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No-regret optimal control redefinition and consequences

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RÉSUMÉ.

ABSTRACT. In this paper, we redefine the notion of no-regret control introduced by J.L.Lions in [2](original idea by Savage in statistics [4]), this new definition is based on taking spermium upon the states $y(0, g)$ instead of taking spermium on missing data g . The main interest is that this definition gives a more simple characterization of no-regret optimal control comparing to the optimality systems in [2], [5] and [6].

MOTS-CLÉS :

KEYWORDS : No-regret optimal control, low-regret optimal control, missing data problems

1. Introduction

Consider the following state equation described by

$$Ay(v, g) = Bv + \beta g \quad (1)$$

where $A \in L(V, V')$ is an isomorphism, V is a Hilbert space with dual V' , $B \in L(U, V')$ is the control operator with U is a Hilbert space of controls, $\beta \in L(G, V')$, G is also a Hilbert space of uncertainties and $v \in U$ is the control function. Suppose that (1) is well posed in V and denote by $y(v, g)$ its unique solution that depends on the control v and on the missing data g . Associate to (1) the objective quadratic function of the form (see [3])

$$J_0(v, g) = \|Cy(v, g) - z_d\|_H^2 + N \|v\|_U^2 \quad (2)$$

where $C \in L(V, H)$, H is another Hilbert space and z_d is a desired state in H , $N > 0$. Our goal is to characterize the optimal control of (1) subject to the cost function (2) whatever the value of the uncertainty g , in other words we are looking to solve

$$\inf_{v \in U} J_0(v, g) \quad \text{for every } g \in G$$

This definition doesn't make any sense when $G \neq \{0\}$. One thinks to look for (see [2])

$$\inf_{v \in U} \left(\sup_{g \in G} J_0(v, g) \right) \quad (3)$$

but we can get $\sup_{g \in G} J_0(v, g) = +\infty$.

2. No-regret control redefinition

The last difficulty leads J.L.Lions to think about looking for controls such that

$$J_0(v, g) \leq J_0(0, g) \quad \text{for every } g \in G$$

and to define [2] :

Definition 1 We say that $u \in U$ is a no-regret control for (1) – (2) if u is a solution of

$$\inf_{v \in U} \left(\sup_{g \in G} (J_0(v, g) - J_0(0, g)) \right)$$

Here, we propose another way to define a no-regret control based on the following idea : Remark that $y(v, g) = y(v, 0) + y(0, g)$ (because of linearity in (1)) which allows us to write

$$J_0(v, g) = \|Cy(v, 0) + Cy(0, g) - z_d\|_H^2 + N \|v\|_U^2$$

Now, look to J_0 as a function of v and $y(0, g)$ in other words $J_0(v, g) = J(v, y(0, g))$ where

$$J(v, y(0, g)) = \|Cy(v, 0) - z_d\|_H^2 + N \|v\|_U^2$$

this allows us to say that $\sup_{g \in G} J_0(v, g) = \sup_{y(0, g) \in Y} J(v, y(0, g))$ where $Y = \{y(0, g), g \in G\} \subset V$, then solving (3) is equivalent to solve

$$\inf_{v \in U} \left(\sup_{y(0, g) \in Y} J(v, y(0, g)) \right) \quad (4)$$

and to redefine the no-regret control by :

Definition 2 We say that $u \in U$ is a no-regret control for (1) – (2) if u is a solution of

$$\inf_{v \in U} \left(\sup_{y(0, g) \in Y} (J(v, y(0, g)) - J(0, y(0, g))) \right)$$

Now, we'll try to rewrite the last quantity under inf-sup to separate the roles of v and $y(0, g)$ by using the following lemma :

Lemma 3 For every $(v, g) \in U \times G$, we have

$$\begin{aligned} J(v, y(0, g)) - J(0, y(0, g)) &= J(v, 0) - J(0, 0) + 2(Cy(v, 0), Cy(0, g))_H \quad (4) \\ &= J(v, 0) - J(0, 0) + 2(C^*Cy(v, 0), y(0, g))_{V'} \end{aligned}$$

Proof. See [6]. ■

The main difficulty arises in no-regret control characterization, where we do not know the structure of the set $\{v \in U : (Cy(v, 0), Cy(0, g))_H = 0\}$, this problem required to take another way like :

3. Low-regret control redefinition

Relax our problem by looking for controls such that

$$J(v, y(0, g)) - J(0, y(0, g)) \leq \gamma \|y(0, g)\|_V^2 \text{ for every } g \in G, \text{ with } \gamma > 0$$

to get a sequence of controls u_γ expected to be convergent to the no-regret control u .

