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ORBITAL STABILITY VIA THE ENERGY-MOMENTUM METHOD: THE CASE OF HIGHER DIMENSIONAL SYMMETRY GROUPS

STEPHAN DE BIÈVRE^{1,2} AND SIMONA ROTA NODARI³

ABSTRACT. We consider the orbital stability of the relative equilibria of Hamiltonian dynamical systems on Banach spaces, in the presence of a multi-dimensional invariance group for the dynamics. We present a generalization of the Vakhitov-Kolokolov slope condition to this higher dimensional setting, and show how it allows to prove the local coercivity of the Lyapunov function, which in turn implies orbital stability. The method is applied to study the orbital stability of the plane waves of a system of two coupled nonlinear Schrödinger equations. We provide a comparison of our approach to the one by Grillakis-Shatah-Strauss.

1. INTRODUCTION

The *Vakhitov-Kolokolov slope condition* [12, 26, 28] is an often used ingredient in the proof of orbital stability of relative equilibria via the energy-momentum method for Hamiltonian systems with a one-dimensional symmetry group. For example, it has been applied in the proofs of stability of stationary or traveling waves of a variety of nonlinear partial differential equations ([4, 5, 9, 10, 11, 14, 26, 29] and references therein). It is our goal in this paper to present a natural generalization of this condition to the case where the Hamiltonian system admits a higher dimensional invariance group and to show how to obtain orbital stability from it.

The overall strategy underlying the energy-momentum method is well understood. Simply stated, it is a generalization of the standard Lyapunov method for proving the stability of fixed points to Hamiltonian systems having a Lie symmetry group G . Indeed, relative equilibria can be seen as fixed points “modulo symmetry”: they are fixed points of the dynamics induced on the space obtained by quotienting the phase space by the action of an appropriate subgroup of the invariance group. For finite dimensional systems, the theory goes back to the nineteenth century. It is concisely explained in [3, 16], in the modern language of Hamiltonian systems with symmetry, through the use of the properties of the momentum map, notably. This theory first of all gives a simple geometric characterization of all relative equilibria. It also naturally provides a candidate Lyapunov function as well as subgroup of G with respect to which the relative equilibria can be hoped to be relatively stable. More recent developments in the finite dimensional setting can be found in [18, 19, 21, 22, 25].

When the Hamiltonian system is infinite dimensional, such as is the case for nonlinear Hamiltonian PDE’s, the general philosophy of the energy-momentum method remains the same, but many technical complications arise, as expected. In [12] and [26], the theory is worked out in a Hilbert space setting, and when the symmetry group G is a *one-dimensional* Lie group. More recently, in [7], a version

of the energy-momentum method has been presented for Hamiltonian dynamical systems on a Banach space E having as invariance group a Lie group G of *arbitrary finite dimension*. What is shown there is that the proof of orbital stability can be reduced to a “local coercivity estimate” on an appropriately constructed Lyapunov function \mathcal{L} . It is shown in [7] that, in the above infinite dimensional setting, the construction of the latter follows naturally from the Hamiltonian structure and basic properties of the momentum map, in complete analogy with the finite dimensional situation. In specific models, it then remains to show the appropriate coercivity estimate on \mathcal{L} .

When the invariance group G of the system is one-dimensional, one way to obtain such an estimate is via the aforementioned *Vakhitov-Kolokolov slope condition*. Our main result here is a generalization of this condition to situations with a higher dimensional invariance group G , and the proof that it implies the desired coercivity of the Lyapunov function (Theorem 3.1 and Theorem 6.3). Using this property, orbital stability can then be obtained using the techniques described in [7].

The energy-momentum method has previously been adapted to the case of higher symmetry groups in [13], and a generalization of the *Vakhitov-Kolokolov slope condition* was given there as well. A detailed analysis of the differences between our work here and the approach of [13] will be given in Section 8.

The rest of this paper is organized as follows. The elements of [7] needed here are summarized in Section 2. In Section 3, we state our main result (Theorem 3.1) in the simplest setting, when the phase space of the system is a Hilbert space. Some preliminary lemmas are proven in Section 4 and the proof of Theorem 3.1 is given in Section 5. In Section 6, we generalize our result to the Banach space setting, see Theorem 6.3. In Section 7, we use our approach to study the stability of the plane waves of a class of coupled nonlinear Schrödinger equations on the circle used to model Bose-Einstein condensate, birefringence in nonlinear optics, and freak ocean waves. In Section 8, finally, we compare our method and results to those of [13].

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2. THE ENERGY-MOMENTUM METHOD

To make this paper self-contained and to fix our notation, we summarize in this section the energy-momentum method as described in [7]. We refer there for more details and for examples of the structures introduced here.

2.1. Hamiltonian systems with symmetry. Let E be a Banach space, \mathcal{D} a domain in E (*i.e.* a dense subset of E) and \mathcal{J} a symplector, that is an injective continuous linear map $\mathcal{J} : E \rightarrow E^*$ such that $(\mathcal{J}u)(v) = -(\mathcal{J}v)(u)$. We will refer to $(E, \mathcal{D}, \mathcal{J})$ as a *symplectic Banach triple*. Next, let $H : E \rightarrow \mathbb{R}$ be differentiable on $\mathcal{D} \subset E$. In other words, H is globally defined on E , and differentiable at each point $u \in \mathcal{D}$. We say that the function H has a \mathcal{J} -compatible derivative if, for all $u \in \mathcal{D}$, $D_u H \in \mathcal{R}_{\mathcal{J}}$, where $\mathcal{R}_{\mathcal{J}}$ is the range of \mathcal{J} . In that case we write $H \in \text{Dif}(\mathcal{D}, \mathcal{J})$.

We define a *Hamiltonian flow* for $H \in \text{Dif}(\mathcal{D}, \mathcal{J})$ as a separately continuous map $\Phi^H : \mathbb{R} \times E \rightarrow E$ with the following properties:

- (i) For all $t, s \in \mathbb{R}$, $\Phi_{t+s}^H = \Phi_t^H \circ \Phi_s^H$, $\Phi_0^H = \text{Id}$;

- (ii) For all $t \in \mathbb{R}$, $\Phi_t^H(\mathcal{D}) = \mathcal{D}$;
- (iii) For all $u \in \mathcal{D}$, the curve $t \in \mathbb{R} \rightarrow u(t) := \Phi_t^H(u) \in \mathcal{D} \subset E$ is differentiable and is the unique solution of

$$\mathcal{J}\dot{u}(t) = D_{u(t)}H, \quad u(0) = u. \quad (2.1)$$

We refer to (2.1) as the Hamiltonian differential equation associated to H and to its solutions as Hamiltonian flow lines.

Next, let G be a Lie group, \mathfrak{g} the Lie algebra of G and $\Phi : (g, x) \in G \times E \rightarrow \Phi_g(x) \in E$, an action of G on E . In what follows we will suppose all Lie groups are connected. We will say Φ is a *globally Hamiltonian action* if the following conditions are satisfied:

- (i) For all $g \in G$, $\Phi_g \in C^1(E, E)$ is symplectic.
- (ii) For all $g \in G$, $\Phi_g(\mathcal{D}) = \mathcal{D}$.
- (iii) For all $\xi \in \mathfrak{g}$, there exists $F_\xi \in C^1(E, \mathbb{R}) \cap \text{Dif}(\mathcal{D}, \mathcal{J})$ such that $\Phi_{\exp(t\xi)} = \Phi_t^{F_\xi}$, and the map $\xi \rightarrow F_\xi$ is linear.

Here and in what follows, we say $\Psi \in C^0(E, E) \cap C^1(\mathcal{D}, E)$ is a symplectic transformation if

$$\forall u \in \mathcal{D}, \forall v, w \in E, (\mathcal{J}D_u\Psi(v))(D_u\Psi(w)) = (\mathcal{J}v)(w). \quad (2.2)$$

Note that, in the above definition of globally Hamiltonian action, $\Psi = \Phi_g \in C^1(E, E)$. For further reference, we introduce, for all $u \in \mathcal{D}$ and for all $\xi \in \mathfrak{g}$,

$$X_\xi(u) = \frac{d}{dt}\Phi_{\exp(\xi t)}(u)|_{t=0}. \quad (2.3)$$

It follows from the preceding definitions that

$$X_\xi(u) = \mathcal{J}^{-1}D_uF_\xi. \quad (2.4)$$

We will always suppose G is a matrix group, in fact, a subgroup of $\text{GL}(\mathbb{R}^N)$. We can then think of the Lie algebra \mathfrak{g} as a sub-algebra of $\mathcal{M}(N, \mathbb{R})$ and define the adjoint action of G on \mathfrak{g} via

$$\text{Ad}_g\xi = g\xi g^{-1},$$

where in the right hand side we have a product of matrices. We will write $m = \dim\mathfrak{g} = \dim\mathfrak{g}^*$, where \mathfrak{g}^* designates the vector space dual of the Lie algebra \mathfrak{g} . For details, we refer to Appendix A.2 of [7]. Note that, for each $u \in E$ fixed, one can think of $\xi \in \mathfrak{g} \rightarrow F_\xi(u) \in \mathbb{R}$ as an element of \mathfrak{g}^* . Hence, if we identify (as we always will) \mathfrak{g} and \mathfrak{g}^* with \mathbb{R}^m and view F as a map $F : E \rightarrow \mathbb{R}^m \simeq \mathfrak{g}^*$, we can write

$$F_\xi = \xi \cdot F,$$

where \cdot refers to the canonical inner product on \mathbb{R}^m . The map F is called the momentum map of the symplectic group action and, in what follows, we will suppose that F is Ad^* -equivariant which means that for all $g \in G$, for all $\xi \in \mathfrak{g}$

$$F_\xi \circ \Phi_g = F_{\text{Ad}_g^{-1}\xi},$$

or equivalently, $F \circ \Phi_g = \text{Ad}_g^*F$. Here Ad^* is the co-adjoint action of G on \mathfrak{g}^* .

Now, for all $\mu \in \mathfrak{g}^*$, we define the isotropy group or stabilizer of μ as

$$G_\mu = \{g \in G \mid \text{Ad}_g^*\mu = \mu\};$$

\mathfrak{g}_μ is the Lie algebra of G_μ , and \mathfrak{g}_μ^* its dual. Finally, for all $\mu \in \mathfrak{g}^* \simeq \mathbb{R}^m$, let

$$\Sigma_\mu = \{u \in E \mid F(u) = \mu\}.$$

We will say μ is a *regular value of F* if $\Sigma_\mu \neq \emptyset$ and if, for all $u \in \Sigma_\mu$, $D_u F$ is surjective (maximal rank). Then Σ_μ is a codimension m sub-manifold of E and the tangent space to Σ_μ at $u \in \Sigma_\mu$ is given by

$$T_u \Sigma_\mu = \text{Ker} D_u F. \quad (2.5)$$

Finally, since the momentum map is Ad^* -equivariant, it is easy to see $G_\mu = G_{\Sigma_\mu}$, where G_{Σ_μ} is the subgroup of G leaving Σ_μ invariant.

Below, G will be an invariance group of H , in the sense that $H \circ \Phi_g = H$, for all $g \in G$. This implies G is an invariance group for the dynamics generated by H , meaning that for all $g \in G, t \in \mathbb{R}$, $\Phi_g \circ \Phi_t^H = \Phi_t^H \circ \Phi_g$ (See Theorem 2.1 (i) below). Nöther's Theorem then implies that the components F_i of the moment map are constants of the motion (See Theorem 2.1 (ii)) and hence that, for any $\mu \in \mathbb{R}^m \simeq \mathfrak{g}^*$, the level set Σ_μ is invariant under the dynamics Φ_t^H . We refer to Sections 7 and 8.4 for examples; see also [7].

2.2. Relative equilibria and orbital stability. We now recall the definition of a relative equilibrium. Let G be an invariance group for the dynamics Φ_t^H , as above, and let \tilde{G} be a subgroup of G . Let $u \in E$ and let $\mathcal{O}_u^{\tilde{G}} = \Phi_{\tilde{G}}(u)$ be the \tilde{G} -orbit of u . We say u is a relative \tilde{G} -equilibrium of the dynamics if, for all $t \in \mathbb{R}$, $\Phi_t^H(u) \in \mathcal{O}_u^{\tilde{G}}$. In other words, if the dynamical trajectory through u lies in the group orbit $\mathcal{O}_u^{\tilde{G}}$.

The goal is to investigate under which circumstances these relative equilibria are orbitally stable. Recall that a relative \tilde{G} -equilibrium $u \in E$ is orbitally stable if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall v \in E, \left(d(v, u) \leq \delta \Rightarrow \forall t \in \mathbb{R}, d(v(t), \mathcal{O}_u^{\tilde{G}}) \leq \varepsilon \right),$$

with $v(t)$ the solution of the Hamiltonian equation of motion with initial condition $v(0) = v$. Note that the definitions of relative equilibrium and of orbital stability are increasingly restrictive as the subgroup \tilde{G} is taken smaller. Sharper statements are therefore obtained by choosing smaller subgroups \tilde{G} .

It turns out that, if H is G invariant and the action of G is Ad^* -equivariant, then u is a G -relative equilibrium if and only if u is a G_μ -relative equilibrium, where $\mu = F(u)$ (See Theorem 7.1 in [7]). This observation, familiar from the finite dimensional theory (See for instance [3, 16]), explains why it is natural to try to prove G_μ -orbital stability. This is the approach we shall adopt here. It differs from the one in [13], where orbital stability is studied with respect to an a priori different subgroup, as we will explain in detail in Section 8. We will also show there that in many situations of interest, the two subgroups actually coincide.

We will write

$$\mathcal{O}_u = \Phi_{G_\mu}(u), \quad (2.6)$$

where $\mu = F(u)$. And

$$T_u \mathcal{O}_u = \{X_\xi(u) \mid \xi \in \mathfrak{g}_\mu\} \subset E. \quad (2.7)$$

To understand what follows, it is helpful to keep in mind that in practice, the action of the invariance group G is well known explicitly, and typically linear and isometric. Whereas the dynamical flow Φ_t^H is a complex object one tries to better understand using the invariance properties of H .

We now collect some results from [7] which give a characterization of the relative equilibria of Hamiltonian systems with symmetry and which also yield the candidate Lyapunov function that can be used to study their stability.

Theorem 2.1. *Let $(E, \mathcal{D}, \mathcal{J})$ be a symplectic Banach triple. Let $H \in C^1(E, \mathbb{R}) \cap \text{Dif}(\mathcal{D}, \mathcal{J})$ and suppose H has a Hamiltonian flow Φ_t^H . Let furthermore G be a Lie group, and Φ a globally Hamiltonian action on E with Ad^* -equivariant momentum map F . Suppose that,*

$$\forall g \in G, \quad H \circ \Phi_g = H. \quad (2.8)$$

- (i) *Then G is an invariance group for Φ_t^H .*
- (ii) *For all $t \in \mathbb{R}$, $F \circ \Phi_t^H = F$.*
- (iii) *u is a relative G -equilibrium if and only if u is a relative G_μ -equilibrium.*
- (iv) *Let $u \in \mathcal{D} \subset E$. If there exists $\xi \in \mathfrak{g}$ so that*

$$D_u H - \xi \cdot D_u F = 0, \quad (2.9)$$

then u is a relative G_μ -equilibrium. Let $\mu = F(u) \in \mathbb{R}^m \simeq \mathfrak{g}^$; if μ is a regular value of F , then u is a critical point of H_μ on Σ_μ , where $H_\mu = H|_{\Sigma_\mu}$.*

We will refer to relative equilibria u for which $D_u F$ is of maximal rank, as *regular relative equilibria*. This is the case when μ is a *regular value* of F . Equation (2.9) is referred to as the stationary equation in the PDE literature. The theorem states that its solutions determine relative G - and hence relative G_μ -equilibria.

We now turn to the stability analysis of those relative equilibria. Suppose we are given $\xi \in \mathfrak{g}$ and u_ξ , solution of (2.9). We first note that the fact that u_ξ is a critical point of the restriction H_{μ_ξ} of the Hamiltonian H to Σ_{μ_ξ} ($\mu_\xi = F(u_\xi)$) is an immediate consequence of the observation that u_ξ is a critical point of the Lagrange function

$$\mathcal{L}_\xi = H - \xi \cdot F : E \rightarrow \mathbb{R}. \quad (2.10)$$

The goal is now to prove that these relative equilibria are orbitally stable. As pointed out in [7], the basic idea underlying the energy-momentum method is that, modulo technical problems, a relative equilibrium is expected to be stable if it is not only a critical point but actually a local minimum of H_μ . To establish such a result, it is natural to use the second variation of the Lagrange multiplier theory and to establish that the Hessian of \mathcal{L}_ξ is positive definite when restricted to $T_{u_\xi} \Sigma_{\mu_\xi} \cap T_{u_\xi} \mathcal{O}_{u_\xi}^\perp$. The precise statement is given in Proposition 2.2 below.

