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# CONVEX REAL PROJECTIVE STRUCTURES ON MANIFOLDS AND ORBIFOLDS

SUHYOUNG CHOI, GYE-SEON LEE, AND LUDOVIC MARQUIS

ABSTRACT. In this survey, we study deformations of finitely generated groups into Lie groups, focusing on the case underlying convex projective structures on manifolds and orbifolds, with an excursion on projective structures on surfaces.

We survey the basics of the theory of deformations,  $(G, X)$ -structures on orbifolds, Hilbert geometry and Coxeter groups. The main examples of finitely generated groups for us will be Fuchsian groups, 3-manifold groups and Coxeter groups.

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## 1. INTRODUCTION

The goal of this paper is to survey convex real projective structures on manifolds and orbifolds. Some may prefer to speak of discrete subgroups of the group  $\mathrm{PGL}_{d+1}(\mathbb{R})$  of projective transformations of the real projective space  $\mathbb{R}\mathbb{P}^d$  which preserves a properly convex open set of  $\mathbb{R}\mathbb{P}^d$ , and some other prefer to speak of Hilbert geometries.

Some motivations for studying this object are the following:

**1.1. Hitchin representations.** Let  $S$  be a closed surface of genus  $g \geq 2$  and let  $\Gamma$  be the fundamental group of  $S$ . There is a unique irreducible representation  $i_m : \mathrm{PSL}_2(\mathbb{R}) \rightarrow \mathrm{PSL}_m(\mathbb{R})$ , up to conjugation. A representation  $\rho : \Gamma \rightarrow \mathrm{PSL}_m(\mathbb{R})$  is called a *Hitchin representation* if there is a discrete and faithful representation  $\tau : \Gamma \rightarrow \mathrm{PSL}_2(\mathbb{R})$  such that  $\rho$  can be continuously

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deformed to a representation  $i_m \circ \tau : \Gamma \rightarrow \mathrm{PSL}_m(\mathbb{R})$ . The space  $\mathrm{Hit}_m(\Gamma)$  of conjugacy classes of Hitchin representations of  $\Gamma$  in  $\mathrm{PSL}_m(\mathbb{R})$  has a lot of interesting properties: Each connected component is homeomorphic to an open ball of dimension  $2(g-1)(m^2-1)$  [Hit92], and every Hitchin representation is discrete, faithful, irreducible and purely loxodromic [Lab06].

When  $m = 3$ , the first author and Goldman [CG93] show that each Hitchin representation preserves a properly convex open set of  $\mathbb{RP}^2$ , in other words,  $\mathrm{Hit}_3(\Gamma)$  is the space  $\mathcal{C}(S)$  of marked convex real projective structures on the surface  $S$ .

To understand the geometric properties of Hitchin representations, Labourie [Lab06] introduces the concept of Anosov representation for surface groups. Later on, this notion is generalized for finitely generated Gromov-hyperbolic groups by Guichard and Wienhard [GW12]. For example, if  $M$  is a closed manifold whose fundamental group is Gromov-hyperbolic, then the holonomy representations of convex real projective structures on  $M$  are Anosov.

**1.2. Deformations of hyperbolic structures.** Let  $M$  be a closed hyperbolic manifold of dimension  $d \geq 3$ , and let  $\Gamma = \pi_1(M)$ . By Mostow rigidity, up to conjugation, there is a unique faithful and discrete representation  $\rho_{\mathrm{geo}}$  of  $\Gamma$  in  $\mathrm{SO}_{d,1}(\mathbb{R})$ . The group  $\mathrm{SO}_{d,1}(\mathbb{R})$  is canonically embedded inside  $\mathrm{PGL}_{d+1}(\mathbb{R})$ . We use the same notation  $\rho_{\mathrm{geo}} : \Gamma \rightarrow \mathrm{PGL}_{d+1}(\mathbb{R})$  to denote the composition of  $\rho_{\mathrm{geo}} : \Gamma \rightarrow \mathrm{SO}_{d,1}(\mathbb{R})$  with the canonical inclusion. Now, there is no reason that  $\rho_{\mathrm{geo}}$  is the unique faithful and discrete representation of  $\Gamma$  in  $\mathrm{PGL}_{d+1}(\mathbb{R})$ , up to conjugation.

In fact, there are examples of closed hyperbolic manifold  $M$  of dimension  $d$  such that  $\Gamma$  admits discrete and faithful representations in  $\mathrm{PGL}_{d+1}(\mathbb{R})$  which are not conjugate to  $\rho_{\mathrm{geo}}$  (see Theorem 5.2). We can start looking at the connected component  $\mathrm{Ben}(M)$  of the space of representations of  $\Gamma$  into  $\mathrm{PGL}_{d+1}(\mathbb{R})$  containing  $\rho_{\mathrm{geo}}$ , up to conjugation. The combination of a theorem of Koszul and a theorem of Benoist implies that every representation in  $\mathrm{Ben}(M)$  is discrete, faithful, irreducible and preserves<sup>1</sup> a properly convex open set  $\Omega$  of  $\mathbb{RP}^d$ . Hence  $\mathrm{Ben}(M)$  is indeed the connected component of the space  $\mathcal{C}(M)$  of marked convex projective structures on  $M$  containing the hyperbolic structure.

At the moment of writing, there is no known necessary and sufficient condition on  $M$  to decide if  $\mathrm{Ben}(M)$  consists of exactly one element, which is the hyperbolic structure. There are infinitely many closed hyperbolic 3-manifolds  $M$  such that  $\mathrm{Ben}(M)$  is the singleton (see [HP11]), and there are infinitely many closed hyperbolic 3-orbifolds  $M$  such that  $\mathrm{Ben}(M)$  is homeomorphic to an open  $k$ -ball, for any  $k \in \mathbb{N}$  (see [Mar10a]).

**1.3. Building block for projective surfaces.** Let  $S$  be a closed surface. We can wake up one day with a wish to understand all the possible real projective structures on  $S$ , not necessarily only the convex one. The first author has shown that convex projective structures are the main building blocks to construct all the possible projective structures on the surface  $S$  (see Theorem 6.9 for details).

**1.4. Geometrization.** Let  $\Omega$  be a properly convex open set of  $\mathbb{RP}^d$ , and let  $\mathrm{Aut}(\Omega)$  be the subgroup of  $\mathrm{PGL}_{d+1}(\mathbb{R})$  preserving  $\Omega$ . There is an  $\mathrm{Aut}(\Omega)$ -invariant metric  $d_\Omega$  on  $\Omega$ , called

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<sup>1</sup>The action of  $\Gamma$  on  $\Omega$  is automatically proper and cocompact for general reason.

the *Hilbert metric*, that make  $(\Omega, d_\Omega)$  a complete proper geodesic metric space, called a *Hilbert geometry*. We will discuss about these in Section 4.2. The flavour of the metric space  $(\Omega, d_\Omega)$  really depends on the geometry of the boundary of  $\Omega$ . For example, on one side, the interior of an ellipse equipped with the Hilbert metric is isometric to the hyperbolic plane, forming the projective model of the hyperbolic plane, and on the other side, the interior of a triangle is isometric to the plane with the norm whose unit ball is the regular hexagon (see [dlH91]).

Unfortunately, Hilbert geometries are almost never CAT(0): A Hilbert geometry  $(\Omega, d_\Omega)$  is CAT(0) if and only if  $\Omega$  is an ellipsoid [KS58]. However, the idea of Riemannian geometry of non-positive curvature is a good guide toward the study of the metric property of Hilbert geometry.

An irreducible symmetric space  $X = G/K$  is "Hilbertizable" if there exist a properly convex open set  $\Omega$  of  $\mathbb{RP}^d$  for some  $d$  and an irreducible representation  $\rho : G \rightarrow \mathrm{SL}_{d+1}(\mathbb{R})$  such that  $\rho(G)$  acts transitively on  $\Omega$  and the stabilizer of a point of  $\Omega$  is conjugate to  $K$ . The symmetric spaces for  $\mathrm{SO}_{d,1}(\mathbb{R})$ ,  $\mathrm{SL}_m(\mathbb{K})$  for  $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ , and the exceptional Lie group  $E_{6,-26}$  are exactly the symmetric spaces that are Hilbertizable (see Vinberg [Vin63b, Vin65] or Koecher [Koe99]).

Nevertheless we can ask the following question to start with:

“Which manifold or orbifold  $M$  can be realized as the quotient of a properly convex open set  $\Omega$  by a discrete subgroup  $\Gamma$  of  $\mathrm{Aut}(\Omega)$ ?”

If so, we say that  $M$  admits a convex real projective structure.

In dimension 2, the answer is easy: a closed surface  $S$  admits a convex projective structure if and only if its Euler characteristic is non-positive. The universal cover of a convex projective torus is a triangle, and a closed surface of negative Euler characteristic admits a hyperbolic structure, which is an example of a convex projective structure.

In dimension greater than or equal to 3, no definite answer is known; see Section 5 for a description of our knowledge. To arouse the reader’s curiosity we just mention that there exist manifolds which can be geometrized thanks to Hilbert geometry but cannot be geometrized thanks to symmetric spaces.

**1.5. Coxeter groups.** A Coxeter group is a finitely presented group that “resembles” the groups generated by reflections; see Section 7 for a precise definition, and [dlH91] for a beautiful invitation if you want more. An important object to study Coxeter group, denoted  $W$ , is a representation  $\rho_{\mathrm{geo}} : W \rightarrow \mathrm{GL}(V)$  introduced by Tits [Bou68]. The representation  $\rho_{\mathrm{geo}}$ , in fact, preserves a convex open set of the real projective space  $\mathbb{P}(V)$ . For example, Margulis and Vinberg use this property of  $\rho_{\mathrm{geo}}$  to show that an irreducible Coxeter group is either finite, virtually abelian or large<sup>2</sup> [MV00].

From our point of view, Coxeter groups are a great source for building groups acting on properly convex open sets of  $\mathbb{P}(V)$ . Benoist uses them to construct the first example of closed 3-manifold that admits a convex projective structure  $\Omega/\Gamma$  such that  $\Omega$  is not strictly convex,

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<sup>2</sup>It contains a subgroup of finite index that admits an onto morphism to a non-abelian free group.

or to build the first example of a closed 4-manifold that admits a convex projective structure  $\Omega/\Gamma$  such that  $(\Omega, d_\Omega)$  is Gromov-hyperbolic but not quasi-isometric to the hyperbolic space (see Section 5).

## 2. CHARACTER VARIETIES

All along this article, we study this kind of objects:

- a group  $\Gamma$  of finite type which we are thinking of as the fundamental group of a complete real hyperbolic manifold/orbifold or its siblings,
- a Lie group  $G$  which is also the set of real points of an algebraic group  $\mathbb{G}$ , and
- a real algebraic variety  $\text{Hom}(\Gamma, G)$ .

We want to understand the space  $\text{Hom}(\Gamma, G)$ . First, the group  $G$  acts on  $\text{Hom}(\Gamma, G)$  by conjugation. We can notice that the quotient space is not necessarily Hausdorff since the orbit of the action of  $G$  on  $\text{Hom}(\Gamma, G)$  may not be closed. But, the situation is not bad since each orbit closure contains at most one closed orbit. Hence, the solution to the problem is to forget the representations whose orbit are not closed. Let's recall the characterization of the closedness on the orbit:

**Lemma 2.1** (Richardson [Ric88]). *Assume that  $G$  is the set of real points of a reductive<sup>3</sup> algebraic group defined over  $\mathbb{R}$ . Let  $\rho : \Gamma \rightarrow G$  be a representation. Then the orbit  $G \cdot \rho$  is closed if and only if the Zariski closure of  $\rho(\Gamma)$  is a reductive subgroup of  $G$ . Such a representation is called a reductive representation.*

Define

$$R(\Gamma, G) = \text{Hom}(\Gamma, G)/G \quad \text{and} \quad \chi(\Gamma, G) = \{[\rho] \in R(\Gamma, G) \mid \rho \text{ is reductive}\}.$$

These are given with the quotient topology and the subspace topology, respectively.

**Theorem 2.2** (Topological, geometric and algebraic view point, Luna [Lun75, Lun76]).

*Assume that  $G$  is as in Lemma 2.1. Let  $\rho : \Gamma \rightarrow G$  be a representation. Then there exists a unique reductive representation  $\vec{\rho} \in \overline{G \cdot \rho}$ , up to conjugation.*

*The space  $\chi(\Gamma, G)$  is Hausdorff and it identifies with the Hausdorff quotient of  $R(\Gamma, G)$ . Also, it is a real semi-algebraic variety which is the GIT-quotient<sup>4</sup> of the action of  $G$  on  $\text{Hom}(\Gamma, G)$ .*

The real semi-algebraic Hausdorff set  $\chi(\Gamma, G)$  is called the *character variety* of the pair  $(\Gamma, G)$ .

**A baby example:**  $\mathbb{Z}$ . The space  $\chi(\mathbb{Z}, G)$  is the set of semi-simple elements of  $G$  modulo conjugation.

- If  $G = \text{SL}_m(\mathbb{C})$  then  $\chi(\mathbb{Z}, G) = \mathbb{C}^{m-1}$ .
- If  $G = \text{SL}_2(\mathbb{R})$  then  $\chi(\mathbb{Z}, G)$  is a circle  $\{e^{i\theta} \mid 0 \leq \theta < 2\pi\}$  with two half-line that are glued on the circle at the points  $\{1\}$  and  $\{-1\}$ .

<sup>3</sup>Its unipotent radical is trivial.

<sup>4</sup>The Geometric Invariant Theory quotient; see the lecture note [Bri10] to go in that direction.

## 3. GEOMETRIC STRUCTURES ON ORBIFOLDS

In this section, we recall the vocabulary of orbifolds and of geometric structures on orbifolds. You can skip this section if you are familiar with these notions. A classical reference is the lecture note of Thurston [Thu02]. See also [Gol88a, Cho12, BMP03]. For the theory of orbifolds itself, we suggest the book of Adem, Leida, and Ruan [ALR07] and Bridson and Haefliger [BH99].

**3.1. Orbifolds.** An orbifold is a topological space which is locally homeomorphic to the quotient space of  $\mathbb{R}^d$  by a finite subgroup of  $\text{Diff}(\mathbb{R}^d)$ , the diffeomorphism group of  $\mathbb{R}^d$ . Here is a formal definition: A  $d$ -dimensional *orbifold*  $\mathcal{O}$  consists of a second countable, Hausdorff space  $X_{\mathcal{O}}$  with the following additional structure:

- (1) A collection of open sets  $\{U_i\}_{i \in I}$ , for some index  $I$ , which is a covering of  $X_{\mathcal{O}}$  and is closed under finite intersections.
- (2) To each  $U_i$  are associated a finite group  $\Gamma_i$ , a smooth action of  $\Gamma_i$  on an open subset  $\tilde{U}_i$  in  $\mathbb{R}^d$  and a homeomorphism  $\phi_i : \tilde{U}_i/\Gamma_i \rightarrow U_i$ .
- (3) Whenever  $U_i \subset U_j$ , there is an injective homomorphism  $f_{ij} : \Gamma_i \rightarrow \Gamma_j$  and a smooth embedding  $\tilde{\phi}_{ij} : \tilde{U}_i \rightarrow \tilde{U}_j$  equivariant with respect to  $f_{ij}$ , i.e.  $\tilde{\phi}_{ij}(\gamma x) = f_{ij}(\gamma)\tilde{\phi}_{ij}(x)$  for  $\gamma \in \Gamma_i$  and  $x \in \tilde{U}_i$ , such that the following diagram commutes:

$$\begin{array}{ccc}
 \tilde{U}_i & \xrightarrow{\tilde{\phi}_{ij}} & \tilde{U}_j \\
 \downarrow & & \downarrow \\
 \tilde{U}_i/\Gamma_i & \longrightarrow & \tilde{U}_j/f_{ij}(\Gamma_i) \\
 \downarrow \phi_i & & \downarrow \\
 U_i & \xrightarrow{\subset} & U_j \\
 & & \downarrow \phi_j \\
 & & \tilde{U}_j/\Gamma_j
 \end{array}$$

- (4) The collection  $\{U_i\}$  is maximal relative to the conditions (1) – (3).

This additional structure is called an *orbifold structure*, and the space  $X_{\mathcal{O}}$  is the *underlying space* of  $\mathcal{O}$ . For example, if  $M$  is a smooth manifold and  $\Gamma$  is a subgroup of  $\text{Diff}(M)$  acting properly discontinuously on  $M$ , then the quotient space  $M/\Gamma$  has an obvious orbifold structure.

An orbifold is said to be *connected*, *compact* or *noncompact* according to whether the underlying space is connected, compact or noncompact, respectively.

