}{\delta} + j_0 K \\ &= K\left(\frac{1}{\delta} + j_0\right). \end{aligned}$$

Hence $Rl_{i_0}t_{i_0} + j_0 K \leq K\left(\dfrac{1}{\delta} + AW\right)$. From the fact that $\varepsilon WK \cong 0$; A and δ are two appreciable numbers, we have $\varepsilon\left(Rl_{i_0}t_{i_0} + j_0 K\right) \cong 0$. On the other hand for $i = n+1$ we have from (2.16) $x_{n+1} = \dfrac{H_{n+1}}{K} + \varepsilon a$ and from (2.18) and (2.19)

$$\Delta_{j_0} = \frac{RT_{n+1,i_0} + j_0 H_{n+1}}{Rl_{i_0}t_{i_0} + j_0 K} - \frac{H_{n+1}}{K} = \varepsilon a + \varepsilon\phi.$$

Hence $\dfrac{RT_{n+1,i_0} + j_0 H_{n+1}}{Rl_{i_0}t_{i_0} + j_0 K} - \varepsilon\phi = \dfrac{H_{n+1}}{K} + \varepsilon a = x_{n+1}$, this means that

$$x_{n+1} = \frac{RT_{n+1,i_0} + j_0 H_{n+1}}{Rl_{i_0}t_{i_0} + j_0 K} + \varepsilon\phi.$$

For $i = 1, 2, ..., n$ we know from (2.17) that:

$$\left|\frac{RT_{i,i_0}}{Rl_{i_0}t_{i_0}} - x_i\right| = \varepsilon\pounds. \tag{2.21}$$

Hence

$$\begin{aligned} \left|\frac{RT_{i,i_0}}{Rl_{i_0}t_{i_0}} - \frac{H_i}{K}\right| &= \left|\frac{RT_{i,i_0}}{Rl_{i_0}t_{i_0}} - x_i + x_i - \frac{H_i}{K}\right| \\ &\leq \left|\frac{RT_{i,i_0}}{Rl_{i_0}t_{i_0}} - x_i\right| + \left|x_i - \frac{H_i}{K}\right| = \varepsilon\pounds + \varepsilon\phi = \varepsilon\pounds \quad . \end{aligned}$$

Therefore

$$\left|\frac{RT_{i,i_0}}{Rl_{i_0}t_{i_0}}-\frac{H_i}{K}\right|=\left|\frac{RT_{i,i_0}K-H_iRl_{i_0}t_{i_0}}{KRl_{i_0}t_{i_0}}\right|=\epsilon\pounds. \tag{2.22}$$

Then we have: $\left|\dfrac{RT_{i,i_0}+j_0H_i}{Rl_{i_0}t_{i_0}+j_0K}-\dfrac{H_i}{K}\right|=\left|\dfrac{RT_{i,i_0}K-H_iRl_{i_0}t_{i_0}}{KRl_{i_0}t_{i_0}\left(1+j_0.\dfrac{K}{Rl_{i_0}t_{i_0}}\right)}\right|=\dfrac{\epsilon\pounds}{1+j_0\delta}.$

Since $j_0\cong+\infty$, then

$$\left|\frac{RT_{i,i_0}+j_0H_i}{Rl_{i_0}t_{i_0}+j_0K}-\frac{H_i}{K}\right|=\varepsilon\phi$$

and seen that for $i=1,2,...,n$, the rational numbers $\dfrac{H_i}{K}$ are, respectively, in the $\varepsilon-$halos of $x_1,x_2,...,x_n$ then:

$$x_i=\frac{RT_{i,i_0}+j_0H_i}{Rl_{i_0}t_{i_0}+j_0K}+\varepsilon\phi.$$

So the lemma is proved.