Definition 4 We say that $u_\gamma \in U$ is a low-regret control for (1) – (2) if u_γ is a solution of

$$\inf_{v \in U} \left(\sup_{y(0, g) \in Y} J(v, y(0, g)) - J(0, y(0, g)) - \gamma \|y(0, g)\|_V^2 \right) \quad (6)$$

Use (5) to get

$$\begin{aligned} & \sup_{y(0, g) \in Y} J(v, y(0, g)) - J(0, y(0, g)) - \gamma \|y(0, g)\|_V^2 \\ = & J(v, 0) - J(0, 0) + \sup_{y(0, g) \in Y} \left(2(C^*Cy(v, 0), y(0, g))_V - \gamma \|y(0, g)\|_V^2 \right) \\ \leq & J(v, 0) - J(0, 0) + \sup_{y \in V} \left(2(C^*Cy(v, 0), y)_V - \gamma \|y\|_V^2 \right) \\ = & J(v, 0) - J(0, 0) + \frac{1}{\gamma} \|C^*Cy(v, 0)\|_{V'}^2 \end{aligned}$$

Identify V and V' to obtain a new optimal control problem

$$\inf_{v \in U} J^\gamma(v) \text{ with } J^\gamma(v) = J(v, 0) - J(0, 0) + \frac{1}{\gamma} \|C^*Cy(v, 0)\|_{V'}^2 \quad (7)$$

Finally, we are inside a classical optimal control problem that depends only on the control v .

4. Low-regret control and no-regret control characterization

(optimality systems)

Proposition 5 The problem (1) – (7) has one solution u_γ .

Proof. We have $J^\gamma(v) \geq -J(0, 0)$ for every $v \in U$ then $d^\gamma = \inf_{v \in U} J^\gamma(v)$ exists. Let $v_n = v_n(\gamma)$ be a minimizing sequence with $J^\gamma(v_n) \rightarrow d^\gamma$ then

$$-J(0, 0) \leq J(v_n, 0) - J(0, 0) + \frac{1}{\gamma} \|C^*Cy(v_n, 0)\|_V^2 \leq d^\gamma + 1$$

from this we deduce $\|v_n\|_U \leq C_\gamma$ independent of n . There exists $u_\gamma \in U$ such $v_n \rightharpoonup u_\gamma$ in U . Also, $y(v_n, 0) \rightarrow y(u_\gamma, 0)$ by continuity w.r.t the data and from strict convexity of J^γ we deduce that u_γ is unique. ■

It stays to prove that u_γ converges to the no-regret control u when $\gamma \rightarrow 0$.

Theorem 6 *The sequence of low-regret control solution to (1) – (7) converges to the no-regret control u weakly in U when $\gamma \rightarrow 0$.*

Proof. u_γ is a low-regret control, then for every $v \in U$ we have

$$J(u_\gamma, 0) - J(0, 0) + \frac{1}{\gamma} \|C^*Cy(u_\gamma, 0)\|_V^2 \leq J(v, 0) - J(0, 0) + \frac{1}{\gamma} \|C^*Cy(v, 0)\|_V^2$$

take $v = 0$ to find

$$\|Cy(u_\gamma, g) - z_d\|_H^2 + N \|u_\gamma\|_U^2 + \frac{1}{\gamma} \|C^*Cy(u_\gamma, 0)\|_V^2 \leq J(0, 0) = \|z_d\|_H^2 \quad (8)$$

which implies

$$\|Cy(u_\gamma, 0)\|_H \leq C, \|u_\gamma\|_U \leq C, \frac{1}{\sqrt{\gamma}} \|C^*Cy(u_\gamma, 0)\|_V \leq C$$

which means that the sequence u_γ is bounded in U then we can extract a subsequence still be denoted u_γ that converges weakly to $u \in U$, it stays to prove that u is a no-regret control. It's clear that for every $v \in U$

$$J(v, g) - J(0, g) - \gamma \|y(0, g)\|_V^2 \leq J(v, g) - J(0, g) \quad \text{for every } g \in G$$

then

$$J(u_\gamma, g) - J(0, g) - \gamma \|y(0, g)\|_V^2 \leq \sup_{y(0, g) \in Y} (J(v, g) - J(0, g))$$

Make $\gamma \rightarrow 0$ to find

$$J(u, g) - J(0, g) \leq \sup_{y(0, g) \in Y} (J(v, g) - J(0, g))$$

■

Theorem 7 *The low-regret u_γ control is characterized by*

$$\begin{cases} Ay_\gamma = Bu_\gamma \\ A^*\zeta_\gamma = C^*Cy_\gamma - z_d + \frac{1}{\gamma}(C^*C)^2 y_\gamma \\ B^*\zeta_\gamma + Nu_\gamma = 0 \text{ in } U \end{cases} \quad (9)$$

where $y_\gamma = y(u_\gamma, 0)$.