**3.2.  $(G, X)$ -orbifolds.** Let  $X$  be a real analytic manifold and let  $G$  be a Lie group acting faithfully and analytically on  $X$ . A  $(G, X)$ -*structure* is an orbifold structure so that  $\Gamma_i$  is a subgroup of  $G$ ,  $\tilde{U}_i$  is an open subset of  $X$ , and  $\tilde{\phi}_{ij}$  is locally an element of  $G$  (compare to the definition of an orbifold). A  $(G, X)$ -*orbifold* is an orbifold with a  $(G, X)$ -structure. A  $(G, X)$ -*manifold* is a  $(G, X)$ -orbifold with  $\Gamma_i$  being trivial.

Here are some examples: Let  $\mathbb{E}^d$ ,  $\mathbb{S}^d$ ,  $\mathbb{H}^d$  be the Euclidean, spherical, hyperbolic space of dimension  $d$ , respectively. Having an  $(\text{Isom}(\mathbb{E}^d), \mathbb{E}^d)$ -structure (or *Euclidean structure*) on an orbifold  $M$  is equivalent to having a Riemannian metric on  $M$  of sectional curvature zero.

We can also define a *spherical structure* or a *hyperbolic structure* on  $M$  and we have similar characterisation.

Let  $\mathbb{A}^d$  be  $\mathbb{R}^d$  and  $\text{Aff}(\mathbb{A}^d)$  be the group of affine transformations, i.e. transformations of the form  $x \mapsto Ax + b$  for  $A \in \text{GL}_d(\mathbb{R})$  and  $b \in \mathbb{R}^d$ . An  $(\text{Aff}(\mathbb{A}^d), \mathbb{A}^d)$ -structure (or *affine structure*) on an orbifold  $M$  is equivalent to a flat torsion-free affine connection on the tangent space of  $M$  (see Kobayashi [KN96] for the details). Similarly, a  $(\text{PGL}_{d+1}(\mathbb{R}), \mathbb{RP}^d)$ -structure (or *real projective structure*) on  $M$  is equivalent to a projectively flat torsion-free affine connection on the tangent space of  $M$  (see Eisenhart [Eis90] for the details).

**3.3. A tool kit for orbifolds.** To each point  $x$  in an orbifold  $\mathcal{O}$  is associated a group  $\Gamma_x$  called *the isotropy group* of  $x$ : In a local coordinate system  $U_i \approx \tilde{U}_i/\Gamma_i$ , it is the isomorphism class of the stabilizer  $\Gamma_{\tilde{x}} \leq \Gamma_i$  of any inverse point  $\tilde{x}$  of  $x$  in  $\tilde{U}_i$ . The set  $\{x \in X_{\mathcal{O}} \mid \Gamma_x \neq \{1\}\}$  is the *singular locus* of  $\mathcal{O}$ .

In general, the underlying space of an orbifold is not even a manifold. However, in dimension two, it is homeomorphic to a surface with/without boundary. Moreover, the singular locus of a 2-orbifold can be classified into three families because there are only three types of finite subgroups in the orthogonal group  $O_2$  :

- Mirror:  $\mathbb{R}^2/\mathbb{Z}_2$  when  $\mathbb{Z}_2$  acts by reflection.
- Cone points of order  $n \geq 2$ :  $\mathbb{R}^2/\mathbb{Z}_n$  when  $\mathbb{Z}_n$  acts by rotations of angle  $2\pi/n$ .
- Corner reflectors of order  $n \geq 2$ :  $\mathbb{R}^2/D_n$  when  $D_n$  is the dihedral group of order  $2n$  generated by reflections on two lines meeting at angle  $\pi/n$ .

In the definition of an orbifold, if we allow  $\tilde{U}_i$  to be an open set in the closed half-space  $\mathbb{R}_+^d$  of  $\mathbb{R}^d$ , then it gives us the structure of an *orbifold with boundary*. To make a somewhat redundant remark, we should not confuse the boundary  $\partial\mathcal{O}$  of an orbifold  $\mathcal{O}$  with the boundary  $\partial X_{\mathcal{O}}$  of the underlying space  $X_{\mathcal{O}}$ , when  $X_{\mathcal{O}}$  is a manifold with boundary.

**Example.** A manifold  $M$  with boundary  $\partial M$  can have an orbifold structure in which  $\partial M$  becomes a *mirror*, i.e. a neighborhood of any point  $x$  in  $\partial M$  is isomorphic to  $\mathbb{R}^d/\mathbb{Z}_2$  such that  $\mathbb{Z}_2$  acts by reflection. Notice that the singular locus is then  $\partial M$  and the boundary of the orbifold is empty.

Given a compact orbifold  $\mathcal{O}$ , we can find a cellular decomposition of the underlying space  $X_{\mathcal{O}}$  such that the isotropy group of each open cell is constant. Define the orbifold Euler characteristic to be

$$\chi(\mathcal{O}) := \sum_{c_i} (-1)^{\dim c_i} \frac{1}{|\Gamma(c_i)|}.$$

Here,  $c_i$  ranges over the cells and  $|\Gamma(c_i)|$  is the order of the isotropy group  $\Gamma(c_i)$  of any point in the relative interior of  $c_i$ .

A *covering orbifold* of an orbifold  $\mathcal{O}$  is an orbifold  $\mathcal{O}'$  with a continuous surjective map between the underlying spaces  $p : X_{\mathcal{O}'} \rightarrow X_{\mathcal{O}}$  such that each point  $x \in X_{\mathcal{O}}$  lies in a coordinate neighborhood  $U \approx \tilde{U}/\Gamma$  and each component  $V_i$  of  $p^{-1}(U)$  is isomorphic to  $\tilde{U}/\Gamma_i$  with  $\Gamma_i$  a subgroup of  $\Gamma$ . The map  $p$  is called a *covering map*.

**Example.** If a group  $\Gamma$  acts properly discontinuously on a manifold  $M$  and  $\Gamma'$  is a subgroup of  $\Gamma$ , then  $M/\Gamma'$  is a covering orbifold of  $M/\Gamma$  with the covering map  $M/\Gamma' \rightarrow M/\Gamma$ . In particular,  $M$  is a covering orbifold of  $M/\Gamma$ .

Now we can define the universal covering orbifold of an orbifold  $\mathcal{O}$  in the same way as we define the universal covering space of a manifold: The *universal covering orbifold* of  $\mathcal{O}$  is a

covering orbifold  $\tilde{\mathcal{O}}$  with the covering map  $p : \tilde{\mathcal{O}} \rightarrow \mathcal{O}$  such that for every covering orbifold  $\tilde{\mathcal{O}}'$  with  $p' : \tilde{\mathcal{O}}' \rightarrow \mathcal{O}$ , there is a covering map  $q : \tilde{\mathcal{O}} \rightarrow \tilde{\mathcal{O}}'$  which satisfies the following commutative diagram:

$$\begin{array}{ccc}
 \tilde{\mathcal{O}} & & \\
 \downarrow p & \searrow q & \\
 & & \tilde{\mathcal{O}}' \\
 & \swarrow p' & \\
 \mathcal{O} & & 
 \end{array}$$

It is important to remark that any orbifold has a unique universal covering orbifold (up to isomorphism). The (*orbifold*) *fundamental group*  $\pi_1(\mathcal{O})$  of  $\mathcal{O}$  is the group of deck transformations of the universal covering orbifold  $\tilde{\mathcal{O}}$ .

**Example.** If  $\Gamma$  is a cyclic group of rotations acting on the sphere  $\mathbb{S}^2$  fixing the north and south poles, then the orbifold  $\mathbb{S}^2/\Gamma$  is a sphere with two cone points. So its (orbifold) fundamental group is  $\Gamma$ , even though the fundamental group of the sphere, which is the underlying topological space of  $\mathbb{S}^2/\Gamma$ , is trivial.

An orbifold  $\mathcal{O}$  is *good* if a covering orbifold of  $\mathcal{O}$  is a manifold. In this case, the universal covering orbifold  $\tilde{\mathcal{O}}$  is a simply connected manifold and the group  $\pi_1(\mathcal{O})$  acts properly discontinuously on  $\tilde{\mathcal{O}}$ . In other words, a good orbifold is simply a manifold  $M$  with a properly discontinuous group action on  $M$ . Moreover, we have a good news:

**Theorem 3.1** (Chapter 3 of Thurston [Thu02]). *Every  $(G, X)$ -orbifold is good.*

**3.4. Geometric structures on orbifolds.** Suppose that  $M$  and  $N$  are  $(G, X)$ -orbifolds. A map  $f : M \rightarrow N$  is a  $(G, X)$ -*map* if, for each pair of charts  $\phi : \tilde{U}_i/\Gamma_i \rightarrow U_i \subset M$  from the  $(G, X)$ -structure of  $M$  and  $\psi : \tilde{V}_j/\Gamma_j \rightarrow V_j \subset N$  from the  $(G, X)$ -structure of  $N$ , the composition  $\psi^{-1} \circ f \circ \phi$  restricted to  $\phi^{-1}(U_i \cap f^{-1}(V_j))$  lifts to the restriction of an element of  $G$  on the inverse image in  $\tilde{U}_i$  of  $\phi^{-1}(U_i \cap f^{-1}(V_j))$ .

We will discuss the deformation space of geometric structures on an orbifold  $\mathcal{O}$  as Goldman [Gol88a] exposes the theory for manifolds.

A  $(G, X)$ -*structure* on an orbifold  $\mathcal{O}$  is a homeomorphism  $f : \mathcal{O} \rightarrow S$  such that  $S$  is a  $(G, X)$ -orbifold. Two  $(G, X)$ -structures  $f : \mathcal{O} \rightarrow S$  and  $f' : \mathcal{O} \rightarrow S'$  on  $\mathcal{O}$  are *equivalent* if the map  $f' \circ f^{-1} : S \rightarrow S'$  is isotopic to a  $(G, X)$ -map from  $S$  to  $S'$ . The set of  $(G, X)$ -structures on  $\mathcal{O}$ , up to the equivalence relation, is denoted by  $\text{Def}(\mathcal{O})$ .

There is a topology on  $\text{Def}(\mathcal{O})$  informally defined by saying that two  $(G, X)$ -structures  $f$  and  $f'$  are close if the map  $f' \circ f^{-1} : S \rightarrow S'$  is isotopic to a map close to a  $(G, X)$ -map. Below is a formal definition.

The construction of the developing map and the holonomy representation extends to orbifolds without difficulty; see [Gol88a] for manifolds and [Cho12] for orbifolds. For a  $(G, X)$ -orbifold  $\mathcal{O}$ , there exists a pair  $(D, \rho)$  of an immersion  $D : \tilde{\mathcal{O}} \rightarrow X$  and a homomorphism



$\rho: \pi_1(\mathcal{O}) \rightarrow G$  such that for each  $\gamma \in \pi_1(\mathcal{O})$ , the following diagram commutes:

$$\begin{array}{ccc} \tilde{\mathcal{O}} & \xrightarrow{D} & X \\ \gamma \downarrow & & \downarrow \rho(\gamma) \\ \tilde{\mathcal{O}} & \xrightarrow{D} & X \end{array}$$

We call  $D$  a *developing map* and  $\rho$  a *holonomy representation*. Moreover if  $(D', \rho')$  is another such pair, then there exists  $g \in G$  such that

$$D' = g \circ D \text{ and } \rho'(\gamma) = g\rho(\gamma)g^{-1} \text{ for each } \gamma \in \pi_1(\mathcal{O}).$$

In other words, a *developing pair*  $(D, \rho)$  is uniquely determined up to the action of  $G$ :

$$(1) \quad g \cdot (D, \rho(\cdot)) = (g \circ D, g\rho(\cdot)g^{-1}), \text{ for each } g \in G.$$

Consider the space

$$\text{Def}'_2(\mathcal{O}) = \{(D, \rho) \mid D: \tilde{\mathcal{O}} \rightarrow X \text{ is an immersion}$$

$$\text{equivariant with respect to a homomorphism } \rho: \pi_1(\mathcal{O}) \rightarrow G\} / \sim.$$

Here  $(D, \rho) \sim (D', \rho')$  if  $D' = D \circ \tilde{\iota}$  for the lift  $\tilde{\iota}$  of an isotopy  $\iota: \mathcal{O} \rightarrow \mathcal{O}$  satisfying  $\gamma \circ \tilde{\iota} = \tilde{\iota} \circ \gamma$  for every  $\gamma \in \pi_1(\mathcal{O})$ . We topologize this space naturally using the compact-open topology before taking the quotient, and denote by  $\text{Def}_2(\mathcal{O})$  the quotient space of  $\text{Def}'_2(\mathcal{O})$  by the action of  $G$  (see Equation (1)).

We can define a map  $\mu: \text{Def}_2(\mathcal{O}) \rightarrow \text{Def}(\mathcal{O})$  from  $[(D, \rho)]$  to a  $(G, X)$ -structure on  $\mathcal{O}$  by pulling back the canonical  $(G, X)$ -structure on  $X$  to  $\tilde{\mathcal{O}}$  by  $D$  and taking the orbifold quotient. The inverse map is derived from the construction of the developing pair, hence  $\mu$  is a bijection. This gives a topology on  $\text{Def}(\mathcal{O})$ .

**3.5. Ehresmann-Thurston principle.** One of the most important result in this area is the following theorem first stated for closed manifolds. However, it easily generalizes to closed orbifolds. There exist many proofs of this theorem for manifolds; see Canary-Epstein [CEG06], Lok [Lok84] following John Morgan, Bergeron-Gelander [BG04], Goldman [Gol88a]. For a proof for orbifolds, see the first author [Cho04], which is a slight modification of the proof for manifolds.

Suppose that  $G$  is the real point of a reductive algebraic group defined over  $\mathbb{R}$ . A representation  $\rho: \pi_1(\mathcal{O}) \rightarrow G$  is *stable* when  $\rho$  is reductive and the centralizer of  $\rho$  is finite<sup>5</sup>. Let  $\text{Hom}^{st}(\pi_1(\mathcal{O}), G)$  be the space of stable representations. It is shown that it is an open subset of  $\text{Hom}(\pi_1(\mathcal{O}), G)$  and that the action of  $G$  on  $\text{Hom}^{st}(\pi_1(\mathcal{O}), G)$  is proper, in Johnson-Millson [JM87]. Let  $\text{Def}_2^{st}(\mathcal{O})$  be the space of  $(G, X)$ -structures on  $\mathcal{O}$  whose holonomy representation is stable.

**Theorem 3.2** (Ehresmann-Thurston principle). *Let  $\mathcal{O}$  be a closed orbifold. The map*

$$\text{Def}'_2(\mathcal{O}) \rightarrow \text{Hom}(\pi_1(\mathcal{O}), G) \quad \text{and} \quad \text{Def}_2^{st}(\mathcal{O}) \rightarrow \text{Hom}^{st}(\pi_1(\mathcal{O}), G)/G$$

*induced by  $(D, \rho) \rightarrow \rho$  are local homeomorphisms.*

<sup>5</sup>This is equivalent to the fact that the image of  $\rho$  is not contained in any parabolic subgroup of  $G$  (see [JM87])

This principle means that nearby  $(G, X)$ -structures are completely determined by their holonomy representations.

#### 4. A STARTING POINT FOR CONVEX PROJECTIVE STRUCTURES

##### 4.1. Convexity in the projective sphere or in the projective space.

Let  $V$  be a real vector space  $\mathbb{R}^{d+1}$ . Consider the action of  $\mathbb{R}_+^*$  on  $V$  by homothety, and the *projective sphere*

$$\mathbb{S}(V) = (V \setminus \{0\}) / \mathbb{R}_+^* = \{\text{rays of } V\}.$$

Of course,  $\mathbb{S}(V)$  is the 2-fold cover of the real projective space  $\mathbb{P}(V)$ . The canonical projection map  $V \setminus \{0\} \rightarrow \mathbb{S}(V)$  is denoted by  $\mathbb{S}$ .

A convex cone  $\mathcal{C}$  is *sharp* if  $\mathcal{C}$  does not contain an affine line. A subset  $C$  of  $\mathbb{S}(V)$  is *convex* (resp. *properly convex*) if the subset  $\mathbb{S}^{-1}(C) \cup \{0\}$  of  $V$  is a convex cone (resp. sharp convex cone). Given a hyperplane  $H$  of  $\mathbb{S}(V)$ , we call the two connected components of  $\mathbb{S}(V) \setminus H$  *affine charts*. An open set  $\Omega \neq \mathbb{S}(V)$  is convex (resp. properly convex) if and only if there exists an affine chart  $\mathbb{A}$  such that  $\Omega \subset \mathbb{A}$  (resp.  $\overline{\Omega} \subset \mathbb{A}$ ) and  $\Omega$  is convex in the usual sense in  $\mathbb{A}$ . A properly convex set  $\Omega$  is *strictly convex* if every line segment in  $\partial\Omega$  is a point. All those definitions can be made for subset of  $\mathbb{P}(V)$ . The projective space is more common but the projective sphere allows to get ride of some technical issue. It will be clear from the context whether our convex open set is inside  $\mathbb{S}(V)$  or  $\mathbb{P}(V)$ .

