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# Impulse output rapid stabilization for heat equations

Kim Dang Phung\*, Gengsheng Wang<sup>†</sup>, Yashan Xu<sup>‡</sup>

## Abstract

In this paper, we study the output rapid stabilization for heat equations with lower terms. Our controls are active in a subdomain and at discrete time points, while our observations are made in another subdomain and at discrete time points which are ahead of control time points. Through studying a kind of minimal norm impulse control problems, we not only build up, for each decay rate, a feedback law, but also provide bounds for feedback laws in terms of decay rates. In the studies of the above-mentioned minimal norm impulse control problems, the unique continuation estimate at one time point, as well as some new observations on it, plays an important role.

**Keywords:** Rapid stabilization, impulse control, heat equation.

**AMS subject classifications:** 34H15, 34K45, 49J20, 93D15.

## 1 Introduction

Let  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 1$ , be a bounded domain, with a  $C^2$  boundary  $\partial\Omega$ . Let  $V$  be a function in  $L^\infty(\Omega)$  with its norm  $\|\cdot\|_\infty$ . Define

$$A \triangleq \Delta - V, \quad \text{with } D(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

Write  $\{e^{tA}, t \geq 0\}$  for the semigroup generated by  $A$  on  $L^2(\Omega)$ . It is well-known that when  $V = 0$ , the semigroup  $\{e^{tA}, t \geq 0\}$  has the exponential decay with the rate  $\alpha_1$ , which is the first eigenvalue of  $-\Delta$  with the homogeneous Dirichlet boundary condition. The aim of this study is to build up, for each  $\gamma > 0$ , an output feedback law  $\mathcal{F}_\gamma$  so that any solution to the closed-loop controlled heat equation, associated with  $A$  and  $\mathcal{F}_\gamma$ , has an exponential decay with the rate  $\gamma$ . Two requirements on this aim are as follows: First, our controls are active in a subdomain  $\omega \subset \Omega$  and at discrete time points  $\tau + nT$  (with  $n = 0, 1, \dots$ ), while our observations are made in another subdomain  $\omega_1 \subset \Omega$

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and at discrete time points  $s + nT$  (with  $n = 0, 1, \dots$ ). (Here and throughout the paper,  $T$ ,  $\tau$  and  $s$  are arbitrarily fixed three time points with  $0 < s < \tau < T$ .) Second, at all control time points, we take the same feedback law. Therefore, the closed-loop controlled equation under consideration reads:

$$\begin{cases} y'(t) - Ay(t) = 0, & t \in \mathbb{R}^+ \setminus (\tau + \bar{\mathbb{N}}T), \\ y(\tau + nT) = y((\tau + nT)_-) + 1_\omega \mathcal{F} (1_{\omega_1}^* y(s + nT)) & \text{for } n \in \bar{\mathbb{N}}. \end{cases} \quad (1.1)$$

Here,  $\mathbb{R}^+ \triangleq (0, \infty)$ ;  $\bar{\mathbb{N}} \triangleq \mathbb{N} \cup \{0\}$ , with  $\mathbb{N}$  the set of all natural numbers;  $y((\tau + nT)_-)$  denotes the left limit of the function:  $t \rightarrow y(\cdot)$  (from  $\mathbb{R}^+$  to  $L^2(\Omega)$ ) at time  $\tau + nT$ ;  $\omega$  is an open and nonempty subset of  $\Omega$ ;  $1_\omega$  denotes the zero-extension operator from  $L^2(\omega)$  to  $L^2(\Omega)$  (i.e., for each  $f \in L^2(\omega)$ ,  $1_\omega(f)$  is defined to be the zero-extension of  $f$  over  $\Omega$ );  $1_{\omega_1}^*$  stands for the adjoint operator of  $1_{\omega_1}$ ;  $\mathcal{F}$  is a linear and bounded operator from  $L^2(\omega_1)$  to  $L^2(\omega)$ . This operator is what we will build up.

Throughout this paper, we denote by  $\|\cdot\|$  and  $\langle \cdot, \cdot \rangle$  the norm and the inner product of  $L^2(\Omega)$  respectively; denote by  $\|\cdot\|_\omega$  and  $\langle \cdot, \cdot \rangle_\omega$  the norm and the inner product of  $L^2(\omega)$ ; set  $\mathcal{L}(H_1, H_2)$  be the space consisting of all bounded linear operators from one Hilbert space  $H_1$  to another Hilbert space  $H_2$ ; write  $\{\lambda_j\}_{j=1}^\infty$  for the family of all eigenvalues of  $-A$  so that

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m \leq 0 < \lambda_{m+1} \leq \dots \text{ and } \lim_{j \rightarrow \infty} \lambda_j = \infty, \quad (1.2)$$

and let  $\{\xi_j\}_{j=1}^\infty$  be the family of the corresponding normalized eigenfunctions.

The main theorem of this paper will be precisely presented in section 4. It can be simply stated as follows: For each  $\gamma > 0$ , there is  $\mathcal{F}_\gamma \in \mathcal{L}(L^2(\omega_1); L^2(\omega))$  so that each solution  $y_\gamma(\cdot)$  to the equation (1.1) (with  $\mathcal{F} = \mathcal{F}_\gamma$ ) satisfies the inequality:

$$\|y_\gamma(t)\| \leq C_\gamma e^{-\gamma t} \|y_\gamma(0)\| \quad \text{for any } t \in \mathbb{R}^+. \quad (1.3)$$

We now give two comments on this result. First, the aforementioned  $\mathcal{F}_\gamma$  has the form:

$$\mathcal{F}_\gamma(w) = - \sum_{j=1}^{K_\gamma} e^{2\lambda_j s} \langle w, h_j \rangle_{\omega_1} f_j \quad \text{for any } w \in L^2(\omega_1). \quad (1.4)$$

Here,  $K_\gamma \in \mathbb{N}$  is the number of all eigenvalues  $\lambda_j$  which are less than  $\gamma + \frac{\ln 3}{T}$  (where  $\frac{\ln 3}{T}$  is not optimal);  $h_j$  and  $f_j$  are vectors in  $L^2(\omega_1)$  and  $L^2(\omega)$  respectively. These vectors are minimal norm controls for a kind of minimal norm problems which can be given by constructive methods. Second, the operator norm of  $\mathcal{F}_\gamma$  is bounded by  $C_1 e^{C_2 \gamma}$ , with  $C_1 > 0$  and  $C_2 > 0$  independent of  $\gamma$ .

We next explain our strategy and key points to prove the above-mentioned results. First, we realize that if a solution  $y_\gamma$  to the equation (1.1), (with  $\mathcal{F}$  being replaced by some  $\mathcal{F}_\gamma \in \mathcal{L}(L^2(\omega_1); L^2(\omega))$ ) satisfies that

$$\|y_\gamma((n+1)T)\| \leq e^{-\gamma T} \|y_\gamma(nT)\| \quad \text{for all } n \in \bar{\mathbb{N}},$$

then this solution satisfies the inequality (1.3). From this and the time translation invariance of

the equation, we can focus our study on the controlled equation:

$$\begin{cases} y'(t) - Ay(t) = 0, & t \in [0, T], \\ y(\tau) = y((\tau)_-) + 1_\omega f, \\ y(0) \in L^2(\Omega), \end{cases} \quad (1.5)$$

where  $f$  is a control taken from  $L^2(\omega)$ . Thus, our aim turns to find, for each  $\gamma > 0$ , an operator  $\mathcal{F}_\gamma \in \mathcal{L}(L^2(\omega_1); L^2(\omega))$  so that each solution  $y_{\gamma, T}(\cdot)$  to the equation (1.5) (with  $f = \mathcal{F}_\gamma(1_{\omega_1}^* y(s))$ ) satisfies the inequality:

$$\|y_{\gamma, T}(T)\| \leq e^{-\gamma T} \|y_{\gamma, T}(0)\|.$$

Second, we project the equation (1.5) with the null control into subspaces  $\text{span}\{\xi_1, \dots, \xi_{K_\gamma}\}$  and  $\text{span}\{\xi_{K_\gamma+1}, \xi_{K_\gamma+2}, \dots\}$  respectively. Write  $P_{K_\gamma}$  for the orthogonal projection of  $L^2(\Omega)$  onto the first subspace. We only need control the projected equation on the first subspace, since the projected equation on the second subspace is already stable. (Though the feedback term may cause some "bad" influence on the second equation, such influence, compared with the natural decay of the second equation, can be ignored when the feedback law is built up in a right way.) Third, we consider the special case that  $\omega_1 = \Omega$  and  $s = \tau$ . In this case, we construct a linear and bounded operator  $\mathcal{C}$  from  $\text{span}\{\xi_1, \dots, \xi_{K_\gamma}\}$  to  $L^2(\omega)$  in the following manner:  $\mathcal{C}(z) \triangleq \sum_{j=1}^{K_\gamma} \langle z, \xi_j \rangle f_j$ . Here,  $f_j$  is the minimal norm control to a minimal norm control problem. In this minimal norm problem, the equation is (1.5), with the initial datum  $\xi_j$ ; the target is a ball in  $L^2(\Omega)$ , centered at the origin and with a sufficiently small radius. Then the desired feedback law  $\mathcal{F}_\gamma$  reads:  $\mathcal{C}P_{K_\gamma}$  with a suitable  $K_\gamma$ . The key to make the above method work is the unique continuation estimate at one time for heat equations. Such an estimate was built up in [PWZ] (see also [PW] and [PW1]). Indeed, this estimate is equivalent to the null approximate controllability at one time with a cost. Such kind of controllability not only ensures the existence of minimal norm controls but also provides a bound for the minimal norm of the above-mentioned minimal norm problem. Based on these, the above-mentioned  $\mathcal{C}$  can be constructed, and furthermore, the bound of its operator norm can be estimated. Finally, we back to the general case where  $\omega_1 \neq \Omega$  and  $s \leq \tau$ . In this case, with the aid of the above-mentioned unique continuation estimate at one point, we can approximately recover  $P_{K_\gamma}(z_0)$  from  $1_{\omega_1}^* e^{sA} z_0$  for each  $z_0 \in L^2(\Omega)$ . More precisely, for each  $\varepsilon > 0$ , we can build up a linear and bounded operator  $\mathcal{R}^\varepsilon : L^2(\omega_1) \rightarrow \text{span}\{\xi_1, \dots, \xi_{K_\gamma}\}$  so that

$$\|\mathcal{R}^\varepsilon(1_{\omega_1}^* e^{sA} z_0) - P_{K_\gamma} z_0\| \lesssim \varepsilon \|z_0\| \quad \text{uniformly w.r.t. } z_0 \in L^2(\Omega).$$

Furthermore, the bound of  $\mathcal{R}^\varepsilon$  can be estimated. The construction of  $\mathcal{R}^\varepsilon$  is based on a kind of minimal norm control problems which are essentially the same as those used in the construction of the operator  $\mathcal{C}$ . Then, the desired feedback law  $\mathcal{F}_\gamma$  reads:  $\mathcal{C}\mathcal{R}^\varepsilon$  with a suitable  $\varepsilon$  depending on  $\gamma$ .

Several remarks are given in order. (a) Impulse control belongs to a class of important control and has wide applications. There are many studies on optimal control and controllability for impulse controlled equations (see, for instance, [BL1], [R], [BL2], [LM], [Z], [LY], [BC], [DS], [MR], [OS], [Be] and references there in). However, we have not found any published paper on stabilization for

impulse controlled equations. From this perspective, the problem studied in the current paper is new. (b) Stabilization is one of the most important subject in control theory. In most studies of this subject, the aim of stabilization is to ask for a feedback law so that the closed loop equation decays exponentially. The current work aims to find, for each decay rate  $\gamma$ , a feedback law so that the closed loop equation has an exponential decay with rate  $\gamma$ . Such kind of stabilization is called the rapid stabilization. About this subject, we would like to mention the works [K], [U], [CCr], [V], [CCo], [CL2], and [CL1]. (c) When observation region is not the whole  $\Omega$ , the corresponding stabilization is a kind of output stabilization. Such stabilization is very useful in applications. Unfortunately, there is no systematic study on this subject, even for the simplest case when the controlled system is time-invariant linear ODE (see [Br]). Most of publications on this subject focus on how to construct an output feedback law for a single special equation. (see, for instance, [I], [Cu], [YY], [Co], [NS] and references therein). Our study also only provides an output feedback law for a special equation. (d) In most studies on stabilization, the structures of feedback laws are based on LQ theory or Lyapunov functions (see, for instance, [Ba] and [C]). In this paper, we present another way to build up the feedback law. (e) As mentioned above, one of the keys to build up our feedback law is the use of the unique continuation estimate at one time, built up in [PWZ] (see also [PW] and [PW1]). Some new observations are made on it in this paper (see Theorem 2.1, Remark 2.2) and Remark 3.6). (f) The following extensions of the current work should be interesting: The first case is that  $V$  depends on both  $x$  and  $t$  variables; The second case is that the equation is semi-linear; The third case is that the equation is other types of PDEs.

The rest of this paper is organized as follows. Section 2 provides several inequalities which are equivalent to the unique continuation estimate at one time. Section 3 introduces several impulse approximately controllability problems, as well as a kind of minimal norm problems. Section 4 presents the main result, as well as its proof.

## 2 Observation at one time

In this section, we present several equivalent inequalities. One of them is the unique continuation estimate at one time built up in [PWZ] (see also [PW] and [PW1]).

**Theorem 2.1** *Let  $\hat{\omega}$  be an open and nonempty subset of  $\Omega$ . Then the following propositions are equivalent and are true:*

(i) *There are two constants  $C_1 > 0$  and  $\beta \in (0, 1)$ , which depend only on  $\Omega$  and  $\hat{\omega}$ , so that for all  $t > 0$  and  $\xi \in L^2(\Omega)$ ,*

$$\|e^{tA}\xi\| \leq e^{C_1(1+\frac{1}{t}+t\|V\|_\infty+\|V\|_\infty^{2/3})} \|\xi\|^\beta \|1_{\hat{\omega}}^* e^{tA}\xi\|_{\hat{\omega}}^{1-\beta}.$$

(ii) *There is a positive constant  $C_2$ , depending only on  $\Omega$  and  $\hat{\omega}$ , so that for each  $\lambda \geq 0$  and each sequence of real numbers  $\{a_j\} \subset \mathbb{R}$ ,*

$$\sum_{\lambda_j < \lambda} |a_j|^2 \leq e^{C_2(1+\|V\|_\infty^{2/3}+\sqrt{\lambda})} \int_{\hat{\omega}} \left| \sum_{\lambda_j < \lambda} a_j \xi_j \right|^2 dx.$$

(iii) There is a positive constant  $C$ , depending only on  $\Omega$  and  $\hat{\omega}$ , so that for all  $\theta \in (0, 1)$ ,  $t > 0$  and  $\xi \in L^2(\Omega)$

$$\|e^{tA}\xi\| \leq e^{C(1+\frac{1}{\theta}+t\|V\|_\infty+\|V\|_\infty^{2/3})} \|\xi\|^\theta \|1_{\hat{\omega}}^* e^{tA}\xi\|_{\hat{\omega}}^{1-\theta}.$$

(iv) There is a positive constant  $C$ , depending only on  $\Omega$  and  $\hat{\omega}$ , so that for all  $\varepsilon > 0$ ,  $\theta \in (0, 1)$ ,  $t > 0$  and  $\xi \in L^2(\Omega)$ ,

$$\|e^{tA}\xi\|^2 \leq \frac{1}{\varepsilon^{1-\theta}} e^{\frac{2C}{1-\theta}(1+\frac{1}{\theta}+t\|V\|_\infty+\|V\|_\infty^{2/3})} \|1_{\hat{\omega}}^* e^{tA}\xi\|_{\hat{\omega}}^2 + \varepsilon \|\xi\|^2.$$

(v) There is a positive constant  $C$ , depending only on  $\Omega$  and  $\hat{\omega}$ , so that for all  $\varepsilon, \beta > 0$ ,  $t > 0$  and  $\xi \in L^2(\Omega)$ ,

$$\|e^{tA}\xi\|^2 \leq \frac{1}{\varepsilon^\beta} e^{2C(1+\beta)(1+\frac{1+\beta}{\beta t}+t\|V\|_\infty+\|V\|_\infty^{2/3})} \|1_{\hat{\omega}}^* e^{tA}\xi\|_{\hat{\omega}}^2 + \varepsilon \|\xi\|^2.$$

(vi) There is a positive constant  $C_3$ , depending only on  $\Omega$  and  $\hat{\omega}$ , so that for all  $\varepsilon, \beta \in (0, 1]$ ,  $s$  and  $\tilde{T}$ , with  $0 \leq s < \tilde{T}$ , and  $\xi \in L^2(\Omega)$

$$\|e^{\tilde{T}A}\xi\|^2 \leq \frac{1}{\varepsilon^\beta} e^{C_3\left(1+\frac{1}{\beta(\tilde{T}-s)}+\tilde{T}\|V\|_\infty+\|V\|_\infty^{2/3}\right)} \|1_{\hat{\omega}}^* e^{(\tilde{T}-s)A}\xi\|_{\hat{\omega}}^2 + \varepsilon \|\xi\|^2.$$

Moreover, constants  $C$  in (iii), (v) and (iv) can be chosen as the same number, and  $C_3$  in (vi) can be chosen as  $8\max\{1, C\}$ .