Let $\langle \cdot, \cdot \rangle$ be a scalar product on E , which is continuous in the sense that

$$\forall v, w \in E, \quad |\langle v, w \rangle| \leq \|v\|_E \|w\|_E,$$

where $\|\cdot\|_E$ is our notation for the Banach norm on E . Note that E is not necessarily a Hilbert space for this inner product, and even if E is in fact a Hilbert space, the inner product $\langle \cdot, \cdot \rangle$ is not necessarily the Hilbert space inner product. As an example, if $E = H^1(\mathbb{R}^d, \mathbb{C})$ and depending on the problem considered, one may want to use either the L^2 inner product or the H^1 inner product. A similar situation occurs for the system of coupled nonlinear Schrödinger equations treated in Section 7. One then has:

Proposition 2.2. *Let E be a Banach space and $\langle \cdot, \cdot \rangle$ be a continuous scalar product on E , \mathcal{D} a domain in E and \mathcal{J} a symplector. Let $H \in C^2(E, \mathbb{R}) \cap \text{Dif}(\mathcal{D}, \mathcal{J})$. Let G be a Lie group, and Φ a globally Hamiltonian G -action on E with Ad^* -equivariant momentum map F . Suppose that $H \circ \Phi_g = H$ for all $g \in G$ and that*

- (i) *Φ_g is linear and preserves both the structure $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|_E$ for all $g \in G$.*

Let $\xi \in \mathfrak{g}$ and suppose $u_\xi \in \mathcal{D}$ satisfies (2.9), i.e. $D_{u_\xi} \mathcal{L}_\xi = 0$, with $\mathcal{L}_\xi = H - \xi \cdot F$. Let $\mu_\xi = F(u_\xi) \in \mathbb{R}^m \simeq \mathfrak{g}^*$ and suppose μ_ξ is a regular value of F . Suppose in addition that

- (ii) $\text{Ad}_g^* \in \text{O}(m)$ for all $g \in G_{\mu_\xi}$.
- (iii) $\forall j = 1, \dots, m,$

$$\exists \nabla F_j(u_\xi) \in E \text{ such that } D_{u_\xi} F_j(w) = \langle \nabla F_j(u_\xi), w \rangle \quad \forall w \in E; \quad (2.11)$$
- (iv) There exists $C > 0$ so that

$$\forall w \in E, \quad D_{u_\xi}^2 \mathcal{L}_\xi(w, w) \leq C \|w\|_E^2;$$

- (v) There exists $c > 0$ so that

$$\forall w \in T_{u_\xi} \Sigma_{\mu_\xi} \cap (T_{u_\xi} \mathcal{O}_{u_\xi})^\perp, \quad D_{u_\xi}^2 \mathcal{L}_\xi(w, w) \geq c \|w\|_E^2, \quad (2.12)$$

where

$$(T_{u_\xi} \mathcal{O}_{u_\xi})^\perp = \{z \in E \mid \langle z, y \rangle = 0, \forall y \in T_{u_\xi} \mathcal{O}_{u_\xi}\}. \quad (2.13)$$

Then there exist $\epsilon > 0, c > 0$ so that

$$\forall u \in \mathcal{O}_{u_\xi}, \forall u' \in \Sigma_{\mu_\xi}, \quad d(u, u') \leq \epsilon \Rightarrow H(u') - H(u) \geq cd^2(u', \mathcal{O}_{u_\xi}). \quad (2.14)$$

This result constitutes a slight generalization of Proposition 5 in [7]. In fact, if G_μ is commutative, the latter result applies immediately. We will give the proof of Proposition 2.2 in the next subsection. The basic message of this result is the following. If G is an invariance group for H that has a globally Hamiltonian action on E and if u_ξ satisfies the stationary equation $D_{u_\xi} \mathcal{L}_\xi = 0$ for some $\xi \in \mathfrak{g}$, then, modulo the technical conditions (i)-(iv), the coercive estimate (2.12) on the Hessian of \mathcal{L}_ξ implies that the restriction of the Hamiltonian H to the constraint surface Σ_{μ_ξ} attains a local minimum on the G_{μ_ξ} -orbit \mathcal{O}_{u_ξ} . As explained in Section 8 of [7], modulo some further technical conditions, (2.14) allows one to show that u_ξ is G_{μ_ξ} -orbitally stable. (See in particular Theorem 10 and Theorem 11 in [7]).

The difficulty in proving (2.12) comes from the fact that, in general, the bilinear symmetric form $D_{u_\xi}^2 \mathcal{L}_\xi$ is not positive on E , but has instead a non-trivial negative cone

$$\mathcal{C}_- = \{v \in E \mid D_{u_\xi}^2 \mathcal{L}_\xi(v, v) < 0\}.$$

The estimate (2.12) implies that $T_{u_\xi} \Sigma_{\mu_\xi}$ does not intersect \mathcal{C}_- . To prove this, we shall show that there exists a maximally negative subspace of E for $D_{u_\xi}^2 \mathcal{L}_\xi$ which is $D_{u_\xi}^2 \mathcal{L}_\xi$ -orthogonal to $T_{u_\xi} \Sigma_{\mu_\xi}$.

The goal of this paper is to give a condition (see Theorem 3.1 (iv) and Theorem 6.3 (iv)), which is a generalization to the *Vakhitov-Kolokolov slope condition*, that implies the coercivity estimate (2.12). This condition is in general easier to verify than the coercivity estimate itself and allows one to prove the orbital stability of relative equilibria of general Hamiltonian system. As an example of this method we study in Section 7 the stability of the plane waves of a system of two coupled nonlinear Schrödinger equations.

2.3. Proof of Proposition 2.2. The general strategy of the proof is identical to the one of Proposition 5 in [7]. First, we need some simple preliminary results.

Lemma 2.3. *Suppose the hypotheses of Proposition 2.2 hold. Then $\forall u \in \mathcal{O}_{u_\xi}$, for all $g \in G_{\mu_\xi}$, we have*

$$(a) \quad \Phi_g(T_u \mathcal{O}_{u_\xi}) = T_v \mathcal{O}_{u_\xi}, \quad \Phi_g \left((T_u \mathcal{O}_{u_\xi})^\perp \right) = (T_v \mathcal{O}_{u_\xi})^\perp,$$

$$(b) \quad \Phi_g(T_u \Sigma_{\mu_\xi}) = T_v \Sigma_{\mu_\xi}, \quad \Phi_g \left((T_u \Sigma_{\mu_\xi})^\perp \right) = (T_v \Sigma_{\mu_\xi})^\perp,$$

where $v = \Phi_g(u)$. In addition, defining $W_u = \text{span}\{\nabla_j F(u) \mid j = 1 \dots m\}$, we have

$$W_u^\perp = T_u \Sigma_{\mu_\xi} \quad \text{and} \quad E = T_u \Sigma_{\mu_\xi} \oplus W_u.$$

Proof. (a), respectively (b), follows from the observation that $\Phi_g(\mathcal{O}_{u_\xi}) = \mathcal{O}_{u_\xi}$, respectively $\Phi_g(\Sigma_\mu) = \Sigma_\mu$, and the fact that Φ_g preserves the inner product $\langle \cdot, \cdot \rangle$. That $W_u^\perp = T_u \Sigma_\mu$ follows from the definitions and the second statement is easily verified. \square

Proof. (of Proposition 2.2) We start with some preliminaries. It follows from Lemma 6 in [7] that there exists $R > 0$ so that, for all $u' \in E$ for which $d(u', \mathcal{O}_{u_\xi}) \leq R$, there exists $v' \in \mathcal{O}_{u_\xi}$ so that $u' - v' \in (T_{v'} \mathcal{O}_{u_\xi})^\perp$. Now, let $g \in G_\mu$ so that $\Phi_g(v') = u_\xi$ and define $u'' = \Phi_g(u')$. Then by the previous lemma, $u'' - u_\xi \in (T_{u_\xi} \mathcal{O}_{u_\xi})^\perp$. Hence we can write

$$u'' - u_\xi = (u'' - u_\xi)_1 + (u'' - u_\xi)_2,$$

where $(u'' - u_\xi)_1 \in T_{u_\xi} \Sigma_\mu \cap (T_{u_\xi} \mathcal{O}_{u_\xi})^\perp$ and $(u'' - u_\xi)_2 \in W_{u_\xi}$. Since $u'', u_\xi \in \Sigma_\mu$, one has

$$0 = F(u'') - F(u_\xi) = D_{u_\xi} F((u'' - u_\xi)_2) + O(\|u'' - u_\xi\|_E^2).$$

As $D_{u_\xi} F$ is of maximal rank, it has no kernel in W_{u_ξ} , and we can conclude there exists c_0 so that

$$\|(u'' - u_\xi)_2\|_E \leq O(\|u'' - u_\xi\|_E^2).$$

Hence, provided $\|u'' - u_\xi\|$ is small enough,

$$\|(u'' - u_\xi)_1\|_E \geq C \|u'' - u_\xi\|_E.$$

We can then conclude the proof as follows. Let $\epsilon > 0$ be small enough so that the previous inequalities hold. Then compute

$$\begin{aligned} \mathcal{L}_\xi(u') - \mathcal{L}_\xi(u_\xi) &= \mathcal{L}_\xi(u'') - \mathcal{L}_\xi(u_\xi) \\ &= D_{u_\xi} \mathcal{L}_\xi(u'' - u_\xi) + \frac{1}{2} D_{u_\xi}^2 \mathcal{L}_\xi(u'' - u_\xi, u'' - u_\xi) \\ &\quad + o(\|u'' - u_\xi\|_E^2) \\ &= \frac{1}{2} D_{u_\xi}^2 \mathcal{L}_\xi((u'' - u_\xi)_1, (u'' - u_\xi)_1) + O(\|u'' - u_\xi\|_E^3) \\ &\quad + o(\|u'' - u_\xi\|_E^2) \\ &= \frac{1}{2} D_{u_\xi}^2 \mathcal{L}_\xi((u'' - u_\xi)_1, (u'' - u_\xi)_1) + o(\|u'' - u_\xi\|_E^2) \\ &\geq \frac{c}{2} \|(u'' - u_\xi)_1\|_E^2 + o(\|u'' - u_\xi\|_E^2) \\ &\geq \tilde{c} \|u'' - u_\xi\|_E^2 \geq \tilde{c} d^2(u'', \mathcal{O}_{u_\xi}) = \tilde{c} d^2(u', \mathcal{O}_{u_\xi}). \end{aligned}$$

Note that, in the first equality above, we used the observation that, for all $g \in G_{\mu_\xi}$, for all $u' \in \Sigma_{\mu_\xi}$, one has

$$\mathcal{L}_\xi(\Phi_g(u')) = \mathcal{L}_\xi(u').$$

This follows from the G -invariance of H and from the fact that

$$\xi \cdot F(\Phi_g(u')) = \xi \cdot \mu_\xi = \xi \cdot F(u')$$

since both $\Phi_g(u')$ and u' belong to Σ_{μ_ξ} . \square

3. MAIN RESULT: THE HILBERT SPACE SETTING

In this section, we state our main result in the setting where E is a Hilbert space (Theorem 3.1). This makes for a simpler statement and proof than in the more general Banach space setting (See Section 6), and allows one to understand the philosophy of the result more easily. We point out that the result we obtain in Theorem 3.1 is of interest also in finite dimensional problems ($\dim E < +\infty$). Indeed, the usual orbital stability results in the literature on finite dimensional Hamiltonian dynamical systems reduce the proof to the coercivity estimate (2.12), which is generally not easy to check directly.

Let $(E, \mathcal{D}, \mathcal{J})$ be a symplectic Banach triple and suppose moreover that $(E, \langle \cdot, \cdot \rangle)$ is a Hilbert space. Here, we will use the norm associated to the scalar product $l \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$ as Banach norm and we drop the subscript E in the notation.

Let $H \in C^2(E, \mathbb{R}) \cap \text{Dif}(\mathcal{D}, \mathcal{J})$. Let G be a Lie group, and Φ a globally Hamiltonian action on E with Ad^* -equivariant momentum map F . Suppose that $H \circ \Phi_g = H$ for all $g \in G$.

Let Ω be an open set in $\mathfrak{g} \simeq \mathbb{R}^m$ and define the following objects

$$\begin{aligned} \tilde{u} : \Omega \subset \mathfrak{g} &\rightarrow E \\ \xi &\mapsto u_\xi \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \hat{F} : \Omega \subset \mathfrak{g} &\rightarrow \mathfrak{g}^* \simeq \mathbb{R}^m \\ \xi &\mapsto F(u_\xi) \end{aligned} \tag{3.2}$$

with u_ξ a family of solutions to the stationary equation:

$$D_{u_\xi} H - \xi \cdot D_{u_\xi} F = 0. \tag{3.3}$$

As recalled in section 2, if u_ξ is a solution to (3.3), then u_ξ is G_μ -relative equilibrium with $\mu = \mu_\xi = F(u_\xi)$. Suppose that each μ_ξ is a regular value of F and $\tilde{u} \in C^1(\Omega \subset \mathfrak{g}, E)$.

So our starting point is equation (3.3), which in PDE applications is often an elliptic partial differential equation and we suppose we have an m -parameter family of solutions, indexed by ξ . One of the major difficulties to apply the theory is of course to find such families of solutions.

Next, consider the Lyapunov function \mathcal{L}_ξ defined by (2.10) and remark that each u_ξ solution to (3.3) is a critical point of \mathcal{L}_ξ . Moreover, define for all $\xi \in \Omega \subset \mathfrak{g}$, the map

$$\begin{aligned} W : \Omega \subset \mathfrak{g} &\rightarrow \mathbb{R} \\ \xi &\mapsto W(\xi) = \mathcal{L}_\xi(u_\xi). \end{aligned} \tag{3.4}$$

Note that

$$W(\xi) = H(u_\xi) - \xi \cdot \hat{F}(\xi).$$

For each $\xi \in \Omega$, the Hessian $D_\xi^2 W$ of W is a bilinear form on \mathbb{R}^m . Hence, we can consider the following decomposition

$$\mathbb{R}^m = W_- \oplus W_0 \oplus W_+,$$

where W_0 is the kernel of $D_\xi^2 W$ and where $D_\xi^2 W$ is positive (negative) definite on W_+ (W_-). Let $d_0(D_\xi^2 W)$, $p(D_\xi^2 W)$, $n(D_\xi^2 W)$ be the dimensions of these spaces. Note that the decomposition is not unique, but the respective dimensions are. In

other words, W_{\pm} are maximal positive/negative definite spaces for W . Also, in order not to burden the notation, we have not made the ξ -dependence of the spaces W_0, W_{\pm} explicit. Recall that, given a symmetric bilinear form B on a Banach space E , a subspace \mathcal{X} of E is said to be a positive (negative) definite subspace for B on E if $B|_{\mathcal{X} \times \mathcal{X}}$ is positive (negative) definite. A subspace is maximally positive (negative) definite if it is positive (negative) definite and it is not contained in any other positive (negative) definite subspace.

Similarly, the Hessian $D_u^2 \mathcal{L}_{\xi}$ of \mathcal{L}_{ξ} is a symmetric bilinear form on E . For each u_{ξ} solution to (3.3), we define as usual the Morse index $n(D_{u_{\xi}}^2 \mathcal{L}_{\xi})$ of $u_{\xi} \in E$ for \mathcal{L}_{ξ} as the dimension of a maximally negative definite subspace for $D_{u_{\xi}}^2 \mathcal{L}_{\xi}$ in E .

Finally, since E is a Hilbert space, we can define for each $u \in E$ a bounded self-adjoint operator $\nabla^2 \mathcal{L}_{\xi}(u)$ by

$$\langle v, \nabla^2 \mathcal{L}_{\xi}(u)w \rangle = D_u^2 \mathcal{L}_{\xi}(v, w). \quad (3.5)$$

As a consequence, we can consider the spectral decomposition of E for $\nabla^2 \mathcal{L}_{\xi}(u_{\xi})$

$$E = E_- \oplus E_0 \oplus E_+ \quad (3.6)$$

with $E_0 = \text{Ker} \nabla^2 \mathcal{L}_{\xi}(u_{\xi}) = \text{Ker} D_{u_{\xi}}^2 \mathcal{L}_{\xi}$, and E_{\pm} the positive and negative spectral subspaces of $\nabla^2 \mathcal{L}_{\xi}(u_{\xi})$. Clearly E_{\pm} are maximally positive/negative subspaces for $\nabla_{u_{\xi}}^2 \mathcal{L}_{\xi}$ so that $n(D_{u_{\xi}}^2 \mathcal{L}_{\xi}) = \dim E_-$. We can now state our main result.

Theorem 3.1. *Under the previous hypotheses, let $\xi \in \Omega$ and suppose*

- (i) $D_{\xi}^2 W$ is non-degenerate,
- (ii) $\text{Ker} D_{u_{\xi}}^2 \mathcal{L}_{\xi} = T_{u_{\xi}} \mathcal{O}_{u_{\xi}}$,
- (iii) $\inf(\sigma(\nabla^2 \mathcal{L}_{\xi}(u_{\xi})) \cap (0, +\infty)) > 0$,
- (iv) $p(D_{\xi}^2 W) = n(D_{u_{\xi}}^2 \mathcal{L}_{\xi})$.

Then there exists $\delta > 0$ such that

$$\forall v \in T_{u_{\xi}} \Sigma_{\mu_{\xi}} \cap (T_{u_{\xi}} \mathcal{O}_{u_{\xi}})^{\perp}, \quad D_u^2 \mathcal{L}_{\xi}(v, v) \geq \delta \|v\|^2. \quad (3.7)$$

Since (3.7) is the same as (2.12), it follows that u_{ξ} is orbitally stable. It is the fourth condition of the above theorem that generalizes the Vakhitov-Kolokolov slope condition, as we now explain. Suppose the group G is 1-dimensional, so that $m = 1$. Then W is a scalar function of $\xi \in \mathbb{R} \simeq \mathfrak{g}$. We will see below (See (4.12)) that

$$W''(\xi) = -\hat{F}'(\xi).$$

Hence the proof of orbital stability for u_{ξ} reduces to verifying that the spectral conditions on $\nabla_{\xi}^2 \mathcal{L}_{\xi}$ are satisfied and notably that $\dim E_- = 1$, and that

$$\hat{F}'(\xi) < 0. \quad (3.8)$$

This is the Vakhitov-Kolokolov slope condition. In applications to the Schrödinger equation, where $F(u) = \frac{1}{2} \|u\|^2$, it says that the norm of u_{ξ} decreases as a function of ξ . In the case $m = 1$, the above result is proven in [12] (Corollary 3.3.1) and in [26] (Proposition 5.2).

The setup of the Hamiltonian dynamics with higher dimensional symmetry on a Hilbert space we used in this section is similar to the one proposed in [13] where the decomposition (3.6) of the bounded self-adjoint operator $\nabla_{u_{\xi}}^2 \mathcal{L}_{\xi}$ as well as condition (iii) of Theorem 3.1 are also used to obtain a coercivity result of the type (3.7). Nevertheless, in [13] a different constraint surface and orbit are used:

for a complete comparison between Theorem 3.1 and the coercivity results of [13], we refer to Section 8 below.

We finally note that, when E is infinite dimensional, and the equation under study a PDE, the Banach space formulation of Section 6 is often more convenient than the Hilbert space formulation proposed here. Indeed, the operator $\nabla^2 \mathcal{L}_\xi$ introduced in Theorem 3.1 is not a partial differentiable operator (it is bounded) making the analysis of its spectrum generally less convenient than for the operator $\nabla^2 \mathcal{L}_\xi$ in Theorem 6.4, which in applications is a self-adjoint partial differential operator.

4. USEFUL LEMMAS

The following lemmas collect some basic properties of the objects introduced above, that are essential in the further analysis of the Hessian of the Lyapunov function. We define, for $\xi \in \Omega$,

$$\mathcal{U}_\xi = \{\eta \cdot \nabla_\xi u_\xi \in E \mid \eta \in \mathbb{R}^m\}, \quad (4.1)$$

where we used the notation

$$\eta \cdot \nabla_\xi u_\xi := D_\xi \tilde{u}(\eta). \quad (4.2)$$

Lemma 4.1. *Let E be a Banach space, Ω an open set in \mathfrak{g} . Let $\tilde{u} \in C^1(\Omega \subset \mathfrak{g}, E)$. Let $\xi \in \Omega$ and consider the following statements:*

- (1) \hat{F} is a local diffeomorphism;
- (2) $D_\xi \tilde{u}$ is injective.
- (3) $\mathcal{U}_\xi \cap \text{Ker} D_{u_\xi} F = \{0\}$.
- (4) There is a neighbourhood of u_ξ where the moment map F is regular (i.e. $D_{u_\xi} F$ has maximal rank).
- (5)

$$\mathcal{U}_\xi \oplus \text{Ker} D_{u_\xi} F = E; \quad (4.3)$$

Then (1) \Leftrightarrow ((2) and (3)) \Leftrightarrow ((4) and (5)).

