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# VANISHING PRESSURE IN GAS DYNAMICS EQUATIONS

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**Abstract.** This work is devoted to the analysis of the behaviour of solutions of gas dynamics equations as the pressure goes to 0 in the context of regular solutions. We obtain in this way a first justification of the connection to pressureless gases model.

**Key Words.** One-dimensional degenerate hyperbolic system, Zero pressure gas dynamics

## 1. INTRODUCTION

This work is a first attempt to deal with vanishing pressure in the following system of gas dynamics equations

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 & \text{in } \mathbb{R}_t^+ \times \mathbb{R}_x, \\ \partial_t(\rho u) + \partial_x(\rho u^2/2 + \varepsilon^2 p(\rho)) = 0 & \text{in } \mathbb{R}_t^+ \times \mathbb{R}_x. \end{cases} \quad (1)$$

In (1), the pressure law  $p : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is required to satisfy

$$p'(\rho) > 0 \text{ for } \rho > 0, \quad (2)$$

$$p''(\rho) \geq 0, \quad (3)$$

$$2p'(\rho) \geq \rho p''(\rho). \quad (4)$$

In the sequel, we will denote by  $P(\rho)$  a function verifying

$$P'(\rho) = \sqrt{p'(\rho)}/\rho.$$

Notice in particular that the power laws  $p(\rho) = \rho^\gamma$ , with  $\gamma > 1$  as well as the isothermal law  $p(\rho) = \rho$  fulfill (2-3), but (4) restricts to  $1 \leq \gamma \leq 3$ . We obtain in these cases  $P(\rho) = 2\sqrt{\gamma}/(\gamma - 1) \rho^{(\gamma-1)/2}$  for

$\gamma > 1$  and  $P(\rho) = \ln(\rho)$  for  $\gamma = 1$ . The problem is supplemented by prescribing Cauchy data for the density  $\rho(0, x) = \rho_0(x) \geq 0$  and the velocity  $u(0, x) = u_0(x)$ . Formally, as  $\varepsilon$  vanishes, we are led to the following pressureless gases equations

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 & \text{in } \mathbb{R}_t^+ \times \mathbb{R}_x, \\ \partial_t(\rho u) + \partial_x(\rho u^2/2) = 0 & \text{in } \mathbb{R}_t^+ \times \mathbb{R}_x. \end{cases} \quad (5)$$

The system (5) was introduced as a simplified model of the dynamics of galaxies by Zeldovich, see [9] and a first mathematical analysis is due to Bouchut in [1]. Existence of solutions and connection to scalar conservation laws are studied by Grenier [5], Brenier-Grenier [4] and, also by Bouchut-James who used in [3] the notion of duality solutions of [2] (see also Poupaud-Rascle [8] where an equivalent formulation is proposed). Regular solutions, even in the multidimensional context, are investigated by Poupaud [7].

In this work, we restrict to the case of regular solutions: crucial bounds are obtained by considering the Riemann invariants associated to (1), and, then we pass to the limit as  $\varepsilon$  goes to 0. In this way, we obtain the following

**Theorem 1** *Let  $\rho_0^\varepsilon, u_0^\varepsilon$  be a sequence of initial data for (1) satisfying*

$$\begin{cases} \rho_0^\varepsilon \geq d_\varepsilon > 0, & \partial_x(u_0^\varepsilon \pm \varepsilon P(\rho_0^\varepsilon)) \geq 0, \\ \rho_0^\varepsilon \rightharpoonup \rho_0 & \text{weakly } * \text{ in } L_{loc}^\infty(\mathbb{R}), \\ u_0^\varepsilon \rightharpoonup u_0 & \text{in } W_{loc}^{1,\infty}(\mathbb{R}). \end{cases} \quad (6)$$

where  $d_\varepsilon$  is a sequence of positive reals, possibly tending to 0. Then the associated solutions of (1) satisfy

$$\begin{aligned} \rho^\varepsilon &\rightharpoonup \rho \text{ weakly } * \text{ in } L_{loc}^\infty(\mathbb{R}^+ \times \mathbb{R}), \\ u^\varepsilon &\rightarrow u \text{ in } C_{loc}^0(\mathbb{R}^+ \times \mathbb{R}), \end{aligned}$$

where  $(\rho, u)$  is the unique solution to (5) with initial data  $(\rho_0, u_0)$ .