*Proof.* We organize the proof by several steps.

*Step 1: On the proposition (i)*

The conclusion (i) has been proved in [PWZ] (see also [PW] and [PW1]).

*Step 2: To prove that (i)  $\Rightarrow$  (ii)*

Let  $C_1 > 0$  and  $\beta \in (0, 1)$  be given by (i). Arbitrarily fix  $\lambda \geq 0$  and  $\{a_j\} \subset \mathbb{R}$ . By applying the inequality in (i), with  $\xi = \sum_{\lambda_j < \lambda} a_j e^{\lambda_j t} \xi_j$ , we get that

$$\sum_{\lambda_j < \lambda} |a_j|^2 \leq e^{2C_1(1+\frac{1}{t}+t\|V\|_\infty+\|V\|_\infty^{2/3})} \left( \sum_{\lambda_j < \lambda} |a_j e^{\lambda_j t}|^2 \right)^\beta \left( \int_{\hat{\omega}} \left| \sum_{\lambda_j < \lambda} a_j \xi_j \right|^2 dx \right)^{1-\beta},$$

which implies that

$$\sum_{\lambda_j < \lambda} |a_j|^2 \leq e^{\frac{2}{1-\beta}C_1(1+\frac{1}{t}+t\|V\|_\infty+\|V\|_\infty^{2/3})} e^{\frac{2\beta}{1-\beta}\lambda t} \int_{\hat{\omega}} \left| \sum_{\lambda_j < \lambda} a_j \xi_j \right|^2 dx \text{ for each } t > 0. \quad (2.1)$$

Meanwhile, since  $\|V\|_\infty^{1/2} \leq 1 + \|V\|_\infty^{2/3}$  and  $\beta \in (0, 1)$ , we see that

$$\begin{aligned} & \inf_{t>0} \left[ C_1 \left( 1 + \frac{1}{t} + t\|V\|_\infty + \|V\|_\infty^{2/3} \right) + \beta\lambda t \right] \\ &= \inf_{t>0} \left[ C_1 (1 + \|V\|_\infty^{2/3}) + \frac{C_1}{t} + (C_1\|V\|_\infty + \beta\lambda)t \right] \\ &= C_1 (1 + \|V\|_\infty^{2/3}) + 2\sqrt{C_1(C_1\|V\|_\infty + \beta\lambda)} \leq C_1 (1 + \|V\|_\infty^{2/3}) + 2(\sqrt{C_1^2\|V\|_\infty} + \sqrt{C_1\beta\lambda}) \\ &= C_1 (1 + \|V\|_\infty^{2/3} + 2\|V\|_\infty^{1/2}) + 2\sqrt{C_1}\sqrt{\beta\lambda} \leq \max\{3C_1, 2\sqrt{C_1}\}(1 + \|V\|_\infty^{2/3} + \sqrt{\lambda}). \end{aligned}$$

This, along with (2.1), leads to the conclusion (ii), with  $C_2 = \max\{\frac{6C_1}{1-\beta}, \frac{4\sqrt{C_1}}{1-\beta}\}$ .

*Step 3: To show that (ii)  $\Rightarrow$  (iii)*

Arbitrarily fix  $\lambda \geq 0$ ,  $t > 0$  and  $\xi = \sum_{j \geq 1} a_j \xi_j$  with  $\{a_j\} \subset l^2$ . Write

$$e^{tA}\xi = \sum_{\lambda_j < \lambda} a_j e^{-\lambda_j t} \xi_j + \sum_{\lambda_j \geq \lambda} a_j e^{-\lambda_j t} \xi_j.$$

Then by (ii), we find that

$$\begin{aligned} \|e^{tA}\xi\| &\leq \left\| \sum_{\lambda_j < \lambda} a_j e^{-\lambda_j t} \xi_j \right\| + \left\| \sum_{\lambda_j \geq \lambda} a_j e^{-\lambda_j t} \xi_j \right\| \\ &\leq \left( \sum_{\lambda_j < \lambda} |a_j e^{-\lambda_j t}|^2 \right)^{1/2} + e^{-\lambda t} \|\xi\| \\ &\leq \left( e^{C_2(1+\|V\|_\infty^{2/3}+\sqrt{\lambda})} \int_{\hat{\omega}} \left| \sum_{\lambda_j < \lambda} a_j e^{-\lambda_j t} \xi_j \right|^2 dx \right)^{1/2} + e^{-\lambda t} \|\xi\|. \end{aligned}$$

This, along with the triangle inequality for the norm  $\|\cdot\|_{\hat{\omega}}$ , yields that

$$\begin{aligned} \|e^{tA}\xi\| &\leq \left( e^{C_2(1+\|V\|_\infty^{2/3}+\sqrt{\lambda})} \int_{\hat{\omega}} \left| \sum_{j \geq 1} a_j e^{-\lambda_j t} \xi_j \right|^2 dx \right)^{1/2} \\ &\quad + \left( e^{C_2(1+\|V\|_\infty^{2/3}+\sqrt{\lambda})} \int_{\hat{\omega}} \left| \sum_{\lambda_j \geq \lambda} a_j e^{-\lambda_j t} \xi_j \right|^2 dx \right)^{1/2} + e^{-\lambda t} \|\xi\|. \end{aligned}$$

Hence, it follows that

$$\begin{aligned} \|e^{tA}\xi\| &\leq e^{\frac{C_2}{2}(1+\|V\|_\infty^{2/3}+\sqrt{\lambda})} \|1_{\hat{\omega}}^* e^{tA}\xi\|_{\hat{\omega}} + e^{\frac{C_2}{2}(1+\|V\|_\infty^{2/3}+\sqrt{\lambda})} e^{-\lambda t} \|\xi\| + e^{-\lambda t} \|\xi\| \\ &\leq 2e^{\frac{C_2}{2}(1+\|V\|_\infty^{2/3}+\sqrt{\lambda})} \left[ \|1_{\hat{\omega}}^* e^{tA}\xi\|_{\hat{\omega}} + e^{-\lambda t} \|\xi\| \right]. \end{aligned}$$

This indicates that for all  $\varepsilon \in (0, 2)$ ,

$$\|e^{tA}\xi\| \leq 2e^{\frac{C_2}{2}(1+\|V\|_\infty^{2/3})} e^{\frac{1}{2t\varepsilon}(\frac{C_2}{2})^2} \left( e^{\frac{\varepsilon}{2}\lambda t} \|1_{\hat{\omega}}^* e^{tA}\xi\|_{\hat{\omega}} + e^{-\frac{2-\varepsilon}{2}\lambda t} \|\xi\| \right). \quad (2.2)$$

Here, we used the following inequality:

$$\frac{C_2}{2}\sqrt{\lambda} \leq \frac{\varepsilon}{2}\lambda t + \frac{1}{2t\varepsilon} \left( \frac{C_2}{2} \right)^2 \quad \text{for any } \varepsilon > 0.$$

Since  $\lambda$  was arbitrarily taken from  $[0, \infty)$ , by (2.2), we deduce that for all  $\varepsilon \in (0, 2)$  and  $\mu \in (0, 1)$ ,

$$\|e^{tA}\xi\| \leq 2e^{\frac{C_2}{2}(1+\|V\|_\infty^{2/3})} e^{\frac{1}{2t\varepsilon}(\frac{C_2}{2})^2} \left( \frac{1}{\mu^{\frac{\varepsilon}{2-\varepsilon}}} \|1_{\hat{\omega}}^* e^{tA}\xi\|_{\hat{\omega}} + \mu \|\xi\| \right). \quad (2.3)$$

Because

$$\|e^{tA}\xi\| \leq e^{t\|V\|_\infty} (\mu \|\xi\|) \quad \text{for each } \mu \in [1, +\infty),$$

we see from (2.3) that for all  $\varepsilon \in (0, 2)$  and  $\mu > 0$ ,

$$\begin{aligned}
\|e^{tA}\xi\| &\leq \max\left\{2e^{\frac{C_2}{2}(1+\|V\|_\infty^{2/3})}e^{\frac{1}{2t\varepsilon}(\frac{C_2}{2})^2}, e^{t\|V\|_\infty}\right\}\left(\frac{1}{\mu^{\frac{\varepsilon}{2-\varepsilon}}}\|1_\omega^*e^{tA}\xi\|_{\hat{\omega}} + \mu\|\xi\|\right) \\
&\leq 2e^{\frac{C_2}{2}(1+\|V\|_\infty^{2/3})}e^{\frac{1}{2t\varepsilon}(\frac{C_2}{2})^2}e^{t\|V\|_\infty}\left(\frac{1}{\mu^{\frac{\varepsilon}{2-\varepsilon}}}\|1_\omega^*e^{tA}\xi\|_{\hat{\omega}} + \mu\|\xi\|\right) \\
&\leq 2e^{(\frac{C_2}{4}+1)^2(1+\frac{1}{t^{\frac{1}{2}}}+t\|V\|_\infty+\|V\|_\infty^{2/3})}\left(\frac{1}{\mu^{\frac{\varepsilon}{2-\varepsilon}}}\|1_\omega^*e^{tA}\xi\|_{\hat{\omega}} + \mu\|\xi\|\right). \tag{2.4}
\end{aligned}$$

Meanwhile, one can directly check that for each  $\varepsilon \in (0, 2)$ :

$$\begin{aligned}
\inf_{\mu>0}\left[\frac{1}{\mu^{\frac{\varepsilon}{2-\varepsilon}}}\|1_\omega^*e^{tA}\xi\|_{\hat{\omega}} + \mu\|\xi\|\right] &= \frac{1}{(\frac{\varepsilon}{2})^{\frac{\varepsilon}{2}}(1-\frac{\varepsilon}{2})^{1-\frac{\varepsilon}{2}}}\|1_\omega^*e^{tA}\xi\|_{\hat{\omega}}^{1-\frac{\varepsilon}{2}}\|\xi\|^{\frac{\varepsilon}{2}} \\
&\leq e\|1_\omega^*e^{tA}\xi\|_{\hat{\omega}}^{1-\frac{\varepsilon}{2}}\|\xi\|^{\frac{\varepsilon}{2}}.
\end{aligned}$$

From this and (2.4), after some simple computations, we get the inequality in (iii).

*Step 4: To show that (iii)  $\Rightarrow$  (iv)*

Let  $C > 0$  be given by (iii). Arbitrarily fix  $\theta \in (0, 1)$ ,  $t > 0$  and  $\xi \in L^2(\Omega)$ . Write

$$\hat{C} \triangleq e^{2C(1+\frac{1}{\theta t}+t\|V\|_\infty+\|V\|_\infty^{2/3})}, \quad \alpha \triangleq \|\xi\| \quad \text{and} \quad \gamma \triangleq \|1_\omega^*e^{tA}\xi\|_{\hat{\omega}}. \tag{2.5}$$

By the Young inequality, we see that for all  $\varepsilon > 0$ ,

$$\begin{aligned}
\hat{C}\alpha^{2\theta}\gamma^{2(1-\theta)} &= (\varepsilon^\theta\alpha^{2\theta})\left(\hat{C}\varepsilon^{-\theta}\gamma^{2(1-\theta)}\right) \\
&\leq \theta(\varepsilon^\theta\alpha^{2\theta})^{1/\theta} + (1-\theta)\left(\hat{C}\varepsilon^{-\theta}\gamma^{2(1-\theta)}\right)^{1/(1-\theta)} \\
&\leq (\varepsilon^\theta\alpha^{2\theta})^{1/\theta} + \hat{C}^{1/(1-\theta)}\left(\varepsilon^{-\theta}\gamma^{2(1-\theta)}\right)^{1/(1-\theta)} \\
&= \varepsilon\alpha^2 + \hat{C}^{1/(1-\theta)}\varepsilon^{-\frac{\theta}{1-\theta}}\gamma^2.
\end{aligned}$$

This, along with (2.5) and the inequality in (iii), implies that the inequality in (iv) with the same constant  $C$  as that in (iii).

*Step 5: To show that (iv)  $\Rightarrow$  (v).*

By taking  $\beta = \frac{\theta}{1-\theta}$  in the inequality in (iv), we are led to the inequality in (v) with the same constant  $C$  as that in (iv).

*Step 6: To show that (v)  $\Rightarrow$  (vi)*

Let  $C > 0$  be given by (v). Arbitrarily fix  $s$  and  $\tilde{T}$  so that  $0 \leq s < \tilde{T}$ . Then fix  $\xi \in L^2(\Omega)$ . Since

$$\|e^{\tilde{T}A}\xi\| \leq e^{s\|V\|_\infty}\|e^{(\tilde{T}-s)A}\xi\|,$$



by taking  $t = \tilde{T} - s$  in the inequality in (v), we find that for all  $\varepsilon > 0$  and  $\beta \in (0, 1]$ ,

$$\begin{aligned}
& \|e^{\tilde{T}A}\xi\|^2 \leq e^{2s\|V\|_\infty} \|e^{(\tilde{T}-s)A}\xi\|^2 \\
& \leq e^{2s\|V\|_\infty} \left[ \frac{1}{\varepsilon^\beta} e^{2C(1+\beta)\left(1 + \frac{1+\beta}{\beta(\tilde{T}-s)} + (\tilde{T}-s)\|V\|_\infty + \|V\|_\infty^{2/3}\right)} \|1_{\hat{\omega}}^* e^{(\tilde{T}-s)A}\xi\|_{\hat{\omega}}^2 + \varepsilon \|\xi\|^2 \right] \\
& = \frac{e^{2(1+\beta)s\|V\|_\infty}}{(\varepsilon e^{2s\|V\|_\infty})^\beta} e^{2C(1+\beta)\left(1 + \frac{1+\beta}{\beta(\tilde{T}-s)} + (\tilde{T}-s)\|V\|_\infty + \|V\|_\infty^{2/3}\right)} \|1_{\hat{\omega}}^* e^{(\tilde{T}-s)A}\xi\|_{\hat{\omega}}^2 \\
& \quad + (\varepsilon e^{2s\|V\|_\infty}) \|\xi\|^2.
\end{aligned} \tag{2.6}$$

Meanwhile, we have that for all  $\beta \in (0, 1]$ ,

$$\begin{aligned}
& 2(1+\beta)s\|V\|_\infty + 2C(1+\beta)\left(1 + \frac{1+\beta}{\beta(\tilde{T}-s)} + (\tilde{T}-s)\|V\|_\infty + \|V\|_\infty^{2/3}\right) \\
& = 2(1+\beta) \left[ C + \frac{C(1+\beta)}{\beta(\tilde{T}-s)} + (s + C(\tilde{T}-s))\|V\|_\infty + C\|V\|_\infty^{2/3} \right] \\
& \leq 2(1+\beta)^2 \max\{1, C\} \left[ 1 + \frac{1}{\beta(\tilde{T}-s)} + \tilde{T}\|V\|_\infty + \|V\|_\infty^{2/3} \right] \\
& \leq 8 \max\{1, C\} \left[ 1 + \frac{1}{\beta(\tilde{T}-s)} + \tilde{T}\|V\|_\infty + \|V\|_\infty^{2/3} \right].
\end{aligned}$$

This, along with (2.6), implies that the inequality in (vi), with  $C_3 = 8 \max\{1, C\}$ .

*Step 7: to show that (vi)  $\Rightarrow$  (i)*

Let  $C_3 > 0$  be given by (vi). Arbitrarily fix  $t > 0$  and  $\xi \in L^2(\Omega)$ . Write

$$D \triangleq e^{\frac{C_3}{2}\left(1 + \frac{1}{\beta t} + t\|V\|_\infty + \|V\|_\infty^{2/3}\right)} \|1_{\hat{\omega}}^* e^{tA}\xi\|_{\hat{\omega}}. \tag{2.7}$$

Then it follows from the inequality in (vi) (where  $\beta = 1$ ,  $s = 0$  and  $\tilde{T} = t$ ) that

$$\|e^{tA}\xi\|^2 \leq \inf_{\varepsilon > 0} \left( \frac{1}{\varepsilon} D^2 + \varepsilon \|\xi\|^2 \right),$$

from which, it follows that

$$\|e^{tA}\xi\| \leq \sqrt{2D\|\xi\|} \leq \sqrt{2} e^{\frac{C_3}{4}\left(1 + \frac{1}{\beta t} + t\|V\|_\infty + \|V\|_\infty^{2/3}\right)} \|1_{\hat{\omega}}^* e^{tA}\xi\|_{\hat{\omega}}^{1/2} \|\xi\|^{1/2}.$$

This, along with (2.7), leads to the inequality in (i).