Note that the lemma does not use the fact that the u_ξ are solutions to the stationary equation: \tilde{u} takes values in E , without further condition. The lemma therefore strings together some useful facts on compositions of maps.

It is easy to see that, whenever u_ξ is a solution to (3.3) for every $\xi \in \Omega$, the map \tilde{u} is injective provided the u_ξ are regular relative equilibria. Indeed, if $u_{\xi_1} = u_{\xi_2}$ are both solutions of (3.3), then

$$(\xi_1 - \xi_2) \cdot D_{u_{\xi_1}} F = 0.$$

Hence, if the u_ξ are regular relative equilibria, one has $\xi_1 = \xi_2$. It is natural in that context to assume it is in fact an immersion, meaning that its derivative is injective, as in condition (2) of Lemma 4.1. One can then think of $\tilde{u}(\Omega)$ as an m -parameter surface in E .

Proof. First note that, for all $\eta_1, \eta_2 \in \mathfrak{g}$,

$$D_\xi(\eta_2 \cdot \hat{F})(\eta_1) = D_{u_\xi}(\eta_2 \cdot F)(\eta_1 \cdot \nabla_\xi u_\xi). \quad (4.4)$$

(1) \Rightarrow ((2) and (3)) Let $\eta_1 \in \mathfrak{g}$ and suppose $\eta_1 \cdot \nabla_\xi u_\xi = 0$. It follows from the above that $D_\xi \hat{F}(\eta_1) = 0$. But since \hat{F} is a local diffeomorphism at ξ , this implies $\eta_1 = 0$. Hence $D_\xi \tilde{u}$ is injective, which shows (2). To show (3), let $\eta_1 \in \mathfrak{g}$ and suppose $\eta_1 \cdot \nabla_\xi u_\xi \in \text{Ker} D_{u_\xi} F$. Then, by definition, $D_{u_\xi}(\eta_2 \cdot F)(\eta_1 \cdot \nabla_\xi u_\xi) = 0$ for all $\eta_2 \in \mathfrak{g}$.

It follows from the previous equation that $\eta_1 \in \text{Ker} D_\xi \hat{F}$ so that, by (1), $\eta_1 = 0$. This proves (3).

((2) and (3)) \Rightarrow (1) Let $\eta_1 \in \text{Ker} D_\xi \hat{F}$. Then according to the above equality, $\eta_1 \cdot \nabla_\xi u_\xi \in \text{Ker} D_{u_\xi} F$. So, by (3), $\eta_1 \cdot \nabla_\xi u_\xi = 0$ and by (2), $\eta_1 = 0$. This proves $D_\xi \hat{F}$ is injectif, hence surjectif, which proves (1).

((2) and (3)) \Rightarrow ((4) and (5)) According to (3), the map

$$D_{u_\xi} F : \mathcal{U}_\xi \rightarrow \mathfrak{g}^* \simeq \mathbb{R}^m$$

is injective. But since by (2), $D_\xi \tilde{u}$ is injectif, the dimension of \mathcal{U}_ξ is m . Hence this map is a bijection. The rank of $D_{u_\xi} F$ is therefore maximal. By continuity of $D_u F$ in u , this remains true in a neighbourhood of u_ξ , which proves (4). It follows from (4) that locally, Σ_{μ_ξ} is a co-dimension m submanifold of E . Since, by definition, $T_{u_\xi} \Sigma_{\mu_\xi} = \text{Ker} D_{u_\xi} F$, we know from (3) that $\mathcal{U}_\xi \cap T_{u_\xi} \Sigma_{\mu_\xi} = \{0\}$. Since, by (2), the dimension of \mathcal{U}_ξ is m , (5) follows.

((4) and (5)) \Rightarrow ((2) and (3)). This is obvious and concludes the proof of the lemma. \square

We introduce

$$G_\xi = \{g \in G \mid \text{Ad}_g \xi = \xi\}, \quad (4.5)$$

which is the subgroup of G for which ξ is a fixed point under the adjoint action. We will write \mathfrak{g}_ξ for its Lie-algebra. We furthermore need

$$Z_\xi = \{X_\eta(u_\xi) \mid \eta \in \mathfrak{g}_\xi\} \subset E \quad (4.6)$$

where $X_\eta(u_\xi)$ is defined in (2.3).

Lemma 4.2. *Let E be a Banach space, Ω an open set in \mathfrak{g} , and \tilde{u} defined by (3.1) and satisfying (3.3). Suppose $\tilde{u} \in C^1(\Omega \subset \mathfrak{g}, E)$. Let $H \in C^2(E, \mathbb{R})$. Let G be a Lie group, and Φ a globally Hamiltonian G -action on E with Ad^* -equivariant momentum map F . Suppose that $H \circ \Phi_g = H$ for all $g \in G$. Let $\xi \in \Omega$. Then, one has:*

(1) For all $\eta \in \mathbb{R}^m$,

$$\eta \in W_0 = \text{Ker} D_\xi^2 W \Leftrightarrow \eta \in \text{Ker} D_\xi \hat{F} \Leftrightarrow \eta \cdot \nabla_\xi u_\xi \in \text{Ker} D_{u_\xi} F. \quad (4.7)$$

In particular, $D_\xi^2 W$ is non-degenerate if and only if \hat{F} is a local diffeomorphism at ξ .

(2) For all $\eta_1, \eta_2 \in \mathbb{R}^m$,

$$D_{u_\xi}^2 \mathcal{L}_\xi(\eta_1 \cdot \nabla_\xi u_\xi, \eta_2 \cdot \nabla_\xi u_\xi) = -D_\xi^2 W(\eta_1, \eta_2). \quad (4.8)$$

(3) For all $v \in \text{Ker} D_{u_\xi} F$, for all $\eta \in \mathfrak{g}$,

$$D_{u_\xi}^2 \mathcal{L}_\xi(\eta \cdot \nabla_\xi u_\xi, v) = 0. \quad (4.9)$$

(4) $T_{u_\xi} \mathcal{O}_{u_\xi}$ is a subspace of the kernel of $(D_{u_\xi}^2 \mathcal{L}_\xi \mid \text{Ker} D_{u_\xi} F)$, which is the restriction of $D_{u_\xi}^2 \mathcal{L}_\xi$ to $\text{Ker} D_{u_\xi} F \times \text{Ker} D_{u_\xi} F$.

(5) $Z_\xi \subset \text{Ker}(D_{u_\xi}^2 \mathcal{L}_\xi) \subset \text{Ker}(D_{u_\xi} F)$.

Note that, combining (4.7) with Lemma 4.1, we can conclude that the directions $\eta \cdot \nabla_\xi u_\xi$ form a complementary subspace to $\text{Ker} D_{u_\xi} F$ when $D_\xi^2 W$ is non-degenerate. Hence, if in addition u_ξ is a regular relative equilibrium, the subspace \mathcal{U}_ξ is complementary to the tangent space $T_{u_\xi} \Sigma_{\mu_\xi}$.

Expression (4.8) is of interest since it identifies part of the Hessian of the Lyapunov function \mathcal{L}_ξ in terms of the Hessian of the function W . More precisely, it is useful to determine a subspace of negative directions of $D_{u_\xi}^2 \mathcal{L}_\xi$. Indeed, if $n_+ = p(D_\xi^2 W)$ and if $\{\eta_1, \dots, \eta_{n_+}\}$ is a family of linearly independent elements of \mathbb{R}^m such that $\text{span}\{\eta_1, \dots, \eta_{n_+}\}$ is a positive definite subspace for $D_\xi^2 W$, then $\text{span}\{\eta_1 \cdot \nabla_\xi u_\xi, \dots, \eta_{n_+} \cdot \nabla_\xi u_\xi\}$ is a negative definite subspace for $D_{u_\xi}^2 \mathcal{L}_\xi$. As a consequence, the dimension of a maximally negative definite subspace for $D_{u_\xi}^2 \mathcal{L}_\xi$ in E is at least $p(D_\xi^2 W)$:

$$n(D_{u_\xi}^2 \mathcal{L}_\xi) \geq p(D_\xi^2 W).$$

Expression (4.9) turns out to be crucial in what follows: it expresses the fact that $\mathcal{U}_\xi = \{\eta \cdot \nabla_\xi u_\xi \mid \eta \in \mathbb{R}^m\}$ is $D_{u_\xi}^2 \mathcal{L}_\xi$ -orthogonal to $\text{Ker } D_{u_\xi} F$.

Proof. First of all, note that, since $u_{\xi'}$ is a solution to the stationary equation (3.3) for all $\xi' \in \Omega$, for all $\eta \in \mathbb{R}^m$

$$D_\xi W(\eta) = -\eta \cdot F(u_\xi). \quad (4.10)$$

Then a straightforward calculation gives, for all $\eta_1, \eta_2 \in \mathbb{R}^m$,

$$D_\xi^2 W(\eta_1, \eta_2) = -D_{u_\xi} \eta_1 \cdot F(\eta_2 \cdot \nabla_\xi u_\xi). \quad (4.11)$$

In other words,

$$D_\xi^2 W = -D_\xi \hat{F}. \quad (4.12)$$

Note that, as \hat{F} is a map from $\mathbb{R}^m \simeq \mathfrak{g}$ to $\mathbb{R}^m \simeq \mathfrak{g}^*$, $D_\xi \hat{F}$ is linear from $\mathbb{R}^m = \mathfrak{g}$ to $\mathbb{R}^m = \mathfrak{g}^*$. It therefore naturally defines a bilinear map on $\mathbb{R}^m \simeq \mathfrak{g}$. In our notation here, we identify \mathfrak{g} with \mathfrak{g}^* using an Euclidean structure, but even without the latter, the above is natural.

The first statement of (4.7) is now obvious and for the second, note that $\eta \cdot \nabla_\xi u_\xi \in \text{Ker } D_{u_\xi} F$ if and only if, for all $\eta' \in \mathbb{R}^m$, $D_{u_\xi} \eta' \cdot F(\eta \cdot \nabla_\xi u_\xi) = 0$, which yields the conclusion, thanks to (4.4) and (4.11).

To obtain (4.8), it is sufficient to take the derivative of the stationary equation with respect to $\xi \in \mathfrak{g}$ and to use (4.11). More precisely, by taking the derivative of (3.3) with respect to ξ in the direction η , we obtain for all $\eta \in \mathfrak{g}$,

$$D_{u_\xi}^2 \mathcal{L}_\xi(\eta \cdot \nabla_\xi u_\xi) = D_{u_\xi} \eta \cdot F \in E^*. \quad (4.13)$$

Hence, using (4.11),

$$D_{u_\xi}^2 \mathcal{L}_\xi(\eta_1 \cdot \nabla_\xi u_\xi, \eta_2 \cdot \nabla_\xi u_\xi) = D_{u_\xi} \eta_1 \cdot F(\eta_2 \cdot \nabla_\xi u_\xi) = -D_\xi^2 W(\eta_1, \eta_2).$$

Next, (4.9) follows directly from (4.13). Indeed, for all $v \in \text{Ker } D_{u_\xi} F$ and for all $\eta \in \mathfrak{g}$, $D_{u_\xi}^2 \mathcal{L}_\xi(\eta \cdot \nabla_\xi u_\xi, v) = D_{u_\xi} \eta \cdot F(v) = 0$.

From $F = (F \circ \Phi_{g^{-1}}) \circ \Phi_g$ and $H = H \circ \Phi_g$ one finds, for all $u \in E, g \in G$,

$$D_u F = (D_{\Phi_g(u)}(F \circ \Phi_{g^{-1}})) D_u \Phi_g, \quad D_u H = D_{\Phi_g(u)} H D_u \Phi_g.$$

Hence, by (3.3),

$$\begin{aligned} D_{\Phi_g(u_\xi)} H &= \xi \cdot D_{u_\xi} F (D_{u_\xi} \Phi_g)^{-1} = D_{\Phi_g(u_\xi)}(\xi \cdot F \circ \Phi_{g^{-1}}) \\ &= D_{\Phi_g(u_\xi)}(Ad_g \xi \cdot F) \end{aligned}$$

and therefore

$$D_{\Phi_g(u_\xi)}(H - \xi \cdot F) = D_{\Phi_g(u_\xi)}((Ad_g \xi - \xi) \cdot F).$$

Now let $\eta \in \mathfrak{g}$, consider $g = \exp(t\eta)$ and take the derivative at $t = 0$ in the previous relation. One finds, for all $v \in E$,

$$D_{u_\xi}^2(H - \xi \cdot F)(X_\eta(u_\xi), v) = D_{u_\xi}[\eta, \xi] \cdot F(v). \quad (4.14)$$

Taking $v \in \text{Ker } D_{u_\xi} F$, the right hand side above vanishes for any $\eta \in \mathfrak{g}$, and one finds (4) follows.

To prove (5), note that, taking $\eta \in \mathfrak{g}_\xi$ so that $[\eta, \xi] = 0$, we see that $Z_\xi \subset \text{Ker } D_{u_\xi}^2 \mathcal{L}_\xi$. Finally, let $v \in E$, then (4.13) yields, for all $\eta \in \mathfrak{g}$

$$D_{u_\xi}^2 \mathcal{L}_\xi(\eta \cdot \nabla_\xi u_\xi, v) = \eta \cdot D_{u_\xi} F(v).$$

Hence, if $v \in \text{Ker}(D_{u_\xi}^2 \mathcal{L}_\xi)$, it follows that $v \in \text{Ker}(D_{u_\xi} F)$. \square

Finally, we state some properties of symmetric bilinear forms and their associated quadratic forms in the form of a short lemma. In what follows, if B is a bilinear form on some vector space E , and \mathcal{Y} is a subspace of E , then we write $(B | \mathcal{Y})$ for the restriction of B to $\mathcal{Y} \times \mathcal{Y}$.

Lemma 4.3. *Let E be a vector space and B a symmetric bilinear form on E .*

- (i) *Let \mathcal{X}_- be a maximally negative definite subspace for B in E . Suppose $\mathcal{Y} \subset E$ is a subspace of E with the property that $\mathcal{X}_- \cap \mathcal{Y} = \{0\}$ and such that $B(\mathcal{X}_-, \mathcal{Y}) = 0$. Then \mathcal{Y} is a positive subspace for B .*
- (ii) *Let $\mathcal{Y}_1, \mathcal{Y}_2$ be two subspaces of \mathcal{Y} , such that $B(\mathcal{Y}_1, \mathcal{Y}_2) = 0$. Then*

$$\text{Ker}(B | \mathcal{Y}_1 + \mathcal{Y}_2) = \text{Ker}(B | \mathcal{Y}_1) + \text{Ker}(B | \mathcal{Y}_2). \quad (4.15)$$

- (iii) *Let \mathcal{Y} be a positive subspace for B . Suppose $u \in \mathcal{Y}$ satisfies $B(u, u) = 0$. Then $u \in \text{Ker}(B | \mathcal{Y})$.*

Note that the B -orthogonality of the subspaces is crucial in parts (i) and (ii).

Proof. (i) Suppose the statement is false, then there exists $y \in \mathcal{Y}$, so that $B(y, y) < 0$. Clearly, $y \neq 0$ and hence, by assumption, $y \notin \mathcal{X}_-$. Now consider $\mathcal{Z} = \text{vect}\{y, \mathcal{X}_-\}$. Let $0 \neq z \in \mathcal{Z}$. Then, there exist $\lambda \in \mathbb{R}$ and $z_- \in \mathcal{X}_-$, not both zero, so that $z = \lambda y + z_-$. It follows from the B orthogonality of \mathcal{X}_- and \mathcal{Y} that

$$B(z, z) = \lambda^2 B(y, y) + B(z_-, z_-) < 0.$$

Hence B is negative definite on \mathcal{Z} . Since $\dim \mathcal{X}_- \subsetneq \mathcal{Z}$ this is a contradiction.

(ii) Immediate.

(iii) One has, for all $v \in \mathcal{Y}$, and for all $\lambda \in \mathbb{R}$,

$$0 \leq B(\lambda u + v, \lambda u + v) = 2\lambda B(u, v) + B(v, v).$$

If $B(u, v) \neq 0$, this is a contradiction. \square

5. PROOF OF THEOREM 3.1

Let $n_- = \dim E_-$ and $\{\eta_1, \dots, \eta_{n_-}\} \subset \mathbb{R}^m$ a family of linearly independent elements of \mathbb{R}^m such that $D_\xi^2 W(\eta, \eta) > 0$ for all $\eta \in \text{span}\{\eta_1, \dots, \eta_{n_-}\}$. As a consequence of (4.8), $\mathcal{X}_- := \text{span}\{\eta_1 \cdot \nabla_\xi u_\xi, \dots, \eta_{n_-} \cdot \nabla_\xi u_\xi\}$ is a negative definite subspace for $D_{u_\xi}^2 \mathcal{L}_\xi$ in E . Moreover, since $\dim \mathcal{X}_- = n_-$, \mathcal{X}_- is a maximally negative definite subspace.

Next, since $D_\xi^2 W$ is non-degenerate by hypothesis, Lemma 4.2 (1) implies that \hat{F} is a local diffeomorphism. Hence, thanks to Lemma 4.1, $\mathcal{X}_- \cap T_{u_\xi} \Sigma_{\mu_\xi} = \{0\}$. Moreover, thanks to (4.9), \mathcal{X}_- and $T_{u_\xi} \Sigma_{\mu_\xi}$ are $D_{u_\xi}^2 \mathcal{L}_\xi$ -orthogonal. As a consequence,

we can apply Lemma 4.3 (i) and conclude that $T_{u_\xi} \Sigma_{\mu_\xi}$ is a positive subspace for $D_{u_\xi}^2 \mathcal{L}_\xi$.