The group  $\text{SL}^\pm(V)$  of linear automorphisms of  $V$  of determinant  $\pm 1$  identifies to the group of automorphisms of  $\mathbb{S}(V)$ . A *convex projective structure* on an orbifold  $\mathcal{O}$  is a  $(\text{PGL}(V), \mathbb{P}(V))$ -structure (or a  $(\text{SL}^\pm(V), \mathbb{S}(V))$ -structure) such that the developing map is injective and its image is a properly convex subset of  $\mathbb{RP}^d$  (or  $\mathbb{S}^d$ ). We send the reader to Section 1 of [Mar14a] to see the equivalence between properly convex  $(\text{PGL}(V), \mathbb{P}(V))$ -structures and properly convex  $(\text{SL}^\pm(V), \mathbb{S}(V))$ -structures.

From now on,  $\mathfrak{C}(\mathcal{O})$  will denote the space of convex projective structures on an orbifold  $\mathcal{O}$ , which is a subspace of  $\text{Def}(\mathcal{O})$ .

##### 4.2. Hilbert geometries.

On every properly convex open set  $\Omega$ , there exists a distance  $d_\Omega$  on  $\Omega$  thanks to the cross ratio: take two points  $x \neq y \in \Omega$  and draw the line between them. This line intersects the boundary  $\partial\Omega$  of  $\Omega$  in two points  $p$  and  $q$ . We assume that  $x$  is between  $p$  and  $y$ . If  $[p : x : y : q]$  denote the cross ratio of  $p, x, y, q$ , then the following formula defines a metric (see Figure 1):

$$d_\Omega(x, y) = \frac{1}{2} \log \left( [p : x : y : q] \right), \quad \text{for every } x, y \in \Omega.$$

This metric gives to  $\Omega$  the same topology as the one inherited from  $\mathbb{S}(V)$ . The metric space  $(\Omega, d_\Omega)$  is complete, the closed balls in  $\Omega$  are compact, the group  $\text{Aut}(\Omega)$  acts on  $\Omega$  by isometries, and therefore acts properly.

The metric is called the *Hilbert metric* and can be also defined by a Finsler norm on the tangent space  $T_x\Omega$  at each point  $x$  of  $\Omega$ : Let  $v$  be a vector of  $T_x\Omega$ . The quantity  $\frac{d}{dt} \Big|_{t=0} d_\Omega(x, x + tv)$  defines a Finsler norm  $F_\Omega(x, v)$  on  $T_x\Omega$ . Let's choose an affine chart  $\mathbb{A}$  containing  $\Omega$  and a Euclidean norm  $|\cdot|$  on  $\mathbb{A}$ . If  $p^+$  (resp.  $p^-$ ) is the intersection point of  $\partial\Omega$

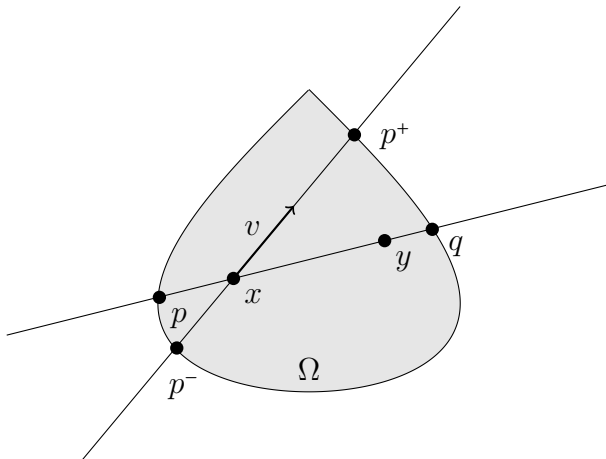


FIGURE 1. The Hilbert metric

and the half-line determined by  $x$  and  $v$  (resp.  $-v$ ), and  $|ab|$  is the distance between two points  $a, b$  of  $\mathbb{A}$  (see Figure 1), then we obtain

$$F_{\Omega}(x, v) = \frac{d}{dt} \Big|_{t=0} d_{\Omega}(x, x + tv) = \frac{|v|}{2} \left( \frac{1}{|xp^{-}|} + \frac{1}{|xp^{+}|} \right).$$

The regularity of the Finsler norm is same as the regularity of the boundary  $\partial\Omega$ , and the Finsler structure gives rise to an absolutely continuous measure  $\mu_{\Omega}$  with respect to Lebesgue measure, called the *Busemann measure*.

If  $\Omega$  is the ellipsoid then  $(\Omega, d_{\Omega})$  is the projective model of the hyperbolic space. More generally, if  $\Omega$  is *round*, i.e. strictly convex with  $\mathcal{C}^1$ -boundary, then the metric space  $(\Omega, d_{\Omega})$  exhibits some hyperbolic behaviour even though  $(\Omega, d_{\Omega})$  is not Gromov-hyperbolic <sup>6</sup>.

A properly convex open set  $\Omega$  is a polytope if and only if  $(\Omega, d_{\Omega})$  is bi-Lipschitz to the Euclidean space [Ver15]. If  $\Omega$  is the projectivization of the space of real positive definite symmetric matrices of size  $m \times m$  then  $\text{Aut}(\Omega) = \text{SL}_m(\mathbb{R})$ .

Convex projective structures are therefore a special kind of  $(G, X)$ -structures, whose golden sisters are hyperbolic structures and whose iron cousins are higher-rank structures, i.e. the  $(G, G/K)$ -structures where  $G$  is a semi-simple Lie group without compact factor of (real) rank  $\geq 2$ ,  $K$  is the maximal compact subgroup of  $G$  and  $G/K$  is the symmetric space of  $G$ .

The above fact leads to initiate an interest in convex projective structures, however, probably it is not the main reason why the convex projective structures are interesting. The main justification is the following theorem.

<sup>6</sup>If  $\Omega$  is strictly convex and  $\partial\Omega$  is of class  $\mathcal{C}^2$  with a positive Hessian then  $(\Omega, d_{\Omega})$  is Gromov-hyperbolic (see Colbois-Verovic [CV04]), but unfortunately the convex set we are interested in will be at most round, except the ellipsoid.

**4.3. Koszul-Benoist's theorem.** Recall that the *virtual center* of a group  $\Gamma$  is the subgroup consisting of the element whose centralizer is of finite index in  $\Gamma$ . So, the virtual center of a group is trivial if and only if every subgroup of finite index has a trivial center.

**Theorem 4.1** (Koszul-Benoist, [Kos68, Ben05]). *Let  $\mathcal{O}$  be a closed orbifold of dimension  $d$  admitting a convex real projective structure. Suppose that the group  $\pi_1(\mathcal{O})$  has a trivial virtual center. Then the space  $\mathfrak{C}(\mathcal{O})$  of convex projective structures on  $\mathcal{O}$  is a union of connected components of the character variety  $\chi(\pi_1(\mathcal{O}), \mathrm{SL}_{d+1}^{\pm}(\mathbb{R}))$ .*

*History of the proof.* As stated in Theorem 4.1, Koszul and Benoist proved the openness and closedness, respectively, but in dimension 2, it was proved by the first author and Goldman [CG93] and when  $\Gamma$  is a uniform lattice of  $\mathrm{SO}_{3,1}$ , it was proved by Kim [Kim01].  $\square$

The condition that the virtual center of a group is trivial is transparent as follows:

**Proposition 4.2** (Corollary 2.13 [Ben05]). *Let  $\Gamma$  be a discrete subgroup of  $\mathrm{PGL}_{d+1}(\mathbb{R})$ . Suppose that  $\Gamma$  acts on a properly convex open subset  $\Omega$  in  $\mathbb{RP}^d$  and  $\Omega/\Gamma$  is compact. Then the following are equivalent.*

- *The virtual center of  $\Gamma$  is trivial.*
- *Every subgroup of finite index of  $\Gamma$  has a finite center.*
- *Every subgroup of finite index of  $\Gamma$  is irreducible in  $\mathrm{PGL}_{d+1}(\mathbb{R})$ <sup>7</sup>.*
- *The Zariski closure of  $\Gamma$  is semisimple.*
- *The group  $\Gamma$  does not contain an infinite nilpotent normal subgroup.*
- *The group  $\Gamma$  does not contain an infinite abelian normal subgroup.*

**4.4. Duality between convex real projective orbifolds.** We start from the linear duality. Every sharp convex open cone  $C$  of a real vector space  $V$  gives rise to a *dual* convex cone

$$C^* = \{\varphi \in V^* \mid \varphi|_{\overline{C} \setminus \{0\}} > 0\}.$$

It can be easily verified that  $C^*$  is also a sharp convex open cone and that  $C^{**} = C$ . Hence, the duality leads to an involution between the sharp convex open cones.

Now, consider the “projectivization” of linear duality: If  $\Omega$  is a properly convex open subset of  $\mathbb{P}(V)$  and  $C_{\Omega}$  is the cone of  $V$  over  $\Omega$ , then *the dual  $\Omega^*$  of  $\Omega$*  is  $\mathbb{P}(C_{\Omega}^*)$ , which is a properly convex open subset of  $\mathbb{P}(V^*)$ . In a more intrinsic way, the dual  $\Omega^*$  is the set of the hyperplanes  $H$  of  $V$  such that  $\mathbb{P}(H) \cap \overline{\Omega} = \emptyset$ . Since there is a correspondence between hyperplanes and affine charts of the projective space,  $\Omega^*$  can be defined as the space of affine charts  $\mathbb{A}$  of  $\mathbb{RP}^d$  such that  $\Omega$  is a bounded subset of  $\mathbb{A}$ .

The second interpretation offers us a map  $\Omega^* \rightarrow \Omega$ : namely to  $\mathbb{A} \in \Omega^*$  we can associate the center of mass of  $\Omega$  in  $\mathbb{A}$ . The map is well-defined since  $\Omega$  is a bounded convex open subset of  $\mathbb{A}$ . In fact, Vinberg shows that this map is an analytic diffeomorphism (See [Vin63a, Gol88b] for details), so we call it the *Vinberg duality diffeomorphism*.

Finally, we can bring the group in the playground. Recall that the dual representation  $\rho^* : \Gamma \rightarrow \mathrm{PGL}(V^*)$  of a representation  $\rho : \Gamma \rightarrow \mathrm{PGL}(V)$  is defined by  $\rho^*(\gamma) = {}^t\rho(\gamma^{-1})$ , i.e. the dual projective transformation of  $\rho(\gamma)$ . All the constructions happen in projective

<sup>7</sup>That is,  $\Gamma$  is *strongly irreducible*.

geometry, therefore if a representation  $\rho$  preserves a properly convex open set  $\Omega$  then the dual representation  $\rho^*$  preserves the dual properly convex set  $\Omega^*$ . Even more, if we assume the representation  $\rho$  to be discrete, then the Vinberg duality diffeomorphism will induce a diffeomorphism between the quotient orbifolds  $\Omega/\rho(\Gamma)$  and  $\Omega^*/\rho^*(\Gamma)$ .

## 5. THE EXISTENCE OF DEFORMATION OR EXOTIC STRUCTURES

**5.1. Bending construction.** Johnson and Millson [JM87] found an important class of deformations of convex real projective structures on an orbifold. The bending construction was introduced by Thurston to deform  $(\mathrm{SO}_{2,1}(\mathbb{R}), \mathbb{H}^2)$ -structures on a surface into  $(\mathrm{SO}_{3,1}(\mathbb{R}), \mathbb{S}^2)$ -structures, i.e. conformally flat structures, and so in particular to produce quasi-Fuchsian groups. It was extended by Kourouniotis [Kou85] to deform  $(\mathrm{SO}_{d,1}(\mathbb{R}), \mathbb{H}^d)$ -structures on a manifold into  $(\mathrm{SO}_{d+1,1}(\mathbb{R}), \mathbb{S}^d)$ -structures.

Johnson and Millson indicated several other deformations, all starting from a hyperbolic structure on an orbifold, however we will focus only on real projective deformations. Just before that we stress that despite the simplicity of the argument, the generalisation is not easy. Goldman and Millson [GM88], for instance, show that there exists no such deformation of a uniform lattice of  $\mathrm{SU}_{d,1}$  into  $\mathrm{SU}_{d+1,1}$ .

Let  $\mathfrak{sl}_{d+1}(\mathbb{R})$  be the Lie algebra of  $\mathrm{SL}_{d+1}(\mathbb{R})$ , and let  $\mathcal{O}$  be a closed convex projective orbifold. Suppose that  $\mathcal{O}$  contains a two-sided totally geodesic hypersurface  $\Sigma$ . For example, all the hyperbolic manifolds obtained from standard arithmetic lattices of  $\mathrm{SO}_{d,1}(\mathbb{R})$ , up to a finite cover, admit such a totally geodesic hypersurface (see Section 7 of [JM87] for the construction of *standard* arithmetic lattices). Let  $\Gamma = \pi_1(\mathcal{O})$  and  $A = \pi_1(\Sigma)$ . Recall that the Lie group  $\mathrm{SL}_{d+1}(\mathbb{R})$  acts on the Lie algebra  $\mathfrak{sl}_{d+1}(\mathbb{R})$  by the adjoint action.

**Lemma 5.1** (Johnson-Millson [JM87]). *Let  $\rho \in \mathrm{Hom}(\Gamma, \mathrm{SL}_{d+1}(\mathbb{R}))$ . Suppose that  $\rho(A)$  fixes an element  $x_1$  in  $\mathfrak{sl}_{d+1}(\mathbb{R})$  and that  $x_1$  is not invariant under  $\Gamma$ . Then  $\mathrm{Hom}(\Gamma, \mathrm{SL}_{d+1}(\mathbb{R}))$  contains a non-trivial curve  $(\rho_t)_{t \in (-\varepsilon, \varepsilon)}$ ,  $\varepsilon > 0$ , with  $\rho_0 = \rho$ , i.e. the curve is transverse to the conjugation action of  $\mathrm{SL}_{d+1}(\mathbb{R})$ .*

**Theorem 5.2** (Johnson-Millson [JM87], Koszul [Kos68], Benoist [Ben04, Ben05]). *Suppose that a closed hyperbolic orbifold  $\mathcal{O}$  contains  $r$  disjoint, embedded, totally geodesic suborbifold  $\Sigma_1, \dots, \Sigma_r$  of codimension-one. Then the dimension of the space  $\mathfrak{C}(\mathcal{O})$  at the hyperbolic structure is greater than or equal to  $r$ . Moreover, the bending curves lie entirely in  $\mathfrak{C}(\mathcal{O})$  and all the properly convex structure on  $\mathcal{O}$  are strictly convex.*

We abbreviate a connected open set to a *domain*, and say that a convex domain of the real projective space is *divisible* (resp. *quasi-divisible*) if there exists a discrete subgroup  $\Gamma$  of  $\mathrm{Aut}(\Omega)$  such that the action of  $\Gamma$  on  $\Omega$  is cocompact (resp. of finite Busemann covolume).

**Theorem 5.3** (Johnson-Millson [JM87], Koszul [Kos68], Benoist [Ben04]). *For every integer  $d \geq 2$ , there exists a non-symmetric divisible strictly convex domain of dimension  $d$ .*

*Remark 5.4.* Kac and Vinberg [KV67] made the first examples of non-symmetric divisible convex domains of dimension 2 using Coxeter groups (see Section 7).

*Remark 5.5.* The third author with Sam Ballas [Mar12a, BM] extends Theorem 5.3 to non-symmetric *quasi-divisible* (not divisible) convex domains.

**5.2. The nature of the exotics.** Now that we have seen the existence of non-symmetric divisible convex domains in all dimensions  $d \geq 2$ . We first remark that so far all the divisible convex domains built are round<sup>8</sup> if there are not the product of lower dimensional convex domains. So we might want to know if we can go further, specially if we can find indecomposable divisible convex domains that are not strictly convex. The first result in dimension 2 is negative<sup>9</sup>:

5.2.1. *Dimension 2: Kuiper-Benzécri's Theorem.*

**Theorem 5.6** (Kuiper [Kui54], Benzécri [Ben60], M. [Mar12b]). *A quasi-divisible convex domain of dimension 2 is round, except the triangle.*

The next result in dimension 3 is positive:

5.2.2. *Dimension 3: Benoist's Theorem.* First, we can classify the possible topology for closed convex projective 3-manifold or 3-orbifold.

**Theorem 5.7** (Benoist [Ben06a]). *If a closed 3-orbifold  $\mathcal{O}$  admits an indecomposable<sup>10</sup> convex projective structure, then  $\mathcal{O}$  is topologically the union along the boundaries of finitely many 3-orbifolds each of which admits a finite volume hyperbolic structure on its interior.*

Second, these examples do exist.