Finally, we see from Steps 4-5 that the constants  $C$  in (iii), (iv) and (v) can be chosen as the same number, while  $C_3$  in (vi) can be taken as  $8 \max\{1, C\}$  (see Step 6). This ends the proof.  $\square$

**Remark 2.2** (a) *The inequality in (i) of Theorem 2.1 is indeed equivalent to a kind of impulse controllability, i.e., the impulse null approximate controllability, which will be explained in the next section. Such controllability is the base for our study on the stabilization. (b) In the following studies of this paper, we will only use the inequality in (v) of Theorem 2.1. However, other inequalities seem to be interesting independently. For instance, when  $V = 0$ , the inequality in (ii) of Theorem 2.1 is*

exactly the Lebeau-Robbiano spectral inequality (see [LR]). Here, we get, in the case that  $V \neq 0$ , the same inequality, and find how the constant (on the right hand side of the inequality) depends on  $\|V\|_\infty$ . (c) It deserves to mention that all constant terms on the right hand side of inequalities in Theorem 2.1 have explicit expressions in terms of the norm of  $V$  and time, but not  $\Omega$  and  $\hat{\omega}$ . (d) It was realized that when  $V = 0$ , the inequality in (i) of Theorem 2.1 can imply the inequality (ii) of Theorem 2.1 (see Remark 1 in [AEWZ])

### 3 Impulse null approximate controllability

The main purposes of this section are to introduce impulse null approximate controllability problems and to show that such controllability is a consequence of the inequality in (v) of Theorem 2.1. Throughout this section, we arbitrarily fix numbers  $\hat{T}$ ,  $\hat{s}$  and  $\hat{t}$ , with  $\hat{T} > \hat{s} \geq \hat{t} \geq 0$ , and fix an open and non-empty subset  $\hat{\omega} \subset \Omega$ .

#### 3.1 Several impulse approximate controllability problems

Consider the following two equations:

$$\begin{cases} y'(t) - Ay(t) = 0, & t \in [\hat{t}, \hat{T}], \\ y(\hat{s}) = y((\hat{s})_-) + 1_\omega f, \\ y(\hat{t}) = z, \end{cases} \quad (3.1)$$

and

$$\begin{cases} \varphi'(t) + A\varphi(t) = 0, & t \in [\hat{t}, \hat{T}], \\ \varphi(\hat{T}) = \xi. \end{cases} \quad (3.2)$$

Here,  $z \in L^2(\Omega)$ ,  $f \in L^2(\hat{\omega})$ , and  $\xi \in L^2(\Omega)$ . The first equation is an impulse controlled equation over  $[\hat{t}, \hat{T}]$ , while the second one is the dual equation of the first one. Write  $y(\cdot; f, z)$  and  $\varphi(\cdot; \xi)$  for the solutions to (3.1) and (3.2), respectively. Then we have that

$$y(t; f, z) = \begin{cases} e^{(t-\hat{t})A}z, & \text{when } t \in [\hat{t}, \hat{s}) \text{ and } \hat{t} < \hat{s}, \\ e^{(t-\hat{t})A}z + e^{(t-\hat{s})A}1_\omega f & \text{when } t \in [\hat{s}, \hat{T}], \end{cases} \quad (3.3)$$

and

$$\varphi(t; \xi) = e^{(\hat{T}-t)A}\xi \quad \text{for any } t \in [\hat{t}, \hat{T}]. \quad (3.4)$$

Several concepts on the impulse approximate controllability for Equation (3.1) are given in order. The first one is the impulse approximate controllability for Equation (3.1), i.e., for each  $z \in L^2(\Omega)$ ,

$$\overline{\{y(\hat{T}; f, z) \mid f \in L^2(\hat{\omega})\}}^{\|\cdot\|} = L^2(\Omega). \quad (3.5)$$

The second one is the impulse null approximate controllability, i.e., for each  $z \in L^2(\Omega)$  and each  $\delta > 0$ , there is a control  $f \in L^2(\omega)$  so that  $y(\hat{T}; f, z) \in B(\delta\|z\|, 0)$ . Here and in what follows,  $B(\delta\|z\|, 0)$  denotes the closed ball in  $L^2(\Omega)$ , centered at the origin and of radius  $\delta\|z\|$ . The third

one is the impulse null approximate controllability with a cost, i.e., for each  $\delta > 0$ , there is a constant  $C > 0$  so that for each  $z \in L^2(\Omega)$ , there is a control  $f \in L^2(\hat{\omega})$  satisfying that  $y(\hat{T}; f, z) \in B(\delta\|z\|, 0)$  and  $\|f\|_{\hat{\omega}} \leq C\|z\|$ .

We now introduce two problems and one property related to the above-mentioned impulse null approximate controllability and impulse null approximate controllability with a cost. The first one is the following minimal norm impulse control problem  $(NP)_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$  (where  $z \in L^2(\Omega) \setminus \{0\}$  and  $\delta > 0$  are arbitrarily fixed):

$$N^{z, \delta}(\hat{T}) \triangleq \inf \left\{ \|f\|_{\hat{\omega}} \mid \|y(\hat{T}; f, z)\| \leq \delta\|z\| \right\}. \quad (3.6)$$

This problem is to ask for a control which has the minimal norm among all controls (in  $L^2(\hat{\omega})$ ) driving solutions of Equation (3.1) from the initial state  $z$  to  $B(\delta\|z\|, 0)$  at the ending time  $\hat{T}$ . In this problem,  $N^{z, \delta}(\hat{T})$  is called the minimal norm, while  $f^* \in L^2(\hat{\omega})$  is called a minimal norm control, if

$$y(\hat{T}; f^*, z) \leq \delta\|z\| \quad \text{and} \quad \|f^*\|_{\hat{\omega}} = N^{z, \delta}(\hat{T}). \quad (3.7)$$

The second one is the following problem  $(NP)_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{\delta}$  (where  $\delta > 0$  are arbitrarily fixed):

$$N^{\delta}(\hat{T}) \triangleq \sup_{\|z\| \leq 1} N^{z, \delta}(\hat{T}) = \sup_{\|z\| \leq 1} \inf \left\{ \|f\|_{\hat{\omega}} \mid \|y(\hat{T}; f, z)\| \leq \delta\|z\| \right\}. \quad (3.8)$$

The quantity  $N^{\delta}(\hat{T})$  is called the value of the problem  $(NP)_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{\delta}$ . The last one is following property  $(C)^{C, \delta}$  (where  $C > 0$  and  $\delta > 0$ ): For any  $z \in L^2(\Omega)$ , there is a control  $f \in L^2(\hat{\omega})$  so that

$$\max \left\{ \frac{1}{C} \|f\|_{\hat{\omega}}, \frac{1}{\delta} \|y(\hat{T}; f, z)\| \right\} \leq \|z\|. \quad (3.9)$$

We would like to mention that the property  $(C)^{C, \delta}$  may not hold for arbitrary  $C > 0$  and  $\delta > 0$ . However, we will see that given  $\delta > 0$ , there is  $C > 0$  so that the property  $(C)^{C, \delta}$  is true.

The following lemma gives the impulse approximate controllability for (3.1). This result will not be used in our study on the stabilization, but for the completeness of the introduction on the impulse approximate controllability, we present it here. It deserves to mention that such result may have existed already. Unfortunately, we did not find the exact reference.

**Theorem 3.1** *Equation (3.1) is impulse approximate controllable.*

*Proof.* Define a linear and bounded operator  $\mathcal{G} : L^2(\hat{\omega}) \rightarrow L^2(\Omega)$  in the following manner:

$$\mathcal{G} \triangleq e^{(\hat{T}-\hat{s})A} \mathbf{1}_{\hat{\omega}}.$$

Then we find that for each  $z \in L^2(\Omega)$  and each  $f \in L^2(\hat{\omega})$ ,

$$y(\hat{T}; f, z) = e^{(\hat{T}-\hat{t})A} z + \mathcal{G} f.$$

From this and (3.5), we see that Equation (3.1) is impulse approximate controllable if and only if

$$\overline{\mathcal{R}(\mathcal{G})}^{\|\cdot\|} = L^2(\Omega), \quad (3.10)$$

It is clear that (3.10) is equivalent to

$$\mathcal{N}(\mathcal{G}^*) = \{0\}. \quad (3.11)$$

Because

$$\mathcal{G}^* = 1_{\hat{\omega}}^* e^{(\hat{T}-\hat{s})A} \in \mathcal{L}(L^2(\Omega), L^2(\hat{\omega})),$$

we see that (3.11) is equivalent to the following property:

$$\chi_{\hat{\omega}}\varphi(\hat{s}; \xi) = 0 \implies \xi = 0.$$

(Here and in what follows,  $\chi_{\hat{\omega}}$  denotes the characteristic function of  $\hat{\omega}$ .) The later is exactly the unique continuation property of heat equations built up in [L] (see also [EFV]). Hence, Equation (3.1) is impulse approximate controllable. This ends the proof.  $\square$

### 3.2 Minimal norm impulse control

This subsection presents some properties on the minimal norm impulse control problem  $(NP)_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$ .

**Lemma 3.2** *For each  $z \in L^2(\Omega) \setminus \{0\}$  and each  $\delta > 0$ , the problem  $(NP)_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$  has a unique minimal norm control.*

*Proof.* Arbitrarily fix  $z \in L^2(\Omega) \setminus \{0\}$  and  $\delta > 0$ . Write

$$F_{ad}^{z, \delta} \triangleq \left\{ f \in L^2(\hat{\omega}) \mid \|y(\hat{T}; f, z)\| \leq \delta \|z\| \right\}.$$

By Theorem 3.1, we see that  $F_{ad}^{z, \delta} \neq \emptyset$ . Meanwhile, one can easily check that  $F_{ad}^{z, \delta}$  is weakly closed in  $L^2(\hat{\omega})$ . From these, it follows that  $(NP)_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$  has a minimal norm control.

Suppose that  $f_1$  and  $f_2$  are two minimal norm controls to  $(NP)_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$ . Then we have that

$$0 \leq \|f_1\|_{\hat{\omega}} = \|f_2\|_{\hat{\omega}} = N^{z, \delta}(\hat{T}) < \infty, \quad (3.12)$$

where  $N^{z, \delta}(\hat{T})$  is given by (3.6). Meanwhile, one can easily check that  $(f_1 + f_2)/2$  is also a minimal norm control to  $(NP)_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$ . This, along with the Parallelogram Law, yields that

$$\left( N^{z, \delta}(\hat{T}) \right)^2 = \|(f_1 + f_2)/2\|_{\hat{\omega}}^2 = \frac{1}{2} (\|f_1\|_{\hat{\omega}}^2 + \|f_2\|_{\hat{\omega}}^2) - \|(f_1 - f_2)/2\|_{\hat{\omega}}^2. \quad (3.13)$$

From (3.12) and (3.13), we find that  $f_1 = f_2$ . Thus, the minimal norm control to  $(NP)_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$  is unique. This ends the proof.  $\square$

To characterize the minimal norm control of  $(NP)_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$ , we introduce its dual problem  $(DP)_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$  in the following manner (which is inspired by [FPZ]):

$$\inf_{\xi \in L^2(\Omega)} J(\xi; z, \delta, \hat{T}, \hat{s}, \hat{t}, \hat{\omega}), \quad (3.14)$$

where

$$J(\xi; z, \delta, \hat{T}, \hat{s}, \hat{t}, \hat{\omega}) \triangleq \frac{1}{2} \|1_{\hat{\omega}}^* \varphi(\hat{s}; \xi)\|_{\hat{\omega}}^2 + \left\langle z, \varphi(\hat{t}; \xi) \right\rangle + \delta \|z\| \|\xi\|, \quad \xi \in L^2(\Omega). \quad (3.15)$$

**Lemma 3.3** For each  $z \in L^2(\Omega) \setminus \{0\}$  and each  $\delta > 0$ , the functional  $J(\cdot; z, \delta, \hat{T}, \hat{s}, \hat{t}, \hat{\omega})$  has the following properties:

(i) It satisfies that

$$\lim_{r \rightarrow \infty} \inf_{\|\xi\|=r} \frac{J(\xi; z, \delta, \hat{T}, \hat{s}, \hat{t}, \hat{\omega})}{\|\xi\|} \geq \delta \|z\|. \quad (3.16)$$

(ii) It has a unique minimizer over  $L^2(\Omega)$ .

(iii) Write  $\xi_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$  for its minimizer. Then

$$\xi_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta} = 0 \quad \text{if and only if} \quad \left\| y(\hat{T}; 0, z) \right\| \leq \delta \|z\|. \quad (3.17)$$

*Proof.* Arbitrarily fix  $z \in L^2(\Omega) \setminus \{0\}$  and  $\delta > 0$ . We will prove conclusions (i) – (iii) one by one.

(i) By contradiction, suppose that (3.16) was not true. Then there would be an  $\varepsilon \in (0, \delta)$  and a sequence  $\{\eta_n\}_{n=1}^\infty$  in  $L^2(\Omega)$  so that

$$\lim_{n \rightarrow \infty} \|\eta_n\| = \infty \quad (3.18)$$

and

$$\frac{J(\eta_n; z, \delta, \hat{T}, \hat{s}, \hat{t}, \hat{\omega})}{\|\eta_n\|} \leq (\delta - \varepsilon) \|z\| \quad \text{for all } n \in \mathbb{N}. \quad (3.19)$$

From (3.18), we can assume, without loss of generality, that  $\eta_n \neq 0$  for all  $n$ . Thus we can set

$$\phi_n = \frac{\eta_n}{\|\eta_n\|} \quad \text{for all } n \in \mathbb{N}. \quad (3.20)$$

From (3.20), we see that  $\{\varphi(\hat{t}; \phi_n)\}_{n=1}^\infty$  is bounded in  $L^2(\Omega)$ . Then, from (3.15), (3.20), (3.18) and (3.19), we find that

$$\begin{aligned} & \overline{\lim}_{n \rightarrow \infty} \frac{1}{2} \|1_{\hat{\omega}}^* \varphi(\hat{s}; \phi_n)\|_{\hat{\omega}}^2 \\ &= \overline{\lim}_{n \rightarrow \infty} \frac{1}{\|\eta_n\|} \cdot \left[ \frac{J(\eta_n; z, \delta, \hat{T}, \hat{s}, \hat{t}, \hat{\omega})}{\|\eta_n\|} - \langle z, \varphi(\hat{t}; \phi_n) \rangle - \delta \|z\| \right] \\ &\leq \overline{\lim}_{n \rightarrow \infty} \frac{-\varepsilon \|z\|}{\|\eta_n\|} + \overline{\lim}_{n \rightarrow \infty} \frac{-\langle z, \varphi(\hat{t}; \phi_n) \rangle}{\|\eta_n\|} = 0. \end{aligned} \quad (3.21)$$

Meanwhile, by (3.20), there is a subsequence of  $\{\phi_n\}$ , denoted in the same manner, so that

$$\phi_n \rightarrow \phi \quad \text{weakly in } L^2(\Omega),$$

for some  $\phi \in L^2(\Omega)$ . Since the semigroup  $\{e^{At}\}_{t \geq 0}$  is compact, the above convergence leads to

$$\varphi(\hat{s}; \phi_n) \rightarrow \varphi(\hat{s}; \phi) \quad \text{strongly in } L^2(\Omega); \quad 1_{\hat{\omega}}^* \varphi(\hat{s}; \phi_n) \rightarrow 1_{\hat{\omega}}^* \varphi(\hat{s}; \phi) \quad \text{strongly in } L^2(\hat{\omega}). \quad (3.22)$$

From (3.21) and the second convergence in (3.22), we find that

$$1_{\hat{\omega}}^* \varphi(\hat{s}; \phi) = 0,$$

which, along with the unique continuation property of heat equations built up in [L], yields that  $\phi = 0$ . Then by (3.15) and the first convergence in (3.22), we see that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{J(\eta_n; z, \delta, \hat{T}, \hat{s}, \hat{t}, \hat{\omega})}{\|\eta_n\|} &\geq \liminf_{n \rightarrow \infty} \left[ \langle z, \varphi(\hat{t}; \phi_n) \rangle + \delta \|z\| \right] \\ &= \langle z, \varphi(\hat{t}; 0) \rangle + \delta \|z\| = \delta \|z\|. \end{aligned}$$

This, along with (3.19), leads to a contradiction. Therefore, (3.16) is true.

