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Polynomial chaos expansion and steady-state response of a class of random dynamical systems

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Abstract

The first two moments of the steady-state response of a dynamical random system are determined through a polynomial chaos expansion (PCE) and a Monte-Carlo simulation (MCS) that gives the reference solution. It is observed that the PCE may not be suitable to describe the steady-state response of a random system harmonically excited at a frequency close to a deterministic eigenfrequency: many peaks appear around the deterministic eigenfrequencies. It was proved that the PCE coefficients are the responses of a deterministic dynamical system, the so-called PC-system. As a consequence these coefficients are subjected to resonances associated to the eigenfrequencies of the PC-system: the spurious resonances are located around the deterministic eigenfrequencies of the actual system. It is shown that the polynomial order required to obtain some good results may be very high, especially when the damping is low. These results were shown on a multi-dof (degree-of-freedom) system with a random stiffness matrix. A 1-dof system was also studied and new analytical expressions that make the polynomial chaos expansion possible even for high order were derived. The influence of the PC order was also highlighted. The results obtained in the paper improve the understanding and scope of applicability of PCE for some structural dynamical systems while harmonically excited around the deterministic eigenfrequencies.

Keywords: Random systems; structural dynamics; polynomial chaos expansion; steady-state response.

1. Introduction

The consideration of uncertainties plays a crucial role in accessing dynamic performance of a structure. A primary reason for this is the increasing parametric sensitivity of the dynamic response around the resonance frequencies. Response sensitivity to variations of the structure, such as parametric variation in the Young's modulus, Poisson's ratio, density, or other kind of error sources, e.g. errors in the model of damping, can be evaluated through uncertainty quantification. It is generally considered that, at low frequencies, the study of the response is best addressed by a parametric approach. For this case, the stochastic finite element methods (SFEM) [12] can be applied to obtain statistics of response or the eigenvalues and eigenvectors of the system.

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Stochastic finite element methods are principally divided into simulation-based methods (e.g. Monte Carlo Simulation (MCS), see [17]) and expansion-based methods (perturbation method, spectral approach and stochastic reduced basis method). Reviews on SFEM applied to calculate the response statistics are given, for example, in [31, 32]. Application of perturbation methods to calculate response can be found in [19]. Spectral approach methods are reviewed in [24], where the most widely used spectral approach method is polynomial chaos (PC) using a Galerkin scheme [12] or collocation methods [4, 36, 37]. A different approach, followed in [33, 34], proposed exact analytical expressions for the response statistics for a single-degree-of-freedom system. They were obtained from the pdf of the eigenvalues, related to the pdf of the random parameters. Also, Laplace's integral has been used to calculate moments and reliability of response, where the maximum and Hessian of the logarithm of the integrated function are obtained through different numerical methods [26]. In the context of random skeletal structures, a doubly spectral stochastic finite element [1] approach was developed in the frequency domain. This can be viewed as a stochastic dynamic stiffness formulation [3, 22, 25].

The dynamic response of linear stochastic systems can be considered either in the modal domain or directly in the time or frequency domain. The review papers [29, 31, 32] give an account of the various techniques available. In the modal domain, many authors have used polynomial-chaos based spectral projection approaches [14, 27, 35] for the eigensolutions of the system. However, spectral methods have not been used widely for the direct computation of the response statistics. Pichler et al. [28] proposed a mode-based meta-model for the frequency response functions of stochastic structural systems. Adhikari and Manohar [2] proposed a random eigenfunction expansion method based on Galerkin projection for the frequency domain response of stochastic dynamical systems. Nieuwenhof and Coyette [5] proposed modal approaches for the stochastic dynamic response of structures with material and geometric uncertainties in the frequency domain. Impollonia and Ricciardi [18] proposed a procedure to derive, in explicit form, the stationary response of a linear structure subjected to Gaussian white noise stochastic excitation. Their method is based on the solution of complex eigenvalue problems for each fluctuating parameters. Muscolino et al. [23] discussed improved dynamic analysis of structures with parametric uncertainties under deterministic input forces. Fang [10] proposed a transfer matrix based approach in conjunction with a perturbation method for dynamical systems. Falsone and Ferro [8, 9] proposed the Exact Principal Deformation Mode (EPDM) approach for the frequency domain response of a general linear dynamic system. Their results show excellent agreement with direct Monte Carlo simulation results. Recently Kundu and Adhikari [20] proposed a spectral-function approach for the time-domain response of stochastic systems. The present work is not related to the time integration issues with PCE as the excitation as well as the response are harmonic: so the calculations are in the frequency domain and then, different issues arise. Note that Lucor and Karniadakis have also worked on nonlinear random systems [21].

Few authors have discussed the convergence [30], [11] of the polynomial-chaos expansion for static problems (elliptic equations in general). Practical works on dynamic problems show that some of the nice and well known features may not be preserved. There is a lack of a detailed investigation in literature on the behaviour of the polynomial chaos expansion (PCE) in dynamic context. The aim of this paper is to contribute in this direction so that PCE can be applied to uncertain dynamic systems in an accurate and computational efficient manner. First we observe in this paper that the PCE may not be suitable to describe the steady-state response of a random system harmonically excited at a frequency close to a deterministic eigenfrequency. Second, an explanation of this undesirable behaviour is given. Third, the quantities of interest are studied when the number of terms of the PCE is increased: the behavior of the PCE around the deterministic resonances is highlighted. Finally, it is shown that a PCE may be possible even for high PC order.

2. Response of a random dynamical system harmonically excited

2.1. Uncertainties in a dynamic system

The dynamic system is assumed to be described by its $n \times n$ mass, damping and stiffness matrices, $\mathbf{M}$, $\mathbf{D}$, $\mathbf{K}$, where n is the number of degrees of freedom (dof), the forces acting on this system are described by force vector $\mathbf{F}(t)$, and $\mathbf{x}(t)$ denotes the degree-of-freedom (dof) vector.