**Theorem 5.8** (Benoist [Ben06a], M. [Mar10a], Ballas-Danciger-L. [BDL]). *There exists an indecomposable divisible properly convex domain  $\Omega$  of dimension 3 which is not strictly convex. Moreover, every line segment in  $\partial\Omega$  is contained in the boundary of a properly embedded triangle<sup>11</sup>.*

At the time of writing, Theorem 5.7 is valid only for divisible convex domain. However, Theorem 5.8 is true also for quasi-divisible convex domains which are not divisible [Mar14b].

The presence of properly embedded triangles in the convex domain links to the existence of incompressible Euclidean orbifolds on the quotient orbifold. Benoist and the third author made the examples using Coxeter groups and a work of Vinberg [Vin71]. We will explore more this technique in Section 7. Ballas, Danciger and the second author [BDL] found a sufficient condition under which the double of a cusped hyperbolic three-manifold admits a convex projective structure, to produce the examples.

In order to obtain the quasi-divisible convex domains in [Mar14b], the second author essentially keeps the geometry of the cusps. In other words, the holonomy of the cusps preserve an ellipsoid as does the cusps of finite volume hyperbolic orbifolds. From this perspective, there is a different type of exotic, thanks to Ballas:

**Theorem 5.9** (Ballas [Bal15]). *There exists an indecomposable quasi-divisible (necessarily not divisible) properly convex domain  $\Omega$  of dimension 3 which is not strictly convex nor with  $\mathcal{C}^1$ -boundary such that the quotient is homeomorphic to a hyperbolic manifold of finite volume.*

More precisely, the Ballas's example is an explicit convex projective deformation of the hyperbolic structure on the figure-eight knot complement.

<sup>8</sup>Strictly convex with  $\mathcal{C}^1$ -boundary.

<sup>9</sup>The word "positive/negative" in this subsection reflects only the feeling of the authors.

<sup>10</sup>A convex projective orbifold is *indecomposable* if its holonomy representation is strongly irreducible.

<sup>11</sup>A simplex  $\Delta$  in  $\Omega$  is *properly embedded* if  $\partial\Delta \subset \partial\Omega$ .

5.2.3. *Orbifolds of dimension 4 and beyond.* Until now, there are only three sources for non-symmetric divisible convex domains of dimension  $d \geq 4$ : from the “standard” bending of Johnson-Millson [JM87], from the “clever” bending of Kapovich [Kap07], and using Coxeter groups. The last method was initiated by Benoist [Ben06a, Ben06b] and extended by the three authors [CLMb]:

**Theorem 5.10** (Benoist [Ben06a], C. L. M. [CLMb]). *For  $d = 4, \dots, 7$ , there exists an indecomposable divisible convex domain  $\Omega$  of dimension  $d$  which is not strictly convex nor with  $C^1$ -boundary such that  $\Omega$  contains a properly embedded  $(d-1)$ -simplex. Moreover, the quotient is homeomorphic to the union along the boundaries of finitely many  $d$ -orbifolds each of which admits a finite volume hyperbolic structure on its interior.*

Other exotics are built by Benoist in dimension 4 and by Kapovich in every dimension:

**Theorem 5.11** (Benoist [Ben06b], Kapovich [Kap07]). *For  $d \geq 4$ , there exists a divisible convex domain  $\Omega$  of dimension  $d$  such that  $(\Omega, d_\Omega)$  is Gromov-hyperbolic but it is not quasi-isometric to a symmetric space. In particular,  $\Omega$  is strictly convex, however it is not quasi-isometric to the hyperbolic space  $\mathbb{H}^d$ .*

The three authors recently construct somehow different exotics:

**Theorem 5.12** (C. L. M. [CLMa]). *For  $d = 4, \dots, 7$ , there exists an indecomposable divisible convex domain  $\Omega$  of dimension  $d$  which is not strictly convex nor with  $C^1$ -boundary such that  $\Omega$  contains a properly embedded  $(d-2)$ -simplex but it does not contain a properly embedded  $(d-1)$ -simplex.*

## 6. REAL PROJECTIVE SURFACES

Another motivation for studying convex projective structures is that these structures are just the right building blocks to construct all the projective structures on closed surfaces.

### 6.1. Affine and projective structures on tori.

6.1.1. *Classification of affine surfaces.* Compact affine surfaces are topologically restrictive:

**Theorem 6.1** (Benzécri [Ben60]). *If  $S$  is a compact affine surface with empty or geodesic<sup>12</sup> boundary, then  $\chi(S) = 0$ .*

Afterwards, in early 1980s, Nagano and Yagi [NY74] classified the affine structures on a torus and an annulus with geodesic boundary; see [Ben94] for a modern viewpoint and [Bau14] for an extensive survey on this topic.

Let  $Q$  be the closed positive quadrant of  $\mathbb{R}^2$ . An *elementary affine annulus* is the quotient of  $Q \setminus \{0\}$  by the group generated by a diagonal matrix with all eigenvalues greater than 1. It is indeed an affine annulus with geodesic boundary.

An affine torus  $A$  is *complex* if there are complex numbers  $a, b$  independent over  $\mathbb{R}$  such that  $A$  is the quotient of  $\mathbb{C} \setminus \{0\}$  by the group generated by the elements  $z \mapsto az$  and  $z \mapsto bz$ .

**Theorem 6.2** (Nagano-Yagi [NY74]). *If  $A$  is a compact affine surface with empty or geodesic boundary, then one of the following hold:*

<sup>12</sup>An arc of a projective (or affine) surface is *geodesic* if it has a lift which is developed into a line segment.

- The universal cover of  $A$  is either a complete affine space, a half-affine space, a closed parallel strip or a quadrant.
- The surface  $A$  is a complex affine torus.
- The surface  $A$  decomposes into elementary affine annuli along simple closed geodesics.

6.1.2. *Classification of projective tori.* Let  $\gamma$  be an element of  $\mathrm{SL}_3(\mathbb{R})$ . A matrix  $\gamma$  is *positive hyperbolic* if  $\gamma$  has three distinct positive eigenvalues, that is,  $\gamma$  is conjugate to

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix} \quad (\lambda\mu\nu = 1 \text{ and } 0 < \lambda < \mu < \nu).$$

A matrix  $\gamma$  is *planar* if  $\gamma$  is diagonalizable and it has only two distinct eigenvalues. A matrix  $\gamma$  is *quasi-hyperbolic* if  $\gamma$  has only two distinct positive eigenvalues and it is not diagonalizable, that is,  $\gamma$  is conjugate to

$$(2) \quad \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1/\lambda^2 \end{pmatrix} \quad (0 < \lambda \neq 1).$$

A matrix  $\gamma$  is a *projective translation* (resp. *parabolic*) if  $\gamma$  is conjugate to

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (\text{resp.} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}).$$

These represent all conjugacy classes of non-trivial elements of  $\mathrm{SL}_3(\mathbb{R})$  with positive eigenvalues.

Let  $\vartheta$  be a positive hyperbolic element of  $\mathrm{SL}_3(\mathbb{R})$  with eigenvalues  $\lambda < \mu < \nu$ . It is easy to describe the action of  $\vartheta$  on the projective plane preserves three lines meeting at the three fixed points. The fixed point  $r$  (resp.  $s$ ,  $a$ ) corresponding to the eigenvector for  $\lambda$  (resp.  $\mu$ ,  $\nu$ ) is said to be *repelling* (resp. *saddle*, *attracting*).

Let  $\Delta$  be the open triangle with vertices  $r, s, a$ . An *elementary annulus of type I* is one of the two<sup>13</sup> real projective annuli  $(\Delta \cup \overline{ar}^\circ \cup \overline{as}^\circ)/\langle \vartheta \rangle$  and  $(\Delta \cup \overline{ar}^\circ \cup \overline{rs}^\circ)/\langle \vartheta \rangle$  (see Figure 2). These annuli are in fact compact since we can find a compact fundamental domain. We call the image of  $\overline{ar}^\circ$  in the annulus a *principal* closed geodesic, and the image of  $\overline{as}^\circ$  and the image of  $\overline{rs}^\circ$  *weak* closed geodesics.

Now, let  $\vartheta$  be a quasi-hyperbolic element conjugate to the matrix (2) with  $\lambda < 1$ . The fixed point  $r$  (resp.  $a$ ) corresponding to the eigenvalues  $\lambda$  (resp.  $1/\lambda^2$ ) is repelling (resp. attracting). Let  $C$  be the line on which the action of  $\vartheta$  is parabolic, and let  $\beta$  be the invariant segment in  $C$  with endpoints  $a$  and  $-a$ , the antipodal point of  $a$ , such that, for each point  $x \in \beta^\circ$ , the sequence  $\vartheta^n(x)$  converges to  $a$ .

Let  $L$  be an open lune bounded by  $\beta$ ,  $\overline{ar}$ ,  $\overline{-ar}$ . An *elementary annulus of type II* is one of two real projective annuli  $(L \cup \overline{ar}^\circ \cup \overline{-ar}^\circ)/\langle \vartheta \rangle$  and  $(L \cup \beta^\circ \cup \overline{ar}^\circ)/\langle \vartheta \rangle$  (see Figure 2). These annuli are also compact with geodesic boundary. We call the image of  $\overline{ar}^\circ$  and of  $\overline{-ar}^\circ$  in

<sup>13</sup>Note that the space  $(\Delta \cup \overline{sa}^\circ \cup \overline{rs}^\circ)/\langle \vartheta \rangle$  is not Hausdorff.



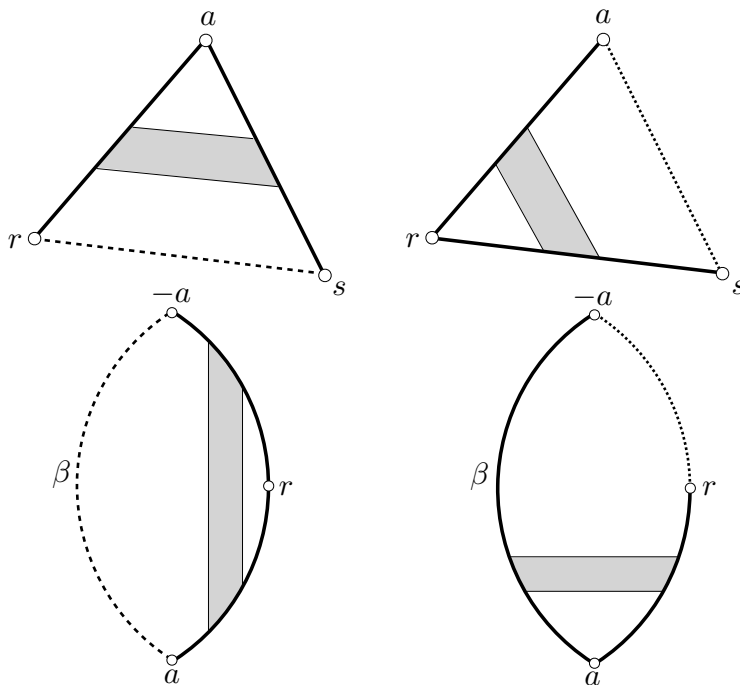


FIGURE 2. Elementary annuli of type I and type II

the annulus a *principal* closed geodesic, and the image of  $\beta^\circ$  a *weak* closed geodesic.

By pasting the boundaries of finitely many compact elementary annuli, we obtain an annulus or a torus. The gluing of course requires that boundaries are either both principal or both weak, and their holonomies are conjugate to each other.

**Theorem 6.3** (Goldman [Gol77]). *If  $T$  is a projective torus or a projective annulus with geodesic boundary, then  $T$  is an affine torus or  $T$  decompose into elementary annuli.*

**6.2. Automorphisms of convex 2-domain.** A *finite-type* 2-orbifold is a 2-orbifold whose underlying space is a finite-type surface and the singular locus is a union of finitely many suborbifolds of dimension 0 or 1. A *peripheral element* of the fundamental group of an orbifold is a loop that can be isotope outside any compact suborbifold.

Let  $S = \mathcal{C}/\Gamma$  be an orientable properly convex real projective 2-orbifold of finite type with empty or geodesic boundary, for a properly convex subset  $\mathcal{C} \subset \mathbb{S}^2$ . We denote by  $\Omega$  the interior of  $\mathcal{C}$ . A closed geodesic  $g$  in  $S$  is *principal* when the holonomy of  $g$  is positive hyperbolic or quasi-hyperbolic and the lift  $\tilde{g}$  of  $g$  is the geodesic segment in  $\mathcal{C}$  connecting the attracting and repelling fixed points of  $\gamma$ .

The following theorem generalizes well-known results of hyperbolic structures on surfaces, and is essential to understand the convex real projective orbifolds. The nonorientable orbifold version exists but is a bit more complicated to state (See C. -Goldman [CG05]).

**Theorem 6.4** (Kuiper [Kui54], C. [Cho94a], M. [Mar12b]). *Let  $S = \mathcal{C}/\Gamma$  be an orientable properly convex real projective 2-orbifold of finite type with empty or geodesic boundary, for a properly convex subset  $\mathcal{C} \subset \mathbb{S}^2$ . We denote by  $\Omega$  the interior of  $\mathcal{C}$ . Suppose that  $\Omega$  is not a triangle.*

- *An element has a finite order if and only if it fixes a point in  $\Omega$ .*

- Each infinite order element  $\gamma$  of  $\Gamma$  is positive hyperbolic, quasi-hyperbolic or parabolic.
- If an infinite order element  $\gamma$  is nonperipheral, then  $\gamma$  is positive hyperbolic and a unique closed geodesic  $g$  realizes the homotopy class of  $\gamma$ . Moreover,  $g$  is principal and any lift of  $g$  is inside  $\Omega$ .
- If  $\gamma$  is represented by a nonperipheral simple closed curve in  $S$ , then  $\gamma$  is positive hyperbolic and realized by a simple closed geodesic in  $S$ .
- A peripheral positive hyperbolic element is realized by a unique principal geodesic  $g$ , and either all the lifts of  $g$  are inside  $\Omega$  or all the lifts of  $g$  are in  $\mathcal{C} \cap \partial\Omega$ .
- A quasi-hyperbolic element is peripheral and is realized by a unique geodesic in  $\mathcal{C}$ . Moreover,  $g$  is principal and any lift of  $g$  is in  $\mathcal{C} \cap \partial\Omega$ .
- A parabolic element  $\gamma$  is peripheral and is realized by the projection of  $\mathcal{E} \setminus \{p\}$  where  $\mathcal{E}$  is a  $\gamma$ -invariant ellipse whose interior is inside  $\Omega$  and  $p \in \partial\Omega$  is the unique fixed point of  $\gamma$ .

Note for the fifth item, the closed geodesics homotopic to  $g$  may not be unique.

**6.3. Convex projective structures on surfaces.** Let  $S$  be a compact surface with or without boundary. When  $S$  has a boundary, in general, the holonomy of a convex projective structure on  $S$  does not determine the structure. More precisely, there exists convex projective structure whose holonomy preserves more than one convex domains. So we need to make some assumptions on the convex projective structure in order to avoid this issue and therefore we consider the subspace, denoted by  $\mathcal{C}^{pgb}(S)$ , in  $\mathcal{C}(S)$  of convex projective structures on  $S$  for which each boundary component is positive hyperbolic and has a lift which is developed into the principal axis. We call such a boundary *principal*, hence  $\mathcal{C}^{pgb}(S)$  is the space of convex projective surface with principal geodesic boundary.

**Theorem 6.5** (Goldman [Gol90]). *If  $S$  is a closed surface of genus  $> 1$ , then the space  $\mathcal{C}(S)$  is homeomorphic to an open cell of dimension  $-8\chi(S)$ .*

The following two propositions illustrate the proof of Theorem 6.5.

**Proposition 6.6** (Goldman [Gol90], M. [Mar10b]). *Let  $S$  be a compact surface with or without boundary such that  $\chi(S) < 0$ , and let  $c$  be a simple closed curve in  $S$ .*

- *Suppose that  $S \setminus c$  is a disjoint union of  $S_1$  and  $S_2$ . Let  $S'_i$ ,  $i = 1, 2$ , be the completion of  $S_i$ , and let  $\mathcal{C}^{pgb}(S'_1) \boxtimes_c \mathcal{C}^{pgb}(S'_2)$  the subspace of structures  $(P_1, P_2)$  in  $\mathcal{C}^{pgb}(S'_1) \times \mathcal{C}^{pgb}(S'_2)$  whose holonomies corresponding to  $c$  are conjugate each other. Then the forgetful map  $\mathcal{C}^{pgb}(S) \rightarrow \mathcal{C}^{pgb}(S'_1) \boxtimes_c \mathcal{C}^{pgb}(S'_2)$  is a  $\mathbb{R}^2$ -principal fiber bundle.*
- *Suppose that  $S \setminus c$  is connected. If  $S'$  is the completion of  $S - c$ , then the forgetful map  $\mathcal{C}^{pgb}(S) \rightarrow \mathcal{C}^{pgb}(S_1)$  is a  $\mathbb{R}^2$ -principal fiber bundle.*

Proposition 6.6 should be compared to Lemma 5.1. For each simple closed curve, since the holonomy of  $c$  is positive hyperbolic, its centralizer is 2-dimensional. The first gluing parameter is the twist parameter like in hyperbolic geometry, and the second gluing parameter is the bending parameter we get a sight of in Lemma 5.1.