Since $\varepsilon\left(Rl_{i_0}t_{i_0}+j_0K\right)\cong 0$, then if for $i=1,2,...,n,n+1$ one takes $\dfrac{RT_{i,i_0}+j_0H_i}{Rl_{i_0}t_{i_0}+j_0K}$ for $\dfrac{P_i}{Q}$ then

$$\begin{cases} x_i = \dfrac{P_i}{Q}+\varepsilon\phi,\ i=1,2,...,n+1 \\ \varepsilon Q\cong 0 \end{cases}. \tag{2.23}$$

In the case where $a<0$ we take ξ the element of S that precedes x_{n+1} i.e. $\xi<x_{n+1}$ (S is ordered) and by doing, to a symmetry near, as we did for the case $a>0$.
From (2.8), (2.9) and (2.23) we have $A(n+1)$. Hence, according to the external recurrence principle, the lemma 2.8 is proved. □

Let us return to the proof of theorem 2.1
Define for $Z=\{x_1,x_2,...,x_s\}\subset[0,1]$, the formula:

$$B(Z)=\text{"}\exists\left(\frac{P_i}{Q}\right)_{i=1,2,...,s}\text{such that :}\forall^{st}m\in\mathbb{N}^*\ G\left(Z,\ \left(\frac{P_i}{Q}\right)_{i=1,2,...,s},\ m\right)\text{"} \tag{2.24}$$

where $G\left(Z,\ \left(\frac{P_i}{Q}\right)_{i=1,2,...,s},\ m\right) \equiv \left\{ \begin{array}{c} \frac{1}{\varepsilon}\left|x_i - \frac{P_i}{Q}\right| \leq \frac{1}{m} \quad ;=1,2,...,s \\ |\varepsilon Q| \leq \frac{1}{m} \end{array} \right.$ is internal.
Consider the set

$$L = \{n \in \mathbb{N}^* : n \leq |S| \ \ \& \ \forall s \in \{1,...,n\} \ \ \forall Z = \{x_1, x_2, ..., x_s\} \subset S \ \ : B(Z)\}. \tag{2.25}$$

where S is the set that has been constructed in the lemma 2.6 .Then

$$L = \left\{ \begin{array}{c} n \in \mathbb{N}^* : n \leq |S| \ \ \& \forall s \in \{1,...,n\} \ \ \forall Z = \{x_1, x_2, ..., x_s\} \subset S, \\ \exists \left(\frac{P_i}{Q}\right)_{i=1,2,...,s} \ \ \forall^{st} m \in \mathbb{N}^* G\left(Z,\ \left(\frac{P_i}{Q}\right)_{i=1,2,...,s},\ m\right) \end{array} \right\}.$$

According to lemma 2.8, $L \supset (\mathbb{N}^*)^\sigma$. If L is internal then, according to the Cauchy principle, it must contain $(\mathbb{N}^*)^\sigma$ strictly and therefore there is an integer $\omega \cong +\infty$ and $\omega \in L$. If L is external then by the idealization principle (I) we can write L as follows:

$$L = \left\{ \begin{array}{c} n \in \mathbb{N}^* : n \leq |S| \ \ \& \forall s \in \{1,...,n\} \ \ \forall Z = \{x_1, x_2, ..., x_s\} \subset S, \\ \forall^{stfini}\ M\ \exists \left(\frac{P_i}{Q}\right)_{i=1,2,...,s} \ \ \forall m \in M\ G\left(Z,\ \left(\frac{P_i}{Q}\right)_{i=1,2,...,s},\ m\right) \end{array} \right\}.$$

where M belongs to the set of finite parts of $\mathbb{N}^*$. Therefore, L is an halo ([4], [6]). Of the fact that $(\mathbb{N}^*)^\sigma \subset L$ and no halo is a galaxy (Fehrele principle), then $(\mathbb{N}^*)^\sigma \underset{\neq}{\subset} L$. Hence it exists an integer $\omega \cong +\infty$ and $\omega \in L$.

Consequently in the two cases (L internal or external) we finds that it exists an integer $\omega \cong +\infty$ and $\omega \in L$, this signifies that $\omega \leq |S|$.