The stiffness matrix is assumed to be uncertain and given by

$$\mathbf{K} \;=\; \overline{\mathbf{K}}\,(1 + \delta_K\,\xi_K) \tag{1}$$

where ξ_K is a standard normal deviate and, $\overline{\mathbf{K}}$ is a deterministic matrix, which represents the mean stiffness matrix; the covariance matrix of $\mathbf{K}$ is controlled by parameter δ_K: note that the deterministic dynamic system corresponds to $\delta_K = 0$. This uncertainty is very simple. This choice was motivated by the objective of the paper that is not to find a new method to derive the moments, but to highlight and to explain a specific feature of a PCE around the deterministic eigenfrequencies: this class of uncertainty makes the equations and the explanations clearer. From a theoretical point of view the normal distribution may not appear relevant as a stiffness is nonnegative. However, due to the value of the mean and the standard deviation of the proposed distribution (standard deviation equals to 0.05 for a unit mean), a negative stiffness is an extremely rare event: for a 10,000 independent samples of the random variable, no negative realization is achieved and accordingly the probability that the stiffness matrix defined by Eq. (1) is not positive definite is extremely weak; so from a practical point of view this distribution is relevant. Further, in a manufacturing context, the mean and the standard deviation are the only known statics: so, if the constraint of positiveness is disregarded, the distribution specified by the principle of maximum entropy is the normal distribution. It must also be specified that all the developments made in this paper are applicable to other distributions such as the uniform distribution.

In this study the mass and damping matrices are assumed to be deterministic. However, this is not a restriction and this study may be done with random mass and damping matrices. In fact conceptually the natural frequencies could have been perturbed. This way our results would have been immediately generalised to proportionally damped systems.

The force vector is assumed to be harmonic $\mathbf{F}(t) \;=\; \mathbf{F_0}\; e^{\,I\omega t}$ and the steady-state response of the dynamic system is then $\mathbf{x}(t) = \mathbf{X}\, e^{\,I\omega t}$, where $I = \sqrt{-1}$. $\mathbf{X}$ is the solution of the following equation:

$$\big(-\omega^2\,\mathbf{M} \;+\; I\,\omega\,\mathbf{D} \;+\; \mathbf{K}\big)\,\mathbf{X} \;=\; \mathbf{F_0}. \tag{2}$$

As $\mathbf{K}$ is a random matrix, $\mathbf{X}$ is a random vector, which can be described by its moments. In this study the first two moments are estimated. Note that, due to the class of system under study, these moments may also be obtained by solving deterministic differential equations [15] or by conditional Monte Carlo [15], [16].

As discussed in the introduction section, several methods may be used to derive these moments such as Monte-Carlo simulation (MCS) or the polynomial chaos (PC) expansion (PCE). In this paper, the MCS results will be the reference results and the PCE results will be compared to them.

The PC method is now a well-known and well-studied alternative to MCS that expands the solution $\mathbf{X}$ in terms of polynomials Ψ_j whose variables are a set of mutually independent standard normal deviates [12]. In our case, polynomial chaos Ψ_j is a Hermite polynomial and depends on one random variable ξ_K. So the response can be expanded on the Polynomial Chaos basis such that

$$\mathbf{X}(\omega, \xi_K) = \sum_{i=0}^{\infty} \mathbf{X_i}(\omega)\Psi_i(\xi_K) \tag{3}$$

This expression for $\mathbf{X}(\omega, \xi_K)$ can be truncated to a finite number of terms P for the numerical study. P defines the number of polynomials that is given by $P+1 = (m+r)!/(m!r!)$ where r is the number of random variables and m is the chaos order. So, by truncating the previous infinite expansion, an approximation of $\mathbf{X}(\omega, \xi_K)$ is obtained as

$$\mathbf{X^P}(\omega, \xi_K) = \sum_{i=0}^{P} \mathbf{X_i}(\omega)\Psi_i(\xi_K) \tag{4}$$

In the following the argument ξ_K of $\mathbf{X^P}$ is dropped for convenience.

3. PC component dynamic equation

As already mentioned, dof vector $\mathbf{x}(t)$ is a random vector and, $\mathbf{x}(t)$ is a solution of the equation

$$\mathbf{M}\,\ddot{\mathbf{x}}(t) \;+\; \mathbf{D}\,\dot{\mathbf{x}}(t) \;+\; \mathbf{K}\,\mathbf{x}(t) \;=\; \mathbf{F}(t) \tag{5}$$

where $\mathbf{K}$ is defined by relation (1).

This random response vector may be expanded in terms of Hermite polynomials Ψ_j [6, 12]. An approximation is obtained by truncating the infinite series:

$$\mathbf{x^P}(t) \;=\; \sum_{i=0}^{P} Y_i(t)\;\Psi_i(\xi_k) \tag{6}$$

t will be dropped to shorten the equations.

