



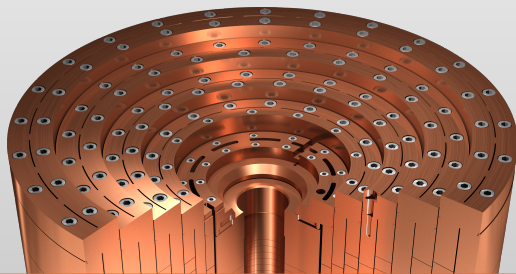
# Full 3D non-linear multi-physics model for High Field polyhelix Magnets

HiFiMagnet Project

C. Daversin<sup>1</sup>, R. Hild<sup>1</sup>, C. Prudhomme<sup>1</sup> and C. Trophime<sup>2</sup>

<sup>1</sup> IRMA, Cemosis – Strasbourg

<sup>2</sup> Laboratoire National des Champs Magnétiques Intenses – Grenoble



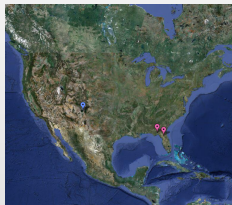


- 1 HiFiMagnet project
- 2 From optimization to CAD
- 3 Full 3D multi-physics model
  - Finite element model
  - Validation and use at LNCMI
  - Reduced order modeling



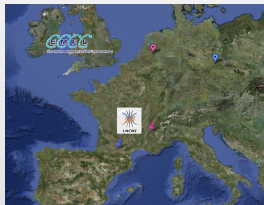


## High fields facilities in the world



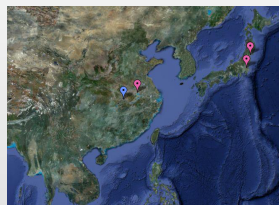
### United States

Thallahassee (FL)  
Los Alamos (NM)  
Gainesville (FL)



### Europe

Grenoble / Toulouse  
Nijmegen (Netherlands)  
Dresden (Germany)

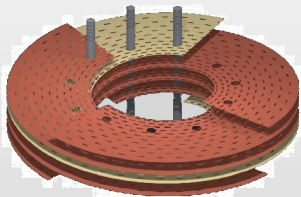


### Asia

Tsukuba (Japan)  
Sendai (Japan)  
Hefei (China)  
Wuhan (China)  
?? (Korea)



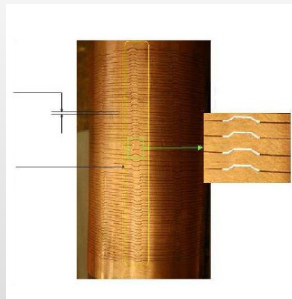
## Beyond 25 tesla... : Resistive high field Magnets



**Bitter**



**Long. cooled Helices**



**Radial. Helices**

### Some Figures

up to 30 kA, up to 24 MW,  
 water cooled 30 m/s, pressure drop 25 bars  
 up to 36 tesla in 34 mm bore

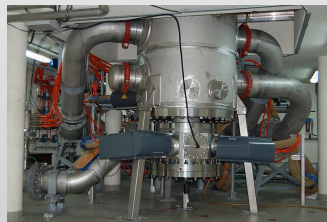
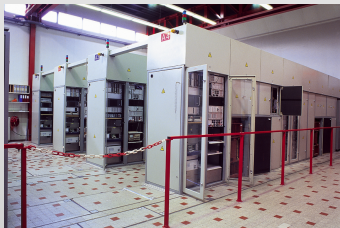


## LNCMI a User Facility run by the CNRS

Pulsed field installation TOULOUSE : 14 MJ, 24 kV, 1 GW, 80 Tesla



Continuous field installation GRENOBLE: 24 MW, 35 Tesla





# LNCMI a User Facility run by the CNRS Research

Solid State Physics

(Bio)Chemistry

MagnetoSciences

App. Superconductivity

Magnetic Levitation (high  $\nabla B^2$ )

Magneto-HydroDynamics

Magnet development

Facilities (10000 hours / year)

EPR, NMR

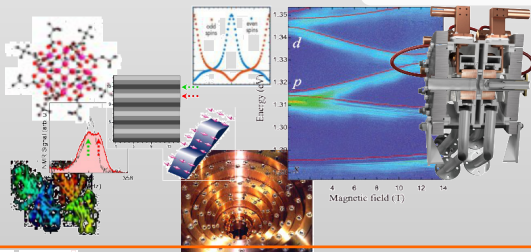
Lasers

Temp : [20 mK, 1600 °C]