(ii) From (3.16), we see that the functional  $J(\cdot; z, \delta, \hat{T}, \hat{s}, \hat{t}, \hat{\omega})$  is coercive on  $L^2(\Omega)$ . From (3.15), we find that this functional is continuous and convex on  $L^2(\Omega)$ . Thus, it has a minimizer on  $L^2(\Omega)$ .

Next, we show the uniqueness of the minimizer. It suffices to prove that the functional  $J(\cdot; z, \delta, \hat{T}, \hat{s}, \hat{t}, \hat{\omega})$  is strictly convex. For this purpose, we arbitrarily fix  $\xi_1, \xi_2 \in L^2(\Omega) \setminus \{0\}$ , with  $\xi_1 \neq \xi_2$ . There are only three possibilities: (a)  $\xi_1 \neq \mu \xi_2$  for any  $\mu \in \mathbb{R}$ ; (b)  $\xi_1 = -\mu_0 \xi_2$  for some  $\mu_0 > 0$ ; (c)  $\xi_1 = \mu_0 \xi_2$  for some  $\mu_0 > 0$ . In the cases (a) and (b), one can easily check that

$$\|\lambda \xi_1 + (1 - \lambda) \xi_2\| < \lambda \|\xi_1\| + (1 - \lambda) \|\xi_2\| \quad \text{for all } \lambda \in (0, 1). \quad (3.23)$$

In the case (c), we let

$$H(\lambda) \triangleq J(\lambda \xi_2; z, \delta, \hat{T}, \hat{s}, \hat{t}, \hat{\omega}), \quad \lambda > 0.$$

Since  $\xi_2 \neq 0$  in  $L^2(\Omega)$ , it follows by the unique continuation property of heat equations (built up in [L]) that  $\|1_{\hat{\omega}}^* \varphi(\hat{s}; \xi_2)\|_{\hat{\omega}} \neq 0$ . Thus,  $H(\cdot)$  is a quadratic function with a positive leading coefficient. Hence,  $H(\cdot)$  is strictly convex. This, along with (3.23), yields the strict convexity of  $J(\cdot; z, \delta, \hat{T}, \hat{s}, \hat{t}, \hat{\omega})$ .

(iii) We first show that

$$\left\| y(\hat{T}; 0, z) \right\| \leq \delta \|z\| \implies \xi_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta} = 0. \quad (3.24)$$

In fact, by (3.3) and (3.4), we see that

$$\langle z, \varphi(\hat{t}; \xi) \rangle = \langle y(\hat{T}; 0, z), \xi \rangle \quad \text{for all } z, \xi \in L^2(\Omega).$$

This, along with (3.15) and the inequality on the left hand side of (3.24), yields that

$$J(\xi; z, \delta, \hat{T}, \hat{s}, \hat{t}, \hat{\omega}) \geq \langle y(\hat{T}; 0, z), \xi \rangle + \delta \|z\| \|\xi\| \geq 0 = J(0; z, \delta, \hat{T}, \hat{s}, \hat{t}, \hat{\omega}) \quad \text{for all } \xi \in L^2(\Omega).$$

This implies the equality on the right hand side of (3.24). Hence, (3.24) is true.

We next show that

$$\xi_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta} = 0 \implies \left\| y(\hat{T}; 0, z) \right\| \leq \delta \|z\|. \quad (3.25)$$

By contradiction, suppose that (3.25) were not true. Then we would have that

$$\left\| y(\hat{T}; 0, z) \right\| > \delta \|z\| \quad \text{and} \quad \xi_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta} = 0. \quad (3.26)$$

Set  $\psi \triangleq -y(\hat{T}; 0, z)$ , which clearly belongs to  $\psi \in L^2(\Omega) \setminus \{0\}$ . Then we have that

$$\langle z, \varphi(\hat{t}; \psi) \rangle = \langle y(\hat{T}; 0, z), \psi \rangle = -\|y(\hat{T}; 0, z)\| \|\psi\|.$$

This, along with the first inequality in (3.26), yields that

$$\langle z, \varphi(\hat{t}; \psi) \rangle + \delta \|z\| \|\psi\| < 0.$$

Thus, there is an  $\varepsilon > 0$  so that

$$J(\varepsilon\psi; z, \delta, \hat{T}, \hat{s}, \hat{t}, \hat{\omega}) = \varepsilon^2 \frac{1}{2} \|1_{\hat{\omega}}^* \varphi(\hat{s}; \psi)\|_{\hat{\omega}}^2 + \varepsilon \left[ \langle z, \varphi(\hat{t}; \psi) \rangle + \delta \|z\| \|\psi\| \right] < 0.$$

This, along with the second equation in (3.26), indicates that

$$0 = J(0; z, \delta, \hat{T}, \hat{s}, \hat{t}, \hat{\omega}) = \min_{\xi \in L^2(\Omega)} J(\xi; z, \delta, \hat{T}, \hat{s}, \hat{t}, \hat{\omega}) < 0,$$

which leads to a contradiction. So we have proved (3.25).

Finally, (3.17) follows from (3.24) and (3.25) at once. This ends the proof of Lemma 3.3.  $\square$

Now we will present some properties on the minimal norm control of the problem  $(NP)_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$ .

**Theorem 3.4** *Let  $z \in L^2(\Omega) \setminus \{0\}$  and  $\delta > 0$ . The following conclusions are true:*

(i) *The minimal norm control  $f_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$  to  $(NP)_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$  satisfies that*

$$f_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta} = 0 \quad \text{if and only if} \quad \|y(\hat{T}; 0, z)\| \leq \delta \|z\|.$$

(ii) *The minimal norm control  $f_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$  to  $(NP)_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$  and the minimizer  $\xi_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$  to  $(DP)_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$  have the following connection:*

$$f_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta} = 1_{\hat{\omega}}^* \left( \varphi \left( \hat{s}; \xi_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta} \right) \right).$$

*Proof.* Arbitrarily fix  $z \in L^2(\Omega) \setminus \{0\}$  and  $\delta > 0$ . According to Lemma 3.2 and the conclusion (ii) in Lemma 3.3,  $(NP)_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$  has a unique minimal norm control  $f_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$  and  $(DP)_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$  has a unique minimizer  $\xi_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$ .

The first conclusion in this theorem follows from the definition of Problem  $(NP)_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$  (see (3.6)) at once.

To show the second conclusion of this theorem, we notice that when  $\|y(\hat{T}; 0, z)\| \leq \delta \|z\|$ , it follows respectively from the first conclusion of this theorem and the conclusion (iii) of Lemma 3.3 that

$$f_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta} = 0 \quad \text{and} \quad \xi_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta} = 0.$$

Hence, the second of this theorem is true in this case.

We now consider the case where

$$\|y(\hat{T}; 0, z)\| > \delta \|z\|. \tag{3.27}$$

Write

$$\hat{\xi} \triangleq \xi_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}, \quad \text{and} \quad \hat{f} \triangleq 1_{\hat{\omega}}^* \left( \varphi \left( \hat{s}; \hat{\xi} \right) \right). \quad (3.28)$$

We first claim that

$$\hat{f} \in F_{ad}^{z, \delta} \triangleq \left\{ f \in L^2(\hat{\omega}) \mid \|y(\hat{T}; f, z)\| \leq \delta \|z\| \right\}. \quad (3.29)$$

In fact, by (3.27) and (iii) of Lemma 3.3, we find that  $\hat{\xi} \neq 0$ . Then the Euler-Lagrange equation associated to  $\hat{\xi}$  reads:

$$e^{(\hat{T}-\hat{s})A} \chi_{\hat{\omega}} e^{(\hat{T}-\hat{s})A} \hat{\xi} + e^{(\hat{T}-\hat{t})A} z + \delta \|z\| \frac{\hat{\xi}}{\|\hat{\xi}\|} = 0. \quad (3.30)$$

Since  $1_{\hat{\omega}} 1_{\hat{\omega}}^* = \chi_{\hat{\omega}}$ , it follows from (3.3), (3.28) and (3.4) that

$$y(\hat{T}; \hat{f}, z) = e^{(\hat{T}-\hat{s})A} \chi_{\hat{\omega}} e^{(\hat{T}-\hat{s})A} \hat{\xi} + e^{(\hat{T}-\hat{t})A} z.$$

This, together with (3.30), indicates that

$$y(\hat{T}; \hat{f}, z) = -\delta \|z\| \frac{\hat{\xi}}{\|\hat{\xi}\|}, \quad (3.31)$$

from which, (3.29) follows at once.

We next claim that

$$\|\hat{f}\|_{\hat{\omega}} \leq \|f\|_{\hat{\omega}} \quad \text{for all } f \in F_{ad}^{z, \delta}. \quad (3.32)$$

To this end, we arbitrarily fix an  $f \in F_{ad}^{z, \delta}$ . Then we have that

$$\|y(\hat{T}; f, z)\| \leq \delta \|z\|. \quad (3.33)$$

Since  $1_{\hat{\omega}} 1_{\hat{\omega}}^* = \chi_{\hat{\omega}}$  and  $\chi_{\hat{\omega}} 1_{\hat{\omega}} = 1_{\hat{\omega}}$ , we find from (3.28), (3.3) and (3.33) that

$$\begin{aligned} \|f\|_{\hat{\omega}} \cdot \|\hat{f}\|_{\hat{\omega}} &\geq \left\langle 1_{\hat{\omega}} f, 1_{\hat{\omega}} \hat{f} \right\rangle = \left\langle 1_{\hat{\omega}} f, \chi_{\hat{\omega}} e^{(\hat{T}-\hat{s})A} \hat{\xi} \right\rangle \\ &= \left\langle e^{(\hat{T}-\hat{s})A} 1_{\hat{\omega}} f, \hat{\xi} \right\rangle = \left\langle y(\hat{T}; f, z) - e^{(\hat{T}-\hat{t})A} z, \hat{\xi} \right\rangle \\ &\geq -\delta \|z\| \|\hat{\xi}\| - \left\langle e^{(\hat{T}-\hat{t})A} z, \hat{\xi} \right\rangle. \end{aligned}$$

This, along with (3.30), yields that

$$\begin{aligned} &\|f\|_{\hat{\omega}} \cdot \|\hat{f}\|_{\hat{\omega}} \\ &\geq -\delta \|z\| \|\hat{\xi}\| + \left\langle e^{(\hat{T}-\hat{s})A} \chi_{\hat{\omega}} e^{(\hat{T}-\hat{s})A} \hat{\xi} + \delta \|z\| \frac{\hat{\xi}}{\|\hat{\xi}\|}, \hat{\xi} \right\rangle \\ &= -\delta \|z\| \|\hat{\xi}\| + \left\| 1_{\hat{\omega}}^* e^{(\hat{T}-\hat{s})A} \hat{\xi} \right\|_{\hat{\omega}}^2 + \delta \|z\| \|\hat{\xi}\| \\ &= \left\| 1_{\hat{\omega}}^* e^{(\hat{T}-\hat{s})A} \hat{\xi} \right\|_{\hat{\omega}}^2 = \|\hat{f}\|_{\hat{\omega}}^2. \end{aligned} \quad (3.34)$$

Since  $\hat{\xi} \neq 0$  in this case, we see from (3.28) and the unique continuation estimate at one time (given by (i) of Theorem 2.1), combined with the classical backward uniqueness for parabolic equations, that  $\hat{f} \neq 0$ . This, along with (3.34), leads to (3.32). From (3.29) and (3.32), we find that  $\hat{f}$  is a



minimal norm control to  $(\text{NP})_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$ . Since the minimal norm control of  $(\text{NP})_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$  is unique, we have that  $\hat{f} = f_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$ . So the second conclusion of this theorem is true.

In summary, we end the proof of this theorem. □

### 3.3 Impulse null approximate controllability with a cost

This subsection presents connections among the problem  $(NP)_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{\delta}$ , the property  $(C)^{C, \delta}$  and an inequality on solutions of (3.2). The later can be treated as a kind of observability estimate. Such connections, along with the estimate in (v) of Theorem 2.1, provide a bound for the minimal norm of  $(\text{NP})_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$ . With the aid of this bound, we can get a bound of the norm for the desired feedback law.

**Theorem 3.5** *Let  $\delta > 0$  and  $C > 0$ . The following statements are equivalent:*

- (i) *Let  $N^{\delta}(\hat{T})$  be given by (3.8). Then  $C \geq N^{\delta}(\hat{T})$ .*
- (ii) *The property  $(C)^{C, \delta}$  (defined by (3.9)) is true.*
- (iii) *The following property  $(O)^{C, \delta}$  is true: For each  $\xi \in L^2(\Omega)$ , the solution  $\varphi(\cdot; \xi)$  to (3.2) satisfies*

$$\|\varphi(\hat{t}; \xi)\| \leq C \|1_{\hat{\omega}}^* \varphi(\hat{s}; \xi)\|_{\hat{\omega}} + \delta \|\xi\|.$$

*Proof.* We organize the proof by three steps as follows:

*Step 1. To show that (i)  $\Leftrightarrow$  (ii)*

We first prove that (i)  $\Rightarrow$  (ii). Assume that (i) is true. When  $z = 0$  in  $L^2(\Omega)$ , we find that (3.9) holds for  $f = 0$ . Thus, it suffices to show (ii) with an arbitrarily fixed  $z \in L^2(\Omega) \setminus \{0\}$ . For this purpose, we write  $\hat{z} = z/\|z\|$ . Let  $\hat{f}$  be the solution to  $(\text{NP})_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{\hat{z}, \delta}$ . Then by (3.6), we see that

$$\|y(\hat{T}; \hat{f}, \hat{z})\| \leq \delta \|\hat{z}\| = \delta.$$

Setting  $f = \|z\|\hat{f}$  in the above leads to that

$$\|y(\hat{T}; f, z)\| = \|z\| \|y(\hat{T}; \hat{f}, \hat{z})\| \leq \delta \|z\|.$$

Thus, to show that the above  $f$  satisfies (3.9), we only need to prove that  $\|f\|_{\hat{\omega}} \leq C\|z\|$ . This will be done in what follows: Since  $\hat{f}$  is the solution to  $(\text{NP})_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{\hat{z}, \delta}$ , we have that  $\|\hat{f}\|_{\hat{\omega}} = N^{\hat{z}, \delta}(\hat{T})$ . This, along with (3.8), (3.6) and (i), yields that

$$\|f\|_{\hat{\omega}} = \|z\| \|\hat{f}\|_{\hat{\omega}} = \|z\| N^{\hat{z}, \delta}(\hat{T}) \leq \|z\| N^{\delta}(\hat{T}) \leq C\|z\|.$$

Hence, (ii) is true.

We next show that (ii)  $\Rightarrow$  (i). Assume that (ii) is true. By contradiction, we suppose that (i) were false. Then there would be  $z \in B(1, 0)$  so that  $N^{z, \delta}(\hat{T}) > C$ . It is clear that  $z \neq 0$ . Let  $\hat{z} = z/\|z\|$ . Then

$$N^{\hat{z}, \delta}(\hat{T}) = \frac{1}{\|z\|} N^{z, \delta}(\hat{T}) > N^{z, \delta}(\hat{T}) > C.$$

From this and (3.6), we see that there is no  $f \in L^2(\Omega)$  so that

$$\left\| y(\hat{T}; f, \hat{z}) \right\| \leq \delta \|\hat{z}\| \quad \text{and} \quad \|f\|_{\hat{\omega}} \leq C = C\|\hat{z}\|.$$

This contradicts (ii). Hence, (i) stands.

*Step 2. To show that (ii)  $\Rightarrow$  (iii)*

Suppose that (ii) holds. Then, given  $z \in L^2(\Omega)$ , there is  $f_z \in L^2(\Omega)$  so that (3.9), with  $f = f_z$ , holds. Meanwhile, it follows by (3.3) and (3.4) that

$$\left\langle y(\hat{T}; f_z, z), \xi \right\rangle - \left\langle z, \varphi(\hat{t}; \xi) \right\rangle = \left\langle 1_{\hat{\omega}} f_z, \varphi(\hat{s}; \xi) \right\rangle \quad \text{for all } z, \xi \in L^2(\Omega).$$

This, along with the inequality (3.9), yields that for each  $\xi \in L^2(\Omega)$ ,

$$\begin{aligned} \|\varphi(\hat{t}; \xi)\| &= \sup_{\|z\| \leq 1} \langle \varphi(\hat{t}; \xi), z \rangle \\ &= \sup_{\|z\| \leq 1} \left[ \langle y(\hat{T}; f_z, z), \xi \rangle - \langle 1_{\hat{\omega}} f_z, \varphi(\hat{s}; \xi) \rangle \right] \\ &\leq \sup_{\|z\| \leq 1} \left[ \|y(\hat{T}; f_z, z)\| \cdot \|\xi\| + \|f_z\|_{\hat{\omega}} \cdot \|1_{\hat{\omega}}^* \varphi(\hat{s}; \xi)\|_{\hat{\omega}} \right] \\ &\leq \sup_{\|z\| \leq 1} \left[ (\delta \|\xi\| + C \|1_{\hat{\omega}}^* \varphi(\hat{s}; \xi)\|_{\hat{\omega}}) \|z\| \right] \\ &= C \|1_{\hat{\omega}}^* \varphi(\hat{s}; \xi)\|_{\hat{\omega}} + \delta \|\xi\|, \end{aligned}$$

which leads to  $(O)^{C, \delta}$ . Hence, (iii) is true.