By replacing $\mathbf{x}$ by its expression (6) in Eq. (5), one has:

$$\sum_{i=0}^{P} \Psi_j(\xi_K)\left(\mathbf{M}\,\ddot{\mathbf{Y}}_\mathbf{i} \;+\; \mathbf{D}\,\dot{\mathbf{Y}}_\mathbf{i} \;+\; \mathbf{K}\,\mathbf{Y}_\mathbf{i}\right) = \;\mathbf{F} \tag{7}$$

By using the Hermite polynomial properties and by taking account of expression (1), the following equations are obtained:

$$\forall j = 0\cdots P,\;\; <0,j,j>\left(\mathbf{M}\,\ddot{\mathbf{Y}}_\mathbf{j} \;+\; \mathbf{D}\,\dot{\mathbf{Y}}_\mathbf{j} \;+\; \overline{\mathbf{K}}\,\mathbf{Y}_\mathbf{j}\right) \;+\; \sum_{i=0}^{n_{PC}} \delta_K <1,i,j>\;\overline{\mathbf{K}}\,\mathbf{Y}_\mathbf{j} = \;\delta_{0j}\mathbf{F} \tag{8}$$

where δ_{ij} is the Kronecker delta and

$$<k,i,j> = \int_{-\infty}^{+\infty} \xi_K^k \Psi_i(\xi_K)\Psi_j(\xi_K)\,\frac{e^{-\xi_K^2/2}}{\sqrt{2\pi}}\,\mathrm{d}\xi_K \tag{9}$$

Due to the Hermite polynomial properties, the following relations may be easily derived:

$$<0,i,j> \;=\; \int_{-\infty}^{+\infty} \Psi_i(\xi_K)\Psi_j(\xi_K)\,\frac{e^{-\xi_K^2/2}}{\sqrt{2\pi}}\,\mathrm{d}\xi_K \;=\; j\,!\,\delta_{ij} \tag{10}$$

$$<1,i,j> \;=\; \int_{-\infty}^{+\infty} \xi_K\Psi_i(\xi_K)\Psi_j(\xi_K)\,\frac{e^{-\xi_K^2/2}}{\sqrt{2\pi}}\,\mathrm{d}\xi_K \;=\; \delta_{j\;i+1}\,j\,! \;+\; \delta_{j\;i-1}\,(j+1)! \tag{11}$$

Define

$$\mathbf{Mom^0} \,/\, Mom^0_{ij} \;=\; <0,i,j> \;\text{ and }\; \mathbf{Mom^0} \in \mathbb{R}^{(P+1)\times(P+1)} \tag{12}$$

$$\mathbf{Mom^1} \,/\, Mom^1_{ij} \;=\; <1,i,j> \;\text{ and }\; \mathbf{Mom^1} \in \mathbb{R}^{(P+1)\times(P+1)} \tag{13}$$

$$\widetilde{\mathbf{M}} \;=\; \mathbf{Mom^0} \otimes \mathbf{M} \in \mathbb{R}^{2(P+1)\times 2(P+1)} \tag{14}$$

$$\widetilde{\mathbf{D}} \;=\; \mathbf{Mom^0} \otimes \mathbf{D} \in \mathbb{R}^{2(P+1)\times 2(P+1)} \tag{15}$$

$$\widetilde{\mathbf{K}} \;=\; (\mathbf{Mom^0} + \delta_K\, \mathbf{Mom^1}) \otimes \overline{\mathbf{K}} \in \mathbb{R}^{2(P+1)\times 2(P+1)} \tag{16}$$

$$\mathbf{Y} \;=\; [\mathbf{Y_0}^T\, \mathbf{Y_1}^T \,\cdots\, \mathbf{Y_P}^T\,]^T \in \mathbb{R}^{2(P+1)} \tag{17}$$

$$\widetilde{\mathbf{F}}(t) \;=\; [\mathbf{F}^T(t)\, 0\, 0 \,\cdots\, 0\,]^T \in \mathbb{R}^{2(P+1)} \tag{18}$$

where $\otimes$ denotes the Kronecker product and $(\bullet)^T$ denotes the transpose of $(\bullet)$. Then the components of the PC expansion satisfy the following equation:

$$\widetilde{\mathbf{M}}\, \ddot{\mathbf{Y}}(t) \;+\; \widetilde{\mathbf{D}}\, \dot{\mathbf{Y}}(t) \;+\; \widetilde{\mathbf{K}}\, \mathbf{Y}(t) \;=\; \widetilde{\mathbf{F}}(t) \tag{19}$$

It should be remarked that Eq. (19) shows that the change of variables provided by relation (6) leads to a $2(P+1)$ dof dynamical system that will be referred to PC-system in the following. So the PCE transforms the study of a random dynamical system to the study of a deterministic dynamical system. Accordingly the PC-system has resonant frequencies and the steady-state response response to a harmonic force shows peaks related to these spurious resonances, as it will be illustrated with a 2-dof system. This is an important result of this paper; the spurious resonances will be referred to as PC-resonances.

4. Example

4.1. Two degree-of-freedom system

MCS and PCE will be used to evaluate the mean and the standard deviation of $\mathbf{X}$ for the example shown in 1 [7]. Stiffness k is assumed to be uncertain:

$$k \;=\; \overline{k}\,(1 + \delta_K\, \xi_K) \tag{20}$$

Thus, the mean stiffness matrix is

$$\overline{\mathbf{K}} \;=\; \overline{k} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \tag{21}$$

The characteristics of the system are listed in Tables 1 and 2. The dof vector was found for 501 values of $\omega/(2\pi)$ in the range $[10-35]$ Hz ($\Delta f = 0.05$ Hz).

$\overline{k}$ (Nm^{-1})	m (kg)	c (Nm^{-1}s^{-1})	δ_K (%)	F_{01} (N)	F_{02} (N)
15000	1	1	5	1	0

Table 1: System characteristics

The MCS results were obtained with 10,000 samples of the random variable ξ_K. The first two moments (mean and standard deviation) are plotted in 2. The deterministic response has been also plotted in 2.

The PC expansion is calculated for two PC orders: 2 and 15. The results are plotted in Figs. 3 and 4, and they are in a perfect agreement with the MCS results, except around the resonances, even for $P = 2$. Around the resonances some spurious oscillations arise and it seems that the curves oscillate more as P increases.