Proof. A first order optimality condition gives for every $w \in U$

$$(Cy_\gamma - z_d, Cy(w, 0))_H + N(u_\gamma, w)_U + \frac{1}{\gamma}(C^*Cy_\gamma, C^*Cy(w, 0))_V \geq 0 \quad (10)$$

or

$$\left(C^*Cy_\gamma - z_d + \frac{1}{\gamma}(C^*C)^2 y_\gamma, y(w, 0) \right)_V + N(u_\gamma, w)_U \geq 0$$

Introduce $\zeta_\gamma \in V$

$$A^*\zeta_\gamma = C^*Cy_\gamma - z_d + \frac{1}{\gamma}(C^*C)^2 y_\gamma$$

then rewrite (10) as

$$(B^*\zeta_\gamma + Nu_\gamma, w)_V \geq 0$$

But U is a vector space so $B^*\zeta_\gamma + Nu_\gamma = 0$. ■

And now, we could give the optimality system that characterize the no-regret control u .

Theorem 8 *The no-regret control u solution to (1) – (7) is characterized by*

$$\begin{cases} Ay = Bu \\ A^*\zeta = C^*Cy(u, 0) - z_d + \lambda \\ B^*\zeta + Nu = 0 \text{ in } U \end{cases} \quad (11)$$

with $\lambda \in V$.

Proof. From (8) we know that $u_\gamma \rightharpoonup u$ in U with $B \in L(U, V')$ we conclude that $Bu_\gamma \rightharpoonup Bu$ in V' , and by the optimality system (9) Ay_γ is bounded in V' then weakly convergent to Ay (because A is an isomorphism), pass to limit to get $Ay = Bu$. By the same way Cy_γ is bounded in H so $Cy(u_\gamma, 0) \rightharpoonup Cy(u, 0)$ in H and $C^*Cy(u_\gamma, 0) \rightharpoonup C^*Cy(u, 0)$ in V then $C^*Cy_\gamma + \frac{1}{\gamma}(C^*C)^2 y_\gamma = \left(I + \frac{1}{\gamma}C^*C \right) C^*Cy_\gamma$ converges weakly to $C^*Cy + \lambda$

in V . We know also that $B^* \zeta_\gamma = -Nu_\gamma$, the right side converges weakly to $-Nu$, for the left side ζ_γ is bounded in V and $B^* \in L(V, U)$ (we identify U to U') to get $B^* \zeta_\gamma \rightharpoonup B^* \zeta$.

■

5. Application to some optimal control problems with incomplete data

In this section, we apply the above method to a various kinds of problems (elliptic, parabolic, hyperbolic) with incomplete data, and we give an optimality system for each case.

Example 9 Here, we consider the following elliptic optimal control problem with a distributed control and a missed values on boundary

$$\begin{cases} -\Delta y + y = v & \text{in } \Omega \\ \frac{\partial y}{\partial \nu} = g & \text{on } \Gamma \end{cases} \quad (12)$$

where Ω is bounded set in \mathbb{R}^n with a regular boundary Γ , $v \in U = L^2(\Omega)$, $g \in G = L^2(\Gamma)$, then $y(v, g)$ is unique in $H^{\frac{3}{2}}(\Omega)$. With the boundary cost function

$$J(v, g) = |y(v, g) - z_d|_{L^2(\Gamma)}^2 + N \|v\|_{L^2(\Omega)}^2 \quad (13)$$

Note that

$$J(v, g) - J(0, g) = J(v, 0) - J(0, 0) + 2(y(v, 0), y(0, g))_{L^2(\Gamma)}$$

The low-regret control is the solution of

$$\inf_{v \in L^2(\Omega)} J^\gamma(v) \text{ with } J^\gamma(v) = J(v, 0) - J(0, 0) + \frac{1}{\gamma} |y(v, 0)|_{L^2(\Gamma)}^2$$

A first order optimality condition gives for every $w \in L^2(\Omega)$

$$\left(y(u_\gamma, 0) + \frac{1}{\gamma} y(u_\gamma, 0), y(w, 0) \right)_{L^2(\Gamma)} + N (u_\gamma, w)_{L^2(\Omega)} \geq 0$$

Then, we have the following proposition.