*Step 3. To show that (iii)  $\Rightarrow$  (ii)*

Suppose that (iii) is true. Arbitrarily fix  $z \in L^2(\Omega)$ . In the case that  $\|y(\hat{T}; 0, z)\| \leq \delta \|z\|$ , (3.9) holds for  $f = 0$ . Thus, we only need to consider the case that

$$\|y(\hat{T}; 0, z)\| > \delta \|z\|. \quad (3.35)$$

In this case, we let  $f_z \triangleq 1_{\hat{\omega}}^* \varphi(\hat{s}; \xi^{z, \delta})$ , where  $\xi^{z, \delta} \triangleq \xi_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$  is the solution to  $(\text{DP})_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$ . Then according to (ii) of Theorem 3.4,  $f^*$  is the minimal norm control to  $(\text{NP})_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$ . By Lemma 3.3 and (3.35), we see that  $\xi^{z, \delta} \neq 0$ . Then using (3.4) and (3.30) (with  $\hat{\xi} = \xi^{z, \delta}$ ), and noticing that  $\chi_{\hat{\omega}} = 1_{\hat{\omega}} 1_{\hat{\omega}}^*$ , we find that

$$\begin{aligned} \left\langle z, \varphi(\hat{t}; \xi^{z, \delta}) \right\rangle &= \left\langle e^{(\hat{T} - \hat{t})A} z, \xi^{z, \delta} \right\rangle \\ &= - \left\langle e^{(\hat{T} - \hat{s})A} \chi_{\hat{\omega}} e^{(\hat{T} - \hat{s})A} \xi^{z, \delta} + \delta \|z\| \frac{\xi^{z, \delta}}{\|\xi^{z, \delta}\|}, \xi^{z, \delta} \right\rangle \\ &= - \left\| 1_{\hat{\omega}}^* \varphi(\hat{s}; \xi^{z, \delta}) \right\|_{\hat{\omega}}^2 - \delta \|z\| \|\xi^{z, \delta}\|. \end{aligned}$$

Since  $f_z = 1_{\hat{\omega}}^* \varphi(\hat{s}; \xi^{z, \delta})$ , the above equality, along with (3.15), shows that

$$J(\xi^{z, \delta}; z, \delta, \hat{T}, \hat{s}, \hat{t}, \hat{\omega}) = \frac{1}{2} \left\| 1_{\hat{\omega}}^* \varphi(\hat{s}; \xi^{z, \delta}) \right\|_{\hat{\omega}}^2 + \left\langle z, \varphi(\hat{t}; \xi^{z, \delta}) \right\rangle + \delta \|z\| \|\xi^{z, \delta}\| = -\frac{1}{2} \|f_z\|_{\hat{\omega}}^2. \quad (3.36)$$

Meanwhile, it follows from (3.15) and the inequality in (iii) in  $(O)^{C,\delta}$  that

$$\begin{aligned}
J(\xi^{z,\delta}; z, \delta, \hat{T}, \hat{s}, \hat{t}, \hat{\omega}) &= \frac{1}{2} \left\| 1_{\hat{\omega}}^* \varphi(\hat{s}; \xi^{z,\delta}) \right\|_{\hat{\omega}}^2 + \left\langle z, \varphi(\hat{t}; \xi^{z,\delta}) \right\rangle + \delta \|z\| \|\xi^{z,\delta}\| \\
&\geq \frac{1}{2} \left\| 1_{\hat{\omega}}^* \varphi(\hat{s}; \xi^{z,\delta}) \right\|_{\hat{\omega}}^2 + \delta \|z\| \|\xi^{z,\delta}\| - \|\varphi(\hat{t}; \xi^{z,\delta})\| \|z\| \\
&\geq \frac{1}{2} \left\| 1_{\hat{\omega}}^* \varphi(\hat{s}; \xi^{z,\delta}) \right\|_{\hat{\omega}}^2 + \delta \|z\| \|\xi^{z,\delta}\| - \left[ C \left\| 1_{\hat{\omega}}^* \varphi(\hat{s}; \xi^{z,\delta}) \right\|_{\hat{\omega}} + \delta \|\xi^{z,\delta}\| \right] \|z\| \\
&\geq \frac{1}{2} \|f_z\|_{\hat{\omega}}^2 - C \|f_z\|_{\hat{\omega}} \|z\|.
\end{aligned} \tag{3.37}$$

From (3.36) and (3.37), it follows that

$$\|f_z\|_{\hat{\omega}} \leq C \|z\|. \tag{3.38}$$

On the other hand, since  $f_z$  is the minimal norm control to  $(\text{NP})_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z,\delta}$ , it follows from (3.6) that  $\|y(\hat{T}; f_z, z)\| \leq \delta \|z\|$ . From this and (3.38), we find that (3.9) is true. Hence, (ii) stands.

In summary, we complete the proof.  $\square$

**Remark 3.6** *The value  $N^\delta(\hat{T})$  (given by (3.8)) is indeed the optimal coefficient  $C$  so that the following inequality holds:*

$$\|\varphi(\hat{t}; \xi)\| \leq C \|1_{\hat{\omega}}^* \varphi(\hat{s}; \xi)\|_{\hat{\omega}} + \delta \|\xi\| \text{ for any } \xi \in L^2(\Omega). \tag{3.39}$$

Indeed, the above-mentioned optimal coefficient reads:

$$C^* \triangleq \inf \left\{ C > 0 \mid C \text{ satisfies (3.39)} \right\}.$$

We now claim that

$$C^* = N^\delta(\hat{T}). \tag{3.40}$$

In fact, if  $C$  satisfies (3.39), then it follows from Theorem 3.5 that  $C \geq N^\delta(\hat{T})$ . From this, we find that

$$C^* \geq N^\delta(\hat{T}). \tag{3.41}$$

On the other hand, if we can prove that the property  $(C)^{N^\delta(\hat{T}),\delta}$  holds, then by making use of Theorem 3.5, we see that

$$C^* \leq N^\delta(\hat{T}),$$

which, along with (3.41) leads to (3.40).

Therefore, the remainder is that for each  $z \in L^2(\Omega)$ , there is a control  $f \in L^2(\hat{\omega})$  satisfying that

$$\max \left\{ \frac{1}{N^\delta(\hat{T})} \|f\|_{\hat{\omega}}, \frac{1}{\delta} \|y(\hat{T}; f, z)\| \right\} \leq \|z\|. \tag{3.42}$$

When  $z = 0$ , we can easily get (3.42) by taking  $f = 0$ . So it suffices to prove (3.42) for an arbitrarily fixed  $z \in L^2(\Omega) \setminus \{0\}$ . To this end, we let  $\hat{z} = z/\|z\|$ . It follows from (3.8) that

$$\inf \left\{ \|f\|_{\hat{\omega}} \mid \|y(\hat{T}; f, \hat{z})\| \leq \delta \|\hat{z}\| \right\} \leq N^\delta(\hat{T}).$$

Because the infimum on the left hand side of the above inequality can be reached, there is  $\hat{f} \in L^2(\hat{\omega})$  so that

$$y(\hat{T}; \hat{f}, \hat{z}) \leq \delta \|\hat{z}\| \quad \text{with } \|\hat{f}\|_{\hat{\omega}} \leq N^\delta(\hat{T}).$$

From these, we see that (3.42) holds for  $f = \|z\| \hat{f}$ . Thus we finish the proof of the claim.

### 3.4 Bound of minimal norm

The next theorem provides a bound of the minimal norm for  $(NP)_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$ , defined by (3.6). Its proof is based on Theorem 3.5 and (v) of Theorem 2.1.

**Theorem 3.7** *Let  $\hat{t}$ ,  $\hat{s}$  and  $\hat{T}$  be so that  $0 \leq \hat{t} < \hat{s} < \hat{T}$ . Let  $\delta > 0$ ,  $z \in L^2(\Omega)$  and  $\hat{\omega}$  be a nonempty open subset of  $\Omega$ . Then the minimal norm control  $f_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$  to Problem  $(NP)_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}$  satisfies the following estimate:*

$$\|f_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}\|_{\hat{\omega}} \leq e^{C \left( B_{\hat{T}, \hat{s}, \hat{t}, \|V\|_\infty} + \sqrt{\frac{[\ln(1/\delta)]^+}{\hat{T} - \hat{s}}} \right)} \|z\|, \quad (3.43)$$

where  $C > 1$  is a constant (depending only on  $\Omega$  and  $\hat{\omega}$ ),  $[\ln(1/\delta)]^+ \triangleq \max\{\ln \frac{1}{\delta}, 0\}$  and

$$B_{\hat{T}, \hat{s}, \hat{t}, \|V\|_\infty} \triangleq 1 + \frac{1}{\hat{T} - \hat{s}} + (\hat{T} - \hat{t})\|V\|_\infty + \|V\|_\infty^{2/3}. \quad (3.44)$$

*Proof.* Arbitrarily fix  $\hat{t}$ ,  $\hat{s}$ ,  $\hat{T}$ ,  $z$  and  $\hat{\omega}$  as required. First of all, we notice that  $\|f_{\hat{T}, \hat{s}, \hat{t}, \hat{\omega}}^{z, \delta}\|_{\hat{\omega}}$ , as a function of  $\delta \in (0, \infty)$ , is decreasing. Thus, it suffices to show (3.43) in the case when  $\delta \in (0, 1)$ . To this end, we arbitrarily fix  $\delta \in (0, 1)$  and  $\xi \in L^2(\Omega)$ . Since

$$\sqrt{a+b} \leq \sqrt{a} + \sqrt{b} \quad \text{for all } a, b \geq 0, \quad (3.45)$$

by taking  $t = \hat{T} - \hat{s}$  in (v) of Theorem 2.1, we see that for all  $\varepsilon > 0$  and  $\beta > 0$ ,

$$\begin{aligned} \|e^{(\hat{T}-\hat{s})A}\xi\| &\leq \left[ \frac{1}{\varepsilon^\beta} e^{2\hat{C}(1+\beta) \left( 1 + \frac{1+\beta}{\beta(\hat{T}-\hat{s})} + (\hat{T}-\hat{s})\|V\|_\infty + \|V\|_\infty^{2/3} \right)} \|1_{\hat{\omega}}^* e^{(\hat{T}-\hat{s})A}\xi\|_{\hat{\omega}}^2 + \varepsilon \|\xi\|^2 \right]^{1/2} \\ &\leq \frac{1}{(\sqrt{\varepsilon})^\beta} e^{\hat{C}(1+\beta) \left( 1 + \frac{1+\beta}{\beta(\hat{T}-\hat{s})} + (\hat{T}-\hat{s})\|V\|_\infty + \|V\|_\infty^{2/3} \right)} \|1_{\hat{\omega}}^* e^{(\hat{T}-\hat{s})A}\xi\|_{\hat{\omega}} + \sqrt{\varepsilon} \|\xi\|, \end{aligned}$$

where  $\hat{C} \triangleq 1 + C$  and  $C$  is given by (v) of Theorem 2.1. From this, we find that

$$\begin{aligned} \|e^{(\hat{T}-\hat{t})A}\xi\| &\leq e^{(\hat{s}-\hat{t})\|V\|_\infty} \|e^{(\hat{T}-\hat{s})A}\xi\| \\ &\leq \frac{e^{(\hat{s}-\hat{t})\|V\|_\infty}}{(\sqrt{\varepsilon})^\beta} e^{\hat{C}(1+\beta) \left( 1 + \frac{1+\beta}{\beta(\hat{T}-\hat{s})} + (\hat{T}-\hat{s})\|V\|_\infty + \|V\|_\infty^{2/3} \right)} \|1_{\hat{\omega}}^* e^{(\hat{T}-\hat{s})A}\xi\|_{\hat{\omega}} + e^{(\hat{s}-\hat{t})\|V\|_\infty} \sqrt{\varepsilon} \|\xi\|. \end{aligned}$$

Since  $\xi$  was arbitrarily taken from  $L^2(\Omega)$ , by taking  $\varepsilon = \left( \frac{\delta}{e^{(\hat{s}-\hat{t})\|V\|_\infty}} \right)^2$  in the above inequality and then using (3.4), we see that for each  $\beta > 0$ ,

$$\|\varphi(\hat{t}; \xi)\| \leq C_{\beta, \delta} \|1_{\hat{\omega}}^* \varphi(\hat{s}; \xi)\|_{\hat{\omega}} + \delta \|\xi\| \quad \text{for all } \xi \in L^2(\Omega), \quad (3.46)$$

where

$$C_{\beta,\delta} \triangleq \frac{1}{\delta^\beta} e^{(1+\beta)(\hat{s}-\hat{t})\|V\|_\infty} e^{\hat{C}(1+\beta)\left(1+\frac{1+\beta}{\beta(\hat{T}-\hat{s})}+(\hat{T}-\hat{s})\|V\|_\infty+\|V\|_\infty^{2/3}\right)}. \quad (3.47)$$

By (3.46), we see that for each  $\beta > 0$ , the property  $(O)^{C_{\beta,\delta}}$  (given in (i) of Theorem 3.5) is true. Then we can apply Theorem 3.5 to find that (i) of Theorem 3.5 holds. Thus, the minimal norm control  $f_{\hat{T},\hat{s},\hat{t},\hat{\omega}}^{z,\delta}$  to Problem  $(NP)_{\hat{T},\hat{s},\hat{t},\hat{\omega}}^{z,\delta}$  satisfies

$$\|f_{\hat{T},\hat{s},\hat{t},\hat{\omega}}^{z,\delta}\|_{\hat{\omega}} \leq C_{\beta,\delta}\|z\| \quad \text{for each } \beta > 0. \quad (3.48)$$

Next, we will prove that

$$\inf_{\beta>0} C_{\beta,\delta} \leq \exp \left[ 4\hat{C} \left( B_{\hat{T},\hat{s},\hat{t},\|V\|_\infty} + \sqrt{\frac{[\ln(1/\delta)]^+}{\hat{T}-\hat{s}}} \right) \right], \quad (3.49)$$

where  $B_{\hat{T},\hat{s},\hat{t},\|V\|_\infty}$  is given by (3.44). In fact, since  $\hat{C} > 1$ , we find from (3.47) that for all  $\beta > 0$ ,

$$\begin{aligned} C_{\beta,\delta} &\leq \frac{1}{\delta^\beta} e^{\hat{C}(1+\beta)(\hat{s}-\hat{t})\|V\|_\infty} e^{\hat{C}(1+\beta)\left(1+\frac{1+\beta}{\beta(\hat{T}-\hat{s})}+(\hat{T}-\hat{s})\|V\|_\infty+\|V\|_\infty^{2/3}\right)} \\ &= e^{(\ln \frac{1}{\delta})\beta} e^{\hat{C}(1+\beta)\left(1+\frac{1+\beta}{\beta(\hat{T}-\hat{s})}+(\hat{T}-\hat{t})\|V\|_\infty+\|V\|_\infty^{2/3}\right)} \\ &= \exp \left[ \left( \hat{C} B_{\hat{T},\hat{s},\hat{t},\|V\|_\infty} + \ln \frac{1}{\delta} \right) \beta + \frac{\hat{C}}{\beta(\hat{T}-\hat{s})} + \hat{C} \left( B_{\hat{T},\hat{s},\hat{t},\|V\|_\infty} + \frac{1}{\hat{T}-\hat{s}} \right) \right], \end{aligned}$$

which, along with (3.45), yields that

$$\begin{aligned} \inf_{\beta>0} C_{\beta,\delta} &= \exp \left\{ 2\sqrt{\frac{\hat{C}^2 B_{\hat{T},\hat{s},\hat{t},\|V\|_\infty}{\hat{T}-\hat{s}} + \frac{\hat{C} \ln \frac{1}{\delta}}{\hat{T}-\hat{s}}} + \hat{C} \left( B_{\hat{T},\hat{s},\hat{t},\|V\|_\infty} + \frac{1}{\hat{T}-\hat{s}} \right) \right\} \\ &\leq \exp \left\{ 2\hat{C} \sqrt{\frac{B_{\hat{T},\hat{s},\hat{t},\|V\|_\infty}{\hat{T}-\hat{s}}} + 2\sqrt{\frac{\hat{C} \ln \frac{1}{\delta}}{\hat{T}-\hat{s}}} + \hat{C} \left( B_{\hat{T},\hat{s},\hat{t},\|V\|_\infty} + \frac{1}{\hat{T}-\hat{s}} \right) \right\}. \quad (3.50) \end{aligned}$$