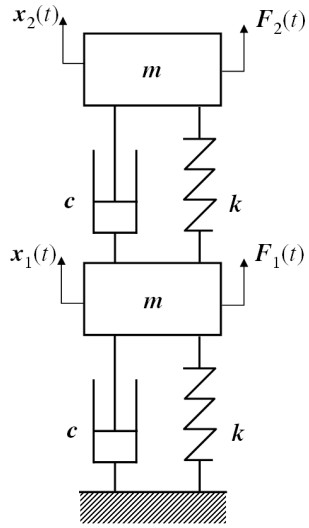


Figure 1: A two degree-of-freedom system with stochastic stiffness coefficients

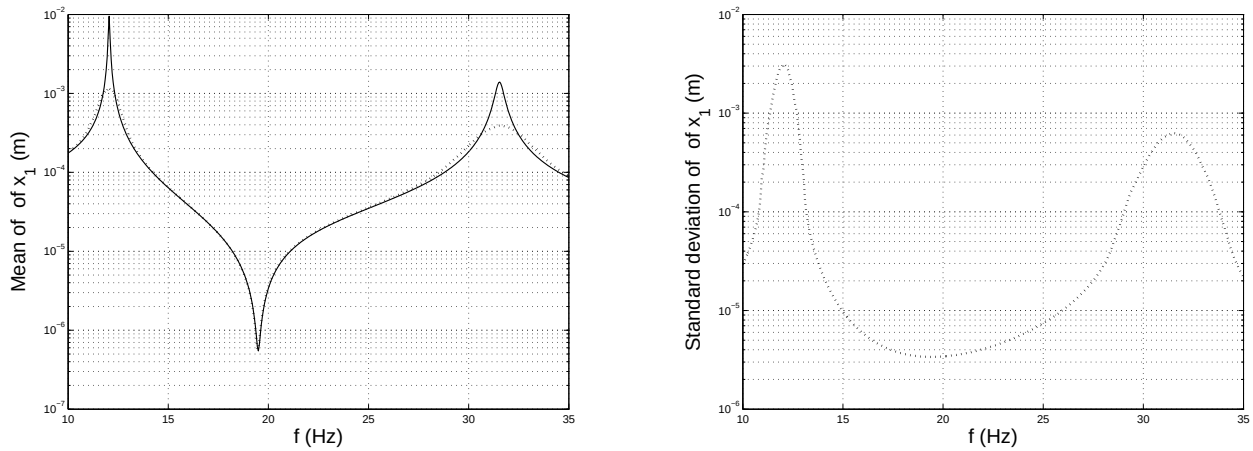
Eigenfrequencies f (Hz)	12.05	31.54
Damping ratio ξ (%)	0.25	0.66

Table 2: Modal characteristics of the deterministic system

As explained above, the PC coefficients are the solution of a deterministic equation of motion of a mechanical system, the PC-system. The dof number of the PC-system is a multiple of the dof number of the deterministic real system: so the eigenfrequencies of the PC-system are noticeable in the steady-state response and are like spurious PC-eigenfrequencies. By solving the PC-eigenproblem associated with matrices $(\widetilde{\mathbf{K}},\ \widetilde{\mathbf{M}})$, the eigenfrequencies were determined and then the modal damping ratios were calculated: the eigenfrequencies are listed in Table 3 as well as the modal damping ratios for several PC expansion degrees.

Table 3 shows that the PC-resonant frequencies may be divided into two sets which are centered around the eigenfrequencies of the deterministic system. This explains why there seems to be more and more "oscillations" around the deterministic frequencies when P increases, in Figs 3 and 4. In fact, there are no oscillations but the peaks are associated with resonances that have no physical meaning and the number of resonances increases with P. Similarly the damping ratios associated with a set of eigenfrequencies are similar to the corresponding damping ratio: this explain why the curve is smoother around the second deterministic eigenfrequency than the first one.

Moreover, it turns out that the deterministic frequencies are in the set of eigenfrequencies only when P is even: when P is odd the deterministic frequencies are antiresonant frequencies. This fact is illustrated in Figure 5 around the first and second deterministic frequencies of the system. Not only are fluctuations of the mean amplitudes observed when P increases, but also there are decreases in the amplitudes and a switch between high and low amplitudes at the first and second deterministic frequencies. This will be explained in the following.



(a) (b)

Figure 2: Deterministic responses (solid lines); Monte-Carlo simulation with 10,000 samples (dotted lines); (a): mean of x_1; (b): standard deviation of x_1

4.2. One degree-of-freedom system

The study of a random 1-dof system is now required to go further in the calculation, and it shows representative behaviour and the underlying physics that would not change significantly with higher dimension. So to keep working on the same example, a modal analysis is performed and the modal coordinates are developed on PCs.