High Pressure

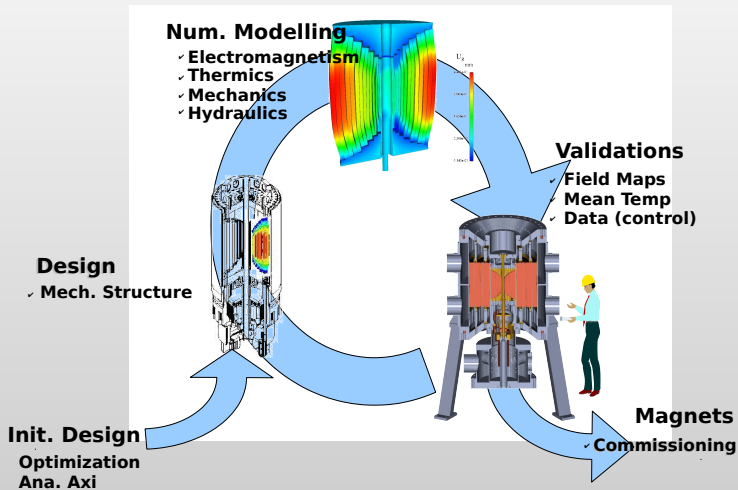
2 Calls for Magnet Time / year

140 projects



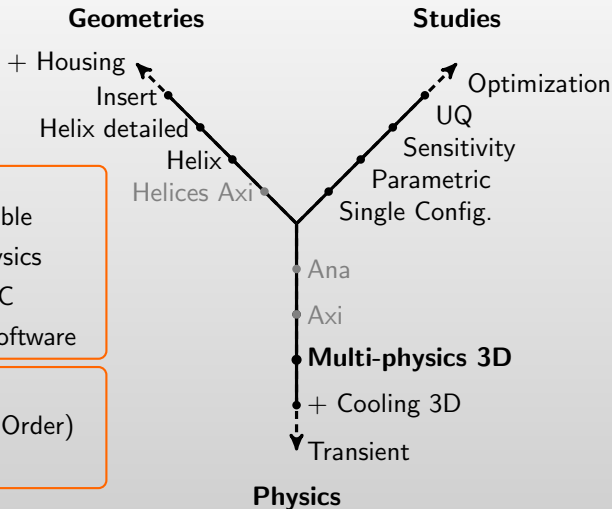


# From Design to Commissioning





# Project HiFiMagnet : Build a software chain for High Field Magnets



## Requirements

- Generic and extensible
- ROM for Multi-Physics
- From laptop to HPC
- Use Open Source software

## Choices

- FE Methods (High Order)
- Feel++ library



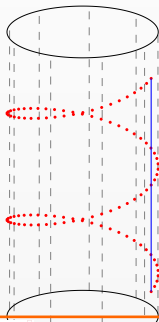
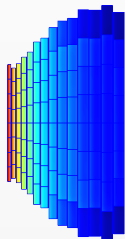
- 1 HiFiMagnet project
- 2 From optimization to CAD
- 3 Full 3D multi-physics model
  - Finite element model
  - Validation and use at LNCMI
  - Reduced order modeling



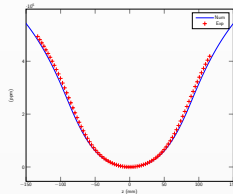
# MagnetTools

## Optimization under constraints

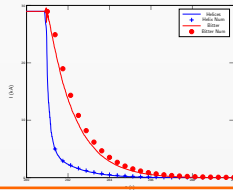
$$\text{Find } \{j_i\} \text{ s.t. } \begin{cases} \sum_i B_i(0,0) j_i & \text{is max} \\ P < P_{max} \\ T < T_{max} \\ r_i j_i B(r_i) < \sigma_{max} \end{cases}$$



## B profile : M9 36 tesla



## Transient : M9 powerfailure







# From Optimization to CAD

## Motivations

- Easier and Proper Insert Geometry generation
- Create Mesh within Salome
- Automatic BC groups selection
- Towards a GUI interface

## Salome Python scripts

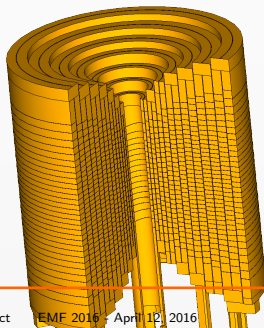
```
import salome

salome.salome_init()
theStudy = salome.myStudy

from LNCMIBuilder import *
Assembly = yaml.load(file(yamfile, 'r'))
(Helices, Rings, Leads) = LoadCfgs.Assembly(Assembly)
(Insert, BCs) = CreateAssembly(theStudy, Assembly)

(smash, Mesh) = MeshAssembly(theStudy, \
                             Insert, BCs, Assembly, \
                             Helices, Rings, Leads, addair)
```

## Insert





# A HiFiMagnet Module for Sa1ome

**SALOME 7** Geometry

Object Browser

- Name
- ▶ iRext361
- ▶ iRint362
- ▶ iRext362
- ▶ iRint363
- ▶ iRext363
- ▶ iRint364
- ▶ iRext364
- ▶ iRint365
- ▶ iRext365
- ▶ iRint366
- ▶ iRext366
- ▶ iRint367
- ▶ iRext367
- ▶ iRint368
- ▶ iRext368
- ▶ iRint369
- ▶ iRext369
- ▶ iRint370
- ▶ iRext370
- ▶ iRint371
- ▶ iRext371
- ▶ iRint372
- ▶ iRext372
- ▶ HChannel
- ▶ **muChannel**

Information

**Creation operation**

Create Group

Parameter	Value
Shape Type	Face
Main Shape	Compound[C...
Indices	2976 items: 4

NoteBook

Variable	Value
1	

Remove Update Study

Python Console

>>>



## A HiFiMagnet Module for Salome

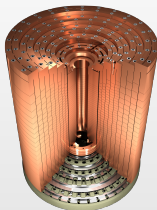
The screenshot displays the Salome 7 CAD software interface. The top toolbar includes icons for file operations, geometry, and meshing. The Object Browser on the left shows a hierarchical tree structure:

- Geometry
  - HL-31\_H1
  - Mesh
    - Hypotheses
    - Algorithms
    - HL-31\_H1
      - HL-31\_H1
      - Applied hypotheses
      - Applied algorithms
      - SubMeshes on Face
      - SubMeshes on Compound
      - Groups of Faces
        - V0
        - V1
        - rInt
        - rExt (highlighted)
        - rInt0
        - rExt0
        - Interface0
        - rInt1
        - rExt1
        - Interface1
      - Groups of Volumes
        - Domain\_1
        - Domain\_2
        - Domain\_3

The main 3D viewer shows a cylindrical magnet assembly with a yellow mesh on the top ring and a green mesh on the bottom ring. The Python Console at the bottom is empty.