Meanwhile, we have that

$$\sqrt{\frac{B_{\hat{T},\hat{s},\hat{t},\|V\|_\infty}{\hat{T}-\hat{s}}} \leq \frac{1}{2} B_{\hat{T},\hat{s},\hat{t},\|V\|_\infty} + \frac{1}{2} \frac{1}{\hat{T}-\hat{s}}.$$

This, together with (3.50), indicates that

$$\inf_{\beta>0} C_{\beta,\delta} \leq \exp \left\{ 2\hat{C} \left( B_{\hat{T},\hat{s},\hat{t},\|V\|_\infty} + \frac{1}{\hat{T}-\hat{s}} \right) + 2\sqrt{\frac{\hat{C} \ln \frac{1}{\delta}}{\hat{T}-\hat{s}}} \right\}.$$

Since  $\hat{C} > 1$ , the above inequality, as well as (3.44), implies that

$$\inf_{\beta>0} C_{\beta,\delta} \leq e^{4\hat{C} \left( B_{\hat{T},\hat{s},\hat{t},\|V\|_\infty} + \sqrt{\frac{\ln \frac{1}{\delta}}{\hat{T}-\hat{s}}} \right)}.$$

This leads to (3.49) (in the case that  $\delta \in (0, 1)$ ). Now, (3.43), with  $\delta \in (0, 1)$ , follows from (3.48) and (3.49). This ends the proof.  $\square$

**Remark 3.8** From Theorem 3.7, we see that the bound of the minimal norm control depends on several quantities, such as time, domains, potential and  $\delta$  (radius of the target ball). Among these quantities,  $\delta$  may be the most important. The way how it depends on  $\delta$  is as:  $e^{\tilde{C}\sqrt{[\ln(1/\delta)]^+}}$  for some  $\tilde{C} > 0$  independent of  $\delta$ . Meanwhile, from the proof of Theorem 3.7, we find that this bound is from the minimization of  $C_{\beta,\delta}$  with respect to  $\beta$ . (Here,  $C_{\beta,\delta}$  is given by (3.47).) Thus, it follows by (3.46) that

$$\|\varphi(\hat{t}; \xi)\| \leq \tilde{C}e^{\tilde{C}\sqrt{[\ln(1/\delta)]^+}} \|1_{\hat{\omega}}^* \varphi(\hat{s}; \xi)\|_{\hat{\omega}} + \delta \|\xi\| \quad \text{for all } \xi \in L^2(\Omega). \quad (3.51)$$

Meanwhile, in the special case when  $V = 0$  and  $0 \leq \hat{t} = \hat{s} < \hat{T}$ , the following property is true: there exists  $C_0 > 0$ ,  $\{\eta_n\} \subset L^2(\Omega) \setminus \{0\}$  and  $\{\delta_n\} \subset (0, 1)$ , with  $\lim_{n \rightarrow \infty} \delta_n = 0$ , so that

$$\|\varphi(\hat{s}; \eta_n)\| \geq C_0 e^{C_0 \sqrt{\ln(1/\delta_n)}} \|1_{\hat{\omega}}^* \varphi(\hat{s}; \eta_n)\|_{\hat{\omega}} + \delta_n \|\eta_n\| \quad \text{for all } n. \quad (3.52)$$

From (3.51) and (3.52), we see that when  $V = 0$  and  $0 \leq \hat{t} = \hat{s} < \hat{T}$ , the quantity  $\sqrt{[\ln(1/\delta)]^+}$  is optimal in the following sense: For any function  $\delta \rightarrow \alpha(\delta)$ ,  $\delta > 0$ , with  $\overline{\lim}_{\delta \rightarrow 0^+} \frac{\alpha(\delta)}{\sqrt{[\ln(1/\delta)]^+}} = 0$ , the following inequality is not true:

$$\|\varphi(\hat{s}; \xi)\| \leq \alpha(\delta) e^{\alpha(\delta)} \|1_{\hat{\omega}}^* \varphi(\hat{s}; \xi)\|_{\hat{\omega}} + \delta \|\xi\| \quad \text{for all } \xi \in L^2(\Omega) \text{ and } \delta > 0.$$

We now prove (3.52). According to Proposition 5.5 in [LL], there exists  $C'_0 > 0$  and  $n_0 > 0$  so that for each  $n \geq n_0$ , there is  $\{a_{n,j}\}_{j=1}^\infty \subset l^2 \setminus \{0\}$  satisfying that

$$\left\| \sum_{\lambda_j \leq n} a_{n,j} \xi_j \right\| \geq C'_0 e^{C'_0 \sqrt{n}} \|1_{\hat{\omega}}^* \sum_{\lambda_j \leq n} a_{n,j} \xi_j\|_{\hat{\omega}}. \quad (3.53)$$

Define, for each  $n \in \mathbb{N}$ ,

$$\eta_n \triangleq \sum_{\lambda_j \leq n} e^{\lambda_j(\hat{T}-\hat{s})} a_{n,j} \xi_j \quad \text{and} \quad \delta_n \triangleq \frac{1}{2} e^{-n(\hat{T}-\hat{s})}. \quad (3.54)$$

From (3.54) and (3.53), we find that for each  $n \geq n_0$ ,

$$\begin{aligned} \|\varphi(\hat{s}; \eta_n)\| &= \left\| \sum_{\lambda_j \leq n} a_{n,j} \xi_j \right\| = (1 - \delta_n e^{n(\hat{T}-\hat{s})}) \left\| \sum_{\lambda_j \leq n} a_{n,j} \xi_j \right\| + \delta_n e^{n(\hat{T}-\hat{s})} \left\| \sum_{\lambda_j \leq n} a_{n,j} \xi_j \right\| \\ &\geq \frac{1}{2} \left\| \sum_{\lambda_j \leq n} a_{n,j} \xi_j \right\| + \delta_n \left\| \sum_{\lambda_j \leq n} e^{\lambda_j(\hat{T}-\hat{s})} a_{n,j} \xi_j \right\| \\ &\geq \frac{1}{2} C'_0 e^{C'_0 \sqrt{n}} \|1_{\hat{\omega}}^* \sum_{\lambda_j \leq n} a_{n,j} \xi_j\|_{\hat{\omega}} + \delta_n \|\eta_n\|. \end{aligned} \quad (3.55)$$

Meanwhile, it follows from (3.54) that for each  $n \in \mathbb{N}$ ,

$$\sqrt{\ln(1/\delta_n)} = \sqrt{\ln 2 + n(\hat{T} - \hat{s})} \leq 1 + \sqrt{n} \sqrt{\hat{T} - \hat{s}}. \quad (3.56)$$

By (3.55) and (3.56), we see that

$$\|\varphi(\hat{s}; \eta_n)\| \geq \frac{1}{2} C'_0 e^{-\frac{C'_0}{\sqrt{\hat{T}-\hat{s}}} e^{\frac{C'_0}{\sqrt{\hat{T}-\hat{s}}}} \sqrt{\ln(1/\delta_n)}} \|1_{\hat{\omega}}^* \sum_{\lambda_j \leq n} a_{n,j} \xi_j\|_{\hat{\omega}} + \delta_n \|\eta_n\|,$$

which leads to (3.52).

Next, we would like to mention the following fact: The order  $\sqrt{[\ln(1/\varepsilon)]^+}$  is also optimal in the inequality in (v) of Theorem 2.1 (with  $V = 0$  and  $\hat{t} = \hat{s}$ ) in the above-mentioned sense. Then according to Theorem 3.5, the following conclusion is not true: For each function  $\alpha(\cdot)$  over  $\mathbb{R}^+$ , with  $\lim_{\eta \rightarrow 0^+} \frac{\alpha(\eta)}{\sqrt{[\ln(1/\eta)]^+}} = 0$ , each  $\delta > 0$  and each  $z \in L^2(\Omega)$ , the minimal norm control  $f_{\hat{T}, \hat{s}, \hat{s}, \hat{\omega}}^{z, \delta}$  to  $(NP)_{\hat{T}, \hat{s}, \hat{s}, \hat{\omega}}^{z, \delta}$  (with  $V = 0$ ) satisfies the following estimate:

$$\|f_{\hat{T}, \hat{s}, \hat{s}, \hat{\omega}}^{z, \delta}\|_{\hat{\omega}} \leq e^{\alpha(\delta)} \|z\|.$$

Finally, it deserves mentioning that for any  $\beta > 0$ ,

$$e^{\sqrt{[\ln(1/\delta)]}} \leq \frac{1}{\delta^\beta}, \quad \text{when } \delta > 0 \text{ is small.} \quad (3.57)$$

Indeed, we have that

$$\frac{1}{\delta^\beta} = e^{\beta \ln(1/\delta)} \quad \text{for all } \beta > 0 \text{ and } \delta > 0$$

and

$$\sqrt{[\ln(1/\delta)]} \leq \ln(1/\delta), \quad \text{when } \delta > 0 \text{ is small.}$$

From these, (3.57) follows at once.

## 4 Main result

This section presents the main result of this paper, as well as its proof. We first recall that  $s$ ,  $\tau$  and  $T$ , with  $0 < s \leq \tau < T$ , are three arbitrarily fixed time points and that  $\omega_1$  and  $\omega$  are two arbitrarily fixed open and non-empty subsets of  $\Omega$ . we next recall that  $\{\lambda_j\}_{j=1}^\infty$  is the family of all eigenvalues of  $-A$  so that (1.2) holds, and that  $\{\xi_j\}_{j=1}^\infty$  is the family of the corresponding normalized eigenfunctions. For each  $\gamma > 0$ , we define a natural number  $K_\gamma$  in the following manner:

$$K_\gamma \triangleq \text{card} \left\{ j \in \mathbb{N}, \lambda_j < \gamma + \frac{\ln 3}{T} \right\}. \quad (4.1)$$

Then we define numbers  $\epsilon_\gamma$  and  $\epsilon_{j,\gamma}$ , with  $1 \leq j \leq K_\gamma$ , as follows:

$$\epsilon_\gamma \triangleq \frac{1}{3K_\gamma^{1/2}} e^{-\gamma T}, \quad \epsilon_{j,\gamma} \triangleq e^{-2\lambda_j s} \frac{1}{6K_\gamma^{1/2}} e^{-(\gamma + \|V\|_\infty)T}. \quad (4.2)$$

Write respectively  $\xi_{T,\tau,0,\omega}^{\xi_j, \epsilon_\gamma}$  and  $\xi_{2s,s,0,\omega_1}^{\xi_j, \epsilon_{j,\gamma}}$  for the solutions to problems  $(DP)_{T,\tau,0,\omega}^{\xi_j, \epsilon_\gamma}$  and  $(DP)_{2s,s,0,\omega_1}^{\xi_j, \epsilon_{j,\gamma}}$ . (These two problems are defined by (3.14), with  $(z, \delta, \hat{T}, \hat{s}, \hat{t}, \hat{\omega})$  being replaced by  $(\xi_j, \epsilon_\gamma, T, \tau, 0, \omega)$  and  $(\xi_j, \epsilon_{j,\gamma}, 2s, s, 0, \omega_1)$  respectively.) Then from (ii) of Theorem 3.4, we find that for each  $\gamma > 0$  and each  $1 \leq j \leq K_\gamma$ ,

$$f_{j,\gamma} \triangleq f_{T,\tau,0,\omega}^{\xi_j, \epsilon_\gamma} = 1_\omega^* \left( e^{(T-\tau)A} \xi_{T,\tau,0,\omega}^{\xi_j, \epsilon_\gamma} \right) \quad \text{and} \quad h_{j,\gamma} \triangleq f_{2s,s,0,\omega_1}^{\xi_j, \epsilon_{j,\gamma}} = 1_{\omega_1}^* \left( e^{sA} \xi_{2s,s,0,\omega_1}^{\xi_j, \epsilon_{j,\gamma}} \right) \quad (4.3)$$

are respectively the minimal norm controls to problems  $(NP)_{T,\tau,0,\omega}^{\xi_j, \epsilon_\gamma}$  and  $(NP)_{2s,s,0,\omega_1}^{\xi_j, \epsilon_{j,\gamma}}$ . (These two problems are defined by (3.6), with  $(z, \delta, \hat{T}, \hat{s}, \hat{t}, \hat{\omega})$  being replaced by  $(\xi_j, \epsilon_\gamma, T, \tau, 0, \omega)$  and

( $\xi_j, \epsilon_{j,\gamma}, 2s, s, 0, \omega_1$ ) respectively.) Now for each  $\gamma > 0$ , we define a linear bounded operator  $\mathcal{F}_\gamma$  from  $L^2(\omega_1)$  into  $L^2(\omega)$  in the following manner:

$$\mathcal{F}_\gamma(v) = - \sum_{j=1}^{K_\gamma} e^{2\lambda_j s} \langle 1_{\omega_1} v, 1_{\omega_1} h_{j,\gamma} \rangle f_{j,\gamma} \quad \text{for each } v \in L^2(\omega_1). \quad (4.4)$$

The closed-loop equation under consideration reads:

$$\begin{cases} y'(t) - Ay(t) = 0 & \text{in } \mathbb{R}^+ \setminus (\tau + \bar{\mathbb{N}}T), \\ y(\tau + nT) = y((\tau + nT)_-) + 1_\omega \mathcal{F}_\gamma(1_{\omega_1}^* y(s + nT)) & \text{for } n \in \bar{\mathbb{N}}, \\ y(0) \in L^2(\Omega). \end{cases} \quad (4.5)$$

The main result of this paper is the following theorem:

**Theorem 4.1** *For each  $\gamma > 0$ , let  $\mathcal{F}_\gamma$  be given by (4.4). Then the following conclusions are true:*

(i) *Each solution  $y_\gamma(\cdot)$  to the equation (4.5) satisfies that*

$$\|y_\gamma(t)\| \leq e^{T(\gamma + \|V\|_\infty)} \left(1 + \|\mathcal{F}_\gamma\|_{\mathcal{L}(L^2(\omega_1), L^2(\omega))}\right) e^{-\gamma t} \|y_\gamma(0)\| \quad \text{for all } t \geq 0.$$

(ii) *The operator  $\mathcal{F}_\gamma$  satisfies the estimate:*

$$\|\mathcal{F}_\gamma\|_{\mathcal{L}(L^2(\omega_1), L^2(\omega))} \leq C_1 e^{C_2 \gamma},$$

where  $C_1$  and  $C_2$  are two positive constants depending on  $T, \tau, s, d$  (dimension of the space),  $\|V\|_\infty, \Omega, \omega$  and  $\omega_1$ , but independent of  $\gamma$ . Moreover, the manner how it depends on  $T, \tau, s, d$  (dimension of the space),  $\|V\|_\infty$  is explicitly given.

*Proof.* For each  $k \in \bar{\mathbb{N}}$ , let  $V_k \triangleq \text{span}\{\xi_1, \dots, \xi_k\}$  and let  $P_k$  be the orthogonal projection of  $L^2(\Omega)$  onto  $V_k$ . Arbitrarily fix  $\gamma > 0$ . We organize the proof by several steps.

In Step 1, we will show that if any solution  $y_\gamma(\cdot)$  to the equation (4.5) satisfies that

$$\|y_\gamma((n+1)T)\| \leq e^{-\gamma T} \|y_\gamma(nT)\| \quad \text{for any } n \in \bar{\mathbb{N}}, \quad (4.6)$$

then the conclusion (i) in the theorem stands.