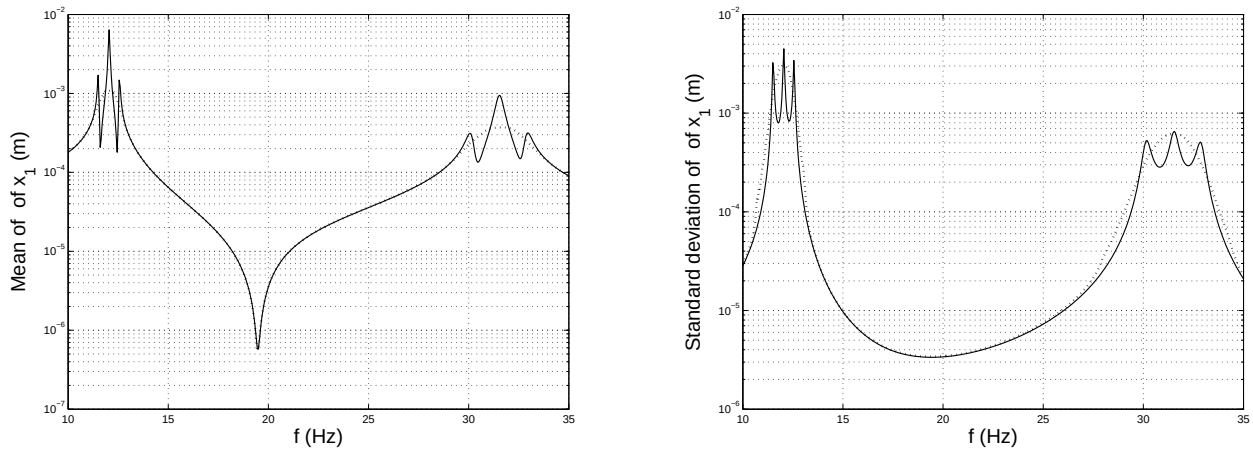
Let's note $(\omega_{det}^{(i)}, \mathbf{\Phi_{det}^{(i)}})$ the deterministic i-th mode associated with $(\overline{\mathbf{K}}, \mathbf{M})$: $\mathbf{\Phi_{det}^{(i)}}$ is also an eigenvector of $(\mathbf{K}, \mathbf{M})$, due to relation (20).

In the following, modal quantity $(\bullet)$ related to mode i will be denoted $(\bullet)^{(i)}$. The associated modal coordinate, $q^{(i)}(t)$, verifies:

$$m^{(i)}\ \ddot{q}^{(i)}\ +\ c^{(i)}\ \dot{q}^{(i)}\ +\ k^{(i)}\ q^{(i)}\ =\ F^{(i)} \tag{22}$$

where $\mathbf{F} = \mathbf{\Phi_{det}}^T\ \mathbf{F}$, $m^{(i)}$, $c^{(i)}$, and $k^{(i)}$ may be easily determined from the system characteristics. Moreover relation (20) means that $k^{(i)} = \overline{k}^{(i)}\ (1 + \delta_K\ \xi_K)$

Suppose a harmonic force ($\mathbf{F_0}\ \exp(I\omega t)$) acts on the mass, and the steady-state response is $q^{(i)}(t) =$



(a) (b)

Figure 3: PC expansion of order 2 (solid lines) vs. Monte-Carlo simulation (dotted lines); (a): mean of x_1; (b): standard deviation of x_1

$Q^{(i)}\ \exp(I\omega t)$. $Q^{(i)}$ is expanded on the PCs

$$Q^{(i)} = \sum_{j=0}^{P} Y_j^{(i)}\ \Psi_j(\xi_K) \tag{23}$$

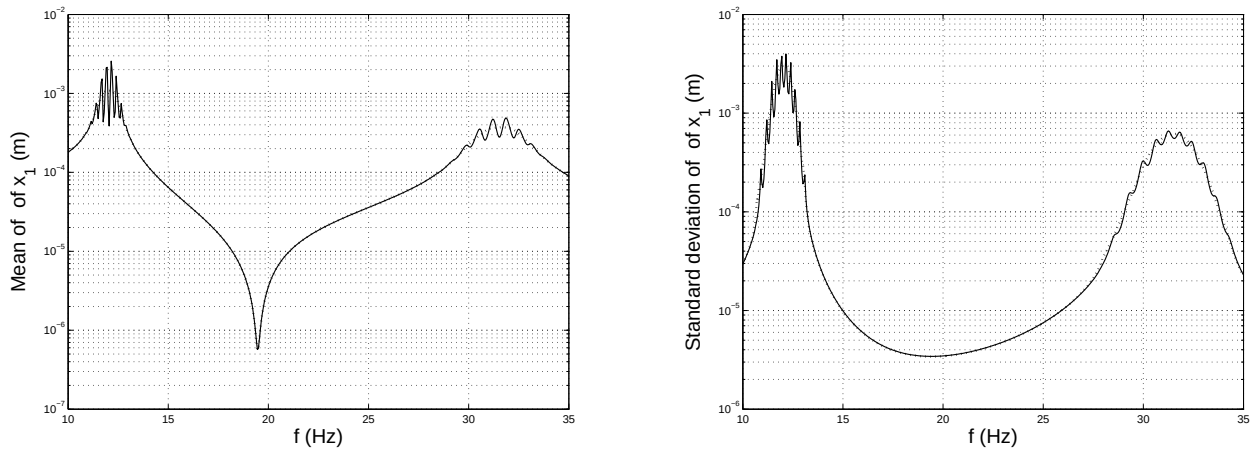
and $Q^{(i)}$ is governed by the following equation:

$$(a^{(i)}\ +\ \overline{k}^{(i)}\ \delta_K\ \xi_K)\ Q^{(i)}\ =\ F^{(i)} \tag{24}$$

where $a^{(i)} = (\overline{k}^{(i)} - \omega^2 m^{(i)}\ +\ I\omega c^{(i)})$. Then the PC expansion components satisfy

$$a^{(i)}\ \sum_{j=0}^{P} Y_j\ \Psi_j(\xi_K)\ +\ \overline{k}^{(i)}\delta_K \sum_{j=0}^{P} Y_j\ \xi_K \Psi_j(\xi_K)\ =\ F^{(i)} \tag{25}$$

Using the Hermite polynomial orthogonality properties given by relations (10) and (11), the following $P+1$



(a) (b)