Furthermore, since by hypothesis (ii) of the theorem, $E_0 = \text{Ker } D_{u_\xi}^2 \mathcal{L}_\xi = T_{u_\xi} \mathcal{O}_{u_\xi}$, it follows from Lemma 4.3 (ii) that

$$\mathcal{Y} := T_{u_\xi} \Sigma_{\mu_\xi} \cap (T_{u_\xi} \mathcal{O}_{u_\xi})^\perp \quad (5.1)$$

is a positive definite subspace for $D_{u_\xi}^2 \mathcal{L}_\xi$, meaning that

$$D_{u_\xi}^2 \mathcal{L}_\xi(v, v) > 0, \quad \forall v \in \mathcal{Y} = T_{u_\xi} \Sigma_{\mu_\xi} \cap (T_{u_\xi} \mathcal{O}_{u_\xi})^\perp, v \neq 0. \quad (5.2)$$

To obtain the desired coercive bound, we now use the spectral information on $\nabla^2 \mathcal{L}_\xi(u_\xi)$ provided by the hypotheses of the theorem. Note first that, since $\nabla^2 \mathcal{L}_\xi(u_\xi)$ is self-adjoint, $E_0^\perp = (T_{u_\xi} \mathcal{O}_{u_\xi})^\perp \subset E$ is an invariant subspace for $\nabla^2 \mathcal{L}_\xi(u_\xi)$: it is the spectral subspace of $\nabla^2 \mathcal{L}_\xi(u_\xi)$ corresponding to \mathbb{R}^* .

Let $\nabla F_j(u_\xi) \in E$ be defined by $D_{u_\xi} F_j(v) = \langle \nabla F_j(u_\xi), v \rangle_E$ for $j = 1, \dots, m$. Then, since μ_ξ is a regular value,

$$(T_{u_\xi} \Sigma_{\mu_\xi})^\perp = \text{span} \{ \nabla F_j(u_\xi) \}_{j=1, \dots, m}.$$

and $\dim (T_{u_\xi} \Sigma_{\mu_\xi})^\perp = m$. Moreover, since $T_{u_\xi} \mathcal{O}_{u_\xi} \subset T_{u_\xi} \Sigma_{\mu_\xi}$, one obtains the following orthogonal decomposition of E_0^\perp :

$$E_0^\perp = (T_{u_\xi} \mathcal{O}_{u_\xi})^\perp = \mathcal{Y} \oplus (T_{u_\xi} \Sigma_{\mu_\xi})^\perp.$$

Note that $\nabla^2 \mathcal{L}_\xi(u_\xi)$ does not leave this decomposition invariant: we are interested in controlling it on \mathcal{Y} . For that purpose, let P be the projection on $\mathcal{Y} \subset E_0^\perp$ and consider the following decomposition of the restriction of the operator $\nabla^2 \mathcal{L}_\xi(u_\xi)$ to E_0^\perp ,

$$\begin{aligned} \nabla^2 \mathcal{L}_\xi(u_\xi)|_{E_0^\perp} &= P \nabla^2 \mathcal{L}_\xi(u_\xi) P + P \nabla^2 \mathcal{L}_\xi(u_\xi) (I_{E_0^\perp} - P) \\ &\quad + (I_{E_0^\perp} - P) \nabla^2 \mathcal{L}_\xi(u_\xi) P + (I_{E_0^\perp} - P) \nabla^2 \mathcal{L}_\xi(u_\xi) (I_{E_0^\perp} - P). \end{aligned}$$

Since $(T_{u_\xi} \Sigma_{\mu_\xi})^\perp$ is finite dimensional, the projector $I_{E_0^\perp} - P$ is finite rank. Hence $\nabla^2 \mathcal{L}_\xi(u_\xi)|_{E_0^\perp} = P \nabla^2 \mathcal{L}_\xi(u_\xi) P + K$ with K a compact operator and it follows that $\sigma_{\text{ess}}(P \nabla^2 \mathcal{L}_\xi(u_\xi) P) = \sigma_{\text{ess}}(\nabla^2 \mathcal{L}_\xi(u_\xi)|_{E_0^\perp})$. In particular, $0 \notin \sigma_{\text{ess}}(P \nabla^2 \mathcal{L}_\xi(u_\xi) P)$ by hypotheses (iii) and (iv) of Theorem 3.1.

Now, note that $\text{Ker } P \nabla^2 \mathcal{L}_\xi(u_\xi) P = (T_{u_\xi} \Sigma_{\mu_\xi})^\perp \subset E_0^\perp$. Indeed, let $u \in E_0^\perp$ be such that $P \nabla^2 \mathcal{L}_\xi(u_\xi) P u = 0$. Then

$$D_{u_\xi}^2 \mathcal{L}_\xi(u_\xi)(Pu, Pu) = \langle u, P \nabla^2 \mathcal{L}_\xi(u_\xi) P u \rangle = 0.$$

Since $D_{u_\xi}^2 \mathcal{L}_\xi(u_\xi)$ is strictly positive on \mathcal{Y} (see (5.2)), it follows that $Pu = 0$, which means that $u \in (T_{u_\xi} \Sigma_{\mu_\xi})^\perp$.

We now consider $P \nabla^2 \mathcal{L}_\xi(u_\xi) P$ as an operator on \mathcal{Y} . We have just shown that $0 \notin \sigma(P \nabla^2 \mathcal{L}_\xi(u_\xi) P) = \sigma_{\text{ess}}(P \nabla^2 \mathcal{L}_\xi(u_\xi) P) \cup \sigma_{\text{d}}(P \nabla^2 \mathcal{L}_\xi(u_\xi) P)$ and that $P \nabla^2 \mathcal{L}_\xi(u_\xi) P$ is a strictly positive operator on \mathcal{Y} (see (5.2)). It therefore has a spectral gap:

$$\delta = \inf_{v \in \mathcal{Y} \setminus \{0\}} \frac{\langle P \nabla^2 \mathcal{L}_\xi(u_\xi) P v, v \rangle}{\langle v, v \rangle} > 0.$$

Finally, for all $v \in \mathcal{Y}$, $v \neq 0$, one finds

$$D_{u_\xi}^2 \mathcal{L}_\xi(v, v) = \langle \nabla^2 \mathcal{L}_\xi(u_\xi)v, v \rangle = \langle P \nabla^2 \mathcal{L}_\xi(u_\xi) P v, v \rangle \geq \delta \|v\|^2$$

which is the desired estimate.

6. MAIN RESULT: THE BANACH SPACE SETTING

In this section, we extend the result of Theorem 3.1 to a more general Banach space setting that we now describe.

Let $(E, \|\cdot\|_E)$ be a Banach space and $\langle \cdot, \cdot \rangle$ be a scalar product on E which is continuous in the sense that

$$|\langle v, w \rangle| \leq \|v\|_E \|w\|_E$$

for all $v, w \in E$. Let $\|\cdot\|$ be the norm associated to the scalar product $\langle \cdot, \cdot \rangle$ and define by \hat{E} the closure of E with respect to the $\|\cdot\|$ -norm, that is $\hat{E} = \overline{E}^{\|\cdot\|}$. Note that \hat{E} is a Hilbert space and $E \subset \hat{E}$. As an example, one can think of $E = H^1(\mathbb{R}^n)$ and $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^n)}$ so that $\hat{E} = L^2(\mathbb{R}^n)$ in that case. This is the typical situation for the nonlinear Schrödinger equation; we refer to [7] and Section 7 for details and further examples.

Let $(E, \mathcal{D}, \mathcal{J})$ be a symplectic Banach triple and H, G, Φ and F as in Section 3.

Let Ω be an open set in $\mathfrak{g} \simeq \mathbb{R}^m$ and \tilde{u}, \hat{F} and W defined by (3.1), (3.2) and (3.4) respectively. As before, for each $\xi \in \Omega$, u_ξ is a solution to (3.3), $\mu_\xi = F(u_\xi)$ is a regular value of F and $\tilde{u} \in C^1(\Omega \subset \mathfrak{g}, E)$. Define $p(D_\xi^2 W)$, $n(D_\xi^2 W)$ and $n(D_{u_\xi}^2 \mathcal{L}_\xi)$ as in Section 3.

To associate to $D_{u_\xi}^2 \mathcal{L}_\xi$ a self-adjoint operator on \hat{E} we proceed as follow.

Lemma 6.1. *Let $(E, \|\cdot\|_E)$ be a Banach space and $\langle \cdot, \cdot \rangle$ be a continuous scalar product on E . Let $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$ and $\hat{E} = \overline{E}^{\|\cdot\|}$. Let $D_{u_\xi}^2 \mathcal{L}_\xi(v, w)$ be defined as above for all $v, w \in E$. If there exist $\varepsilon, C > 0$ such that*

$$D_{u_\xi}^2 \mathcal{L}_\xi(v, v) \geq \varepsilon \|v\|_E^2 - C \|v\|^2 \quad (6.1)$$

for all $v \in E$, then there is a self-adjoint operator $\nabla^2 \mathcal{L}_\xi(u_\xi) : \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi)) \subset \hat{E} \rightarrow \hat{E}$ defined by

$$\begin{aligned} \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi)) &= \{z \in E \mid \exists w \in \hat{E} \text{ such that } D_{u_\xi}^2 \mathcal{L}_\xi(z, v) = \langle w, v \rangle \text{ for all } v \in E\}, \\ \nabla^2 \mathcal{L}_\xi(u_\xi)z &= w \text{ for all } z \in \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi)). \end{aligned} \quad (6.2)$$

Moreover, $\mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi))$ is a form core for $D_{u_\xi}^2 \mathcal{L}_\xi$.

Remark 6.2. Note that

- (1) E is the form domain of the operator $\nabla^2 \mathcal{L}_\xi(u_\xi)$,
- (2) Since $\mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi))$ is a form core for $D_{u_\xi}^2 \mathcal{L}_\xi$ and condition (6.1) holds, $\mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi))$ is dense in E .

The existence and the uniqueness of the operator $\nabla^2 \mathcal{L}_\xi(u_\xi)$ is a consequence of the First Representation theorem in Kato [15, Theorem 2.1 and 2.6 in Chapter VI]. Condition (6.1) ensures that the hypotheses of the First Representation theorem are satisfied (see [26, Lemma 3.3]). See also [24, 27] for more details.

We can now state our main result.

Theorem 6.3. *Under the previous hypotheses, in particular those of Lemma 6.1, suppose*

- (i) $D_\xi^2 W$ is non-degenerate,
- (ii) $\text{Ker } D_{u_\xi}^2 \mathcal{L}_\xi = T_{u_\xi} \mathcal{O}_{u_\xi}$,
- (iii) $\inf(\sigma(\nabla^2 \mathcal{L}_\xi(u_\xi)) \cap (0, +\infty)) > 0$,
- (iv) $p(D_\xi^2 W) = n(D_{u_\xi}^2 \mathcal{L}_\xi)$,
- (v) for all $u \in \mathcal{O}_{u_\xi}$ and for all $j = 1, \dots, m$, there exists $\nabla F_j(u) \in \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi)) \subset E$ such that

$$D_u F_j(w) = \langle \nabla F_j(u), w \rangle, \quad \forall w \in E. \quad (6.3)$$

Then there exists $\delta > 0$ such that

$$\forall v \in T_{u_\xi} \Sigma_{\mu_\xi} \cap (T_{u_\xi} \mathcal{O}_{u_\xi})^\perp, \quad D_{u_\xi}^2 \mathcal{L}_\xi(v, v) \geq \delta \|v\|_E \quad (6.4)$$

with $(T_{u_\xi} \mathcal{O}_{u_\xi})^\perp = \{v \in \hat{E} \mid \langle v, w \rangle = 0 \ \forall w \in T_{u_\xi} \mathcal{O}_{u_\xi}\}$.

Let us point out that the hypotheses on the bilinear form $D_{u_\xi}^2 \mathcal{L}_\xi$ in Theorem 6.3 can be re-expressed in terms of spectral hypotheses on the unbounded self-adjoint operator $\nabla^2 \mathcal{L}_\xi(u_\xi)$, as shown in the following lemma. This is important in applications, since it allows one to use the tools of spectral analysis for partial differential operators to establish those conditions.

Lemma 6.4. *Under the hypotheses of Lemma 6.1, $\text{Ker } \nabla^2 \mathcal{L}_\xi(u_\xi) = \text{Ker } D_{u_\xi}^2 \mathcal{L}_\xi$.*

If, in addition $\dim \text{Ker } \nabla^2 \mathcal{L}_\xi(u_\xi) < +\infty$, the negative spectral subspace of $\nabla^2 \mathcal{L}_\xi(u_\xi)$ is finite dimensional, and hypothesis (iii) of Theorem 6.3 is satisfied then the dimension of the negative spectral subspace of $\nabla^2 \mathcal{L}_\xi(u_\xi)$ in \hat{E} is equal to the Morse index $n(D_{u_\xi}^2 \mathcal{L}_\xi)$ of $u_\xi \in E$ for \mathcal{L}_ξ .

This lemma constitutes a slight generalization of Lemma 5.4 in [26] and its proof follows along the same lines. We give it for completeness.

Proof. By definition $\text{Ker } \nabla^2 \mathcal{L}_\xi(u_\xi) = \{v \in \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi)) \mid \nabla^2 \mathcal{L}_\xi(u_\xi)v = 0\}$ and a straightforward calculation leads to $\text{Ker } \nabla^2 \mathcal{L}_\xi(u_\xi) = \text{Ker } D_{u_\xi}^2 \mathcal{L}_\xi \cap \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi))$. Moreover, using the definition (6.2), it is easy to see that $\text{Ker } D_{u_\xi}^2 \mathcal{L}_\xi \subset \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi))$. As a consequence, $\text{Ker } \nabla^2 \mathcal{L}_\xi(u_\xi) = \text{Ker } D_{u_\xi}^2 \mathcal{L}_\xi$.

Now, we know that

$$\begin{aligned} \dim \text{Ker } \nabla^2 \mathcal{L}_\xi(u_\xi) &= n_0 < +\infty \\ \inf(\sigma(\nabla^2 \mathcal{L}_\xi(u_\xi)) \cap (0, +\infty)) &> 0 \end{aligned}$$

and we denote by $0 \leq n_- < +\infty$ the dimension of the negative spectral subspace of $\nabla^2 \mathcal{L}_\xi(u_\xi)$ in \hat{E} . It follows that there exists $\Gamma > 0$ such that $\sigma(\nabla^2 \mathcal{L}_\xi(u_\xi)) \cap (0, \Gamma] = \emptyset$. Let $P_{(-\infty, 0]} \in \mathcal{B}(\hat{E}, \hat{E})$ denote the orthogonal projection onto the finite dimensional span of all the eigenvectors of $\nabla^2 \mathcal{L}_\xi(u_\xi)$ corresponding to the eigenvalues in $(-\infty, 0]$, and let $Q = I - P_{(-\infty, 0]}$. We have that $P_{(-\infty, 0]}(\hat{E}) \subset \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi))$ and that $Qz \in \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi))$ if and only if $z \in \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi))$. Also $\dim P_{(-\infty, 0]}(\hat{E}) = n_0 + n_-$ and $\langle \nabla^2 \mathcal{L}_\xi(u_\xi) Qz, Qz \rangle \geq \Gamma \|Qz\|^2$ for all $z \in \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi))$. Thus, for any $z \in$

$\mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi))$

$$\begin{aligned} D_{u_\xi}^2 \mathcal{L}_\xi(z, z) &= \langle \nabla^2 \mathcal{L}_\xi(u_\xi)z, z \rangle = \langle Q \nabla^2 \mathcal{L}_\xi(u_\xi)z, z \rangle + \langle P_{(-\infty, 0]} \nabla^2 \mathcal{L}_\xi(u_\xi)z, z \rangle \\ &= \langle \nabla^2 \mathcal{L}_\xi(u_\xi)Qz, Qz \rangle + \langle P_{(-\infty, 0]} \nabla^2 \mathcal{L}_\xi(u_\xi)z, z \rangle \\ &\geq \Gamma \|Qz\|^2 + \langle P_{(-\infty, 0]} \nabla^2 \mathcal{L}_\xi(u_\xi)z, z \rangle \\ &\geq \Gamma \|z\|^2 - \Gamma \|P_{(-\infty, 0]}z\|^2 + \langle \nabla^2 \mathcal{L}_\xi(u_\xi)P_{(-\infty, 0]}z, z \rangle \\ &\geq \Gamma \|z\|^2 + \langle (\nabla^2 \mathcal{L}_\xi(u_\xi) - I\Gamma)P_{(-\infty, 0]}z, z \rangle. \end{aligned}$$

As a consequence, using (6.1) and the fact that $\mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi))$ is dense in E , we obtain

$$\left(1 + \frac{\Gamma}{C}\right) D_{u_\xi}^2 \mathcal{L}_\xi(z, z) - \langle (\nabla^2 \mathcal{L}_\xi(u_\xi) - I\Gamma)P_{(-\infty, 0]}z, z \rangle \geq \frac{\varepsilon\Gamma}{C} \|z\|_E^2$$

for all $z \in E$, which implies

$$D_{u_\xi}^2 \mathcal{L}_\xi(z, z) - \frac{C}{\Gamma + C} \langle (\nabla^2 \mathcal{L}_\xi(u_\xi) - I\Gamma)P_{(-\infty, 0]}z, z \rangle \geq \frac{\varepsilon\Gamma}{\Gamma + C} \|z\|_E^2$$

for all $z \in E$. Moreover, for all $z \in Q(E)$,

$$D_{u_\xi}^2 \mathcal{L}_\xi(z, z) \geq \frac{\varepsilon\Gamma}{\Gamma + C} \|z\|_E^2 \quad (6.5)$$

since $P_{(-\infty, 0]}z = 0$. But $Q(E) \subset E$ since $P_{(-\infty, 0]}(E) \subset P_{(-\infty, 0]}(\hat{E}) \subset \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi)) \subset E$. So we have shown that $Q(E)$ is a positive subspace of E , for $D_{u_\xi}^2 \mathcal{L}$. Now consider the direct sum decomposition of E given by

$$E = Q(E) \oplus P_{(-\infty, 0)}(E) \oplus \text{Ker}(D^2 \mathcal{L}_\xi).$$

Here $P_{(-\infty, 0)}$ is the projector onto the n_- -dimensional space spanned by the eigenvectors of $\nabla^2 \mathcal{L}_\xi(u_\xi)$ with strictly negative eigenvalue. Clearly, $P_{(-\infty, 0)}(E)$ is a negative definite space for $D_{u_\xi}^2 \mathcal{L}_\xi$. We now show it is maximal. For that purpose, suppose $z_* \in E$, $z_* \notin P_{(-\infty, 0)}(E)$ and suppose $\text{span}\{z_*, P_{(-\infty, 0)}(E)\}$ is a negative definite subspace of E for $D_{u_\xi}^2 \mathcal{L}_\xi$ of dimension $n_- + 1$. We can suppose, without loss of generality, that $z_* \in \text{Ker}(D^2 \mathcal{L}_\xi) \oplus Q(E)$. Writing $z_* = z_{*, 0} + z_{*, +}$, we see

$$D_{u_\xi}^2 \mathcal{L}_\xi(z_*, z_*) = D_{u_\xi}^2 \mathcal{L}_\xi(z_{*, +}, z_{*, +}) \geq 0,$$

where we used (6.5). This contradicts the fact that $\text{span}\{z_*, P_{(-\infty, 0)}(E)\}$ is negative definite space for $D_{u_\xi}^2 \mathcal{L}_\xi$ and shows that $P_{(-\infty, 0)}(E)$ is a maximally negative definite subspace for $D_{u_\xi}^2 \mathcal{L}_\xi$. Thus $n(D_{u_\xi}^2 \mathcal{L}_\xi) = n_-$. \square

For the proof of Theorem 6.3, we will need the following two lemmas.