Theorem 10 The low-regret control u_γ is unique and characterized by

$$\begin{cases} -\Delta y_\gamma + y_\gamma = u_\gamma, -\Delta \zeta_\gamma + \zeta_\gamma = 0 & \text{in } \Omega \\ \frac{\partial y_\gamma}{\partial \nu} = 0, \frac{\partial \zeta_\gamma}{\partial \nu} = y_\gamma - z_d + \frac{1}{\gamma} y_\gamma & \text{on } \Gamma \\ \zeta_\gamma + Nu_\gamma = 0 & \text{in } L^2(\Omega) \end{cases}$$

To obtain no-regret control optimality system, adapt the proof of theorem 8 to find.

Theorem 11 *The no-regret control u is unique and characterized by*

$$\begin{cases} -\Delta y + y = u; & -\Delta \zeta + \zeta = 0 & \text{in } \Omega \\ \frac{\partial y}{\partial \nu} = 0; & \frac{\partial \zeta}{\partial \nu} = y - z_d + \lambda & \text{on } \Gamma \\ \zeta + Nu = 0 & & \text{in } L^2(\Omega) \end{cases}$$

where $\lambda \in L^2(\Gamma)$

Example 12 *A high order parabolic problem : it's a fourth order equation with a distributed control and an uncertainty in the initial state, given by*

$$\begin{cases} y + \Delta(a(x, t) \Delta y) = v & \text{in } Q \\ y = 0, \frac{\partial y}{\partial \nu} = 0 & \text{on } \Sigma \\ y(0) = g & \text{in } \Omega \end{cases} \quad (14)$$

Where Ω is an open in \mathbb{R}^n with a smooth boundary Γ , $t \in [0; T], T > 0$, $Q = \Omega \times]0; T[$, $\Sigma = \Gamma \times]0; T[$, with $a \in L^\infty(Q)$, $a \geq \alpha > 0$ almost every where, $v \in U = L^2(Q)$, $g \in G = L^2(\Omega)$. The problem (14) has a unique solution in $L^2(0, T; H_0^2(\Omega))$ (see [3]). Associate to the cost function

$$J(v, g) = \|y(v, g) - z_d\|_{L^2(Q)}^2 + N \|v\|_{L^2(Q)}^2$$

We have

$$J(v, g) - J(0, g) = J(v, 0) - J(0, 0) + 2(y(v, 0), y(0, g))_{L^2(Q)}$$

The low-regret control is the solution of

$$\inf_{v \in L^2(Q)} J^\gamma(v) \text{ with } J^\gamma(v) = J(v, 0) - J(0, 0) + \frac{1}{\gamma} \|y(v, 0)\|_{L^2(Q)}^2$$

Theorem 13 *The low-regret u_γ control is unique and characterized by*

$$\begin{cases} y'_\gamma + \Delta(a(x, t) \Delta y_\gamma) = u_\gamma; & -\zeta'_\gamma + \Delta(a(x, t) \Delta y_\gamma) = y_\gamma - z_d + \frac{1}{\gamma} y_\gamma & \text{in } Q \\ y_\gamma = 0, \frac{\partial y_\gamma}{\partial \nu} = 0; & \zeta_\gamma = 0, \frac{\partial \zeta_\gamma}{\partial \nu} = 0 & \text{on } \Sigma \\ y_\gamma(0) = 0; & \zeta_\gamma(T) = 0 & \text{in } \Omega \end{cases}$$

where $y_\gamma = y(u_\gamma, 0)$, with

$$\zeta_\gamma + Nu_\gamma = 0 \text{ in } L^2(Q)$$

Proof. Again, for every $w \in L^2(Q)$

$$\left(y(u_\gamma, 0) - z_d + \frac{1}{\gamma} y(u_\gamma, 0), y(v - u_\gamma, 0) \right)_{L^2(Q)} + N(u_\gamma, v - u_\gamma)_{L^2(Q)} \geq 0$$

Introduce

$$\begin{cases} -\zeta'_\gamma - \Delta \zeta_\gamma = y_\gamma - z_d + \frac{1}{\gamma} y_\gamma & \text{in } Q \\ \zeta_\gamma = 0, \frac{\partial \zeta_\gamma}{\partial \nu} = 0 & \text{on } \Sigma \\ \zeta_\gamma(T) = 0 & \text{in } \Omega \end{cases}$$

to get

$$\zeta_\gamma + Nu_\gamma = 0 \text{ in } L^2(Q)$$

■

Theorem 14 *The no-regret u control is unique and characterized by*

$$\begin{cases} y' + \Delta(a(x, t) \Delta y) = u; -\zeta' + \Delta(a(x, t) \Delta y) = y(u, 0) - z_d + \lambda & \text{in } Q \\ y = 0, \frac{\partial y}{\partial \nu} = 0; \zeta = 0, \frac{\partial \zeta}{\partial \nu} = 0 & \text{on } \Sigma \\ y(0) = 0; \zeta(T) = 0 & \text{in } \Omega \end{cases}$$