By lemma 2.3, there is a finite part $F \subset [0,1]$ containing all standard elements of $[0,1]$ such that $|F| = \omega' \cong +\infty$ and $\omega' < \omega$. Then $F \cap S$ is a finite part of S containing all standard elements of $[0,1]$ with $|F \cap S| \leq |F| = \omega' < \omega$. Put $F \cap S = \{x_1, x_2, ..., x_{n_0}\}$. Then $\exists \left(\frac{P_i}{Q}\right)_{i=1,2,...,n_0}$ such

that $:\begin{cases} x_i - \dfrac{P_i}{Q} = \varepsilon\phi \\ \varepsilon Q \cong 0 \ ; \ i = 1, 2, ..., n_0 \end{cases}$. It follows that if $x \in \mathbb{R}$ is a standard then $x - E(x) = \dfrac{P_{i_1}}{Q} + \varepsilon\phi$ where $i_1 \in \{1, 2, ..., n_0\}$ since $x - E(x)$ is a standard of $[0, 1]$. Hence

$$\left\{ \ x \ = \ E(x) + \frac{P_{i_1}}{Q} + \varepsilon\phi \ = \ \frac{P_x}{Q} + \varepsilon\phi \right.$$

where $\varepsilon Q \cong 0$. Thus the proof is complete.

3 Deduction of the classical equivalent of the main result

The theorem 2.1. can be written as follows

$$\forall\varepsilon \left\{ \left(\forall^{st} r \ \ (0 < \varepsilon \leq r) \right) \Longrightarrow \exists q \ \forall^{st} x \ \forall^{st} t \ \ (\parallel qx \parallel < \varepsilon qt \ \& \ \varepsilon q \leq t) \right\}$$

where ε, $r \in \mathbb{R}^{*+}$, $q \in \mathbb{N}$, $x \in \mathbb{R}$ and $t \in \mathbb{R}^{*+}$. By using the idealization principle (I), the last formula is equivalent to

$$\forall\varepsilon \left\{ \left(\forall^{st} r \ \ (0 < \varepsilon \leq r) \right) \Longrightarrow \forall^{st \ fini} X \ \exists q \ \forall (x, t) \in X \ \ (\parallel qx \parallel < \varepsilon qt \ \& \ \varepsilon q \leq t) \right\}$$

where X belongs to the set of finite parts of $\mathbb{R} \times \mathbb{R}^{*+}$. This last formula is equivalent to

$$\forall^{st \ fini} X \forall\varepsilon \exists^{st} r \left\{ (0 < \varepsilon \leq r) \Longrightarrow \exists q \ \forall (x, t) \in X \ \ (\parallel qx \parallel < \varepsilon qt \ \& \ \varepsilon q \leq t) \right\}.$$

Again, by using the idealization principle (I), the last formula is equivalent to

$$\forall^{st \ fini} X \ \exists^{st \ fini} R \ \forall\varepsilon \ \exists r \in R \ \left\{ (0 < \varepsilon \leq r) \Longrightarrow \exists q \ \forall (x, t) \in X \ \ (\parallel qx \parallel < \varepsilon qt \ \& \ \varepsilon q \leq t) \right\}.$$

where R belongs to the set of finite parts of $\mathbb{R}^{*+}$. By the transfer principle (T), this last formula is equivalent to

$$\forall^{fini} X\ \exists^{fini}\ R\ \forall\varepsilon\ \exists r \in R\ \left\{(0 < \varepsilon \leq r) \Longrightarrow \exists q\ \forall\,(x,\,t) \in X\ \ (\parallel qx \parallel < \varepsilon qt\ \&\ \varepsilon q \leq t)\right\}.$$

This last formula is exactly the main theorem announced in the abstract. Indeed, if $X = \{(x_1,\,t_1)\,,\ (x_2,\,t_2)\,,\ ...,\ (x_n,\,t_n)\}$ is a finite part of $\mathbb{R} \times \mathbb{R}^{*+}$, then there exist a finite part R of $\mathbb{R}^{*+}$ such that for all $\varepsilon > 0$ there exists $r \in R$ such that if $0 < \varepsilon \leq r$ then there exist rational numbers $\left(\frac{p_i}{q}\right)_{i=1,2,...,n}$ such that:

$$\left\{ \begin{array}{c} \left| x_i - \dfrac{p_i}{q} \right| \leq \varepsilon t \\ \varepsilon q \leq t \end{array} \right| , = 1, 2, ..., n.$$

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