Lemma 6.5. *Let $(\mathcal{H}, \|\cdot\|)$ be a Hilbert space and $W \subset \mathcal{K} \subset \mathcal{H}$ with W a closed subspace of \mathcal{H} and \mathcal{K} a dense subspace of \mathcal{H} ($\overline{\mathcal{K}^{\|\cdot\|}} = \mathcal{H}$). Then*

$$\overline{W^\perp \cap \mathcal{K}^{\|\cdot\|}} = W^\perp \quad (6.6)$$

Proof. Let $u \in W^\perp$. There exists a sequence $k_n \in \mathcal{K}$ so that $k_n \rightarrow u$. Since W is closed, we can write $k_n = w_n + v_n$, with $w_n \in W$, $v_n \in W^\perp$. Moreover, since $k_n \in \mathcal{K}$ and $w_n \in W \subset \mathcal{K}$, $v_n \in W^\perp \cap \mathcal{K}$. Clearly both sequences w_n and v_n converge, respectively to $w \in W$, $v \in W^\perp$. Since $u = w + v \in W^\perp$, we find $w = 0$ and $v = u$. Hence $v_n \in \mathcal{K} \cap W^\perp$ converges to u . \square

We introduce, as before $E_0 = \text{Ker } \nabla_{u_\xi}^2 \mathcal{L}_\xi$. We know from Lemma 6.4 that $\text{Ker } \nabla_{u_\xi}^2 \mathcal{L}_\xi = \text{Ker } D_{u_\xi}^2 \mathcal{L}_\xi$. Hypothesis (ii) of the theorem then implies

$$E_0 = T_{u_\xi} \mathcal{O}_{u_\xi} \subset \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi)). \quad (6.7)$$

We define furthermore

$$V = \text{span}\{\nabla F_j(u_\xi), j = 1, \dots, m\}. \quad (6.8)$$

Note that, by hypothesis (v) of Theorem 6.3, $V \subset \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi)) \subset E$.

Lemma 6.6. *Under the hypotheses of Theorem 6.3, we have*

- (i) $E_0^\perp \cap \overline{T_{u_\xi} \Sigma_{\mu_\xi}}^{\|\cdot\|} = \overline{E_0^\perp \cap T_{u_\xi} \Sigma_{\mu_\xi}}^{\|\cdot\|}$.
- (ii) $\overline{T_{u_\xi} \Sigma_{\mu_\xi} \cap \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi))}^{\|\cdot\|} = \overline{T_{u_\xi} \Sigma_{\mu_\xi}}^{\|\cdot\|}$.
- (iii) Define

$$\mathcal{Y} = T_{u_\xi} \Sigma_{\mu_\xi} \cap \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi)) \cap E_0^\perp. \quad (6.9)$$

Then

$$\hat{\mathcal{Y}} := \overline{\mathcal{Y}}^{\|\cdot\|} = E_0^\perp \cap V^\perp, \quad (6.10)$$

where V is defined in (6.8). Hence

$$E_0^\perp = T_{u_\xi} \mathcal{O}_{u_\xi}^\perp = \hat{\mathcal{Y}} \oplus_\perp V, \quad \hat{E} = E_0 \oplus_\perp \hat{\mathcal{Y}} \oplus_\perp V. \quad (6.11)$$

(iv) Let P be the orthogonal projector onto $\hat{\mathcal{Y}}$. Let $u \in \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi)) \cap E_0^\perp$. Then $Pu \in \mathcal{Y}$.

We use the notation \oplus_\perp to indicate a direct sum that is orthogonal for the inner product $\langle \cdot, \cdot \rangle$.

Proof. (i) Note that $E_0 \subset T_{u_\xi} \Sigma_{\mu_\xi} \subset \overline{T_{u_\xi} \Sigma_{\mu_\xi}}^{\|\cdot\|}$. We now apply Lemma 6.5 with $W = E_0, \mathcal{K} = T_{u_\xi} \Sigma_{\mu_\xi}, \mathcal{H} = \overline{T_{u_\xi} \Sigma_{\mu_\xi}}^{\|\cdot\|}$ to obtain

$$E_0^{\perp\sigma} = \overline{E_0^{\perp\sigma} \cap T_{u_\xi} \Sigma_{\mu_\xi}}^{\|\cdot\|}.$$

Here we wrote $E_0^{\perp\sigma}$ for the orthogonal complement to E_0 in $\overline{T_{u_\xi} \Sigma_{\mu_\xi}}^{\|\cdot\|}$, i.e.

$$E_0^{\perp\sigma} = E_0^\perp \cap \overline{T_{u_\xi} \Sigma_{\mu_\xi}}^{\|\cdot\|}. \quad (6.12)$$

The last two equations imply the result.

(ii) Since V is a closed finite dimensional subspace of \hat{E} , we have $\hat{E} = V^\perp \oplus_\perp V$ with

$$V^\perp = \{w \in \hat{E} \mid \langle v, w \rangle = 0 \ \forall v \in V\}.$$

Since by hypothesis (v) of Theorem 6.3, for all $w \in E$, $D_{u_\xi} F_j(w) = \langle \nabla F_j(u_\xi), w \rangle$ for $j = 1, \dots, m$, we see that

$$T_{u_\xi} \Sigma_{\mu_\xi} = V^\perp \cap E \quad (6.13)$$

and hence

$$E = T_{u_\xi} \Sigma_{\mu_\xi} \oplus_\perp V. \quad (6.14)$$

Using Lemma 6.5, and the fact that $V \subset E \subset \hat{E}$, with E dense in \hat{E} , (6.13) implies $\overline{T_{u_\xi} \Sigma_{\mu_\xi}}^{\|\cdot\|} = \overline{V^\perp \cap E}^{\|\cdot\|} = V^\perp$ meaning that

$$\hat{E} = \overline{T_{u_\xi} \Sigma_{\mu_\xi}}^{\|\cdot\|} \oplus_\perp V. \quad (6.15)$$

From (6.14) one concludes

$$\mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi)) = (\mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi)) \cap T_{u_\xi} \Sigma_{\mu_\xi}) \oplus_\perp V,$$

and hence

$$\hat{E} = \overline{\mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi)) \cap T_{u_\xi} \Sigma_{\mu_\xi}}^{\|\cdot\|} \oplus_\perp V$$

Comparing this to (6.15), one concludes

$$\overline{T_{u_\xi} \Sigma_{\mu_\xi}}^{\|\cdot\|} = \overline{\mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi)) \cap T_{u_\xi} \Sigma_{\mu_\xi}}^{\|\cdot\|}. \quad (6.16)$$

In other words, $\mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi)) \cap T_{u_\xi} \Sigma_{\mu_\xi}$ is dense in $\overline{T_{u_\xi} \Sigma_{\mu_\xi}}^{\|\cdot\|}$. This proves (ii).

(iii) Note that $E_0 \subset \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi)) \cap T_{u_\xi} \Sigma_{\mu_\xi} \subset \overline{T_{u_\xi} \Sigma_{\mu_\xi}}^{\|\cdot\|}$. Then we can, in view of part (ii) of the Lemma, apply Lemma 6.5 with $W = E_0$, $\mathcal{K} = \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi)) \cap T_{u_\xi} \Sigma_{\mu_\xi}$, and $\mathcal{H} = \overline{T_{u_\xi} \Sigma_{\mu_\xi}}^{\|\cdot\|}$ to obtain:

$$E_0^{\perp\sigma} = \overline{E_0^{\perp\sigma} \cap \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi)) \cap T_{u_\xi} \Sigma_{\mu_\xi}}^{\|\cdot\|} = \overline{E_0^{\perp\sigma} \cap \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi)) \cap T_{u_\xi} \Sigma_{\mu_\xi}}^{\|\cdot\|} = \hat{\mathcal{Y}},$$

where $E_0^{\perp\sigma}$ is defined in (6.12). Since

$$E_0^{\perp\sigma} = E_0^{\perp\sigma} \oplus_\perp V,$$

part (iii) follows.

(iv) Let $u \in \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi)) \cap E_0^{\perp\sigma}$. Since $E_0^{\perp\sigma} = \hat{\mathcal{Y}} \oplus_\perp V$, $u = Pu + v$ with $Pu \in \hat{\mathcal{Y}}$ and $v \in V$. Moreover, since $u \in \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi))$ and $V \subset \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi))$, it follows $Pu \in \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi)) \cap \hat{\mathcal{Y}}$. To conclude, we observe that

$$\mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi)) \cap \hat{\mathcal{Y}} = \mathcal{Y}.$$

Indeed, using (6.10) and (6.13),

$$\begin{aligned} \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi)) \cap \hat{\mathcal{Y}} &= \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi)) \cap E_0^{\perp\sigma} \cap V^\perp = \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi)) \cap E_0^{\perp\sigma} \cap V^\perp \cap E \\ &= \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi)) \cap E_0^{\perp\sigma} \cap T_{u_\xi} \Sigma_{\mu_\xi} = \mathcal{Y}. \end{aligned}$$

Finally, $Pu \in \mathcal{Y}$. □

We can now proceed with the proof of Theorem 6.3.

Proof of Theorem 6.3. Let $n_- = n(D_{u_\xi}^2 \mathcal{L}_\xi)$. As in the proof of Theorem 3.1, $\mathcal{X}_- = \text{span} \{ \eta_1 \cdot \nabla_\xi u_\xi, \dots, \eta_{n_-} \cdot \nabla_\xi u_\xi \}$ is a maximally negative definite subspace for $D_{u_\xi}^2 \mathcal{L}_\xi$, $\mathcal{X}_- \cap T_{u_\xi} \Sigma_{\mu_\xi} = \{0\}$, and $T_{u_\xi} \Sigma_{\mu_\xi}$ is a positive subspace for $D_{u_\xi}^2 \mathcal{L}_\xi$. Furthermore, note that by hypothesis $\text{Ker } D_{u_\xi}^2 \mathcal{L}_\xi = T_{u_\xi} \mathcal{O}_{u_\xi} \subset T_{u_\xi} \Sigma_{\mu_\xi}$. Hence, by Lemma 4.3 (ii), $T_{u_\xi} \Sigma_{\mu_\xi} \cap (T_{u_\xi} \mathcal{O}_{u_\xi})^\perp$ is a positive definite subspace for $D_{u_\xi}^2 \mathcal{L}_\xi$, meaning

$$D_{u_\xi}^2 \mathcal{L}_\xi(v, v) > 0, \quad \forall v \in T_{u_\xi} \Sigma_{\mu_\xi} \cap (T_{u_\xi} \mathcal{O}_{u_\xi})^\perp, v \neq 0. \quad (6.17)$$

We note for further reference that, by Lemma 6.4, $\text{Ker } D_{u_\xi}^2 \mathcal{L}_\xi = \text{Ker } \nabla^2 \mathcal{L}_\xi(u_\xi)$, so that $T_{u_\xi} \mathcal{O}_{u_\xi} \subset \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi))$.

Recall that, since E_0 is the kernel of $\nabla^2 \mathcal{L}_\xi(u_\xi)$, $E_0^{\perp\sigma}$ is the spectral space associated to \mathbb{R}^* , and hence invariant under $\nabla^2 \mathcal{L}_\xi(u_\xi)$. Now, let P be the projection on $\hat{\mathcal{Y}}$ and consider the following decomposition of the operator $\nabla^2 \mathcal{L}_\xi(u_\xi)$ on

$$E_0^\perp = \hat{\mathcal{Y}} \oplus_\perp V,$$

$$\begin{aligned} \nabla^2 \mathcal{L}_\xi(u_\xi)|_{E_0^\perp} &= P\nabla^2 \mathcal{L}_\xi(u_\xi)P + P\nabla^2 \mathcal{L}_\xi(u_\xi)(I_{E_0^\perp} - P) \\ &\quad + (I_{E_0^\perp} - P)\nabla^2 \mathcal{L}_\xi(u_\xi)P + (I_{E_0^\perp} - P)\nabla^2 \mathcal{L}_\xi(u_\xi)(I_{E_0^\perp} - P). \end{aligned}$$

We claim that $P\nabla^2 \mathcal{L}_\xi(u_\xi)P$ is a self-adjoint operator on E_0^\perp with domain $E_0^\perp \cap \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi))$. Indeed, since V is finite dimensional, we can easily show that $P\nabla^2 \mathcal{L}_\xi(u_\xi)(I_{E_0^\perp} - P) + (I_{E_0^\perp} - P)\nabla^2 \mathcal{L}_\xi(u_\xi)P$ and $(I_{E_0^\perp} - P)\nabla^2 \mathcal{L}_\xi(u_\xi)(I_{E_0^\perp} - P)$ are bounded self-adjoint operators on E_0^\perp . Hence $P\nabla^2 \mathcal{L}_\xi(u_\xi)P$ is the sum of a self-adjoint operator with domain $E_0^\perp \cap \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi))$ and a bounded operator on E_0^\perp and, by the Kato-Rellich theorem, it is self-adjoint on E_0^\perp with domain $E_0^\perp \cap \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi))$. In particular, $\nabla^2 \mathcal{L}_\xi(u_\xi) = P\nabla^2 \mathcal{L}_\xi(u_\xi)P + K$ with K a finite rank operator and $\sigma_{\text{ess}}(P\nabla^2 \mathcal{L}_\xi(u_\xi)P) = \sigma_{\text{ess}}(\nabla^2 \mathcal{L}_\xi(u_\xi))$. As a consequence, $0 \notin \sigma_{\text{ess}}(P\nabla^2 \mathcal{L}_\xi(u_\xi)P)$ by hypotheses (iii) and (iv) of Theorem 6.3.

Now note that $\ker P\nabla^2 \mathcal{L}_\xi(u_\xi)P = V \subset E_0^\perp \cap \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi))$. Indeed, let $u \in E_0^\perp \cap \mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi))$ be such that $P\nabla^2 \mathcal{L}_\xi(u_\xi)Pu = 0$. Then

$$D_{u_\xi}^2 \mathcal{L}_\xi(Pu, Pu) = \langle u, P\nabla^2 \mathcal{L}_\xi(u_\xi)Pu \rangle = 0$$

with $Pu \in \mathcal{Y}$ by Lemma 6.6. Since $D_{u_\xi}^2 \mathcal{L}_\xi$ is strictly positive on \mathcal{Y} (see (6.17)), it follows that $Pu = 0$, which means that $u \in V$.

We now consider $P\nabla^2 \mathcal{L}_\xi(u_\xi)P$ on \mathcal{Y} . We have just shown that $0 \notin \sigma(P\nabla^2 \mathcal{L}_\xi(u_\xi)P) = \sigma_{\text{ess}}(P\nabla^2 \mathcal{L}_\xi(u_\xi)P) \cup \sigma_{\text{d}}(P\nabla^2 \mathcal{L}_\xi(u_\xi)P)$ and that $P\nabla^2 \mathcal{L}_\xi(u_\xi)P$ is strictly positive on \mathcal{Y} (see (6.17)). It therefore has a spectral gap:

$$\tilde{\delta} = \inf_{v \in \mathcal{Y} \setminus \{0\}} \frac{\langle P\nabla^2 \mathcal{L}_\xi(u_\xi)Pv, v \rangle}{\langle v, v \rangle} > 0.$$

Next, using the inequality (6.1), we obtain for all $v \in \mathcal{Y}$

$$\left(1 + \frac{\tilde{\delta}}{C}\right) D_{u_\xi}^2 \mathcal{L}_\xi(v, v) \geq \frac{\epsilon \tilde{\delta}}{C} \|v\|_E^2$$

which implies

$$D_{u_\xi}^2 \mathcal{L}_\xi(v, v) \geq \frac{\epsilon \tilde{\delta}}{\tilde{\delta} + C} \|v\|_E^2.$$

Finally, the density of $\mathcal{D}(\nabla^2 \mathcal{L}_\xi(u_\xi))$ in E for $\|\cdot\|_E$ yields (6.4). \square

7. EXAMPLES

7.1. Plane wave solutions of a system of nonlinear Schrödinger equations.

As an application of our main theorem, we investigate the orbital stability of plane waves of a cubic nonlinear system of coupled nonlinear Schrödinger equations on the one-dimensional torus, defined by

$$\begin{cases} i\partial_t u_1(t, x) + \beta \Delta u_1(t, x) + (\gamma_1 |u_1(t, x)|^2 + \gamma_{12} |u_2(t, x)|^2)u_1(t, x) = 0 \\ i\partial_t u_2(t, x) + \beta \Delta u_2(t, x) + (\gamma_{12} |u_1(t, x)|^2 + \gamma_2 |u_2(t, x)|^2)u_2(t, x) = 0 \\ u(0, x) = u(x) \end{cases} \quad (7.1)$$

with $u(t, x) = \begin{pmatrix} u_1(t, x) \\ u_2(t, x) \end{pmatrix} : \mathbb{R} \times \mathbb{T} \rightarrow \mathbb{C}^2$. Here $\mathbb{T} = \mathbb{T}^1$ is the one-dimensional torus of length $L > 0$. The constants $\gamma_1, \gamma_2, \gamma_{12} \in \mathbb{R}$ and $\beta \in \mathbb{R}_+$ are parameters of the model.

The coupled NLSE (7.1) have been used to model nonlinear wave propagation in a variety of physical systems. In nonlinear optics, they describe light propagation in birefringent fibers [2]. In the study of ocean waves, they have been proposed as a model for the generation of rogue waves in crossing sea states : these are two-component wave systems with different directions of propagation (See [20] and references therein). They also appear in the study of two-component Bose-Einstein condensates ([1, 23]). A central topic in each of these situations is the stability or instability of the plane wave solutions of those equations. Linear instability is in this context referred to as *modulational instability* and was studied for various parameter ranges in [2, 6, 8], among others. We will use the methods exposed in this paper to show that, in the parameter regimes where linear stability can be established, orbital stability also holds.