Figure 4: PC expansion of order 15 (solid lines) vs. Monte-Carlo simulation (dotted lines); (a): mean of x_1; (b): standard deviation of x_1

equations are derived:

$$a^{(i)}\ 0!\ Y_0^{(i)} + \overline{k}^{(i)}\delta_K\ 1!\ Y_1^{(i)} = F^{(i)} \tag{26}$$

$$a^{(i)}\ 1!\ Y_1^{(i)} + \overline{k}^{(i)}\delta_K\ 1!\ Y_0^{(i)} + \overline{k}^{(i)}\delta_K\ 2!\ Y_2^{(i)} = 0 \tag{27}$$

$$\cdots = \cdots \tag{28}$$

$$a^{(i)}\ (P-1)!\ Y_{P-1}^{(i)} + \overline{k}^{(i)}\delta_K\ (P-1)!\ Y_{P-2}^{(i)} + \overline{k}^{(i)}\delta_K\ P!\ Y_P^{(i)} = 0 \tag{29}$$

$$a^{(i)}\ P!\ Y_P^{(i)} + \overline{k}^{(i)}\delta_K\ P!\ Y_{P-1}^{(i)} = 0 \tag{30}$$

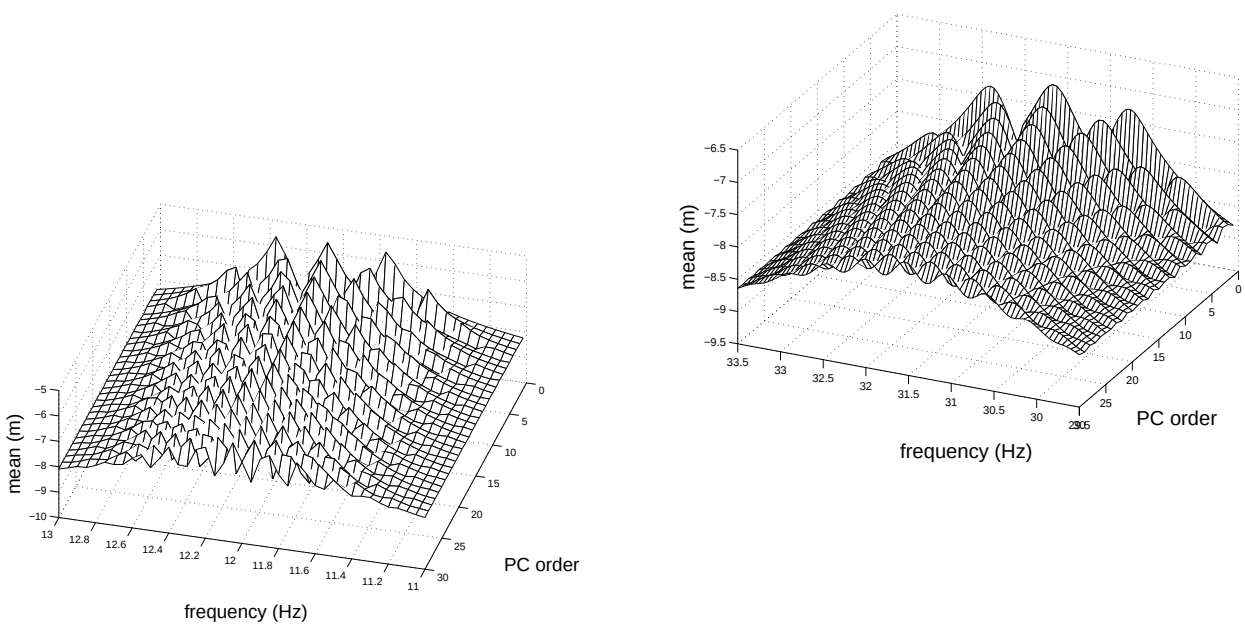
Note that the j-th equation may be simplified by divided this equation by $(j-1)!$.

Define $N_j^{(i)}$ and $D_j^{(i)}$ as

$$Y_j^{(i)} = \frac{N_j^{(i)}}{D_j^{(i)}}\ Y_{j-1}^{(i)} \tag{31}$$

P=1		P=2		P=3		P=4	
f (Hz)	ξ (%)	f (Hz)	ξ (%)	f (Hz)	ξ (%)	f (Hz)	ξ (%)
11.74	0.26	11.51	0.26	11.32	0.27	11.15	0.27
12.34	0.25	12.05	0.25	11.82	0.26	11.63	0.26
		12.56	0.24	12.27	0.25	12.05	0.25
				12.73	0.24	12.45	0.24
						12.88	0.24
30.74	0.67	30.14	0.69	29.64	0.70	29.20	0.71
32.32	0.64	31.54	0.66	30.95	0.67	30.45	0.68
		32.88	0.63	32.12	0.65	31.54	0.66
				33.33	0.63	32.59	0.64
						33.72	0.62

Table 3: Modal characteristics of the PC-component system



(a) (b)

Figure 5: Behaviour around the first (a) and the second (b) deterministic eigenfrequencies for several PC orders.

Then the following recurrence relations hold:

$$\forall j = 2 \cdots P, \;\; N_{j-1}^{(i)} \; = \; - \, \delta_K \, D_j^{(i)} \;\; \text{and} \;\; D_{j-1}^{(i)} \; = \; \frac{a^{(i)}}{\overline{k}^{(i)}} \, D_j^{(i)} \; + \; \delta_K \, j \, N_j^{(i)} \tag{32}$$

with the initial values:

$$N_P^{(i)} \;=\; -\,\delta_K \quad \text{and} \quad D_P^{(i)} \;=\; \frac{a^{(i)}}{\overline{k}^{(i)}} \tag{33}$$

Then, $N_1^{(i)}$ and $D_1^{(i)}$ may be derived easily and finally

$$Y_0^{(i)} \;=\; \frac{D_1^{(i)}}{a^{(i)}\; D_1^{(i)} \;+\; \overline{k}^{(i)}\delta_K\; N_1^{(i)}}\; F^{(i)} \tag{34}$$