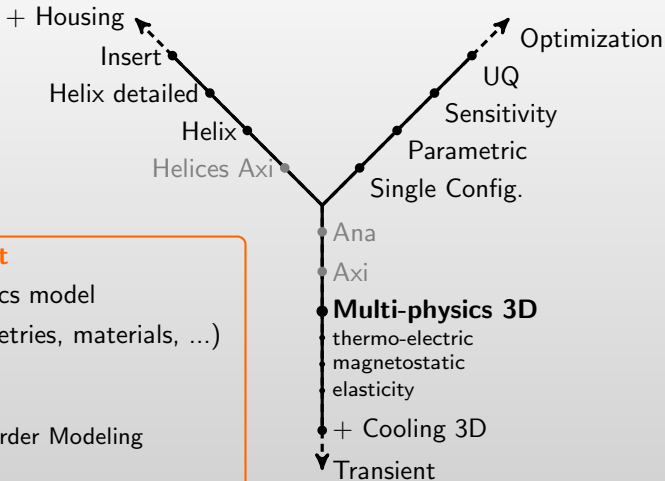


- 1 HiFiMagnet project
- 2 From optimization to CAD
- 3 Full 3D multi-physics model
  - Finite element model
  - Validation and use at LNCMI
  - Reduced order modeling



## Geometries

## Studies



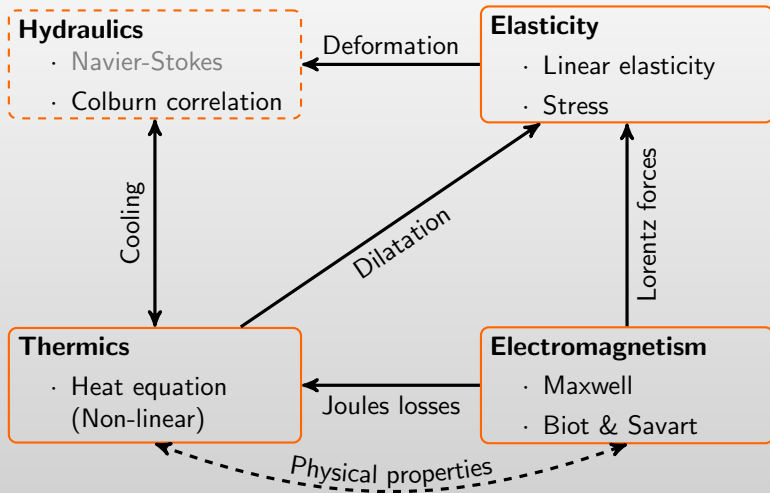
### HifiMagnet project

- 3D Multi-physics model
- Generic (Geometries, materials, ...)
- Efficient
  - HPC
  - Reduced Order Modeling
- Reliable
  - Uncertainty quantification

## Physics



## Involved physics - Coupling





## A 3D non-linear multi-physics FE model

### Feel++ (Finite Element Embedded Language)

- Arbitrary order Galerkin methods in 1D, 2D and 3D
- $H_{\text{div}}$  and  $H_{\text{curl}}$  conforming elements
- Syntax very close to mathematical language
- Simplex, hypercube and high order meshes
- Seamless parallel computing
- Large scale parallel linear and non-linear solvers



## De-Rham diagram

$$\begin{array}{ccccccc}
 \mathbb{R} & \xrightarrow{id} & H_1(\Omega) & \xrightarrow{grad} & H_{curl}(\Omega) & \xrightarrow{curl} & H_{div}(\Omega) & \xrightarrow{div} & L_2(\Omega) \\
 & & \downarrow \pi_{H_1} & & \downarrow \pi_{H_{curl}} & & \downarrow \pi_{H_{div}} & & \downarrow \pi_{L_2} \\
 \mathbb{R} & \xrightarrow{id} & U_{\mathcal{N}} & \xrightarrow{grad} & V_{\mathcal{N}} & \xrightarrow{curl} & W_{\mathcal{N}} & \xrightarrow{div} & Z_{\mathcal{N}}
 \end{array}$$

De-Rham diagram is fully supported in Feel++

$$\mathbb{R} \xrightarrow{id} H1 \xrightarrow{Igrad} Hcurl \xrightarrow{Icurl} Hdiv \xrightarrow{Idiv} L2$$

`H1 = Pch<k>(mesh); Hcurl = Nh<k=0>(mesh);`

`L2 = Pdh<0>(mesh); Hdiv = Dh<k=0>(mesh);`

`Igrad = Grad( _domainSpace=H1, _imageSpace=Hcurl);`

`Icurl = Curl( _domainSpace=Hcurl, _imageSpace=Hdiv);`

`Idiv = Div( _domainSpace=Hdiv, _imageSpace=L2);`

`B = curl(A) → B=Icurl(A);`





- 1 HiFiMagnet project
- 2 From optimization to CAD
- 3 Full 3D multi-physics model
  - Finite element model
  - Validation and use at LNCMI
  - Reduced order modeling