To prove this, we arbitrarily fix a solution  $y_\gamma(\cdot)$  to the equation (4.5). Then by (4.6), we see that

$$\|y_\gamma(nT)\| \leq e^{-\gamma nT} \|y_\gamma(0)\| \quad \text{for all } \bar{\mathbb{N}}. \quad (4.7)$$

Arbitrarily fix an  $n \in \bar{\mathbb{N}}$ . From (4.5) and (4.7), we find that

$$\begin{aligned} \|y_\gamma(t)\| &\leq e^{(t-nT)\|V\|_\infty} \|y_\gamma(nT)\| \leq e^{(t-nT)\|V\|_\infty} e^{-\gamma nT} \|y_\gamma(0)\| \\ &\leq e^{\tau\|V\|_\infty} e^{-\gamma nT} \|y_\gamma(0)\| \leq e^{T(\gamma + \|V\|_\infty)} e^{-\gamma t} \|y_\gamma(0)\|, \quad \text{when } t \in [nT, \tau + nT], \end{aligned}$$

and

$$\begin{aligned} \|y_\gamma(t)\| &\leq e^{(t-\tau-nT)\|V\|_\infty} \|y_\gamma(\tau + nT)\| \\ &\leq e^{(t-\tau-nT)\|V\|_\infty} \|y_\gamma((\tau + nT)_-)\| + e^{(t-\tau-nT)\|V\|_\infty} \|\mathcal{F}_\gamma(1_{\omega_1}^* y(s + nT))\|_\omega \\ &\leq e^{(t-nT)\|V\|_\infty} \|y_\gamma(nT)\| + \|\mathcal{F}_\gamma\|_{\mathcal{L}(L^2(\omega_1), L^2(\omega))} e^{(t-\tau-nT)\|V\|_\infty} e^{s\|V\|_\infty} \|y_\gamma(nT)\| \\ &\leq e^{T(\gamma + \|V\|_\infty)} \left(1 + \|\mathcal{F}_\gamma\|_{\mathcal{L}(L^2(\omega_1), L^2(\omega))}\right) e^{-\gamma t} \|y_\gamma(0)\|, \quad \text{when } t \in [\tau + nT, (n+1)T]. \end{aligned}$$



From these, we see that the conclusion (i) in the theorem is true.

In Step 2, we will prove that for each  $\varepsilon > 0$  and each  $k \in \mathbb{N}$  with  $k > m$ , the linear operator  $\mathcal{C}^{\varepsilon,k} : V_k \rightarrow L^2(\omega)$ , constructed in the following manner:

$$\mathcal{C}^{\varepsilon,k}(z) = \sum_{j=1}^k \langle z, \xi_j \rangle f_j^\varepsilon, \quad z \in V_k, \quad (4.8)$$

with  $f_j^\varepsilon \triangleq f_{T,\tau,0,\omega}^{\xi_j,\varepsilon}$  the minimal norm control to  $(\text{NP})_{T,\tau,0,\omega}^{\xi_j,\varepsilon}$ , satisfies that for each  $z_0 \in L^2(\Omega)$ ,

$$\left\| e^{TA} z_0 + e^{(T-\tau)A} \mathbf{1}_\omega \mathcal{C}^{\varepsilon,k}(P_k z_0) \right\| \leq \varepsilon \sqrt{k} \|P_k z_0\| + e^{-\lambda_{k+1}T} \|z_0 - P_k z_0\|. \quad (4.9)$$

To prove this, we first consider the following controlled equation:

$$\begin{cases} y'(t) - Ay(t) = 0 & \text{in } [nT, (n+1)T] \setminus (\tau + nT), \\ y(\tau + nT) = y((\tau + nT)_-) + \mathbf{1}_\omega f \\ y(nT) = z_0, \end{cases} \quad (4.10)$$

where  $z_0 \in L^2(\Omega)$ ,  $f \in L^2(\omega)$  and  $n \in \bar{\mathbb{N}}$ . Write  $y_n(\cdot; f, z_0)$  for the solution of (4.10). The relationship between  $y_n(\cdot; f, z_0)$  and the solution  $y(\cdot; f, z_0)$  to the equation (3.1), where

$$(\hat{s}, \hat{t}, \hat{T}, \hat{\omega}, z) = (\tau, nT, (n+1)T, \omega, z_0),$$

is as follows:

$$y_n(t; f, z_0) = y(t - nT; f, z_0) \quad \text{for each } t \in [nT, (n+1)T]. \quad (4.11)$$

Arbitrarily fix  $\varepsilon > 0$ ,  $k \in \mathbb{N}$  with  $k > m$ , and  $z_0 \in L^2(\Omega)$ . Since  $f_j^\varepsilon$  is the minimal norm control to  $(\text{NP})_{T,\tau,0,\omega}^{\xi_j,\varepsilon}$ , from (3.7), where

$$(z, \delta, \hat{T}, \hat{s}, \hat{t}, \hat{\omega}) = (\xi_j, \varepsilon, T, \tau, 0, \omega),$$

we find that

$$\|y(T; f_j^\varepsilon, \xi_j)\| \leq \varepsilon \quad \text{for all } 1 \leq j \leq k.$$

This, along with (4.11), yields that

$$\|y_n((n+1)T; f_j^\varepsilon, \xi_j)\| \leq \varepsilon \quad \text{for all } 1 \leq j \leq k. \quad (4.12)$$

Define

$$f_{z_0}^{\varepsilon,k} \triangleq \sum_{j=1}^k \langle z_0, \xi_j \rangle f_j^\varepsilon, \quad (4.13)$$

Since

$$\|y_n((n+1)T; 0, z_0 - P_k z_0)\| \leq e^{-\lambda_{k+1}T} \|z_0 - P_k z_0\|,$$

it follows by (4.12) and (4.13) that

$$\begin{aligned}
& \|y_n((n+1)T; f_{z_0}^{\varepsilon,k}, z_0)\| \\
&= \|y_n((n+1)T; f_{z_0}^{\varepsilon,k}, P_k z_0)\| + \|y_n((n+1)T; 0, z_0 - P_k z_0)\| \\
&\leq \sum_{j=1}^k \left[ |\langle z_0, \xi_j \rangle| \cdot \|y_n((n+1)T; f_j^\varepsilon, \xi_j)\| \right] + e^{-\lambda_{k+1}T} \|z_0 - P_k z_0\| \\
&\leq \varepsilon \sum_{j=1}^k |\langle z_0, \xi_j \rangle| + e^{-\lambda_{k+1}T} \|z_0 - P_k z_0\| \\
&\leq \varepsilon \sqrt{k} \|P_k z_0\| + e^{-\lambda_{k+1}T} \|z_0 - P_k z_0\|.
\end{aligned} \tag{4.14}$$

Meanwhile, we have from (4.10) and (4.13) that

$$y_n((n+1)T; f_{z_0}^{\varepsilon,k}, z_0) = e^{TA} z_0 + e^{(T-\tau)A} 1_\omega \sum_{j=1}^k \langle z_0, \xi_j \rangle f_j^\varepsilon,$$

while, we see from (4.8) that

$$\mathcal{C}^{\varepsilon,k}(P_k z_0) = \sum_{j=1}^k \langle z_0, \xi_j \rangle f_j^\varepsilon.$$

These, along with (4.14), leads to (4.9).

It deserves mentioning what follows: From (4.9), one can easily check that when  $\omega_1 = \Omega$  and  $s = \tau$ , we can suitably choose  $\varepsilon$  and  $k$  in terms of  $\gamma$  and  $N_\gamma$  so that  $\hat{\mathcal{F}}_\gamma \triangleq \mathcal{C}^{\varepsilon,k} P_k$  plays the role of such a feedback law that each solution  $\hat{y}_\gamma(\cdot)$  to the equation (4.5), where

$$(\omega_1, s, \mathcal{F}_\gamma) = (\Omega, \tau, \hat{\mathcal{F}}_\gamma),$$

satisfies that

$$\|\hat{y}_\gamma(t)\| \leq \hat{C}_\gamma e^{-\gamma t} \|\hat{y}_\gamma(0)\|, \quad \text{when } t \geq 0,$$

for some positive constant  $\hat{C}_\gamma$ . However, in the current case,  $\omega_1$  may not be  $\Omega$  and  $s < \tau$ . Thus we need recover approximately  $P_k z_0$  from  $1_{\omega_1}^* e^{sA} z_0$ . This is the job in the next step.

In Step 3, we will construct, for each  $k > m$  and  $\hat{\varepsilon} > 0$ , a linear and bounded operator  $\mathcal{R}^{\hat{\varepsilon},k} : L^2(\omega_1) \rightarrow V_k$  so that

$$\|\mathcal{R}^{\hat{\varepsilon},k} (1_{\omega_1}^* e^{sA} z_0) - P_k z_0\|_{L^2(\Omega)} \leq \sqrt{k} \hat{\varepsilon} \|z_0\|_{L^2(\Omega)} \quad \text{for any } z_0 \in L^2(\Omega). \tag{4.15}$$

To this end, we arbitrarily fix  $k > m$  and  $\hat{\varepsilon} > 0$ . Write

$$\hat{\varepsilon}_j \triangleq \hat{\varepsilon} e^{-2\lambda_j s}, \quad j = 1, \dots, k. \tag{4.16}$$

Let

$$\hat{h}_j \triangleq f_{2s, s, 0, \omega_1}^{\xi_j, \hat{\varepsilon}_j}, \quad 1 \leq j \leq k.$$

Then each  $\hat{h}_j$  is the minimal norm control to (NP) $_{2s, s, 0, \omega_1}^{\xi_j, \hat{\varepsilon}_j}$ . Write  $\tilde{y}(\cdot; \hat{h}_j, \xi_j)$  for the solution to the equation (3.1), with

$$(\hat{T}, \hat{s}, \hat{t}, \hat{\omega}, z, f) = (2s, s, 0, \omega_1, \xi_j, \hat{h}_j).$$

Then by (3.7), it follows that

$$\|\tilde{y}(2s; \hat{h}_j, \xi_j)\| \leq e^{-2\lambda_j s \hat{\varepsilon}}.$$

Meanwhile, one can easily check that for each  $j$  with  $1 \leq j \leq k$ ,

$$\tilde{y}(2s; \hat{h}_j, \xi_j) = e^{-2\lambda_j s} \xi_j + e^{sA} 1_{\omega_1} \hat{h}_j.$$

From these and (4.16), we see that the vector  $\zeta_j$  (with  $1 \leq j \leq k$ ), defined by

$$\zeta_j \triangleq e^{-2\lambda_j s} \xi_j + e^{sA} 1_{\omega_1} \hat{h}_j, \quad (4.17)$$

satisfies that

$$\|\zeta_j\| \leq \hat{\varepsilon}_j \|\xi_j\| = e^{-2\lambda_j s} \hat{\varepsilon}. \quad (4.18)$$

Define a linear and bounded operator  $\mathcal{R}^{\hat{\varepsilon}, k} : L^2(\omega_1) \rightarrow V_k$  in the following manner:

$$\mathcal{R}^{\hat{\varepsilon}, k}(\eta) = - \sum_{j=1}^k \left\langle 1_{\omega_1} \eta, 1_{\omega_1} \hat{h}_j \right\rangle e^{2\lambda_j s} \xi_j \quad \text{for any } \eta \in L^2(\omega_1). \quad (4.19)$$

By a direct calculation, we see that

$$\mathcal{R}^{\hat{\varepsilon}, k} (1_{\omega_1}^* e^{sA} z_0) - P_k z_0 = - \sum_{j=1}^k \left\langle z_0, e^{2\lambda_j s} \zeta_j \right\rangle \xi_j. \quad (4.20)$$

Now, (4.15) follows from (4.20) and (4.18) at once.

In Step 4, we prove each solution  $y_\gamma(\cdot)$  to (4.5) satisfies (4.6).

To this end, we first let

$$k = K_\gamma, \quad \varepsilon = \varepsilon_\gamma \quad \text{and} \quad \hat{\varepsilon} = \frac{1}{6K_\gamma^{1/2}} e^{-(\gamma + \|V\|_\infty)T}, \quad (4.21)$$

where  $K_\gamma$  and  $\varepsilon_\gamma$  are given by (4.1) and (4.2) respectively. From (4.4), (4.8), (4.19) and (4.21), one can directly check that

$$\mathcal{F}_\gamma = \mathcal{C}^{\varepsilon, k} \circ \mathcal{R}^{\hat{\varepsilon}, k}. \quad (4.22)$$

Next, we arbitrarily fix  $n \in \bar{\mathbb{N}}$  and fix a solution  $y_\gamma(\cdot)$  to (4.5). Then from (4.22), (4.8) and (4.19), we find that

$$\begin{aligned} y_\gamma((n+1)T) &= e^{TA} y_\gamma(nT) + e^{(T-\tau)A} 1_\omega \mathcal{C}^{\varepsilon, k} \mathcal{R}^{\hat{\varepsilon}, k} (1_{\omega_1}^* e^{sA} y_\gamma(nT)) \\ &= e^{TA} [P_k y_\gamma(nT) - \mathcal{R}^{\hat{\varepsilon}, k} (1_{\omega_1}^* e^{sA} y_\gamma(nT))] \\ &\quad + \left[ e^{TA} \left( y_\gamma(nT) - P_k y_\gamma(nT) + \mathcal{R}^{\hat{\varepsilon}, k} (1_{\omega_1}^* e^{sA} y_\gamma(nT)) \right) + e^{(T-\tau)A} 1_\omega \mathcal{C}^{\varepsilon, k} \mathcal{R}^{\hat{\varepsilon}, k} (1_{\omega_1}^* e^{sA} y_\gamma(nT)) \right] \\ &\triangleq Z_{1,n} + Z_{2,n} \end{aligned} \quad (4.23)$$

On one hand, we see from (4.15), (4.21) and (4.1) that

$$\left\| P_k y_\gamma(nT) - \mathcal{R}^{\hat{\varepsilon}, k} (1_{\omega_1}^* e^{sA} y_\gamma(nT)) \right\| \leq \frac{1}{6} e^{-T(\gamma + \|V\|_\infty)} \|y_\gamma(nT)\|. \quad (4.24)$$

From (4.24), we find that

$$\|Z_{1,n}\| \leq e^{T\|V\|_\infty} \left\| P_k y(nT) - \mathcal{R}^{\hat{\varepsilon},k}(1_{\omega_1}^* e^{sA} y_\gamma(nT)) \right\| \leq \frac{1}{6} e^{-\gamma T} \|y_\gamma(nT)\|. \quad (4.25)$$

On the other hand, we let

$$\hat{z}_0 \triangleq y_\gamma(nT) - P_k y_\gamma(nT) + \mathcal{R}^{\hat{\varepsilon},k}(1_{\omega_1}^* e^{sA} y_\gamma(nT)). \quad (4.26)$$

Because

$$\mathcal{R}^{\hat{\varepsilon},k}(1_{\omega_1}^* e^{sA} y_\gamma(nT)) \in V_k,$$

we have that

$$P_k z_0 = \mathcal{R}^{\hat{\varepsilon},k}(1_{\omega_1}^* e^{sA} y_\gamma(nT)) \in V_k. \quad (4.27)$$

From (4.26), (4.27) and (4.9), it follows that

$$\begin{aligned} \|Z_{2,n}\| &= \|e^{TA} \hat{z}_0 + e^{(T-\tau)A} 1_\omega \mathcal{C}^{\varepsilon,k} \mathcal{R}^{\hat{\varepsilon},k} P_k \hat{z}_0\| \\ &\leq \varepsilon \sqrt{k} \|\mathcal{R}^{\hat{\varepsilon},k}(1_{\omega_1}^* e^{sA} y_\gamma(nT))\| + e^{-\lambda_{k+1}T} \|y(nT) - P_k y_\gamma(nT)\| \\ &\leq \varepsilon \sqrt{k} \|P_k y_\gamma(nT)\| + \varepsilon \sqrt{k} \|P_k y_\gamma(nT) - \mathcal{R}^{\hat{\varepsilon},k}(1_{\omega_1}^* e^{sA} y_\gamma(nT))\| + e^{-\lambda_{k+1}T} \|y_\gamma(nT)\| \\ &\leq \left( \varepsilon \sqrt{k} + e^{-\lambda_{k+1}T} \right) \|y_\gamma(nT)\| + \varepsilon \sqrt{k} \|P_k y_\gamma(nT) - \mathcal{R}^{\hat{\varepsilon},k}(1_{\omega_1}^* e^{sA} y_\gamma(nT))\|. \end{aligned}$$

This, along with (4.24), (4.21) and (4.2), yields that

$$\|Z_{2,n}\| \leq \left( \frac{2}{3} + \frac{1}{18} \right) e^{-\gamma T} \|y_\gamma(nT)\|. \quad (4.28)$$

Now, (4.6) follows from (4.23), (4.25) and (4.28). Then by the conclusion in Step 1, we see that the conclusion (i) of the theorem holds.