The four-parameter family of plane waves

$$\tilde{u}_\xi(t, x) = \begin{pmatrix} \alpha_1 e^{ik_1 \cdot x} e^{-i\xi_1 t} \\ \alpha_2 e^{ik_2 \cdot x} e^{-i\xi_2 t} \end{pmatrix} \quad (7.2)$$

with $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, $(k_1, k_2) \in \frac{2\pi}{L}\mathbb{Z}^2$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ such that

$$\begin{cases} \xi_1 = \beta k_1^2 - (\gamma_1 \alpha_1^2 + \gamma_{12} \alpha_2^2) \\ \xi_2 = \beta k_2^2 - (\gamma_{12} \alpha_1^2 + \gamma_2 \alpha_2^2). \end{cases} \quad (7.3)$$

are solution to the equation (7.1) and we are interested in study their orbital stability. Using Galilean invariance of the equation (see [7]), the stability of these plane waves is seen to be equivalent to that of

$$\tilde{u}_\xi(t, x) = \begin{pmatrix} \alpha_1 e^{ikx} e^{-i\xi_1 t} \\ \alpha_2 e^{-ikx} e^{-i\xi_2 t} \end{pmatrix} \quad (7.4)$$

with $k = k_1 = -k_2$. Furthermore, we can easily remark that $\tilde{u}_\xi(t, x)$ can be written in the form

$$\tilde{u}_\xi(t, x) = \begin{pmatrix} e^{ikx} & 0 \\ 0 & e^{-ikx} \end{pmatrix} u_\xi(t, x)$$

with

$$u_\xi(t, x) = \begin{pmatrix} \alpha_1 e^{-i\xi_1 t} \\ \alpha_2 e^{-i\xi_2 t} \end{pmatrix} \quad (7.5)$$

a solution to the system of coupled nonlinear Schrödinger equations

$$\begin{cases} i\partial_t u_1 + \beta \Delta u_1 + 2\beta ik \nabla u_1 + (\gamma_1 |u_1|^2 + \gamma_{12} |u_2|^2) u_1 - \beta k^2 u_1 = 0 \\ i\partial_t u_2 + \beta \Delta u_2 - 2\beta ik \nabla u_2 + (\gamma_{12} |u_1|^2 + \gamma_2 |u_2|^2) u_2 - \beta k^2 u_2 = 0 \\ u(0, x) = u(x) \end{cases} \quad (7.6)$$

It is easy to show that the Cauchy problem (7.6) is globally well-posed in $H^1(\mathbb{T}, \mathbb{C}^2)$ (since we consider here only the dimension $d = 1$).

Let $E = H^1(\mathbb{T}, \mathbb{C}^2)$ be viewed as a real Banach space with the usual H^1 -norm and let $\langle \cdot, \cdot \rangle$ be the scalar product on E defined by

$$\begin{aligned} \langle u, v \rangle &= \operatorname{Re} \int_0^L (u_1(x)\bar{v}_1(x) + u_2(x)\bar{v}_2(x)) \, dx \\ &= \int_0^L (\operatorname{Re}(u_1(x))\operatorname{Re}(v_1(x)) + \operatorname{Im}(u_1(x))\operatorname{Im}(v_1(x))) \, dx \end{aligned} \quad (7.7)$$

$$+ \int_0^L (\operatorname{Re}(u_2(x))\operatorname{Re}(v_2(x)) + \operatorname{Im}(u_2(x))\operatorname{Im}(v_2(x))) \, dx. \quad (7.8)$$

This scalar product is clearly continuous with respect to the H^1 -norm. In this case we find that $\hat{E} = L^2(\mathbb{T}, \mathbb{C}^2)$, viewed as a real Hilbert space.

Now, equation (7.6) is the Hamiltonian differential equation associated to the function H defined by

$$\begin{aligned} H(u) &= \frac{\beta}{2} \int_0^L (|\nabla + ik u_1(x)|^2 + |\nabla - ik u_2(x)|^2) \, dx \\ &\quad - \frac{1}{4} \int_0^L (\gamma_1 |u_1(x)|^4 + 2\gamma_{12} |u_1(x)|^2 |u_2(x)|^2 + \gamma_2 |u_2(x)|^4) \, dx. \end{aligned} \quad (7.9)$$

Next, let $G = \mathbb{R} \times \mathbb{R}$ and define its action on E via

$$\forall u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in H^1(\mathbb{T}, \mathbb{C}^2), \quad (\Phi_{\gamma_1, \gamma_2}(u))(x) = \begin{pmatrix} e^{-i\gamma_1} u_1(x) \\ e^{-i\gamma_2} u_2(x) \end{pmatrix}. \quad (7.10)$$

The group G is an invariance group for the dynamics and the quantities

$$F_1(u) = \frac{1}{2} \int_0^L |u_1(x)|^2 \, dx \quad (7.11)$$

$$F_2(u) = \frac{1}{2} \int_0^L |u_2(x)|^2 \, dx \quad (7.12)$$

are the corresponding constants of the motion.

The two-parameter family of plane waves

$$u_\xi(x) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \quad (7.13)$$

with $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}$ such that

$$\begin{cases} \xi_1 = \beta k^2 - (\gamma_1 \alpha_1^2 + \gamma_{12} \alpha_2^2) \\ \xi_2 = \beta k^2 - (\gamma_{12} \alpha_1^2 + \gamma_2 \alpha_2^2). \end{cases} \quad (7.14)$$

are solutions to the stationary equation (2.9). As a consequence, u_ξ is G_{μ_ξ} -relative equilibria of (7.6) with $\mu_\xi = F(u_\xi) = \frac{L}{2} \begin{pmatrix} \alpha_1^2 \\ \alpha_2^2 \end{pmatrix}$.

The G_{μ_ξ} -orbit of the initial condition $u_\xi(x)$ is given by

$$\mathcal{O}_{u_\xi} = \left\{ \begin{pmatrix} \alpha_1 e^{-i\gamma_1} \\ \alpha_2 e^{-i\gamma_2} \end{pmatrix}, (\gamma_1, \gamma_2) \in G_{\mu_\xi} \right\}. \quad (7.15)$$

Our goal is to investigate the orbital stability of these plane wave solutions by applying Theorem 6.3. From now, assume that $\gamma_1 \gamma_2 \neq \gamma_{12}^2$. This is the necessary

and sufficient condition for the map

$$\mu = \left(\frac{L}{2} |\alpha_1|^2, \frac{L}{2} |\alpha_2|^2 \right) \rightarrow (\xi_1, \xi_2)$$

defined in (7.14) to be invertible. Its inverse is \hat{F} , which is a diffeomorphism. Note that this condition corresponds to the case in which the system is not completely integrable.

By definition (3.4) and by using the dispersion relation (7.14), we have

$$\begin{aligned} W(\xi) &= H(u_\xi) - \xi_1 F_1(u_\xi) - \xi_2 F_2(u_\xi) \\ &= -\frac{L}{4} (\alpha_1^2 (\xi_1 - \beta k^2) + \alpha_2^2 (\xi_2 - \beta k^2)) \\ &= \frac{L}{4(\gamma_1 \gamma_2 - \gamma_{12}^2)} [\gamma_2 (\xi_1 - \beta k^2)^2 - 2\gamma_{12} (\xi_1 - \beta k^2) (\xi_2 - \beta k^2) + \gamma_1 (\xi_2 - \beta k^2)^2] \end{aligned} \quad (7.16)$$

As a consequence,

$$D_\xi^2 W = \frac{L}{2(\gamma_1 \gamma_2 - \gamma_{12}^2)} \begin{pmatrix} \gamma_2 & -\gamma_{12} \\ -\gamma_{12} & \gamma_1 \end{pmatrix} \quad (7.17)$$

It is clear that $D_\xi^2 W$ is non-degenerate. Moreover a straightforward calculation shows that

- (1) If $\gamma_1 \gamma_2 - \gamma_{12}^2 > 0$ and $\min(\gamma_1, \gamma_2) > 0$, then $p(D_\xi^2 W) = 2$;
- (2) If $\gamma_1 \gamma_2 - \gamma_{12}^2 > 0$ and $\max(\gamma_1, \gamma_2) < 0$, then $p(D_\xi^2 W) = 0$;
- (3) If $\gamma_1 \gamma_2 - \gamma_{12}^2 < 0$, then $p(D_\xi^2 W) = 1$.

Next, we have to compute $D_{u_\xi}^2 \mathcal{L}_\xi(v, v)$ with $\mathcal{L}_\xi(u) = H(u) - \xi_1 F_1(u) - \xi_2 F_2(u)$. A straightforward calculation gives

$$\begin{aligned} D_{u_\xi}^2 \mathcal{L}_\xi(v, v) &= \beta \int_0^L (-\Delta \operatorname{Re}(v_1) \operatorname{Re}(v_1) - \Delta \operatorname{Re}(v_2) \operatorname{Re}(v_2)) \, dx \\ &\quad + \beta \int_0^L (-\Delta \operatorname{Im}(v_1) \operatorname{Im}(v_1) - \Delta \operatorname{Im}(v_2) \operatorname{Im}(v_2)) \, dx \\ &\quad + \beta \int_0^L (-2k \nabla \operatorname{Re}(v_1) \operatorname{Im}(v_1) + 2k \operatorname{Re}(v_1) \nabla \operatorname{Im}(v_1)) \, dx \\ &\quad + \beta \int_0^L (2k \nabla \operatorname{Re}(v_2) \operatorname{Im}(v_2) - 2k \operatorname{Re}(v_2) \nabla \operatorname{Im}(v_2)) \, dx \\ &\quad - 2 \int_0^L (\gamma_1 \alpha_1^2 \operatorname{Re}(v_1)^2 + 2\gamma_{12} \alpha_1 \alpha_2 \operatorname{Re}(v_1) \operatorname{Re}(v_2) + \gamma_2 \alpha_2^2 \operatorname{Re}(v_2)^2) \, dx \\ &= \langle \nabla^2 \mathcal{L}_\xi(u_\xi) v, v \rangle \end{aligned} \quad (7.18)$$

with

$$\nabla^2 \mathcal{L}_\xi(u_\xi) = \begin{pmatrix} -\beta \Delta - 2\gamma_1 \alpha_1^2 & -2\gamma_{12} \alpha_1 \alpha_2 & 2\beta k \nabla & 0 \\ -2\gamma_{12} \alpha_1 \alpha_2 & -\beta \Delta - 2\gamma_2 \alpha_2^2 & 0 & -2\beta k \nabla \\ -2\beta k \nabla & 0 & -\beta \Delta & 0 \\ 0 & 2\beta k \nabla & 0 & -\beta \Delta \end{pmatrix}. \quad (7.19)$$

In particular, in this functional space setting the hypotheses of Lemma 6.1 are clearly satisfied by $D_{u_\xi}^2 \mathcal{L}_\xi(\cdot, \cdot)$.

7.1.1. *Case $k = 0$.* If $k = 0$, that is u_1 and u_2 have the same wave number, by using Fourier series it is easy to show that the eigenvalues of $\nabla^2 \mathcal{L}_\xi(u_\xi)$ are of the form

$$\rho_n = \beta \left(\frac{2\pi}{L} n \right)^2, \quad n \in \mathbb{N} \quad (7.20)$$

with purely imaginary eigenvectors, and for all $n \in \mathbb{N}$,

$$\lambda_{\pm, n} = \beta \left(\frac{2\pi}{L} n \right)^2 - (\gamma_1 \alpha_1^2 + \gamma_2 \alpha_2^2) \pm \sqrt{(\gamma_1 \alpha_1^2 + \gamma_2 \alpha_2^2)^2 - 4\alpha_1^2 \alpha_2^2 (\gamma_1 \gamma_2 - \gamma_{12}^2)} \quad (7.21)$$

with real eigenvectors. By analyzing the sign of the eigenvalues $\lambda_{\pm, 0}$, we obtain the following situation

- (1) If $\gamma_1 \gamma_2 - \gamma_{12}^2 > 0$ and $\min(\gamma_1, \gamma_2) > 0$, then $n(D_{u_\xi}^2 \mathcal{L}_\xi) \geq 2$ since $\lambda_{+, 0}$ and $\lambda_{-, 0}$ are both negative.
- (2) If $\gamma_1 \gamma_2 - \gamma_{12}^2 > 0$ and $\max(\gamma_1, \gamma_2) < 0$, then $n(D_{u_\xi}^2 \mathcal{L}_\xi) = 0$ since $\lambda_{+, 0}$ and $\lambda_{-, 0}$ are both positive.
- (3) If $\gamma_1 \gamma_2 - \gamma_{12}^2 < 0$, then $n(D_{u_\xi}^2 \mathcal{L}_\xi) \geq 1$ since $\lambda_{-, 0} < 0$ and $\lambda_{+, 0} > 0$. As a consequence $\lambda_{+, n} > 0$ for all $n \in \mathbb{N}$.

So if we assume that

$$\beta \left(\frac{2\pi}{L} \right)^2 - (\gamma_1 \alpha_1^2 + \gamma_2 \alpha_2^2) - \sqrt{(\gamma_1 \alpha_1^2 + \gamma_2 \alpha_2^2)^2 - 4\alpha_1^2 \alpha_2^2 (\gamma_1 \gamma_2 - \gamma_{12}^2)} > 0 \quad (7.22)$$

which correspond to $\lambda_{-, 1} > 0$ we can conclude that

$$p(D_\xi^2 W) = n(D_{u_\xi}^2 \mathcal{L}_\xi).$$

Moreover, it is clear that

$$\inf(\sigma(\nabla^2 \mathcal{L}_\xi(u_\xi)) \cap (0, +\infty)) > 0.$$

Finally, since ρ_0 is the unique eigenvalue equal to 0, we have that

$$\ker(D_{u_\xi}^2 \mathcal{L}_\xi) = \ker(\nabla^2 \mathcal{L}_\xi(u_\xi)) = \text{span} \left\{ \begin{pmatrix} i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ i \end{pmatrix} \right\} = T_{u_\xi} \mathcal{O}_{u_\xi}.$$

Hence Theorem 6.3 applies and, together with Proposition 2.2, leads to the following result.

Proposition 7.1. *Let $k = 0$, $\alpha_1, \alpha_2 \in \mathbb{R}^*$ and $\gamma_1, \gamma_2, \gamma_{12} \in \mathbb{R}$ such that $\gamma_1 \gamma_2 \neq \gamma_{12}^2$. If*

$$\beta \left(\frac{2\pi}{L} \right)^2 - (\gamma_1 \alpha_1^2 + \gamma_2 \alpha_2^2) - \sqrt{(\gamma_1 \alpha_1^2 + \gamma_2 \alpha_2^2)^2 - 4\alpha_1^2 \alpha_2^2 (\gamma_1 \gamma_2 - \gamma_{12}^2)} > 0 \quad (7.23)$$

then $u_\xi = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ is an orbitally stable G_{μ_ξ} -relative equilibrium.

We remark that condition (7.23) is a necessary and sufficient condition for linear stability. To see this, note that the linearization of (7.6) around $u_\xi = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ is given by the system

$$\begin{cases} i\partial_t v_1 + \beta \Delta v_1 + 2\beta i k \nabla v_1 + (\gamma_1 \alpha_1^2 (v_1 + \bar{v}_1) + \gamma_{12} \alpha_1 \alpha_2 (v_2 + \bar{v}_2)) = 0 \\ i\partial_t v_2 + \beta \Delta v_2 - 2\beta i k \nabla v_2 + (\gamma_{12} \alpha_1 \alpha_2 (v_1 + \bar{v}_1) + \gamma_2 \alpha_2^2 (v_2 + \bar{v}_2)) = 0 \end{cases} \quad (7.24)$$

that can be written as

$$\partial_t \begin{pmatrix} \operatorname{Re}(v_1) \\ \operatorname{Re}(v_2) \\ \operatorname{Im}(v_1) \\ \operatorname{Im}(v_2) \end{pmatrix} = \mathbb{L} \begin{pmatrix} \operatorname{Re}(v_1) \\ \operatorname{Re}(v_2) \\ \operatorname{Im}(v_1) \\ \operatorname{Im}(v_2) \end{pmatrix}$$

with

$$\mathbb{L} = \begin{pmatrix} -2\beta k \nabla & 0 & -\beta \Delta & 0 \\ 0 & 2\beta k \nabla & 0 & -\beta \Delta \\ \beta \Delta + 2\gamma_1 \alpha_1^2 & 2\gamma_{12} \alpha_1 \alpha_2 & -2\beta k \nabla & 0 \\ 2\gamma_{12} \alpha_1 \alpha_2 & \beta \Delta + 2\gamma_2 \alpha_2^2 & 0 & 2\beta k \nabla \end{pmatrix}. \quad (7.25)$$

A solution to (7.6) is said to be linearly stable if all the eigenvalues of \mathbb{L} are purely imaginary. By using Fourier series, the eigenvalues of (7.25) can be seen to be the zeros of the characteristic polynomial

$$\begin{aligned} P_n(\lambda) &= \lambda^4 - 2\lambda^2 \beta n_L^2 (-\beta n_L^2 + (\gamma_1 \alpha_1^2 + \gamma_2 \alpha_2^2) - 4\beta k^2) + i\lambda 8\beta n_L^2 \beta k n_L (\gamma_1 \alpha_1^2 - \gamma_2 \alpha_2^2) \\ &\quad + (\beta n_L^2)^3 (\beta n_L^2 - 2(\gamma_1 \alpha_1^2 + \gamma_2 \alpha_2^2)) + 4(\beta n_L^2)^2 \alpha_1^2 \alpha_2^2 (\gamma_1 \gamma_2 - \gamma_{12}^2) \\ &\quad + 8\beta n_L^2 \beta^2 k^2 n_L^2 (-\beta n_L^2 + (\gamma_1 \alpha_1^2 + \gamma_2 \alpha_2^2) + 2\beta k^2) \end{aligned}$$

with $n_L = (\frac{2\pi}{L}n)$. So, whenever $k = 0$, $P_n(\lambda)$ reduces to

$$\begin{aligned} P_n(\lambda) &= \lambda^4 - 2\lambda^2 \beta n_L^2 (-\beta n_L^2 + (\gamma_1 \alpha_1^2 + \gamma_2 \alpha_2^2)) \\ &\quad + (\beta n_L^2)^3 (\beta n_L^2 - 2(\gamma_1 \alpha_1^2 + \gamma_2 \alpha_2^2)) + 4(\beta n_L^2)^2 \alpha_1^2 \alpha_2^2 (\gamma_1 \gamma_2 - \gamma_{12}^2). \end{aligned}$$

and, for all $n \in \mathbb{N}$ the eigenvalues of \mathbb{L} are

$$\tilde{\lambda}_{\pm, n}^2 = \beta n_L^2 \left(-\beta n_L^2 + (\gamma_1 \alpha_1^2 + \gamma_2 \alpha_2^2) \pm \sqrt{(\gamma_1 \alpha_1^2 - \gamma_2 \alpha_2^2)^2 + 4\alpha_1^2 \alpha_2^2 \gamma_{12}^2} \right).$$

Now it is clear that $\tilde{\lambda}_{+, 1}^2 < 0$ if and only if condition (7.23) holds. In that case, for all $n \in \mathbb{N}^*$, $\tilde{\lambda}_{+, n}^2 \leq \tilde{\lambda}_{+, 1}^2$. Moreover, $\tilde{\lambda}_{-, n}^2 \leq \tilde{\lambda}_{+, n}^2$ for all $n \in \mathbb{N}^*$. As a consequence, all the eigenvalues of \mathbb{L} are purely imaginary and the corresponding plane wave is linearly stable if and only if (7.23) holds.