## Electro-thermal model

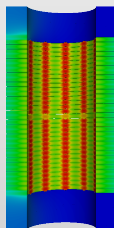
$V$  : electric potential [V]

$T$  : temperature [K]

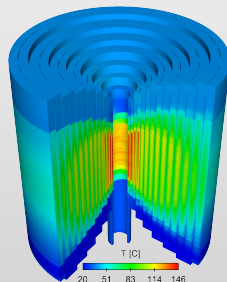
$$\begin{cases} -\nabla \cdot (\sigma(T) \nabla V) = 0 & \text{in } \Omega \\ -\nabla \cdot (k(T) \nabla T) = \sigma(T) \nabla V \cdot \nabla V & \text{in } \Omega \end{cases}$$

$$\sigma(T) = \frac{\sigma_0}{1 + \alpha(T - T_0)} \quad \xleftrightarrow[\text{Non linearity}]{\text{Material properties}} \quad k(T) = LT\sigma(T)$$

- **Applied potential**  
 $\rightarrow V = 0$  (in)  $V = V_D$  (out)
- **Water / Glue electrically isolant**  
 $\rightarrow -\sigma(T) \nabla V \cdot \bar{n} = 0$
- **No thermic exchange with air / glue**  
 $\rightarrow -k(T) \nabla T \cdot \bar{n} = 0$
- **Thermic exchange with cooling water**  
 $\rightarrow -k(T) \nabla T \cdot \bar{n} = h(T - T_w)$



(a) Radial



(b) Insert

Non-Linear Temperature



## Magnetostatic model

$\vec{j}$  : current density                       $\vec{H}$  : magnetic field intensity                       $\vec{B}$  : magnetic flux

$$\begin{cases} \nabla \cdot \vec{B} = 0 \\ \nabla \wedge \vec{H} = \vec{j} = -\sigma \nabla V \end{cases} \Rightarrow \exists \vec{A} \mid \begin{cases} \vec{B} = \nabla \wedge \vec{A} \\ \vec{B} = \mu \vec{H} \end{cases}$$

$$\nabla \wedge ( \mu(\vec{B})^{-1} \nabla \wedge \vec{A} ) = \vec{j}$$

Regularized formulation

$$\xrightarrow[\text{Time harmonic equation}]{\text{Fourier transform}} \nabla \wedge ( \mu(\vec{B})^{-1} \nabla \wedge \vec{A} ) + (\sigma i\omega - \epsilon\omega^2)\vec{A} = \vec{j}$$

$$\nabla \wedge ( \mu(\vec{B})^{-1} \nabla \wedge \vec{A}_\alpha ) + \alpha \vec{A}_\alpha = \vec{j} \quad \text{with} \quad \vec{A}_\alpha \xrightarrow{\alpha \rightarrow 0} \vec{A}$$

Saddle-point formulation

$$\xrightarrow[\text{Lagrange multiplier } p]{\text{Divergence - free condition}} \begin{cases} \nabla \wedge ( \mu(\vec{B})^{-1} \nabla \wedge \vec{A} ) + \nabla p = \vec{j} \\ \nabla \cdot \vec{A} = 0 \end{cases}$$

$$\nabla \wedge ( \mu(\vec{B})^{-1} \nabla \wedge \vec{A} ) + \nabla p = \vec{j} \quad \text{with} \quad \nabla \cdot \vec{A} = 0 \quad \text{and} \quad p = 0 \quad \text{on boundary}$$

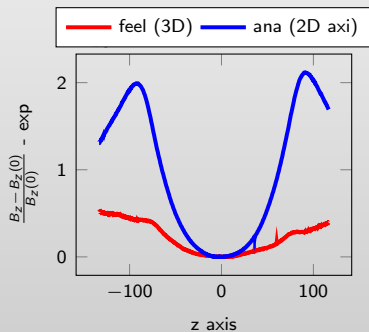
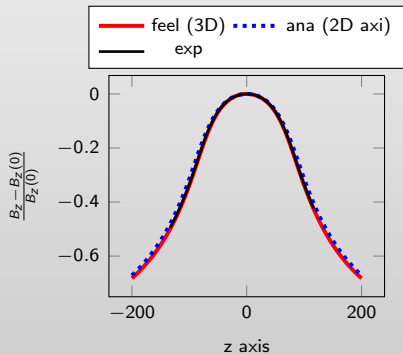
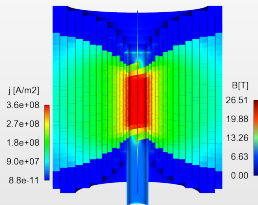
Auxiliary space preconditioner AMS (Hypre)



## Magnetostatic : Biot & Savart

Magnetic field computation outside the conductor

$$\vec{B}(r) = \frac{\mu_0}{4\pi} \int_{\Omega_{cond}} \frac{\vec{j}(r') \wedge (r - r')}{|r - r'|^3} dr' \text{ with } r \text{ in } \Omega_{mgn}$$