**Remark 1** Note that (6) implies that  $\partial_x u_0^\varepsilon \geq \varepsilon |\partial_x(P(\rho_0^\varepsilon))|$ , thus  $\partial_x u_0^\varepsilon$  is non-negative, and, passing to the limit,  $\partial_x u_0$  is too. Moreover, in terms of  $\rho_0^\varepsilon$ , this yields  $|\partial_x(\rho_0^\varepsilon)| \leq 1/\varepsilon \rho_0^\varepsilon / \sqrt{p'(\rho_0^\varepsilon)} \partial_x u_0^\varepsilon$  which essentially means that  $\partial_x \rho_0^\varepsilon$  may blow up as  $(\varepsilon \sqrt{p'(d_\varepsilon)})^{-1}$ .

In Section 2, we perform some computations on the Riemann invariants  $u \pm \varepsilon P(\rho)$  associated to solutions of (1), following essentially ideas of P. D. Lax [6]. This allows us to discuss crucial bounds on the solutions in Section 3 and we conclude in Section 4.

## 2. Riemann Invariants

Let  $c = \sqrt{p'(\rho)}$ . We set  $w = u + \varepsilon P(\rho)$  and  $z = u - \varepsilon P(\rho)$ , where  $(\rho, u)$  satisfies (1). Then, assuming that  $\rho$  and  $u$  are regular enough, some elementary computations leads to

$$\begin{cases} \partial_t w + (u + \varepsilon c) \partial_x w = 0, \\ \partial_t z + (u - \varepsilon c) \partial_x z = 0. \end{cases} \quad (7)$$

Following P. D. Lax, [6], possible loss of regularity of the solutions can be observed by considering the equations satisfied by the first derivative of  $w$  and  $z$ . Let  $W = \partial_x w$ ,  $Z = \partial_x z$  and set  $\alpha = \frac{\rho p''(\rho)}{2p'(\rho)} = c'(\rho)/P'(\rho)$ .

By deriving (7) we get

$$\begin{cases} \partial_t W + (u + \varepsilon c) \partial_x W + (1 + \alpha)/2 W^2 + (1 - \alpha)/2 ZW = 0, \\ \partial_t Z + (u - \varepsilon c) \partial_x Z + (1 + \alpha)/2 Z^2 + (1 - \alpha)/2 ZW = 0, \end{cases} \quad (8)$$

since  $\partial_x u = \partial_x(\frac{w+z}{2}) = 1/2(W+Z)$  and  $\partial_x(\varepsilon P(\rho)) = 1/2(W-Z)$ .

We introduce the characteristics associated to the velocities  $u \pm \varepsilon c$ , namely

$$\begin{cases} \frac{d}{dt} X(t, x) = u(t, X(t, x)) + \varepsilon c(t, X(t, x)), \\ \frac{d}{dt} Y(t, x) = u(t, Y(t, x)) - \varepsilon c(t, Y(t, x)), \end{cases}$$

and  $X(0, x) = x = Y(0, x)$ . For  $\phi$  a real-valued function depending on time and position, we set  $\phi^\sharp(t, x) = \phi(t, X(t, x))$  and  $\phi^\flat(t, x) =$