7.1.2. *Case $k \neq 0$.* If $k \neq 0$, using Fourier series, we can show that the eigenvalues of $\nabla^2 \mathcal{L}_\xi(u_\xi)$, for all $n \in \mathbb{N}$, are of the form

$$\lambda_{\pm, n}^+ = \beta \left(\frac{2\pi}{L}n \right)^2 + \frac{1}{2} \left(C_+ \pm \sqrt{C_+^2 + 16\beta^2 k^2 n_L^2} \right) \quad (7.26)$$

$$\lambda_{\pm, n}^- = \beta \left(\frac{2\pi}{L}n \right)^2 + \frac{1}{2} \left(C_- \pm \sqrt{C_-^2 + 16\beta^2 k^2 n_L^2} \right) \quad (7.27)$$

with

$$\begin{aligned} C_+ &= -(\gamma_1 \alpha_1^2 + \gamma_2 \alpha_2^2) + \sqrt{(\gamma_1 \alpha_1^2 + \gamma_2 \alpha_2^2)^2 - 4\alpha_1^2 \alpha_2^2 (\gamma_1 \gamma_2 - \gamma_{12}^2)}, \\ C_- &= -(\gamma_1 \alpha_1^2 + \gamma_2 \alpha_2^2) - \sqrt{(\gamma_1 \alpha_1^2 + \gamma_2 \alpha_2^2)^2 - 4\alpha_1^2 \alpha_2^2 (\gamma_1 \gamma_2 - \gamma_{12}^2)}. \end{aligned}$$

As before, by analyzing the sign of the eigenvalues for $n = 0$, we obtain the following situation

- (1) If $\gamma_1 \gamma_2 - \gamma_{12}^2 > 0$ and $\min(\gamma_1, \gamma_2) > 0$, then $\lambda_{-, 0}^+$ and $\lambda_{-, 0}^-$ are both negative and $n(D_{u_\xi}^2 \mathcal{L}_\xi) \geq 2$.

- (2) If $\gamma_1\gamma_2 - \gamma_{12}^2 > 0$ and $\max(\gamma_1, \gamma_2) < 0$, then $\lambda_{+,0}^+$ and $\lambda_{+,0}^-$ are both positive. This implies $\lambda_{+,n}^+ > 0$ and $\lambda_{+,n}^- > 0$ for all $n \in \mathbb{N}$.
- (3) If $\gamma_1\gamma_2 - \gamma_{12}^2 < 0$, then $\lambda_{-,0}^- < 0$, $\lambda_{+,0}^+ > 0$ and $n(D_{u_\xi}^2 \mathcal{L}_\xi) \geq 1$. As a consequence $\lambda_{+,n}^+ > 0$ for all $n \in \mathbb{N}$.

In all the cases, the two remaining eigenvalues are both 0 with purely imaginary eigenvectors.

Next, a straightforward calculation shows that if we assume

$$C_\pm^2 + 16\beta^2 k^2 \left(\frac{2\pi}{L}\right)^2 > 16\beta^2 k^4 \quad (7.28)$$

then $\lambda_{-,n}^\pm$ is increasing as a function of n for all $n \in \mathbb{N}^*$. Hence, it enough to suppose that $\lambda_{-,1}^\pm > 0$, that is

$$\beta \left(\frac{2\pi}{L}\right)^2 + C_\pm > 4\beta k^2, \quad (7.29)$$

to conclude that

$$p(D_\xi^2 W) = n(D_{u_\xi}^2 \mathcal{L}_\xi).$$

Note that condition (7.29) implies condition (7.28). Moreover, since $C_+ \geq C_-$, it is enough to assume that

$$\beta \left(\frac{2\pi}{L}\right)^2 + C_- > 4\beta k^2.$$

Moreover, as before, it is clear that

$$\inf(\sigma(\nabla^2 \mathcal{L}_\xi(u_\xi)) \cap (0, +\infty)) > 0.$$

and

$$\ker(D_{u_\xi}^2 \mathcal{L}_\xi) = \ker(\nabla^2 \mathcal{L}_\xi(u_\xi)) = \text{span} \left\{ \begin{pmatrix} i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ i \end{pmatrix} \right\} = T_{u_\xi} \mathcal{O}_{u_\xi}.$$

Hence Theorem 6.3 applies and, together with Proposition 2.2, leads to the following result.

Proposition 7.2. *Let $k \in \frac{2\pi}{L}\mathbb{Z}^*$, $\alpha_1, \alpha_2 \in \mathbb{R}^*$ and $\gamma_1, \gamma_2 \in \mathbb{R}, \gamma_{12} \in \mathbb{R}^*$ such that $\gamma_1\gamma_2 \neq \gamma_{12}^2$. If*

$$\beta \left(\frac{2\pi}{L}\right)^2 + C_- > 4\beta k^2 \quad (7.30)$$

then $u_\xi = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ is an orbitally stable G_{μ_ξ} -relative equilibrium.

In the particular case $\gamma_1\alpha_1^2 = \gamma_2\alpha_2^2$ which is a generalization of the set of parameters treated in [6], we can show that condition (7.30) is a necessary and sufficient condition for linear stability.

Indeed, in this case, $P_n(\lambda)$ the characteristic polynomial of (7.25) reduces to

$$\begin{aligned} P_n(\lambda) &= \lambda^4 - 2\lambda^2 \beta n_L^2 (-\beta n_L^2 + (\gamma_1 \alpha_1^2 + \gamma_2 \alpha_2^2) - 4\beta k^2) \\ &\quad + (\beta n_L^2)^3 (\beta n_L^2 - 2(\gamma_1 \alpha_1^2 + \gamma_2 \alpha_2^2)) + 4(\beta n_L^2)^2 \alpha_1^2 \alpha_2^2 (\gamma_1 \gamma_2 - \gamma_{12}^2) \\ &\quad + 8(\beta n_L^2)^2 \beta k^2 (-\beta n_L^2 + (\gamma_1 \alpha_1^2 + \gamma_2 \alpha_2^2) + 2\beta k^2) \end{aligned}$$

and, for all $n \in \mathbb{N}$, the eigenvalues are given by

$$\begin{aligned} \tilde{\lambda}_{\pm,n}^2 &= \beta n_L^2 \left[-\beta n_L^2 + (\gamma_1 \alpha_1^2 + \gamma_2 \alpha_2^2) - 4\beta k^2 \right. \\ &\quad \left. \pm \sqrt{4\alpha_1^2 \alpha_2^2 \gamma_{12}^2 + 16\beta k^2 (\beta n_L^2 - (\gamma_1 \alpha_1^2 + \gamma_2 \alpha_2^2))} \right]. \end{aligned}$$

Now, a tedious but straightforward calculation shows that $\tilde{\lambda}_{+,n}^2 < 0$ for all $n \in \mathbb{N}^*$ if and only if condition (7.30) holds. Moreover, $\tilde{\lambda}_{-,n}^2 \leq \tilde{\lambda}_{+,n}^2$ for all $n \in \mathbb{N}^*$. As a consequence, all the eigenvalues of \mathbb{L} are purely imaginary and the corresponding plane wave is linearly stable if and only if (7.30) holds.

7.1.3. Physical interpretation. To sum up, we can conclude that, given $k = k_1 - k_2 \in \frac{2\pi}{L}\mathbb{Z}$, $\alpha_1, \alpha_2 \in \mathbb{R}^*$ and $\gamma_1, \gamma_2, \gamma_{12} \in \mathbb{R}^*$ such that $\gamma_1 \gamma_2 \neq \gamma_{12}^2$, if

$$\beta \left(\frac{2\pi}{L} \right)^2 - (\gamma_1 \alpha_1^2 + \gamma_2 \alpha_2^2) - \sqrt{(\gamma_1 \alpha_1^2 + \gamma_2 \alpha_2^2)^2 - 4\alpha_1^2 \alpha_2^2 (\gamma_1 \gamma_2 - \gamma_{12}^2)} > 4\beta k^2 \quad (7.31)$$

then the plane waves

$$u_\xi(t, x) = \begin{pmatrix} \alpha_1 e^{ik_1 \cdot x} e^{-i\xi_1 t} \\ \alpha_2 e^{ik_2 \cdot x} e^{-i\xi_2 t} \end{pmatrix}$$

are orbitally stable G_{μ_ξ} -relative equilibria. Moreover, for $k = 0$, if this condition is not satisfied the plane wave is unstable (at least linearly). For $k \neq 0$ this remains true whenever $\gamma_1 \alpha_1^2 = \gamma_2 \alpha_2^2$.

We know from [7] that plane waves solutions to a cubic defocusing nonlinear Schrödinger equation on the one-dimensional torus are orbitally stable. This means that whenever, $\gamma_{12} = 0$, $\gamma_1 < 0$ and $\gamma_2 < 0$, all the plane waves of the form

$$u_\xi(t, x) = \begin{pmatrix} \alpha_1 e^{ik_1 \cdot x} e^{-i\xi_1 t} \\ \alpha_2 e^{ik_2 \cdot x} e^{-i\xi_2 t} \end{pmatrix}$$

are orbitally stable. It is natural to ask what happens if $|\gamma_{12}| \neq 0$. We have two different situations : $k = 0$ (*i.e.* the plane waves have the same wave number $k_1 = k_2$) and $k \neq 0$. If $k = 0$ and $\gamma_{12}^2 < \gamma_1 \gamma_2$, which means that the coupling is weak, then $C_- > 0$ and condition (7.31) remains true. This means that the plane waves with $k = 0$ then remain stable. If $k = 0$ and $\gamma_{12}^2 > \gamma_1 \gamma_2$, which means that the coupling is strong, then $C_- < 0$ and condition (7.31) fails at least if L is large enough. Then the plane waves considered become unstable.

In the case $k \neq 0$, note that condition (7.31) can be satisfied only in the case (2) above, namely when $\gamma_1 \gamma_2 - \gamma_{12}^2 > 0$, and $\max(\gamma_1, \gamma_2) < 0$, since otherwise $C_- < 0$. This corresponds to a relatively small perturbation of two uncoupled defocusing Schrödinger equations with orbitally stable plane wave solutions. Condition (7.31) can then be satisfied for a finite number of values of k , provided C_- is large enough, but it fails for larger ones. The size of C_- depends in particular on the “power” of the plane wave, determined by $|\alpha_1|$ and $|\alpha_2|$. For larger values of k , the plane wave becomes linearly unstable, on the other hand, even at weak coupling. In other words, high k plane waves show modulational instability, even at arbitrarily low γ_{12} .

8. ON THE LINK WITH GRILLAKIS-SHATAH-STRAUSS

8.1. The main coercivity estimate of Grillakis-Shatah-Strauss. We will now compare the results in this paper to the work of [13]. Since in [13] the phase space E on which the dynamics takes place is taken to be a Hilbert space, we place ourselves for this discussion in the Hilbert space setting of Section 3 and consider the situation described by (3.1)-(3.6).

To state the coercivity estimate of [13] which is the analog of our Theorem 3.1, we need some additional notation. We define

$$\tilde{W} : \Omega \cap \mathfrak{g}_\xi \rightarrow \mathcal{L}_\xi(u_\xi) \in \mathbb{R}, \quad (8.1)$$

which is the restriction of the W-function (3.4) to the sub-Lie-algebra \mathfrak{g}_ξ of \mathfrak{g} , defined in (4.5). Also

$$\tilde{\mathcal{O}}_{u_\xi} = \Phi_{G_\xi}(u_\xi), \quad (8.2)$$

is the G_ξ orbit through u_ξ . Since a priori G_ξ differs from G_{μ_ξ} , one should not confuse $\tilde{\mathcal{O}}_{u_\xi}$ with \mathcal{O}_{u_ξ} , which is the G_{μ_ξ} -orbit through u_ξ . We introduce furthermore

$$\tilde{\Sigma}_\xi = \{v \in E \mid \eta \cdot F(v) = \eta \cdot \mu_\xi(u_\xi), \forall \eta \in \mathfrak{g}_\xi\}. \quad (8.3)$$

In other words, $\tilde{\Sigma}_\xi$ is the constraint surface corresponding to the constants of the motion $\eta \cdot F$ for $\eta \in \mathfrak{g}_\xi$. Note that $\Sigma_{\mu_\xi} \subset \tilde{\Sigma}_\xi$. In fact, when the moment map is regular at μ_ξ , then $\tilde{\Sigma}_\xi$ is a submanifold of E of co-dimension $\dim \mathfrak{g}_\xi$ which contains the submanifold Σ_{μ_ξ} , itself of codimension $\dim \mathfrak{g}$. The following theorem, which is the analog of Theorem 3.1, can be inferred from [13].

Theorem 8.1. *Under the previous hypotheses, let $\xi \in \Omega$ and suppose*

- (i) $D_\xi^2 \tilde{W}$ is non-degenerate, i.e. $\text{Ker}(D_\xi^2 \tilde{W}) = \{0\}$,
- (ii) $\text{Ker} D_{u_\xi}^2 \mathcal{L}_\xi = Z_\xi$,
- (iii) $\inf(\sigma(\nabla^2 \mathcal{L}_\xi(u_\xi)) \cap (0, +\infty)) > 0$,
- (iv) $p(D_\xi^2 \tilde{W}) = n(D_{u_\xi}^2 \mathcal{L}_\xi)$.

Then there exists $\delta > 0$ such that

$$\forall v \in T_{u_\xi} \tilde{\Sigma}_\xi \cap \left(T_{u_\xi} \tilde{\mathcal{O}}_{u_\xi}\right)^\perp, \quad D_{u_\xi}^2 \mathcal{L}_\xi(v, v) \geq \delta \|v\|^2. \quad (8.4)$$

It is clear that, when the invariance group G is one-dimensional, i.e. $\dim \mathfrak{g} = 1$, this theorem is identical to Theorem 3.1. Indeed, then $G = G_\xi = G_{\mu_\xi}$ and hence $W = \tilde{W}$ so that both the assumptions and the conclusions of both theorems are identical. This is the situation studied in [12] and [26]. The same conclusions hold true more generally when the group G is abelian, since then again, $G_\xi = G_{\mu_\xi} = G$. In general, however, the groups G_ξ and G_{μ_ξ} may be distinct, and so may therefore be the orbits $\tilde{\mathcal{O}}_{u_\xi}$ and \mathcal{O}_{u_ξ} . Hence, a priori, the two approaches could yield different coercivity estimates and hence different stability results. Their comparison therefore needs to be done with care, a task we turn to in the next subsection.

Remark 8.2. A proof of Theorem 8.1 can be given along the same lines as the proof of Theorem 3.1 and we don't reproduce it here. We point out that in fact only the bound $D_{u_\xi}^2 \mathcal{L}_\xi(v, v) \geq 0$ is shown in [13]; the argument leading from that bound to (8.4) is the same as in the proof of Theorem 3.1 above.

8.2. Comparing Theorem 3.1 to Theorem 8.1. Let us first compare the respective conclusions (3.7) and (8.4) as follows. Writing

$$\mathcal{C}_- = \{u \in E \mid D_{u_\xi}^2 \mathcal{L}(u, u) < 0\} \quad (8.5)$$

for the negative cone of $D_{u_\xi}^2 \mathcal{L}$, we see that they imply that

$$T_{u_\xi} \Sigma_{\mu_\xi} \cap \mathcal{C}_- = \emptyset, \quad \text{respectively} \quad T_{u_\xi} \tilde{\Sigma}_\xi \cap \mathcal{C}_- = \emptyset, \quad (8.6)$$

meaning that $T_{u_\xi} \Sigma$, respectively $T_{u_\xi} \tilde{\Sigma}_\xi$ are non-negative subspaces of E for $D_{u_\xi}^2 \mathcal{L}$. Since $T_{u_\xi} \Sigma_{\mu_\xi} \subset T_{u_\xi} \tilde{\Sigma}_\xi$ the second of these statements implies the first and should in general be harder to obtain. Indeed, the cone \mathcal{C}_- may avoid $T_{u_\xi} \Sigma_{\mu_\xi}$ but have a non-trivial intersection with $T_{u_\xi} \tilde{\Sigma}_\xi$. This is further reflected in the fact that (8.6) implies that

$$\text{codim}(T_{u_\xi} \Sigma_{\mu_\xi}) \geq n(D_{u_\xi}^2 \mathcal{L}_\xi) \quad \text{respectively} \quad \text{codim}(T_{u_\xi} \tilde{\Sigma}_\xi) \geq n(D_{u_\xi}^2 \mathcal{L}_\xi).$$

When u_ξ is a regular relative equilibrium, one has $\dim \mathfrak{g} = \text{codim}(T_{u_\xi} \Sigma_{\mu_\xi}) \geq \dim \mathfrak{g}_{\mu_\xi} = \text{codim}(T_{u_\xi} \tilde{\Sigma}_\xi)$.

To understand how the stronger conclusion comes about, one may note that condition (iv) of Theorem 8.1 has a more limited range of applicability than condition (iv) of Theorem 3.1 since in general

$$p(D_\xi^2 \tilde{W}) \leq p(D_\xi^2 W) \leq n(D_{u_\xi}^2 \mathcal{L}_\xi). \quad (8.7)$$

In particular, condition (iv) of Theorem 8.1 cannot be satisfied when $p(D_\xi^2 \tilde{W}) < p(D_\xi^2 W)$. Below, we will give an example where

$$p(D_\xi^2 \tilde{W}) < p(D_\xi^2 W) = n(D_{u_\xi}^2 \mathcal{L}_\xi),$$

so that Theorem 3.1 applies, but Theorem 8.1 does not.

The following corollary further clarifies the link between the two results.

Corollary 8.3. *Suppose the hypotheses of Theorem 8.1 are satisfied. Then*

$$T_{u_\xi} \tilde{\mathcal{O}}_{u_\xi} = T_{u_\xi} \mathcal{O}_{u_\xi}$$

so that there exists $\delta > 0$ such that

$$\forall v \in T_{u_\xi} \tilde{\Sigma}_\xi \cap (T_{u_\xi} \mathcal{O}_{u_\xi})^\perp, \quad D_{u_\xi}^2 \mathcal{L}_\xi(v, v) \geq \delta \|v\|^2. \quad (8.8)$$

Moreover hypotheses (ii), (iii) and (iv) of Theorem 3.1 are satisfied. One has $\mathfrak{g}_\xi \subset \mathfrak{g}_{\mu_\xi}$ and if, in addition, u_ξ is a regular relative equilibrium, then $\mathfrak{g}_{\mu_\xi} = \mathfrak{g}_\xi$.