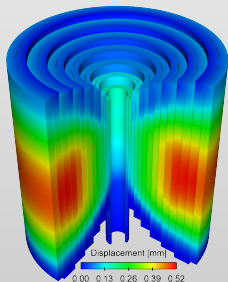
## Elasticity model

$$- \underbrace{\nabla \cdot \bar{\bar{\sigma}}(\bar{\bar{\epsilon}})}_{\text{includes thermal dilatation}} = \bar{\bar{f}} \text{ with } \bar{\bar{f}} = \underbrace{\bar{\bar{j}} \times \bar{\bar{B}}}_{\text{Lorentz forces}} \text{ and } \bar{\bar{\epsilon}} = \frac{1}{2}(\nabla \bar{\bar{u}} + \nabla \bar{\bar{u}}^T)$$

$$\bar{\bar{\sigma}}(\bar{\bar{\epsilon}}) = \frac{E}{1+\nu} \left( \bar{\bar{\epsilon}} + \frac{\nu}{1-2\nu} \text{Tr}(\bar{\bar{\epsilon}}) I \right) - \frac{E}{1-2\nu} \delta(T - T_0) I$$

Displacement condition :  $\bar{\bar{u}} = \bar{\bar{u}}_D$

Pressure condition :  $\bar{\bar{\sigma}} \cdot \bar{\bar{n}} = \bar{\bar{p}}$

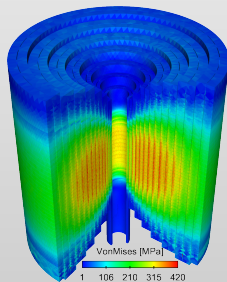


Yield strength - Tresca

$$tr = \max_{1 \leq i < j \leq Dim} (|\bar{\bar{\sigma}}_{ii}^d - \bar{\bar{\sigma}}_{jj}^d|)$$

Yield strength - Von-Mises

$$vm = \sqrt{\sum_{1 \leq i < j \leq Dim} \frac{1}{2} \left( \bar{\bar{\sigma}}_{ii}^d - \bar{\bar{\sigma}}_{jj}^d \right)^2}$$





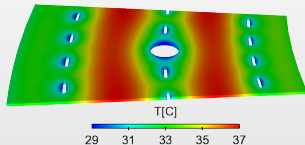
- 1 HiFiMagnet project
- 2 From optimization to CAD
- 3 Full 3D multi-physics model
  - Finite element model
  - Validation and use at LNCMI
  - Reduced order modeling



## Cooling of Bitter magnet

Heat transfer coefficient  $h$  :

- Standard hydraulic correlation
- Hard to measure accurately
- $-k(T)\nabla T \cdot \vec{n} = h(T - T_w)$  on cooled surfaces



Available measures :

- Intensity  $I_{ex} = \int_S -\sigma(T)\nabla V \cdot \mathbf{n}$
- Voltage  $V_D$

Electro-thermal model with boundary conditions

$$V = 0 \text{ (in)}, V = V_D \text{ (out)}$$

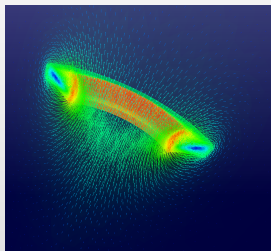
$$-k(T)\nabla T \cdot \vec{n} = h(T - T_w) \text{ on cooled surfaces}$$

Find  $h$  such that electro-thermal model solution  $(V, T)$  gives  $I(V, T) = I_{ex}$

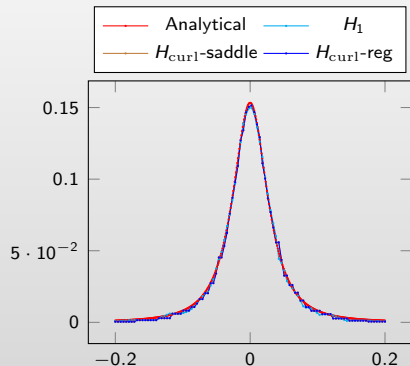
→ Compare  $h$  with  $h_{ex}$  given by hydraulic correlation



# Magnetostatic model



	$B_z(0)$	$\frac{ B_z(0)^{ana} - B_z(0) }{B_z(0)^{ana}}$
Biot & Savart	0.1533	$6,76 \cdot 10^{-5}$
$H_1$	0.1498	$2.24 \cdot 10^{-2}$
$H_{curl-saddle}$	0.1516	$1.12 \cdot 10^{-2}$
$H_{curl-reg}$	0.1515	$1.18 \cdot 10^{-2}$