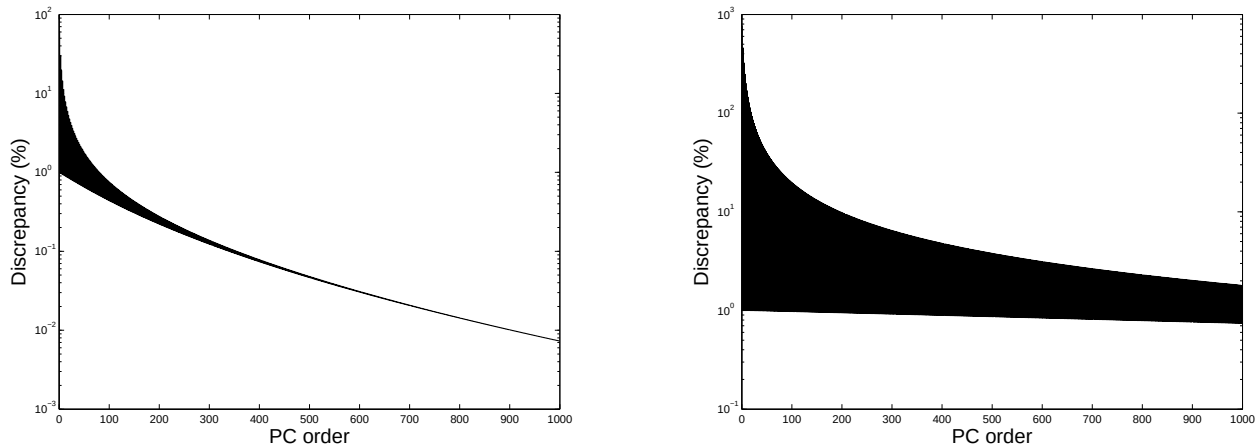
By mathematical induction, it is easy to prove that $a^{(i)}$ may be factorized either in $N_j^{(i)}$ or in $D_j^{(i)}$ for each j. More precisely, it may be shown that if $a^{(i)}$ is a factor of $D_j^{(i)}$ (resp. $N_j^{(i)}$), then $a^{(i)}$ is a factor of $N_{j-1}^{(i)}$ (resp. $D_{j-1}^{(i)}$). However $\omega_{det}^{(i)}$ is a resonant frequency of the FRF $1/a^{(i)}$, accordingly, $\omega_{det}^{(i)}$ is a resonant frequency of $Y_P^{(i)}$, $Y_{P-2}^{(i)}$, $Y_{P-4}^{(i)}$, and so on; similarly $\omega_{det}^{(i)}$ is an antiresonant frequency of $Y_{P-1}^{(i)}$, $Y_{P-3}^{(i)}$, $Y_{P-5}^{(i)}$, and so on. Then if P is even, $\omega_{det}^{(i)}$ is a resonant frequency of $Y_0^{(i)}$, which corresponds to the mean of the i-th modal response; if P is odd, $\omega_{det}^{(i)}$ is an antiresonant frequency of $Y_0^{(i)}$. The change of variable $\mathbf{X^P} = \mathbf{\Phi_{det}}^T\,\mathbf{Q}$ leads to the same conclusions on the steady-state response.

This shows also that the behavior of the response expanded on the PC depends on the parity of P. However, PCE converges in probability and in distribution [13], [11], so $\mathbf{X^{2P}}(\omega = \omega_{det})$ and $\mathbf{X^{2P+1}}(\omega = \omega_{det})$ tend to the same limit as P tends to infinity. Define the following discrepancy for the first dof:

$$\mathcal{D}(P) \;=\; \frac{||X_1^{P+1}{}_{mean} - X_1^{P}{}_{mean}||}{||X_1^{P}{}_{mean}||} \tag{35}$$

This discrepancy provides a valuable information about the required number of terms to include in the expansion. Fig. 6(a) shows that for low P this discrepancy varies significantly from P to $P+1$ due to the alternating resonances and antiresonances. However, for P greater than 300, $\mathcal{D}(P)$ is about 10 %. Fig. 7(a) shows that for $P = 100$, the peaks are still noticeable around the first eigenfrequency whereas the agreement with the MCS results are excellent for $P = 300$ (cf Fig. 7(b)). It is also interesting to note the PC damping effect: as the PC damping ratio associated with the second deterministic mode is higher than for the first mode, the agreement between PCE and MCS around the second mode is good for a PC order lower than 300. Indeed, no oscillations are noticeable around the second mode in 7(a), that is for $P = 100$.

However, in practice, a convergence is not purely theoretical: it must be possible to be close enough to the limit from a numerical point of view. This may be a real problem for a very low damping ratio. In that case, when the system is excited at a frequency equal to a deterministic eigenfrequency, the alternating resonance and antiresonance will produce a sequence with alternatively very low and very high values: then a very high P will be required. In Fig. 6(b), discrepancy $\mathcal{D}$ was plotted when the damping is divided by 5, that is for a first mode damping ratio equal to 0.05 %: it can be remarked that even for P equal to 1500, the limit is not reached. Unfortunately, for P higher than 1800, some numerical problems occurred and then it was not possible to have a good estimation of the limit.



