Lower Bounds for the Height in Galois Extensions

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F. AMOROSO AND D. MASSER

1. Introduction

For an algebraic number α denote by $h(\alpha) \geq 0$ the absolute logarithmic Weil height; recall that $h(\alpha) = 0$ if and only if $\alpha = 0$ or α is a root of unity. The well-known Lehmer Problem from 1933 asks if there is a positive constant c such that

$$h(\alpha) \geq cd^{-1}$$

whenever $\alpha \neq 0$ has degree d and is not a root of unity. This is still unknown, but the celebrated result of Dobrowolski [8] implies that for any $\varepsilon > 0$ there is $c(\varepsilon) > 0$ such that $h(\alpha) \geq c(\varepsilon)d^{-1-\varepsilon}$ (we will not worry about logarithmic refinements in this note).

The inequality in the Lehmer Problem has been established for various classes of α. Thus Smyth [14] proved it for non-reciprocal α (in particular whenever d is odd), and David with the first author [1] (see *Corollaire 1.7*) proved it when $\mathbb{Q}(\alpha)/\mathbb{Q}$ is a Galois extension. See also their *Corollaire 1.8* for a generalization to extensions that are "almost Galois".

In this note we improve the result in the Galois case, and we even show that for any $\varepsilon > 0$ there is $c(\varepsilon) > 0$ such that

$$h(\alpha) \geq c(\varepsilon)d^{-\varepsilon}$$

when $\mathbb{Q}(\alpha)/\mathbb{Q}$ is a Galois extension. This is related to a problem posed by Smyth during a recent BIRS workshop (see [12], problem (21) at page 17), who asks for small positive values of $h(\alpha)$ for $\alpha \in \overline{\mathbb{Q}}$ with $\mathbb{Q}(\alpha)/\mathbb{Q}$ Galois.

2. A lemma

We start with a result whose proof is implicit in *Corollaire 6.1* of [1].

Lemma 2.1. *Let $\mathbb{F}/\mathbb{Q}$ be a Galois extension and $\alpha \in \mathbb{F}^*$. Let ρ be the multiplicative rank of the conjugates $\alpha_1, \ldots, \alpha_d$ of α over $\mathbb{Q}$, and suppose $\rho \geq 1$. Then there exists a subfield $\mathbb{L} \subseteq \mathbb{F}$ which is Galois over $\mathbb{Q}$ of degree $[\mathbb{L} : \mathbb{Q}] = n \leq n(\rho)$ and an integer $e \geq 1$ such that $\mathbb{Q}(\zeta_e) \subseteq \mathbb{F}$ (for a primitive eth root of unity ζ_e) and $\alpha^e \in \mathbb{L}$.*

Proof. Let e be the order of the group of roots of unity in $\mathbb{F}$, so that $\mathbb{F}$ contains $\mathbb{Q}(\zeta_e)$. Define $\beta_i = \alpha_i^e$ and $\mathbb{L} = \mathbb{Q}(\beta_1, \ldots, \beta_d)$. The $\mathbb{Z}$-module

$$\mathcal{M} = \{\beta_1^{a_1} \cdots \beta_d^{a_d}, \mid a_1, \ldots, a_d \in \mathbb{Z}\}$$

is free (by the choice of e) and of rank ρ. This shows that the action of $\mathrm{Gal}(\mathbb{L}/\mathbb{Q})$ over $\mathcal{M}$ defines an injective representation $\mathrm{Gal}(\mathbb{L}/\mathbb{Q}) \to \mathrm{GL}_\rho(\mathbb{Z})$. Thus $\mathrm{Gal}(\mathbb{L}/\mathbb{Q})$

Date: May 3, 2016.

identifies to a finite subgroup of $\mathrm{GL}_\rho(\mathbb{Z})$. But, by well-known results (see Remark 2.2 below), the cardinality of the finite subgroups of $\mathrm{GL}_\rho(\mathbb{Z})$ is uniformly bounded by, say, $n = n(\rho)$.