$\phi(t, Y(t, x))$ . Now, we can rewrite (8) as the following ode

$$\frac{d}{dt}S + AS^2 + BS = 0, \quad (9)$$

where  $S$  stands for  $W^\sharp$  and  $A = (1 + \alpha^\sharp)/2$ ,  $B = (1 - \alpha^\sharp)/2 Z^\sharp$  or  $S = Z^\flat$ ,  $A = (1 + \alpha^\flat)/2$ ,  $B = (1 - \alpha^\flat)/2 W^\flat$ , respectively. In (9), the function  $B$  can be viewed as the time derivative of another function. We introduce  $h$  and  $k$  as functions of the two variables  $(w, z)$  verifying  $\partial_z h(w, z) = (1 - \alpha(\rho))/(2\varepsilon c(\rho)) = \partial_w k(w, z)$  with  $\rho = P^{-1}((w - z)/2\varepsilon)$ ,  $P^{-1}$  being the inverse of  $P$ . Indeed, we remark that

$$\begin{aligned} \frac{d}{dt}(h^\sharp(t, x)) &= \frac{d}{dt}(h(w^\sharp(t, x), z^\sharp(t, x))) \\ &= (\partial_z h)^\sharp(t, x) (2\varepsilon c \partial_x z)(t, X(t, x)) \\ &= ((1 - \alpha)Z)^\sharp = +2B \end{aligned} \quad (10)$$

by (7). Similarly, we get

$$\frac{d}{dt}(k^\flat) = (\partial_w k)^\flat (-2\varepsilon c(\partial_x w)(t, Y(t, x))) = -((1 - \alpha)W)^\flat = -2B. \quad (11)$$

Combining (10) and (11) to (9) leads to

$$\frac{d}{dt}S + AS^2 + \left(\frac{d}{dt}C\right)S = 0 \quad (12)$$

with  $C = h^\sharp/2$  or  $-k^\flat/2$ . This Ricatti-like equations can be integrated easily; we obtain

$$\begin{aligned} S(t, x) &= \left( 1/S(0, x) \exp((C(t, x) - C(0, x))) \right. \\ &\quad \left. + \int_0^t A(\tau, x) \exp(C(t, x) - C(\tau, x)) d\tau \right)^{-1}. \end{aligned} \quad (13)$$

Since  $A = (1 + \alpha)/2 \geq 1/2 > 0$  by (2-3), for  $S(0, x) \geq 0$  the solution of (13) exists globally in time which in turn leads to global existence for (1).

### 3. Estimates

The computations made in the previous Section allow us to derive some estimates on the solution of (1). We have

**Proposition 1** *Let  $\rho_0^\varepsilon, u_0^\varepsilon$  be the initial data of (1) satisfying the requirements of Theorem 1. Then there exists a global regular solution  $(\rho^\varepsilon, u^\varepsilon)$  of (1) and the sequences  $(u_\varepsilon, \varepsilon P(\rho^\varepsilon))_{\varepsilon>0}$  are bounded in  $L^\infty(\mathbb{R}^+; W_{loc}^{1,\infty}(\mathbb{R}))$  while  $(\rho_\varepsilon)_{\varepsilon>0}$  is bounded in  $L_{loc}^\infty(\mathbb{R}^+ \times \mathbb{R})$ .*

We turn to the Riemann invariant. Assumption (6) means that  $W(0, x)$  and  $Z(0, x)$  are non-negative. By (13), one deduces that  $W$  and  $Z$  remain non-negative and are defined for all time. In turn, this implies that  $t \mapsto h^\sharp(t, x)$  is non-decreasing by (10) and  $t \mapsto k^\flat(t, x)$  is non-increasing by (11), since (4) means that  $1 - \alpha \geq 0$ . Thus  $t \mapsto C(t, x)$  is non-decreasing and it follows that