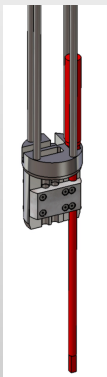


$$B_z(0)^{ana} = 0.1533$$

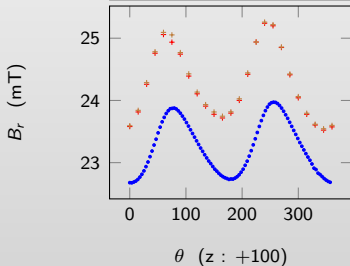




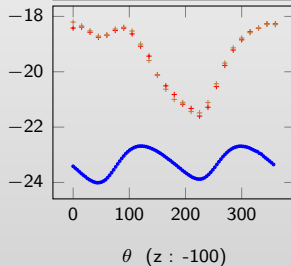
## Measure magnetic field out of z axis



+ 16/12/15    + 17/12/15  
 • Biot & Savart



+ 16/12/15    + 17/12/15  
 • Biot & Savart



- Experimental process/device to be improved
- To be tested on a new magnet



## Full 3D multi-physics model on polyhelix magnet

### Simulation setup

Nb elements :  $27 \cdot 10^6$

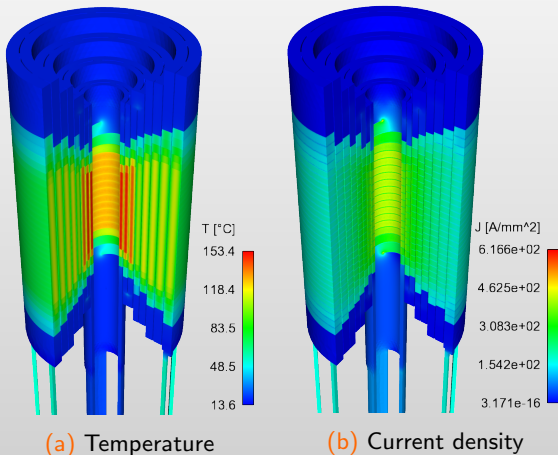
Edges :  $31,6 \cdot 10^6$

Vertices :  $4,3 \cdot 10^6$

48 procs

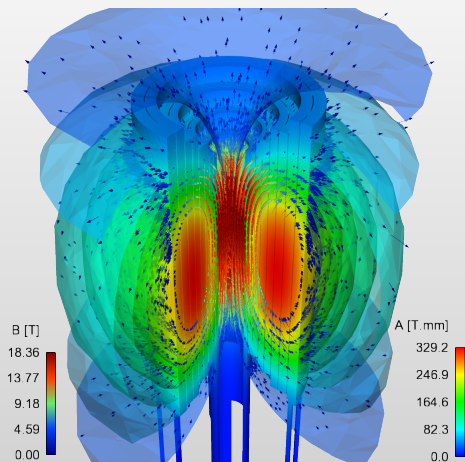
### Ressources

- Global time
  - $H_{\text{curl}}$ : 4695 s
  - $H_1$ : 1640 s
- Memory (RAM) : 785 Gb

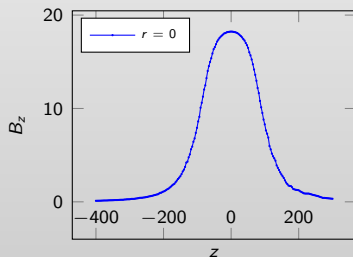
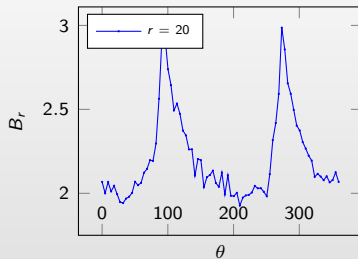




## Full 3D multi-physics model on polyhelix magnet

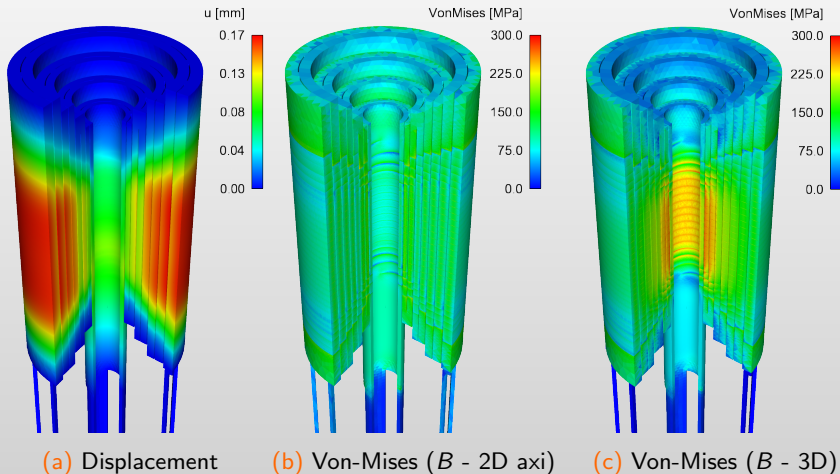


(a) 3D magnetic field





## Full 3D multi-physics model on polyhelix magnet





- 1 HiFiMagnet project
- 2 From optimization to CAD
- 3 Full 3D multi-physics model
  - Finite element model
  - Validation and use at LNCMI
  - Reduced order modeling



## Improve model efficiency : Reduced Basis Method

Weak formulation of the model :  $a(u(\mu), v; \mu) = f(v; \mu)$

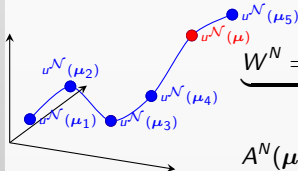
Quantity of interest :  $s(\mu) = \ell(u(\mu)) \implies$  **Evaluation for several  $\mu \in \mathcal{D} \subset \mathbb{R}^p$**

### FEM approximation

$$\underbrace{X^{\mathcal{N}} = \text{span}\{\phi_1, \dots, \phi_{\mathcal{N}}\}}_{\text{FEM approximation space}} \longrightarrow \underbrace{u^{\mathcal{N}}(\mu) = \sum_{i=1}^{\mathcal{N}} u_i^{\mathcal{N}} \phi_i}_{\mathcal{N} \approx 10^6} \longrightarrow \underbrace{A^{\mathcal{N}}(\mu) u^{\mathcal{N}}(\mu) = F^{\mathcal{N}}(\mu)}_{\substack{\mathcal{N} \times \mathcal{N} \text{ system } \mathbf{expensive} \text{ to solve} \\ \approx 10 \text{ min on } 32 \text{ procs}}}$$