Next, we need to understand the convex projective structures on a pair of pants. Assume that  $\gamma$  is an element of  $\mathrm{SL}_3(\mathbb{R})$  with positive eigenvalues. We denote by  $\lambda(\gamma)$  the smallest eigenvalue of  $\gamma$  and by  $\tau(\gamma)$  the sum of the two other eigenvalues, and we call the pair  $(\lambda(\gamma), \tau(\gamma))$  the *invariant* of  $\gamma$ . The map  $[\gamma] \mapsto (\lambda(\gamma), \tau(\gamma))$  is a homeomorphism between

the space of conjugacy classes of positive hyperbolic, quasi-hyperbolic or parabolic element  $\gamma$  of  $\mathrm{SL}_3(\mathbb{R})$  and the space

$$\mathcal{R} := \{(\lambda, \tau) \in \mathbb{R}^2 \mid 0 < \lambda \leq 1, 2/\sqrt{\lambda} \leq \tau \leq \lambda + \lambda^{-2}\}$$

(see Figure 3). Note that  $(1, 2) \in \mathcal{R}$  corresponds to the conjugacy class of parabolic elements.

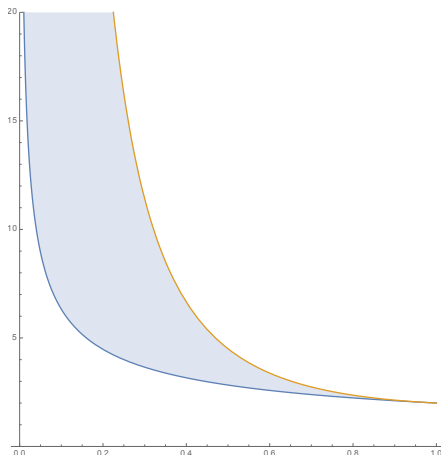


FIGURE 3. The region  $\mathcal{R}$  between the graphs  $y = \frac{2}{\sqrt{x}}$  and  $y = x + \frac{1}{x^2}$

Let  $S$  be a compact surface with  $n$  (oriented) boundary components  $c_i$ ,  $i = 1, \dots, n$ . Let's define a map

$$R_{\partial S} : \mathcal{C}^{pgb}(S) \rightarrow \mathcal{R}^n$$

from each structure to the  $n$ -tuples of invariants  $(\lambda(\gamma_i), \tau(\gamma_i))$  of the holonomy  $\gamma_i$  of  $c_i$ .

**Proposition 6.7** (Goldman [Gol90], M. [Mar10b]). *If  $P$  is a pair of pants, then the map  $R_{\partial P} : \mathcal{C}^{pgb}(P) \rightarrow \mathcal{R}^3$  is a  $\mathbb{R}^2$ -principal fiber bundle, and the interior of  $\mathcal{C}^{pgb}(P)$  is exactly the convex projective structures with principal geodesic boundary and in particular it is an open cell of dimension 8.*

Let  $S$  be a surface with ends. We say that a convex projective structure on  $S$  is of *relatively finite volume*, if for each end of  $S$ , there exists an end neighborhood  $\mathcal{V}$  such that  $\mu_S(\mathcal{V}) < \infty$ . Let  $\mathcal{C}_{gf}^{pgb}(S)$  denote the subspace in  $\mathcal{C}^{pgb}(S)$  of convex projective structures with principal geodesic boundary and of relatively finite volume. The following theorem generalizes to these kind of structures:

**Theorem 6.8** (Fock-Goncharov [FG07], Theorem 3.7 of M. [Mar10b]). *Let  $S$  be a surface of finite-type. If  $b$  is the number of boundary components of  $S$  and  $p$  is the number of ends of  $S$ , then the space  $\mathcal{C}_{gf}^{pgb}(S)$  is a manifold with corner homomorphic to  $\mathbb{R}^{16g-16+6p+7b} \times [0, 1]^b$ , and the interior of  $\mathcal{C}_{gf}^{pgb}(S)$  is exactly the structures whose end holonomies are positive hyperbolic.*

**6.4. The convex decomposition of projective surfaces of genus  $> 1$ .** Now let's consider compact real projective surfaces with geodesic boundary.

**Theorem 6.9** (C. [Cho94a, Cho94b]). *Let  $S$  be a compact projective surface with geodesic boundary. If  $\chi(S) < 0$ , then  $S$  canonically decomposes along simple closed geodesics into convex projective subsurfaces with principal geodesic boundary and elementary annuli.*

The annuli with principal geodesic boundary whose holonomies are positive hyperbolic (resp. quasi-hyperbolic) are classified by Goldman [Gol77] (resp. C. [Cho94b]).

If  $A$  is an annulus with quasi-hyperbolic principal geodesic boundary, then only one boundary can be identified with a boundary of a convex real projective surface with principal geodesic boundary by Proposition 5 of [Cho96]. Hence, if a compact real projective surface  $S$  has quasi-hyperbolic holonomy for a closed curve, then  $S$  must have a boundary.

*Remark 6.10.* Given a convex projective surface  $S = \Omega/\Gamma$  with a positive hyperbolic principal boundary and an elementary annulus  $A$ , we can obtain a new projective surface  $S'$  identifying the boundaries of  $S$  and  $A$ . The surface  $S'$  is still convex since the union of  $\Omega$  and the triangles given by the universal cover of  $A$ , remains convex.

**6.5. Projective structures on a closed surface of genus  $> 1$ .** Let  $S$  be a closed surface of genus greater than 1 and let  $\mathbb{P}(S)$  denote the space of projective structures on  $S$ . For each connected component of  $\mathbb{P}(S)$ , any two elements share the same canonical decomposition up to isotopy, given by Theorem 6.9. Let  $\mathcal{S}(S)$  denote the collection of mutually disjoint isotopy classes of non-trivial simple closed curves, and let  $F_2^{+,even}$  be the set of elements of the free semigroup on two generators whose word-lengths are even.

In [Gol77] Goldman constructs a map

$$\mathbb{P}(S) \rightarrow \mathcal{S}(S) \times F_2^{+,even}$$

that describes the combinatorics of the glued elementary annuli of type I. Finally, by removing all annuli with principal geodesic boundary and reattaching, we obtain a convex projective structure on  $S$ , and get:

**Theorem 6.11** (C.-Goldman [Gol90, Cho94a, Cho94b]). *Let  $S$  be a closed surface with  $\chi(S) < 0$ . Then each fiber of the map  $\mathbb{P}(S) \rightarrow \mathcal{S}(S) \times F_2^{+,even}$  identifies with  $\mathfrak{C}(S)$ . In particular,  $\mathbb{P}(S)$  is homeomorphic to a countable many disjoint union of open cells of dimension  $16(g-1)$ .*

## 6.6. Convex projective orbifolds of negative Euler characteristic.

**Definition 6.12.** A *suborbifold* of an orbifold  $\mathcal{O}$  is an orbifold  $\mathcal{O}'$  such that the underlying space  $X_{\mathcal{O}'}$  of  $\mathcal{O}'$  is an embedded subspace of  $X_{\mathcal{O}}$  and, for each point  $x \in X_{\mathcal{O}'}$ , there are coordinate neighborhoods  $U' \approx \tilde{U}'/\Gamma'$  in  $\mathcal{O}'$  and  $U \approx \tilde{U}/\Gamma$  in  $\mathcal{O}$  such that  $\tilde{U}'$  is a  $\Gamma$ -invariant closed submanifold of  $\tilde{U}$  and  $\Gamma'$  is the restriction of  $\Gamma$  to  $\tilde{U}'$ . Sometimes  $\Gamma' \rightarrow \Gamma$  may not be injective.

Our definition is more restrictive than Adem-Leida-Ruan's [ALR07] and less restrictive than Kapovich's [Kap09], however, this definition seems to be better for studying decompositions of 2-orbifolds along 1-orbifolds. For example, a mirror arc ending at two corner-reflectors of order 2 is a suborbifold.

Every 2-orbifold is obtained from a surface with corners by making some arcs mirrors and putting cone-points and corner-reflectors in a locally finite manner. A mirror arc is either open or its end is a corner-reflector that should be an end of another mirror arc. Moreover, the smooth topology of a 2-orbifold is classified by the underlying topology of the surface with corners, the number of cone-points and corner-reflectors, and the boundary patterns of

the mirror arcs.

A *full* 1-orbifold is a segment with two mirror endpoints. Let  $\Sigma$  be a compact 2-orbifold with  $m$  cone points of order  $q_i$ ,  $i = 1, \dots, m$ , and  $n$  corner-reflectors of order  $r_j$ ,  $j = 1, \dots, n$ , and  $n_\Sigma$  boundary full 1-orbifolds. The orbifold Euler characteristic of  $\Sigma$  is

$$\chi(\Sigma) = \chi(X_\Sigma) - \sum_{i=1}^m \left(1 - \frac{1}{q_i}\right) - \frac{1}{2} \sum_{j=1}^n \left(1 - \frac{1}{r_j}\right) - \frac{1}{2} n_\Sigma$$

called the generalized Riemman-Hurwitch formula (see Section 3.3 for the definition of the orbifold Euler characteristic).

**Theorem 6.13** (Thurston [Thu02]). *Let  $\Sigma$  be a compact 2-orbifold of negative orbifold Euler characteristic with the underlying space  $X_\Sigma$ . Then the deformation space  $\mathcal{T}(\Sigma)$  of hyperbolic structures on  $\Sigma$  is a cell of dimension  $-\chi(X_\Sigma) + 2k + l + 2n$  where  $k$  is the number of cone-points and  $l$  is the number of corner-reflectors and  $n$  is the number of boundary full 1-orbifolds.*

In order to understand the deformation spaces of convex projective structures on surfaces, we have seen that it is important to find out the convex projective structures on a pair of pants, which is the most "elementary" surface. Similarly, in the case of 2-orbifolds, we should firstly understand the elementary 2-orbifolds.

Let's discuss the splitting and sewing of 2-orbifolds. Note that orbifolds always have a path-metric. For example, it has a notion of Riemannian metric, i.e. a Riemannian metric on neighborhoods invariant under the finite group action compatible with other neighborhoods (See C. [Cho12] for details). Let  $l$  be a simple closed curve or a full 1-orbifold in the interior of a 2-orbifold  $\Sigma$ . We complete  $\Sigma - l$  with respect to the path-metric on  $\Sigma - l$  induced from a standard one on  $\Sigma$ . The completion  $\hat{\Sigma}$  of  $\Sigma - l$  is called the *splitting* of  $\Sigma$  along  $l$ . Conversely, if  $\hat{l}$  is the union of boundary components of  $\hat{\Sigma}$  mapping to  $l$ , then  $\Sigma$  is obtained from *sewing*  $\hat{\Sigma}$  along  $\hat{l}$ .

An *elementary 2-orbifold* is a compact 2-orbifold so that we cannot split further along simple closed curves or full 1-orbifolds into 2-orbifolds. We assume in this subsection that our orbifolds have negative orbifold Euler characteristic.

The following is the classification of elementary 2-orbifolds. See Figure 4; Arcs with dotted arcs next to them indicate boundary components and black points indicate singular points. The P-rows can be obtained starting with (P1) and degenerating the boundary components to cusps and then to elliptic points, considering them as hyperbolic surfaces with singularities. The orbifolds in the P-row covers A-row's as a quotient orbifold by an order-two involution switching a pair of boundary components or cone-points. The orbifolds in the P-row covers D-row's by an order-two involution preserving boundary components and cone-points.

- (P1) A pair-of-pants ( $\chi = -1$ )
- (P2) An annulus with a cone-point of order  $n$  ( $\chi = -1 + 1/n$ )
- (P3) A disk with two cone-points of orders  $p, q$  ( $\chi = -1 + 1/p + 1/q$ )
- (P4) A sphere with three cone-points of order  $p, q, r$  ( $\chi = -1 + 1/p + 1/q + 1/r$ )
- (A1) An annulus with a boundary circle, a boundary arc and a mirror arc ( $\chi = -1/2$ )
- (A2) An annulus with a boundary circle and a corner-reflector of order  $n$  ( $\chi = -(n-1)/2n$ )

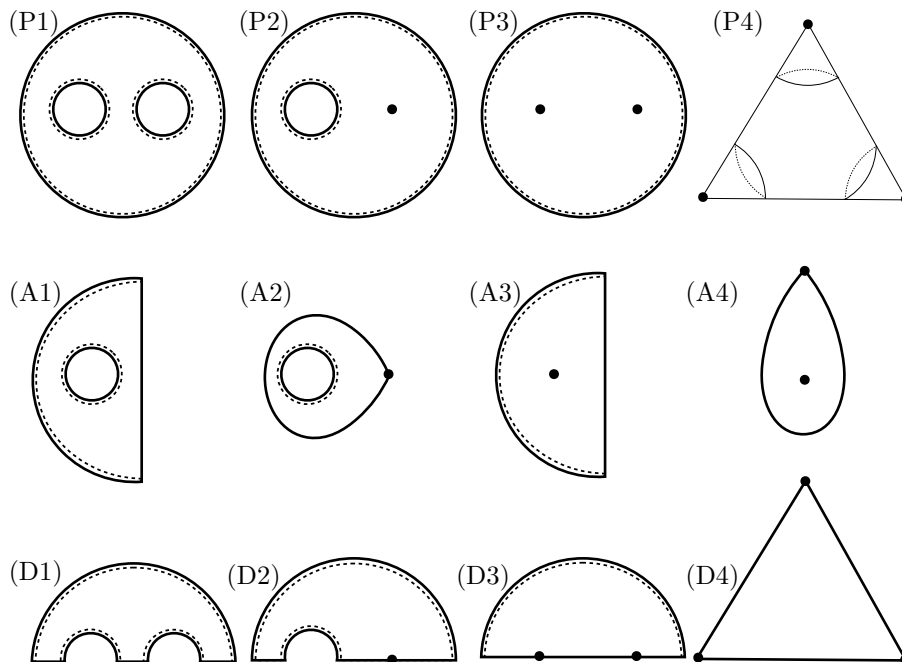


FIGURE 4. Elementary orbifolds

- (A3) A disk with a boundary arc, a mirror arc and a cone-point of order  $n$  ( $\chi = 1/n - 1/2$ )
- (A4) A disk with a corner-reflector of order  $m$  and a cone-point of order  $n$  ( $\chi = 1/n + 1/2m - 1/2$ )
- (D1) A disk with three mirror arcs and three boundary arcs ( $\chi = -1/2$ )
- (D2) A disk with a corner-reflector of order  $n$  at which two mirror arcs meet, one more mirror arc and two boundary arcs ( $\chi = 1/2(1 - 1/n)$ )
- (D3) A disk with two corner-reflectors of order  $p, q$ , and a boundary arc ( $\chi = -1/2 + 1/2p + 1/2q$ )
- (D4) A disk with three corner-reflectors of order  $p, q, r$  and three mirror arcs ( $\chi = -1/2 + 1/2p + 1/2q + 1/2r$ )

A geodesic full 1-orbifold in a convex real projective orbifold  $\Sigma$  is *principal* if the double cover of  $\Sigma$ , which is a simple geodesic, is principal. Let  $\Sigma$  be a convex projective orbifold with principal boundary. For an oriented boundary component  $c$  of  $\Sigma$  homeomorphic to a circle, the space of projective invariants is given by the eigenvalues of the holonomy  $\gamma$  of  $c$ , that is, the smallest eigenvalue  $\lambda$  of  $\gamma$  and the sum  $\tau$  of two other eigenvalues, when  $\gamma$  is positive hyperbolic. Thus,  $\mathfrak{C}(c) = \mathcal{R}^\circ$  (see Section 6.3). For a full 1-orbifold  $c$ ,  $\mathfrak{C}(c)$  is defined to be  $\mathbb{R}$ . The set of convex real projective structures on  $c$  is parametrised by the Hilbert length.