$$0 \leq S(t, x) \leq \left(1/S(0, x) + t/2\right)^{-1}. \quad (14)$$

Therefore  $W = \partial_x w^\varepsilon$  and  $Z = \partial_x z^\varepsilon$  are bounded in  $L^\infty(\mathbb{R}^+ \times \mathbb{R}_{loc})$  while integrating (7) along the characteristics provides a  $L^\infty(\mathbb{R}^+ \times \mathbb{R}_{loc})$  bound on  $w^\varepsilon, z^\varepsilon$ . Thus,  $u^\varepsilon = 1/2(w^\varepsilon + z^\varepsilon)$  and  $\varepsilon P(\rho^\varepsilon) = 1/2(w^\varepsilon - z^\varepsilon)$  are bounded in  $L^\infty(\mathbb{R}; W^{1,\infty}(-R, +R))$  for all  $0 < R < \infty$ . Finally, coming back to (1) we get

$$\partial_t \rho^\varepsilon + u^\varepsilon \partial_x \rho^\varepsilon + \rho^\varepsilon \partial_x u^\varepsilon = 0$$

which can be integrated along the characteristic curves associated to the velocity  $u^\varepsilon$  and provides the estimate on  $\rho^\varepsilon$  in  $L^\infty((0, T) \times (-R, +R))$  for all  $0 < T, R < \infty$ .

**Remark 2** *One needs assumption (4) to obtain a uniform bound on  $\partial_x u^\varepsilon$  and  $\partial_x(\varepsilon P(\rho^\varepsilon))$ . Indeed, for  $\varepsilon > 0$  given, by (13)  $W, Z$  belongs to*

$L^\infty(\mathbb{R} \times \mathbb{R}_{loc})$ , but this estimate is not uniform wrt.  $\varepsilon$ . For instance, considering power law with  $\gamma > 3$ ,  $W, Z$  blow up as  $e^{1/\varepsilon}$ .

#### 4. Passage to the limit

We are ready to end the proof of Theorem 1. Since  $u^\varepsilon$  satisfies

$$\begin{aligned} \partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon + \varepsilon^2 / \rho^\varepsilon \partial_x (p(\rho^\varepsilon)) &= 0 \\ = \partial_t u^\varepsilon + u^\varepsilon \partial_x u^\varepsilon + \varepsilon \sqrt{p'(\rho^\varepsilon)} \partial_x (\varepsilon P(\rho^\varepsilon)) &= 0 \end{aligned} \quad (15)$$

by (1), we deduce from Proposition 1 that  $\partial_t u^\varepsilon$  lies in a bounded set of  $L^\infty_{loc}(\mathbb{R}^+ \times \mathbb{R})$ . Indeed,  $\rho^\varepsilon$  is bounded in  $L^\infty_{loc}(\mathbb{R}^+ \times \mathbb{R})$  and by (2),  $p'$  is non decreasing, thus  $p'(\rho^\varepsilon)$  is bounded in  $L^\infty_{loc}(\mathbb{R}^+ \times \mathbb{R})$ . Therefore, Proposition 1 implies that the last term in (15) is bounded (and actually tends to 0!). One deduces that  $u^\varepsilon$  belongs to a bounded set in  $W^{1,\infty}_{loc}(\mathbb{R}^+ \times \mathbb{R})$  and by Ascoli's theorem, we can assume

$$u^\varepsilon \rightarrow u \text{ in } C^0((0, T) \times (-R, +R)), \quad 0 < T, R < \infty, \quad (16)$$

at least for a subsequence. Furthermore, we can also suppose, say,  $\rho^\varepsilon \rightharpoonup_* \rho$  in  $L^\infty((0, T) \times (-R, +R))$ . These convergence properties permit us to pass to the limit in the  $\mathcal{D}'$  sense in (1) and we are led to (5) as  $\varepsilon$  goes to 0. We check easily that the initial data for  $(\rho, u)$  are the limits  $(\rho_0, u_0)$  of  $(\rho_0^\varepsilon, u_0^\varepsilon)$ . Since  $\rho_0$  has no atomic part, uniqueness for (5) follows, see [3], which ensures that the whole sequence converges and achieves the proof of Theorem 1. Note that we can also pass to the limit in (15) which shows that  $u$  is a regular solution of the Burgers equation.

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