□

Remark 2.2. To quickly see that the order of a finite subgroup of $\mathrm{GL}_\rho(\mathbb{Z})$ is uniformly bounded by some $n(\rho) < \infty$, apply Serre's result [13] which asserts that the reduction mod 3 is injective on the finite subgroups of $\mathrm{GL}_\rho(\mathbb{Z})$. This gives the bound $n(\rho) \leq 3^{\rho^2}$. More precise results are known. Feit [9] (unpublished) shows that the orthogonal group (of order $2^\rho\rho!$) has maximal order for $\rho = 1, 3, 5$ and for $\rho > 10$. For the seven remaining values of ρ, Feit characterizes the corresponding maximal groups. See [10] for more details and for a proof of the weaker statement $n(\rho) \leq 2^\rho\rho!$ for large ρ.

3. Main results

We now state two results about α which merely lie in Galois extensions, so are not necessarily generators.

Theorem 3.1. *For any integer $r \geq 1$ and any $\varepsilon > 0$ there is a positive effective constant $c(r, \varepsilon)$ with the following property. Let $\mathbb{F}/\mathbb{Q}$ be a Galois extension of degree D and $\alpha \in \mathbb{F}^*$. We assume that there are r conjugates of α over $\mathbb{Q}$ which are multiplicatively independent*[1]*. Then*

$$h(\alpha) \geq c(r, \varepsilon) D^{-1/(r+1)-\varepsilon}.$$

Proof. The new ingredient with respect to *Corollaire 1.7* of [1] is the main result of Delsinne [7], which was not available at that time. We use standard abbreviations like $\ll_\varepsilon, \gg_{r,\varepsilon}$.

Let $\alpha_1, \ldots, \alpha_d$ (with $d \leq D$) be the conjugates of α over $\mathbb{Q}$ (so lie in $\mathbb{F}$). Their multiplicative rank is at least r. If it is strictly bigger, then the main result *Théorème 1.6* of [1] applied to $r + 1$ independent conjugates gives

$$h(\alpha) \gg_{r,\varepsilon} D^{-1/(r+1)-\varepsilon} .$$

Thus we may assume that the rank is exactly r.

By Lemma 2.1 there exists a number field $\mathbb{L} \subseteq \mathbb{F}$ of degree $[\mathbb{L} : \mathbb{Q}] = n \leq n(r)$ and an integer $e \geq 1$ such that $\mathbb{Q}(\zeta_e) \subseteq \mathbb{F}$ and $\alpha^e \in \mathbb{L}$.

Let now $\varepsilon > 0$. Since $\alpha^e \in \mathbb{L}$ and $[\mathbb{L} : \mathbb{Q}] \leq n$,

$$h(\alpha) = \frac{1}{e} h(\alpha^e) \gg_r \frac{1}{e}. \tag{3.1}$$

On the other hand, the degree of $\mathbb{F}$ over the cyclotomic extension $\mathbb{Q}(\zeta_e)$ is $D/\phi(e)$ and $\alpha_1, \ldots, \alpha_r \in \mathbb{F}$ are multiplicatively independent. By the main result *Théorème 1.6* of [7] (for $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r)$ and recalling that B has positive codimension) we have

$$h(\alpha) \gg_{r,\varepsilon} (D/\phi(e))^{-1/r-\varepsilon} \gg_{r,\varepsilon} e^{1/r} D^{-1/r-\varepsilon} . \tag{3.2}$$

Combining (3.1) and (3.2) we get

$$h(\alpha)^{r+1} = h(\alpha) h(\alpha)^r \gg_{r,\varepsilon} D^{-1-r\varepsilon}.$$

□

[1] which implies that α is not a root of unity.

Taking $r = 1$ we get

Corollary 3.2. *For any $\varepsilon > 0$ there is a positive effective constant $c(\varepsilon)$ with the following property. Let $\mathbb{F}/\mathbb{Q}$ be a Galois extension of degree D. Then for any $\alpha \in \mathbb{F}^*$ which is not a root of unity we have*

$$h(\alpha) \geq c(\varepsilon) D^{-1/2-\varepsilon} .$$

For a direct proof of this corollary, which uses [5] instead of the more deep result of [7], see [11] exercise 16.23.