Given  $\Sigma$ , if we denote  $B(\Sigma)$  is the set of boundary components of  $\Sigma$ , then we can define

$$\mathfrak{C}(\partial\Sigma) := \prod_{c \in B(\Sigma)} \mathfrak{C}(c) \quad \text{and} \quad \mathfrak{C}(\emptyset) = \{*\} \text{ is a singleton.}$$

**Proposition 6.14** (C.-Goldman [CG05]). *Let  $S$  be an elementary orbifold in each of the above cases. The map  $\mathfrak{C}(S) \rightarrow \mathfrak{C}(\partial S)$  is a fibration of  $n$ -dimensional open cell over the  $k$ -dimensional open cell with  $l$ -dimensional open cell fiber. We list  $(n, k, l)$  below.*

- (P1):  $(8, 6, 2)$  (Goldman [Gol90])  
(P2):  $(6, 4, 2)$  if there is no cone-point of order 2. Otherwise  $(4, 4, 0)$ .  
(P3):  $(4, 2, 2)$  if there is no cone-point of order 2. Otherwise  $(2, 2, 0)$ .  
(P4):  $(2, 0, 2)$  if there is no cone-point of order 2. Otherwise  $(0, 0, 0)$ .  
(A1):  $(4, 3, 1)$ .  
(A2):  $(2, 2, 0)$ .  
(A3):  $(2, 1, 1)$ .  
(A4):  $(1, 0, 0)$  if there is no corner-reflector of order 2. Otherwise  $(0, 0, 0)$ .  
(D1):  $(4, 3, 1)$ .  
(D2):  $(3, 2, 1)$  if there is no corner-reflector of order 2. Otherwise  $(2, 2, 0)$ .  
(D3):  $(2, 1, 1)$  if there is no corner-reflector of order 2. Otherwise  $(1, 1, 0)$ .  
(D4):  $(1, 0, 0)$  if there is no corner-reflector of order 2. Otherwise  $(0, 0, 0)$ .

Finally, we can describe the deformation space of convex projective structures on closed 2-orbifolds.

**Theorem 6.15** (C.-Goldman [CG05], Chapter 8 of the book [Cho12]). *Let  $\Sigma$  be a closed 2-orbifold with  $\chi(\Sigma) < 0$ . Then the space of  $\mathfrak{C}(\Sigma)$  of convex projective structures on  $\Sigma$  is homeomorphic to a cell of dimension*

$$-8\chi(X_\Sigma) + (6k_c - 2b_c) + (3k_r - b_r)$$

where  $X_\Sigma$  is the underlying space of  $\Sigma$ ,  $k_c$  is the number of cone-points,  $b_c$  is the number of cone-points of order 2,  $k_r$  is the number of corner-reflectors, and  $b_r$  is the number of corner-reflectors of order 2.

## 7. CONVEX PROJECTIVE COXETER ORBIFOLDS

### 7.1. Definitions.

7.1.1. *Coxeter groups.* Let  $S$  be a finite set, and denote by  $|S|$  the cardinality of  $S$ . A *Coxeter matrix* on  $S$  is an  $|S| \times |S|$  symmetric matrix  $M = (M_{st})_{s,t \in S}$  with diagonal entries  $M_{ss} = 1$  and other entries  $M_{st} \in \{2, 3, \dots, \infty\}$ . The couple  $(S, M)$  is called a *Coxeter system*.

To a Coxeter system  $(S, M)$  we can associate a *Coxeter group*  $W = W_S$ : it is the group generated by  $S$  with the relations  $(st)^{M_{st}} = 1$  for all  $(s, t) \in S \times S$  such that  $M_{st} \neq \infty$ . If  $(S, M)$  is a Coxeter system, then for each subset  $T$  of  $S$  we can define the Coxeter subsystem  $(T, M_T = (M_{st})_{s,t \in T})$ . The Coxeter group  $W_T$  can be thought of as a subgroup of  $W_S$  since the canonical map from  $W_T$  to  $W_S$  is an embedding. We stress that the last sentence is not an obvious statement and it is, in fact, a corollary of Theorem 7.1 of Tits and Vinberg.

The *Coxeter graph* of a Coxeter system  $(S, M)$  is the *labeled* graph such that the set of vertices is  $S$ , two vertices  $s$  and  $t$  are connected by an edge  $\overline{st}$  if and only if  $M_{st} \neq 2$ , and the label of the edge  $\overline{st}$  is  $M_{st}$ . A Coxeter system  $(S, M)$  is *irreducible* when its Coxeter graph is connected. It is a little abusive but we also say that the Coxeter group  $W$  is irreducible.

7.1.2. *Coxeter orbifolds.* We are interested in  $d$ -dimensional Coxeter orbifolds whose underlying space is homeomorphic to a  $d$ -dimensional polytope<sup>14</sup>  $P$  minus some faces, and whose singular locus is the boundary of  $P$  made up of mirrors. For the sake of clarity, facets are

<sup>14</sup>We implicitly assume that all the polytopes and polygons are convex.

faces of codimension 1, ridges are faces of codimension 2 and proper faces are faces different from  $P$  and  $\emptyset$ . Choose a polytope  $P$  and a Coxeter matrix  $M$  on the set  $S$  of facets of  $P$ , such that if two facets  $s$  and  $t$  are not adjacent<sup>15</sup>, then  $M_{st} = \infty$ . When two facets  $s, t$  are adjacent, the ridge  $s \cap t$  of  $P$  is said to be of order  $M_{st}$ . The first objects we get are the Coxeter system  $(S, M)$  and the Coxeter group  $W$ .

We now build an orbifold whose fundamental group is  $W$  and whose underlying topological space is the starting polytope  $P$  minus some faces: For each proper face  $f$  of  $P$ , let  $S_f = \{s \in S \mid f \subset s\}$ . If  $W_f := W_{S_f}$  is an infinite Coxeter group then the face  $f$  is said to be *undesirable*. Let  $\hat{P}$  be the orbifold obtained from  $P$  with facets as mirrors, with ridges  $s \cap t$  as corner reflectors of orders  $M_{st}$ , and with undesirable faces removed. We call  $\hat{P}$  a *Coxeter  $d$ -orbifold*. Remark that a Coxeter  $d$ -orbifold is closed if and only if for each vertex  $v$  of  $P$ , the Coxeter group  $W_v$  is finite.

For example, let  $P$  be a polytope in  $\mathbb{X} = \mathbb{S}^d, \mathbb{E}^d$  or  $\mathbb{H}^d$  with dihedral angles submultiples of  $\pi$ . The uniqueness of the reflection across a hyperplane of  $\mathbb{X}$  allows us to obtain a Coxeter  $(\text{Isom}(\mathbb{X}), \mathbb{X})$ -orbifold  $\hat{P}$  from  $P$ .

**7.1.3. Deformation spaces.** Recall that  $\mathfrak{C}(\hat{P})$  denote the deformation space of convex real projective structures on the Coxeter orbifold  $\hat{P}$ , that is, the space of projective structures on  $\hat{P}$  such that the image of the developing map is projectively isomorphic to a convex domain in  $\mathbb{R}\mathbb{P}^d$  and the holonomy is a discrete faithful representation.

**7.2. Vinberg's breakthrough.** In this subsection we give a description of Vinberg's results in his article [Vin71]. An alternative treatment is given in Benoist's notes [Ben09].

**7.2.1. Groundwork.** Let  $V$  be the real vector space of dimension  $(d+1)$ . A *projective reflection* (or simply *reflection*)  $\sigma$  is an element of order 2 of  $\text{SL}^\pm(V)$  which is the identity on a hyperplane  $H$ . All reflections are of the form  $\sigma = \text{Id} - \alpha \otimes b$  for some linear functional  $\alpha \in V^*$  and a vector  $b \in V$  with  $\alpha(b) = 2$ . Here, the kernel of  $\alpha$  is the subspace  $H$  of fixed points of  $\sigma$  and  $b$  is the eigenvector corresponding to the eigenvalue  $-1$ .

Let  $P$  be a  $d$ -polytope in  $\mathbb{S}(V)$  and let  $S$  be the set of facets of  $P$ . For each  $s \in S$ , choose a reflection  $\sigma_s = \text{Id} - \alpha_s \otimes b_s$  with  $\alpha_s(b_s) = 2$  which fixes  $s$ . By making a suitable choice of signs, we assume that  $P$  is defined by the inequalities  $\alpha_s \leq 0$ ,  $s \in S$ . Let  $\Gamma \subset \text{SL}^\pm(V)$  be the group generated by all these reflections  $(\sigma_s)_{s \in S}$  and let  $\mathring{P}$  be the interior of  $P$ . A couple  $(P, (\sigma_s)_s)$  is called a *Coxeter polytope* if the family  $\{\gamma \mathring{P}\}_{\gamma \in \Gamma}$  is pairwise disjoint.

The  $|S| \times |S|$  matrix  $A = (A_{st})_{s, t \in S}$ ,  $A_{st} = \alpha_s(b_t)$ , is called the *Cartan matrix* of a Coxeter polytope  $P$ . For each reflection  $\sigma_s$ , the linear functional  $\alpha_s$  and the vector  $b_s$  are defined up to transformations

$$\alpha_s \mapsto \lambda_s \alpha_s \quad \text{and} \quad b_s \mapsto \lambda_s^{-1} b_s \quad \text{with} \quad \lambda_s > 0.$$

Hence the Cartan matrix of  $P$  is defined up to the following equivalence relation: two matrices  $A$  and  $B$  are *equivalent* if  $A = \Lambda B \Lambda^{-1}$  for a diagonal matrix  $\Lambda$  having positive entries. This implies that for  $s \neq t$ , the number  $A_{st} A_{ts}$  is an invariant of the Coxeter polytope  $P$ .

<sup>15</sup>Two facets  $s$  and  $t$  are adjacent if and only if  $s \cap t$  is a ridge of  $P$ .



7.2.2. *Vinberg's results.* Vinberg proved that the following conditions are necessary and sufficient for  $P$  to be a Coxeter polytope:

- (V1)  $A_{st} \leq 0$  for  $s \neq t$ , and  $A_{st} = 0$  if and only if  $A_{ts} = 0$ .
- (V2)  $A_{ss} = 2$ ; and for  $s \neq t$ ,  $A_{st}A_{ts} \geq 4$  or  $A_{st}A_{ts} = 4 \cos^2 \frac{\pi}{m_{st}}$ ,  $m_{st} \in \mathbb{N} \setminus \{0, 1\}$ .

The starting point of the proof is that for every two facets  $s$  and  $t$  of  $P$ , the automorphism  $\sigma_s \sigma_t$  have to be conjugate to one of the following automorphisms of  $V/U$  with  $U = \ker(\alpha_s) \cap \ker(\alpha_t)$ :

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix} (\lambda > 0), \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \left( \theta = \frac{2\pi}{m_{st}} \right).$$

In the third case we call  $\sigma_s \sigma_t$  a *rotation of angle  $\theta$* .

To a Coxeter polytope  $P$ , we can associate the Coxeter matrix  $M = (M_{st})_{s,t \in S}$  with the set  $S$  of facets of  $P$  such that  $M_{st} = m_{st}$  if  $\sigma_s \sigma_t$  is a rotation of angle  $\frac{2\pi}{m_{st}}$ , and  $M_{st} = \infty$  otherwise. Now, from the Coxeter system  $(S, M)$  and the polytope  $P$ , we get the Coxeter group  $W$  and Coxeter orbifold  $\hat{P}$ . Eventually, we are also interested in the subgroup  $\Gamma$  of  $\text{SL}^\pm(V)$  generated by all the reflections across the facets of  $P$ .

**Theorem 7.1** (Tits [Bou68], Vinberg [Vin71]). *Let  $P$  be a Coxeter polytope. Then the following are true:*

- The morphism  $\sigma : W \rightarrow \Gamma$  given by  $\sigma(s) = \sigma_s$  is an isomorphism.
- The group  $\Gamma$  is a discrete subgroup of  $\text{SL}^\pm(V)$ .
- The union of tiles  $\mathcal{C} := \cup_{\gamma \in \Gamma} \gamma P$  is convex.
- The group  $\Gamma$  acts properly discontinuously on  $\Omega$  the interior of  $\mathcal{C}$ .
- An open face  $f$  of  $P$  lies in  $\Omega$  if and only if the Coxeter group  $W_f$  is finite.

7.3. **Convex projective Coxeter 2-orbifolds.** In the previous section, we explain the deformation space  $\mathfrak{C}(\Sigma)$  of convex real projective structures on a closed 2-orbifold  $\Sigma$  of negative orbifold Euler characteristic (Theorem 6.15). As a special case, if  $\Sigma$  is a closed Coxeter 2-orbifold then the underlying space of  $\Sigma$  is a polygon and  $\Sigma$  does not contain cone-points. Let  $v_+$  be the number of corner reflectors of order greater than 2, and let  $\mathcal{T}(\Sigma)$  be the Teichmüller space of  $\Sigma$ . The space  $\mathfrak{C}(\Sigma)$  is homeomorphic to an open cell of dimension

$$-8 + 3v - (v - v_+) = v_+ - 2 + 2(v - 3) = v_+ - 2 + 2 \dim \mathcal{T}(\Sigma).$$

In his undergraduate thesis [Gol77], Goldman proved this result.

7.4. **Hyperbolic Coxeter 3-orbifolds.** The Coxeter 3-orbifolds which admit a finite volume hyperbolic structure have been classified by Andreev [And70a, And70b].

A polytope is naturally a CW complex. The CW complex arose from a polytope is called a *combinatorial polytope*. We abbreviate a 3-dimensional polytope to a *polyhedron*.

Let  $\mathcal{G}$  be a combinatorial polyhedron and  $(\partial\mathcal{G})^*$  be the dual CW complex of the boundary  $\partial\mathcal{G}$ . A simple closed curve  $\gamma$  is called a *k-circuit* if it consists of  $k$  edges of  $(\partial\mathcal{G})^*$ . A circuit  $\gamma$  is *prismatic* if all the edges of  $\mathcal{G}$  intersecting  $\gamma$  are disjoint.

**Theorem 7.2** (Andreev [And70a, And70b]). *Let  $\mathcal{G}$  be a combinatorial polyhedron, and let  $\{s_i\}_{i=1}^N$  be the set of facets of  $\mathcal{G}$ . Suppose that  $\mathcal{G}$  is not a tetrahedron and non-obtuse angles  $\theta_{ij} \in (0, \frac{\pi}{2}]$  are given at each edge  $s_{ij} = s_i \cap s_j$  of  $\mathcal{G}$ . Then the following conditions (A1)–(A4)*

are necessary and sufficient for the existence of a compact hyperbolic polyhedron  $P$  which realizes<sup>16</sup>  $\mathcal{G}$  with dihedral angle  $\theta_{ij}$  at each edge  $s_{ij}$ .

- (A1) If  $s_i \cap s_j \cap s_k$  is a vertex of  $\mathcal{G}$  then  $\theta_{ij} + \theta_{jk} + \theta_{ki} > \pi$ .
- (A2) If  $s_i, s_j, s_k$  form a prismatic 3-circuit, then  $\theta_{ij} + \theta_{jk} + \theta_{ki} < \pi$ .
- (A3) If  $s_i, s_j, s_k, s_l$  form a prismatic 4-circuit, then  $\theta_{ij} + \theta_{jk} + \theta_{kl} + \theta_{li} < 2\pi$ .
- (A4) If  $\mathcal{G}$  is a triangular prism with triangular facets  $s_1$  and  $s_2$ , then

$$\theta_{13} + \theta_{14} + \theta_{15} + \theta_{23} + \theta_{24} + \theta_{25} < 3\pi.$$

The following (F1)–(F6) are necessary and sufficient for the existence of a finite volume hyperbolic polyhedron  $P$  which realizes  $\mathcal{G}$  with dihedral angle  $\theta_{ij} \in (0, \pi/2]$  at each edge  $s_{ij}$ .

- (F1) If  $s_i \cap s_j \cap s_k$  is a vertex of  $\mathcal{G}$  then  $\theta_{ij} + \theta_{jk} + \theta_{ki} \geq \pi$ .
- (F2) (resp. (F3) or (F4)) is the same as (A2) (resp. (A3) or (A4)).
- (F5) If  $s_i \cap s_j \cap s_k \cap s_l$  is a vertex of  $\mathcal{G}$  then  $\theta_{ij} + \theta_{jk} + \theta_{kl} + \theta_{li} = 2\pi$ .
- (F6) If  $s_i, s_j, s_k$  are facets such that  $s_i$  and  $s_j$  are adjacent,  $s_j$  and  $s_k$  are adjacent, and  $s_i$  and  $s_k$  are not adjacent but meet in a vertex not in  $s_j$ , then  $\theta_{ij} + \theta_{jk} < \pi$ .

In both cases, the hyperbolic polyhedron is unique up to hyperbolic isometries.