### Objective : reduce the computational cost

$u^N(\mu) \approx u^{\mathcal{N}}(\mu)$  : linear combination of FEM solution



$$\underbrace{W^{\mathcal{N}} = \text{span}\{u^{\mathcal{N}}(\mu_1), \dots, u^{\mathcal{N}}(\mu_N)\}}_{\text{RB approximation space}} \longrightarrow \underbrace{u^N(\mu) = \sum_{i=1}^N u_i^N(\mu) \underline{u}^{\mathcal{N}}(\mu_i)}_{10 \leq N \leq 100}$$

$A^{\mathcal{N}}(\mu) u^{\mathcal{N}}(\mu) = F^{\mathcal{N}}(\mu) \longrightarrow$   **$N \times N$  system cheaper to solve**

## Efficient offline-online strategy

$$u^N(\boldsymbol{\mu}) = \sum_{i=1}^N u_i^N(\boldsymbol{\mu}) \underline{u}^N(\boldsymbol{\mu}_i)$$

$u^N(\boldsymbol{\mu})$  satisfies weak formulation :  $a(u^N(\boldsymbol{\mu}), v; \boldsymbol{\mu}) = f(v; \boldsymbol{\mu})$

$N \times N$  system to solve :  $\sum_{i=1}^N a(\underline{u}^N(\boldsymbol{\mu}_i), v_k; \boldsymbol{\mu}) u_i^N(\boldsymbol{\mu}) = f(v_k; \boldsymbol{\mu}), 1 \leq k \leq N$

## Affine decomposition

$$a(u, v; \boldsymbol{\mu}) = \sum_{q=1}^{Q_a} \theta_a^q(\boldsymbol{\mu}) a^q(u, v) \quad \text{and} \quad f(v; \boldsymbol{\mu}) = \sum_{q=1}^{Q_f} \theta_f^q(\boldsymbol{\mu}) f^q(v)$$

$$\Rightarrow \sum_{i=1}^N \left[ \sum_{q=1}^{Q_a} \theta_a^q(\boldsymbol{\mu}) \underbrace{a^q(\underline{u}^N(\boldsymbol{\mu}_i), \underline{u}^N(\boldsymbol{\mu}_j))}_{\text{precomputed}} \right] u_i^N(\boldsymbol{\mu}) = \sum_{q=1}^{Q_f} \theta_f^q(\boldsymbol{\mu}) \underbrace{f^q(\underline{u}^N(\boldsymbol{\mu}_j))}_{\text{precomputed}}$$

## What if the problem is non affine ?

Empirical Interpolation Method (EIM)  $\rightarrow w_M(u, \mathbf{x}; \boldsymbol{\mu}) = \sum_{m=1}^M \beta_m^M(u; \boldsymbol{\mu}) q_m(\mathbf{x}) \approx w(u, \mathbf{x}; \boldsymbol{\mu})$



## Uncertainty quantification - Mean temperature in a helix

$$\sigma_0 \in [50; 50.2](MS.m^{-1}) \quad L \in [2.5; 2.9] \times 10^{-8} \quad \alpha \in [0.33\%; 0.35\%](K^{-1})$$

$$I \in [25000; 35000](A) \quad h \in [70000; 90000](W.m^{-2}.K^{-1}) \quad T_w \in [293, 313](K)$$

### Configuration

500 000 degrees of freedom - 12 processors

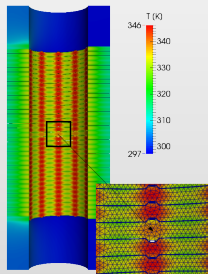
Computational time : Gain  $\times$  150

FE  $\approx$  16min;      RB : 6.7s.

### Variations

Mean temperature :  $T = 375.5K \approx 103^\circ C$

Standard deviation : 12



Quantiles :  $q(\gamma) \mid P(Y < q(\gamma)) > \gamma$

401.3 K  $\approx$  129  $^\circ C$  : 99.0 (quantile)

385.3 K  $\approx$  113  $^\circ C$  : 80.0 (quantile)

### Sensitivity analysis

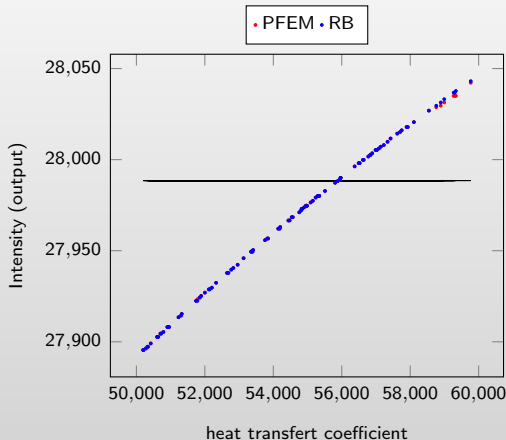
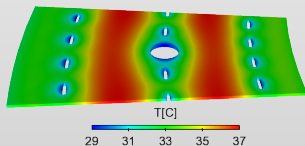
$I$	:	<b>0.75</b>	$L$	:	0.0028
$T_w$	:	<b>0.16</b>	$\alpha$	:	0.00023
$h$	:	<b>0.069</b>	$\sigma_0$	:	0.000019