with

$$\zeta + Nu = 0 \text{ in } L^2(Q)$$

Example 15 *Let's take a hyperbolic example : a wave equation with a boundary control and a missed source*

$$\begin{cases} y'' - \Delta y = g & \text{in } Q \\ \frac{\partial y}{\partial \nu} = v & \text{on } \Sigma \\ y(0) = 0 ; y'(0) = 0 & \text{in } \Omega \end{cases}$$

Where Ω is an open in \mathbb{R}^n with a smooth boundary Γ , $t \in [0; T]$, $T > 0$, $Q = \Omega \times]0; T[$, $\Sigma = \Gamma \times]0; T[$, $v \in U = L^2(\Sigma)$, $g \in G = L^2(Q)$. There is a unique solution in sense of transposition $y \in L^2(Q)$ (see[3]). With the cost function

$$J(v, g) = \|y(v, g) - z_d\|_{L^2(Q)}^2 + N \|v\|_{L^2(\Sigma)}^2$$

Again

$$J(v, g) - J(0, g) = J(v, 0) - J(0, 0) + 2(y(v, 0), y(0, g))_{L^2(Q)}$$

The low-regret control is the solution of

$$\inf_{v \in L^2(\Sigma)} J^\gamma(v) \text{ with } J^\gamma(v) = J(v, 0) - J(0, 0) + \frac{1}{\gamma} \|y(v, 0)\|_{L^2(Q)}^2$$

Theorem 16 The new low-regret control u_γ is characterized by

$$\begin{cases} y_\gamma'' - \Delta y_\gamma = 0; \zeta_\gamma'' - \Delta \zeta_\gamma = y_\gamma - z_d + \frac{1}{\gamma} y_\gamma & \text{in } Q \\ \frac{\partial y_\gamma}{\partial \nu} = u_\gamma; \frac{\partial \zeta_\gamma}{\partial \nu} = 0 & \text{on } \Sigma \\ y_\gamma(0) = 0, y_\gamma'(0) = 0; \zeta_\gamma(T) = 0, \zeta_\gamma'(T) = 0 & \text{in } \Omega \end{cases}$$

where $y_\gamma = y(u_\gamma, 0)$, and

$$\zeta_\gamma + Nu_\gamma = 0 \text{ in } L^2(\Sigma)$$

Proof. First order optimality condition writes for every $v \in L^2(\Sigma)$

$$\left(y_\gamma - z_d + \frac{1}{\gamma} y_\gamma, y(w, 0) \right)_{L^2(Q)} + N(u_\gamma, w)_{L^2(\Sigma)} \geq 0$$

$$\left(y(u_\gamma, 0) - z_d + \frac{1}{\gamma} y(u_\gamma, 0), y(v - u_\gamma, 0) \right)_{L^2(\Gamma)} + N(u_\gamma, v - u_\gamma)_{L^2(\Omega)} \geq 0$$

Define a state ζ_γ by

$$\begin{cases} \zeta_\gamma'' - \Delta \zeta_\gamma = y_\gamma - z_d + \frac{1}{\gamma} y_\gamma & \text{in } Q \\ \frac{\partial \zeta_\gamma}{\partial \nu} = 0 & \text{on } \Sigma \\ \zeta_\gamma(T) = 0, \zeta_\gamma'(T) = 0 & \text{in } \Omega \end{cases}$$

to get

$$\zeta_\gamma + Nu_\gamma = 0 \text{ in } L^2(Q)$$

■

Theorem 17 The no-regret control u is characterized by

$$\begin{cases} y'' - \Delta y = 0; \zeta'' - \Delta \zeta = y - z_d + \lambda & \text{in } Q \\ \frac{\partial y}{\partial \nu} = u; \frac{\partial \zeta}{\partial \nu} = 0 & \text{on } \Sigma \\ y(0) = 0, y'(0) = 0; \zeta(T) = 0, \zeta'(T) = 0 & \text{in } \Omega \end{cases}$$

where $\lambda \in L^2(Q)$, and

$$\zeta + Nu = 0 \text{ in } L^2(\Sigma)$$

Conclusion 18 As we have seen, the redefinition of the no-regret control (an equivalent definition to the original one) and consequently the low-regret control, leads to a more simple characterization of no-regret control i.e. a more simple optimality system.

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