## 7.5. Convex projective Coxeter 3-orbifolds.

7.5.1. *Restricted deformation spaces.* A point  $(P_0, (\sigma_s)_s)$  of  $\mathfrak{C}(\hat{P})$  gives us its underlying polytope  $P \subset \mathbb{RP}^d$ , well defined up to projective automorphisms. We can focus on the subspace of  $\mathfrak{C}(\hat{P})$  with a fixed underlying polytope  $P_0$ . This subspace is called the *restricted deformation space* of  $\hat{P}$  and denoted by  $\mathfrak{C}_{P_0}(\hat{P})$ .

Let  $\hat{P}$  be a Coxeter 3-orbifold. We now give a combinatorial hypothesis on  $\hat{P}$ , called "the orderability", which allow us to say something about the restricted deformation space  $\mathfrak{C}_{P_0}(\hat{P})$  of  $\hat{P}$ . A Coxeter 3-orbifold  $\hat{P}$  is *orderable* if the facets of  $\hat{P}$  can be ordered so that each facet contains at most three edges that are edges of order 2 or edges in a facet of higher index.

Let  $e$  (resp.  $f, e_2$ ) be the number of edges (resp. facets, edges of order 2) of  $P$ , and let  $k(P)$  be the dimension of the group of projective automorphisms of  $P$ . Note that  $k(P) = 3$  if  $P$  is tetrahedron,  $k(P) = 1$  if  $P$  is the cone over a polygon other than a triangle, and  $k(P) = 0$  otherwise.

**Theorem 7.3** (C. [Cho06]). *Let  $\hat{P}$  be a Coxeter 3-orbifold such that  $\mathfrak{C}(\hat{P}) \neq \emptyset$ . Suppose that  $\hat{P}$  is orderable and the Coxeter group  $\pi_1(\hat{P})$  is infinite and irreducible. Then every restricted deformation space  $\mathfrak{C}_P(\hat{P})$  is a smooth manifold of dimension  $3f - e - e_2 - k(P)$ .*

A polyhedron with all facets triangular is orderable. By Andreev's theorem, hyperbolic triangular prisms are orderable. However the cube and dodecahedron do not carry an orderable Coxeter orbifold structure, since the lowest index facet in an orderable orbifold must be triangular.

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<sup>16</sup>There is an isomorphism  $\phi : \mathcal{G} \rightarrow P$  such that the given angle at each edge  $e$  of  $\mathcal{G}$  is the dihedral angle at the edge  $\phi(e)$  of  $P$ .

7.5.2. *Truncation polyhedra.* Andreev’s theorem gives the necessary and sufficient conditions for the existence of a closed or finite volume hyperbolic Coxeter 3-orbifold. We can think of analogous questions for closed or finite volume convex projective Coxeter orbifolds.

The third author [Mar10a] completely answered whether or not a Coxeter 3-orbifold  $\hat{P}$  admits a convex projective structure assuming that the underlying space  $P$  is a truncation polyhedron: a *truncation  $d$ -polytope* is a  $d$ -polytope obtained from the  $d$ -simplex by iterated truncations of vertices. For example, a triangular prism is a truncation polyhedron, however again the cube and the dodecahedron are not truncation polyhedra.

A prismatic 3-circuit formed by the facets  $r, s, t$  is *bad* if

$$\frac{1}{M_{rs}} + \frac{1}{M_{st}} + \frac{1}{M_{tr}} \geq 1 \quad \text{and} \quad 2 \in \{M_{rs}, M_{st}, M_{tr}\}.$$

Let  $e_+$  be the number of edges of order greater than 2 in  $\hat{P}$ .

**Theorem 7.4** (M., [Mar10a]). *Let  $\hat{P}$  be a Coxeter 3-orbifold arising from a truncation polyhedron  $P$ . Assume that  $\hat{P}$  has no bad prismatic 3-circuits. If  $\hat{P}$  is not a triangular prism and  $e_+ > 3$ , then  $\mathfrak{C}(\hat{P})$  is homeomorphic to a finite union of open cells of dimension  $e_+ - 3$ . Moreover, if  $\hat{P}$  admits a hyperbolic structure, then  $\mathfrak{C}(\hat{P})$  is connected.*

The third author actually provides an explicit homeomorphism between  $\mathfrak{C}(\hat{P})$  and the union of  $q$  copies of  $\mathbb{R}^{e_+-3}$  when  $\hat{P}$  is a Coxeter truncation 3-orbifold. Moreover, the integer  $q$  can be computed easily in terms of the combinatorics and the edge orders.

## 7.6. Near the hyperbolic structure.

7.6.1. *Restricted deformation spaces.* C.-Hodgson-L. [CHL12] described the local restricted deformation space for a class of Coxeter orbifolds arising from *ideal* hyperbolic polyhedra, i.e. polyhedra with all vertices on  $\partial\mathbb{H}^3$ .

**Theorem 7.5** (C.-Hodgson-L. [CHL12]). *Let  $P$  be an ideal hyperbolic polyhedron whose dihedral angles are all equal to  $\frac{\pi}{3}$ . If  $P$  is not a tetrahedron, then at the hyperbolic structure the restricted deformation space  $\mathfrak{C}_P(\hat{P})$  is smooth and of dimension 6.*

7.6.2. *Weakly orderable Coxeter orbifolds.* The first and second authors [CL15] found a “large” class of Coxeter 3-orbifolds whose local deformation spaces are understandable. A Coxeter 3-orbifold  $\hat{P}$  is *weakly orderable* if the facets of  $P$  can be ordered so that each facets contains at most 3 edges of order 2 in a facet of higher index. Note that Greene [Gre13] gave an alternative (cohomological) proof of the following theorem.

**Theorem 7.6** (C.-L. [CL15], Greene [Gre13]). *Let  $\hat{P}$  is a closed hyperbolic Coxeter 3-orbifold. If  $\hat{P}$  is weakly orderable, then at the hyperbolic structure  $\mathfrak{C}(\hat{P})$  is smooth and of dimension  $e_+ - 3$ .*

For example, if  $P$  is a truncation polyhedron, then  $\hat{P}$  is always weakly orderable. The cube is not a truncation polyhedron; however, every closed hyperbolic Coxeter 3-orbifold arising from the cube is weakly orderable. On the other hand, there exist closed hyperbolic Coxeter 3-orbifolds arising from the dodecahedron which are not weakly orderable. However, almost all the closed hyperbolic Coxeter 3-orbifolds arising from the dodecahedron are weakly orderable:

**Theorem 7.7** (C.-L. [CL15]). *Let  $P$  be a simple<sup>17</sup> polyhedron. Suppose that  $P$  has no prismatic 3-circuit and has at most one prismatic 4-circuit. Then*

$$\lim_{m \rightarrow \infty} \frac{|\{\text{weakly orderable, closed hyperbolic Coxeter 3-orbifolds } \hat{P} \text{ with edge order } \leq m\}|}{|\{\text{closed hyperbolic Coxeter 3-orbifolds } \hat{P} \text{ with edge order } \leq m\}|} = 1$$

A similar result to Theorem 7.6 is true for higher dimensional closed Coxeter orbifolds  $\hat{P}$  whose underlying polytope  $P$  is a *truncation polytope*:

**Theorem 7.8** (C.-L. [CL15], Greene [Gre13]). *If  $\hat{P}$  be a closed hyperbolic Coxeter orbifold arising from a truncation polytope  $P$ , then at the hyperbolic structure  $\mathfrak{C}(\hat{P})$  is smooth and of dimension  $e_+ - d$ .*

We remark that if  $\hat{P}$  is not weakly orderable then Theorem 7.6 is not true anymore: Let  $m$  be a fixed integer greater than 3. Consider the compact hyperbolic Coxeter polyhedron  $P_1$  shown in Figure 5 (A). Here, if an edge is labeled  $m$  then its dihedral angle is  $\frac{\pi}{m}$ . Otherwise, its dihedral angle is  $\frac{\pi}{2}$ . Let  $\hat{P}_1$  be the corresponding hyperbolic Coxeter 3-orbifold. Then  $e_+ - 3 = 0$ , but  $\mathfrak{C}(\hat{P}_1) = \mathbb{R}$  (see [CL15]). Of course  $\hat{P}_1$  is not weakly orderable, since every facet in  $\hat{P}_1$  contains four edges of order 2.

There is also a compact hyperbolic Coxeter 4-polytope  $P_2$  such that  $\mathfrak{C}(\hat{P}_2)$  is not homeomorphic to a manifold. The underlying polytope  $P_2$  is the product of two triangles and the Coxeter graph of  $\hat{P}_2$  is shown in Figure 5 (B).

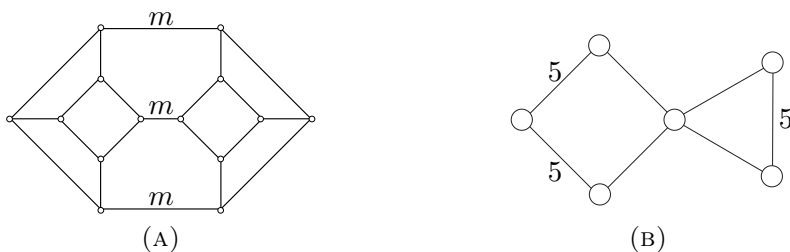


FIGURE 5. (A) Coxeter 3-polytope  $P_1$  (B) Coxeter 4-polytope  $P_2$

The space  $\mathfrak{C}(\hat{P}_2)$  is homeomorphic to the following solution space (see [CL15]):

$$\mathcal{S} = \{(x, y) \in (\mathbb{R}_+)^2 : 8x - (5 + \sqrt{5})y - (6 - 2\sqrt{5})xy - (5 + \sqrt{5})x^2y + 8xy^2 = 0\},$$

which is pictured in Figure 6, and hence  $\mathfrak{C}(\hat{P}_2)$  is not a manifold. Here the singular point  $(1, 1) \in \mathcal{S}$  corresponds to the hyperbolic structure in  $\mathfrak{C}(\hat{P}_2)$ .

## 8. THE TANGENT SPACE TO A REPRESENTATION: INFINITESIMAL DEFORMATIONS

### 8.1. Rigidity or deformability.

**Definition 8.1.** A representation  $\rho : \Gamma \rightarrow G$  is *locally rigid* if the  $G$ -orbit of  $\rho$  in  $\text{Hom}(\Gamma, G)$  contains a neighbourhood of  $\rho$  in  $\text{Hom}(\Gamma, G)$ . Otherwise,  $\rho$  is *locally deformable*.

<sup>17</sup>Each vertex of  $P$  is adjacent to exactly three edges.

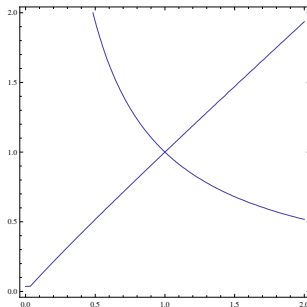


FIGURE 6.  $8x - (5 + \sqrt{5})y - (6 - 2\sqrt{5})xy - (5 + \sqrt{5})x^2y + 8xy^2 = 0$

If  $\rho$  is locally deformable then there exists a sequence of representations  $\rho_n : \Gamma \rightarrow G$  converging to  $\rho$  such that  $\rho_n$  is not conjugate to  $\rho$ . We emphasise that  $\rho_n$  are no reason to be discrete or faithful even if so is  $\rho$ .

**Definition 8.2.** Two representations  $\rho, \rho' : \Gamma \rightarrow G$  are of the *same type* if for all  $\gamma \in \Gamma$ ,  $\rho(\gamma)$  and  $\rho'(\gamma)$  have the same type in the Jordan decomposition. A discrete faithful representation  $\rho : \Gamma \rightarrow G$  is *globally rigid* if every discrete faithful representation in  $\text{Hom}(\Gamma, G)$  whose type is the same as  $\rho$  is conjugate to  $\rho$ .

**8.2. What is an infinitesimal deformation?** In this subsection, we explore the tangent space to a representation. In order to do that, we will exploit the fact that  $\chi(\Gamma, G)$  is a real semi-algebraic variety. Recall that differential geometers and algebraic geometers do not always agree with the concept of the smoothness, e.g. the point  $(0, 0)$  of the curve  $x^3 - y^3 = 0$  in the plane  $\mathbb{R}^2$  is singular for an algebraic geometer, since the Zariski tangent space is all  $\mathbb{R}^2$ , but the curve is a  $\mathcal{C}^\infty$ -smooth submanifold of  $\mathbb{R}^2$ .

To clarify this issue, we want to introduce some terminology: Let  $V$  be a real semi-algebraic subset of  $\mathbb{R}^m$ . To each point  $x \in V$  we associate the dimension of the Zariski tangent space to  $V$  at  $x$ . This map  $V \rightarrow \mathbb{Z}$  attains the minimum value at each point in a Zariski open subset  $W$  of  $V$ . A point in  $W$  (resp.  $V \setminus W$ ) is called *regular* (resp. *singular*). That is the point of view of an algebraic geometer. If we can find a neighbourhood  $\mathcal{U}$  of a point  $x$  in  $\mathbb{R}^m$  such that  $\mathcal{U} \cap V$  is a  $\mathcal{C}^\infty$ -smooth submanifold of  $\mathbb{R}^m$ , then we say that  $x$  is *smooth*. That is the point of view of a differential geometer.

**8.3. First order.** Assume that  $\rho_t$  is a smooth path in  $\text{Hom}(\Gamma, G)$ , i.e. for each  $\gamma \in \Gamma$ , a path  $\rho_t(\gamma)$  in  $G$  is smooth. Then there exists a map  $u_1 : \Gamma \rightarrow \mathfrak{g}$  such that

$$\forall \gamma \in \Gamma, \quad \rho_t(\gamma) = \exp(t u_1(\gamma) + O(t^2)) \rho_0(\gamma)$$

Since  $\rho_t$  is a homomorphism, i.e.  $\rho_t(\gamma\delta) = \rho_t(\gamma)\rho_t(\delta)$ , it follows that  $u_1$  is a 1-cocycle. Conversely, if  $u_1 : \Gamma \rightarrow \mathfrak{g}$  is a 1-cocycle then  $\rho_t$  is a homomorphism up to first order. This computation motivates the following: Given a representation  $\rho : \Gamma \rightarrow G$ , we define the space of 1-cocycles  $\Gamma \rightarrow \mathfrak{g}$ :

$$Z^1(\Gamma, \mathfrak{g})_\rho = \{u_1 : \Gamma \rightarrow \mathfrak{g} \mid u_1(\gamma\delta) = u_1(\gamma) + \text{Ad}_{\rho(\gamma)} u_1(\delta), \forall \gamma, \delta \in \Gamma\}.$$

Moreover, since the Zariski tangent space to an algebraic variety is the space of germs of paths satisfying the equations up to first order, the Zariski tangent space  $T_\rho^{\text{Zar}} \text{Hom}(\Gamma, G)$  to

$\text{Hom}(\Gamma, G)$  at  $\rho$  can be identified with the space of 1-cocycle  $Z^1(\Gamma, \mathfrak{g})_\rho$  thanks to the following:

$$\forall \gamma \in \Gamma, \quad \frac{d}{dt} \rho_t(\gamma) \rho_0(\gamma^{-1})|_{t=0} = u_1(\gamma).$$

In the end, we want to understand the tangent space to the character variety, hence we need to figure out which cocycle comes from the conjugation. We introduce the space of 1-coboundaries:

$$B^1(\Gamma, \mathfrak{g})_\rho = \{v_1 : \Gamma \rightarrow \mathfrak{g} \mid \exists u_0 \in \mathfrak{g} \text{ such that } v_1(\gamma) = \text{Ad}_{\rho(\gamma)} u_0 - u_0\}.$$

Every coboundary  $v_1$ , in fact, is tangent to the conjugation path  $\rho_t = \exp(-tu_0) \rho_0 \exp(tu_0)$ . The first cohomology group with coefficients in  $\mathfrak{g}$  twisted by the adjoint action of  $\rho$  is

$$H^1(\Gamma, \mathfrak{g})_\rho = Z^1(\Gamma, \mathfrak{g})_\rho / B^1(\Gamma, \mathfrak{g})_\rho.$$

**Theorem 8.3** (Weil [Wei64]). *Let  $G$  be the group of real points of an algebraic group defined over  $\mathbb{R}$ . Assume that a representation  $\rho : \Gamma \rightarrow G$  is reductive. Then the map  $T_\rho^{\text{Zar}} \text{Hom}(\Gamma, G) \rightarrow Z^1(\Gamma, \mathfrak{g})_\rho$  is an isomorphism. In addition, under this isomorphism, the Zariski tangent vectors coming from the  $G$ -conjugation of  $\rho$  exactly correspond to the coboundaries. Hence, the Zariski tangent space to the character variety is isomorphic to the cohomology group  $H^1(\Gamma, \mathfrak{g})_\rho$ .*

Theorem 8.3 leads to the following definition and its classical corollary:

**Definition 8.4.** A representation  $\rho : \Gamma \rightarrow G$  is *infinitesimally rigid* if  $H^1(\Gamma, \mathfrak{g})_\rho = 0$ .