# Cooling on Bitter magnet - Find $I(V, T)$ which minimizes $f = I - I_{ex}$

- EIM :  $M = 25$   
with sampling size = 100
- RB :  $N = 25$   
with sampling size = 1000
- Input :  $h$
- Output : Intensity



$$h = 5,58 \cdot 10^4 \approx h_{ex} = 5,46 \cdot 10^4 - \text{mean error on output} \approx 10^{-2}$$



## How to reduce EIM (offline) cost ?

- Number of FE approximation required : proportional to EIM trainset size  
→ EIM trainset can be large → **EIM parameter selection can be costly !**
- Proposed solution : use  $u^N$  to select  $\bar{\mu}_m$  (Greedy algorithm)

$$\bar{\mu}_m = \arg \max_{\mu \in \Xi} \inf_{z \in W_{m-1}} \|w(u^{X \rightarrow N}(\mu); \cdot; \mu) - z\|_{L^\infty(\Omega)}$$

Alternate build of EIM and RB basis functions in offline step

### Simultaneous Empirical interpolation and Reduced basis method (SER)

- Offline (standard basis build) :  $N + (M \times \text{Card}(\Xi_{train}))$  FE approximations
  - Offline (simultaneous basis build) :  **$N + 1$  FE approximations**
- SER efficiency validated on 2D benchmark [Daversin and Prud'Homme, 2015]  
→ Works with 3D electro-thermal model



## Conclusion

- Full 3D multi-physics model
  - Use on real magnet geometries
  - Validation still in progress, needs experimental data
  - Reduced model (electro-thermal) to be completed
- In progress...
  - High order  $H_{\text{div}}/H_{\text{curl}}$ -conforming elements
  - Refactoring of the code towards more user-friendly interface
  - HDG method
- In the future
  - Reproduce at best magnet setting up process
  - Investigate possibility to improve cooling modeling (hydraulic model)
  - Mesh adaptation (parallel) to optimize size of the problem

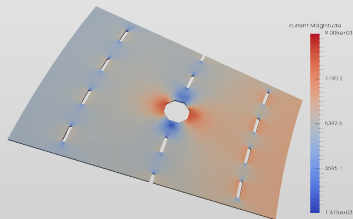


## Hybridized Discontinuous Galerkin (HDG) Methods

Global problem on $\Omega$	+	Boundary conditions on $\delta\Omega$
	$\Downarrow$	
Many local problems ( $\forall K$ )	+	Boundary conditions $\forall F \in \delta\Omega$
	+	Transmission conditions $[[u _{\kappa}]] = 0 \forall F \in \delta\Omega$

### Motivation

- Optimal approximation of both the primal and flux variables.



### Electro-thermal model




- Current density is solution of local problems

→ Better approx. of  $\vec{j}$

→ Better approx. of  $I = \int_S \vec{j} \cdot \mathbf{n}$



## Reference

-  Daversin, C. and Prud'Homme, C. (2015).  
Simultaneous empirical interpolation and reduced basis method for non-linear problems.  
*C. R. Acad. Sci. Paris, Ser. I*, 353:1105–1109.
-  Daversin, C., Prudhomme, C., and Trophime, C. (2016).  
Full three-dimensional multiphysics model of high-field polyhelices magnets.  
*IEEE transactions on applied superconductivity*, 26(4).
-  Daversin, C., Veys, S., Trophime, C., and Prud'Homme, C. (2013).  
A reduced basis framework: Application to large scale nonlinear multi-physics problems.  
*Esaim Proc.*, 43:225–254.

## Non-linear and non-affine problems

Non-linearity : Iterative method  
→ Picard → Newton

**Offline** : Basis build

for  $\mu \in S_N$  do

${}^0 u^N \leftarrow \text{initialGuess}(\mu)$

while increment > tol do

$$A_{i,j}^N = \sum_{q=1}^{Q_a^{eim}} \sum_{m=1}^{M_q^a} \gamma_{a,m}^q({}^k u^N, \mu) a_m^q(\phi_j, \phi_i) + \dots$$

$$F_j^N = \sum_{q=1}^{Q_f^{eim}} \sum_{m=1}^{M_q^f} \gamma_{f,m}^q({}^k u^N, \mu) f_m^q(\phi_j) + \dots$$

$${}^{k+1} u^N(\mu) \leftarrow \text{solve } A^N {}^k u^N(\mu) F^N$$

end while

$$W_N \leftarrow W_N \cup {}^k u^N(\mu)$$

end for

**Offline** : Pre-computations

$$A_{i,j}^{N,q,m} = a_m^q(\underline{u}^N(\mu_j), \underline{u}^N(\mu_i))$$

$$F_j^{N,q,m} = f_m^q(\underline{u}^N(\mu_j))$$

**Online**

${}^0 u^N \leftarrow \text{initialGuess}(\mu), k \leftarrow 0$

while increment > tol do

$$A_{i,j}^N = \sum_{q=1}^{Q_a^{eim}} \sum_{m=1}^{M_q^a} \gamma_{a,m}^q({}^k u^N, \mu) \underbrace{A_{i,j}^{N,q,m}}_{\text{precomputed}}$$

+ ...

$$F_j^N = \sum_{q=1}^{Q_f^{eim}} \sum_{m=1}^{M_q^f} \gamma_{f,m}^q({}^k u^N, \mu) \underbrace{F_j^{N,q,m}}_{\text{precomputed}}$$

+ ...

$${}^{k+1} u^N \leftarrow \text{solve } A^N {}^k u^N(\mu) F^N$$

end while