In Step 5, we will prove the conclusion (ii) of the theorem. To this end, we let  $k$ ,  $\varepsilon$  and  $\hat{\varepsilon}$  be given by (4.21); let  $\hat{\varepsilon}_j$  be given by (4.16) (with aforementioned  $\hat{\varepsilon}$ ). From (4.22), (4.8) and (4.19), we find that

$$\|\mathcal{F}_\gamma\|_{\mathcal{L}(L^2(\omega_1), L^2(\omega))} \leq \left\| \mathcal{C}^{\varepsilon,k} \right\|_{\mathcal{L}(V_k, L^2(\omega))} \left\| \mathcal{R}^{\hat{\varepsilon},k} \right\|_{\mathcal{L}(L^2(\omega_1), V_k)}, \quad (4.29)$$

$$\left\| \mathcal{C}^{\varepsilon,k} \right\|_{\mathcal{L}(V_k, L^2(\omega))} \leq \sum_{j=1}^k \|f_j^\varepsilon\|_\omega \quad (4.30)$$

and

$$\left\| \mathcal{R}^{\hat{\varepsilon},k} \right\|_{\mathcal{L}(L^2(\omega_1), V_k)} \leq \left( \sum_{j=1}^k e^{4\lambda_j s} \left\| \hat{h}_j \right\|_{\omega_1}^2 \right)^{1/2} \leq \sqrt{k} e^{2\lambda_k s} \max_{j \leq k} \left\| \hat{h}_j \right\|_{\omega_1}. \quad (4.31)$$

Since  $f_j^\varepsilon$  and  $\hat{h}_j$  are the minimal norm controls to  $(\text{NP})_{T,\tau,0,\omega}^{\xi_j,\varepsilon}$  and  $(\text{NP})_{2s,s,0,\omega_1}^{\xi_j,\hat{\varepsilon}_j}$  respectively, according to Theorem 3.7, there is a constant  $C > 1$ , depending only on  $\Omega$ ,  $\omega$  and  $\omega_1$ , so that

$$\|f_j^\varepsilon\|_\omega \leq e^{C \left( B_{T,\tau,0,\|V\|_\infty} + \sqrt{\frac{[\ln(1/\varepsilon)]^+}{T-\tau}} \right)} \quad (4.32)$$

and

$$\|\hat{h}_j\|_{\omega_1} \leq e^{C \left( B_{2s,s,0,\|V\|_\infty} + \sqrt{\frac{[\ln(1/\hat{\varepsilon}_j)]^+}{s}} \right)}, \quad (4.33)$$

where  $B_{T,\tau,0,\|V\|_\infty}$  and  $B_{2s,s,0,\|V\|_\infty}$  are given by (3.44), with  $(\hat{T}, \hat{s}, \hat{t}, \|V\|_\infty)$  being replaced by  $(T, \tau, 0, \|V\|_\infty)$  and  $(2s, s, 0, \|V\|_\infty)$  respectively. (Here, we would like to mention that the constant  $C$  in (4.32) only depends on  $\Omega$  and  $\omega$ , while the constant  $C$  in (4.33) only depends on  $\Omega$  and  $\omega_1$ .)

We next estimate the terms on the right hand sides of (4.32) and (4.33). By (4.21), (4.2) and (4.16), it follows that

$$\frac{1}{\varepsilon} = 3K_\gamma^{\frac{1}{2}} e^{\gamma T} \quad \text{and} \quad \frac{1}{\hat{\varepsilon}_j} = \frac{1}{\hat{\varepsilon}} e^{2\lambda_j s} = 6K_\gamma^{\frac{1}{2}} e^{(\gamma + \|V\|_\infty)T} e^{2\lambda_j s}.$$

From these and (4.1), we can apply Weyl's asymptotic formula (see [SV]) to find a constant  $\bar{C} > 0$  (depending only on  $\Omega$ ) so that

$$\frac{1}{\varepsilon} \leq \bar{C} \left( \gamma + \frac{\ln 3}{T} \right)^{d/4} e^{\gamma T} \quad \text{and} \quad \frac{1}{\hat{\varepsilon}_j} \leq \bar{C} \left( \gamma + \frac{\ln 3}{T} \right)^{d/4} e^{(\gamma + \|V\|_\infty)T} e^{2(\gamma + \frac{\ln 3}{T})s}. \quad (4.34)$$

By using the inequality:  $\ln a \leq a$  for all  $a > 0$ , we can derive from (4.34) that

$$\ln(1/\varepsilon) \leq (\ln \bar{C} + d) + d \frac{1}{T - \tau} + (1 + d)T\gamma$$

and

$$\ln(1/\hat{\varepsilon}_j) \leq (\ln \bar{C} + d) + d \frac{1}{T - \tau} + \|V\|_\infty T + 4 \frac{s}{T - \tau} + (1 + d)(T + s)\gamma.$$

Because  $(a + b)^+ \leq a^+ + b^+$  for all  $a, b \in \mathbb{R}$ , the above two inequalities imply that

$$[\ln(1/\varepsilon)]^+ \leq [\ln \bar{C} + d]^+ + d \frac{1}{T - \tau} + (1 + d)T\gamma \quad (4.35)$$

and that

$$[\ln(1/\hat{\varepsilon}_j)]^+ \leq [\ln \bar{C} + d]^+ + d \frac{1}{T - \tau} + \|V\|_\infty T + 4 \frac{s}{T - \tau} + (1 + d)(T + s)\gamma. \quad (4.36)$$

Since  $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$  and  $2\sqrt{a} \leq 1 + a$  for all  $a, b \geq 0$ , we can get from (4.35) and (4.36), after some simple computations, that

$$\sqrt{\frac{[\ln(1/\varepsilon)]^+}{T - \tau}} \leq 1 + \frac{[\ln \bar{C} + d]^+ + \sqrt{d}}{T - \tau} + (1 + d) \sqrt{\frac{T}{T - \tau}} \sqrt{\gamma} \quad (4.37)$$

and

$$\sqrt{\frac{[\ln(1/\hat{\varepsilon}_j)]^+}{s}} \leq 2 + \frac{[\ln \bar{C} + d]^+ + \sqrt{d}}{s} + \sqrt{\|V\|_\infty} \sqrt{\frac{T}{s}} + \frac{1 + \sqrt{d}}{T - \tau} + (1 + d) \sqrt{\frac{T + s}{s}} \sqrt{\gamma}. \quad (4.38)$$

Since it follows by (3.44) (with  $(\hat{T}, \hat{s}, \hat{t}, \|V\|_\infty)$  being replaced by  $(T, \tau, 0, \|V\|_\infty)$  and  $(2s, s, 0, \|V\|_\infty)$  respectively) that

$$\frac{1}{T - \tau} \leq B_{T,\tau,0,\|V\|_\infty} \quad \text{and} \quad \frac{1}{s} \leq B_{2s,s,0,\|V\|_\infty},$$

we find from (4.37) and (4.38) that

$$\begin{aligned} e^{C\left(B_{T,\tau,0,\|V\|_\infty} + \sqrt{\frac{[\ln(1/\varepsilon)]^+}{T-\tau}}\right)} &\leq e^{C\left(B_{T,\tau,0,\|V\|_\infty} + 1 + \frac{[\ln \bar{C} + d]^+ + \sqrt{d}}{T-\tau}\right)} e^{C(1+d)\sqrt{\frac{T}{T-\tau}}\sqrt{\gamma}} \\ &\leq e^{C\left(2 + [\ln \bar{C} + d]^+ + \sqrt{d}\right)B_{T,\tau,0,\|V\|_\infty}} e^{C(1+d)\sqrt{\frac{T}{T-\tau}}\sqrt{\gamma}} \end{aligned} \quad (4.39)$$

and that

$$\begin{aligned} &e^{C\left(B_{2s,s,0,\|V\|_\infty} + \sqrt{\frac{[\ln(1/\varepsilon_j)]^+}{s}}\right)} \\ &\leq e^{C\left(2 + [\ln \bar{C} + d]^+ + \sqrt{d}\right)\left(B_{2s,s,0,\|V\|_\infty} + \sqrt{\|V\|_\infty}\sqrt{\frac{T}{s} + \frac{1}{T-\tau}}\right)} e^{C(1+d)\sqrt{\frac{T+s}{s}}\sqrt{\gamma}} \\ &\leq e^{C\left(2 + [\ln \bar{C} + d]^+ + \sqrt{d}\right)\left(B_{2s,s,0,\|V\|_\infty} + \sqrt{\|V\|_\infty}\sqrt{\frac{T}{s}} + B_{T,\tau,0,\|V\|_\infty}\right)} e^{C(1+d)\sqrt{\frac{T+s}{s}}\sqrt{\gamma}}. \end{aligned} \quad (4.40)$$

Write

$$\begin{aligned} \hat{C}_1 &\triangleq e^{C\left(2 + [\ln \bar{C} + d]^+ + \sqrt{d}\right)\left(B_{T,\tau,0,\|V\|_\infty} + B_{2s,s,0,\|V\|_\infty} + \sqrt{\|V\|_\infty}\sqrt{\frac{T}{s}}\right)}, \\ \hat{C}_2 &\triangleq e^{C(1+d)\left(\sqrt{\frac{T}{T-\tau}} + \sqrt{\frac{T+s}{s}}\right)}. \end{aligned} \quad (4.41)$$

From (4.32), (4.33), (4.39), (4.40) and (4.41), it follows that

$$\|f_j^\varepsilon\|_\omega \leq \hat{C}_1 e^{\hat{C}_2 \sqrt{\gamma}} \quad \text{and} \quad \|\hat{h}_j\|_{\omega_1} \leq \hat{C}_1 e^{\hat{C}_2 \sqrt{\gamma}}. \quad (4.42)$$

We now back to estimate  $\|\mathcal{C}^{\varepsilon,k}\|_{\mathcal{L}(V_k, L^2(\omega))}$  and  $\|\mathcal{R}^{\hat{\varepsilon},k}\|_{\mathcal{L}(L^2(\omega_1), V_k)}$ . To deal with  $\|\mathcal{C}^{\varepsilon,k}\|_{\mathcal{L}(V_k, L^2(\omega))}$ , we see from (4.30), the first estimate in (4.42) and (4.21) that

$$\|\mathcal{C}^{\varepsilon,k}\|_{\mathcal{L}(V_k, L^2(\omega))} \leq \sum_{j=1}^k \|f_j^\varepsilon\|_\omega \leq k \hat{C}_1 e^{\hat{C}_2 \sqrt{\gamma}} = K_\gamma \hat{C}_1 e^{\hat{C}_2 \sqrt{\gamma}} = \hat{C}_1 e^{\ln K_\gamma} e^{\hat{C}_2 \sqrt{\gamma}}. \quad (4.43)$$

Meanwhile, from (4.1), Weyl's asymptotic formula (see [SV]) and the inequality:

$$\ln \sqrt{a+b} \leq \sqrt{a+b} \leq \sqrt{a} + \sqrt{b} \quad \text{for all } a, b \geq 0,$$

we deduce that

$$\begin{aligned} \ln K_\gamma &\leq \ln \left( \bar{C} \left( \gamma + \frac{\ln 3}{T} \right) \right) = \ln \bar{C} + 2 \ln \sqrt{\gamma + \frac{\ln 3}{T}} \\ &\leq \ln \bar{C} + 2 \sqrt{\gamma + \frac{\ln 3}{T}} \leq \ln \bar{C} + 2\sqrt{\gamma} + 2\sqrt{\frac{\ln 3}{T}} \\ &\leq \ln \bar{C} + 2\sqrt{\gamma} + \frac{4}{\sqrt{T-\tau}}, \end{aligned} \quad (4.44)$$

where  $\bar{C} > 0$  is a constant depending only on  $\Omega$ . We can take two  $\bar{C} > 0$  in (4.44) and (4.34), respectively, to be the same number. By (4.43) and (4.44), it follows that

$$\|\mathcal{C}^{\varepsilon,k}\|_{\mathcal{L}(V_k, L^2(\omega))} \leq \hat{C}_1 \bar{C} e^{\frac{4}{\sqrt{T-\tau}}} e^{(\hat{C}_2+2)\sqrt{\gamma}} \leq \hat{C}_1 \bar{C} e^{\frac{4}{\sqrt{T-\tau}}} e^{(\hat{C}_2+2)(1+\gamma)}. \quad (4.45)$$

To estimate  $\|\mathcal{R}^{\hat{\varepsilon},k}\|_{\mathcal{L}(L^2(\omega_1),V_k)}$ , we find from (4.31), (4.21), (4.1) and the second estimate in (4.42) that

$$\begin{aligned} \|\mathcal{R}^{\hat{\varepsilon},k}\|_{\mathcal{L}(L^2(\omega_1),V_k)} &\leq \sqrt{k}e^{2\lambda_k s} \max_{j \leq k} \|\hat{h}_j\|_{\omega_1} \leq \sqrt{K_\gamma} e^{2(\gamma + \frac{\ln 3}{T})s} \hat{C}_1 e^{\hat{C}_2 \sqrt{\gamma}} \\ &= \hat{C}_1 e^{\frac{1}{2} \ln K_\gamma} e^{2 \ln 3 \frac{s}{T}} e^{2s\gamma} e^{\hat{C}_2 \sqrt{\gamma}} \leq \hat{C}_1 e^{\frac{1}{2} \ln K_\gamma} e^{4 \frac{s}{T}} e^{2s\gamma} e^{\hat{C}_2 \sqrt{\gamma}}. \end{aligned}$$

This, along with (4.44), yields that

$$\begin{aligned} \|\mathcal{R}^{\hat{\varepsilon},k}\|_{\mathcal{L}(L^2(\omega_1),V_k)} &\leq \hat{C}_1 \sqrt{\bar{C}} e^{\frac{2}{\sqrt{T-\tau}}} e^{4 \frac{s}{T}} e^{2s\gamma} e^{(\hat{C}_2+1)\sqrt{\gamma}} \\ &\leq \hat{C}_1 \sqrt{\bar{C}} e^{\frac{2}{\sqrt{T-\tau}}} e^{4 \frac{s}{T}} e^{2s\gamma} e^{(\hat{C}_2+1)(1+\gamma)} \\ &= \hat{C}_1 \sqrt{\bar{C}} e^{\hat{C}_2+1} e^{\frac{2}{\sqrt{T-\tau}}} e^{4 \frac{s}{T}} e^{(\hat{C}_2+2s+1)\gamma}. \end{aligned} \quad (4.46)$$

Finally, by (4.29), (4.45) and (4.46), we get that

$$\|\mathcal{F}_\gamma\|_{\mathcal{L}(L^2(\omega_1),L^2(\omega))} \leq C_1 e^{C_2 \gamma},$$

with

$$C_1 \triangleq \hat{C}_1^2 \bar{C}^{3/2} e^{2\hat{C}_2+3+\frac{4s}{T}+\frac{6}{\sqrt{T-\tau}}} \quad \text{and} \quad C_2 \triangleq 2\hat{C}_2 + 2s + 3.$$

In summary, we finish the proof of this theorem.  $\square$

**Remark 4.2** *The feedback law  $\mathcal{F}_\gamma$  constructed by (4.4) is robust with respect to  $V$ . Indeed, for each  $V^\varepsilon \in L^\infty(\Omega)$ , we let  $A^\varepsilon \triangleq \Delta - V + V^\varepsilon$ . Define an operator  $\mathcal{L}^{V^\varepsilon}$  from  $L^2(\Omega)$  to  $L^2(\Omega)$  in the following manner:*

$$\mathcal{L}^{V^\varepsilon} = e^{A^\varepsilon T} + e^{A^\varepsilon(T-\tau)} \mathcal{F}_\gamma 1_{\omega_1}^* e^{A^\varepsilon(T-s)} \quad \text{for any } V^\varepsilon \in L^\infty(\Omega). \quad (4.47)$$

Then by (4.6), we see that

$$\|\mathcal{L}^0\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))} \leq e^{-\gamma T}. \quad (4.48)$$

From (4.47) and (4.48), one can easily check that for any  $\gamma_1 < \gamma$ , there is  $\varepsilon_0 > 0$  so that

$$\|\mathcal{L}^{V^\varepsilon}\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))} \leq e^{-\gamma_1 T}, \quad \text{when } \|V^\varepsilon\|_\infty \leq \varepsilon_0. \quad (4.49)$$

Meanwhile, any solution  $y_{\gamma,\varepsilon}(\cdot)$  to the equation (4.5), where  $A$  is replaced by  $A^\varepsilon$ , satisfies that

$$y_{\gamma,\varepsilon}((n+1)T) = \mathcal{L}^{V^\varepsilon} y_{\gamma,\varepsilon}(nT) \quad \text{for all } n \in \bar{\mathbb{N}}.$$

This, along with (4.49), yields that when  $\|V^\varepsilon\|_\infty \leq \varepsilon_0$ ,

$$\|y_{\gamma,\varepsilon}((n+1)T)\| \leq e^{-\gamma_1 T} \|y_{\gamma,\varepsilon}(nT)\| \quad \text{for all } n \in \bar{\mathbb{N}}. \quad (4.50)$$

From (4.50), we can use the same way as that used in Step 1 of the proof of Theorem 4.1 to prove that when  $\|V^\varepsilon\|_\infty \leq \varepsilon_0$ ,

$$\|y_{\gamma,\varepsilon}(t)\| \leq C_{\gamma_1} e^{-\gamma_1 T} \|y_{\gamma,\varepsilon}(0)\| \quad \text{for all } t \geq 0,$$

where  $C_{\gamma_1}$  is a positive constant depending on  $\gamma_1$ . This proves the desired robust property of the feedback law  $\mathcal{F}_\gamma$ .

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