**Theorem 8.5** (Weil's rigidity theorem [Wei64]). *If  $\rho$  is infinitesimally rigid, then  $\rho$  is locally rigid.*

A nice presentation of Theorem 8.5 can be found in Besson [Bes09]. Weil and Raghunathan also compute the group  $H^1(\Gamma, \mathfrak{g})_\rho$  in a number of important cases and show that it is often trivial.

**Theorem 8.6** (Weil [Wei64], Raghunathan [Rag67, Rag65]). *Suppose that  $G$  is a semi-simple group and  $\Gamma$  is a lattice of  $G$ . Denote by  $i : \Gamma \rightarrow G$  the canonical representation.*

- If  $H^1(\Gamma, \mathfrak{g})_i \neq 0$  then
  - Either  $\mathfrak{g} = \mathfrak{so}_{2,1}(\mathbb{R}) = \mathfrak{su}_{1,1} = \mathfrak{sl}_2(\mathbb{R})$ ,
  - Or  $\mathfrak{g} = \mathfrak{so}_{3,1}(\mathbb{R}) = \mathfrak{sl}_2(\mathbb{C})$  and  $\Gamma$  is a non-uniform lattice.
- Let  $\rho : G \rightarrow H$  be a representation of  $G$  into a semi-simple group  $H$ . If  $\Gamma$  is now a uniform lattice and  $H^1(\Gamma, \mathfrak{h})_{\rho \circ i} \neq 0$  then  $\mathfrak{g} = \mathfrak{so}_{d,1}(\mathbb{R})$  or  $\mathfrak{su}_{d,1}$ . Moreover, if we write  $\mathfrak{h} = V_1 \oplus \dots \oplus V_r$  the decomposition of the  $\mathfrak{g}$ -semi-simple module  $\mathfrak{h}$  into simple modules, then the highest weight of  $V_i$  is a multiple of the highest weight of the standard representation.

**8.4. Higher order.** Given a 1-cocycle  $u_1 \in Z^1(\Gamma, \mathfrak{g})_\rho$ , (i.e a Zariski tangent vector to the representation variety), we may ask if  $u_1$  is *integrable* (i.e. the tangent vector to a smooth deformation). Since solving this question involves an infinite number of computations, we can start up with the simplest investigation: Is the 1-cocycle  $u_1$  *integrable up to second order*? (i.e. the tangent vector to a smooth deformation up to order 2). Writing the expression

$$\forall \gamma \in \Gamma, \quad \rho_t(x) = \exp(t u_1(\gamma) + t^2 u_2(\gamma) + O(t^3)) \rho_0(\gamma)$$

and using the Baker-Campbell-Hausdorff formula, we see that  $\rho_t$  is a homomorphism up to second order if and only if

$$\forall \gamma, \delta \in \Gamma, \quad u_2(\gamma) - u_2(\gamma\delta) + \text{Ad}_{\rho_0(\gamma)}u_2(\delta) = \frac{1}{2}[\text{Ad}_{\rho_0(\gamma)}u_1(\delta), u_1(\gamma)] := \frac{1}{2}[u_1, u_1](\gamma, \delta)$$

Hence, the 1-cocycle  $u_1$  is integrable up to second order if and only if the 2-cocycle  $[u_1, u_1] \in Z^2(\Gamma, \mathfrak{g})$  is a 2-coboundary. We could ask the same question for the third, fourth, etc. order and we would find a sequence of obstructions, which are all lying in  $H^2(\Gamma, \mathfrak{g})_\rho$ . In other words, for each  $n \geq 2$ , there exists a map  $o_n : H^1(\Gamma, \mathfrak{g})_\rho \rightarrow H^2(\Gamma, \mathfrak{g})_\rho$  such that the 1-cocycle  $u_1$  is integrable up to order  $n$  if and only if the obstructions  $o_k(u_1) = 0$  for all  $k = 2, \dots, n$ .

The story ends with a good news. Recall that  $G = \mathbb{G}_{\mathbb{R}}$  and  $\mathbb{R}[[t]]$  is the ring of formal power series. A *formal deformation* of  $\rho : \Gamma \rightarrow G$  is a representation  $\tilde{\rho} : \Gamma \rightarrow \mathbb{G}_{\mathbb{R}[[t]]}$  whose evaluation at  $t = 0$  is  $\rho$ . A 1-cocycle  $u_1$  is, by definition, the formal tangent vector to a formal deformation (or simply *formally integrable*) if and only if the obstructions  $o_n(u_1) = 0$  for all  $n \geq 2$ . A priori, this does not imply that  $u_1$  is the tangent vector to a smooth deformation, but that is in fact true:

**Theorem 8.7** (Artin [Art68]). *If a 1-cocycle  $u_1$  is formally integrable, then  $u_1$  is integrable.*

**8.5. Examples in hyperbolic geometry.** The world of hyperbolic geometry offers a lot of interesting behaviors. Assume that  $M$  is a hyperbolic  $d$ -dimensional manifold with or without boundary and  $\Gamma$  is the fundamental group of  $M$ .

**8.5.1. Hyperbolic surfaces.** Many are known on representations of surface group, and the story about surface groups is different from higher dimensional manifold groups, which we will eventually concentrate on. Hence, we send the readers to their favourite surveys on surface group representations (see for example [Gol84, Gol85, Lab13, Gui11]).

**8.5.2. Finite volume hyperbolic manifolds.** If  $d \geq 3$  and  $M$  has a finite volume, then the famous Mostow-Prasad rigidity theorem [Mos68, Pra73] states that the holonomy  $\rho$  of  $M$  is globally rigid. This (conjugacy class of) representation is the *geometric representation* of  $\Gamma$ .

However this does not imply that  $\rho$  is locally rigid. Indeed, the geometric representation might be deformed to non-faithful or non-discrete representations<sup>18</sup>. It is a theorem of Thurston for the dimension  $d = 3$  and of Garland and Raghunathan for the dimension  $d \geq 4$  that  $\rho$  is locally deformable if and only if  $d = 3$  and  $M$  has a cusp. This wonderful exception in the local rigidity of finite volume hyperbolic manifolds is the starting point of the Thurston's hyperbolic Dehn surgery theorem.

**Theorem 8.8** (Thurston [Thu02], Garland-Raghunathan [GR70]). *The holonomy representation of a finite volume hyperbolic manifold  $M$  of dimension  $d \geq 3$  is infinitesimally rigid except if  $d = 3$  and  $M$  is not compact. In the exceptional case, the geometric representation is a regular point of the character variety of dimension twice the number of cusps.*

Bergeron and Gelander [BG04] give an alternative proof of Theorem 8.8 using the Mostow-Prasad rigidity.

<sup>18</sup>It is easy to see that every discrete and faithful representation of  $\Gamma$  are of the same type.

8.5.3. *Hyperbolic manifolds with boundary.* We also wish to cite this beautiful theorem which push this kind of question beyond the scope of finite volume manifolds.

**Theorem 8.9** (Kerckhoff-Storm [KS12]). *The holonomy representation of a compact hyperbolic manifold with totally geodesic boundary of dimension  $d \geq 4$  is infinitesimally rigid.*

We now get back to the original interest of this survey: convex projective structures on manifold. From the point of view of representations, our problem is to understand deformations  $\rho_t : \pi_1(M) \rightarrow \mathrm{SL}_{d+1}(\mathbb{R})$  from the holonomy  $\rho : \pi_1(M) \rightarrow \mathrm{SO}_{d,1}$  of the hyperbolic structure on  $M$  into representations in  $\mathrm{SL}_{d+1}(\mathbb{R})$ .

## 9. INFINITESIMAL DUALITY TO COMPLEX HYPERBOLIC GEOMETRY

Suppose that  $M$  is a finite volume hyperbolic manifold of dimension  $d \geq 3$  and  $\Gamma$  is the fundamental group of  $M$ . We have seen that there exists a unique discrete faithful representation  $\rho_{\mathrm{geo}}$  of  $\Gamma$  into  $\mathrm{SO}_{d,1}(\mathbb{R})$ , up to conjugation. If  $G$  is a Lie group and  $i : \mathrm{SO}_{d,1} \rightarrow G$  is a representation, then we call the conjugacy class  $[i \circ \rho_{\mathrm{geo}}]$  the *hyperbolic point* of the character variety  $\chi(\Gamma, G)$  and we denote it again by  $\rho_{\mathrm{geo}}$ . We abuse a little bit of notation here, since we ignore  $i$ , but in the following  $i$  will always be the canonical inclusion.

The complex hyperbolic geometry can help us to understand local deformations into  $\mathrm{SL}_{d+1}(\mathbb{R})$ . Indeed, the complex hyperbolic geometry is "dual" to Hilbert geometry, however, only at the hyperbolic point and in the infinitesimal level.

*Remark 9.1.* The groups  $\mathrm{SL}_{d+1}(\mathbb{R})$  and  $\mathrm{SU}_{d,1}$  are non-compact real forms of the complex algebraic group  $\mathrm{SL}_{d+1}(\mathbb{C})$  that both contains the real algebraic group  $\mathrm{SO}_{d,1}(\mathbb{R})$ . Moreover, the Lie algebra  $\mathfrak{sl}_{d+1}(\mathbb{R})$  splits as

$$(3) \quad \mathfrak{sl}_{d+1}(\mathbb{R}) = \mathfrak{so}_{d,1} \oplus \mathfrak{o}$$

where  $\mathfrak{o}$  is the orthogonal Lie subalgebra of  $\mathfrak{so}_{d,1}$  in  $\mathfrak{sl}_{d+1}(\mathbb{R})$  with respect to the Killing form of  $\mathfrak{sl}_{d+1}(\mathbb{R})$ , and the adjoint action preserves the decomposition (3). Hence to study the cohomology group  $H^1(M, \mathfrak{sl}_{d+1}(\mathbb{R}))_\rho$ , we just have to understand  $H^1(M, \mathfrak{o})_\rho$ , since the cohomology group  $H^1(M, \mathfrak{so}_{d,1})_\rho$  is well known. But, since the Lie algebra  $\mathfrak{so}_{d,1} \oplus i\mathfrak{o} = \mathfrak{su}_{d,1}$ , we can find out  $H^1(M, \mathfrak{o})_\rho$  using complex hyperbolic geometry (see [HP11, CLT07] for more details).

Remark 9.1 evolves into the following theorem:

**Theorem 9.2** (Cooper-Long-Thistlethwaite [CLT07]). *Let  $M$  be a closed hyperbolic manifold, and  $\Gamma = \pi_1(M)$ . Then the hyperbolic point  $\rho_{\mathrm{geo}}$  in  $\chi(\Gamma, \mathrm{SL}_{d+1}(\mathbb{R}))$  is regular if and only if the corresponding hyperbolic point in  $\chi(\Gamma, \mathrm{SU}_{d,1})$  is regular. Moreover, in that case, the dimensions of two character varieties at the hyperbolic point are the same.*

**Theorem 9.3** (Koszul [Kos68], Guichard [Gui04]). *In both  $\chi(\Gamma, \mathrm{SL}_{d+1}(\mathbb{R}))$  and  $\chi(\Gamma, \mathrm{SU}_{d,1})$ , the representations close to the hyperbolic point  $\rho_{\mathrm{geo}}$  are faithful and discrete.*

The local pictures of the character variety for  $G = \mathrm{SL}_{d+1}(\mathbb{R})$  and  $\mathrm{SU}_{d,1}$  are therefore the same, however the global pictures can be very different. Morgan-Shalen-Bestvina-Paulin's work [MS84, Bes88, Pau88] shows that the space of discrete and faithful representations of  $\Gamma$  in  $\mathrm{SU}_{d,1}$  is compact (if  $d \geq 2$ ), but it can be false if  $G = \mathrm{SL}_{d+1}(\mathbb{R})$ .



## 10. CONVEX PROJECTIVE STRUCTURE ON 3-MANIFOLDS

**10.1. Computing character varieties.** In [CLT06], Cooper, Long and Thistlethwaite investigate the local structure of the variety  $\chi(\Gamma, \mathrm{SL}_4(\mathbb{R}))$  at the hyperbolic point  $\rho := \rho_{\mathrm{geo}}$  when  $\Gamma$  is the fundamental group of one of the first 4500 closed hyperbolic 3-manifolds with 2-generator groups in the Hodgson-Weeks census:

<http://www.math.uic.edu/t3m/SnapPy/censuses.html>

We summarize their conclusions about the character variety  $\chi(\Gamma, \mathrm{SL}_4(\mathbb{R}))$  around the hyperbolic point:

- 4439 regular points, i.e.  $H^1(\Gamma, \mathrm{SL}_4(\mathbb{R}))_\rho = 0$ .
- 9 singular points, i.e.  $H^1(\Gamma, \mathrm{SL}_4(\mathbb{R}))_\rho \neq 0$  but no Zariski tangent vector is integrable.
- 43 regular curves.
- 7 regular surfaces.
- 1 singular surface such that  $H^1(\Gamma, \mathrm{SL}_4(\mathbb{R}))_\rho$  is of dimension 3.
- 1 singular 3-variety, which has two 3-dimensional branches meeting in a curve.

First, we should mention that these computations are mostly done in floating-point mode, hence this summary is a very good speculation but not a statement. Second, the authors check rigorously their speculations on certain characters varieties of this list.

*Remark 10.1.* If  $\Gamma$  is the fundamental group of a closed hyperbolic 3-manifold, then the first obstruction  $o_2(u_1) = 0$ , for every Zariski tangent vector at the hyperbolic point  $\rho$  (see [CLT07]). Indeed, first, the infinitesimal rigidity of the closed hyperbolic 3-manifold in  $\mathrm{SO}_{3,1}$  implies that  $u_1 \in H^1(\Gamma, \mathfrak{o})$ . Second, since  $[\mathfrak{o}, \mathfrak{o}] \subset \mathfrak{so}_{3,1}$ , it follows that  $o_2(u_1)$  is not only an element of  $H^2(\Gamma, \mathfrak{sl}_4)$  but also an element of  $H^2(\Gamma, \mathfrak{so}_{3,1})$ . Finally, we know from the Poincaré duality that  $H^2(\Gamma, \mathfrak{so}_{3,1}) = H^1(\Gamma, \mathfrak{so}_{3,1}) = 0$ .

So, the singularities of some varieties are more than quadratic, since for example the manifold “Vol3” is locally rigid even if its Zariski tangent space is one-dimensional. Compare to the result of Goldman-Millson [GM88] that the singularity at a reductive representation is at most quadratic if  $\Gamma$  is the fundamental group of a Kähler manifold.

**10.2. Infinitesimal rigidity relative to the boundary.** Heusener and Porti make use of a relative version of the infinitesimal rigidity for finite volume hyperbolic 3-manifolds  $M$  in order to obtain the infinitesimal rigidity for some Dehn fillings of  $M$ .

Let  $M$  be a 3-manifold with a boundary whose interior  $N$  carries a finite volume complete hyperbolic metric, and let  $\rho := \rho_{\mathrm{geo}}$  be the holonomy representation of  $N$ . We say that  $M$  (or  $N$ ) is *infinitesimally rigid rel boundary* if the map  $H^1(M, G)_\rho \rightarrow H^1(\partial M, G)_\rho$  is injective. Roughly speaking, in the infinitesimal level, each deformation must change the geometry of the cusp. The combination of the following two theorems shows in particular that infinitely many closed hyperbolic 3-manifolds are locally rigid in  $G = \mathrm{SL}_4(\mathbb{R})$ .

**Theorem 10.2.** [HP11, Theorem 1.4] *Infinitely many Dehn fillings of a non-compact hyperbolic 3-manifold of finite volume which is infinitesimally rigid rel boundary are infinitesimally rigid.*

**Theorem 10.3.** [HP11] *There exists non-compact hyperbolic 3-manifolds of finite volume which are infinitesimally rigid rel boundary.*

*Remark 10.4.* The figure-eight knot complement and the Whitehead link complement, for example, are infinitesimally rigid rel boundary. A finite volume non-compact hyperbolic 3-manifold which contains an embedded totally geodesic closed hypersurface is not infinitesimally rigid rel boundary. This raises the following question: Can we find a (topological) characterization of the finite volume hyperbolic 3-manifolds which are infinitesimally rigid rel boundary? An answer even for knots or links would already be quite nice.

Surprisingly, the technique of Heusener and Porti, which is extended by Ballas, also produces deformations. A *slope* is a curve in the boundary, and a slope  $\gamma$  is *rigid* if the map  $H^1(M, \mathfrak{o})_\rho \rightarrow H^1(\gamma, \mathfrak{o})_\rho$  is non-trivial.

**Theorem 10.5.** [HP11, Bal14] *Infinitely many (generalized) Dehn fillings of an amphicheiral knot whose longitude is a rigid slope are deformable.*

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