(a) (b)

Figure 6: Discrepancy for increasing PC order; (a) damping ratio ξ=0.25 %; (b) damping ratio ξ=0.05 %

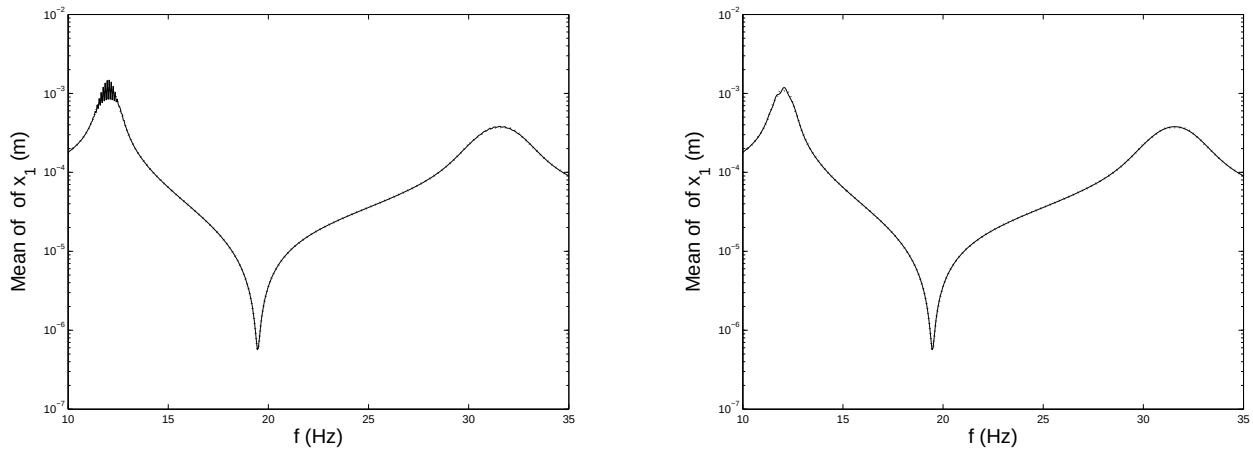
The use of the recurrence relation (32) also provides an interesting expression for $Y_j^{(i)}$:

$$\forall j \geq 1,\ Y_j^{(i)} = \frac{N_j^{(i)}}{D_j^{(i)}}\, Y_{j-1}^{(i)} = \frac{N_j^{(i)}}{D_j^{(i)}}\, \frac{N_{j-1}^{(i)}}{D_{j-1}^{(i)}} \cdots \frac{N_1^{(i)}}{D_1^{(i)}}\, Y_0^{(i)} \tag{36}$$

$$= \frac{N_j^{(i)}}{D_j^{(i)}}\, \frac{N_{j-1}^{(i)}}{D_{j-1}^{(i)}} \cdots \frac{N_1^{(i)}}{D_1^{(i)}}\, \frac{D_1^{(i)}}{a^{(i)}\, D_1^{(i)} + \delta_K\, \overline{k}_i^{mod} N_1^{(i)}}\, F_i^{mod} \tag{37}$$

$$= \frac{N_j^{(i)}}{D_j^{(i)}}\, \frac{-\delta_K\, D_j^{(i)}}{D_{j-1}^{(i)}} \cdots \frac{-\delta_K\, D_2^{(i)}}{D_1^{(i)}}\, \frac{D_1^{(i)}}{a^{(i)}\, D_1^{(i)} + \delta_K\, \overline{k}_i^{mod} N_1^{(i)}}\, F_i^{mod} \tag{38}$$

$$= (-\delta_K)^{j-1}\, \frac{N_j^{(i)}}{a^{(i)}\, D_1^{(i)} + \delta_K\, \overline{k}_i^{mod} N_1^{(i)}}\, F_i^{mod} \tag{39}$$



(a) (b)

Figure 7: Mean of x_1 for $P = 100$ (a), and $P = 300$ (b): PC expansion (black lines) vs. Monte-Carlo simulation (red lines)

Then the modal coordinates may be obtained as

$$Q^{(i)} \;=\; \frac{F_i^{mod}}{a^{(i)}\; D_1^{(i)} + \delta_K\; \overline{k}_i^{mod} N_1^{(i)}} \left(D_1^{(i)}\; \Psi_0 \;+\; \sum_{j=1}^{P} (-\delta_K)^{j-1}\; N_j^{(i)}\; \Psi_j \right) \tag{40}$$

5. Conclusion

The comparison between MCS and a PCE of a random system steady-state response statistics showed that the PCE results tend quickly to the MCS results except around the deterministic eigenfrequencies. This study proved that these oscillations are due to peaks that correspond to resonances. Indeed the PCE coefficients turn to be the response of a deterministic dynamical system that have $P + 1$ eigenfrequencies located around each deterministic eigenfrequency of the actual system: that explains why the spurious oscillations increase with the PC order. This result shows that the PCE coefficients are calculated by transforming the initial deterministic dynamical system into another dynamical system.

Moreover, it was observed that the PCE tends in a not monotonic way to the MCS around the deterministic frequencies: this depends on the parity of the PC order. More precisely, when the random system is excited at a deterministic frequency, the response moments are associated either to a resonance for even

PC order or to an antiresonance for odd PC order. Accordingly, it may be very slow for low damping ratios to reach the MCS results and, may not be achieved numerically. A discrepancy was defined and provides information about the number of PCs that must be used.

The response of a harmonically excited random system can take advantage of modal analysis to determine a PCE. Indeed, the equations may be simplified and some recurrence relations were determined. It was then possible to achieve a PCE even for high PC order that provided results very quickly, which are in an excellent agreement with the MCS results.

The results are related to a class of uncertain dynamical systems. This may be extended easily to an uncertain mass, damping, stiffness matrices that have a Karhunen-Loeve expansion, due to the orthogonality properties of the multidimensional PC. In this study the number of dof was small. Further work that is submitted for publication is dealing with more realistic systems. Moreover an effort must be done to make the PCE efficient at each excitation frequency: ongoing work is done to deal with that issue.

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