We remark that Corollary 3.2 is optimal: take for $\mathbb{F}$ the splitting field of $x^d - 2$, with $D = d\phi(d)$, and $\alpha = 2^{1/d}$. Nevertheless, as mentioned above, this result can be strengthened for a *generator* α of a Galois extension.

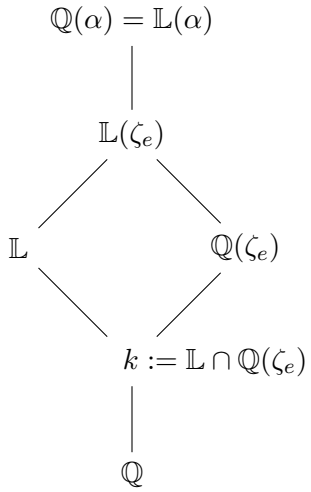
Theorem 3.3. *For any $\varepsilon > 0$ there is a positive effective constant $c(\varepsilon)$ with the following property. Let $\alpha \in \overline{\mathbb{Q}}^*$ be of degree d, not a root of unity, such that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois. Then we have*

$$h(\alpha) \geq c(\varepsilon) d^{-\varepsilon} .$$

Proof. Let r be the smallest integer $> 1/\varepsilon$. If $r \geq d$ then $d < 1/\varepsilon$ and $h(\alpha) \gg_\varepsilon 1$. So we can assume $r < d$. If r among the conjugates of α are multiplicatively independent, by [1] we have

$$h(\alpha) \gg_\varepsilon D^{-1/r-\varepsilon} \gg_\varepsilon D^{-2\varepsilon} .$$

Otherwise, the multiplicative rank $\rho \geq 1$ of the conjugates of α is at most $r - 1 \leq 1/\varepsilon$. By Lemma 2.1 there exists a number field $\mathbb{L} \subseteq \mathbb{Q}(\alpha)$ of degree $[\mathbb{L} : \mathbb{Q}] = n \leq n(\varepsilon)$ and an integer $e \geq 1$ such that $\mathbb{Q}(\zeta_e) \subseteq \mathbb{Q}(\alpha)$ and $\alpha^e \in \mathbb{L}$. As a consequence $\mathbb{L}(\alpha)/\mathbb{L}$ is of degree $e' \leq e$. The diagram



shows that the degree of α over $\mathbb{Q}(\zeta_e)$ is

$$[\mathbb{Q}(\alpha) : \mathbb{L}(\zeta_e)] \cdot [\mathbb{L}(\zeta_e) : \mathbb{Q}(\zeta_e)] = e' \frac{[\mathbb{L}(\zeta_e) : \mathbb{Q}(\zeta_e)]}{[\mathbb{L}(\zeta_e) : \mathbb{L}]}$$

which is

$$e' \frac{[\mathbb{L} : k]}{[\mathbb{Q}(\zeta_e) : k]} = e' \frac{[\mathbb{L} : \mathbb{Q}]}{[\mathbb{Q}(\zeta_e) : \mathbb{Q}]} = \frac{e'}{\phi(e)} n \leq \frac{e}{\phi(e)} n \ll_\varepsilon d^\varepsilon .$$

By the relative Dobrowolski lower bound of [5] we get

$$h(\alpha) \gg_\varepsilon d^{-2\varepsilon} .$$

□

Remark 3.4. *The proof above may be made completely explicit using* [4] *and* [2] *respectively instead of* [1] *and* [5]. *This would of course lead to a lower bound depending only on d.*

We note that Theorem 3.3 is best possible in the sense that an inequality $h(\alpha) \gg d^\delta$ would be false for any fixed $\delta > 0$. For example $\alpha = 1 + \zeta_e$ with $d = \phi(e)$ has $h(\alpha) \leq \log 2$. Or $\alpha = 2^{1/e} + \zeta_e$, whose degree is easily seen to be $e\phi(e)$, with $h(\alpha) \leq 2 \log 2$. But Smyth in [12] quoted above asked if even $h(\alpha) \gg 1$ is true, a kind of "Galois-Lehmer Problem". We do not know, but it would imply the main result of Amoroso-Dvornicich [3] on abelian, and a slightly weaker result of Amoroso-Zannier [6] (Corollary 1.3) on dihedral.

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