We can conclude from the previous discussion and the corollary that, under the non-degeneracy hypothesis $\text{Ker}(D_\xi^2 W) = \{0\}$, Theorem 3.1 provides the desired coercivity estimate (3.7) under weaker conditions than Theorem 8.1. As a result, to find a situation where Theorem 8.1 does apply, whereas Theorem 3.1 does not, one has to suppose $\text{Ker}(D_\xi^2 W) \neq \{0\}$, whereas $\text{Ker}(D_\xi^2 \tilde{W}) = \{0\}$. We did not find an example of such a situation.

Proof. We know from Lemma 4.2 (5) that $\text{Ker}(D_{u_\xi}^2 \mathcal{L}_\xi) \subset T_{u_\xi} \Sigma_{\mu_\xi}$, so that hypothesis (ii) of the theorem implies $T_{u_\xi} \tilde{\mathcal{O}}_{u_\xi} \subset T_{u_\xi} \Sigma_{u_\xi}$. Since $\tilde{\mathcal{O}}_{u_\xi} \cap \Sigma_{\mu_\xi} \subset \Phi_G(u_\xi) \cap \Sigma_{\mu_\xi} = \mathcal{O}_{u_\xi}$, it follows that $T_{u_\xi} \tilde{\mathcal{O}}_{u_\xi} \subset T_{u_\xi} \mathcal{O}_\xi$. Now, we know from Lemma 4.2 (4) that

$T_{u_\xi} \mathcal{O}_{u_\xi} \subset \text{Ker}(D_{u_\xi}^2 \mathcal{L}_\xi | T_{u_\xi} \Sigma_{\mu_\xi})$. So, if $v \in T_{u_\xi} \mathcal{O}_{u_\xi} \subset T_{u_\xi} \Sigma_{\mu_\xi} \subset T_{u_\xi} \tilde{\Sigma}_\xi$, then $D_{u_\xi}^2 \mathcal{L}_\xi(v, v) = 0$. Writing

$$v = v_{\parallel} + v_{\perp}, \quad v_{\parallel} \in T_{u_\xi} \tilde{\mathcal{O}}_{u_\xi}, \quad v_{\perp} \in T_{u_\xi} \tilde{\mathcal{O}}_{u_\xi}^{\perp}$$

we have,

$$\begin{aligned} 0 &= D_{u_\xi}^2 \mathcal{L}(v, v) = D_{u_\xi}^2 \mathcal{L}_\xi(v_{\parallel}, v_{\parallel}) + D_{u_\xi}^2 \mathcal{L}_\xi(v_{\perp}, v_{\perp}) + 2D_{u_\xi}^2 \mathcal{L}_\xi(v_{\parallel}, v_{\perp}) \\ &= D_{u_\xi}^2 \mathcal{L}_\xi(v_{\perp}, v_{\perp}), \end{aligned}$$

since $v_{\parallel} \in \text{Ker}(D_{u_\xi}^2 \mathcal{L}_\xi)$. It follows from (8.4) that $v_{\perp} = 0$ so that $v \in T_{u_\xi} \tilde{\mathcal{O}}_{u_\xi}$. We conclude that $T_{u_\xi} \tilde{\mathcal{O}}_{u_\xi} = T_{u_\xi} \mathcal{O}_{u_\xi}$ and hence (8.8) follows from (8.4). It also follows that hypothesis (ii) of Theorem 3.1 is satisfied. Hypothesis (iii) is the same in both theorems and hypothesis (iv) of Theorem 8.1, together with (8.7), implies hypothesis (iv) of Theorem 3.1.

Now let $\eta \in \mathfrak{g}_\xi$. Since $T_{u_\xi} \tilde{\mathcal{O}}_{u_\xi} = T_{u_\xi} \mathcal{O}_{u_\xi}$ there exists $\eta' \in \mathfrak{g}_{\mu_\xi}$ so that $X_\eta(u_\xi) = X_{\eta'}(u_\xi)$. Hence there exists $\eta'' \in \mathfrak{g}_{u_\xi}$ (which is the Lie-algebra of the isotropy group G_{u_ξ} of u_ξ) so that $\eta = \eta' + \eta''$. Since $\mathfrak{g}_{u_\xi} \subset \mathfrak{g}_{\mu_\xi}$, it follows that $\eta \in \mathfrak{g}_{\mu_\xi}$. Hence $\mathfrak{g}_\xi \subset \mathfrak{g}_{\mu_\xi}$. Finally, recall that $X_{\eta''}(u_\xi) = 0$ is equivalent to $\eta'' \cdot D_{u_\xi} F = 0$ and hence, if u_ξ is a regular relative equilibrium, then $\mathfrak{g}_{u_\xi} = \{0\}$; by the same argument, one then shows that $\mathfrak{g}_{\mu_\xi} \subset \mathfrak{g}_\xi$, so that the result follows. \square

To complete our comparative analysis of those two theorems, we further analyse the conditions on the kernel of $D_{u_\xi}^2 \mathcal{L}_\xi$ they impose. Similarly to the non-degeneracy condition (i), those conditions are also not in a clear logical relation, in particular because they refer to two a priori different subgroups of G , namely G_ξ and G_{μ_ξ} . The following lemma sheds further light on the situation.

Lemma 8.4. *Suppose the hypotheses of Lemma 4.2 are satisfied. Let $\xi \in \Omega$ and suppose $\text{Ker}(D_\xi^2 W) = \{0\}$. Then*

$$\text{Ker}(D_{u_\xi}^2 \mathcal{L}_\xi) = \text{Ker}(D_{u_\xi}^2 \mathcal{L}_\xi | \text{Ker} D_{u_\xi} F). \quad (8.9)$$

In addition, the following two statements are equivalent:

- (i) $\text{Ker}(D_{u_\xi}^2 \mathcal{L}_\xi) = T_{u_\xi} \tilde{\mathcal{O}}_\xi$;
- (ii) $\text{Ker}(D_{u_\xi}^2 \mathcal{L}_\xi) = T_{u_\xi} \mathcal{O}_\xi$ and $\mathfrak{g}_\xi = \mathfrak{g}_{\mu_\xi}$.

The lemma shows that, if u_ξ is a regular and non-degenerate relative equilibrium, then the condition on the kernel of $D_{u_\xi}^2 \mathcal{L}_\xi$ of Theorem 8.1 implies not only the kernel condition in Theorem 3.1, but in addition that $\mathfrak{g}_\xi = \mathfrak{g}_{\mu_\xi}$. This statement is independent of the other spectral conditions of these theorems on $D_{u_\xi}^2 \mathcal{L}_\xi$.

Proof. It follows from $\text{Ker}(D_\xi^2 W) = \{0\}$, together with Lemma 4.2 (1) and Lemma 4.1 that $E = \mathcal{U}_\xi \oplus T_{u_\xi} \Sigma_{\mu_\xi}$. Hence (4.9) implies that

$$\text{Ker}(D_{u_\xi}^2 \mathcal{L}_\xi) = \text{Ker}(D_{u_\xi}^2 \mathcal{L}_\xi | \mathcal{U}_\xi) \oplus \text{Ker}(D_{u_\xi}^2 \mathcal{L}_\xi | \text{Ker} D_{u_\xi} F) \quad (8.10)$$

On the other hand, it follows from (4.8) that

$$\text{Ker}(D_{u_\xi}^2 \mathcal{L}_\xi | \mathcal{U}_\xi) = D_\xi \tilde{u}(\text{Ker}(D_\xi^2 W)) = \{0\},$$

so that the first statement of the Lemma follows.

(i) \Rightarrow (ii) From Lemma 4.2 (4), together with (8.9) and hypothesis (i), we conclude

that

$$T_{u_\xi} \mathcal{O}_{u_\xi} \subset \text{Ker}(D_{u_\xi}^2 \mathcal{L}_\xi | \text{Ker} D_{u_\xi} F) = \text{Ker}(D_{u_\xi}^2 \mathcal{L}_\xi) = T_{u_\xi} \tilde{\mathcal{O}}_{u_\xi} \subset T_{u_\xi} \Sigma_{\mu_\xi}.$$

On the other hand, as in the proof of Corollary 8.3, we have $T_{u_\xi} \tilde{\mathcal{O}}_{u_\xi} \subset T_{u_\xi} \mathcal{O}_{u_\xi}$. Hence $T_{u_\xi} \tilde{\mathcal{O}}_{u_\xi} = T_{u_\xi} \mathcal{O}_{u_\xi}$ and the first statement of (ii) follows. The second statement is now proven as in the proof of Corollary 8.3.

(ii) \Rightarrow (i) This is obvious. \square

8.3. Proving orbital stability. The above coercivity estimate (8.4) (or, equivalently (8.8)) is used in [13] as an essential input to show the G_ξ -orbital stability of u_ξ . Note that this distinguishes their approach from the rest of the literature on orbital stability, including this paper and [7], where instead G_{μ_ξ} -stability is proven. The argument given in [13] leading from the above coercivity estimate to G_ξ -orbital stability of u_ξ is based on an implicit assumption on F , referred to as Hypothesis F in [7]. It was explained in that last paper that it is typically not satisfied when F takes values in \mathbb{R}^m , with $m > 1$. As explained in Section 2 above, it is instead possible to use arguments provided in [7] to prove G_{μ_ξ} -relative stability, using (3.7) (or, a fortiori (8.8)) as a starting point. One may wonder if it would be possible to prove G_ξ -relative stability as well, which is potentially stronger, since according to the corollary it is possible that $\tilde{\mathcal{O}}_{u_\xi} \subset \mathcal{O}_{u_\xi}$ since $\mathfrak{g}_\xi \subset \mathfrak{g}_{\mu_\xi}$. We consider first the case where G_{μ_ξ} is commutative. Then \mathfrak{g}_ξ is commutative and both for G_ξ and G_{μ_ξ} the exponential map is surjective, so that $G_\xi \subset G_{\mu_\xi}$. On the other hand, when G_{μ_ξ} is commutative, the definition of G_ξ implies that $G_{\mu_\xi} \subset G_\xi$, so that then $G_\xi = G_{\mu_\xi}$ and hence $\tilde{\mathcal{O}}_{u_\xi} = \mathcal{O}_{u_\xi}$. So $\tilde{\mathcal{O}}_{u_\xi} \subsetneq \mathcal{O}_{u_\xi}$ is only possible under the hypotheses of Theorem 8.1 provided G_{μ_ξ} is not commutative. In addition, note that if $\tilde{\mathcal{O}}_{u_\xi}$ and \mathcal{O}_{u_ξ} are submanifolds of E , then the corollary furthermore implies they have the same dimension, so that $\tilde{\mathcal{O}}_{u_\xi}$ is an open subset of \mathcal{O}_{u_ξ} . In conclusion, we are not aware of a situation where the conditions of Theorem 8.1 hold, and where $\tilde{\mathcal{O}}_{u_\xi} \subsetneq \mathcal{O}_{u_\xi}$. In addition, should such a situation exist, it is not clear that the arguments provided in the cited works would yield G_ξ -orbital stability.

It remains to see if there exist circumstances where the conditions of Theorem 8.1 are satisfied, while those of Theorem 3.1 are not. Note that, according to the corollary, conditions (ii), (iii) and (iv) of Theorem 3.1 are implied by the hypotheses of Theorem 8.1. Concerning the non-degeneracy conditions (i), it is clear that one has neither that $\text{Ker}(D_\xi^2 W) = \{0\}$ implies $\text{Ker}(D_\xi^2 \tilde{W}) = \{0\}$, nor the opposite. It is therefore possible that the latter holds while the former is violated, but we are not aware of such a situation.

8.4. An example. We end this section with a simple example in $E = \mathbb{R}^6$ where Theorem 3.1 applies, but Theorem 8.1 does not. As already pointed out, G must be non-commutative for this to happen. Consider the $\text{SO}(3)$ -invariant Hamiltonian

$$H_\alpha(q, p) = H_0(q, p) - \alpha F^2(q, p), \quad \text{with} \quad H_0(q, p) = \frac{\|p\|^2}{2} + V(\|q\|), \quad (8.11)$$

and $F(q, p) = q \wedge p$. Note that H_0 is the Hamiltonian of a particle in a central potential V and that the components of the angular momentum vector F generate rotations. Since H_0 Poisson commutes with F^2 and since F^2 generates rotations about $q \wedge p$, it is easy to see that the circular orbits of H_0 are also flow lines of H_α

and that they are relative equilibria. These are the ones whose orbital stability we shall study. Consider for that purpose the stationary equation, with $\xi \in \mathbb{R}^3 \simeq \mathfrak{so}(3)$,

$$D_{u_\xi} H_\alpha - \xi \cdot D_{u_\xi} F = 0,$$

where $u_\xi = (q_\xi, p_\xi)$. A simple computation shows that any solution u_ξ is of the form $u_\xi = (q_\xi, p_\xi) = (\rho_\xi \hat{q}_\xi, \sigma_\xi \hat{p}_\xi)$ with

$$\sigma_\xi^2 = \rho_\xi V'(\rho_\xi), \quad \hat{q}_\xi \cdot \hat{p}_\xi = 0.$$

Here $\rho_\xi > 0, \sigma_\xi > 0$ and we write $\hat{a} = a/\|a\|$ for each $a \in \mathbb{R}^3$. We have

$$\mu_\xi = F(u_\xi) = q_\xi \wedge p_\xi = \rho_\xi \sigma_\xi \hat{q}_\xi \wedge \hat{p}_\xi, \text{ with } \xi = \eta_{\alpha, \xi} \mu_\xi, \quad \eta_{\alpha, \xi} = \frac{1 - 2\alpha\rho_\xi^2}{\rho_\xi^2}.$$

Clearly, in this situation $G_\xi = G_{\mu_\xi} \simeq \text{SO}(2)$. One has

$$\hat{F}(\xi) = \rho_\xi \sigma_\xi \hat{\xi} = \rho_\xi^{3/2} [V'(\rho_\xi)]^{1/2} \hat{\xi}$$

$$\|\xi\| = \rho_\xi \sigma_\xi \frac{|1 - 2\alpha\rho_\xi^2|}{\rho_\xi^2} = \rho_\xi^{3/2} [V'(\rho_\xi)]^{1/2} \frac{|1 - 2\alpha\rho_\xi^2|}{\rho_\xi^2} = \left[\rho_\xi^{-1} V'(\rho_\xi) \right]^{1/2} |1 - 2\alpha\rho_\xi^2|.$$

For illustrative purposes, it is sufficient to consider $V(\rho_\xi) = \frac{1}{2}\omega\rho_\xi^2$. Then it is clear that, provided $2\alpha > \rho_\xi^{-2}$, \hat{F} is a local diffeomorphism and

$$\|\xi\| = \omega^{1/2}(2\alpha\rho_\xi^2 - 1).$$

A simple computation then yields

$$W(\xi) = H_\alpha(u_\xi) - \xi \cdot F(u_\xi) = \alpha\omega\rho_\xi^4 = \frac{1}{4} \frac{\omega}{\alpha} \left(1 + \frac{\|\xi\|}{\sqrt{\omega}} \right)^2,$$

and furthermore that

$$D_\xi^2 W(v, v) = \frac{1}{2\alpha} \left[1 + \frac{\sqrt{\omega}}{\|\xi\|} \right] v^2 - \frac{\sqrt{\omega}}{2\alpha\|\xi\|} (v \cdot \hat{\xi})^2 \geq \frac{v^2}{2\alpha},$$

which is positive definite so that $p(D_\xi^2 W) = 3$. Hence, $n(D^2 \mathcal{L}_\xi) \geq 3$. A further lengthy but straightforward computation shows that

$$n(D^2 \mathcal{L}_\xi) = 3,$$

and that $\text{Ker}(D_{u_\xi}^2 \mathcal{L}_\xi) = T_{u_\xi} \mathcal{O}_{u_\xi}$. Hence Theorem 3.1 implies that the circular orbits are orbitally stable.

On the other hand the assumptions of [13] are too strong to apply in this simple example. Indeed, in [13] the authors consider the Hessian of function W restricted to \mathfrak{g}_ξ . The main hypothesis of their stability theorem is that $p(D_\xi^2 \tilde{W}) = n(D_{u_\xi}^2 \mathcal{L}_{\alpha, \xi})$. In the present situation this condition is not satisfied. Indeed, since $\mathfrak{g}_\xi = \mathfrak{so}(2)$, $p(D_\xi^2 \tilde{W}) \leq 1$. In fact, it is easy to see it is equal to 1. As a consequence, $p(D_\xi^2 \tilde{W}) < n(D_{u_\xi}^2 \mathcal{L}_{\alpha, \xi}) = 3$ and so the hypothesis of Theorem 8.1 are not satisfied.

8.5. Conclusions. In this paper we studied the stability of relative equilibria u_ξ , meaning solutions of (2.9). Those can be classified as follows: *regular* equilibria are those for which $D_{u_\xi}F$ has maximal rank. Relative equilibria that are not regular are called *singular*. *Non-degenerate* relative equilibria are those for which $\text{Ker}(D_\xi^2 W) = \{0\}$. We have seen that non-degenerate equilibria are regular. Our main results, Theorem 3.1 and Theorem 6.3, give sufficient conditions for non-degenerate relative equilibria to be G_{μ_ξ} stable. They are stated in terms of the hessian $D_\xi^2 W$, and as such seem to be novel even in finite dimensional systems, where sufficient conditions for orbital stability are usually given directly in terms of $D_{u_\xi}^2 \mathcal{L}_\xi$ itself. We have further analysed the link between our results and the ones of Grillakis-Shatah-Strauss in [13], thereby clarifying the latter.

It would be of interest to extend our results to degenerate but regular equilibria. Results in that direction can be found in [13]. Singular equilibria are known to be more difficult to deal with, even in finite dimensional systems, where their stability has been studied in [17, 19].

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¹LABORATOIRE PAUL PAINLEVÉ, CNRS, UMR 8524 ET UFR DE MATHÉMATIQUES, UNIVERSITÉ LILLE 1, SCIENCES ET TECHNOLOGIES F-59655 VILLENEUVE D’ASCQ CEDEX, FRANCE.

E-mail address: Stephan.De-Bievre@math.univ-lille1.fr

²EQUIPE-PROJET MEPHYSTO, CENTRE DE RECHERCHE INRIA FUTURS, PARC SCIENTIFIQUE DE LA HAUTE BORNE, 40, AVENUE HALLEY B.P. 70478, F-59658 VILLENEUVE D’ASCQ CEDEX, FRANCE.

³INSTITUT DE MATHÉMATIQUES DE BOURGOGNE (IMB), CNRS, UMR 5584, UNIVERSITÉ BOURGOGNE FRANCHE-COMTÉ, F-21000 DIJON, FRANCE.

E-mail address: simona.rota-nodari@u-bourgogne.fr